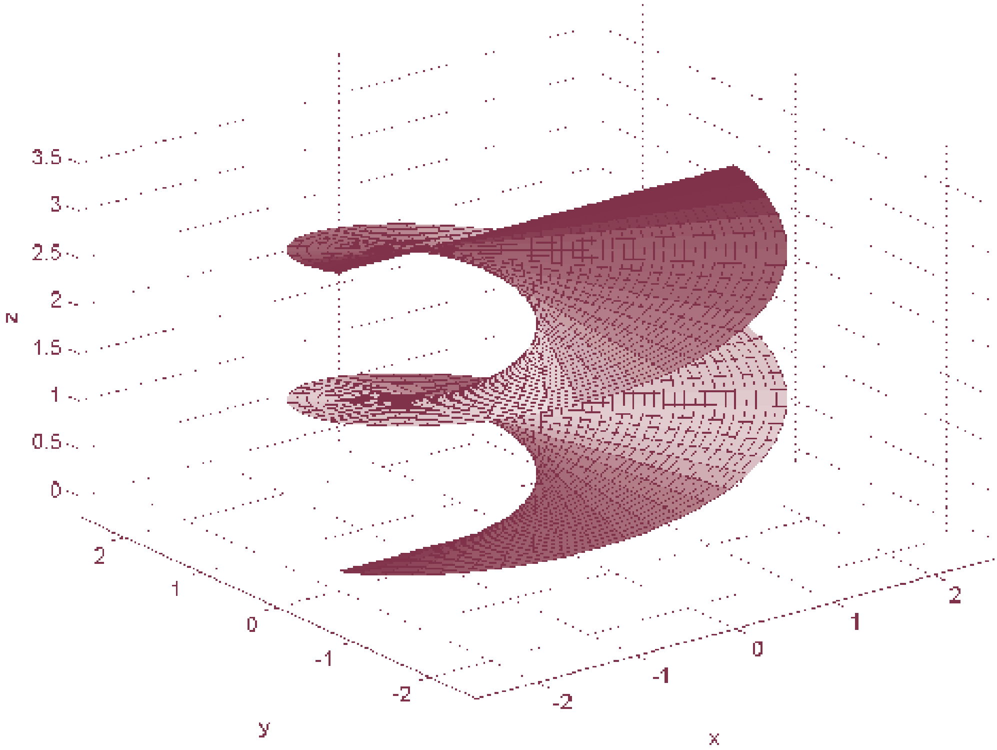
STUDIA UNIVERSITATIS
BABEŞ-BOLYAI



MATHEMATICA

3/2017

STUDIA
UNIVERSITATIS BABEŞ-BOLYAI
MATHEMATICA

3/2017

YEAR (LXII) 2017
MONTH SEPTEMBER
ISSUE 3

S T U D I A
UNIVERSITATIS BABEŞ-BOLYAI

MATHEMATICA
3

Redacţia: 400084 Cluj-Napoca, str. M. Kogălniceanu nr. 1
Telefon: 0264 405300

CONTENTS

Stud. Univ. Babeş-Bolyai Math. 62(2017), No. 3, 275–286
DOI: 10.24193/subbmath.2017.3.01

Spectral characterization of new classes of multicone graphs

Seyed Morteza Mirafzal and Ali Zeydi Abdian

Abstract. This paper deals with graphs that are known as multicone graphs. A multicone graph is a graph obtained from the join of a clique and a regular graph. Let w, l, m be natural numbers and k is a natural number. It is proved that any connected graph cospectral with multicone graph $K_w \bigtriangledown mECP_l^k$ is determined by its adjacency spectra as well as its Laplacian spectra, where $ECP_l^k = K_{\underbrace{3^k,\ 3^k,\ \ldots,\ 3^k}_{l\ times}}$. Also, we show that complements of some of these multicone graphs are determined by their adjacency spectra. Moreover, we prove that any connected graph cospectral with these multicone graphs must be perfect. Finally, we pose two problems for further researches.

Mathematics Subject Classification (2010): 05C50.

Keywords: Adjacency spectrum, Laplacian spectrum, DS graph.

1. Introduction

All graphs considered here are simple and undirected. All notions on graphs that are not defined here can be found in [4, 5, 10, 12, 19]. Let Γ be a graph with n vertices, $V(\Gamma)$ and $E(\Gamma)$ be the sets of vertices and edges of Γ, respectively. The complement of a graph Γ, denoted by $\overline{\Gamma}$, is the graph on the vertices set of Γ such that two vertices of $\overline{\Gamma}$, are adjacent if and only if they are not adjacent in Γ. The union of (disjoint) graphs Γ_1 and Γ_2 is denoted by $\Gamma \cup \Gamma_2$, is the graph whose vertices (respectively, edges) set is the union of vertices (respectively, edges) set of Γ_1 and Γ_2. A graph consisting of k disjoint copies of an arbitrary graph Γ will be denoted by $k\Gamma$. The join of two vertex disjoint graphs Γ_1 and Γ_2 is the graph obtained from $\Gamma_1 \cup \Gamma_2$ by joining each vertex in Γ_1 with every vertex in Γ_2. It is denoted by $\Gamma_1 \bigtriangledown \Gamma_2$. Let Γ be a graph with adjacency matrix $A(\Gamma)$. The characteristic polynomial of Γ is $\det(\lambda I - A(\Gamma))$, and denoted by $P_\Gamma(\lambda)$. The roots of $P_\Gamma(\lambda)$ are called the adjaceny eigenvalues of $A(\Gamma)$. The eigenvalues and the spectrum of $A(\Gamma)$ are also called the eigenvalues and the spectrum of Γ, respectively. If we consider a matrix $L = D - A$ instead of A, where D is the diagonal matrix of degree of vertices (in Γ), we get the Laplacian eigenvalues and the Laplacian spectrum, while in the case of matrix $SL(G) = D(\Gamma)+A(\Gamma)$, we get the signless Laplacian eigenvalues and the signless Laplacian spectrum, respectively. Since both matrices $A(\Gamma)$ and $L(\Gamma)$ are real symmetric matrices, their eigenvalues are all real numbers. Let $\lambda_1, \lambda_2, \ldots, \lambda_s$ be the distinct eigenvalues of Γ with multiplicities $m_1, m_2, \ldots, m_s$, respectively. We denote the adjacency spectrum of Γ by $Spec(\Gamma) = \{[\lambda_1]^{m_1}, [\lambda_2]^{m_2}, ..., [\lambda_s]^{m_s}\}$. Two graphs Γ and Λ are called cospectral, if $Spec(\Gamma) = Spec(\Lambda)$. A graph Γ is said to be determined by its spectrum or DS for short, if $Spec(\Gamma) = Spec(\Lambda)$, follows that $\Gamma \cong \Lambda$. About the background of the guestion "which graphs are determined by their spectrums?", we refer to [15]. The friendship graph F_n consists of n edge-disjoint triangles that all of them meeting in one vertex, where n is a natural number (see Figure 1). The friendship (or Dutch windmill or n-fan) graph F_n is the graph that can be constructed by coalesencing n copies of the cycle graph C_3 of length 3 with a common vertex. By construction, the friendship graph F_n is isomorphic to the windmill graph Wd(3,n) [11]. The friendship theorem of Paul Erdös, Alfred Réyni and Vera T. Sós [12], states that graphs with the property that every two vertices have exactly one neighbour in common are exactly the friendship graphs. In [17, 18], it has been proposed that the friendship graph is DS with respect to its adjacency spectrum. This conjecture studied in [2, 8]. It is claimed in [8] that conjecture is valid. In [7], it is proved that if Γ is any graph cospectral with F_n $(n \neq 16)$, then $\Gamma \cong F_n$. Abdollahi and Janbaz [3] precented a proof in special case of this topic. They proved that any connected graph cospectral with F_n is isomorphic to F_n. Abdian and Mirafzal [1] characterized new classes of multicone graphs. In this paper, we present new classes of multicone graphs that friendship graphs are special classes of them and we show these graphs are DS with respect to their spectra. The plan of the present paper is as follows. In Section 2, we review some basic information and preliminaries. In Subsection 3.1, we show that any connected graph cospectral with multicone graph $K_w \bigtriangledown mECP_l^k$ (see Figures 1 and 2, for example) must be regular or bidegreed (Lemma 3.2). In Subsection 3.2, we prove that any connected graphs cospectral with $K_w \bigtriangledown mECP_l^k$ is determined by its adjacency spectra (Theorem 3.4). In Subsection 3.3, we prove that complement of $K_w \bigtriangledown mECP_l^k$ is DS with respect to their adjacency spectra (Theorem 3.7). In Subsection 3.4, we show that graphs $K_w \bigtriangledown mECP_l^k$ are DS with respect to their Laplacian spectra (Theorem 3.8). In Subsection 3.5, we show that any connected graph cospectral with multicone graph $K_w \bigtriangledown mECP_l^k$ must be perfect. We conclude with final remarks and open problems in Section 4.

2. Preliminaries

In this section, we give some facts that will be used in the proof of the main results.

A walk of length m in a graph $\Gamma(V, E)$ is an alternating sequence:

$$v_1 l_1 v_2 l_2 v_3 v_n l_m v_{m+1}$$

of vertices and edges that begins and ends with a vertex and has the added property that l_j is incident with both v_i and v_{i+1}, where $1 \leq i \leq m+1$ and $1 \leq j \leq m$. In graph $\Gamma(V, E)$ a walk of length m is closed, if $v_1 = v_{m+1}$.

Lemma 2.1. ([2, 14]) *Let Γ be a graph. For the adjacency matrix and Laplacian matrix, the following can be obtained from the spectrum:*
(i) The number of vertices,
(ii) The number of edges.
For the adjacency matrix, the following follows from the spectrum:
(iii) The number of closed walks of any length.
(iv) Being regular or not and the degree of regularity.
(v) Being bipartite or not.
For the Laplacian matrix, the following follows from the spectrum:
(vi) The number of spanning trees.
(vii) The number of components.
$(viii)$ The sum of squares of degrees of vertices.

Theorem 2.2. ([5]) *If Γ_1 is r_1-regular with n_1 vertices, and Γ_2 is r_2-regular with n_2 vertices, then the characteristic polynomial of the join $\Gamma_1 \triangledown \Gamma_2$ is given by:*

$$P_{\Gamma_1 \triangledown \Gamma_2(\lambda)} = \frac{P_{\Gamma_1}(\lambda) P_{\Gamma_2}(\lambda)}{(\lambda - r_1)(\lambda - r_2)}((\lambda - r_1)(\lambda - r_2) - n_1 n_2).$$

Proposition 2.3. ([5]) *Let $\Gamma - j$ be the graph obtained from Γ by deleting the vertex j and all edges containing j. Then $P_{\Gamma - j}(\lambda) = P_\Gamma(\lambda) \sum_{i=1}^{m} \frac{\alpha_{ij}^2}{\lambda - \mu_i}$, where m, α_{ij}^2 and $P_\Gamma(\lambda)$ are the number of distinct eigenvalues of graph Γ, the main angle of Γ and the characteristic polynomial of Γ.*

A graph is bidegreed if the set of degrees of its vertices consists of exactly two distinct elements. Also, the spectral radius $\varrho(\Gamma)$ of Γ is the largest eigenvalue of its adjacency matrix $A(\Gamma)$.

Theorem 2.4. ([3]) *Let Γ be a simple graph with n vertices and m edges. Let $\delta = \delta(\Gamma)$ be the minimum degree of vertices of Γ and $\varrho(\Gamma)$ be the spectral radius of the adjacency matrix of Γ. Then*

$$\varrho(\Gamma) \leq \frac{\delta - 1}{2} + \sqrt{2m - n\delta + \frac{(\delta + 1)^2}{4}}.$$

Equality holds if and only if Γ is either a regular graph or a bidegreed graph in which each vertex is of degree either δ or $n - 1$.

A t-multipartite graph of order n is $K_{b_1,\ldots,b_t}$, where $b_1 + \ldots + b_t = n$. D. Cvetković, Doob and S. Simić [6] defined a generalized cocktail-party graph, denoted by GCP, as a complete graph with some independent edges removed. A special case of this graph is the well-known cocktail-party graph $CP(t)$ obtained from K_{2t} by removing t disjoint edges.

Theorem 2.5. ([1]) *A graph has exactly one positive eigenvalue if and only if its non-isolated vertices form a complete multipartite graph.*

Lemma 2.6. ([1]) *Let Γ be a connected non-regular graph with three distinct eigenvalues $\theta_0 > \theta_1 > \theta_2$. Then the following hold:*
(i) Γ has diameter two.
(ii) If θ_0 is not an integer, then Γ is complete bipartite.
(iii) $\theta_1 \geq 0$ with equality if and only if Γ is complete bipartite.
(iv) $\theta_2 \leq -\sqrt{2}$ with equality if and only if Γ is the path of length 2.

Proposition 2.7. ([12]) *For a graph Γ, the following statements are equivalent:*
(i) Γ is d-regular.
(ii) $\varrho(\Gamma) = d_\Gamma$, the average vertex degree.
(iii) G has $v = (1, 1, ..., 1)^t$ as an eigenvector for $\varrho(\Gamma)$.

Proposition 2.8. ([16]) *Let Γ be a disconnected graph that is determined by the Laplacian spectrum. Then the cone over Γ, the graph Λ; that is, obtained from Γ by adding one vertex that is adjacent to all vertices of Γ, is also determined by its Laplacian spectrum.*

Lemma 2.9. ([13]) *Let Γ be a graph on n vertices. Then n is Laplacian eigenvalue of Γ if and only if Γ is the join of two graphs.*

Theorem 2.10. ([13]) *Let Γ and Λ be two graphs with Laplacian spectrum $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$ and $\mu_1 \geq \mu_2 \geq .. \geq \mu_m$, respectively. Then the Laplacian spectra of $\overline{\Gamma}$ and $\Gamma \bigtriangledown \Lambda$ are $n-\lambda_1, n-\lambda_2, ..., n-\lambda_{n-1}, 0$ and $n+m, m+\lambda_1, ..., m+\lambda_{n-1}, n+\mu_1, ..., n+\mu_{m-1}, 0$, respectively.*

Lemma 2.11. ([12]) *Let $G \neq K_1$ be connected with $P_\Gamma(\lambda) = \sum_{i=0}^{n} a_i \lambda^{n-i}$ and $\lambda = \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n = \varrho(\Gamma)$, where $P_\Gamma(\lambda)$ is the characteristic polynomial of graph Γ and λ_i ($1 \leq i \leq n$) is eigenvalue of Γ. The following are equivalent:*
(i) G is bipartite.
(ii) $a_{2i-1} = 0$ for all $1 \leq i \leq \lceil \frac{n}{2} \rceil$.
(iii) $\lambda_i = -\lambda_{n+1-i}$ for $1 \leq i \leq n$.
(iv) $\varrho(\Gamma) = -\lambda$.
Moreover, $m(\lambda_i) = m(-\lambda_i)$, where $m(\lambda_i)$ denote the multiplicities of λ_i.

3. Main results

In the following, we show that any connected graph cospectral with multicone graphs $K_w \bigtriangledown mECP_l^k$ are regular or bidegreed.

3.1. Connected bidegreed graph cospectral with multicone graphs $K_w \bigtriangledown mECP_l^k$

Proposition 3.1. *Let G be a graph cospectral with multicone graphs $K_w \bigtriangledown mECP_l^k$. Then*

$$Spec(G) = \left\{ [0]^{(3^k l - l)m}, [-1]^{w-1}, \left[3^k l - 3^k\right]^{m-1}, \left[-3^k\right]^{lm-m}, \left[\frac{\chi + \sqrt{\chi^2 - 4\Theta}}{2}\right]^1, \left[\frac{\chi - \sqrt{\chi^2 - 4\Theta}}{2}\right]^1 \right\},$$

where $\chi = w - 1 + 3^k l - 3^k$ and $\Theta = (w-1)(3^k l - 3^k) - 3^k lwm$.

Proof. By Theorem 2.2 and $Spec(mECP_l^k) = \left\{ \left[3^k l - 3^k\right]^m, [0]^{3^k lm - lm}, \left[-3^k\right]^{lm-m} \right\}$ the proof is completed. □

In the following, we show that any graph cospectral with a multicone graph $K_w \bigtriangledown mECP_l^k$ must be bidegreed.

Lemma 3.2. *Let* Γ *be a connected graph cospectral with multicone graph* $K_w \bigtriangledown mECP_l^k$. *Then* Γ *is bidegreed in which any vertex of* Γ *is of degree* $w - 1 + 3^k lm$ *or* $3^k l - 3^k + w$.

Proof. It is obvious that Γ cannot be regular; since regularity of a graph can be determined by its spectrum. By contrary, we suppose that the sequence of degrees of vertices of graph Γ consists of at least three number. Hence the equality in Theorem 2.4 cannot happen for any δ. But, if we put $\delta = 3^k l - 3^k + w$, then the equality in Theorem 2.4 holds. So, Γ must be bidegreed. Now, we show that $\Delta = \Delta(\Gamma) = w - 1 + 3^k lm$. By contrary, we suppose that $\Delta < w - 1 + 3^k lm$. Therefore, the equality in Theorem 2.4 cannot hold for any δ. But, if we put $\delta = 3^k l - 3^k + w$, then this equality holds. This is a contradiction and so $\Delta = 3^k l - 3^k + w$. Now, $\delta = 3^k l - 3^k + w$, since Γ is bidegreed and Γ has $w + 3^k lm$, $\Delta = w - 1 + 3^k lm$ and

$$w(w - 1 + 3^k lm) + 3^k lm(3^k l - 3^k + w) = w\Delta + 3^k lm(3^k l - 3^k + w) = \sum_{i=1}^{w+3^k lm} \deg v_i.$$

This completes the proof. □

3.2. Spectral characterization of connected graphs cospectral with multicone graphs $K_1 \bigtriangledown mECP_l^k$.

In this subsection, we show that multicone graphs $K_1 \bigtriangledown mECP_l^k$ are DS.

Lemma 3.3. *Any connected graph cospectral with multicone graph* $K_1 \bigtriangledown mECP_l^k$ *is isomorphic to* $K_1 \bigtriangledown mECP_l^k$.

Proof. Let Γ be a graph cospectral with multicone graph $K_1 \bigtriangledown mECP_l^k$. If $m = 1$ there is nothing to prove. Hence we suppose that $m \neq 1$. It is obvious that in this case Γ cannot be regular. First we show that Γ has one vertex of degree $\Delta = 3^k lm$ and $3^k lm$ vertices of degree $\delta = 3^k l - 3^k + 1$. Let G has t vertex of degree $\Delta = 3^k lm$. Hence

$$t3^k lm + (3^k lm + 1 - t)(3^k l - 3^k + 1) = 3^k lm + 3^k lm(3^k l - 3^k + 1) = \sum_{i=1}^{1+3^k lm} \deg v_i$$

and so $t = 1$. Therefore, Γ has one vertex of degree $\Delta = 3^k lm$, say j. It follows from Proposition 2.3 that

$$\begin{aligned} P_{\Gamma - j}(\lambda) = {} & (\lambda - \mu_3)^{m-2}(\lambda - \mu_4)^{lm-m-1}(\lambda - \mu_5)^{3^k lm - lm - 1} \\ & \times[\alpha_{1j}^2 F + \alpha_{2j}^2 G + \alpha_{3j}^2 H + \alpha_{4j}^2 I + \alpha_{5j}^2 J], \end{aligned}$$

where

$$\begin{aligned} F &= (\lambda - \mu_2)(\lambda - \mu_3)(\lambda - \mu_4)(\lambda - \mu_5), \\ G &= (\lambda - \mu_1)(\lambda - \mu_3)(\lambda - \mu_4)(\lambda - \mu_5), \end{aligned}$$

$$H = (\lambda - \mu_1)(\lambda - \mu_2)(\lambda - \mu_4)(\lambda - \mu_5),$$

$$I = (\lambda - \mu_1)(\lambda - \mu_2)(\lambda - \mu_3)(\lambda - \mu_5),$$

$$J = (\lambda - \mu_1)(\lambda - \mu_2)(\lambda - \mu_3)(\lambda - \mu_4),$$

where

$$\mu_1 = \frac{3^k l - 3^k + \sqrt{(3^k l - 3^k)^2 + 4(2^k lm)}}{2},$$

$$\mu_2 = \frac{3^k l - 3^k - \sqrt{(3^k l - 3^k)^2 + 4(2^k lm)}}{2},$$

$$\mu_3 = 3^k l - 3^k, \ \mu_4 = -3^k \text{ and } \mu_5 = 0.$$

It is clear that $P_{\Gamma - j}(\lambda)$ has $3^k lm$ roots. So, we have:

$$\alpha + \beta + \gamma + 3^k l - 3^k = -[(m-2)\mu_3 + (lm - m - 1)\mu_4],$$

$$\alpha^2 + \beta^2 + \gamma^2 + (3^k l - 3^k)^2 = 3^k lm(3^k l - 3^k) - [(m-2)\mu_3^2 + (lm - m - 1)\mu_4^2],$$

$$\alpha^3 + \beta^3 + \gamma^3 + (3^k l - 3^k)^3 = 6m(3^{3k})\binom{l}{3} - [(m-2)\mu_3^2 + (lm - m - 1)\mu_4^2],$$

where α, β and γ are the eigenvalues of graph $\Gamma - j$. If we solve the above equations, then we will have: $\alpha = -3^k$, $\beta = 0$ and $\gamma = 3^k l - 3^k$. Therefore,

$$spec(\Gamma - j) = \left\{ \left[3^k l - 3^k\right]^m, [0]^{3^k lm - lm}, \left[-3^k\right]^{lm - m} \right\}.$$

Graph $\Gamma - j$ is regular and degree of its regularity is $3^k l - 3^k$. It follows from Theorem 2.4 that $\Gamma - j = mK_{\underbrace{3^k, \ldots, 3^k}_{l\ times}}$ and so $G - j = mECP_l^k$. Hence $\Gamma = K_1 \bigtriangledown mECP_l^k$. This follows the result. $\square$

Up to now, we have shown that the multicone graphs $K_1 \bigtriangledown mECP_l^k$ are DS. The natural question is; what happen for multicone graphs $K_w \bigtriangledown mECP_l^k$? we answer to this question in the following theorem.

Theorem 3.4. *Any connected graph cospectral with multicone graph $K_w \bigtriangledown mECP_l^k$ is isomorphic to $K_w \bigtriangledown mECP_l^k$.*

Proof. We solve the problem by induction on w. If $w = 1$, there is nothing to prove. Let the claim be true for w; that is, if $Spec(\Gamma_1) = Spec(K_w \bigtriangledown mECP_l^k)$, then $\Gamma_1 \cong K_w \bigtriangledown mECP_l^k$, where Γ_1 is a graph. We show that, if $Spec(\Gamma) = Spec(K_{w+1} \bigtriangledown mECP_l^k)$, then $\Gamma \cong K_{w+1} \bigtriangledown mECP_l^k$, where Γ is a graph. By Lemma 3.2, Theorem 2.4, Lemma 2.1 (iii) and in a similar manner of Lemma 3.3 for $\Gamma - j$, where j is a vertex of degree $w + 3^k lm$ belonging to Γ, we obtain $Spec(\Gamma - j) = Spec(K_w \bigtriangledown mECP_l^k)$. Therefore, the assertion holds. $\square$

In the following, we give another proof of the above theorem.

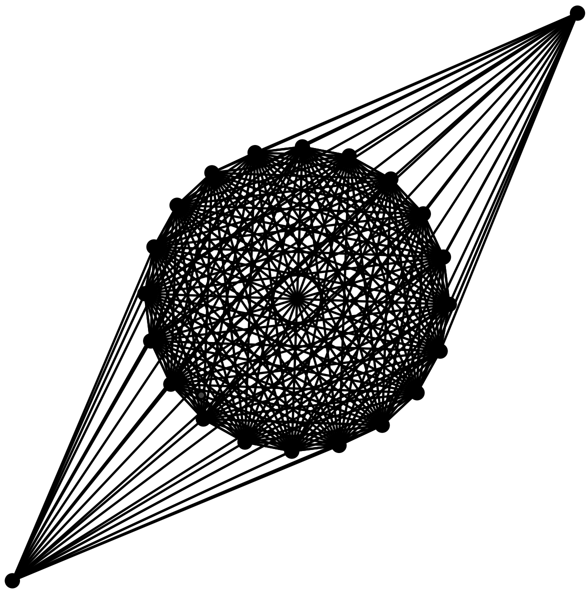


FIGURE 1. Multicone graph $K_{20} \bigtriangledown 2ECP_1^0$

Proof. Let Γ be a connected graph cospectral with multicone graph $K_w \bigtriangledown mECP_l^k$. By Lemma 3.2, Γ has subgraph L in which degree of any vertex of L is $w-1+3^k lm$. In other words, $\Gamma \cong K_w \bigtriangledown H$, where H is a subgraph of Γ. Now, we remove the vertices of K_w and we consider $3^k lm$ another vertices. Consider H consisting of these $3^k lm$ vertices. H is regular and degree of its regularity is $3^k l - 3^k$ and multiplicity of $3^k l - 3^k$ is m. By Theorem 2.2, $Spec(H) = \left\{ \left[3^k l - 3^k\right]^m, [0]^{(3^k l - l)m}, \left[-3^k\right]^{(l-1)m} \right\}$. Now, it follows from Theorem 2.5 that $Spec(H) = Spec(mECP_l^k)$. This implies the result. □

Corollary 3.5. *Any connected graph cospectral with multicone graph*

$$K_w \bigtriangledown mECP_1^k = K_w \bigtriangledown mK_{3^k}$$

is DS with respect to their adjacency spectrums.

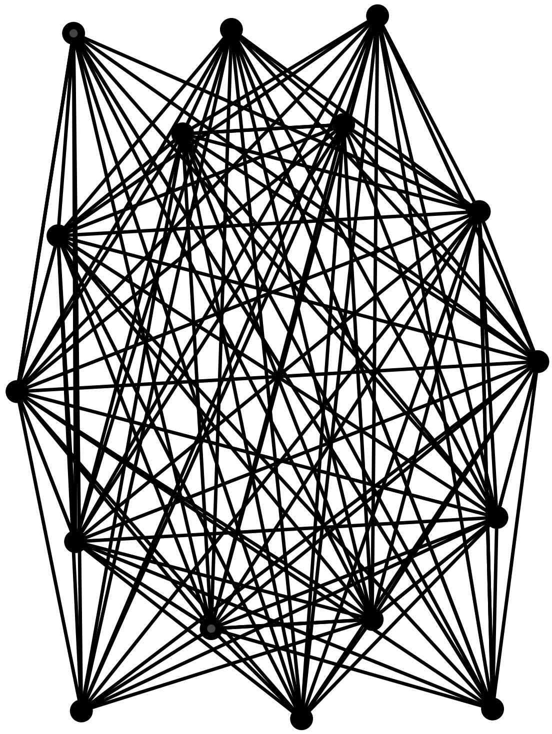


FIGURE 2. Multicone graph $K_{10} \bigtriangledown 2ECP_2^1$

3.3. Some complements of multicone graphs $K_w \bigtriangledown mECP_l^k$ are DS with respect to their spectra.

In this subsection, we show that the complement of multicone graphs $K_w \bigtriangledown mECP_l^k$ are DS with respect to their adjacency spectrum.

Proposition 3.6. *Let Γ be cospectral with complement of multicone graphs $K_w \bigtriangledown mECP_l^k$. Then*

$$Spec(\Gamma) = \left\{ \left[3^k lm - 3^k l + 3^k - 1\right]^m, \, [-1]^{(3^k-1)lm}, \, \left[3^k - 1\right]^{(l-1)m}, [0]^w \right\}.$$

Proof. Straightforward. □

Theorem 3.7. *The complement of multicone graph $K_w \bigtriangledown ECP_l^k$ are DS with respect to their adjacency spectrum.*

Proof. The proof of this theorem is the similar of Theorm 5.2 of [1]. Let

$$Spec(\Gamma) = Spec(\overline{K_w \bigtriangledown ECP_l^k}) = \left\{ [-1]^{(3^k-1)l}, \, \left[3^k - 1\right]^l, [0]^w \right\}.$$

If $l = 1$, by Lemma 2.1 ((i), (ii) and (iii)) the proof is clear (Also, by Theorem 2.5 the proof follows). Hence we suppose that $l \neq 1$. It is easy to see that Γ cannot be regular, since regularity of a graph can be determined by its spectrum. By contrary, we suppose that Γ is connected. So, we from Lemma 2.6 and Lemma 2.11 conclude that $k = l = 1$. This is a contradiction. Hence $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \ldots \cup \Gamma_h$, where Γ_s is a connected component of Γ and $1 \leq s \leq h$. Now, we show that Γ_s cannot have three distinct eigenvalues. By contrary, we suppose that Γ_i has three distinct eigenvalues. In this case, if we also suppose Γ_s is non-regular, then it follows from Lemma 2.6 that Γ_s is a complete bipartite graph. Hence $l = k = 1$. This is a contradiction. Therefore, if Γ_s has three distinct eigenvalues, then it must be regular. Now, it follows from Theorem 2.5 that $\Gamma_s \cong K_{\underbrace{1,\, 1,\, \ldots,\, 1}_{3^k\ times}} \cong K_{3^k}$. This is a contradiction. So, Γ_s cannot have three distinct eigenvalues. Therefore, it has one or two eigenvalue(s). Hence, any connected component of Γ is either isolated vertex or a complete graph. Hence $\Gamma \cong wK_1 \cup lK_{3^k}$. This follows the result. $\square$

3.4. The multicone graphs $K_w \triangledown mECP_l^k$ are determined by their Laplacian spectra

In this subsection, we show that any graph cospectral with multicone graph $K_w \triangledown mECP_l^k$ is DS with respect to its Laplacian spectrum.

Theorem 3.8. *Multicone graphs $K_w \triangledown mECP_l^k$ are DS with respect to their Laplacian spectrum.*

Proof. We solve the problem by induction on w. If $w = 1$, there is nothing to prove. Let the claim be true for w; that is,

$$Spec(L(H)) = Spec(L(K_w \triangledown mECP_l^k)$$

$$= \left\{ \left[3^k lm + w\right]^{w},\ [w]^{m-1},\ \left[3^k l - 3^k + w\right]^{3^k lm - lm},\ \left[3^k l + w\right]^{lm-m},\ [0]^1 \right\}$$

follows that $H \cong K_w \triangledown mECP_l^k$. We show that the problem is true for $w + 1$; that is, we show that

$$Spec(L(G)) = Spec(L(K_{w+1} \triangledown mECP_l^k))$$

$$= \left\{ \left[3^k lm + w + 1\right]^{w+1}, [w+1]^{m-1}, \left[3^k l - 3^k + w + 1\right]^{3^k lm - lm}, \left[3^k l + w + 1\right]^{lm-m}, [0]^1 \right\}$$

follows that $G \cong K_{w+1} \triangledown mECP_l^k$. It follows from Lemma 2.9 that H and G are the join of two graphs. On the other hand,

$$Spec(L(K_1 \triangledown H)) = Spec(L(G)) = spec(L(K_{w+1} \triangledown mECP_l^k)).$$

Therefore, we must have $G \cong K_1 \triangledown H$. Because, G is the join of two graphs and also according to spectrum of G, must K_1 be joined to H and this is only available state. This completes the proof. $\square$

Corollary 3.9. *Multicone graphs $K_w \triangledown mECP_1^k = K_w \triangledown mK_{3^k}$ are DS with respect to their Laplacian spectrums.*

3.5. Some results about multicone graphs $K_w \bigtriangledown mECP_l^k$

In this subsection, we show that any graph cospectral with multicone graph $K_w \bigtriangledown mECP_l^k$ must be perfect. Also, we prove that any graph cospectral with multicone graph $K_w \bigtriangledown mECP_l^k$ with respect to Laplacian spectrum is perfect. In addition, we show that any graph cospectral with complement of multicone graph $K_w \bigtriangledown mECP_l^k$ is perfect.

Suppose $\chi(\Gamma)$ and $\omega(\Gamma)$ are chromatic number and clique number of graph G, respectively. A graph is perfect if $\chi(H) = \omega(H)$ for every induced subgraph H of Γ. It is proved that a graph G is perfect if and only if Γ is Berge; that is, it contains no odd hole or antihole as induced subgraph, where odd hole and antihole are odd cycle, C_m for $m \geq 5$, and its complement, respectively. Also, in 1972 Lovász proved that, a graph is perfect if and only if its complement is perfect (see [22] of [2]). Now, by Theorem 3.4, Theorem 3.7, Theorem 3.8 and by what was said in the previous sections we can conclude the following results.

Theorem 3.10. *Let graph Γ be cospectral with multicone graph $K_w \bigtriangledown mECP_l^k$. Then Γ and $\overline{\Gamma}$ are perfect.*

Proof. By what was said in the beginning of this section and Theorem 3.4 the proof is completed. □

Theorem 3.11. *Let Γ be a graph and $Spec(L(\Gamma)) = Spec(L(K_w \bigtriangledown mECP_l^k))$. Then Γ and $\overline{\Gamma}$ are perfect.*

Proof. The proof is straightforwad. □

Theorem 3.12. *Let Γ be a graph and $Spec(\Gamma) = Spec(\overline{K_w \bigtriangledown mECP_l^k})$. Then Γ and $\overline{\Gamma}$ are perfect.*

Proof. It is obvious. □

In the following, we pose two conjectures.

4. Final remarks and open problems

In this paper, we have shown any connected graph cospectral with multicone graph $K_w \bigtriangledown mECP_l^k$ is DS with respect to its spectra. Also, we have shown in special cases complement of these graphs are DS. In addition, we have proved any connected graph cospectral with these graph is perfect. On the other hand, It is obvious that, F_n are special classes of multicone graphs $K_w \bigtriangledown mECP_l^k$(one can also consider $k = 0$). In addition, F_n are DS with respect to:

(i) Their adjaccency spectrum (if $n \neq 16$).

(ii) Their Laplacian spectrum.

(iii) Their signless Laplacian spectrum. Also, $\overline{F_n}$ are DS with respect to their adjacency spectrum, where $n \neq 2$.

Hence we pose the following conjectures.

Conjecture 4.1. *Multicone graphs $K_w \bigtriangledown mECP_l^k$ are DS with respect to their signless Laplacian spectrum.*

Conjecture 4.2. *The complement of multicone graphs $K_w \bigtriangledown mECP_l^k$ are DS with respect to their adjacency spectrum.*

References

[1] Abdian, A.Z., Mirafzal S.M., *On new classes of multicone graphs determined by their spectrums*, Alg. Struc. Appl., **2**(1)(2015), 21-32.

[2] Abdollahi, A., Janbaz, S., Oubodi, M., *Graphs Cospectral with A Friendship Graph Or its Complement*, Trans. Comb., **2**(4)(2013), 37-52.

[3] Abdollahi, A., Janbaz, S., *Connected graphs cospectral with a friendship graph*, Trans. Comb., **3**(2)(2014), 17-20.

[4] Biggs, N.L., *Algebraic Graph Theory*, Second ed., Cambridge University Press, Cambridge, 1933.

[5] Cvetković, D., Rowlinson, P., Simić, S., *An Introduction to the Theory of graph Spectra*, London Mathematical Society Student Texts, 75, Cambridge University Press, Cambridge, 2010.

[6] Cvetković, D., Doob, M., Simić, S.K., *Generalized line graphs*, J. Graph Theory, **5**(4)(1981), 385-399.

[7] Cioaba, S.M., Haemers, W.H., Vermette, J.R., Wong, W., *The graphs with all but two eigenvalues equal to* ± 1, Journal of Algebraic Combinatorics, **44**(3)(2013), 887-897.

[8] Das, K.C., *Proof of conjectures on adjacency eigenvalues of graphs*, Discret. Math., **313**(1)(2013), 19-25.

[9] Erdös, P., Réyni A., Sós V.T., *On a problem of graph theory*, Sci. Math. Hungarica, **1**(966), 215-235.

[10] Godsil, C.D., Royle, G., *Algebraic Graph Theory*, Graduate Texts in Mathematics 207, 2001.

[11] Gallian, J.A., *A dynamic survey of graph labeling*, Electron. J. Combin., **16**(6)(2009), 1-219.

[12] Kanauer, U., *Algebraic Graph Theory*, Morphisms, Monoids and Matrices, **41**, Walter de Gruyter, 2011.

[13] Mohammadian, A., Tayfeh-Rezaie, B., *Graphs with four distinct Laplacian eigenvalues*, Journal of Algebraic Combinatorics, **34**(4)(2011), 671-682.

[14] Omidi, G.R., *On graphs with largest Laplacian eignnvalues at most* 4, Australasian J. Combin., **44**(2009), 163-170.

[15] van Dam E.R., Haemers, W.H., *Which graphs are determined by their spectrum?*, Linear Alg. Appl., **373**(2003), 241-272.

[16] van Dam, E.R., Haemers, W.H., *Developments on spectral characterizations of graphs*, Discrete Math., **309**(2009), 576-586.

[17] Wang, J., Belardo, Huang, Q., Borovicanin, B., *On the two largest Q-eigenvalues of graphs*, Discrete Math., **310**(2010), 2858-2866.

[18] Wang, J., Zhao, H., Huang, Q., *Spectral characterization of multicone graphs*, Czechoslovak Math. J., **62**(1)(2012), 117-126.

[19] West, D.B., *Introduction to Graph Theory*, Upper Saddle River, Prentice Hall, 2001.

Seyed Morteza Mirafzal
Lorestan University, Faculty of Science
Lorestan, Khorramabad, Iran
e-mail: Morteza_mirafzal@yahoo.com; mirafzal.m@lu.ac.ir

Ali Zeydi Abdian
Lorestan University, College of Science
Lorestan, Khorramabad, Iran
e-mail: azeydiabdi@gmail.com; aabdian67@gmail.com; abdian.al@fs.lu.ac.ir

Stud. Univ. Babeş-Bolyai Math. 62(2017), No. 3, 287–293
DOI: 10.24193/subbmath.2017.3.02

Uniquely clean 2×2 invertible integral matrices

Dorin Andrica and Grigore Călugăreanu

Abstract. While units in any unital ring are strongly clean by definition, which units are uniquely clean, is a far from being simple question, even in particular rings. In this paper, the question is solved for 2×2 integral matrices. It turns out that uniquely clean invertible matrices are scarce: only the matrices similar to $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. The study is splitted into three cases: the elliptic, the parabolic and the hyperbolic cases, according to the discriminant of their characteristic polynomial. In the first two cases, units are not uniquely clean.

Mathematics Subject Classification (2010): 15B36, 16U99, 11D09, 11D45.

Keywords: Clean, uniquely clean, class number, Diophantine equation, reduced matrix.

1. Introduction

Let R be a ring with identity. An element $r \in R$ is called *clean* if $r = e + u$ with idempotent e and unit u. It is called *uniquely* clean if it has only one clean decomposition, and *strongly* clean if the components of the decomposition commute.

Clean elements which use trivial idempotents (hereafter called *trivial* clean) are obviously strongly clean. That is, units and sums $1 + u$ with unit u are strongly clean.

However, when are such elements (also) uniquely clean turns out to be a difficult question even for particular unital rings.

In this paper we give a complete answer to this question for $R = \mathcal{M}_2(\mathbf{Z})$, that is, we show that only matrices U (with determinant -1 and trace 0) similar to $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ are uniquely clean invertible 2×2 integral matrices.

Since units already have the (strongly) clean 0_2-decomposition, a unit $U \in \mathcal{M}_2(\mathbf{Z})$ is uniquely clean iff U is not nontrivial clean and $U - I_2$ is not a unit. Notice that $\det(U - I_2) = \det U - \mathrm{Tr}(U) + 1$ and so

Lemma 1.1. *Suppose U is a unit. Then*

(a) for $\det U = 1$*, $U - I_2$ is a unit iff* $\mathrm{Tr}(U) \in \{1, 3\}$*, and*

(b) for $\det U = -1$, $U - I_2$ *is a unit iff* $\mathrm{Tr}(U) \in \{\pm 1\}$.

Any 2×2 integral matrix U has a characteristic polynomial $X^2 - \mathrm{Tr}(U)\cdot X + \det U$, whose discriminant is $\Delta = \mathrm{Tr}^2(U) - 4\det U$.

If U is a unit, then $\det U \in \{\pm 1\}$. In what follows we separately deal with the *elliptic, parabolic and hyperbolic cases* according to $\Delta < 0$, $\Delta = 0$ and $\Delta > 0$ respectively.

Definition 1.2. Two 2×2 matrices A, B over any unital ring R, are *similar* (or *conjugate*) if there is an invertible matrix U such that $B = U^{-1}AU$. Since similarity is obviously an equivalence relation, a partition of $\mathcal{M}_2(R)$ corresponds to it. The subsets in this partition are called *similarity classes*.

Such classes may consist only in one matrix, for instance, 0_2 respectively I_2. So is every scalar matrix (since it belongs to the center), and generally, a matrix A forms a singleton class iff $AU = UA$ for every invertible matrix U.

If A is idempotent (or unit) and B is similar to A then B is also idempotent (respectively unit). This similarity invariance clearly extends to clean matrices and it also restricts to uniquely or strongly clean matrices, respectively. Rephrasing, the notions of clean, uniquely clean and strongly clean are similarity invariants. So is the clean index.

Further, recall that for $R = \mathbf{Z}$, if $f(t) = t^n + a_1t^{n-1} + ... + a_n$ is irreducible in $\mathbf{Q}[t]$ and ω is a root of $f(t) = 0$ then, according to Latimer and MacDuffee theorem (see e. g. [7]), in the elliptic case, there is a one-to-one correspondence between ideal classes in the ring of integers of the field $\mathbf{Q}[\omega]$ and $\mathbf{Z}$-similarity classes of $n \times n$ matrices A of integers which satisfy $f(A) = 0$. The common number is (finite and called) *the class number* of $\mathbf{Z}[\omega]$.

The answer to our question above amounts to several results from Number Theory related to (positive) quadratic forms. However, it was not necessary to use such results because of the transfer done directly to similarity classes of integral 2×2 matrices done in Behn, Van der Merwe paper (see [4]). From this paper we recall the following definitions and results.

Definition 1.3. A 2×2 integral matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $\Delta = \mathrm{Tr}(A)^2 - 4\det(A) < 0$ is *reduced* if $|d - a| \le c \le -b$ and, $d \ge a$ if at least one is equality, i.e. $|d - a| = c$ or $c = -b$. Notice that if $|d - a| < c < -b$ then $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $\begin{bmatrix} d & b \\ c & a \end{bmatrix}$ are different reduced matrices.

An integral matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $\Delta = \mathrm{Tr}^2(A) - 4\det(A) > 0$ but not a square in $\mathbf{Z}$ is *reduced* if $c > 0$ and $\left|\sqrt{\Delta} - 2c\right| < d - a < \sqrt{\Delta}$.

If Δ is a square (e.g. $\det(A) = 0$), that is, the characteristic polynomial of the matrix factors over the integers, say, $f(x) = (x - a)(x - d)$, where $a \ge d$, then, for $a \ne d$ the matrix $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ is *reduced* if $0 \le b < a - d$, and, for $a = d$, if $b \ge 0$. While our results are up to a similarity, in this case it is sufficient to define *upper triangular* *reduced* matrices because, if $f(x) = (x-a)(x-d)$ (and $a \geq d$) then A is similar to a matrix $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ with $0 \leq b \leq |a-d|$, and for $a = d$, with $b \geq 0$.

For integers x, y and $y \neq 0$, $r(x,y)$ will denote the unique integer such that $r \equiv x \bmod 2y$ and $-|y| < r \leq |y|$ if $|y| > \sqrt{\Delta}$, and $\sqrt{\Delta} - 2|y| < r < \sqrt{\Delta}$ if $|y| < \sqrt{\Delta}$.

Theorem 1.4. ([4], *Theorem 3.3*) *Consider matrices in $\mathcal{M}_2(\mathbf{Z})$ with a fixed trace and determinant and $\Delta = \mathrm{Tr}^2 - 4\det < 0$. Then there is precisely one reduced matrix in each matrix class.*

Theorem 1.5. ([4], *Theorem 5.2*) *Let $M \in \mathcal{M}_2(\mathbf{Z})$, and assume that the characteristic polynomial of M factors over $\mathbf{Z}$. Then M is equivalent to a reduced matrix. Moreover, this class representative is unique thus no two different reduced matrices are equivalent.*

Theorem 1.6. ([4], *Theorem 4.3*) *Consider all matrices A in $\mathcal{M}_2(\mathbf{Z})$ with a fixed trace and determinant. If $\Delta = \mathrm{Tr}^2(A) - 4\det(A) > 0$ is not a square in $\mathbf{Z}$ then there is precisely one cycle of reduced matrices in each matrix class. Thus for each matrix class there is a matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ in the class and a positive integer n such that $P^i \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ for $0 \leq i \leq n$ are all the reduced matrices in the class and*

$$P^{n+1} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

where P denotes a reduction operator on the matrix, namely a conjugation *with $\begin{bmatrix} 0 & -1 \\ 1 & -n \end{bmatrix}$, if $-b > 0$ and with $\begin{bmatrix} 0 & 1 \\ 1 & n \end{bmatrix}$, if $b > 0$, where $n = \dfrac{r(a-d,b)+d-a}{2b}$.*

Finally recall the following characterization (partly hidden in [2]).

Theorem 1.7. *A 2×2 integral matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is nontrivial clean iff the system*

$$x^2 + x + yz = 0 \tag{1.1}$$

$$(a-d)x + cy + bz + \det(A) - d = \pm 1 \tag{1.2}$$

with unknowns x, y, z, has at least one solution over $\mathbf{Z}$. If $b \neq 0$ and (1.2) holds, then (1.1) is equivalent to

$$bx^2 - (a-d)xy - cy^2 + bx + (d - \det(A) \pm 1)y = 0. \tag{1.3}$$

The equation (1.3) is a quadratic Diophantine equation in x an y, and its type (elliptic, parabolic, or hyperbolic) is defined by its discriminant ([3, p.119-120]). In our case we have $\Delta = (a-d)^2 + 4bc = \mathrm{Tr}^2(A) - 4\det(A)$.

2. The elliptic case

Theorem 2.1. *Units in the elliptic case are not uniquely clean.*

Proof. First notice that in this case, $\mathrm{Tr}^2(U) - 4\det U < 0$. This happens only if $\det U = 1$ and $\mathrm{Tr}^2(U) < 4$.

Hence units U in the elliptic case have $\det U = 1$ and $\mathrm{Tr}(U) \in \{-1, 0, 1\}$. Comparing with Lemma 1.1, for $\det U = 1$ only $\mathrm{Tr}(U) \in \{-1, 0\}$ are suitable. Therefore we go into 2 cases.

(i) If $\mathrm{Tr}(U) = -1$, the characteristic polynomial for such matrices is $X^2 + X + 1$. Such matrices are of form

$$U = \begin{bmatrix} a & b \\ c & -a-1 \end{bmatrix} \tag{2.1}$$

with $a(a+1) + bc = -1$. The discriminant $\Delta = \mathrm{Tr}^2(U) - 4\det(U) = -3$ which has class number 1 (see e.g. [5], p. 229).

To find the reduced matrix it suffices to reduce any representative of this similarity class, say $\begin{bmatrix} 4 & -7 \\ 3 & -5 \end{bmatrix}$. All matrices of type (2.1) are (not) uniquely clean iff the reduced representative is so. This is (see [4], p. 7) $\begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$ and it is readily seen that this matrix is **not** uniquely clean. It has 3 nontrivial clean decompositions:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -2 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}.$$

Hence the matrices of type (2.1) are not uniquely clean.

(ii) If $\mathrm{Tr}(U) = 0$, by Cayley-Hamilton theorem, $U^2 + I_2 = 0_2$, i.e. $U^2 = -I_2$ and so

$$U = \begin{bmatrix} a & b \\ -\dfrac{a^2+1}{b} & -a \end{bmatrix}$$

for integers a, b with b a nonzero divisor of $a^2 + 1$.

Again, the discriminant $\Delta = \mathrm{Tr}^2(U) - 4\det(U) = -4$ which has (also) class number 1, and we argue as in the previous case. The reduced representative of this similarity class is $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Alternatively, it suffices to notice that matrices with $b \in \{\pm 1\}$ in this class, are **not** uniquely clean:

$$\begin{aligned} \begin{bmatrix} a & \pm 1 \\ \mp(a^2+1) & -a \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ \mp a & 0 \end{bmatrix} + \begin{bmatrix} a-1 & \pm 1 \\ \mp a^2 \pm a \mp 1 & -a \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ \pm a & 1 \end{bmatrix} + \begin{bmatrix} a & \pm 1 \\ \mp(a^2+a+1) & -a-1 \end{bmatrix}. \end{aligned}$$

This includes the reduced representative above. □

3. The parabolic case

A unit u is called *unipotent* if $u = 1 + t$ with nilpotent t. The units, in the parabolic case, are precisely the unipotents (including I_2) and negatives of unipotents.

Indeed, in this case we have $\det U = 1$ and $\mathrm{Tr}(U) \in \{-2, 2\}$. The characteristic polynomial is now $X^2 \pm 2X + 1 = (X \pm 1)^2$ and so by Cayley-Hamilton theorem, we have to consider two cases: either $(U - I_2)^2 = 0_2$, i.e., $U = I_2 + T$ is unipotent (with nilpotent T), or else $(U + I_2)^2 = 0_2$, i.e., $-U = I_2 - T$ is unipotent.

Since we intend to prove that units in the parabolic case are not uniquely clean, in the proof of the next theorem, we deal with the first case, i.e. $\det = 1$ and $\mathrm{Tr}(U) = -2$. Matrices in this case are of form $\begin{bmatrix} a & b \\ c & -a-2 \end{bmatrix}$ with $a(a+2) + bc = -1$, i.e. $bc = -(a+1)^2$. The discriminant is now $\Delta = \mathrm{Tr}^2(U) - 4\det(U) = 0$. The proof in the second case is analogous.

Theorem 3.1. *Units in the parabolic case are not uniquely clean.*

Proof. The proof follows the same lines as the proof of Theorem 2.1. The characteristic polynomial for such matrices is $(X-1)^2$, so factors over $\mathbf{Z}$ and it suffices to deal with the reduced representative, which is now $V = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$ (we just use the algorithm described by [4], in the proof of Theorem 1.5, for example for $\begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$). Thus

$$V = \begin{bmatrix} n+1 & n^2+n \\ -1 & -n \end{bmatrix} + \begin{bmatrix} -n-2 & -n^2-n+1 \\ 1 & n-1 \end{bmatrix}$$

for every integer n (infinite clean index) is not uniquely clean, and nor are all units in the parabolic case. □

4. The hyperbolic case

Theorem 4.1. *The only units in the hyperbolic case which are uniquely clean are the matrices similar to* $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

Proof. We have to distinguish two cases.

1. *For a unit U we have* $\det(U) = -1$.

Here also we go into 2 subcases.

(i) $\mathrm{Tr}(U) = 0$. In this subcase $\Delta = 2^2$ is a square, the characteristic polynomial factors over $\mathbf{Z}$ (i.e. $X^2 - 1 = (X-1)(X+1)$) and the proof follows the same lines as in the parabolic case. Again it suffices to deal with the reduced representatives which are now $S = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $T = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$. Since for S, equations (1.2) (see Theorem 1.7) are $2x = \pm 1$, with no integer solutions, this unit has no nontrivial clean decomposition. Since $S - I_2$ is not a unit, we deduce that S is indeed a uniquely clean matrix. So are all matrices similar to S.

Hence all units (with $\det(U) = -1$ and $\mathrm{Tr}(U) = 0$) similar to $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ are uniquely clean.

Notice that not all units U with $\det(U) = -1$ and $\mathrm{Tr}(U) = 0$ are uniquely clean. Indeed, the matrices similar to T have nontrivial clean decompositions and so, are not uniquely clean. An example:

$$\begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ -1 & -2 \end{bmatrix},$$

is a nontrivial clean decomposition. 4).

(ii) $\mathrm{Tr}(U) \neq 0$. In this subcase $\Delta = \mathrm{Tr}^2(U) - 4\det(U) = \mathrm{Tr}^2(U) + 4 > 0$ is never a square over $\mathbf{Z}$ (otherwise 2 would be component of a Pythagorean triple) and we use Theorem 1.6. In doing so, notice that it suffices to show that any reduced matrix (from the cycle) in any given similarity class is not uniquely clean. Denote $\mathrm{Tr}(U) = t$. If $t > 0$ then a reduced representative is

$$W_t = \begin{bmatrix} 0 & 1 \\ 1 & t \end{bmatrix},$$

which is not uniquely clean since

$$W_t = \begin{bmatrix} 1 & t \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -1 & 1-t \\ 1 & t \end{bmatrix}.$$

If $t < 0$, a reduced representative is

$$V_t = \begin{bmatrix} t & 1 \\ 1 & 0 \end{bmatrix},$$

also not uniquely clean, having a symmetric nontrivial clean decomposition.

2. *For a unit U we have* $\det(U) = 1$ *and* $|\mathrm{Tr}(U)| > 2$.

Here $\Delta = \mathrm{Tr}^2(U) - 4\det(U) = \mathrm{Tr}^2(U) - 4 > 0$ is never a square over $\mathbf{Z}$ (otherwise 2 would be component of a Pythagorean triple) and we use Theorem 1.6. We argue as in the previous subcase: now a reduced representative is $\begin{bmatrix} 0 & -1 \\ 1 & t-2 \end{bmatrix}$, if $t > 2$ and $\begin{bmatrix} t+2 & -1 \\ 1 & 0 \end{bmatrix}$, if $t < -2$. Both are not uniquely clean. Indeed, a nontrivial clean decomposition for the first is

$$\begin{bmatrix} 1 & t-4 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} -1 & -t+3 \\ 1 & t-2 \end{bmatrix},$$

and is

$$\begin{bmatrix} 0 & t+2 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} t+2 & -t-3 \\ 1 & -1 \end{bmatrix}$$

for the second. □

Therefore, the final conclusion of our paper is

Theorem 4.2. *An invertible 2×2 integral matrix U is uniquely clean iff it is similar to* $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, *i.e., there exists a unit K such that*

$$KU = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} K.$$

Acknowledgement. Thanks are due to the referee whose careful reading and suggestions improved this paper.

References

[1] Alpern, D., *Quadratic Equation Solver*, www.alpertron.com.ar/quad.htm.

[2] Andrica, D., Călugăreanu, G., *A nil-clean* 2×2 *matrix over integers which is not clean*, J. Algebra Appl., **13**(6)(2014), 9 pp.

[3] Andreescu, T., Andrica, D., Cucurezeanu, I., *An Introduction to Diophantine Equations*, A Problem-Based Approach, Birkhäuser, 2010.

[4] Behn, A., Van der Merwe, A.B., *An algorithmic version of the theorem by Latimer and MacDuffee for 2 times 2 integral matrices*, Linear Algebra Appl., **346**(2002), 1-14.

[5] Cohen, H., *A Course in Computational Number Theory*, Third Edition, Springer Verlag, 1996.

[6] Khurana, D., Lam, T.Y., Nielsen, P., Zhou, Y., *Uniquely Clean Elements in Rings*, Comm. Algebra, **43**(5)(2015), 1742-1751.

[7] Newman, M., *Integral matrices*, Academic Press, New York - London, 1972, Pure Appl. Math., Vol. 45.

Dorin Andrica
Babeş-Bolyai University
Faculty of Mathematics and Computer Science
Cluj-Napoca, Romania
e-mail: `dandrica@math.ubbcluj.ro`

Grigore Călugăreanu
Babeş-Bolyai University
Faculty of Mathematics and Computer Science
Cluj-Napoca, Romania
e-mail: `calu@math.ubbcluj.ro`

Stud. Univ. Babeş-Bolyai Math. 62(2017), No. 3, 295–307
DOI: 10.24193/subbmath.2017.3.03

Finite valuated groups as modules over their endomorphism ring

Ulrich Albrecht

Abstract. This paper discusses the structure of a finite valuated p-group when viewed as a module over its endomorphism ring. A category equivalence between full subcategories of the category of valuated p-groups and the category of right modules over the endomorphism ring of A is used to investigate the interaction between this module structure and homological properties of the underlying group. Examples are given throughout the paper.

Mathematics Subject Classification (2010): 20K30, 20K40, 20K10.

Keywords: Valuated p-group, endomorphism ring, Ulmer's theorem, projective module.

1. Introduction

Consider a prime p and a p-local Abelian group G. A *valuation* v on G assigns a value $v(g)$ to each $g \in G$ which is either an ordinal or ∞ subject to the rules

i) $v(px) > v(x)$ for all $x \in G$ where $\infty > \infty$,
ii) $v(x+y) \geq min\{v(x), v(y)\}$ for all $x, y \in G$, and
iii) $v(nx) = v(x)$ whenever n and p are relatively prime [11].

The third condition is redundant whenever G is a p-group. The valuated p-local groups are the objects of the category $\mathcal{V}_p$ studied extensively by Hunter, Richman and Walker (e.g. see [7], [8] and [11]). A group homomorphism $\alpha : (G, v) \to (H, w)$ is a $\mathcal{V}_p$*-morphism* if $w(\alpha(x)) \geq v(x)$ for all $x \in G$, and we write $\alpha \in \mathrm{Mor}(G, H)$ in this case. The category $\mathcal{V}_p$ is pre-Abelian, i.e. all maps have kernels and cokernels. While the kernel and cokernel of a $\mathcal{V}_p$-map $G \to H$ are its kernel and cokernel in the category $\mathcal{A}b$ of Abelian groups, their valuations are induced by those on G and H respectively. Consequently, monomorphisms and epimorphisms need not be kernels and cokernels; and $\mathcal{V}_p$ is not Abelian. Finally, the forgetful functor $\mathcal{F} : \mathcal{V}_p \to \mathcal{A}b$ strips a valuated group (G, v) of its valuation.

In this paper, all valuated groups are assumed to be finite valuated p-groups. Although the group structure of a finite valuated p-group is well understood, the addition of a valuation directly impacts its homological properties. In addition, Arnold discovered a surprising connection between finite valuated p-groups and torsion-free Abelian groups of finite rank in [3] by demonstrating that representation theory can be used to investigate finite rank Butler groups as well as finite valuated p-groups. Moreover, both classes of groups are equally difficult to describe.

This paper follows Arnold's approach by investigating valuated p-groups using tools which have traditionally been used in the discussion of torsion-free groups of finite rank. For instance, homological properties of Abelian groups A of finite torsion-free rank have been successfully studied by viewing A as a left module over its endomorphism ring. This paper extends this approach to finite valuated p-groups by considering such a group A as a module over its $\mathcal{V}_p$-endomorphism ring $R = \mathrm{Mor}(A, A)$ and by studying how this module structure affects the homological properties of A. Section 2 focuses on the case that A is projective as an R-module, while Section 3 considers the case that R has specific ring-theoretic properties.

2. Valuated p-Groups Projective as R-modules

A finite valuated p-group *A-free* if it is isomorphic to A^n for some $n < \omega$, and *A-projective* if it is a $\mathcal{V}_p$-direct summand of an A-free group. Since A is a left R-module, $H_A = \mathrm{Mor}(A, -)$ can be viewed as a functor from $\mathcal{V}_p$ to the category $\mathcal{M}_R$ of right R-modules, with the property that $H_A(P)$ is free (projective) if P is A-free (A-projective).

We begin our discussion with a few technical results. If α is a kernel in $\mathcal{V}_p$, then $\alpha = \ker(coker(\alpha))$ [12]; and a similar result holds for cokernels. However, composition of kernels (cokernels) in $\mathcal{V}_p$ need not be kernels (cokernels) [10]. Therefore, the usual homological constructions may not carry over from Abelian categories. Nevertheless, it is still possible to develop a homological algebra for pre-Abelian categories as Yakovlev showed in [14].

Lemma 2.1. *Let A, B and C be valuated p-groups. If $\alpha \in \mathrm{Mor}(A, B)$ is an epimorphism and $\beta \in Mor(B, C)$ such that $\beta\alpha$ is a cokernel of a $\mathcal{V}_p$-map δ, then β is a cokernel for $\alpha\delta$.*

Proof. Suppose that ϕ satisfies $\phi\alpha\delta = 0$. Since $\beta\alpha$ is a cokernel for δ, there is a map ψ such that $\psi\beta\alpha = \phi\alpha$. Because α is an epimorphism, $\phi = \psi\beta$. Since β is an epimorphism, ψ is unique with this property. $\square$

A sequence $A \overset{\alpha}{\to} B \overset{\beta}{\to} C$ of valuated p-groups is is *left-exact* if α is a kernel for β, and *right-exact* if β is a cokernel for α. It is *exact in $\mathcal{V}_p$* if α is a kernel for β and β is a cokernel for α [11]. The functor $H_A : \mathcal{V}_p \to \mathcal{M}_R$ is left-exact since

$$0 \to H_A(U) \overset{H_A(\alpha)}{\longrightarrow} H_A(B) \overset{H_A(\beta)}{\longrightarrow} H_A(C) \ \ (*)$$

is an exact sequence of right R-modules whenever

$$0 \to U \xrightarrow{\alpha} B \xrightarrow{\beta} C$$

is a left-exact sequence of valuated p-groups.

Consider the functor $t_A : \mathcal{M}_R \to \mathcal{A}b$ defined by $t_A = - \otimes_R A$ for all $M \in \mathcal{M}_R$. If F is a free right R-module with basis $\{x_i \mid i \in I\}$, then

$$v(\Sigma_{i \in I} x_i \otimes a_i) = min\{v(a_i) \mid i \in I\}$$

defines a valuation on $t_A(F)$, and the resulting valuated group is denoted by $T_A(F)$ [1]. To define a valuation on $t_A(M)$ for an arbitrary right R-module M, we choose a free resolution

$$F_1 \xrightarrow{\alpha} F_0 \xrightarrow{\beta} M \to 0$$

of M. Applying t_A induces an exact sequence

$$T_A(F_1) \xrightarrow{t_A(\alpha)} T_A(F_0) \xrightarrow{t_A(\beta)} t_A(M) \to 0$$

where $t_A(\alpha)$ is a $\mathcal{V}_p$-map, which we denote as $T_A(\alpha)$, by [1]. Since $\mathcal{V}_p$ is pre-Abelian, there is a unique valuation v on $t_A(M)$ such that $t_A(\beta)$ becomes the $\mathcal{V}_p$-cokernel of $T_A(\alpha)$ [11]. We define $T_A(M) = (t_A(M), v)$, and observe $t_A = \mathcal{F}T_A$. The next result summarizes the basic properties of T_A which were established in [2, Section 2]:

Theorem 2.2. [2] *Let A be a finite valuated p-group.*

a) *$T_A : \mathcal{M}_R \to \mathcal{V}_p$ is a right exact functor.*
b) *The evaluation map $\theta_G : T_A H_A(G) \to G$ defined by $\theta_G(\alpha \otimes a) = \alpha(a)$ is a natural $\mathcal{V}_p$-map for all valuated p-groups G such that θ_P is an isomorphism for all A-projective groups P.*
c) *The natural map $\Phi_M : M \to \mathrm{Hom}(A, T_A(M))$ defined by $[\Phi_M(x)](a) = x \otimes a$ is a natural transformation such that $\theta_{T_A(M)} T_A(\Phi_M) = 1_{T_A(M)}$ for all right R-modules M. Moreover, Φ_P is an isomorphism for all finitely generated projective right R-modules P.*

An epimorphism $G \to H$ of valuated p-groups is *A-balanced* if the induced map $H_A(\alpha) : H_A(G) \to H_A(H)$ is onto. A valuated p-group G is *weakly A-generated* if we can find an A-balanced epimorphism

$$\oplus_I A \xrightarrow{\beta} G \to 0$$

for some index-set I. It is A-generated if β can be chosen to be a cokernel in $\mathcal{V}_p$. Although there is no need to distinguish between A-generated and weakly A-generated objects in an Abelian category, it is necessary to do this in the pre-Abelian case as was shown in [2].

A valuated p-group G is *A-presented* if there is an exact sequence

$$0 \to U \to F \to G \to 0$$

of valuated p-groups such that F is A-free and U is weakly A-generated. If this sequence can be chosen to be A-balanced, then G is called *A-solvable*. A valuated p-group G is A-presented if and only if $G \cong T_A(M)$ for some right R-module M.

Moreover, it is A-solvable if and only if θ_G is an isomorphism [2]. In particular, every A-projective group is A-solvable.

In a pre-Abelian category like $\mathcal{V}_p$, neither the 5-Lemma nor the Snake-Lemma need to hold [11]. The next result is frequently used in this paper as a substitute for the 5-Lemma throughout this paper:

Lemma 2.3. *Let A be a finite valuated p-groups. If $0 \to U \xrightarrow{\alpha} H \xrightarrow{\beta} G \to 0$ is a $\mathcal{V}_p$-exact sequence such that θ_H is an isomorphism, then there exists a commutative $\mathcal{V}_p$-diagram*

$$\begin{tikzcd} & T_AH_A(U) \arrow[r, "T_AH_A(\alpha)"] \arrow[d, "\theta_U"] & T_AH_A(H) \arrow[r, "T_AH_A(\beta)"] \arrow[d, "\theta_H", "\wr"'] & T_A(M) \arrow[r] \arrow[d, "\theta"] & 0 \\ 0 \arrow[r] & U \arrow[r, "\alpha"] & H \arrow[r, "\beta"] & G \arrow[r] & 0 \end{tikzcd}$$

with $\mathcal{V}_p$-exact rows in which $M = imH_A(\beta) \subseteq H_A(G)$ and $\theta : T_A(M) \to G$ is the evaluation map. Moreover, θ is a cokernel, and $\theta = \theta_G T_A(\iota)$ where $\iota : M \to H_A(G)$ is the inclusion map.

Proof. Since H_A is left-exact, every exact sequence

$$0 \to U \xrightarrow{\alpha} H \xrightarrow{\beta} G \to 0$$

of valuated groups induces an exact sequence

$$0 \to H_A(U) \xrightarrow{H_A(\alpha)} H_A(H) \xrightarrow{H_A(\beta)} M \to 0$$

of right R-modules where $M = im(H_A(\beta))$ is a submodule of $H_A(G)$. By Part a) of Theorem 2.2, the induced sequence

$$T_AH_A(U) \xrightarrow{T_AH_A(\alpha)} T_AH_A(H) \xrightarrow{T_AH_A(\beta)} T_A(M) \to 0$$

is right exact. Part b) of same result yields that θ_U and θ_G are $\mathcal{V}_p$-maps, and the commutativity of the diagram follows directly. Since $T_A(\iota)$ is a $\mathcal{V}_p$-map by another application of Theorem 2.2, the same holds for $\theta = T_A(\iota)\theta_G$. Using the fact that θ_H is a $\mathcal{V}_p$-isomorphism, we obtain $\theta[T_A(\beta)\theta_H^{-1}] = \beta$. Because $T_A(\beta)$ is a cokernel, θ is a cokernel by Lemma 2.1. □

Ulmer described the objects of an Abelian Groethendick category which are flat over their endomorphism ring [13]. When discussing the validity of Ulmer's result in $\mathcal{V}_p$, one immediately realizes that his original arguments need to be modified extensively because this category is only pre-Abelian. In particular, we want to remind the reader that a finite valuated p-group is flat as an R-module if and only if it is projective.

Theorem 2.4. *The following conditions are equivalent for a finite valuated p-group A:*

a) *A is projective as a left R-module.*
b) *Whenever $\phi \in \mathrm{Mor}(A^n, A)$ for some $n < \omega$, then* ker ϕ *is weakly A-generated.*
c) *Whenever $\phi \in \mathrm{Mor}(G, H)$ for A-solvable valuated p-groups G and H, then* ker ϕ *is weakly A-generated.*

Proof. *a*) $\Rightarrow$ *c*): For $K = \ker\ \phi$, consider the exact sequence

$$0 \to H_A(K) \longrightarrow H_A(G) \xrightarrow{\phi} M \to 0$$

of right R-modules in which $M = im(H_A(\phi))$ is a submodule of $H_A(H)$. Let ι denote embedding $M \subseteq H_A(H)$. By Proposition 2.3, we obtain a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & T_AH_A(K) \arrow[r] \arrow[d, "\theta_K"] & T_AH_A(G) \arrow[r, "T_AH_A(\phi)"] \arrow[d, "\theta_{\oplus_I G}", "\wr"'] & T_A(M) \arrow[r] \arrow[d, "\theta"] & 0 \\
0 \arrow[r] & K \arrow[r] & G \arrow[r, "\phi"] & H &
\end{tikzcd}$$

of $\mathcal{V}_p$-maps whose top-row is right exact in $\mathcal{V}_p$. Moreover, it is exact in $\mathcal{A}b$ since A is projective as a left R-module. Using the projectivity of A once more yields that $T_A(\iota)$ is a monomorphism, and the same holds for $\theta = \theta_H T_A(\iota)$ since H is A-solvable. Thus, θ is an isomorphism of Abelian groups. Because the 3-Lemma is valid in $\mathcal{A}b$, we obtain that θ_K is an epimorphism in $\mathcal{A}b$, and hence in $\mathcal{V}_p$.

Since *c*) $\Rightarrow$ *b*) is obvious, it remains to show *b*) $\Rightarrow$ *a*):

It suffices to establish that the inclusion map $\iota : I \to R$ induces a monomorphism $t_A(\iota) : t_A(I) \to t_A(R)$ of Abelian groups for all right ideals I of R. Since R is finite, $I = \{r_1, \ldots, r_n\}$. We define a map $\phi_1 : F = R^n \to I$ by $\phi_1(e_i) = r_i$ where $\{e_1, \ldots, e_n\}$ is an R-basis of F. Set $\phi = \iota\phi_1 : F \to R$. By b), the kernel K of the $\mathcal{V}_p$-map $T_A(\phi) : T_A(F) \to T_A(R)$ is weakly A-generated. Since A is finite, we can select a finite A-projective group P and an A-balanced epimorphism $\lambda : P \to K$. Because

$$0 \to K \to T_A(F) \xrightarrow{T_A(\phi)} T_A(R)$$

is $\mathcal{V}_p$-exact, the induced sequence

$$0 \to H_A(K) \to H_AT_A(F) \xrightarrow{H_AT_A(\phi)} H_AT_A(R)$$

is exact. Combining this sequence with $H_A(\lambda)$ yields that the top-row of the commutative diagram

$$\begin{tikzcd}
H_A(P) \arrow[r, "H_A(\lambda)"] & H_AT_A(F) \arrow[r, "H_AT_A(\phi)"] & H_AT_A(R) \\
& F \arrow[u, "\Phi_F"', "\wr"] \arrow[r, "\phi"] & R \arrow[u, "\Phi_R"', "\wr"]
\end{tikzcd}$$

of right R-modules is exact. In view of $\phi(F) = I$, the diagram gives us the exact sequence

$$(E)\ \ H_A(P) \xrightarrow{H_A(\lambda)} H_AT_A(F) \xrightarrow{\phi_1\Phi_F^{-1}} I \to 0$$

of right R-modules. Since $\theta_{T_A(M)}T_A(\Phi_M) = 1_{T_A(M)}$ for all right R-modules M, we obtain $\theta_{T_A(X)} = T_A(\Phi_X^{-1})$ for all finitely generated projective right R-modules X. Hence,

$$T_A(\phi)\theta_{T_A(F)} = T_A(\phi\Phi_F^{-1}) = T_A(\Phi_R^{-1}H_AT_A(\phi)) = \theta_{T_A(R)}T_AH_AT_A(\phi).$$

Because of this and Theorem 2.2, an application of T_A yields the commutative diagram

$$\begin{array}{ccccc} T_AH_A(P) & \xrightarrow{T_AH_A(\lambda)} & T_AH_AT_A(F) & \xrightarrow{T_AH_AT_A(\phi)} & T_AH_AT_A(R) \\ \wr\downarrow\theta_P & & \wr\downarrow\theta_{T_A(F)} & & \wr\downarrow\theta_{T_A(R)} \\ P & \xrightarrow{\lambda} & T_A(F) & \xrightarrow{T_A(\phi)} & T_A(R) \end{array}$$

of Abelian groups. Since it suffices to show that $t_A(\iota)$ is a monomorphism of Abelian groups, our computations are done from this point only in $\mathcal{A}b$ instead of in $\mathcal{V}_p$. In particular, we use the fact that the $\mathcal{V}_p$-kernel of a map is its kernel in $\mathcal{A}b$ with a valuation added. The symbols t_A and T_A can be used interchangeably when computing in $\mathcal{A}b$.

Observe that the bottom row of the last diagram is exact at $T_A(F)$ as a sequence of Abelian groups by the choice of P and λ. Since the vertical maps are isomorphisms, the top-row is exact at $T_AH_AT_A(F)$. Moreover, (E) induces the exact sequence

$$T_AH_A(P) \stackrel{T_AH_A(\lambda)}{\longrightarrow} T_AH_AT_A(F) \stackrel{T_A(\phi_1\Phi_F^{-1})}{\longrightarrow} T_A(I) \to 0$$

of Abelian groups. Therefore, the map $T_A(\phi_1\Phi_F^{-1})$ is a cokernel in $\mathcal{A}b$ for the left top-map $T_AH_A(\lambda)$. On the other hand, the projection

$$\pi : T_A(F) \to G = T_A(F)/K$$

is a cokernel of λ in $\mathcal{A}b$. Hence, there is an isomorphism $\sigma : T_A(I) = t_A(I) \to G$ of Abelian groups such that $\pi\theta_{T_A(F)} = \sigma T_A(\phi_1\Phi_F^{-1})$. Since the bottom row of the last diagram is exact at $T_A(F)$, there is a map $\tau : G \to T_A(R)$ with $\tau\pi = T_A(\phi)$ using the exactness of the bottom row of the last diagram once more. For $g \in \ker\, \tau$, select $x \in T_A(F)$ with $\pi(x) = g$. Then $0 = \tau\pi(x) = T_A(\phi)(x)$ yields $x = \lambda(y)$ for some $y \in P$. Hence, $g = \pi\lambda(y) = 0$, and τ is a monomorphism.

Because $H_ATA(\phi_1)\Phi_F = \Phi_I\phi_1$, we have

$$\begin{aligned} \theta_{T_A(R)}T_AH_AT_A(\iota)T_A(\Phi_I)T_A(\phi_1) &= \theta_{T_A(R)}T_AH_AT_A(\iota)T_AH_AT_A(\phi_1)T_A(\Phi_F) \\ &= \theta_{T_A(R)}T_AH_AT_A(\phi)T_A(\Phi_F) \\ &= T_A(\phi)\theta_{T_A(F)}T_A(\Phi_F) \\ &= \tau\pi\theta_{T_A(F)}T_A(\Phi_F) \\ &= \tau\sigma T_A(\phi_1\Phi_F^{-1})T_A(\Phi_F) \\ &= \tau\sigma T_A(\phi_1). \end{aligned}$$

Since $T_A(\phi_1)$ is an epimorphism, we obtain that

$$\theta_{T_A(R)}T_AH_AT_A(\iota)T_A(\Phi_I) = \tau\sigma$$

is a monomorphism since the maps on the right are monomorphisms, and the same holds for

$$T_A(\Phi_R)t_A(\iota) = T_AH_AT_A(\iota)T_A(\Phi_I)$$

using the fact that $T_A(R) \cong A$. Because $T_A(\Phi_R)$ is an isomorphism, $t_A(\iota)$ is one-to-one as desired. $\square$

For a finite p-group G, let $e(A)$ denote the smallest $n < \omega$ such that $p^nG = 0$.

Corollary 2.5. *Every finite valuated p-group A is a direct summand of a finite valuated p-group B such that $e(A) = e(B)$ and B is flat as a module over its endomorphism ring.*

Proof. Choose $n < \omega$ minimal with the property that $p^n A = 0$, and consider the group $B = \mathbb{Z}/p^n\mathbb{Z} \oplus A$ where $\mathbb{Z}/p^n$ carries the height valuation h. Since h is the smallest valuation on $\mathbb{Z}/p^n\mathbb{Z}$, and every B-generated group is bounded by p^n, the kernel of every map between any two B-generated groups is a $\mathcal{V}_p$-epimorphic image of $(\mathbb{Z}/p^n\mathbb{Z}, h)$. By Theorem 2.4, B is projective over its endomorphism ring. □

We continue our discussion by looking at simply presented groups. A *(p-)valuated tree* is a set X, on which a partial multiplication by p is defined, together with a function v assigning a value $v(x)$ to each $x \in X$ which is either an ordinal or ∞ subject to the rules

i) If $p^n x = x$ for some $0 < n < \omega$, then $px = x$, and there is exactly one element in X with this property, called the *root of* X.
ii) $v(px) > v(x)$ whenever px is defined.

Moreover, if $X_1, \ldots, X_n$ are rooted valuated trees, then the co-product $\cup_{i=1}^n X_i$ in the category of valuated p-tree is the tree that is obtained by joining $X_1, \ldots, X_n$ at their roots.

Associated with any rooted tree X is a *simply presented* valuated p-group $S(X)$ defined as F_X/R_X where F_X is a free $\mathbb{Z}_p$-module with basis $\{\langle x\rangle | x \in X\}$ and R_X is generated by the elements $p\langle x\rangle - \langle px\rangle$. If we set $\overline{x} = \langle x\rangle + R_X$, then every $g \in S(X)$ has a unique presentation $g = \Sigma_{x\in X} n_x \overline{x}$ with $0 \leq n_x < p$, and the valuation on $S(X)$ is defined by

$$v(g) = min\{v(x) \mid n_x \neq 0\}.$$

Finally, a valuated cyclic p-group G of order p^n is of the form $G = S(X)$ for a valuated p-tree $X = \{x_0, \ldots, x_{n-1}\}$ such that $G = \langle x_0\rangle$ and $x_i = px_{i-1}$ for $i = 1, \ldots, n$.

A map $\psi : X \to Y$ between valuated trees is a *tree map* if $\psi(px) = p\psi(x)$ if px exists and $v(\psi(x)) \geq v(x)$. A tree map $r : X \to X$ is a *retraction* if $r^2 = r$. Hunter, Richman and Walker showed that there is an order preserving retraction from $S(X)$ onto X for all valuated trees [7]. Moreover, every tree map $\psi : X \to Y$ induces a $\mathcal{V}_p$-map $\overline{\psi} : S(X) \to S(Y)$.

Corollary 2.6. *The following conditions are equivalent for a finite valuated p-group A:*

a) *A is a cyclic group.*
b) *A is an indecomposable simply presented group which is projective as an R-module.*

Proof. It remains to show that an indecomposable simply presented group A is cyclic if it is projective as an R-module. Since A is indecomposable, R is a local ring. Therefore, all projective R-modules are free. Consequently, we can find $a \in A$ such that $A = Ra$, and $ra \neq 0$ for all non-zero $r \in R$.

Write $A = S(X)$ for some valuated tree X. Since A is indecomposable, X is irretractable and has a unique element y of order p. Let $x_1, \ldots, x_n$ be the elements of maximal order of X, and select $r_1, \ldots, r_n \in R$ such that $x_i = r_i a$ for $i = 1, \ldots, n$.

If $r_1, \ldots, r_n \in J(R)$, then $A = J(R)A$ because $x_1, \ldots, x_n$ generate A as an Abelian group, which is impossible by Nakayama's Lemma. Therefore, we may, without loss of generality, assume $r_1 \notin J(R)$. Thus, r_1 is a unit in R, and

$$A = Ra = Rr_1a = Rx_1.$$

Moreover, if $sx_1 = 0$, then

$$0 = sx_1 = sr_1(r_1^{-1}x_1) = sr_1a$$

from which we obtain $sr_1 = 0$. Then $s = 0$ since r_1 is a unit of R. Therefore, $\phi(x_1) \neq 0$ for all non-zero $\phi \in R$.

Suppose that $n > 1$, and define a map $r : X \to X$ by $r(x) = 0$ if $x \neq x_2$ and $r(x_2) = y$. Observe that $v(x_2) \leq v(y)$ by the choice of x_2 and y. For $x \neq x_2$, $px \neq x_2$ because x_2 is an element of maximal order. Thus, $r(px) = 0$. On the other hand $pr(x_2) = py = 0$ while $r(px_2) = 0$ since $px_2 \neq x_2$. Therefore, r is a map of valuated trees, and induces an endomorphism α of the valuated group A with $\alpha(x_1) = 0$ and $\alpha(x_2) = y \neq 0$, a contradiction. Consequently, X has only one element x_1 of maximal order, and $A = \langle x_1 \rangle$. □

However, Corollary 2.5 shows that a simply presented group which is flat as a module over its endomorphism ring need not be a direct sum of cyclic groups. Moreover, there are infinitely many isomorphism classes of indecomposable finite valuated p-groups G such that $p^4G = 0$ and $v(g) \leq 9$ for all $0 \neq g \in G$ [3, Example 8.2.5]. Furthermore, the category of indecomposable finite valuated p-groups G such that $p^5G = 0$ and $v(g) \leq 11$ for all $0 \neq g \in G$ has wild representation type [3, Example 8.2.6].

Example 2.7. Let $A_1 = \langle a_1 \rangle$, $A_2 = \langle a_2 \rangle$ and $A_3 = \langle a_3 \rangle$ be cyclic groups of order p^3, and define a valuation on A_1 by $v(a_1) = 1$, $v(pa_1) = 4$ and $v(p^2a_1) = 5$ and on A_2 by $v(a_2) = 2$, $v(pa_2) = 3$ and $v(p^2a_2) = 5$. Finally, set $v(a_3) = \infty$.

To see that $A = A_1 \oplus A_2 \oplus A_3$ is not flat as an R-module, consider the map $\delta : A_1 \oplus A_2 \to A_3$ defined by $\delta((na_1, ma_2)) = (n - m)a_3$. It is easy to see that $K = \ker\ \delta =< (a_1, a_2) >$ and $v(a_1, a_2) = 1$, $v(pa_1, pa_2) = 3$, and $v(p^2a_1, p^2a_2) = 5$.

If $\phi \in \mathrm{Mor}(A_1, K)$, then $\phi(a_1) \in pK$ for otherwise

$$4 = v(pa_1) \leq v(\phi(pa_1)) = v(pa_1, pa_2) = 3.$$

Similarly, if $\psi \in \mathrm{Mor}(A_2, K)$, then $\psi(a_2) \in pK$ since otherwise

$$2 = v(a_2) \leq v(\psi(a_2)) = v(a_1, a_2) = 1.$$

Since $\mathrm{Mor}(A_3, A_1 \oplus A_2) = 0$, we have $im\ \theta_K \subseteq pK$, and K is not weakly A-generated. By Theorem 2.4, A is not projective as an R-module.

Example 2.8. If $A = \langle x \rangle$ is a cyclic group of order p^2 with the height valuation, then A is free as a module over its endomorphism ring $E = \mathbb{Z}/p^2\mathbb{Z}$. Moreover, $v(px) = 1$. On the other hand, $M = \mathbb{Z}/p\mathbb{Z}$ is a left E-module which fits into the exact sequence

$$E \xrightarrow{\alpha} E \xrightarrow{\beta} M \to 0$$

where $\alpha(1 + p^2\mathbb{Z}) = p + p^2\mathbb{Z}$ and $\beta(1 + p^2\mathbb{Z}) = 1 + p\mathbb{Z}$. Then $T_A(M) \cong \mathbb{Z}/p\mathbb{Z}$ and setting $v(1 + p\mathbb{Z}) = 0$ yields the cokernel valuation on $T_A(M)$. On the other hand, the map $\gamma : M \to E$ defined by $\gamma(1 + p\mathbb{Z}) = p + p^2\mathbb{Z}$ induces a monomorphism $T_A(\gamma) : T_A(M) \to A$ such that $im(T_A(\gamma)) = \langle px \rangle$. Since

$$0 = v(1 + p\mathbb{Z}) < v(px) = 1,$$

the map $T_A(\gamma)$ does not preserve valuations. If we consider the sequence

$$0 \to M \xrightarrow{\gamma} E \xrightarrow{\beta} M \to 0,$$

then $T_A(\gamma) : T_A(M) \to T_A(E)$ is not a kernel for $T_A(\beta)$.

Therefore, the class of A-solvable groups may behave quite different from the case that A is either a torsion-free or mixed Abelian group even if A is a finite valuated p-group which is projective over its endomorphism ring. For instance, the kernel of a map between two A-solvable groups need not be A-solvable, nor is a weakly A-generated subgroup U of an A-solvable group necessarily A-solvable.

Corollary 2.9. *Let A be a finite valuated p-group which is projective as an R-module. An A-generated subgroup U of an A-solvable group G is A-solvable.*

Proof. By Proposition 2.3, it remains to show that θ_U is an isomorphism in $\mathcal{V}_p$. Since A is projective as an R-module, one can argue as in the case of torsion-free groups that θ_U is an isomorphism of Abelian groups. Select an A-free group F and an A-balanced exact sequence $0 \to V \xrightarrow{\alpha} F \xrightarrow{\beta} U \to 0$. It induces the commutative diagram

$$\begin{tikzcd}
 & & T_AH_A(F) \arrow[r, "T_AH_A(\beta)"] \arrow[d, "\theta_F", "\wr"'] & T_AH_A(U) \arrow[r] \arrow[d, "\theta_U"] & 0 \\
0 \arrow[r] & V \arrow[r, "\alpha"] & F \arrow[r, "\beta"] & U \arrow[r] & 0.
\end{tikzcd}$$

Since θ_U is an isomorphism of Abelian groups, $T_AH_A(\beta)\theta_F^{-1}\alpha = 0$. There is a $\mathcal{V}_p$-map $\lambda : U \to T_AH_A(U)$ such that $T_AH_A(\beta)\theta_F^{-1} = \lambda\beta$ because β is a cokernel of α in $\mathcal{V}_p$. Then

$$\theta_U\lambda\beta = \theta_U T_AH_A(\beta)\theta_F^{-1} = \beta$$

yields $\theta_U\lambda = 1_U$. Thus, $\lambda\theta_U = 1_{T_AH_A(U)}$ since θ_U is an isomorphism of Abelian groups. Hence

$$v(x) = v(\lambda\theta_U(x)) \geq v(\theta_U(x)) \geq v(x)$$

for all $x \in T_AH_A(U)$. Thus, θ_U is a $\mathcal{V}_p$-isomorphism. $\square$

Corollary 2.10. *The following conditions are equivalent for a finite valuated p-group A:*

a) *A is a progenerator for ${}_R\mathcal{M}$.*

b) i) *Whenever $\phi \in \mathrm{Mor}(G, H)$ for A-solvable valuated p-groups G and H, then $\ker\ \phi$ is weakly A-generated.*

   ii) *Whenever $\phi \in \mathrm{Mor}(G, H)$ is an epimorphism of A-solvable valuated p-groups G and H, then $H_A(\phi)$ is an epimorphism.*

Proof. *a)* $\Rightarrow$ *b)*: It remains to show that ii) holds. For this, consider the submodule $M = im\ H_A(\phi)$ of $H_A(H)$, and denote the inclusion map $M \to H_A(H)$ by ι. The evaluation map $\theta : T_A(M) \to H$ is a $\mathcal{V}_p$-map since it satisfies $\theta = \theta_H T_A(\iota)$. Moreover, it is one-to-one since A is a projective as a right R-module guarantees that $T_A(\iota)$ is a monomorphism of Abelian groups and θ_H is an isomorphism. On the other hand, it also fits into the commutative diagram

$$\begin{array}{ccccc} T_AH_A(G) & \xrightarrow{T_A(\phi)} & T_A(M) & \longrightarrow & 0 \\ \wr\big\downarrow \theta_G & & \big\downarrow \theta & & \\ G & \xrightarrow{\phi} & H & \longrightarrow & 0. \end{array}$$

Hence, θ is an isomorphism of Abelian groups, and the same holds for $T_A(\iota)$. However, the latter fits into the exact sequence

$$T_A(M) \overset{T_A(\iota)}{\longrightarrow} T_AH_A(H) \to H_A(H)/M \to 0.$$

Therefore, $T_A(H_A(H)/M) = 0$. Since A is a projective generator, $M = H_A(H)$.

b) $\Rightarrow$ *a)*: By [9, Proposition 2.4], every faithful projective module is a generator. Since A is a projective left R-module by Theorem 2.4, it remains to show that it is faithful. Let M be a right R-module with $t_A(M) = 0$, and consider an exact sequence $P \to F \to M \to 0$ in which P and F are projective module. By Theorem 2.2, we obtain a right exact sequence $T_A(P) \to T_A(F) \to 0$ of valuated p-groups. By ii), the top sequence in the diagram

$$\begin{array}{ccccccc} H_AT_A(P) & \longrightarrow & H_AT_A(F) & \longrightarrow & 0 & & \\ \wr\big\uparrow \Phi_P & & \wr\big\uparrow \Phi_F & & & & \\ P & \longrightarrow & F & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

is exact. Thus, $M = 0$. $\square$

3. Hereditary and Quasi-Frobenius Endomorphism Rings

We conclude our discussion by considering finite valuated p-groups A whose endomorphism ring has specific ring-theoretic properties. We focus particularly on the cases that R is either hereditary or self-injective. We want to remind the reader that there is no need to deal with right/left conditions since R is finite [4].

A finite valuated p-group G is *A-torsion-less* if there is a monomorphism $G \to A^\ell$ for some $\ell < \omega$. We say that an exact sequence of valuated groups is *A-cobalanced* if A is injective with respect to it.

Theorem 3.1. *Let R be a finite valuated p-group A:*

a) *R is hereditary if and only if A is a direct sum of cyclic groups of order p.*
b) *R is (semi-)simple Artinian if and only if $A \cong B^m$ where B is a cyclic group of order p.*

c) *If R is a quasi-Frobenius ring, then every exact sequence $0 \to U \to G$ in which U is weakly A-generated and G is A-solvable is A-cobalanced. If A is a projective R-module, then the converse holds, and every A-presented group is A-torsion-less.*

Proof. a) If R is hereditary, then so is eRe for any idempotent e of R [4]. If B is an indecomposable summand of A, then there is a primitive idempotent e of R such that eRe is the $\mathcal{V}_p$-endomorphism ring of B. Since eRe is a hereditary local ring, all right ideals of eRe are free eRe-modules. However, this means that eRe is a field since it is finite. Because, $pE(B)$ is a proper ideal of $E(B)$, we have $pB = 0$. By [8], B is a cyclic group. Hence, A is a direct sum of cyclic groups of order p.

Conversely, if A has the described form, then $A = A_1 \oplus \ldots \oplus A_n$ where $A_i \cong B_i^{\ell_i}$ and each B_i is a cyclic group of order p. If $B_i = \langle b_i \rangle$, then no generality is lost if we assume $v(b_i) < v(b_j)$ for $i < j$ and $v(b_i) \neq \infty$ for $i < n$. Then $\mathrm{Mor}(B_i, B_j) \cong \mathbb{Z}/p\mathbb{Z}$ if $i \leq j$, and $\mathrm{Mor}(B_i, B_j) = 0$ otherwise. Therefore, R is Morita-equivalent to a lower triangular matrix ring over $\mathbb{Z}/p\mathbb{Z}$. By [5], R is hereditary.

b) We continue using the notation from a). If $A = A_1 \oplus \ldots \oplus A_n$ and $n > 1$, then $\mathrm{Mor}(A_i, A_j) = 0$ for $i > j$, but $\mathrm{Mor}(A_i, A_j) \neq 0$ for $i < j$. In particular, $N(R) \neq 0$. b) now follows immediately.

c) If R is quasi-Frobenius, then we consider an exact sequence $0 \to U \xrightarrow{\alpha} G$ in which U is an epimorphic image of an A-projective group and G is A-solvable. For $\phi \in \mathrm{Mor}(U, A)$, we can find a map $\psi : H_A(G) \to R$ such that $\psi H_A(\alpha) = \phi$. Since both, α and ϕ, fit into the commutative diagram

$$\begin{array}{ccc} T_A H_A(U) & \xrightarrow{T_A H_A(.)} & T_A H_A(G) \\ \Big\downarrow \theta_U & & \wr \Big\downarrow \theta_G \\ U & \xrightarrow{\quad . \quad} & G, \end{array}$$

we obtain

$$T_A(\psi)\theta_G^{-1}\alpha\theta_U = \theta_A T_A(\psi) T_A H_A(\alpha) = \theta_A T_A H_A(\phi) = \phi\theta_U.$$

Because θ_U is a $\mathcal{V}_p$-epimorphism, $T_A(\psi)\theta_G^{-1}\alpha = \phi$.

Conversely, let

$$0 \to I \xrightarrow{\alpha} R$$

be an exact sequence and $\phi \in \mathrm{Hom}_R(I, R)$. Because A is a flat R-module,

$$0 \to T_A(I) \xrightarrow{T_A(\alpha)} T_A(R)$$

is a $\mathcal{V}_p$-exact sequence. Since $T_A(I)$ is an image of an A-projective group, there is a map $\psi \in \mathrm{Mor}(T_A(R), T_A(R))$ such that $\psi T_A(\alpha) = T_A(\phi)$. We consider commutative diagrams of the form

$$\begin{array}{ccccc} 0 & \longrightarrow & H_A T_A(I) & \xrightarrow{H_A T_A(.)} & H_A T_A(R) \\ & & \Big\uparrow \Phi_I & & \wr \Big\uparrow \Phi_R \\ 0 & \longrightarrow & I & \xrightarrow{\quad . \quad} & R \end{array}$$

to obtain

$$\begin{aligned}\Phi_R^{-1}H_A(\psi)\Phi_R\alpha &= \Phi_R^{-1}H_A(\psi)H_AT_A(\alpha)\Phi_I\\ &= \Phi_R^{-1}H_AT_A(\phi)\Phi_I\\ &= \Phi_R^{-1}\Phi_R\phi = \phi.\end{aligned}$$

Finally, if G is an A-presented group, then $G \cong T_A(M)$ for some finitely generated right R-module M by [2] as mentioned before. Let E be an injective hull of M. Since R is quasi-Frobenius, E is projective. Thus, M can be embedded into a free R-module F, which can be chosen to be finite since M is finite. Then $T_A(M)$ is isomorphic to a submodule of $T_A(F)$ since A is projective. □

Corollary 3.2. *Let A be a finite valuated p-group whose endomorphism ring is self-injective. Every exact sequence*

$$0 \to P \xrightarrow{\alpha} G$$

such that P is A-projective and G is A-solvable splits.

□

We conclude with two examples that show that the endomorphism ring of a direct sum of cyclic valuated p-groups may or may not be quasi-Frobenius:

Example 3.3. a) Let A_1 be a cyclic group of order p^n, and A_2 a cyclic valuated group of order p^n whose generator x satisfies $v(p^{n-1}x) > n$. Then, the endomorphism ring of $A = A_1 \oplus A_2$ is the lower triangular matrix ring over $\mathbb{Z}/p^n\mathbb{Z}$, which is not self-injective.

b) By [6, Example 1], the ring

$$R = \begin{bmatrix} \mathbb{Z}/p^3\mathbb{Z} & p\mathbb{Z}/p^3\mathbb{Z} \\ p\mathbb{Z}/p^3\mathbb{Z} & \mathbb{Z}/p^3\mathbb{Z} \end{bmatrix}$$

is quasi-Frobenius. Consider two cyclic valuated groups $A_1 = (\langle x_1\rangle, v_1)$ and $A_2 = (\langle x_2\rangle, v_2)$ of order p^3 such that $v_1(x_1) = 1$, $v_1(px_1) = 4$, $v_2(x_2) = 2$, $v_2(px_2) = 3$ and $v_1(p^2x_1) = v_2(p^2x_2) \geq 5$. In view of the fact that $\mathrm{Mor}(A_i, A_j) \cong \mathbb{Z}/p^2\mathbb{Z}$ for $i \neq j$, we obtain that $A = A_1 \oplus A_2$ has R as its $\mathcal{V}_p$-endomorphism ring.

References

[1] Albrecht, U., *Bewertete p-Gruppen und ein Satz von Szele*, J. Algebra, **97**(1985), 201-220.

[2] Albrecht, U., *A-presented valuated groups*, preprint.

[3] Arnold, D.M., *Abelian Groups and Representations of Finite Partially Ordered Sets*, CMS Books in Mathematics, Springer, 2000.

[4] Chatters, A.W., Hajarnavis, C.R., *Rings with Chain Conditions*, Pitman, 1980.

[5] Goodearl, K.R., *Ring Theory*, Pure and Applied Mathematics, No. 33, Marcel Dekker, New York - Basel, 1976.

[6] Hanulla, T.A., *On the construction of quasi-Frobenius rings*, J. Algebra, **25**(1973), 403-414.

[7] Hunter, R., Richman, F., Walker, E.A., *Simply presented valuated p-groups*, J. Algebra, **49**(1977), 125-133.

[8] Hunter, R., Richman, F., Walker, E.A., *Finite direct sums of cyclic valuated p-groups*, Pacific J. Math., **69**(1977), 97-133.

[9] Ishikawa, T., *Faithfully exact functors and their applications to projective modules and injective modules*, Nagoya Math. J., **44**(1964), 29-42.

[10] Richman, F., Walker, E.A., *Ext in pre-Abelian categories*, Pacific J. Math., **71**(1977), 521-535.

[11] Richman, F., Walker, E.A., *Valuated groups*, J. Algebra, **56**(1979), 145-167.

[12] Stenstrom, Bo, *Rings of Quotients*, Die Grundlehren der Mathematischen Wissenschaften, Band 217, Springer-Verlag, New York - Heidelberg, 1975.

[13] Ulmer, F., *A flatness criterrion in Groethendick categories*, Invent. Math., **19**(1973), 331-336.

[14] Yakolev, A.V., *Homological algebra in pre-abelian categories*, Journal of Soviet Mathematics, **19**(1982), 1060-1067.

Ulrich Albrecht
Department of Mathematics
Auburn University
Auburn, AL 36849, U.S.A.
e-mail: `albreuf@mail.auburn.edu`

Stud. Univ. Babeş-Bolyai Math. 62(2017), No. 3, 309–323
DOI: 10.24193/subbmath.2017.3.04

A study on Hermite-Hadamard type inequalities for s-convex functions via conformable fractional integrals

Erhan Set and Abdurrahman Gözpınar

Abstract. In the present note, firstly we established a generalization of Hermite Hadamard's inequality for s-convex functions via conformable fractional integrals which generalized Riemann-Liouville fractional integrals. Secondly, we proved new identity involving conformable fractional integrals via beta and incompleted beta functions.Then, by using this identity, some Hermite Hadamard type integral inequalities for s-convex functions in the second sense are obtained.

Mathematics Subject Classification (2010): 26A33, 26A51, 26D10, 26D15.

Keywords: s-convex functions, Hermite-Hadamard inequality, conformable fractional integrals.

1. Introduction

One of the most famous inequality for convex functions is so called Hermite-Hadamard inequality as follows: Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

This famous inequality discovered by C. Hermite and J. Hadamard is important in the literature. For more studies via Hermite Hadamard type inequalities see [13] in the references.

Definition 1.1. Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a function and $a, b \in I$ with $a < b$, the function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if the inequality

$$f\left(tx + (1-t)\,y\right) \leq tf\left(x\right) + (1-t)\,f\left(y\right)$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

Definition 1.2. [7, 15] A function $f : \mathbb{R}_+ \to \mathbb{R}$ is said to be s-convex in the second sense if

$$f(\alpha x + \beta y) \leq \alpha^s f(x) + \beta^s f(y)$$

for all $x, y \in \mathbb{R}_+$ and all $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.

We denote this by K_s^2. It is obvious that the s-convexity means just the convexity when $s = 1$.

In [12] Dragomir and Fitzpatrick proved a variant of Hermite-Hadamard inequality which holds for s-convex functions in the second sense.

Theorem 1.3. *Suppose that $f : [0, \infty) \to [0, \infty)$ is an s-convex function in the second sense, where $s \in (0, 1]$ and let $a, b \in [0, \infty), a < b$. If $f \in L^1[a, b]$, then the following inequality hold:*

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)\, dx \leq \frac{f(a) + f(b)}{s+1} \tag{1.2}$$

The constant $k = \frac{1}{s+1}$ is the best possible in the second inequality in (1.2). For more study related to s-convexity in the second sense, see, e.g, (for example) ([3], [5], [11]).

Theory of convex functions has great importance in various fields of pure and applied sciences. It is known that theory of convex functions is closely related to theory of inequalities. Many interesting convex functions inequalities established via Riemann-Liouville fractional integrals. Now, lets us give some necessary definition and mathematical preliminaries of fractional calculus theory as follows, which are used lots of study. For more details, one can consult ([8]-[10], [14], [16]-[23], [28]).

Definition 1.4. Let $f \in L_1[a, b]$. The Riemann-Liouville integrals $J_{a+}^{\alpha} f$ and $J_{b-}^{\alpha} f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \qquad x > a$$

and

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, \qquad x < b$$

respectively. Here $\Gamma(t)$ is the Gamma function and its definition is

$$\Gamma(t) = \int_0^{\infty} e^{-x} x^{t-1} dx.$$

It is to be noted that $J_{a+}^0 f(x) = J_{b-}^0 f(x) = f(x)$ and in the case of $\alpha = 1$, the fractional integral reduces to the classical integral.

The beta function defined as follows:

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^1 t^{a-1} (1-t)^{b-1} dt, \qquad a, b > 0,$$

where $\Gamma(\alpha)$ is Gamma function. The incomplete beta function is defined by

$$B_x(a,b) = \int_0^x t^{a-1}(1-t)^{b-1}dt, \ 0 \le x \le 1.$$

For $x = 1$, the incomplete beta function coincides with the complete beta function. For easy understanding the computation in our theorems, let us give some properties of beta and incompleted beta function:

$$\begin{aligned}
&B(a,b) = B_t(a,b) + B_{1-t}(b,a), \ i.e \ B(a,b) = B_{\frac{1}{2}}(a,b) + B_{\frac{1}{2}}(b,a) \\
&B_x(a+1,b) = \frac{aB_x(a,b) - (x)^a(1-x)^b}{a+b} \\
&B_x(a,b+1) = \frac{bB_x(a,b) + (x)^a(1-x)^b}{a+b} \\
&B(a,b+1) + B(a+1,b) = B(a,b)
\end{aligned}$$

In [21] Sarıkaya et al. gave a remarkable integral inequality of Hermite-Hadamard type involving Riemann-Liouville fractional integrals as follows:

Theorem 1.5. *Let $f : [a,b] \to \mathbb{R}$ be a positive function with $0 \le a < b$ and $f \in L^1[a,b]$. If f is convex function on $[a,b]$, then the following inequality for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[(J_{a^+}^\alpha f)(b) + (J_{b^-}^\alpha f)(a)] \le \frac{f(a)+f(b)}{2} \tag{1.3}$$

It is obviously seen that, if we take $\alpha = 1$ in Theorem 1.5, then the inequality (1.3) reduces to well known Hermite-Hadamard inequality as (1.1).

Hermite-Hadamard type inequalities for s-convex functions via Riemann-Liouville fractional integral is given in [22] as follows:

Theorem 1.6. *Let $f : [a,b] \to \mathbb{R}$ be a positive function with $0 \le a < b$ and $f \in L_1[a,b]$. If f is s-convex mapping in the second sense on $[a,b]$, then the following inequality for fractional integral with $\alpha > 0$ and $s \in (0,1]$ hold:*

$$\begin{aligned}
2^{s-1}f\left(\frac{a+b}{2}\right) &\le \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[(J_{a^+}^\alpha f)(b) + (J_{b^-}^\alpha f)(a)] \\
&\le \alpha\Big[\frac{1}{\alpha+s} + B(\alpha,s+1)\Big]\frac{f(a)+f(b)}{2}
\end{aligned} \tag{1.4}$$

where B(a,b) is Euler beta function.

Sarikaya et al. established an identity which we will generalize for conformable fractional integral in section 3 for differentiable convex mappings via Riemann-Liouville fractional integral. Then they gave some results by using this identity.

Lemma 1.7. [21] *Let $f : [a, b] \to \mathbb{R}$ be a differentiable mapping on (a, b) with $a < b$. If $f' \in L[a, b]$, then the following equality for fractional integrals holds:*

$$\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[J^{\alpha}_{a+}f(b) + J^{\alpha}_{b_-}f(a)] \tag{1.5}$$
$$= \frac{b-a}{2}\int_0^1 \left[(1-t)^{\alpha} - t^{\alpha}\right] f'(ta+(1-t)b)dt.$$

Theorem 1.8. [21] *Let $f : [a, b] \to \mathbb{R}$ be a differentiable mapping on (a, b) with $a < b$. If $f' \in L[a, b]$, then the following inequality for fractional integrals holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[J^{\alpha}_{a+}f(b) + I^{\alpha}_{b_-}f(a)]\right| \tag{1.6}$$
$$\leq \frac{b-a}{2(\alpha+1)}(1-\frac{1}{2^{\alpha}})|f'(a)| + |f'(b)|$$

Recently, some authors started to study on conformable fractional integral. In [18], Khalil et al. defined the fractional integral of order $0 < \alpha \leq 1$ only. In [1], Abdeljawad gave the definition of left and right conformable fractional integrals of any order $\alpha > 0$.

Definition 1.9. Let $\alpha \in (n, n+1]$ and set $\beta = \alpha - n$ then the left conformable fractional integral starting at a if order α is defined by

$$(I^a_{\alpha}f)(t) = \frac{1}{n!}\int_a^t (t-x)^n (x-a)^{\beta-1} f(x)dx$$

Analogously, the right conformable fractional integral is defined by

$$({}^bI_{\alpha}f)(t) = \frac{1}{n!}\int_t^b (x-t)^n (b-x)^{\beta-1} f(x)dx.$$

Notice that if $\alpha = n+1$ then $\beta = \alpha - n = n+1-n = 1$ where $n = 0, 1, 2, 3...$ and hence $(I^a_{\alpha}f)(t) = (J^a_{n+1}f)(t)$.

In [24] Set et.al. gave Hermite-Hadamard inequality for conformable fractional integral as follows:

Theorem 1.10. *Let $f : [a, b] \to \mathbb{R}$ be a function with $0 \leq a < b$ and $f \in L_1[a, b]$. If f is a convex function on $[a, b]$, then the following inequalities for conformable fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}\Gamma(\alpha-n)}[(I^a_{\alpha}f)(b) + ({}^bI_{\alpha}f)(a)] \leq \frac{f(a)+f(b)}{2} \tag{1.7}$$

with $\alpha \in (n, n+1]$, where Γ is Euler Gamma function.

For some studies on conformable fractional integral, see ([1], [2], [4], [6]). In papers ([25]-[27]), Set et.al obtained some Hermite-Hadamard, Ostrowski, Chebyshev, Fejer type inequalities by using conformable fractional integrals for various classes of functions. The aim of this study is to establish new Hermite-Hadamard inequalities related to other fractional integral inequalities for conformable fractional integral.

2. Hermite-Hadamard's inequalities for conformable fractional integrals

In this section, using the given properties of conformable fractional integrals, we will establish a generalization of Hermite-Hadamard type inequalities for s-convex functions. We will also noticed the relation with fractional and classical Hermite-Hadamard type integral inequalities.

Theorem 2.1. *Let $f : [a,b] \to \mathbb{R}$ be a function with $0 \leq a < b$, $s \in (0,1]$ and $f \in L_1[a,b]$. If f is an s-convex function on $[a,b]$, then the following inequalities for conformable fractional integrals hold:*

$$\begin{aligned} & \frac{\Gamma(\alpha-n)}{\Gamma(\alpha+1)} f\left(\frac{a+b}{2}\right) \qquad (2.1) \\ \leq\; & \frac{1}{(b-a)^{\alpha} 2^s}\left[(I_\alpha^a f)(b) + ({}^b I_\alpha f)(a)\right] \\ \leq\; & \left[\frac{B(n+s+1,\alpha-n)+B(n+1,\alpha-n+s)}{n!}\right]\frac{f(a)+f(b)}{2^s} \end{aligned}$$

with $\alpha \in (n, n+1]$, $n = 0,1,2,...$ where Γ is Euler Gamma function and $B(a,b)$ is a beta function.

Proof. Let $x, y \in [a,b]$. If f is a s-convex function on [a,b],

$$f\left(\frac{x+y}{2}\right) \leq \left(\frac{1}{2}\right)^s f(x) + \left(\frac{1}{2}\right)^s f(y)$$

if we change the variables with $x = ta + (1-t)b,\; y = (1-t)a + tb$,

$$2^s f\left(\frac{a+b}{2}\right) \leq f(ta+(1-t)b) + f((1-t)a+tb). \qquad (2.2)$$

Multiplying both sides of above inequality with $\frac{1}{n!}t^n(1-t)^{\alpha-n-1}$ and integrating the resulting inequality with respect to t over $[0,1]$, we get

$$\begin{aligned} & \frac{2^s}{n!} f\left(\frac{a+b}{2}\right)\int_0^1 t^n(1-t)^{\alpha-n-1}dt \\ \leq\; & \frac{1}{n!}\int_0^1 t^n(1-t)^{\alpha-n-1} f(ta+(1-t)b)dt \\ & +\frac{1}{n!}\int_0^1 t^n(1-t)^{\alpha-n-1} f((1-t)a+tb)dt \\ =\; & \frac{1}{n!}\int_a^b \left(\frac{b-x}{b-a}\right)^n \left(\frac{x-a}{b-a}\right)^{\alpha-n-1} f(x)\frac{dx}{b-a} \\ & +\frac{1}{n!}\int_a^b \left(\frac{y-a}{b-a}\right)^n \left(\frac{b-y}{b-a}\right)^{\alpha-n-1} f(y)\frac{dy}{b-a} \\ =\; & \frac{1}{(b-a)^{\alpha}}\left[I_\alpha^a f(b) +^b I_\alpha f(a)\right]. \end{aligned}$$

Note that

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2^s(b-a)^\alpha\Gamma(\alpha-n)}[I_\alpha^a f(b) +^b I_\alpha f(a)] \tag{2.3}$$

where

$$\int_0^1 t^n(1-t)^{\alpha-n-1}dt = B(n+1,\alpha-n) = \frac{\Gamma(n+1)\Gamma(\alpha-n)}{\Gamma(\alpha+1)}$$

which means that the left side of (2.1) is proved. Since f is s-convex in the second sense, to prove the right side of (2.1) we have the following inequalities:

$$\begin{aligned} f(ta+(1-t)b) &\leq t^s f(a) + (1-t)^s f(b) \\ f((1-t)a+tb) &\leq (1-t)^s f(a) + t^s f(b). \end{aligned}$$

Adding these two inequalities, we get

$$f(ta+(1-t)b) + f((1-t)a+tb) \leq [t^s + (1-t)^s][f(a)+f(b)].$$

Multiplying both sides of the resulting inequality with $\frac{1}{n!}t^n(1-t)^{\alpha-n-1}$ and integrating with respect to t over $[0,1]$, we have

$$\begin{aligned} &\frac{1}{(b-a)^\alpha}[I_\alpha^a f(b) +^b I_\alpha f(a)] \qquad (2.4) \\ \leq\ &\frac{1}{n!}\int_0^1 t^n(1-t)^{\alpha-n-1}[t^s+(1-t)^s][f(a)+f(b)]dt \\ =\ &\frac{1}{n!}\Big[B(n+s+1,\alpha-n) + B(n+1,\alpha-n+s)\Big][f(a)+f(b)]. \end{aligned}$$

Combining (2.3) and (2.4) completes the proof. □

Remark 2.2. If we choose $s=1$ in Theorem (2.1), by using relation between Γ and B functions, the inequality (2.1) reduced to inequality (1.7).

Remark 2.3. If we choose $\alpha = n+1$ in Theorem 2.1, the inequality (2.2) reduced to inequality (1.4). And also if we choose α, $s=1$ in the inequality (2.2), then we get well-known Hermite-Hadamard inequality as (1.2).

3. Some new Hermite Hadamard type inequalities via conformable integration

In order to achieve our aim, we will give an important identity for differentiable functions involving conformable fractional integrals as follows:

Lemma 3.1. *Let $f:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $f' \in L[a,b]$, then the following inequality for conformable fractional integrals holds:*

$$\begin{aligned} &B(n+1,\alpha-n)\left(\frac{f(a)+f(b)}{2}\right) - \frac{n!}{2(b-a)^\alpha}[I_\alpha^a f(b) +^b I_\alpha f(a)] \qquad (3.1) \\ &= \frac{(b-a)}{2}\left\{\int_0^1 \left[B_{1-t}(n+1,\alpha-n) - B_t(n+1,\alpha-n)\right] f'(ta+(1-t)b)dt\right\} \end{aligned}$$

where $B(a,b)$, $B_t(a,b)$ is Euler beta and incompleted beta functions respectively and $\alpha \in (n, n+1], n = 0, 1, 2, \ldots$.

Proof. Let

$$I = \int_0^1 \left[B_{1-t}(n+1, \alpha-n) - B_t(n+1, \alpha-n)\right] f'(ta+(1-t)b)dt.$$

Then, integrating by parts and changing variables with $x = ta+(1-t)b$, we can write

$$\begin{aligned} I_1 &= \int_0^1 B_{1-t}(n+1, \alpha-n) f'(ta+(1-t)b)dt \qquad (3.2)\\ &= \int_0^1 \left(\int_0^{1-t} x^n(1-x)^{\alpha-n-1}dx\right) f'(ta+(1-t)b)dt \\ &= \left(\int_0^{1-t} x^n(1-x)^{\alpha-n-1}dx\right) \frac{f(ta+(1-t)b)dt}{a-b}\Bigg|_0^1 \\ &\quad + \int_0^1 (1-t)^n t^{\alpha-n-1} f(ta+(1-t)b)\frac{dt}{a-b} \\ &= \left(\int_0^1 x^n(1-x)^{\alpha-n-1}dx\right)\frac{f(b)}{b-a} \\ &\quad + \frac{1}{b-a}\int_a^b \left(\frac{x-a}{b-a}\right)^n \left(\frac{b-x}{b-a}\right)^{\alpha-n-1} f(x)\frac{dx}{a-b} \\ &= B(n+1, \alpha-n)\frac{f(b)}{b-a} - \frac{n!}{(b-a)^{\alpha+1}}({}^bI_\alpha f)(a) \end{aligned}$$

$$\begin{aligned} I_2 &= \int_0^1 B_t(n+1, \alpha-n) f'(ta+(1-t)b)dt \qquad (3.3)\\ &= B_t(n+1, \alpha-n)\frac{f(ta+(1-t)b)}{a-b}\Bigg|_0^1 \\ &\quad - \int_0^1 t^n(1-t)^{\alpha-n-1} f(ta+(1-t)b)\frac{dt}{a-b} \\ &= -B(n+1, \alpha-n)\frac{f(a)}{b-a} + \frac{1}{b-a}\int_a^b \left(\frac{b-x}{b-a}\right)^n \left(\frac{x-a}{b-a}\right)^{\alpha-n-1} f(x)\frac{dx}{b-a} \\ &= -B(n+1, \alpha-n)\frac{f(a)}{b-a} + \frac{n!}{(b-a)^{\alpha+1}}(I_\alpha^a f)(b). \end{aligned}$$

It means that $I = I_1 - I_2$. Thus, by multiplying both sides by $\frac{b-a}{2}$ i.e

$$\frac{b-a}{2}I = \frac{b-a}{2}I_1 - \frac{b-a}{2}I_2$$

we have desired result. □

Remark 3.2. If we choose $\alpha = n+1$ in Lemma 3.1, the equality (3.1) becomes the equality (1.5).

Now, using the obtained identity, we will establish some inequalities connected with the left part of the inequality (2.1)

Theorem 3.3. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b$. If $f' \in L[a,b]$ and $|f'|$ is s-convex in the second sence with $s \in (0,1]$, then the following inequality for conformable fractional integrals holds:*

$$\begin{aligned} &\left|B(n+1,\alpha-n)\left(\frac{f(a)+f(b)}{2}\right)-\frac{n!}{2(b-a)^{\alpha}}[I_{\alpha}^{a}f(b)+^{b}I_{\alpha}f(a)]\right| \qquad (3.4)\\ \leq\ & \frac{b-a}{2}\left[\frac{|f'(a)|+|f'(b)|}{s+1}\right]\\ &\left\{B_{\frac{1}{2}}(\alpha-n+s+1,n+1)-B_{\frac{1}{2}}(n+1,\alpha-n+s+1)\right.\\ &\left.+B_{\frac{1}{2}}(n+s+2,\alpha-n)-B_{\frac{1}{2}}(\alpha-n,n+s+2)+B(n+1,\alpha-n)\right\} \end{aligned}$$

where $B(a,b)$, $B_t(a,b)$ is Euler beta and incompleted beta functions respectively and $\alpha \in (n, n+1], n = 0,1,2,\ldots$.

Proof. Taking modulus on Lemma 3.1 and using s-convexity of $|f'|$ we get:

$$\begin{aligned} &\left|B(n+1,\alpha-n)\left(\frac{f(a)+f(b)}{2}\right)-\frac{n!}{2(b-a)^{\alpha}}[I_{\alpha}^{a}f(b)+^{b}I_{\alpha}f(a)]\right| \qquad (3.5)\\ =\ & \frac{b-a}{2}\left|\int_0^1\big[B_{1-t}(n+1,\alpha-n)-B_t(n+1,\alpha-n)\big]f'(ta+(1-t)b)dt\right|\\ \leq\ & \frac{b-a}{2}\int_0^1\big|\big[B_{1-t}(n+1,\alpha-n)-B_t(n+1,\alpha-n)\big]\big|\big|f'(ta+(1-t)b)\big|dt\\ =\ & \frac{b-a}{2}\int_0^{\frac{1}{2}}\big[B_{1-t}(n+1,\alpha-n)-B_t(n+1,\alpha-n)\big]\big|f'(ta+(1-t)b)\big|dt\\ &+\int_{\frac{1}{2}}^1\big[B_t(n+1,\alpha-n)-B_{1-t}(n+1,\alpha-n)\big]\big|f'(ta+(1-t)b)\big|dt\\ \leq\ & \frac{b-a}{2}\left\{\int_0^{\frac{1}{2}}B_{1-t}(n+1,\alpha-n)\big(t^s|f'(a)|+(1-t)^s|f'(b)|\big)dt\right.\\ &-\int_0^{\frac{1}{2}}B_t(n+1,\alpha-n)\big(t^s|f'(a)|+(1-t)^s|f'(b)|\big)dt\\ &+\int_{\frac{1}{2}}^1B_t(n+1,\alpha-n)\big(t^s|f'(a)|+(1-t)^s|f'(b)|\big)dt\\ &\left.-\int_{\frac{1}{2}}^1B_{1-t}(n+1,\alpha-n)\big(t^s|f'(a)|+(1-t)^s|f'(b)|\big)dt\right\} \end{aligned}$$

$$
\begin{aligned}
&= \frac{b-a}{2}\Bigg\{|f'(a)|\int_0^{\frac{1}{2}}\big[B_{1-t}(n+1,\alpha-n)-B_t(n+1,\alpha-n)\big]t^s dt \\
&+|f'(b)|\int_0^{\frac{1}{2}}\big[B_{1-t}(n+1,\alpha-n)-B_t(n+1,\alpha-n)\big](1-t)^s dt \\
&+|f'(a)|\int_{\frac{1}{2}}^{1}\big[B_t(n+1,\alpha-n)-B_{1-t}(n+1,\alpha-n)\big]t^s\big)dt \\
&+|f'(b)|\int_{\frac{1}{2}}^{1}\big[B_t(n+1,\alpha-n)-B_{1-t}(n+1,\alpha-n)\big](1-t)^s dt.
\end{aligned}
$$

On the other hand, using the properties of incompleted beta function we have:

$$
\begin{aligned}
& B_{1-t}(n+1,\alpha-n)-B_t(n+1,\alpha-n) \\
&= \int_0^{1-t} x^n(1-x)^{\alpha-n-1}dx-\int_0^{t} x^n(1-x)^{\alpha-n-1}dx \\
&= \int_t^{1-t} x^n(1-x)^{\alpha-n-1}dx, \quad where\ 0\le t\le \frac{1}{2}
\end{aligned}
\tag{3.6}
$$

and

$$
\begin{aligned}
& B_t(n+1,\alpha-n)-B_{1-t}(n+1,\alpha-n) \\
&= \int_0^{t} x^n(1-x)^{\alpha-n-1}dx-\int_0^{1-t} x^n(1-x)^{\alpha-n-1}dx \\
&= \int_{1-t}^{t} x^n(1-x)^{\alpha-n-1}dx, \quad where\ \frac{1}{2}\le t\le 1
\end{aligned}
\tag{3.7}
$$

Using (3.6), (3.7) and Newton Leibnitz formula and integrating by parts we can write the following computation:

$$
\begin{aligned}
\Phi_1 &= \int_0^{\frac{1}{2}}\left(\int_t^{1-t} x^n(1-x)^{\alpha-n-1}dx\right)t^s dt \\
&= \left[\left(\int_t^{1-t} x^n(1-x)^{\alpha-n-1}dx\right)\frac{t^{s+1}}{s+1}\right]\Bigg|_0^{\frac{1}{2}} \\
&\quad -\int_0^{\frac{1}{2}}\big(-(1-t)^n t^{\alpha-n-1}-t^n(1-t)^{\alpha-n-1}\big)\frac{t^{s+1}}{s+1}dt \\
&= \frac{1}{s+1}\Big[\int_0^{\frac{1}{2}} t^{\alpha-n+s}(1-t)^n dt+\int_0^{\frac{1}{2}} t^{n+s+1}(1-t)^{\alpha-n-1}dt\Big] \\
&= \frac{1}{s+1}\Big[B_{\frac{1}{2}}(\alpha-n+s+1,n+1)+B_{\frac{1}{2}}(n+s+2,\alpha-n)\Big],
\end{aligned}
\tag{3.8}
$$

$$\begin{aligned}
\Phi_2 &= \int_0^{\frac{1}{2}} \left(\int_t^{1-t} x^n (1-x)^{\alpha-n-1} dx \right) (1-t)^s dt \qquad (3.9)\\
&= \left[\left(\int_t^{1-t} x^n (1-x)^{\alpha-n-1} dx \right) \frac{-(1-t)^{s+1}}{s+1} \right] \Bigg|_0^{\frac{1}{2}} \\
&\quad - \int_0^{\frac{1}{2}} \left(-(1-t)^n t^{\alpha-n-1} - t^n (1-t)^{\alpha-n-1} \right) \frac{-(1-t)^{s+1}}{s+1} dt \\
&= \frac{1}{s+1} \int_0^1 x^n (1-x)^{\alpha-n-1} dx \\
&\quad - \frac{1}{s+1} \left[\int_0^{\frac{1}{2}} t^{\alpha-n-1} (1-t)^{n+s+1} dt + \int_0^{\frac{1}{2}} t^n (1-t)^{\alpha-n+s} dt \right] \\
&= \frac{1}{s+1} \Big[B(n+1, \alpha-n) - B_{\frac{1}{2}}(\alpha-n, n+s+2) \\
&\quad - B_{\frac{1}{2}}(n+1, \alpha-n+s+1) \Big],
\end{aligned}$$

$$\begin{aligned}
\Phi_3 &= \int_{\frac{1}{2}}^1 \left(\int_{1-t}^{t} x^n (1-x)^{\alpha-n-1} dx \right) t^s dt \qquad (3.10)\\
&= \left[\left(\int_{1-t}^{t} x^n (1-x)^{\alpha-n-1} dx \right) \frac{t^{s+1}}{s+1} \right] \Bigg|_{\frac{1}{2}}^{1} \\
&\quad - \frac{1}{s+1} \int_{\frac{1}{2}}^1 \left(t^n (1-t)^{\alpha-n-1} + t^{\alpha-n-1} (1-t)^n \right) t^{s+1} dt \\
&= \frac{1}{s+1} \int_0^1 x^n (1-x)^{\alpha-n-1} dx \\
&\quad - \frac{1}{s+1} \left[\int_{\frac{1}{2}}^1 t^{n+s+1} (1-t)^{\alpha-n-1} dt + \int_{\frac{1}{2}}^1 t^{\alpha-n+s} (1-t)^n dt \right] \\
&= \frac{1}{s+1} \Big[B(n+1, \alpha-n) - B_{\frac{1}{2}}(\alpha-n, n+s+2) \\
&\quad - B_{\frac{1}{2}}(n+1, \alpha-n+s+1) \Big]
\end{aligned}$$

and

$$\begin{aligned}
\Phi_4 &= \int_{\frac{1}{2}}^1 \left(\int_{1-t}^{t} x^n (1-x)^{\alpha-n-1} dx \right) (1-t)^s dt \qquad (3.11)\\
&= \left[\left(\int_{1-t}^{t} x^n (1-x)^{\alpha-n-1} dx \right) \frac{-(1-t)^{s+1}}{s+1} \right] \Bigg|_{\frac{1}{2}}^{1} \\
&\quad + \int_{\frac{1}{2}}^1 \left(t^n (1-t)^{\alpha-n-1} + t^{\alpha-n-1} (1-t)^n \right) \frac{(1-t)^{s+1}}{s+1} dt
\end{aligned}$$

$$= \Big[\int_{\frac{1}{2}}^{1} t^n(1-t)^{\alpha-n+s}dt + \int_{\frac{1}{2}}^{1} t^{\alpha-n-1}(1-t)^{n+s+1}dt\Big]$$
$$= \frac{1}{s+1}\Big[B_{\frac{1}{2}}(\alpha-n+s+1,n+1) + B_{\frac{1}{2}}(n+s+2,\alpha-n)\Big],$$

Using the fact that $B(a,b) = B_{\frac{1}{2}}(a,b) + B_{\frac{1}{2}}(b,a)$ and combining (3.8), (3.9), (3.10), (3.11) with (3.5) completes the proof. $\square$

Corollary 3.4. *Taking $s = 1$ in Theorem 3.3 i.e $|f'|$ is convex, we get the following result:*

$$\begin{aligned}
&\left|B(n+1,\alpha-n)\left(\frac{f(a)+f(b)}{2}\right) - \frac{n!}{2(b-a)^\alpha}[I_\alpha^a f(b) +^b I_\alpha f(a)]\right| \\
\leq\ & \frac{b-a}{2}\left(\frac{|f'(a)|+|f'(b)|}{2}\right) \\
&\times\Big\{B_{\frac{1}{2}}(\alpha-n+2,n+1) - B_{\frac{1}{2}}(n+1,\alpha-n+2) \\
&+B_{\frac{1}{2}}(n+3,\alpha-n) - B_{\frac{1}{2}}(\alpha-n,n+3) + B(n+1,\alpha-n)\Big\}
\end{aligned} \tag{3.12}$$

Remark 3.5. Taking $\alpha = n+1$ in Corollary 3.4, the inequality (3.12) reduces to (1.6).

Theorem 3.6. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b), $a < b$ and $p > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. If $f' \in L[a,b]$ and $|f'|^q$ is s-convex in the second sense, then the following inequality for conformable fractional integrals holds:*

$$\begin{aligned}
&\left|B(n+1,\alpha-n)\left(\frac{f(a)+f(b)}{2}\right) - \frac{n!}{2(b-a)^\alpha}[I_\alpha^a f(b) +^b I_\alpha f(a)]\right| \\
&\leq \frac{b-a}{2}\Psi^{\frac{1}{p}}\left[\frac{|f'(a)|^q+|f'(b)|^q}{s+1}\right]^{\frac{1}{q}}.
\end{aligned} \tag{3.13}$$

where $B(a,b)$ is Euler beta function, $\alpha \in (n,n+1], n = 0,1,2,\dots$ and

$$\Psi = 2\int_0^{\frac{1}{2}}\left(\int_t^{1-t} x^n(1-x)^{\alpha-n-1}dx\right)^p$$

.

Proof. Taking modulus and using Hölder inequality with a function of $|f'|^q$ convexity we get inequalities as follow:

$$\begin{aligned}
&\left|B(n+1,\alpha-n)\left(\frac{f(a)+f(b)}{2}\right)-\frac{n!}{2(b-a)^{\alpha}}[I_{\alpha}^{a}f(b)+^{b}I_{\alpha}f(a)]\right| \qquad (3.14)\\
=\ & \frac{b-a}{2}\left|\int_0^1\big[B_{1-t}(n+1,\alpha-n)-B_t(n+1,\alpha-n)\big]f'(ta+(1-t)b)dt\right|\\
\leq\ & \frac{b-a}{2}\int_0^1\big|B_{1-t}(n+1,\alpha-n)-B_t(n+1,\alpha-n)\big|\big|f'(ta+(1-t)b)\big|dt\\
\leq\ & \frac{b-a}{2}\Big[\int_0^1\big|B_{1-t}(n+1,\alpha-n)-B_t(n+1,\alpha-n)\big|^p dt\Big]^{\frac{1}{p}}\\
&\times\Big[\int_0^1\big|f'(ta+(1-t)b)\big|^q dt\Big]^{\frac{1}{q}}.
\end{aligned}$$

It follows that:

$$\begin{aligned}
\Psi &= \int_0^1\big|B_{1-t}(n+1,\alpha-n)-B_t(n+1,\alpha-n)\big|^p dt \qquad (3.15)\\
&= \int_0^{\frac{1}{2}}\Big(B_{1-t}(n+1,\alpha-n)-B_t(n+1,\alpha-n)\Big)^p dt\\
&\quad+\int_{\frac{1}{2}}^1\Big(B_t(n+1,\alpha-n)-B_{1-t}(n+1,\alpha-n)\Big)^p dt\\
&= \int_0^{\frac{1}{2}}\left(\int_t^{1-t}x^n(1-x)^{\alpha-n-1}dx\right)^p dt\\
&\quad+\int_{\frac{1}{2}}^1\left(\int_{1-t}^{t}x^n(1-x)^{\alpha-n-1}dx\right)^p dt\\
&= 2\int_0^{\frac{1}{2}}\left(\int_t^{1-t}x^n(1-x)^{\alpha-n-1}dx\right)^p dt
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1\big|f'(ta+(1-t)b)\big|^q dt\leq|f'(a)|^q\int_0^1 t^s dt+|f'(b)|^q\int_0^1(1-t)^s dt\\
=\ & \frac{1}{s+1}\big(|f'(a)|^q+|f'(b)|^q\big) \qquad (3.16)
\end{aligned}$$

which completes the proof. □

Corollary 3.7. *If we take $s=1$ in Theorem 3.6, the inequality (3.13) reduces to following inequality:*

$$\begin{aligned}
&\left|B(n+1,\alpha-n)\left(\frac{f(a)+f(b)}{2}\right)-\frac{n!}{2(b-a)^{\alpha}}[I_{\alpha}^{a}f(b)+^{b}I_{\alpha}f(a)]\right|\\
\leq\ & \frac{b-a}{2}\Psi^{\frac{1}{p}}\Big[\frac{|f'(a)|^q+|f'(b)|^q}{2}\Big]^{\frac{1}{q}} \qquad (3.17)
\end{aligned}$$

where $B(a,b)$ is Euler beta function and

$$\Psi = 2\int_0^{\frac{1}{2}}\left(\int_t^{1-t} x^n(1-x)^{\alpha-n-1}dx\right)^p.$$

Corollary 3.8. *If we take $\alpha = n+1$ in corollary 3.7, the inequality (3.17) reduces to following inequality:*

$$\left|B(\alpha,1)\left(\frac{f(a)+f(b)}{2}\right) - \frac{\Gamma(\alpha)}{2(b-a)^\alpha}[J^\alpha_{a+}f(b) + J^\alpha_{b-}f(a)]\right| \tag{3.18}$$
$$\leq \frac{b-a}{2}\Psi_1^{\frac{1}{p}}\left[\frac{|f'(a)|^q+|f'(b)|^q}{2}\right]^{\frac{1}{q}},$$

where $\Psi_1 = 2\int_0^{\frac{1}{2}}\left(\frac{(1-t)^\alpha - t^\alpha}{\alpha}\right)^p dt.$

Remark 3.9. If we take $\alpha = 1$ in Corollary 3.8, the inequality (3.18) reduces to following inequality:

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{(b-a)}\int_a^b f(x)dx\right| \tag{3.19}$$
$$\leq \frac{b-a}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\frac{|f'(a)|^q+|f'(b)|^q}{2}\right]^{\frac{1}{q}},$$

which is the same as Theorem 2.3 in [12].

Remark 3.10. If we take $\alpha \in (0,1]$ in Corollary 3.8, then the inequality (3.18) reduces to special case of Corollary 1 for $s = 1$ in [19], which is the same as

$$\left|\left(\frac{f(a)+f(b)}{2}\right) - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J^\alpha_{a+}f(b) + J^\alpha_{b-}f(a)]\right| \tag{3.20}$$
$$\leq \frac{b-a}{2}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left[\frac{|f'(a)|^q+|f'(b)|^q}{2}\right]^{\frac{1}{q}}.$$

References

[1] Abdeljawad, T., *On conformable fractional calculus*, J. Comput. Appl. Math., **279**(2015), 57-66.

[2] Abdeljawad, T., *On multiplicative fractional calculus*, arxiv:1510, o4176v1 [math.CA], 6 Oct 2015.

[3] Alomari, M., Darus, M., Dragomir, S.S., Cerone, P., *Ostrowski type inequalities for functions whose derivatives are s-convex in the second sense*, Appl. Math. Lett., **23**(9), 1071-1076.

[4] Anderson, O.R., *Taylor's formula and integral inequalities for conformable fractional derivatives*, arxiv:140958888v1 [math. CA], 20 Sep 2014.

[5] Avci, M., Kavurmaci, H., Özdemir, M.E., *New inequalities of Hermite-Hadamard type via s-convex functions in the second sense with applications*, Appl. Math. Comput., **217**(12), 5171-5176.

[6] Benkhettou, N., Hassani, S., Torres, D.E.M., *A conformable fractional calculus on arbitrary time*, J. King Saud Univ. Sci., **28**(2016), 9398.

[7] Breckner, W.W., *Stetigkeitsaussagen fr eine Klasse verallgemeinerter konvexer funktionen in topologischen linearen Raumen*, Publ. Inst. Math., **23**(1978), 13-20.

[8] Chen, F., *Extensions of the Hermite-Hadamard Inequality for convex functions via fractional integrals*, J. Math. Ineq., **10**(1)(2016), 75-81.

[9] Dahmani, Z., *New inequalities in fractional integrals*, Int. J. Nonlinear Sci., **9**(4)(2010), 493-497.

[10] Dahmani, Z., Tabharit, L., Taf, S., *New generalizations of Gruss inequality using RiemannLiouville fractional integrals*, Bull. Math. Anal. Appl., **2**(3)(2010), 93-99.

[11] Dragomir, S.S., Agarval, R.R., *The Hadamard's inequality for s-convex functions in the second sense*, Demonstratio Math., **32**(4)(1999), 687-696.

[12] Dragomir, S.S., Fitzpatrik, S., *Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula*, Appl. Math. Lett., **11**(5)(1998), 91-95.

[13] Dragomir, S.S., Pearce, C.E.M., *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, Victoria University, 2000.

[14] Gorenflo, R., Mainardi, F., *Fractional Calculus: Integral and Differential Equations of Fractional Order*, 2008, arXiv preprint arXiv:0805.3823.

[15] Hudzik, H., Maligranda, L., *Some remarks on s-convex functions*, Aequationes Math., **48**(1994), 100-111.

[16] İşcan, İ., *Generalization of different type integral inequalities for s-convex functions via fractional integrals*, Appl. Anal., **93**(9)(2014), 1846-1862.

[17] İşcan, İ., *Hermite Hadamard type inequalities for harmonically convex functions via fractional integral*, Appl. Math. Comput., **238**(2014), 237-244.

[18] Khalil, R., Al Horani, M., Yousef, A., Sababheh, M., *A new definition of fractional derivative*, J. Comput. Appl. Math., **264**(2014), 65-70.

[19] Özdemir, M.E., Kavurmacı, H., Yıldız, Ç., *Fractional integral inequalities via s-convex functions*, Turkish J. Anal. Number Theory, **5**(1)(2017), 18-22.

[20] Podlubni, I., *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[21] Sarıkaya, M.Z., Set, E., Yaldız, H., Başak, N., *Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities*, Math. Comput. Model., **57**(2013), 2403-2407.

[22] Set, E., Sarıkaya, M.Z., Özdemir, M.E., Yıldırım, H., *The Hermite-Hadamard's inequality for some convex functions via fractional integrals and related results*, J. Appl. Math. Stat. Inform., **10**(2)(2014), 69-83.

[23] Set, E., *New inequalities of Ostrowski type for mappings whose derivatives are s-convex in the second sense via fractional integrals*, Comput. Math. Appl., **63**(7)(2012), 1147-1154.

[24] Set, E., Akdemir, A.O., Mumcu, İ., *The Hermite-Hadamard's inequality and its extentions for conformable fractioanal integrals of any order $\alpha > 0$*, Submitted.

[25] Set, E., Akdemir, A.O., Mumcu, İ., *Ostrowski type inequalities for functions whoose derivatives are convex via conformable fractional integrals*, J. Adv. Math. Stud., **10**(3)(2017), 386-395.

[26] Set, E., Akdemir, A.O., Mumcu, İ., *Chebyshev type inequalities for conformable fractional integrals*, Submitted.

[27] Set, E., Mumcu, İ., *Hermite-Hadamard-Fejer type inequalies for conformable fractional integrals* , Submitted.

[28] Zhu, C., Feckan, M., Wang, J., *Fractional integral inequalities for differentiable convex mappings and applications to special means and a midpoint formula*, J. Math. Stat. Inform., **8**(2)(2012), 21-28.

Erhan Set
Department of Mathematics, Faculty of Arts and Sciences, Ordu University
52200 Ordu, Turkey
e-mail: erhanset@yahoo.com

Abdurrahman Gözpınar
Department of Mathematics, Faculty of Arts and Sciences, Ordu University
52200 Ordu, Turkey
e-mail: abdurrahmangozpinar79@gmail.com

Stud. Univ. Babeş-Bolyai Math. 62(2017), No. 3, 325–329
DOI: 10.24193/subbmath.2017.3.05

A new proof of Ackermann's formula from control theory

Marius Costandin, Petru Dobra and Bogdan Gavrea

Abstract. This paper presents a novel proof for the well known Ackermann's formula, related to pole placement in linear time invariant systems. The proof uses a lemma [3], concerning rank one updates for matrices, often used to efficiently compute the determinants. The proof is given in great detail, but it can be summarised to few lines.

Mathematics Subject Classification (2010): 26D10, 46N30.

Keywords: Eigenvalues placement algorithms, rank one updates, linear systems, matrix determinants.

1. Introduction

Given a matrix $A \in \mathbb{R}^{n\times n}$ and a vector $B \in \mathbb{R}^{n\times 1}$, it is known, see [1] that if the marix $Co(A,B) = [B|A \cdot B| \dots |A^{n-1} \cdot B]$ is invertible then there exists a unique $K \in \mathbb{R}^{n\times 1}$ such that $\hat{A} = A + B \cdot K^T$ has any desired set of eigenvalues $\mathcal{S} = \{\lambda_1^*, \dots, \lambda_n^*\}$, closed under complex conjugation, that is if $\lambda \in \mathcal{S}$ then $\bar{\lambda} \in \mathcal{S}$. Algorithms for finding K are well known in literature among which the algorithm of Bass-Gura (see [2]) and Ackerman (see [1]) are mentioned.

In the following a new demonstration to Ackermann's result is given, using a well known lemma often used for computing the determinant of a certain invertible matrix, see [3]. This lemma relates the determinant of a rank-one update to the determinant of the initial matrix. For an elegant proof of this result we point the reader to [3].

Lemma 1.1 (Matrix determinant lemma, [3]**).** *Suppose that A is an invertible square matrix and u and v are column vectors. Then:*

$$\det(A + uv^T) = \left(1 + v^T A^{-1} u\right) \det(A) \tag{1.1}$$

2. The novel proof for Ackermann's formula

Theorem 2.1 (Ackermann). *Let $\dot{X} = A \cdot X + B \cdot u$ be a linear time invariant dynamical system, with $X, B \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n \times n}$. If $Co(A,B) = [B|A \cdot B| \dots |A^{n-1} \cdot B]$ is invertible, then the matrix $\hat{A} = A - B \cdot K_x^T$ has the user-defined eigenvalues $\{\lambda_1^*, \dots, \lambda_p^*\}$, with algebraic multiplicities $q_1, \dots, q_p$, where*

$$\begin{aligned} K_x &= \left(\prod_{i=1}^{p} (A - \lambda_i^* I)^{q_i} \right)^T \cdot Co(A,B)^{-T} \cdot \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \\ &= P^*(A)^T \cdot Co(A,B)^{-T} \cdot \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \end{aligned}$$

Proof. Let $P^*(\lambda) = \prod_{i=1}^{p} (\lambda - \lambda_i^*)^{q_i} = \det(\lambda I - \hat{A})$ denote the characteristic polynomial of $\hat{A}$ and $P(\lambda) = \det(\lambda I - A)$ the characteristic polynomial of A. Suppose, for start, that the desired eigenvalues are not already eigenvalues for the system matrix, A. Therefore $\det(\lambda_i^* I - A) \neq 0$ for all $i \in \{1, \dots, p\}$. Then, from Lemma 1.1:

$$\begin{aligned} P^*(\lambda) &= \det(\lambda I - \hat{A}) \\ &= \det(\lambda I - (A - BK_x^T)) \\ &= \det((\lambda I - A) + BK_x^T) \\ &= \left(1 + K_x^T (\lambda I - A)^{-1} B\right) \det(\lambda I - A) \\ &= \left(1 + K_x^T (\lambda I - A)^{-1} B\right) \cdot P(\lambda) \end{aligned} \tag{2.1}$$

We are interested in finding K_x such that Equation (2.1) holds. Equation (2.1) is a monic polynomial equality, so it is enough to hold for the roots. Let $\lambda = \lambda_i^*$ in Equation (2.1).

Because λ_i^* has multiplicity q_i, then the folowing relations are obtained:

$$\begin{cases} K_x^T \cdot (\lambda_i^* I - A)^{-1} \cdot B = -1 \\ K_x^T \cdot (\lambda_i^* I - A)^{-2} \cdot B = 0 \\ \vdots \\ K_x^T \cdot (\lambda_i^* I - A)^{-q_i} \cdot B = 0 \end{cases} \qquad \forall i \in \{1, \dots, p\} \tag{2.2}$$

Hence

$$\begin{bmatrix} B^T\cdot(\lambda_1^* I - A^T)^{-1} \\ B^T\cdot(\lambda_1^* I - A^T)^{-2} \\ \vdots \\ B^T\cdot(\lambda_1^* I - A^T)^{-q_1} \\ \vdots \\ B^T\cdot(\lambda_p^* I - A^T)^{-1} \\ B^T\cdot(\lambda_p^* I - A^T)^{-2} \\ \vdots \\ B^T\cdot(\lambda_1^* I - A^T)^{-q_p} \end{bmatrix} \cdot K_x = \begin{bmatrix} -1 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{2.3}$$

Denote

$$C = \begin{bmatrix}(\lambda_1^* I - A)^{-1}\cdot B| & \dots & |(\lambda_1^* I - A)^{-q_1}\cdot B| & \dots\end{bmatrix}$$

and

$$N = \begin{bmatrix}-1 & 0 & \dots & 0 & \dots & -1 & 0 & \dots & 0\end{bmatrix}^T$$

then

$$C^T\cdot K_x = N$$

Looking closely at C one can see:

$$\begin{aligned}\prod_{i=1}^{p}(\lambda_i^* I - A)^{q_i}\cdot C &= \begin{bmatrix}P_1\{\lambda_1^*\}(A)\cdot B| & \dots & |P_{q_1}\{\lambda_1^*\}(A)\cdot B| & \dots\end{bmatrix} \\ &= \bar{C} \end{aligned} \tag{2.4}$$

where $P_j\{\lambda_k^*\}(A) = \left(\prod_{i=1,i\neq k}^{p}(\lambda_i^* I - A)^{q_i}\right)\cdot(\lambda_k^* I - A)^{q_k - j}$ with $k\in\overline{1,p}$ and $j\in\overline{1,q_k}$. If seen as a polynomial over $\mathbb{R}$, then it's roots are $\{\lambda_1^*,\dots,\lambda_k^*,\dots,\lambda_p^*\}$, with the multiplicity $q_1,\dots,q_k - j,\dots,q_p$. The order of the polynomial is $n-j$. Stacking the polynomial's coefficients in a vector, with the coefficient of the smallest power in the first position, and leaving the same name for the vector, one has:

$$\begin{aligned}\bar{C} &= \begin{bmatrix}B| & A\cdot B| & \dots| & A^{n-1}\cdot B\end{bmatrix}\cdot \\ &\cdot\begin{bmatrix}P_1\{\lambda_1^*\}| & \dots| & P_{q_1}\{\lambda_1^*\}| & \dots| & P_1\{\lambda_p^*\}| & \dots| & P_{q_p}\{\lambda_p^*\}\end{bmatrix} \\ &= Co(A,B)\cdot\mathcal{P}\end{aligned} \tag{2.5}$$

Of course, $\mathcal{P}$ is invertible, since it has linearly independent columns. Indeed let

$$\alpha_1^1\cdot P_1\{\lambda_1^*\} + \dots + \alpha_1^p\cdot P_1\{\lambda_p^*\} + \dots = 0$$

be a null linear combination of the columns of $\mathcal{P}$. Suppose the polynomial's variable is X. Let $k\in\overline{1,p}$ and let α_j^k be the the coefficient of the polynomial having λ_k^* as a root with the smallest multiplicity m_k. Differentiating the above linear combination, m_k times, with respect to X, then replacing X with λ_k^*, will yield $\alpha_{q_k}^k = 0$. Repeating the process will conclude that the polynomials are linear independent. Hence:

$$C^{-T} = \left(\prod_{i=1}^{p}(\lambda_i^* I - A)^{q_i}\right)^T\cdot Co(A,B)^{-T}\cdot\mathcal{P}^{-T} \tag{2.6}$$

therefore

$$\begin{aligned} K_x &= \left(\prod_{i=1}^{p}(A-\lambda_i^* I)^{q_i}\right)^T \cdot Co(A,B)^{-T}\cdot(-1)^n\cdot\mathcal{P}^{-T}\cdot N \\ &= P^*(A)^T\cdot Co(A,B)^{-T}\cdot(-1)^n\cdot\mathcal{P}^{-T}\cdot N \end{aligned} \tag{2.7}$$

Denote $V=(-1)^n\cdot\mathcal{P}^{-T}\cdot N$ therefore $(-1)^n\cdot\mathcal{P}^T\cdot V=N$. Because $\mathcal{P}$ is invertible, V is unique.

$$(-1)^n\cdot\begin{bmatrix} P_1\{\lambda_1^*\}^T \\ P_2\{\lambda_1^*\}^T \\ \vdots \\ P_{q_1}\{\lambda_1^*\}^T \\ \vdots \\ P_1\{\lambda_p^*\}^T \\ P_2\{\lambda_p^*\}^T \\ \vdots \\ P_{q_p}\{\lambda_p^*\}^T \end{bmatrix}\cdot\begin{bmatrix} v_1 \\ \vdots \\ v_n\end{bmatrix}=\begin{bmatrix} -1 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ -1 \\ 0 \\ \vdots \\ 0\end{bmatrix} \tag{2.8}$$

Because $P_j\{\lambda_k^*\}$ has the order $n-j$, and the coefficient of the smallest power is on the first position in vector, that is the coefficient of the greatest power is on the last position, follows:

$$(-1)^n\cdot\begin{bmatrix} \dots & (-1)^{n-1} \\ \dots & 0 \\ \vdots & \vdots \\ \dots & 0 \\ \vdots & \vdots \\ \dots & (-1)^{n-1} \\ \dots & 0 \\ \vdots & \vdots \\ \dots & 0 \end{bmatrix}\cdot\begin{bmatrix} v_1 \\ \vdots \\ v_n\end{bmatrix}=\begin{bmatrix} -1 \\ 0 \\ \vdots \\ 0 \\ \vdots \\ -1 \\ 0 \\ \vdots \\ 0\end{bmatrix} \tag{2.9}$$

It is easy to see that $V=[0,\dots,0,1]^T$ is a solution. Therefore

$$K_x=P^*(A)^T\cdot Co(A,B)^{-T}\cdot V \tag{2.10}$$

If $\lambda_i^*=\lambda_i$, for some $i\in\overline{1,p}$, then take $\lambda_i^*(\epsilon)=\epsilon+\lambda_i^*$ to obtain

$$\det(\lambda I-(A-B\cdot K_x(\epsilon)^T))=P^*\{\epsilon\}(\lambda).$$

Letting $\epsilon\longrightarrow 0$, one has $\det(\lambda I-(A-B\cdot K_x^T))=P^*(\lambda)$. $\square$

3. Conclusions

A new proof for the well known Akermann's formula was presented. The proof uses a matrix lemma, giving an in depth look at the mechanics of eigenvalues change using rank one updates. The state feedback matrix K_x is shown to be the unique solution to a system of equations, obtained using a well known matrix lemma. The proof can be summarised as follows:

1. Use Equation (2.1) to obtaing Equation (2.3)
2. Use Equations (2.4) and (2.5) to obtain Equation (2.6) regardind the resolvent matrix
3. Use Equation (2.8) and (2.9) in Equation (2.7) to obtain K_x

References

[1] Ackermann, J., *Der Entwurf linearer Regelungsysteme im Zustandraum*, Regeltech. Proz.-Datenverarb., **7**(1972), 297-300.

[2] Bass, W., Gura, I., *High order system design via state-space considerations*, Preprints, Join Automatic Control Conference, Rensselner Polytechnic Institute, Troy, N.Y., 1965, 311-318.

[3] Ding, J., Zhou, A., *Eigenvalues of rank-one updated matrices with some applications*, Appl. Math. Lett., **20**(2007), 1223-1226.

[4] Ogata, K., *Modern Control Engineering*, 4th Ed. Englewood Cliffs, NJ, Prentice Hall, 2001.

Marius Costandin
Technical University of Cluj-Napoca, Str. Memorandumului nr. 28
400114, Cluj-Napoca, Romania
e-mail: `costandin@mail.utcluj.ro`

Petru Dobra
Technical University of Cluj-Napoca, Str. Memorandumului nr. 28
400114, Cluj-Napoca, Romania
e-mail: `petru.dobra@aut.utcluj.ro`

Bogdan Gavrea
Technical University of Cluj-Napoca, Str. Memorandumului nr. 28
400114, Cluj-Napoca, Romania
e-mail: `bogdan.gavrea@math.utcluj.ro`

Stud. Univ. Babeş-Bolyai Math. 62(2017), No. 3, 331–340
DOI: 10.24193/subbmath.2017.3.06

Third Hankel determinant for reciprocal of bounded turning function has a positive real part of order alpha

Bolineni Venkateswarlu and Nekkanti Rani

Abstract. The objective of this paper is to obtain an upper bound to the third Hankel determinant denoted by $|H_3(1)|$ for certain subclass of univalent functions, using Toeplitz determinants.

Mathematics Subject Classification (2010): 30C45, 30C50.

Keywords: Univalent function, upper bound, function whose reciprocal derivative has a positive real part, Hankel determinant, positive real function, Toeplitz determinants.

1. Introduction

Let A denote the class of all functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

in the open unit disc $E = \{z : |z| < 1\}$. Let S be the subclass of A consisting of univalent functions. For a univalent function in the class A, it is well known that the n^{th} coefficient is bounded by n. The bounds for the coefficients give information about the geometric properties of these functions In particular, the growth and distortion properties of a normalized univalent function are determined by the bound of its second coefficient. The Hankel determinant of f for $q \geq 1$ and $n \geq 1$ was defined by Pommerenke [12] as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}.$$

This determinant has been considered by many authors in the literature. For example, Noor [10] determined the rate of growth of $H_q(n)$ as $n \to \infty$ for the functions in S with bounded boundary. Ehrenborg [4] studied the Hankel determinant of exponential polynomials. The Hankel transform of an integer sequence and some of its properties were discussed by Layman in [7]. In the recent years several authors have investigated bounds for the Hankel determinant of functions belonging to various subclasses of univalent and multivalent analytic functions. In particular for, $q = 2$, $n = 1$, $a_1 = 1$ and $q = 2$, $n = 2$, $a_1 = 1$, the Hankel determinant simplifies respectively to

$$H_2(1) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_3 - a_2^2, \text{ and } H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2 a_4 - a_3^2.$$

For our discussion in this paper, we consider the Hankel determinant in the case of $q = 3$ and $n = 1$, denoted by $H_3(1)$, given by

$$H_3(1) = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix}. \tag{1.2}$$

For $f \in A$, $a_1 = 1$, so that, we have

$$H_3(1) = a_3(a_2 a_4 - a_3^2) - a_4(a_4 - a_2 a_3) + a_5(a_3 - a_2^2)$$

and by applying triangle inequality, we obtain

$$|H_3(1)| \leq |a_3||a_2 a_4 - a_3^2| + |a_4||a_2 a_3 - a_4| + |a_5||a_3 - a_2^2|. \tag{1.3}$$

The sharp upper bound to the second Hankel functional $|H_2(2)|$ for the subclass RT of S, consisting of functions whose derivative has a positive real part, studied by Mac Gregor [9] was obtained by Janteng [6]. It was known that if $f \in RT$ then $|a_k| \leq \frac{2}{k}$, for $k \in \{2, 3, ...\}$. Further, the best possible sharp upper bound for the functional $|a_2 a_3 - a_4|$ and $|a_3 - a_2^2|$ was obtained by Babalola [2] and hence the sharp inequality for $|H_3(1)|$, for the class RT. For $f \in RT(\alpha)$, the sharp upper bound to second Hankel [14] and $|H_3(1)|$ were obtained by Vamshee Krishna et al.[15]. The sharp upper bound to $H_3(1)$ for the subclass of $\widetilde{RT}$ of S consisting of a function whose reciprocal derivative has a positive real part was obtained by Venkateswarlu [16].

Motivated by the result obtained by Babalola [2], we obtain an upper bound to the functional second Hankel determinant, $|a_2 a_3 - a_4|$ and hence $|H_3(1)|$, for the function f given in (1.1), when it belongs to the class $\widetilde{RT}(\alpha)$, defined as follows.

Definition 1.1. A function $f(z) \in A$ is said to be function whose reciprocal derivative has a positive real part of order α, (also called reciprocal of bounded turning function of order α), denoted by $f \in \widetilde{RT}(\alpha)$ $(0 \leq \alpha < 1)$, if and only if

$$Re\Big(\frac{1}{f'(z)}\Big) > \alpha, \forall z \in E. \tag{1.4}$$

Observe that for $\alpha = 0$, we obtain $\widetilde{RT}(0) = \widetilde{RT}$. Some preliminary lemmas required for proving our results are as follows:

2. Preliminary results

Let $\mathscr{P}$ denote the class of functions consisting of p, such that

$$p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + ... = \left[1 + \sum_{n=1}^{\infty} c_n z^n\right], \tag{2.1}$$

which are regular in the open unit disc E and satisfy $\text{Re}\{p(z)\} > 0$ for any $z \in E$. Here $p(z)$ is called the Carathe\`odory function [3].

Lemma 2.1. [11, 13] *If $p \in \mathscr{P}$, then $|c_k| \leq 2$, for each $k \geq 1$ and the inequality is sharp for the function $\frac{1+z}{1-z}$.*

Lemma 2.2. [5] *The power series for $p(z) = 1 + \sum\limits_{n=1}^{\infty} c_n z^n$ given in (2.1) converges in the open unit disc E to a function in $\mathscr{P}$ if and only if the Toeplitz determinants*

$$D_n = \begin{vmatrix} 2 & c_1 & c_2 & \cdots & c_n \\ c_{-1} & 2 & c_1 & \cdots & c_{n-1} \\ c_{-2} & c_{-1} & 2 & \cdots & c_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{-n} & c_{-n+1} & c_{-n+2} & \cdots & 2 \end{vmatrix}, \ n = 1, 2, 3, \cdots$$

and $c_{-k} = \overline{c}_k$, are all non-negative. They are strictly positive except for

$$p(z) = \sum_{k=1}^{m} \rho_k p_0(e^{it_k} z),$$

$\rho_k > 0$, t_k real and $t_k \neq t_j$, for $k \neq j$, where $p_0(z) = \frac{1+z}{1-z}$; in this case $D_n > 0$ for $n < (m-1)$ and $D_n \doteq 0$ for $n \geq m$.

This necessary and sufficient condition found in [5] is due to Carathe\`odory and Toeplitz. We may assume without restriction that $c_1 > 0$. On using Lemma 2.2, for $n = 2$, we have

$$D_2 = \begin{vmatrix} 2 & c_1 & c_2 \\ \overline{c}_1 & 2 & c_1 \\ \overline{c}_2 & \overline{c}_1 & 2 \end{vmatrix} = [8 + 2Re\{c_1^2 c_2\} - 2 \mid c_2 \mid^2 - 4|c_1|^2] \geq 0,$$

which is equivalent to

$$2c_2 \ = \ c_1^2 + x(4 - c_1^2), \text{ for some } x, \ |x| \leq 1. \tag{2.2}$$

For $n = 3$,

$$D_3 = \begin{vmatrix} 2 & c_1 & c_2 & c_3 \\ \overline{c}_1 & 2 & c_1 & c_2 \\ \overline{c}_2 & \overline{c}_1 & 2 & c_1 \\ \overline{c}_3 & \overline{c}_2 & \overline{c}_1 & 2 \end{vmatrix} \geq \ 0$$

and is equivalent to

$$|(4c_3 - 4c_1 c_2 + c_1^3)(4 - c_1^2) + c_1(2c_2 - c_1^2)^2| \leq 2(4 - c_1^2)^2 - 2|(2c_2 - c_1^2)|^2. \tag{2.3}$$

From the relations (2.2) and (2.3), after simplifying, we get

$$4c_3 = c_1^3 + 2c_1(4-c_1^2)x - c_1(4-c_1^2)x^2 + 2(4-c_1^2)(1-|x|^2)z, \tag{2.4}$$

for some z, with $|z| \le 1$.

To obtain our result, we refer to the classical method initiated by Libera and Zlotkiewicz [8] and used by several authors in the literature.

3. Main result

Theorem 3.1. *If $f(z) \in \widetilde{RT}(\alpha)$ $(0 \le \alpha \le \frac{1}{\sqrt{2}})$ then*

$$|\, a_2a_4 - a_3^2| \le \left[\frac{2}{3(\alpha-1)}\right]^2$$

and the inequality is sharp.

Proof. For

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \widetilde{RT}(\alpha),$$

there exists an analytic function $p \in \mathscr{P}$ in the open unit disc E with $p(0) = 1$ and $\mathrm{Re}\{p(z)\} > 0$ such that

$$\frac{1-\alpha f'(z)}{(1-\alpha)f'(z)} = p(z) \;\Leftrightarrow\; 1-\alpha f'(z) \;=\; (1-\alpha)f'(z)p(z). \tag{3.1}$$

Replacing $f'(z)$ and $p(z)$ with their equivalent series expressions in (3.1) , we have

$$1-\alpha\Big(1+\sum_{n=2}^{\infty} na_nz^{n-1}\Big) \;=\; (1-\alpha)\Big(1+\sum_{n=2}^{\infty} na_nz^{n}\Big)\Big(1+\sum_{n=1}^{\infty} c_nz^{n}\Big).$$

Upon simplification, we obtain

$$\begin{aligned}&(1-\alpha) - 2\alpha a_2 z - 3\alpha a_3 z^2 - 4\alpha a_4 z^3 - 5a_5z^4 - \cdots \;=\; (1-\alpha)\\ &+ z(1-\alpha)[2a_2+c_1] + z^2(1-\alpha)[c_2+2a_2c_1+3a_3] + z^3(1-\alpha)\\ &[c_3+2a_2c_2+3a_3c_1+4a_4] + z^4(1-\alpha)[c_4+2a_2c_3+3a_3c_2+4a_4c_1+5a_5] + \cdots .\end{aligned} \tag{3.2}$$

Equating the coefficients of like powers of z, z^2, z^3 and z^4 respectively on both sides of (3.2), after simplifying, we get

$$\begin{aligned}a_2 &= -\frac{1-\alpha}{2}c_1;\; a_3 = -\frac{1-\alpha}{3}\Big[c_2-(1-\alpha)c_1^2\Big];\\ a_4 &= -\frac{1-\alpha}{4}\Big[c_3-2(1-\alpha)c_1c_2+(1-\alpha)^2c_1^3\Big];\\ a_5 &= -\frac{1-\alpha}{5}\Big[c_4-2(1-\alpha)c_1c_3+3(1-\alpha)^2c_1^2c_2-(1-\alpha)c_2^2-(1-\alpha)^3c_1^4\Big].\end{aligned} \tag{3.3}$$

Substituting the values of a_2, a_3 and a_4 from (3.3) in the functional $|a_2a_4 - a_3^2|$ for the function $f \in \widetilde{RT}(\alpha)$, upon simplification, we obtain

$$| a_2a_4 - a_3^2| = \frac{(1-\alpha)^2}{72} \left|9c_1c_3 - 2(1-\alpha)c_1^2c_2 - 8c_2^2 + (1-\alpha)^2c_1^4\right|$$

which is equivalent to

$$| a_2a_4 - a_3^2| = \frac{(1-\alpha)^2}{72} \left|d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4\right|, \tag{3.4}$$

$$\text{where } d_1 = 9;\ d_2 = -2(1-\alpha);\ d_3 = -8;\ d_4 = (1-\alpha)^2. \tag{3.5}$$

Substituting the values of c_2 and c_3 given in (2.2) and (2.4) respectively from Lemma 2.2 on the right-hand side of (3.4), we have

$$\begin{aligned}|d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4| = \Big|&\frac{d_1c_1}{4}\{c_1^3 + 2c_1(4-c_1^2)x - c_1(4-c_1^2)x^2 \\ &+ 2(4-c_1^2)(1-|x|^2)z\} + \frac{d_2c_1^2}{2}\{c_1^2 + x(4-c_1^2)\} \\ &+ \frac{d_3}{4}\{c_1^2 + x(4-c_1^2)\}^2 + d_4c_1^4\Big|.\end{aligned} \tag{3.6}$$

Using triangle inequality and the fact that $|z| < 1$, we get

$$\begin{aligned}4 \mid d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4 \mid \leq \Big|&(d_1 + 2d_2 + d_3 + 4d_4)c_1^4 + 2d_1c_1(4-c_1^2) \\ &+ 2(d_1 + d_2 + d_3)c_1^2(4-c_1^2)|x| \\ &- \left\{(d_1 + d_3)c_1^2 + 2d_1c_1 - 4d_3\right\}(4-c_1^2)|x|^2\Big|.\end{aligned} \tag{3.7}$$

From (3.5), we can now write

$$d_1 + 2d_2 + d_3 + 4d_4 = 4\alpha^2 - 4\alpha + 1;\ 2(d_1 + d_2 + d_3) = -2(1-2\alpha); \tag{3.8}$$

$$(d_1 + d_3)c_1^2 + 2d_1c_1 - 4d_3 = c_1^2 + 18c_1 + 32 = (c_1 + 16)(c_1 + 2). \tag{3.9}$$

Since $c_1 \in [0, 2]$, using the result $(c_1 + a)(c_1 + b) \geq (c_1 - a)(c_1 - b)$, where $a, b \geq 0$ in (3.9), we can have

$$-\{(d_1 + d_3)c_1^2 + 2d_1c_1 - 4d_3\} \leq -(c_1^2 - 18c_1 + 32). \tag{3.10}$$

Substituting the calculated values from (3.8) and (3.10) on the right-hand side of (3.7), we have

$$\begin{aligned}4|d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4| \leq \Big|&(4\alpha^2 - 4\alpha + 1)c_1^4 + 18c_1(4-c_1^2) \\ &- 2(1-2\alpha)c_1^2(4-c_1^2)|x| - (c_1^2 - 18c_1 + 32)(4-c_1^2)|x|^2\Big|.\end{aligned} \tag{3.11}$$

Choosing $c_1 = c \in [0,2]$, applying triangle inequality and replacing $|x|$ by μ on the right-hand side of the above inequality, we get

$$\begin{aligned} 4\ |d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4| \le & \Big[(4\alpha^2 - 4\alpha + 1)c^4 + 18c(4-c^2) \\ & + 2(1-2\alpha)c^2(4-c^2)\mu + (c^2 - 18c + 32)(4-c^2)\mu^2\Big] \\ & = F(c,\mu)\ ,\ 0 \le \mu = |x| \le 1 \text{ and } 0 \le c \le 2. \end{aligned} \tag{3.12}$$

We next maximize the function $F(c,\mu)$ on the closed region $[0,2] \times [0,1]$. Differentiating $F(c,\mu)$ given in (3.12) partially with respect to μ, we obtain

$$\frac{\partial F}{\partial \mu} = 2[(1-2\alpha)c^2 + (c^2 - 18c + 32)\mu](4-c^2). \tag{3.13}$$

For $0 < \mu < 1$ and for fixed c with $0 < c < 2$, from (3.13), we observe that $\frac{\partial F}{\partial \mu} > 0$. Therefore, $F(c,\mu)$ becomes an increasing function of μ and hence it cannot have a maximum value at any point in the interior of the closed region $[0,2]\times[0,1]$. Moreover, for a fixed $c \in [0,2]$, we have

$$\max_{0\le\mu\le1} F(c,\mu) = F(c,1) = G(c).$$

Therefore, replacing μ by 1 in $F(c,\mu)$, upon simplification, we obtain

$$G(c) = 2\Big[-c^4(1-2\alpha^2) - 2c^2(4\alpha+5) + 64)\Big]. \tag{3.14}$$

$$G'(c) = -8c\Big[c^2(1-2\alpha^2) + (4\alpha+5)\Big]. \tag{3.15}$$

From (3.15), we observe that $G'(c) \le 0$, for every $c \in [0,2]$. Therefore, $G(c)$ is a decreasing function of c in the interval $[0,2]$, whose maximum value occurs at $c = 0$ only. From (3.14), the maximum value of $G(c)$ at $c = 0$ is given by

$$G_{max} = G(0) = 128. \tag{3.16}$$

Simplifying the expressions (3.12) and (3.16), we get

$$|d_1c_1c_3 + d_2c_1^2c_2 + d_3c_2^2 + d_4c_1^4| \le\ 32. \tag{3.17}$$

From the relations (3.4) and (3.17), upon simplification, we obtain

$$|\ a_2a_4 - a_3^2|\ \le \Big[\frac{2}{3}(1-\alpha)\Big]^2. \tag{3.18}$$

By setting $c_1 = c = 0$ and selecting $x = 1$ in the expressions (2.2) and (2.4), we find that $c_2 = 2$ and $c_3 = 0$ respectively. Substituting these values in (3.17) together with the values in (3.4), we observe that equality is attained, which shows that our result is sharp. The extremal function in this case is given by

$$\frac{1-\alpha f'(z)}{(1-\alpha)f'(z)}\ =\ 1 + 2z^2 + 2z^4 + \cdots\ =\ \frac{1+z^2}{1-z^2}.$$

This completes the proof of our Theorem. □

Remark 3.2. It is observed that the sharp upper bound to the second Hankel determinant of a function whose derivative has a positive real part of order α, obtained by Vamshee Krishna et al. [14] and a function whose reciprocal derivative has a positive real part of order α is the same. Further, for the choice of $\alpha = 0$, we get $\widetilde{RT}(0) = \widetilde{RT}$, for which from (3.18), we obtain $|a_2a_4 - a_3^2| \leq \frac{4}{9}$. This inequality is sharp and this result coincides with that of Janteng et al. [6] and Venkateswarlu et al. [16]. From this we conclude that the sharp upper bound to the second Hankel determinant of a function whose derivative has a positive real part of order α and a function whose reciprocal derivative has a positive real part of order α is the same.

Theorem 3.3. *If $f(z) \in \widetilde{RT}(\alpha)$ $(0 \leq \alpha \leq \frac{5}{8})$ then $|\ a_2a_3 - a_4| \ \leq \ \frac{1}{6}\left[\frac{5-8\alpha}{3}\right]^{\frac{3}{2}}$.*

Proof. Substituting the values of a_2, a_3 and a_4 from (3.3) in the determinant $|\ a_2a_3 - a_4\ |$ for the function $f \in \widetilde{RT}(\alpha)$, after simplifying, we get

$$|\ a_2a_3 - a_4| \ = \ \frac{(1-\alpha)}{12}\Big|3c_3 - 4(1-\alpha)c_1c_2 + (1-\alpha)^2c_1^3\Big|. \tag{3.19}$$

Substituting the values of c_2 and c_3 from (2.2) and (2.4) respectively from Lemma 2.2 on the right-hand side of (3.19), and using the fact that $|z| < 1$, we have

$$\begin{aligned}4\Big|3c_3 - 4(1-\alpha)c_1c_2 + (1-\alpha)^2c_1^3\Big| \leq \ & \Big| - c_1^3(1-4\alpha^2) + 6(4-c_1^2) \\ & - 2c_1(4-c_1^2)|x|(1-4\alpha) - 3(4-c_1^2)|x|^2(c_1+2)\Big|.\end{aligned}$$

Since $c_1 = c \in [0,2]$, using the result $(c_1+a) \geq (c_1-a)$, where $a \geq 0$, applying triangle inequality and replacing $|x|$ by μ on the right-hand side of the above inequality, we have

$$\begin{aligned}4|3c_3 - 4c_1c_2 + c_1^3| \leq \ & \Big|c^3(1-4\alpha^2) + 6(4-c^2) \\ & + 2(1-\alpha)c(4-c^2)\mu + 3(c-2)(4-c^2)\mu^2\Big| \\ = \ & F(c,\mu)\ ,\ 0 \leq \mu = |x| \leq 1 \text{ and } 0 \leq c \leq 2.\end{aligned} \tag{3.20}$$

Next, we maximize the function $F(c,\mu)$ on the closed square $[0,2] \times [0,1]$. Differentiating $F(c,\mu)$ given in(3.20) partially with respect to μ, we get

$$\frac{\partial F}{\partial \mu} = 2(4-c^2)[(1-4\alpha)c + 3(c-2)\mu] > 0. \tag{3.21}$$

As described in Theorem 3.1, further we obtain

$$G(c) \ = \ -4c^3(1-\alpha)^2 + 4(5-8\alpha)c. \tag{3.22}$$

$$G'(c) \ = \ -12c^2(1-\alpha)^2 + 4(5-8\alpha)c. \tag{3.23}$$

$$G''(c) \ = \ -24c(1-\alpha)^2. \tag{3.24}$$

For optimum value of $G(c)$, consider $G'(c) = 0$, From (3.23), we get

$$c^2 \ = \ \frac{5-8\alpha}{3(1-\alpha)^2}, \ \text{for } 0 \leq \alpha < \frac{5}{8}.$$

Using the obtained value of $c = \sqrt{\frac{5-8\alpha}{3(1-\alpha)^2}} \in [0,2]$ in (3.24). In which simplifies to give

$$G''(c) = -24\sqrt{\frac{5-8\alpha}{3}(1-\alpha)} < 0, \text{ for } 0 \leq \alpha < \frac{5}{8}.$$

Therefore, by the second derivative test, $G(c)$ has maximum value at $c = \sqrt{\frac{5-8\alpha}{3(1-\alpha)^2}}$. Substituting the value of c in the expression (3.22), upon simplification, we obtain the maximum value of $G(c)$ at c, as

$$G_{max} = \frac{8}{1-\alpha}\left[\frac{5-8\alpha}{3}\right]^{\frac{3}{2}}. \tag{3.25}$$

From the expressions (3.20) and (3.25), after simplifying, we get

$$|3c_3 - 4(1-\alpha)c_1c_2 + (1-\alpha)^2c_1^3| \leq \frac{2}{1-\alpha}\left[\frac{5-8\alpha}{3}\right]^{\frac{3}{2}}. \tag{3.26}$$

Simplifying the relations (3.19) and (3.26), upon simplification, we obtain

$$|a_2a_3 - a_4| \leq \frac{1}{6}\left[\frac{5-8\alpha}{3}\right]^{\frac{3}{2}}. \tag{3.27}$$

This completes the proof of our Theorem. □

Remark 3.4. For the choice of $\alpha = 0$, from (3.27), we obtain $|a_2a_3 - a_4| \leq \frac{1}{6}\left(\frac{5}{3}\right)^{\frac{3}{2}}$. This inequality is sharp and this result coincides with that of obtained by Babalola [2] and Venkateswarlu et al. [16]. From this we conclude that for $\alpha = 0$, the sharp upper bound to the $|a_2a_3 - a_4|$ of a function whose derivative has a positive real part of order alpha and a function whose reciprocal derivative has a positive real part or order alpha is the same.

The following theorem is a straight forward verification on applying the same procedure as described in Theorems 3.1 and 3.3 and the result is sharp for the values $c_1 = 0, c_2 = 2$ and $x = 1$.

Theorem 3.5. *If $f \in \widetilde{RT}(\alpha)$ $(0 \leq \alpha < 1)$ then $|a_3 - a_2^2| \leq \frac{2}{3}[1-\alpha]$.*

Using the fact that $|c_n| \leq 2$, $n \in N = \{1, 2, 3, \cdots\}$, with the help of c_2 and c_3 values given in (2.2) and (2.4) respectively together with the values in (3.3), we obtain $|a_k| \leq \frac{2}{k}(1-\alpha)(1-2\alpha)^{k-2}$, for $k \in \{2, 3, 4, 5, \cdots\}$.

Substituting the results of Theorems 3.1, 3.3, 3.5 and $|a_k| \leq \frac{2}{k}(1-\alpha)(1-2\alpha)^{k-2}$, for $k \in \{2, 3, 4, 5, \cdots\}$, for the function $f \in \widetilde{RT}(\alpha)$ in the inequality (1.3), upon simplification, we obtain the following corollary.

Corollary 3.6. *If $f(z) \in \widetilde{RT}(\alpha)$ $(0 \leq \alpha \leq \frac{1}{\sqrt{2}})$ then*

$$|H_3(1)| \leq \frac{(1-\alpha)(1-2\alpha)}{3}\left[\frac{4(1-\alpha)(36\alpha^2 - 46\alpha + 19)}{45} + \frac{(1-2\alpha)}{4}\left(\frac{5-8\alpha}{3}\right)^{\frac{3}{2}}\right]. \tag{3.28}$$

Remark 3.7. We choose $\alpha = 0$, from the expressions (3.28), we obtain $|H_3(1)| \leq 0.742$. These inequalities are sharp and coincide with the results of Babalola [2] and Venkateswarlu et al. [16]. From this we conclude that for $\alpha = 0$, the sharp upper bound to the third Hankel determinant of a function whose derivative has a positive real part or order alpha and a function whose reciprocal derivative has a positive real part of order alpha is the same.

References

[1] Ali, R.M., *Coefficients of the inverse of strongly starlike functions*, Bull. Malays. Math. Sci. Soc., (second series), **26**(2003), no. 1, 63-71.

[2] Babalola, K.O., *On $H_3(1)$ Hankel determinant for some classes of univalent functions*, Inequ. Theory and Appl., **6**(2010), 1-7.

[3] Duren, P.L., *Univalent Functions*, vol. 259, Grundlehren der Mathematischen Wissenschaften, Springer, New York, USA, 1983.

[4] Ehrenborg, R., *The Hankel determinant of exponential polynomials*, Amer. Math. Monthly, **107**(2000), no. 6, 557-560.

[5] Grenander, U., Szegö, G., *Toeplitz Forms and Their Applications,* Second Edition, Chelsea Publishing Co., New York, 1984.

[6] Janteng, A., Halim, S.A., Darus, M., *Coefficient inequality for a function whose derivative has a positive real part*, J. Inequal. Pure Appl. Math., **7**(2006), no. 2, 1-5.

[7] Layman, J.W., *The Hankel transform and some of its properties*, J. Integer Seq., **4**(2001), no. 1, 1-11.

[8] Libera, R.J., Zlotkiewicz, E.J., *Coefficient bounds for the inverse of a function with derivative in $\mathscr{P}$,* Proc. Amer. Math. Soc., **87**(1983), no. 2, 251-257.

[9] Mac Gregor, T.H., *Functions whose derivative have a positive real part*, Trans. Amer. Math. Soc., **104**(1962), no. 3, 532 - 537.

[10] Noor, K.I., *Hankel determinant problem for the class of functions with bounded boundary rotation*, Rev. Roumaine Math. Pures Appl., **28**(1983), no. 8, 731-739.

[11] Pommerenke, Ch., *Univalent Functions,* Vandenhoeck and Ruprecht, Göttingen, 1975.

[12] Pommerenke, Ch., *On the coefficients and Hankel determinants of univalent functions*, J. Lond. Math. Soc., **41**(1966), 111-122.

[13] Simon, B., *Orthogonal polynomials on the unit circle*, Part 1, *Classical theory*, Amer. Math. Soc. Colloq. Publ., 54, Part 1, 2005.

[14] Vamshee Krishna, D., RamReddy, T., *Coefficient inequality for a function whose derivative has a positive real part of order α*, Math. Bohem., **140**(2015), 43-52.

[15] Vamshee Krishna, D., Venkateswarlu, B., RamReddy, T., *Third Hankel determinant for bounded turning functions of order alpha*, J. Nigerian Math. Soc., **34**(2015), 121-127.

[16] Venkateswarlu, B., Vamshee Krishna, D., Rani, N., RamReddy, T., *Third Hankel determinant for reciprocal of bounded turning functions* (Communicated).

Bolineni Venkateswarlu
Department of Mathematics, GIT, GITAM University
Visakhapatnam, 530045, A.P., India
e-mail: `bvlmaths@gmail.com`

Nekkanti Rani
Department of Sciences and Humanities
Praveenya Institute of Marine Engineering and Maritime studies
Modavalasa, 534 002, Visakhapatnam, A.P., India
e-mail: raninekkanti1111@gmail.com

Stud. Univ. Babeş-Bolyai Math. 62(2017), No. 3, 341–352
DOI: 10.24193/subbmath.2017.3.07

A study of the inextensible flows of tube-like surfaces associated with focal curves in Galilean 3-space G_3

Adel H. Sorour

Abstract. In this paper, we study inextensible flows of focal curves associated with tube-like surfaces in Galilean 3-space G_3. We give some characterizations for curvature and torsion of focal curves associated with tube-like surfaces in Galilean 3-space G_3. Furthermore, we show that if flow of this tube-like surface is inextensible then this surface is not developable as well as not minimal. Finally an example of tube-like surface is used to demonstrate our theoretical results and graphed.

Mathematics Subject Classification (2010): 31B30, 58E20.

Keywords: Tube-like surface, Gaussian curvature, mean curvature, Galilean 3-space.

1. Introduction

Curve design using splines is one of the most fundamental topics in CAGD. Inextensible flows of curves possess a beautiful shape preserving connection to their control polygon. They allow us the formulation of algorithms for processing, especially subdivision algorithms. Moreover, at least the curves of odd degree and maximal smoothness also arise as solutions of variational problems.

In the past two decades, for the need to explain certain physical phenomena and to solve practical problems, geometers and geometric analysis have begun to deal with curves and surfaces which are subject to various forces and which flow or evolve with time in response to those forces so that the metrics are changing. Now, various geometric flows have become one of the central topics in geometric analysis. Many authors have studied geometric flow problems. In [9, 10] Kwon et al. studied inextensible flows of curves and developable surface in $\mathbb{R}^3$.

Korpinar et al. [8] studied inextensible flows of developable surfaces associated focal curve of helices in Euclidean 3-space E^3. Differential geometry of the Galilean space G_3 has been largely developed in Kamenarovic [6], Ogrenmiş et al. [13, 14] and Roschel [17].

In this work, we study inextensible flows of focal curves associated with tube-like surfaces in Galilean 3-space G_3. We give some characterizations for curvature and torsion of focal curves associated with tube-like surfaces in Galilean 3-space G_3. Finally, we show that if the flow of a tube-like surface associated to a focal curve is inextensible, then the surface is not developable as well as not minimal for an arbitrary focal curve. We used some idea from Korpinar et al. [8] in this paper.

2. Preliminaries

The geometry of the Galilean space G_3 has been treated in detail in O. Roschl's habilitation in 1984 [17]. More about Galilean space and Pseudo-Galilean space may be found in [20, 1, 3, 7, 11, 12, 21]. The Galilean space G_3 is a Cayley-Klein space equipped with the projective metric of signature $(0, 0, +, +)$, as in [21].

The Galilean space is a three dimensional complex projective space P_3 in which the absolute figure $\{\omega, f, I_1, I_2\}$ consists of a real plane ω (the absolute plane), a real line $f \subset \omega$ (the absolute line) and two complex conjugate points $I_1, I_2 \in f$ (the absolute points) [6]. We shall take, as a real model of the space G_3, a real projective space P_3 with the absolute $\{\omega, f\}$ consisting of a real plane $\omega \subset G_3$ and a real line $f \in \omega$ on which an elliptic involution ε has been defined. In homogeneous coordinates

$$\begin{aligned} &\omega ... x_0 = 0, \quad f ... x_0 = x_1 = 0, \\ &\varepsilon : (0 : 0 : x_2 : x_3) \to (0 : 0 : x_3 : -x_2), \end{aligned} \tag{2.1}$$

while in the nonhomogeneous coordinates, the similarity group H_8 has the form

$$\overline{x} = a_{11} + a_{12}x, \tag{2.2}$$

$$\begin{aligned} \overline{y} &= a_{21} + a_{22}x + a_{23}\Big(y\cos[\phi] + z\sin[\phi]\Big), \\ \overline{z} &= a_{31} + a_{32}x - a_{33}\Big(y\sin[\phi] - z\cos[\phi]\Big), \end{aligned}$$

where a_{ij} and ϕ are real numbers. For $a_{12} = a_{23} = 1$, we have the subgroup B_6 which is the group of Galilean motions:

$$\begin{aligned} \overline{x} &= a + x, \\ \overline{y} &= b + cx + y\cos[\phi] + z\sin[\phi], \\ \overline{z} &= d + ex - y\sin[\phi] + z\cos[\phi]. \end{aligned}$$

It is worth noting that [16]: in G_3 there are four classes of lines:

a): (proper) nonisotropic lines: they do not meet the absolute line f.

b): (proper) isotropic lines: lines that do not belong to the plane ω but meet the absolute line f.

c): (unproper) nonisotropic lines: all lines of ω but f.

d): the absolute line f.

Planes x =constant are Euclidean and so is the plane ω. Other planes are isotropic. In what follows, the coefficients a_{12} and a_{23} will play a special role.

In particular, for $a_{12} = a_{23} = 1$, (2.2) defines the group $B_6 \subset H_8$ of isometries of the Galilean space G_3.

In affine coordinates, the Galilean scalar product between two vectors $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ is defined by [15]

$$(\langle a, b\rangle)_{G_3} = \begin{cases} a_1b_1, & \text{if } a_1 \neq 0 \text{ or } b_1 \neq 0, \\ a_2b_2 + a_3b_3, & \text{if } a_1 = 0 \text{ and } b_1 = 0. \end{cases} \tag{2.3}$$

The Galilean cross product is defined by

$$(a \wedge b)_{G_3} = \begin{cases} \begin{vmatrix} 0 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}, & \text{if } a_1 \neq 0 \text{ or } b_1 \neq 0, \\ \\ \begin{vmatrix} e_1 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}, & \text{if } a_1 = 0 \text{ and } b_1 = 0. \end{cases} \tag{2.4}$$

The unit Galilean sphere is defined by [5]

$$S^2_{\pm} = \{\alpha \in G_3 \mid \langle \alpha, \alpha\rangle_{G_3} = \mp r^2\}.$$

Let $M : \Phi = \Phi(u, v)$ be a surface in Galilean 3-space is given by the parametrization

$$\Phi(u, v) = \Big(x(u, v), y(u, v), z(u, v)\Big), \quad u, v \in \mathbb{R},$$

where $x(u, v), y(u, v), z(u, v) \in C^3$. The isotropic unit normal vector U of the surface M is defined by

$$U(u, v) = \frac{\Phi_u \wedge \Phi_v}{\|\Phi_u \wedge \Phi_v\|}, \quad \Phi_u = \frac{\partial \Phi}{\partial u}, \quad \Phi_v = \frac{\partial \Phi}{\partial v}, \tag{2.5}$$

or equivalently

$$U(u, v) = \frac{\Big(0, x_vz_u - x_uz_v, x_uy_v - x_vy_u\Big)}{\sqrt{(x_vz_u - x_uz_v)^2 + (x_uy_v - x_vy_u)^2}},$$

where $x_u = \frac{\partial x(u,v)}{\partial u}$, $x_v = \frac{\partial x(u,v)}{\partial v}$. Using (2.1) and $W = \|\Phi_u \wedge \Phi_v\|$, we get the isotropic unit vector $\delta(u, v)$ in the tangent plane of the surface as [4]

$$\delta(u, v) = \frac{\Big(0,\ x_vy_u - x_uy_v,\ x_vz_u - x_uz_v\Big)}{W}, \tag{2.6}$$

where

$$\langle \delta, \delta\rangle = 1, \quad \langle U, \delta\rangle = 0,$$

by means of Galilean geometry. Observe that a straightforward computation shows that δ can be expressed by

$$\delta(u, v) = \frac{x_v\Phi_u - x_u\Phi_v}{W}. \tag{2.7}$$

The first fundamental form ds^2 of a surface M in G_3 is given by

$$\begin{aligned} I &= ds^2 \\ &= (g_1du + g_2dv)^2 + \epsilon(h_{11}du^2 + 2h_{12}dudv + h_{22}dv^2) \end{aligned} \tag{2.8}$$

where

$$\begin{aligned} &g_1 = x_u, \quad g_2 = x_v, \quad g_{ij} = g_i g_j, \\ &h_{11} = \langle \Phi_u, \Phi_u \rangle, \ h_{12} = \langle \Phi_u, \Phi_v \rangle, \ h_{22} = \langle \Phi_v, \Phi_v \rangle, \end{aligned} \tag{2.9}$$

and

$$\epsilon = \begin{cases} 0, & \text{if direction } du : dv \text{ is non-isotropic,} \\ 1, & \text{if direction } du : dv \text{ is isotropic.} \end{cases}$$

The coefficients of the second fundamental form can be determined from

$$L_{ij} = \langle \frac{\Phi_{ij} x_u - x_{ij}\Phi_u}{x_u}, U \rangle = \langle \frac{\Phi_{ij} x_v - x_{ij}\Phi_v}{x_v}, U \rangle,$$

where Φ_{ij} denotes the second order partial differentials of M and the indices i, j belong to the parameters u, v respectively. Under this parametrization of the surface M, the Gaussian curvature K and the mean curvature H have the classical expressions, respectively [11]

$$K = \frac{\det(L_{ij})}{W^2} = \frac{L_{11}L_{22} - L_{12}^2}{h_{11}h_{22} - h_{12}^2}, \tag{2.10}$$

$$H = \frac{1}{2} h^{ij} L_{ij} = \frac{h_{11}L_{22} + h_{22}L_{11} - 2h_{12}L_{12}}{2(h_{11}h_{22} - h_{12}^2)}. \tag{2.11}$$

3. Inextensible flows of tube-like surfaces associated with focal curves in G_3

The aim of this section, we will obtain the tube-like surface from the tube surface. Since the tube surfaces are special kinds of the canal surfaces in Galilean 3-space.

If we find the canal surface with taking variable radius $r(u)$ as constant, then the tube surface can be found, since the canal surface is a general case of the tube surface. An envelope of a 1-parameter family of surfaces is constructed in the same way that we constructed a 1-parameter family of curves. The family is described by a differentiable function $F(x, y, z, \lambda) = 0$, where λ is a parameter. When λ can be eliminated from the equations

$$F(x, y, z, \lambda) = 0,$$

and

$$\frac{\partial F(x, y, z, \lambda)}{\partial \lambda} = 0,$$

we get the envelope, which is a surface described implicitly as $G(x, y, z) = 0$. For example, for a 1-parameter family of planes we get a developable surface [18].

Definition 3.1. The envelope of a 1-parameter family $u \to S^2_\pm$ of the spheres in G_3 is called a canal surface in Galilean 3-space. The curve formed by the centers of the Galilean spheres is called center curve of the canal surface.The radius of the canal surface is the function r such that $r(u)$ is the radius of the Galilean sphere $S^2_\pm$. Then, the canal surface can be parametrized as follows

$$C(u,v) = \alpha(u) + r(u)\Big(\cos[v]N(u) + \sin[v]B(u)\Big). \tag{3.1}$$

Definition 3.2. Let $\alpha : (a,b) \to G_3$ be a unit speed curve whose curvature does not vanish. Consider a tube of radius r around α. Since the normal N and binormal B are perpendicular to α, the Galilean circle is perpendicular α and $\alpha(u)$. As this Galilean circle moves along α, it traces out a surface about α which will be the tube about α, provided r is not too large. If the radius function $r(u) = r$ is a constant, then, the canal surface is called a tube (pipe) surface and it parametrized as

$$\mathrm{Tube}(u,v) = \alpha(u) + r\Big(\cos[v]N(u) + \sin[v]B(u)\Big). \tag{3.2}$$

Theorem 3.3. *Let $\alpha : I \to G_3$ be a curve in Galilean 3-space. Assume the center curve of a tube-like surface is a unit speed curve α with nonzero curvature. Then, the tube-like surface can be expressed as follows*

$$X(u,v) = \alpha(v) + r\Big(\cos[u]N(v) - \sin[u]B(v)\Big), \tag{3.3}$$

where T, N and B are the tangent, principal normal and binormal of α.

Proof. Suppose X is a patch that parametrizes the envelope of the Galilean spheres defining tube-like surface. Where the curvature of $\alpha(v)$ is nonzero, the Frenet frame of it is well defined, and we can write

$$X(u,v) - \alpha(v) = p(u,v)T(v) + q(u,v)N(v) - w(u,v)B(v), \tag{3.4}$$

where p, q and w are differentiable on the interval on which α is defined. We have

$$\langle X(u,v) - \alpha(v), X(u,v) - \alpha(v)\rangle_{G_3} = \begin{cases} p^2 = r^2 \text{ if } p(u,v) \neq 0, \\ q^2 + w^2 = r^2 \text{ if } p(u,v) = 0. \end{cases} \tag{3.5}$$

The equation (3.5) expresses analytically the geometric fact that $X(u,v)$ lies on a Galilean sphere $S^2_\pm(v)$ of radius r centrered at $\alpha(v)$. Furthermore, $X(u,v) - \alpha(v)$ is a normal vector to the tube-like surface; this fact implies that

$$\langle X(u,v) - \alpha(v), X_u\rangle_{G_3} = 0, \tag{3.6}$$

$$\langle X(u,v) - \alpha(v), X_v\rangle_{G_3} = 0. \tag{3.7}$$

Equations (3.5), (3.6) and (3.7) say that the vectors X_u and X_v are tangents to $S^2_\pm(v)$. Calculating the partial derivative of (3.4) with respect to u and v respectively, we obtain

$$X_u = p_u T + q_u N - w_u B, \tag{3.8}$$

$$X_v = (1 + p_v)T + (p\kappa + q_v + w\tau)N + (q\tau - w_v)B. \tag{3.9}$$

Case 1. If $p(u,v) \neq 0$, from (3.4) and (3.5), we have

$$\begin{cases} p^2 & = & r^2, \\ pp_v & = & 0. \end{cases} \tag{3.10}$$

Equations (3.5), (3.7), (3.9) and (3.10) imply

$$(1+p_v)p = 0. \tag{3.11}$$

From (3.10) and (3.11), we get

$$r = 0. \tag{3.12}$$

Hence, the equation (3.4) is not surface
Case 2. If $p(u,v) = 0$. From (3.4) and (3.5), we have the following

$$\begin{cases} q^2 + w^2 & = & r^2, \\ qq_v + ww_v & = & 0 \ (r\text{=constant}). \end{cases} \tag{3.13}$$

Then, Eqs. (3.6), (3.8) and (3.13) imply that

$$qq_u + ww_u = 0 \ (r\text{=constant}). \tag{3.14}$$

From (3.13) and (3.14), we obtain

$$\begin{cases} q & = & r\cos[u], \\ w & = & r\sin[u]. \end{cases} \tag{3.15}$$

Thus, (3.4) becomes

$$X(u,v) = \alpha(v) + r\Big(\cos[u]N(v) - \sin[u]B(v)\Big).$$

From the above theorem, one can formulate the following definition:

Definition 3.4. Given a space curve $\alpha(v) = \Big(x(v), y(v), z(v)\Big)$, at each point, there are three directions associated with it, the tangent, normal and binormal directions. The unit tangent vector is denoted by T, i.e., $T(v) = \frac{\alpha'(v)}{\|\alpha'(v)\|}$, the unit normal vector is denoted by N, i.e., $N(v) = \frac{T'(v)}{\|T'(v)\|}$, the unit binormal vector is denoted by B, i.e., $B(v) = T(v) \wedge N(v)$ (cross product). With $\alpha(v)$, $T(v)$, $N(v)$ and $B(v)$, a tube-like surface can be expressed as follows [19]

$$M : X(u,v) = \alpha(v) + r\Big(\cos[u]N(v) - \sin[u]B(v)\Big), \tag{3.16}$$

where r is a parameter corresponding to the radius of the rotation (In general r can be a function of v). For fixed v, when u runs from 0 to 2π, we have a circle around the point $\alpha(v)$ in the T, N plane. As we change v, this circle moves along the space curve α, and we will generate a tube-like surface along α (a special kind of tube surfaces defined by (3.16)).

Let $\alpha : I \subset \mathbb{R} \to G_3$, be an unit speed curve in Galilean space G_3 given by

$$\alpha(v) = \Big(v, y(v), z(v)\Big), \tag{3.17}$$

where v is a Galilean invariant parameter (the arc-length on α). The orthonormal frame in the sense of Galilean space G_3 is defined by

$$\begin{cases} T(v) = \alpha'(v) = \Big(1, y'(v), z'(v)\Big), \\ \\ N(v) = \frac{\alpha''(v)}{\kappa(v)} = \frac{1}{\kappa(v)}\Big(0, y''(v), z''(v)\Big), \\ \\ B(v) = \Big(T(v) \wedge N(v)\Big)_G = \frac{1}{\kappa(v)}\Big(0, -z''(v), y''(v)\Big), \end{cases} \tag{3.18}$$

where $\kappa(v) = \|\alpha''(v)\| = \sqrt{(y''(v))^2 + (z''(v))^2}$ is the curvature and

$$\tau(v) = \frac{1}{\kappa^2(v)} \det\Big[\alpha'(v), \alpha''(v), \alpha'''(v)\Big]$$

is the torsion. The vectors $T(v), N(v)$ and $B(v)$ in (3.18) are called of the tangent vector, the principal normal vector and the binormal vector of $\alpha(v)$, respectively. They satisfy the following Frenet equations [13]

$$\begin{bmatrix} T'(v) \\ N'(v) \\ B'(v) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(v) & 0 \\ 0 & 0 & \tau(v) \\ 0 & -\tau(v) & 0 \end{bmatrix} \begin{bmatrix} T(v) \\ N(v) \\ B(v) \end{bmatrix}, \tag{3.19}$$

where the prime denotes the differentiation with respect to v and we denote by κ, τ the curvature and the torsion of the curve α. We can know that T, N, B are mutually orthogonal vector fields satisfying equations

$$\begin{aligned} &\langle T, T\rangle_G = \langle N, N\rangle_G = \langle B, B\rangle_G = 1, \\ &\langle T, N\rangle_G = \langle T, B\rangle_G = \langle N, B\rangle_G = 0, \\ &\det(T, N, B)_G = 1. \end{aligned}$$

Using the equations (3.16), (3.17) and (3.18), we have

$$X(u, v) = \Big(v, y(v), z(v)\Big) + \frac{r}{\kappa}\Big[\Big(0, y''(v), z''(v)\Big)\cos[u] - \Big(0, -z''(v), y''(v)\Big)\sin[u]\Big]. \tag{3.20}$$

From now on, For a unit speed curve $\alpha = \alpha(v) : I \to G_3$, the curve consisting of the centers of the osculating spheres of α is called the parametrized focal curve of α. The hyperplanes normal to α at a point consist of the set of centers of all spheres tangent to α at that point. Hence the center of the osculating spheres at that point lies in such a normal plane. Therefore, denoting the focal curve by C_α, we can write [2]

$$C_\alpha(v) = \Big(\alpha + c_1 N + c_2 B\Big)(v), \tag{3.21}$$

where the coefficients c_1, c_2 are smooth functions of the parameter of the curve α, called the first and second focal curvatures of α, respectively. Further, the focal curvatures c_1, c_2 are defined by

$$c_1 = \frac{1}{\kappa}, \quad c_2 = \frac{c_1'}{\tau}, \quad \kappa \neq 0, \quad \tau \neq 0. \tag{3.22}$$

Lemma 3.5. *Let $\alpha : I \to G_3$ be a unit speed helix and C_α its focal curve on G_3. Then,*

$$c_1 = \frac{1}{\kappa} = \textit{constant and } c_2 = 0. \tag{3.23}$$

On the other hand, the fundamental quantities h_{ij}, L_{ij} and its evolution of tube-like surface (3.16) are obtained, respectively. Thus the Gaussian, mean curvatures and its evolution of such surface are given. For this purpose, let a tube-like surface generated by sweeping a space curve along another central space curve, moving in 3-dimensional Galilean space G_3, be given at time t by the parametrization

$$\begin{cases} \overline{X}(u,v,t) = C_\alpha(v,t) + r\Big(\cos[u,t]N(v,t) - \sin[u,t]B(v,t)\Big), \\ \\ \text{where } \overline{X}(u,v,0) = X(u,v),\ C_\alpha(v,0) = C_\alpha(v),\ \cos[u,0] = \cos[u], \\ \\ N(v,0) = N(v),\ \sin[u,0] = \sin[u] \text{ and } B(v,0) = B(v). \end{cases} \tag{3.24}$$

Definition 3.6. A smooth surface $X(u,v)$ is called a developable surface if its Gaussian curvature K vanishes everywhere on the surface.

Definition 3.7. [10] A surface evolution $\overline{X}(u,v,t)$ and its flow $\frac{\partial \overline{X}}{\partial t}$ are said to be inextensible if its coefficients first fundamental form $\{h_{11}, h_{12}, h_{22}\}$ satisfies

$$\frac{\partial h_{11}}{\partial t} = \frac{\partial h_{12}}{\partial t} = \frac{\partial h_{22}}{\partial t} = 0. \tag{3.25}$$

This definition states that the surface $\overline{X}(u,v,t)$ is, for all time t, the isometric image of the original surface $\overline{X}(u,v,t_0)$ defined at some initial time t_0. For a tube-like surface, $\overline{X}(u,v,t)$ can be physically pictured as the parametrization of a waving flag. For a given surface that is rigid, there exists no nontrivial inextensible evolution.

Theorem 3.8. *Let $\overline{X}$ be the tube-like surface associated with focal curve in G_3. $\frac{\partial \overline{X}}{\partial t}$ is inextensible, then*

$$\frac{\partial \overline{X}}{\partial t} = 0. \tag{3.26}$$

Proof. Suppose that $\overline{X}(u,v,t)$ be a tube-like surface. We show that $\overline{X}$ is inextensible.

$$\begin{aligned} \overline{X}_u &= -r\Big[\sin[u,t]N + \cos[u,t]B\Big], \\ \overline{X}_v &= T + r\tau\sin[u,t]N + \Big[c_1\tau + c_2' + r\tau\cos[u,t]\Big]B. \end{aligned} \tag{3.27}$$

Equations (2.9) and (3.27) lead to the coefficients of the first fundamental form obtained by

$$h_{11} = r^2, \quad h_{12} = 0, \quad h_{22} = 1. \tag{3.28}$$

Under the previous calculations, we have

$$\frac{\partial h_{11}}{\partial t} = 0, \quad \frac{\partial h_{12}}{\partial t} = 0, \quad \frac{\partial h_{22}}{\partial t} = 0.$$

If $\frac{\partial \overline{X}}{\partial t}$ is inextensible, then we have (3.26).

Theorem 3.9. *Let $\overline{X}(u,v,t)$ be the tube-like surface associated with focal curve in G_3. If flow of this tube-like surface is inextensible then this surface is not developable as well as not minimal.*

Proof. Assume that $\overline{X}$ be a tube-like surface parametrized by (3.24). The vector cross product of $\overline{X}_u$ and $\overline{X}_v$ is given by

$$\overline{X}_u \wedge \overline{X}_v = -r\Big[\cos[u,t]N - \sin[u,t]B\Big]. \tag{3.29}$$

Hence, one can get

$$\|\overline{X}_u \wedge \overline{X}_v\| = r. \tag{3.30}$$

Using equations (3.29) and (3.30), we obtain the isotropic normal vector of tube-like surface as

$$U = \frac{\overline{X}_u \wedge \overline{X}_v}{\|\overline{X}_u \wedge \overline{X}_v\|} = -\cos[u,t]N + \sin[u,t]B. \tag{3.31}$$

The second order partial differentials of $\overline{X}$ are found

$$\begin{cases} \overline{X}_{uu} &= r\Big[-\cos[u,t]N + \sin[u,t]B\Big], \\ \overline{X}_{uv} &= r\tau\Big[\cos[u,t]N - \sin[u,t]B\Big], \\ \overline{X}_{vv} &= \Big[\kappa + r\tau' \sin[u,t] - c_1\tau^2 - c_2'\tau - r\tau^2\cos[u,t]\Big]N + \\ & \quad \Big[r\tau^2 \sin[u,t] + c_1'\tau + c_1\tau' + c_2'' + r\tau'\cos[u,t]\Big]B. \end{cases} \tag{3.32}$$

From the equations (3.31) and (3.32), one can compute the coefficients of the second fundamental form for the surface (3.24) as the following

$$\begin{cases} L_{11} &= r, \\ L_{12} &= -r\tau, \\ L_{22} &= \Big[-\kappa + c_1\tau^2 + c_2'\tau\Big]\cos[u,t] + \\ & \quad \Big[c_1'\tau + c_1\tau' + c_2''\Big]\sin[u,t] + r\tau^2. \end{cases} \tag{3.33}$$

Based on the above calculations, the Gaussian curvature K and the mean curvature H of (3.24) are given by, respectively

$$K = \frac{1}{r}\Big(\Big[-\kappa + c_1\tau^2 + c_2'\tau\Big]\cos[u,t] + \Big[c_1'\tau + c_1\tau' + c_2''\Big]\sin[u,t]\Big), \tag{3.34}$$

$$H = \frac{1}{2}\Big(\Big[-\kappa + c_1\tau^2 + c_2'\tau\Big]\cos[u,t] + \Big[c_1'\tau + c_1\tau' + c_2''\Big]\sin[u,t] + r\tau^2\Big) + \frac{1}{2r}. \tag{3.35}$$

By the use of (3.22) and above equations the proof is complete.

Here, we compute in special case the curvatures of the surface (3.24) as well as the curvatures associated to the focal curve of helix on this surface as follows:

At $\kappa = 1, \tau = 1$, the surface (3.24) has the following

$$K = 0, \quad H = \frac{r^2+1}{2r}.$$

Making use of the data described above, one can formulate the following theorem:

Theorem 3.10. *Let $\overline{X}(u,v,t)$ be a tube-like surface associated with focal curve of helix in G_3. If flow of this surface is inextensible then this surface is developable as well as not minimal.*

4. Applications

In this section, we consider an example to illustrate the main results that we have presented in our paper.

Example 4.1. Let us consider a surface

$$\overline{X}(u,v,t) = C_\alpha(v,t) + r\Big(\cos[u,t]N(v,t) - \sin[u,t]B(v,t)\Big), \tag{4.1}$$

where $\alpha(v)$ is a helix

$$\alpha(v) = (v, \cos[v], \sin[v]),$$

it is easy to see that the Frenet's frame is

$$\begin{cases} T(v) = (1, -\sin[v], \cos[v]), \\ N(v) = (0, -\cos[v], -\sin[v]), \\ B(v) = (0, \sin[v], -\cos[v]). \end{cases}$$

Since $\kappa = 1$ is the curvature and $\tau = 1$ is the torsion of the curve α. Then, the focal curve of helix takes the form

$$C_\alpha = (v, 0, 0).$$

Thus, the surface (4.1) takes the following form

$$\overline{X}(u,v,t) = \Big(v, -r\cos[u,t]\cos[v,t] - r\sin[u,t]\sin[v,t], r\sin[u,t]\cos[v,t] - r\cos[u,t]\sin[v,t]\Big). \tag{4.2}$$

Calculating the partial derivative of (4.2) with respect to u and v respectively, we get

$$\overline{X}_u = \Big(0, r\sin[u,t]\cos[v,t] - r\cos[u,t]\sin[v,t], r\cos[u,t]\cos[v,t] + r\sin[u,t]\sin[v,t]\Big),$$

$$\overline{X}_v = \Big(1, r\cos[u,t]\sin[v,t] - r\sin[u,t]\cos[v,t], -r\sin[u,t]\sin[v,t] - r\cos[u,t]\cos[v,t]\Big).$$

The components of the first and second fundamental forms of the surface (4.2) are given by, respectively

$$h_{11} = r^2, \quad h_{12} = 0, \quad h_{22} = 1, \quad L_{11} = r, \quad L_{12} = -r, \quad L_{22} = r.$$

The unit normal vector of the surface (4.2) takes the form

$$U = \Big(0, \sin[u,t]\sin[v,t] + \cos[u,t]\cos[v,t], -\sin[u,t]\cos[v,t] + \cos[u,t]\sin[v,t]\Big). \tag{4.3}$$

For this surface, the Gaussian curvature K and the mean curvature H are defined by, respectively

$$K = 0, \tag{4.4}$$

$$H = \frac{r^2+1}{2r}. \tag{4.5}$$

Then, the surface (4.2) is a developable and not minimal. One can see the graph of $\overline{X}(u,v,t)$ in Figure 1.

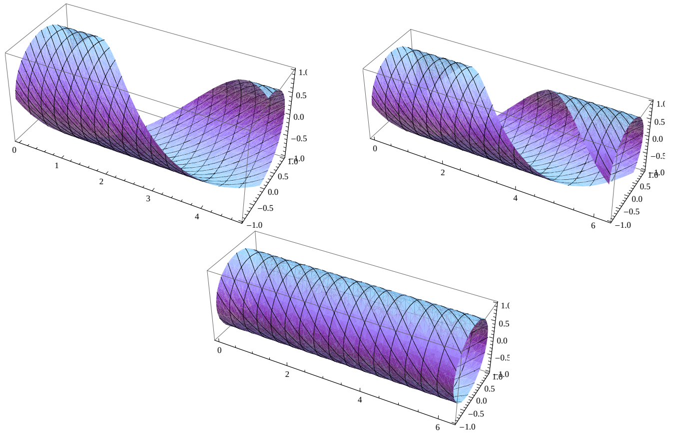


Figure 1. Some tube-like surfaces associated with focal curve of helices with $r=1$, $t=0$, Left: $u \in [0,\pi], v \in [0,\frac{3}{2}\pi]$, Middle: $u \in [0,\frac{13}{10}\pi], v \in [0,2\pi]$ and Right: $u \in [0,2\pi], v \in [0,2\pi]$.

References

[1] Abdel-Aziz, H.S., Khalifa Saad, M., *Weingarten timelike tube surfaces around a spacelike curve*, Int. J. Math. Anal., **5**(2011), 1225–1236.

[2] Alegre, P., Arslan, K., Carriazo, A., Murathan, C., Öztürk, G., *Some special types of developable ruled surface*, Hacettepe Journal of Mathematics and Statistics, **9**(2010), no. 3, 319-325.

[3] Blaga, P.A., *On tubular surfaces in computer graphics*, Stud. Univ. Babeş-Bolyai Inform., **50**(2005), 81–90.

[4] Dede, M., *Tubular surfaces in Galilean space*, Math. Commun., **18**(2013), 209-217.

[5] Ergut, M., Ogrenmis, A.O., *Some characterizations of a spherical curves in Galilean space G_3*, Journal of Advanced Research in Pure Mathematics, **1-2**(2009),18-26.

[6] Kamenarovic, I., *Existence theorems for ruled surfaces in the Galilean space G_3*, Rad Hazu Math., **456**(1991), no. 10, 183-196.

[7] Karacan, M.K., Yayli, Y., *On the geodesics of tubular surfaces in Minkowski 3-Space*, Bull. Malays. Math. Sci. Soc., **31**(2008), 1–10.

[8] Korpinar, T., Turhan, E., Altay, G., *Inextensible flows of developable surfaces associated focal curve of helices in Euclidean 3-space E^3*, Acta Univ. Apulensis, **29**(2012), 235-240.

[9] Kwon, D.Y., Park, F.C., *Evolution of inelastic plane curves*, Appl. Math. Lett., **12**(1999), 115-119.

[10] Kwon, D.Y., Park, F.C., Chi, D.P., *Inextensible flows of curves and developable surfaces*, Appl. Math. Lett., **18**(2005), 1156-1162.

[11] Milin-Sipus, Z., *Ruled Weingarten surfaces in Galilean Space*, Period. Math. Hung., **56**(2008), 213–225.

[12] Milin-Sipus, Z., Divjak, B., *Some special surface in the pseudo-Galilean space*, Acta Math. Hungar., **118**(2008), 209–226.

[13] Ogrenmiş, A.O., Bektaş, M., Ergüt, M., *On helices in the Galilean space G_3*, Iranian Journal of Science & Technology, Transaction A, **31**(A02)(2007), 177-181.

[14] Ogrenmiş, A.O., Ŏztekin, H., Ergüt, M., *Bertrand curves in Galilean space and their characterizations*, Kragujevac J. Math., **32**(2009), 139-147.

[15] Pavkovic, B.J., *The general solution of the Frenet system of differential equations for curves in the Galilean space G_3*, Rad Hazu Math., **450**(1990), 123–128.

[16] Pavkovic, B.J., Kamenarovic, I., *The equiform differential geometry of curves in the Galilean space*, Glasnik Matematicki, **22**(1987), 449–457.

[17] Roschel, O., *Die Geometrie des Galileischen Raumes*, Habilitationsschrift, Leoben, 1984.

[18] Sahin, T., Yilmaz, M., *the rectifying developable and the tangent indicatrix of a curve in Galilean 3-space*, Acta Math. Hungar., **132**(2011), no. 1-2, 154–167.

[19] Sorour, A.H., *Weingarten Tube-like Surfaces in Euclidean 3-Space*, Stud. Univ. Babeş-Bolyai Math., **61**(2016), no. 2, 239-250.

[20] Suk Ro, J., Yoon, D.W., *Tubes of Weingarten types in a Euclidean 3-space*, J. Chungcheong Math. Soc., **22**(2009), 359–366.

[21] Xu, Z., Feng, R., Sun, J., *Analytic and algebraic properties of canal surfaces*, J. Comput. Appl. Math., **195**(2006), 220–228.

Adel H. Sorour
Department of Mathematics, Faculty of Science
Sohag University, 82524, Sohag, Egypt
e-mail: `adel7374@yahoo.com`

Stud. Univ. Babeş-Bolyai Math. 62(2017), No. 3, 353–365
DOI: 10.24193/subbmath.2017.3.08

Integral characterizations for the (h,k)-splitting of skew-evolution semiflows

Claudia Luminiţa Mihiţ and Mihail Megan

Abstract. The main aim of this paper is to give integral characterizations for a general concept of (h,k)-splitting for skew-evolution semiflows in Banach spaces. As consequences, criteria for the properties of (h,k)-dichotomy, nonuniform exponential splitting and exponential splitting are obtained.

Mathematics Subject Classification (2010): 34D09, 40C10.

Keywords: Skew-evolution semiflows, (h,k)-splitting, (h,k)-dichotomy.

1. Introduction

The study of the asymptotic behaviours for dynamical systems represents a research area of large interest, with an impressive development in the last years.

An important starting point for the stability theory is due to E. A Barbashin and R. Datko, who establish integral characterizations for the property of uniform exponential stability in [2], respectively [8].
Recently, P.V. Hai ([10]) obtains discrete and continuous characterizations for the concept of (uniform) exponential stability in terms of Banach sequence (function) spaces. Also, in [20] and [25] are proved generalizations of the results obtained by E. A. Barbashin and R. Datko.

Significant results in the field of exponential dichotomy of skew-product flows are obtained in [7], [11], [13], [14], [22] and for the case of nonlinear differential equations, we emphasize the contributions of S. Elaydi and O. Hajek ([9]).
In [18], respectively [24], the authors give necessary and sufficient conditions for exponential dichotomy with input-output techniques, using spaces of continuous and bounded functions, respectively Lebesgue spaces. Also, the property of (uniform) exponential dichotomy is studied in [23] through the Banach function spaces.

Different concepts of dichotomy of exponential type or more general, with different growth rates, are treated in [4], [5], [6], [12], [16], [19] and the references therein.

As application, we mention the robustness property studied by L. Barreira, J. Chu, C. Valls in [3] and by M. Lizana in [15].

The notion of exponential splitting is a extension of the exponential dichotomy and it is studied for difference equations in [1] and [17]. Important characterizations for various concepts of splitting with growth rates are given in [21].

In this paper we approach the concept of (h, k)-splitting as generalization of (h, k)-dichotomy for skew-evolution semiflows in Banach spaces. Integral conditions of Datko and Barbashin type are given, considering invariant and strongly invariant families of projectors.
Also, we emphasize the results for (h, k)-dichotomy, nonuniform exponential splitting and exponential splitting.

2. Preliminaries

We denote by X a metric space, V a Banach space and $\mathcal{B}(V)$ the Banach algebra of all bounded linear operators on V. The norms on V, respectively $\mathcal{B}(V)$ will be denoted $||\cdot||$.
Also, we consider the sets

$$\Delta = \{(t, t_0) \in \mathbb{R}_+^2 : t \geq t_0\},$$

$$T = \{(t, s, t_0) \in \mathbb{R}_+^3 : t \geq s \geq t_0\}$$

and $Y = X \times V$.

Definition 2.1. A continuous map $\varphi : \Delta \times X \to X$ is said to be *evolution semiflow* on X if it satisfies the following relations:

(es_1) $\varphi(s, s, x) = x,$ for all $(s, x) \in \mathbb{R}_+ \times X$;

(es_2) $\varphi(t, s, \varphi(s, t_0, x)) = \varphi(t, t_0, x),$ for all $(t, s, t_0, x) \in T \times X$.

Definition 2.2. We say that $\Phi : \Delta \times X \to \mathcal{B}(V)$ is an *evolution cocycle* over the evolution semiflow φ if

(ec_1) $\Phi(s, s, x) = I$ (the identity operator on V), for all $(s, x) \in \mathbb{R}_+ \times X$;

(ec_2) $\Phi(t, s, \varphi(s, t_0, x))\Phi(s, t_0, x) = \Phi(t, t_0, x),$ for all $(t, s, t_0, x) \in T \times X$;

(ec_3) $(t, s, x) \mapsto \Phi(t, s, x)v$ is continuous for every $v \in V$.

Definition 2.3. If φ is an evolution semiflow on X and Φ is an evolution cocycle over φ, then the pair $C = (\Phi, \varphi)$ is called *skew-evolution semiflow.*

Example 2.4. Let X be a compact metric space, V a Banach space, φ an evolution semiflow on X and $A : X \to \mathcal{B}(V)$ a continuous map. If $\Phi(t, s, x)v$ is the solution of the equation

$$\dot{v}(t) = A(\varphi(t, s, x))v(t), \quad t \geq s \geq 0,$$

then $C = (\Phi, \varphi)$ is a skew-evolution semiflow.

Definition 2.5. We say that a continuous map $P:\mathbb{R}_+\times X\to\mathcal{B}(V)$ is a *family of projectors* on V if

$$P(s,x)^2=P(s,x),\quad \text{for all } (s,x)\in\mathbb{R}_+\times X.$$

Remark 2.6. If $P:\mathbb{R}_+\times X\to\mathcal{B}(V)$ is a family of projectors for $C=(\Phi,\varphi)$, then $Q:\mathbb{R}_+\times X\to\mathcal{B}(V)$, $Q(t,x)=I-P(t,x)$ is also a family of projectors for C, called the *complementary family of projectors of* P.

Definition 2.7. A family of projectors $P:\mathbb{R}_+\times X\to\mathcal{B}(V)$ is called

(i) *invariant* for the skew-evolution semiflow $C=(\Phi,\varphi)$ if

$$P(t,\varphi(t,s,x))\Phi(t,s,x)=\Phi(t,s,x)P(s,x),\quad \text{for all } (t,s,x)\in\Delta\times X;$$

(ii) *strongly invariant* for the skew-evolution semiflow $C=(\Phi,\varphi)$ if it is invariant for C and for all $(t,s,x)\in\Delta\times X$, the map $\Phi(t,s,x)$ is an isomorphism from *Range* $Q(s,x)$ to *Range* $Q(t,\varphi(t,s,x))$.

Remark 2.8. An example of an invariant family of projectors for a skew-evolution semiflow which is not strongly invariant is given in [21].

Proposition 2.9. *If $P:\mathbb{R}_+\times X\to\mathcal{B}(V)$ is a strongly invariant family of projectors for $C=(\Phi,\varphi)$, then there exists an isomorphism $\Psi:\Delta\times X\to\mathcal{B}(V)$ from Range $Q(t,\varphi(t,s,x))$ to Range $Q(s,x)$, such that:*

(Ψ_1) $\Phi(t,s,x)\Psi(t,s,x)Q(t,\varphi(t,s,x))=Q(t,\varphi(t,s,x))$;
(Ψ_2) $\Psi(t,s,x)\Phi(t,s,x)Q(s,x)=Q(s,x)$;
(Ψ_3) $\Psi(t,s,x)Q(t,\varphi(t,s,x))=Q(s,x)\Psi(t,s,x)Q(t,\varphi(t,s,x))$;
(Ψ_4) $\Psi(t,t_0,x)Q(t,\varphi(t,t_0,x))=\Psi(s,t_0,x)\Psi(t,s,\varphi(s,t_0,x))Q(t,\varphi(t,t_0,x))$,

for all $(t,s,t_0,x)\in T\times X$.

Proof. See [21], Proposition 2. □

Throughout this paper, we will consider two nondecreasing functions $h,k:\mathbb{R}_+\to[1,+\infty)$ with $\lim\limits_{t\to+\infty}h(t)=\lim\limits_{t\to+\infty}k(t)=+\infty$ (*growth rates*).

Let $C=(\Phi,\varphi)$ be a skew-evolution semiflow and $P:\mathbb{R}_+\times X\to\mathcal{B}(V)$ an invariant family of projectors for C.

Definition 2.10. The pair (C,P) admits a (h,k)-*splitting* if there exist two constants $\alpha,\beta\in\mathbb{R}$, $\alpha<\beta$ and a nondecreasing map $N:\mathbb{R}_+\to[1,+\infty)$ such that

(hs_1) $h(s)^\alpha||\Phi(t,t_0,x_0)P(t_0,x_0)v_0||\le N(s)h(t)^\alpha||\Phi(s,t_0,x_0)P(t_0,x_0)v_0||$;
(ks_1) $k(t)^\beta||\Phi(s,t_0,x_0)Q(t_0,x_0)v_0||\le N(t)k(s)^\beta||\Phi(t,t_0,x_0)Q(t_0,x_0)v_0||$,

for all $(t,s,t_0,x_0,v_0)\in T\times Y$, where Q is the complementary family of projectors of P.

The constants α and β are called *splitting constants.*

As particular cases, we have:

(i) if the map N is constant, then we have the property of *uniform* (h,k)*-splitting*;
(ii) if $\alpha<0<\beta$, then we obtain the notion of (h,k)*-dichotomy*;
(iii) if $h(t)=k(t)=e^t$, $t\ge 0$, then we recover the concept of *nonuniform exponential splitting*;

(iv) if $h(t) = k(t) = e^t$ and $N(t) = Se^{\varepsilon t}$, with $t \geq 0$, $S \geq 1$ and $\varepsilon \geq 0$, then we obtain the concept of *exponential splitting.*

Remark 2.11. The pair (C, P) is (h, k)-dichotomic if and only if there are $a, b > 0$ and a nondecreasing mapping $N : \mathbb{R}_+ \to [1, +\infty)$ with

(hd_1) $h(t)^a||\Phi(t, t_0, x_0)P(t_0, x_0)v_0|| \leq N(s)h(s)^a||\Phi(s, t_0, x_0)P(t_0, x_0)v_0||$;

(kd_1) $k(t)^b||\Phi(s, t_0, x_0)Q(t_0, x_0)v_0|| \leq N(t)k(s)^b||\Phi(t, t_0, x_0)Q(t_0, x_0)v_0||$,

for all $(t, s, t_0, x_0, v_0) \in T \times Y$.

Example 2.12. Let $P : \mathbb{R}_+ \times X \to \mathcal{B}(V)$ be a constant family of projectors on V and $Q = I - P$.

Let $h, k : \mathbb{R}_+ \to [1, +\infty)$ be two growth rates and let $\alpha < \beta$ be two real constants.

For every two nondecreasing functions $u, v : \mathbb{R}_+ \to [1, +\infty)$ with

$$\sup_{t\geq 0} u(t) = \alpha \quad \text{and} \quad \sup_{t\geq 0} v(t) = \beta$$

we define $\Phi : \Delta \times X \to \mathcal{B}(V)$ by

$$\Phi(t, s, x) = \frac{u(s)}{u(t)}\left(\frac{h(t)}{h(s)}\right)^{\alpha} P(s, x) + \frac{v(t)}{v(s)}\left(\frac{k(t)}{k(s)}\right)^{\beta} Q(s, x),$$

which is an evolution cocycle over every evolution semiflow on X with

$$\Phi(t, s, x_1) = \Phi(t, s, x_2), \quad \text{for all} \quad (t, s, x_1), (t, s, x_2) \in \Delta \times X,$$

$$\Phi(t, t_0, x_0)P(t_0, x_0) = \frac{u(t_0)}{u(t)}\left(\frac{h(t)}{h(t_0)}\right)^{\alpha} P(t_0, x_0), \quad \text{for all} \quad (t, t_0, x_0) \in \Delta \times X,$$

$$\Phi(t, t_0, x_0)Q(t_0, x_0) = \frac{v(t)}{v(t_0)}\left(\frac{k(t)}{k(t_0)}\right)^{\beta} Q(t_0, x_0), \quad \text{for all} \quad (t, t_0, x_0) \in \Delta \times X.$$

Moreover,

$$h(s)^{\alpha}||\Phi(t, t_0, x_0)P(t_0, x_0)v_0|| = \frac{u(t_0)}{u(t)}\left(\frac{h(s)}{h(t_0)}\right)^{\alpha} h(t)^{\alpha}||P(t_0, x_0)v_0|| \leq$$

$$\leq u(s)h(t)^{\alpha}||\Phi(s, t_0, x_0)P(t_0, x_0)v_0|| \leq N(s)h(t)^{\alpha}||\Phi(s, t_0, x_0)P(t_0, x_0)v_0||$$

and

$$k(t)^{\beta}||\Phi(s, t_0, x_0)Q(t_0, x_0)v_0|| = \frac{v(s)}{v(t)}k(s)^{\beta}||\Phi(t, t_0, x_0)Q(t_0, x_0)v_0|| \leq$$

$$\leq v(t)k(s)^{\beta}||\Phi(t, t_0, x_0)Q(t_0, x_0)v_0|| \leq N(t)k(s)^{\beta}||\Phi(t, t_0, x_0)Q(t_0, x_0)v_0||,$$

for all $(t, s, t_0, x_0, v_0) \in T \times Y$, where $N(t) = u(t) + v(t)$, for every $t \geq 0$.

Finally, we obtain that (C, P) has a (h, k)-splitting, with the splitting constants α and β.

If we suppose that (C, P) is (h, k)-dichotomic, then it results that there exist $\gamma > 0$ and a nondecreasing function $N : \mathbb{R}_+ \to [1, +\infty)$ such that

$$h(t)^{\gamma}||\Phi(t, t_0, x_0)P(t_0, x_0)v_0|| \leq N(s)h(s)^{\gamma}||\Phi(s, t_0, x_0)P(t_0, x_0)v_0||,$$

for all $(t, s, t_0) \in T$ and all $(x_0, v_0) \in Y$.

From here, for $s = t_0 = 0$ we deduce

$$u(0)h(t)^{\alpha+\gamma} \leq N(0)h(0)^{\alpha+\gamma}u(t) \leq \alpha N(0)h(0)^{\alpha+\gamma}$$

and for $t \to +\infty$ we obtain a contradiction.

Remark 2.13. The previous example shows that for every two growth rates h, k and all two real constants $\alpha < \beta$ there is a skew-evolution semiflow which admits a (h, k)-splitting with the splitting constants α, β and which is not (h, k)-dichotomic.

Remark 2.14. The pair (C, P) has a (h, k)-splitting if and only if there exist $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$ and nondecreasing map $N : \mathbb{R}_+ \to [1, +\infty)$ such that

(hs_1') $h(t_0)^{\alpha}||\Phi(t, t_0, x_0)P(t_0, x_0)v_0|| \leq N(t_0)h(t)^{\alpha}||P(t_0, x_0)v_0||$;
(ks_1') $k(t)^{\beta}||Q(t_0, x_0)v_0|| \leq N(t)k(t_0)^{\beta}||\Phi(t, t_0, x_0)Q(t_0, x_0)v_0||$,

for all $(t, t_0, x_0, v_0) \in \Delta \times Y$.

Definition 2.15. We say that (C, P) has a (h, k)-*growth* if there exist two constants $\omega_1, \omega_2 > 0$ and nondecreasing map $M : \mathbb{R}_+ \to [1, +\infty)$ such that

(hg_1) $h(s)^{\omega_1}||\Phi(t, t_0, x_0)P(t_0, x_0)v_0|| \leq M(t_0)h(t)^{\omega_1}||\Phi(s, t_0, x_0)P(t_0, x_0)v_0||$;
(kg_1) $k(s)^{\omega_2}||\Phi(s, t_0, x_0)Q(t_0, x_0)v_0|| \leq M(t)k(t)^{\omega_2}||\Phi(t, t_0, x_0)Q(t_0, x_0)v_0||$,

for all $(t, s, t_0, x_0, v_0) \in T \times Y$.

In particular,

- (neg) for $h(t) = k(t) = e^t$, $t \geq 0$, we have the property of *nonuniform exponential growth*;
- (eg) for $h(t) = k(t) = e^t$ and $M(t) = Ge^{\gamma t}$, $t \geq 0$, $G \geq 1$ and $\gamma \geq 0$, we obtain the notion of *exponential growth.*

Proposition 2.16. *Let $P : \mathbb{R}_+ \times X \to \mathcal{B}(V)$ be a strongly invariant family of projectors for $C = (\Phi, \varphi)$. Then (C, P) admits a (h, k)-splitting if and only if there exist two real constants $\alpha < \beta$ and a nondecreasing mapping $N : \mathbb{R}_+ \to [1, +\infty)$ such that*

(hs_1) $h(s)^{\alpha}||\Phi(t, t_0, x_0)P(t_0, x_0)v_0|| \leq N(s)h(t)^{\alpha}||\Phi(s, t_0, x_0)P(t_0, x_0)v_0||$;
(ks_1'') $k(s)^{\beta}||\Psi(t, t_0, x_0)Q(t, \varphi(t, t_0, x_0))v_0|| \leq$
$\leq N(s)k(t_0)^{\beta}||\Psi(t, s, \varphi(s, t_0, x_0))Q(t, \varphi(t, t_0, x_0))v_0||$,

for all $(t, s, t_0, x_0, v_0) \in T \times Y$.

Proof. See [21], Proposition 3. □

Similarly, we obtain

Remark 2.17. Let $P : \mathbb{R}_+ \times X \to \mathcal{B}(V)$ be a strongly invariant family of projectors for $C = (\Phi, \varphi)$. Then (C, P) has a (h, k)-growth if and only if there exist $\omega_1, \omega_2 > 0$ and nondecreasing function $M : \mathbb{R}_+ \to [1, +\infty)$ with

(hg_1) $h(s)^{\omega_1}||\Phi(t, t_0, x_0)P(t_0, x_0)v_0|| \leq M(t_0)h(t)^{\omega_1}||\Phi(s, t_0, x_0)P(t_0, x_0)v_0||$;
(kg_1') $k(t_0)^{\omega_2}||\Psi(t, t_0, x_0)Q(t, \varphi(t, t_0, x_0))v_0|| \leq$
$\leq M(s)k(s)^{\omega_2}||\Psi(t, s, \varphi(s, t_0, x_0))Q(t, \varphi(t, t_0, x_0))v_0||$,

for all $(t, s, t_0, x_0, v_0) \in T \times Y$.

3. The main results

In this section we will denote with $\mathcal{H}_1$ the set of the growth rates $h : \mathbb{R}_+ \to [1,+\infty)$ with

$$\int_0^{+\infty} h(s)^c ds < +\infty, \quad \text{for all } c < 0.$$

Also, $\mathcal{K}_1$ represents the set of the growth rates $k : \mathbb{R}_+ \to [1,+\infty)$, with the property that there exists a constant $K \geq 1$ such that

$$\int_0^{t} k(s)^c ds \leq Kk(t)^c, \quad \text{for all } c > 0,\ t \geq 0.$$

By $\mathcal{H}$ we denote the set of the growth rates $h : \mathbb{R}_+ \to [1,+\infty)$ with the property that there exists $H \geq 1$ such that

$$h(t)^c \leq Hh(s)^c, \quad \text{for all} \quad (t,s) \in \Delta,\ t \leq s+1,\ c \in \mathbb{R}.$$

Remark 3.1. If we denote by $e(t) = e^t$, $t \geq 0$, then $e \in \mathcal{H}_1 \cap \mathcal{K}_1 \cap \mathcal{H}$.

We consider $C = (\Phi, \varphi)$ a skew-evolution semiflow, $P : \mathbb{R}_+ \times X \to \mathcal{B}(V)$ an invariant family of projectors for C.

A first characterization for the (h,k)-splitting property is given by

Theorem 3.2. *Let (C,P) be a pair with (h,k)-growth, where $h \in \mathcal{H}_1 \cap \mathcal{H}$ and $k \in \mathcal{K}_1 \cap \mathcal{H}$. Then (C,P) admits a (h,k)-splitting if and only if there exist $d_1, d_2 \in \mathbb{R}$, $d_1 < d_2$ and a nondecreasing mapping $D : \mathbb{R}_+ \to [1,+\infty)$ such that the following assertions hold:*

$$(Dhs_1) \quad \int_s^{+\infty} \frac{||\Phi(\tau,t_0,x_0)P(t_0,x_0)v_0||}{h(\tau)^{d_1}} d\tau \leq \frac{D(s)}{h(s)^{d_1}} ||\Phi(s,t_0,x_0)P(t_0,x_0)v_0||,$$

for all $(s,t_0,x_0,v_0) \in \Delta \times Y$;

$$(Dks_1) \quad \int_{t_0}^{t} \frac{||\Phi(\tau,t_0,x_0)Q(t_0,x_0)v_0||}{k(\tau)^{d_2}} d\tau \leq \frac{D(t)}{k(t)^{d_2}} ||\Phi(t,t_0,x_0)Q(t_0,x_0)v_0||,$$

for all $(t,t_0,x_0,v_0) \in \Delta \times Y$.

Proof. Necessity. It is a simple verification for $\alpha < d_1 < d_2 < \beta$ and

$$D(s) = N(s)[K + Hh(s)^{d_1-\alpha}],$$

where $H = \int\limits_0^{+\infty} h(\tau)^{\alpha-d_1} d\tau$.

Sufficiency. We show that the relations from Definition 2.10 are verified.

(hs_1) *Case 1*: Let $t \geq s+1$, $(s,t_0) \in \Delta$ and $(x_0,v_0) \in Y$. Then

$$h(s)^{d_1} ||\Phi(t,t_0,x_0)P(t_0,x_0)v_0|| \leq$$

$$\leq h(s)^{d_1} M(t_0) \int_{t-1}^{t} \left(\frac{h(t)}{h(\tau)}\right)^{\omega_1} ||\Phi(\tau,t_0,x_0)P(t_0,x_0)v_0|| d\tau =$$

$$= M(t_0)h(s)^{d_1}h(t)^{d_1}\int\limits_{t-1}^{t}\left(\frac{h(t)}{h(\tau)}\right)^{\omega_1-d_1}\frac{||\Phi(\tau,t_0,x_0)P(t_0,x_0)v_0||}{h(\tau)^{d_1}}d\tau \leq$$

$$\leq HM(s)h(s)^{d_1}h(t)^{d_1}\int\limits_{s}^{+\infty}\frac{||\Phi(\tau,t_0,x_0)P(t_0,x_0)v_0||}{h(\tau)^{d_1}}d\tau \leq$$

$$\leq N(s)h(t)^{d_1}||\Phi(s,t_0,x_0)P(t_0,x_0)v_0||, \quad \text{for all } t \geq s+1,\ s \geq t_0, (x_0,v_0) \in Y,$$

where $N(s) = HM(s)D(s)$, $s \geq 0$.
Case 2: Let $t \in [s, s+1]$, $s \geq t_0$ and $(x_0, v_0) \in Y$. We obtain

$$h(s)^{d_1}||\Phi(t,t_0,x_0)P(t_0,x_0)v_0|| \leq$$

$$\leq M(t_0)\left(\frac{h(t)}{h(s)}\right)^{\omega_1-d_1}h(t)^{d_1}||\Phi(s,t_0,x_0)P(t_0,x_0)v_0|| \leq$$

$$\leq N(s)h(t)^{d_1}||\Phi(s,t_0,x_0)P(t_0,x_0)v_0||,$$

for all $t \in [s, s+1]$, $s \geq t_0$, $(x_0, v_0) \in Y$.
Then, we obtain that (hs_1) is verified for all $(t, s, t_0, x_0, v_0) \in T \times Y$.
(ks_1) *Case 1*: We consider $(t, s, t_0) \in T$, $t \geq s+1$, $(x_0, v_0) \in Y$. Then,

$$\int\limits_{s}^{s+1}k(t)^{d_2}||\Phi(s,t_0,x_0)Q(t_0,x_0)v_0||d\tau \leq$$

$$\leq k(t)^{d_2}\int\limits_{s}^{s+1}M(\tau)\left(\frac{k(\tau)}{k(s)}\right)^{\omega_2}||\Phi(\tau,t_0,x_0)Q(t_0,x_0)v_0||d\tau \leq$$

$$\leq M(t)k(t)^{d_2}k(s)^{d_2}\int\limits_{s}^{s+1}\left(\frac{k(\tau)}{k(s)}\right)^{\omega_2+d_2}\frac{||\Phi(\tau,t_0,x_0)Q(t_0,x_0)v_0||}{k(\tau)^{d_2}}d\tau \leq$$

$$\leq HM(t)k(s)^{d_2}k(t)^{d_2}\int\limits_{t_0}^{t}\frac{||\Phi(\tau,t_0,x_0)Q(t_0,x_0)v_0||}{k(\tau)^{d_2}}d\tau \leq$$

$$\leq N(t)k(s)^{d_2}||\Phi(t,t_0,x_0)Q(t_0,x_0)v_0||.$$

We obtain

$$k(t)^{d_2}||\Phi(s,t_0,x_0)Q(t_0,x_0)v_0|| \leq N(t)k(s)^{d_2}||\Phi(t,t_0,x_0)Q(t_0,x_0)v_0||,$$

for all $t \geq s+1$, $s \geq t_0$, $(x_0, v_0) \in Y$.
Case 2: Let $t \in [s, s+1]$, $s \geq t_0$ and $(x_0, v_0) \in Y$. We deduce the following:

$$k(t)^{d_2}||\Phi(s,t_0,x_0)Q(t_0,x_0)v_0|| \leq$$

$$\leq M(t)\left(\frac{k(t)}{k(s)}\right)^{\omega_2}k(t)^{d_2}||\Phi(t,t_0,x_0)Q(t_0,x_0)v_0|| =$$

$$= M(t)\left(\frac{k(t)}{k(s)}\right)^{\omega_2+d_2}k(s)^{d_2}||\Phi(t,t_0,x_0)Q(t_0,x_0)v_0|| \leq$$

$$\leq N(t)k(s)^{d_2}||\Phi(t, t_0, x_0)Q(t_0, x_0)v_0||.$$

Thus, the condition (ks_1) holds for all $(t, s, t_0, x_0, v_0) \in T \times Y$.
In conclusion, the pair (C, P) has a (h, k)-splitting. $\square$

As consequences, we obtain

Corollary 3.3. *Let (C, P) be a pair with (h, k)-growth, where $h \in \mathcal{H}_1 \cap \mathcal{H}$ and $k \in \mathcal{K}_1 \cap \mathcal{H}$. Then (C, P) is (h, k)-dichotomic if and only if then there exist $d_1 < 0 < d_2$ and a nondecreasing function $D : \mathbb{R}_+ \to [1, +\infty)$ such that:*

$$(Dhd_1) \quad \int_s^{+\infty} \frac{||\Phi(\tau, t_0, x_0)P(t_0, x_0)v_0||}{h(\tau)^{d_1}} d\tau \leq \frac{D(s)}{h(s)^{d_1}} ||\Phi(s, t_0, x_0)P(t_0, x_0)v_0||,$$

for all $(s, t_0, x_0, v_0) \in \Delta \times Y$;

$$(Dkd_1) \quad \int_{t_0}^{t} \frac{||\Phi(\tau, t_0, x_0)Q(t_0, x_0)v_0||}{k(\tau)^{d_2}} d\tau \leq \frac{D(t)}{k(t)^{d_2}} ||\Phi(t, t_0, x_0)Q(t_0, x_0)v_0||,$$

for all $(t, t_0, x_0, v_0) \in \Delta \times Y$.

Corollary 3.4. *We consider (C, P) a pair with nonuniform exponential growth. Then (C, P) has a nonuniform exponential splitting if and only if there are two constants $d_1, d_2 \in \mathbb{R}$, $d_1 < d_2$ and a nondecreasing map $D : \mathbb{R}_+ \to [1, +\infty)$ with:*

$$(Dnes_1) \quad \int_s^{+\infty} e^{-\tau d_1} ||\Phi(\tau, t_0, x_0)P(t_0, x_0)v_0|| d\tau \leq$$
$$\leq D(s)e^{-sd_1} ||\Phi(s, t_0, x_0)P(t_0, x_0)v_0||,$$

for all $(s, t_0, x_0, v_0) \in \Delta \times Y$;

$$(Dnes_2) \quad \int_{t_0}^{t} e^{-\tau d_2} ||\Phi(\tau, t_0, x_0)Q(t_0, x_0)v_0|| d\tau \leq$$
$$\leq D(t)e^{-td_2} ||\Phi(t, t_0, x_0)Q(t_0, x_0)v_0||,$$

for all $(t, t_0, x_0, v_0) \in \Delta \times Y$.

Corollary 3.5. *If (C, P) is a pair with exponential growth, then it admits an exponential splitting if and only if there exists some real constants $d_1 < d_2$, $D \geq 1$ and $\delta \geq 0$ such* *that:*

$$(Des_1) \qquad \int_s^{+\infty} e^{-\tau d_1}||\Phi(\tau,t_0,x_0)P(t_0,x_0)v_0||d\tau \leq$$
$$\leq De^{(\delta-d_1)s}||\Phi(s,t_0,x_0)P(t_0,x_0)v_0||,$$
for all $(s,t_0,x_0,v_0)\in\Delta\times Y$;

$$(Des_2) \qquad \int_{t_0}^{t} e^{-\tau d_2}||\Phi(\tau,t_0,x_0)Q(t_0,x_0)v_0||d\tau \leq$$
$$\leq De^{(\delta-d_2)t}||\Phi(t,t_0,x_0)Q(t_0,x_0)v_0||,$$
for all $(t,t_0,x_0,v_0)\in\Delta\times Y$.

Remark 3.6. The results given by Theorem 3.2, Corollary 3.3, Corollary 3.4 and Corollary 3.5 are characterizations of Datko-type for the splitting concepts studied in this paper.

Further, $C=(\Phi,\varphi)$ represents a skew-evolution semiflow and $P:\mathbb{R}_+\times X\to\mathcal{B}(V)$ a strongly invariant family of projectors for C.
In this context, we obtain the following characterization for (h,k)-splitting:

Theorem 3.7. *Let (C,P) be a pair with (h,k)-growth,where $h\in\mathcal{H}_1\cap\mathcal{H}$ and $k\in\mathcal{K}_1\cap\mathcal{H}$. Then (C,P) admits a (h,k)-splitting if and only if there exist $d_1,d_2\in\mathbb{R}$, $d_1<d_2$ and a nondecreasing map $D:\mathbb{R}_+\to[1,+\infty)$ such that the following inequalities are verified:*

$$(Dhs_1) \qquad \int_s^{+\infty}\frac{||\Phi(\tau,t_0,x_0)P(t_0,x_0)v_0||}{h(\tau)^{d_1}}d\tau\leq\frac{D(s)}{h(s)^{d_1}}||\Phi(s,t_0,x_0)P(t_0,x_0)v_0||,$$
for all $(s,t_0,x_0,v_0)\in\Delta\times Y$;

$$(Dks_1') \qquad \int_{t_0}^{s}\frac{||\Psi(t,\tau,\varphi(\tau,t_0,x_0))Q(t,\varphi(t,t_0,x_0))v_0||}{k(\tau)^{d_2}}d\tau\leq$$
$$\leq\frac{D(s)}{k(s)^{d_2}}||\Psi(t,s,\varphi(s,t_0,x_0))Q(t,\varphi(t,t_0,x_0))v_0||,$$
for all $(t,s,t_0,x_0,v_0)\in T\times Y$.

Proof. Necessity. It results from Proposition 2.16, for $\alpha<d_1<d_2<\beta$ and
$$D(s)=N(s)[K+Hh(s)^{d_1-\alpha}],$$
where $H=\int\limits_0^{+\infty}h(\tau)^{\alpha-d_1}d\tau$.

Sufficiency. We prove that the inequalities (hs_1) and (ks_1'') from Proposition 2.16 hold.

In a similar manner with the proof of Theorem 3.2 we obtain
$$h(s)^{d_1}||\Phi(t,t_0,x_0)P(t_0,x_0)v_0||\leq N(s)h(t)^{d_1}||\Phi(s,t_0,x_0)P(t_0,x_0)v_0||,$$

for all $(t, s, t_0, x_0, v_0) \in T \times Y$, where $N(s) = HM(s)D(s)$, $s \geq 0$.

Thus, we consider $(t, s, t_0) \in T$, $s \geq t_0 + 1$, $(x_0, v_0) \in Y$ and it results that

$$k(s)^{d_2}||\Psi(t, t_0, x_0)Q(t, \varphi(t, t_0, x_0)v_0|| =$$

$$= k(s)^{d_2} \int_{t_0}^{t_0+1} ||\Psi(\tau, t_0, x_0)\Psi(t, \tau, \varphi(\tau, t_0, x_0))Q(t, \varphi(t, t_0, x_0))v_0||d\tau \leq$$

$$\leq k(s)^{d_2} \int_{t_0}^{t_0+1} M(\tau) \left(\frac{k(\tau)}{k(t_0)}\right)^{\omega_2} ||\Psi(t, \tau, \varphi(\tau, t_0, x_0))Q(t, \varphi(t, t_0, x_0))v_0||d\tau \leq$$

$$\leq M(s)k(s)^{d_2}k(t_0)^{d_2} \int_{t_0}^{t_0+1} \left(\frac{k(\tau)}{k(t_0)}\right)^{\omega_2+d_2} \frac{||\Psi(t, \tau, \varphi(\tau, t_0, x_0))Q(t, \varphi(t, t_0, x_0))v_0||}{k(\tau)^{d_2}} d\tau \leq$$

$$\leq HM(s)k(s)^{d_2}k(t_0)^{d_2} \int_{t_0}^{s} \frac{||\Psi(t, \tau, \varphi(\tau, t_0, x_0))Q(t, \varphi(t, t_0, x_0))v_0||}{k(\tau)^{d_2}} d\tau \leq$$

$$\leq N(s)k(t_0)^{d_2}||\Psi(t, s, \varphi(s, t_0, x_0))Q(t, \varphi(t, t_0, x_0))v_0||.$$

For $t \geq s$, $s \in [t_0, t_0 + 1)$, $(x_0, v_0) \in Y$ we have

$$k(s)^{d_2}||\Psi(t, t_0, x_0)Q(t, \varphi(t, t_0, x_0))v_0|| \leq$$

$$\leq k(s)^{d_2}M(s) \left(\frac{k(s)}{k(t_0)}\right)^{\omega_2} ||\Psi(t, s, \varphi(s, t_0, x_0))Q(t, \varphi(t, t_0, x_0))v_0|| \leq$$

$$\leq M(s)k(t_0)^{d_2} \left(\frac{k(s)}{k(t_0)}\right)^{\omega_2+d_2} ||\Psi(t, s, \varphi(s, t_0, x_0))Q(t, \varphi(t, t_0, x_0))v_0|| \leq$$

$$\leq N(s)k(t_0)^{d_2}||\Psi(t, s, \varphi(s, t_0, x_0))Q(t, \varphi(t, t_0, x_0))v_0||.$$

We deduce that (ks_1'') is verified, for all $(t, s, t_0) \in T$, $(x_0, v_0) \in Y$.

Using Proposition 2.16, it follows that (C, P) admits a (h, k)-splitting. $\square$

In particular, we emphasize the following consequences:

Corollary 3.8. *Let (C, P) be a pair with (h, k)-growth,where $h \in \mathcal{H}_1 \cap \mathcal{H}$ and $k \in \mathcal{K}_1 \cap \mathcal{H}$. Then (C, P) is (h, k)-dichotomic if and only if there exist two constants $d_1 < 0 < d_2$*

and a nondecreasing map $D : \mathbb{R}_+ \to [1, +\infty)$ *with:*

$$(Dhd_1) \quad \int_s^{+\infty} \frac{||\Phi(\tau, t_0, x_0)P(t_0, x_0)v_0||}{h(\tau)^{d_1}} d\tau \leq \frac{D(s)}{h(s)^{d_1}} ||\Phi(s, t_0, x_0)P(t_0, x_0)v_0||,$$

for all $(s, t_0, x_0, v_0) \in \Delta \times Y$;

$$(Dkd_1') \quad \int_{t_0}^{s} \frac{||\Psi(t, \tau, \varphi(\tau, t_0, x_0))Q(t, \varphi(t, t_0, x_0))v_0||}{k(\tau)^{d_2}} d\tau \leq$$

$$\leq \frac{D(s)}{k(s)^{d_2}} ||\Psi(t, s, \varphi(s, t_0, x_0))Q(t, \varphi(t, t_0, x_0))v_0||,$$

for all $(t, s, t_0, x_0, v_0) \in T \times Y$.

Corollary 3.9. *Let* (C, P) *be with nonuniform exponential growth. Then* (C, P) *has a nonuniform exponential splitting if and only if exist two real constants* $d_1 < d_2$ *and a nondecreasing function* $D : \mathbb{R}_+ \to [1, +\infty)$ *such that:*

$$(Dnes_1) \quad \int_s^{+\infty} e^{-\tau d_1} ||\Phi(\tau, t_0, x_0)P(t_0, x_0)v_0|| d\tau \leq$$

$$\leq D(s)e^{-sd_1} ||\Phi(s, t_0, x_0)P(t_0, x_0)v_0||,$$

for all $(s, t_0, x_0, v_0) \in \Delta \times Y$;

$$(Dnes_2') \quad \int_{t_0}^{s} e^{-\tau d_2} ||\Psi(t, \tau, \varphi(\tau, t_0, x_0))Q(t, \varphi(t, t_0, x_0))v_0|| d\tau \leq$$

$$\leq D(s)e^{-sd_2} ||\Psi(t, s, \varphi(s, t_0, x_0))Q(t, \varphi(t, t_0, x_0))v_0||,$$

for all $(t, s, t_0, x_0, v_0) \in T \times Y$.

Corollary 3.10. *If* (C, P) *has an exponential growth, then it admits an exponential splitting if and only if there exist* $d_1, d_2 \in \mathbb{R}$, $d_1 < d_2$, $D \geq 1$ *and* $\delta \geq 0$ *such that:*

$$(Des_1) \quad \int_s^{+\infty} e^{-\tau d_1} ||\Phi(\tau, t_0, x_0)P(t_0, x_0)v_0|| d\tau \leq$$

$$\leq De^{(\delta - d_1)s} ||\Phi(s, t_0, x_0)P(t_0, x_0)v_0||,$$

for all $(s, t_0, x_0, v_0) \in \Delta \times Y$;

$$(Des_2') \quad \int_{t_0}^{s} e^{-\tau d_2} ||\Psi(t, \tau, \varphi(\tau, t_0, x_0))Q(t, \varphi(t, t_0, x_0))v_0|| d\tau \leq$$

$$\leq De^{(\delta - d_2)s} ||\Psi(t, s, \varphi(s, t_0, x_0))Q(t, \varphi(t, t_0, x_0))v_0||,$$

for all $(t, s, t_0, x_0, v_0) \in T \times Y$.

Remark 3.11. Theorem 3.7, Corollary 3.8, Corollary 3.9 and Corollary 3.10 are characterizations of Barbashin-type for the splitting concepts considered in this paper.

References

[1] Aulbach, B., Kalkbrenner, J., *Exponential forward splitting for noninvertible difference equations*, Comput. Math. Appl., **42**(2001), 743-754.

[2] Barbashin, E.A., *Introduction to Stability Theory*, Nauka, Moscow, 1967.

[3] Barreira, L., Chu, J., Valls, C., *Robustness of nonuniform dichotomies with different growth rates,* Sao Paolo J. Math. Sci., **2**(2011), 203-231.

[4] Barreira, L., Valls, C., *Nonuniform exponential dichotomies and admissibility*, Discrete and Contin. Dyn. Syst. Series B., **30**(2011), no. 1, 39-53.

[5] Bento, A.J.G., Lupa, N., Megan, M., Silva, C., *Integral conditions for nonuniform μ-dichotomy on the half-line,* accepted for publication in Discrete and Contin. Dyn. Syst.-Series B.

[6] Bento, A.J.G., Silva, C., *Nonuniform dichotomic behavior: Lipschitz invariant manifolds for ODEs*, Bull. Sci. Math., **138**(2014), no. 1, 89-109.

[7] Chow, S.N., Leiva, H., *Existence and roughness of the exponential dichotomy for skew-product semiflow in Banach spaces*, J. Differential Equation, **120**(1995), 429-477.

[8] Datko, R., *Uniform Asymptotic Stability of Evolutionary Processes in Banach Space*, SIAM J. Math. Anal., **3**(1972), 428-445.

[9] Elaydi, S., Hajek, O., *Exponential dichotomy and trichotomy of nonlinear differential equations*, Differential Integral Equations, **3**(1990), 1201-1204.

[10] Hai, P.V., *Continuous and discrete characterizations for the uniform exponential stability of linear skew-evolution semiflows*, Nonlinear Anal., **72**(2010), 4390-4396.

[11] Huy, N.T., *Existence and robustness of exponential dichotomy of linear skew-product semiflows over semiflows,* J. Math. Anal. Appl., **333**(2007), 731-752.

[12] Kovacs, M.I., Babuţia, M.G., Megan, M., *On (h, k)- dichotomy in Banach spaces,* Bull. Ştiinţ. Univ. Politeh. Timiş. Ser. Mat. Fiz., **59**(2014), 19-27.

[13] Latushkin, Y., Montgomery-Smith, S., Randolph, T., *Evolutionary semigroups and dichotomy of linear skew-product flows on locally compact spaces with Banach fibers*, J. Differential Equations, **125**(1996), 73-116.

[14] Latushkin, Y., Schnaubelt, R., *Evolution semigroups, translation algebras and exponential dichotomy of cocycles*, J. Differential Equations, **159**(1999), 321-369.

[15] Lizana, M., *Roughness of (μ_1, μ_2)-dichotomies under small perturbations in L^∞*, Divulg. Mat., **6**(1998), 133-138.

[16] Megan, M., *On (h, k)-dichotomy of evolution operators in Banach spaces*, Dynam. Systems Appl., **5**(1996), 189-196.

[17] Megan, M., Popa, I.L., *Exponential splitting for nonautonomous linear discrete-time systems in Banach spaces*, J. Computat. Appl. Math., **312**(2017), 181-191.

[18] Megan, M., Sasu, A.L., Sasu, B., *Uniform exponential dichotomy and admissibility for linear skew-product semiflows,* Oper. Theory Adv. Appl., **153**(2004), 185-195.

[19] Megan, M., Stoica, C., Buliga, L., *On asymptotic behaviours for linear skew-evolution semiflows in Banach spaces*, Carpathian J. Math., **23**(2007), 117-125.

[20] Megan, M., Stoica, C., *Exponential instability of skew-evolution semiflows in Banach spaces*, Stud. Univ. Babeş-Bolyai Math., **53**(2008), no. 1, 17-24.

[21] Mihiţ, C.L., Borlea, D., Megan, M., *On some concepts of (h, k)-splitting for skew-evolution semiflows in Banach spaces*, to appear in Ann. Acad. Rom. Sci. Ser. Math. Appl.

[22] Sacker, R.J., Sell, G.R., *Dichotomies for linear evolutionary equations in Banach spaces*, J. Differential Equations, **113**(1994), 17-67.

[23] Sasu, A.L., Sasu, B., *Translation invariant spaces and asymptotic properties of variational equations*, Abstr. Appl. Anal., (2011), Art. ID 539026, 1-36.

[24] Sasu, A.L., Sasu, B., *Integral equations in the study of the asymptotic behavior of skew-product flows*, Asymptot. Anal., **68**(2010), 135-153.

[25] Stoica, C., Megan, M., *On uniform exponential stability for skew-evolution semiflows on Banach spaces*, Nonlinear Anal., **72**(2010), no. 3-4, 1305-1313.

Claudia Luminiţa Mihiţ
West University of Timişoara
Faculty of Mathematics and Computer Science
Department of Mathematics
V. Pârvan Blv. 4, 300223 Timişoara, Romania
e-mail: `mihit.claudia@yahoo.com`

Mihail Megan
Academy of Romanian Scientists
Independenţei 54, 050094 Bucharest, Romania
West University of Timişoara
Faculty of Mathematics and Computer Science
Departament of Mathematics
V. Pârvan Blv. 4, 300223 Timişoara, Romania
e-mail: `mihail.megan@e-uvt.ro`

Stud. Univ. Babeş-Bolyai Math. 62(2017), No. 3, 367–376
DOI: 10.24193/subbmath.2017.3.09

Characterization of q-Cesàro convergence for double sequences

Emre Taş and Cihan Orhan

Abstract. In the present paper we examine the Buck-Pollard property of 4-dimensional q-Cesàro matrices. Indeed we discuss some questions related to the q-Cesàro summability of subsequences of a given double sequence. The main result states that " a bounded double sequence is q-Cesàro summable to L if and only if almost all of its subsequences are q-Cesàro summable to $2^{1-q}L$".

Mathematics Subject Classification (2010): 40B05, 40C05, 28A12.

Keywords: Double sequences, Pringsheim convergence, the Buck-Pollard property, q-Cesàro matrices.

1. Introduction

Buck and Pollard [2] proved that a bounded sequence (s_n) is $(C, 1)$ summable if and only if almost all of its subsequences is $(C, 1)$ summable. Since this idea has been introduced by Buck and Pollard, the property is to be called "Buck-Pollard property". The Buck-Pollard property is related to the convergence or summability of subsequences of a given sequence. Taking into consideration q-Cesàro matrix instead of $(C, 1)$ matrix, similar results have been investigated in [7]. Recently the Buck-Pollard property for $(C, 1, 1)$ summability method has been examined and also provided a new characterization of $(C, 1, 1)$ summability for double sequences with respect to its subsequences [10].

In the present paper we consider similar problems for four dimensional q-Cesàro matrix on double sequences. We first introduce the notions of our interest related to double sequences.

A double sequence $s = (s_{jk})$ is said to be Pringsheim convergent (i.e., it is convergent in Pringsheim's sense) to L if for every $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that $|s_{jk} - L| < \varepsilon$ whenever $j, k \geq N$ ([9]). In this case L is called the Pringsheim limit of s and the space of such sequences is denoted by $c^{(2)}$. A double sequence s is bounded if there exists a positive number H such that $|s_{jk}| < H$ for all j and k, i.e.,

$$\|s\|_{(\infty,2)} = \sup_{j,k} |s_{jk}| < \infty.$$

We will denote the set of all bounded double sequences by $l_{\infty}^{(2)}$. Note that in contrast to the case for single sequences, a convergent double sequence need not to be bounded.

Throughout the paper when there is no confusion, "convergence" means the Pringsheim convergence.

Four dimensional q-Cesàro matrix $(C_q, 1, 1) = \left(c_{jk}^{nm}\right)$ is defined by

$$c_{jk}^{nm} = \begin{cases} \frac{1}{n^q m^q} & , \quad 1 \le j \le n \text{ and } 1 \le k \le m \\ 0 & , \quad \text{otherwise} \end{cases}$$

where $0 < q < \infty$. Observe that the case $q = 1$ reduces to $(C, 1, 1)$, 4-dimensional Cesàro matrix. Also if $q \neq 1$, q-Cesàro matrices $(C_q, 1, 1)$ cannot be RH regular, i.e., it cannot sum every bounded convergent sequence to the same limit.

There exist several versions of the concept of subsequences for double sequences ([3], [8], [12]). We adopt Definition 2 of [3] on subsequences of double sequences throughout the paper.

Let X denote the set of all double sequences of 0's and 1's, that is

$$X = \{x = (x_{jk}) : x_{jk} \in \{0, 1\} \text{ for each } j, k \in \mathbb{N}\}.$$

Let $\Re$ be the smallest σ-algebra of subsets of the set X which contains all sets of the form

$$\{x = (x_{jk}) \in X : \ x_{j_1 k_1} = a_1, ..., x_{j_n k_n} = a_n\}$$

where each $a_i \in \{0, 1\}$ and the pairs $\{(j_i k_i)\}_{i=1}^n$ are pairwise distinct.

There exists a unique probability measure P on the set $\Re$, such that

$$P\left(\{x = (x_{jk}) \in X : \ x_{j_1 k_1} = a_1, ..., x_{j_n k_n} = a_n\}\right) = \frac{1}{2^n}$$

for all choices of n and all pairwise disjoint pairs $\{(j_i k_i)\}_{i=1}^n$, and all choices of $a_1, ..., a_n$ (see, [3]).

Let $s = (s_{jk})$ be a double sequence and $x = (x_{jk}) \in X$. Following [3] we define a subsequence of the sequence s by

$$s_{jk}(x) = \begin{cases} s_{jk} & , \quad if \ x_{jk} = 1 \\ * & , \quad if \ x_{jk} = 0 \end{cases}.$$

Mapping $x \to s(x)$ is a bijection from the set X to the set of all the subsequences of the sequence $s = (s_{jk})$ [3].

An element x of X is said to be normal ([3]) if for each $\varepsilon > 0$ there is a natural number N_{ε} such that for $n, m \ge N_{\varepsilon}$ we have $\left| \frac{1}{nm} \sum_{\substack{j \le n \\ k \le m}} x_{jk} - \frac{1}{2} \right| < \varepsilon$. Let η denote the set of all elements x in X that are normal. This implies that normal elements are $(C, 1, 1)$-summable to $\frac{1}{2}$. It is also known ([3]) that $P(\eta) = 1$.

2. Subsequence Characterization of q-Cesàro Summability

In this section we characterize $(C_q, 1, 1)$ summability of a double sequence. In particular we study conditions under which $(C_q, 1, 1)$ summability of a double sequence carry over to that of its subsequences, and conversely, whether these properties for suitable subsequences imply them for the sequence itself. We begin with the following theorem which is analog to that of Buck and Pollard [2] for single sequences.

Theorem 2.1. *If almost all subsequences of $s = (s_{jk})$ are $(C_q, 1, 1)$-summable to a value L then the sequence $s = (s_{jk})$ is $(C_q, 1, 1)$-summable to $2^{1-q}\ L$.*

Proof. If almost all subsequences of (s_{jk}) are $(C_q, 1, 1)$-summable to a value L then the set $G = \{x \in X :\ s(x) \text{ is } (C_q, 1, 1)\text{-summable to } L\}$ has probability measure 1. We use the technique given in [3]. Now given a sequence $x = (x_{jk}) \in X$ we define a sequence $\bar{x} = (\bar{x}_{jk})$ by

$$\bar{x}_{jk} = \begin{cases} 0 & , \ if\ x_{jk} = 1 \\ 1 & , \ if\ x_{jk} = 0 \end{cases}.$$

Let $Y = G \cap \eta$ and $\overline{Y} = \{(\bar{x}_{jk}) :\ x_{jk} \in Y\}$. Therefore we have $\overline{Y} = \overline{G} \cap \eta$ where $\overline{G}$ is defined in the obvious way. Since the mapping $(x_{jk}) \to (\bar{x}_{jk})$ preserves the measure P, we get $P\left(\overline{Y}\right) = 1$ and hence $P\left(Y \cap \overline{Y}\right) = 1$. So $Y \cap \overline{Y}$ is a non-empty set. If $x = (x_{jk}) \in Y \cap \overline{Y}$, then we have $x \in G$, $x \in \eta$ and $\bar{x} \in G$, $\bar{x} \in \eta$. Hence we obtain

$$s(x) \to L\,(C_q, 1, 1)$$

and

$$s(\bar{x}) \to L\,(C_q, 1, 1)$$

with $x, \bar{x} \in \eta$. That is

$$\lim_{n,m\to\infty} \frac{\sum\limits_{j,k=1,1}^{n,m} s_{jk}x_{jk}}{\left(\sum\limits_{j,k=1,1}^{n,m} x_{jk}\right)^q} = L \text{ and } \lim_{n,m\to\infty} \frac{\sum\limits_{j,k=1,1}^{n,m} s_{jk}\bar{x}_{jk}}{\left(\sum\limits_{j,k=1,1}^{n,m} \bar{x}_{jk}\right)^q} = L.$$

Also since $x, \bar{x} \in \eta$, we have

$$\lim_{n,m\to\infty} \frac{1}{nm} \sum_{j,k=1,1}^{n,m} x_{jk} = \frac{1}{2} \text{ and } \lim_{n,m\to\infty} \frac{1}{nm} \sum_{j,k=1,1}^{n,m} \bar{x}_{jk} = \frac{1}{2}.$$

On the other hand, the $(C_q, 1, 1)$-summability of the sequence (s_{jk}) is equivalent to the existence of the limit of the following expression

$$\frac{\sum\limits_{j,k=1,1}^{n,m} s_{jk}}{n^q m^q} = \frac{\left(\sum\limits_{j,k=1,1}^{n,m} x_{jk}\right)^q}{n^q m^q} \frac{\sum\limits_{j,k=1,1}^{n,m} s_{jk}x_{jk}}{\left(\sum\limits_{j,k=1,1}^{n,m} x_{jk}\right)^q} + \frac{\left(\sum\limits_{j,k=1,1}^{n,m} \bar{x}_{jk}\right)^q}{n^q m^q} \frac{\sum\limits_{j,k=1,1}^{n,m} s_{jk}\bar{x}_{jk}}{\left(\sum\limits_{j,k=1,1}^{n,m} \bar{x}_{jk}\right)^q}.$$

So we get that

$$\lim_{n,m\to\infty} \frac{\sum\limits_{j,k=1,1}^{n,m} s_{jk}}{n^q m^q} = \frac{L}{2^q} + \frac{L}{2^q} = 2^{1-q}L$$

which implies that the sequence (s_{jk}) is $(C_q, 1, 1)$-summable to $2^{1-q}L$. $\square$

In order to get the converse of Theorem 2.1, we need the following two lemmas presented in [10]. The first lemma is an analog of the Khintchine inequality for double sequences.

Lemma 2.2. *Let*

$$t_{nm}(x) = \sum_{j,k=1,1}^{n,m} s_{jk} r_{jk}(x), \quad B_{nm} = \sum_{j,k=1,1}^{n,m} s_{jk}^2.$$

Then the following inequality

$$E\left((t_{nm})^{2r}\right) \le \frac{(2r)!}{2^r r!}(B_{nm})^r$$

is fulfilled, where r is a positive integer.

The next result is an analog of the Marcinkiewicz-Zygmund inequality for double sequences.

Lemma 2.3. *Let*

$$t_{nm}(x) = \sum_{j,k=1,1}^{n,m} s_{jk} r_{jk}(x), \quad B_{nm} = \sum_{j,k=1,1}^{n,m} s_{jk}^2$$

and $t^*_{nm}(x) = \max\limits_{(j,k)\in K_{nm}} |t_{jk}|$, where $K_{nm} := \{(j,k) : 1 \le j \le n,\ 1 \le k \le m\}$. Then for $a > 0$ the following inequality

$$E\left(e^{at^*_{nm}(x)}\right) \le 32e^{a^2 \frac{B_{nm}}{2}}$$

holds.

Now we are ready to provide the converse of Theorem 2.1.

Theorem 2.4. *If the sequence (s_{jk}) is $(C_q, 1, 1)$-summable to a value L and*

$$\sum_{j,k=1,1}^{n,m} s_{jk}^2 = o\left(\frac{n^{2q}m^{2q}}{\log\log n^q m^q}\right)$$

then almost all subsequences of (s_{jk}) are $(C_q, 1, 1)$-summable to $2^{q-1}L$.

Proof. The $(C_q, 1, 1)$-summability of almost all subsequences of (s_{jk}) is equivalent to the convergence of the following expression

$$\frac{\sum\limits_{j,k=1,1}^{n,m} s_{jk} x_{jk}}{\left(\sum\limits_{j,k=1,1}^{n,m} x_{jk}\right)^q} \quad \text{for almost all } x.$$

We can rewrite the above expression as follows for almost all x

$$\frac{\sum\limits_{j,k=1,1}^{n,m} s_{jk}\left(\dfrac{1+r_{jk}(x)}{2}\right)}{\left\{\sum\limits_{j,k=1,1}^{n,m}\left(\dfrac{1+r_{jk}(x)}{2}\right)\right\}^{q}} = \frac{\frac{1}{2n^q m^q}\sum\limits_{j,k=1,1}^{n,m} s_{jk} + \frac{1}{2n^q m^q}\sum\limits_{j,k=1,1}^{n,m} s_{jk} r_{jk}(x)}{\frac{1}{n^q m^q}\left\{\sum\limits_{j,k=1,1}^{n,m}\left(\dfrac{1+r_{jk}(x)}{2}\right)\right\}^{q}} \tag{2.1}$$

where $r_{jk}(x) = 2x_{jk} - 1$. Recall that the functions r_{jk} are the Rademacher functions (see [3]). Since $P(\eta) = 1$, observe that the denumerator of (2.1) converges to $\frac{1}{2^q}$ for almost all x. To complete the proof, it suffices to establish that

$$\frac{1}{n^q m^q}\sum_{j,k=1,1}^{n,m} s_{jk} r_{jk}(x) \to 0, \ (\text{as } n,m\to\infty) \text{ for almost all } x.$$

Let $\varepsilon > 0$ and define

$$M_{jk} := \left\{x : \text{there is } (n,m) \text{ with } 2^{j-1} < n \le 2^j,\ 2^{k-1} < m \le 2^k \text{ such that } |t_{nm}(x)| \geqslant n^q m^q \varepsilon\right\}$$

and let

$$G_{jk} = \left\{x:\ t^{*}_{2^j,2^k}(x) > 2^{q(j-1)}2^{q(k-1)}\varepsilon\right\}.$$

Notice that $M_{jk} \subset G_{jk}$. The proof will be completed if we prove that for every $\varepsilon > 0$,

$$\sum_{j,k=1,1}^{\infty,\infty} P(G_{jk}) < \infty.$$

Now using Lemma 2.3 we have

$$P(G_{jk})\, e^{a2^{q(j-1)}2^{q(k-1)}\varepsilon} \le \int\limits_X e^{at^{*}_{2^j,2^k}(x)} dP(x) = E\left(e^{at^{*}_{2^j,2^k}(x)}\right) \le 32 e^{a^2 \frac{B_{2^j2^k}}{2}}.$$

Hence

$$P(G_{jk}) \le 32 e^{\frac{a^2 B_{2^j2^k}}{2} - a2^{q(j-1)}2^{q(k-1)}\varepsilon}.$$

Taking $a = \frac{2^{q(j-1)}2^{q(k-1)}\varepsilon}{B_{2^j2^k}}$, we have

$$\begin{aligned} P(G_{jk}) &\le 32 e^{-\dfrac{\varepsilon^2 2^{2q(j-1)}2^{2q(k-1)}}{2B_{2^j2^k}}} \\ &= 32 e^{-\dfrac{\varepsilon^2 \left(2^j\right)^{2q}\left(2^k\right)^{2q}}{2.16^q B_{2^j2^k}}}. \end{aligned} \tag{2.2}$$

On the other hand it follows from the hypothesis that

$$\frac{B_{2^j2^k}}{\left(2^j\right)^{2q}\left(2^k\right)^{2q}} = o\left(\frac{1}{\log\log 2^{jq}2^{kq}}\right)$$

which yields

$$\frac{B_{2^j2^k}}{\left(2^j\right)^{2q}\left(2^k\right)^{2q}} \le \frac{\varepsilon^2}{2.16^q \log\log 2^{jq}2^{kq}}.$$

Then (2.2) yields that

$$\begin{aligned} P\left(G_{jk}\right) &\leq 32e^{-\dfrac{\varepsilon^2}{2.16^q}\dfrac{6.16^q \log\log 2^{jq}2^{kq}}{\varepsilon^2}} \\ &= 32e^{-3\log\log 2^{jq}2^{kq}} \\ &= \frac{32}{\left[(j+k)\log 2^q\right]^3}. \end{aligned}$$

Since $\sum\limits_{j,k=1,1}^{\infty,\infty} \frac{1}{\left[(j+k)\log 2^q\right]^3} < \infty$ (*see* [1]),

$$\sum_{j,k=1,1}^{\infty,\infty} P\left(G_{jk}\right) \leq 32 \sum_{j,k=1,1}^{\infty,\infty} \frac{1}{\left[(j+k)\log 2^q\right]^3} < \infty.$$

Hence we obtain $\lim\limits_{j,k\to\infty} P\left(G_{jk}\right) = 0$ and also $\lim\limits_{j,k\to\infty} P\left(M_{jk}\right) = 0$. This completes the proof. □

A criterion for $(C_q,1,1)$ summability of bounded double sequences is provided in the next corollary.

Corollary 2.5. *A bounded double sequence (s_{jk}) is $(C_q,1,1)$-summable if and only if the almost all subsequences are $(C_q,1,1)$-summable.*

Theorem 2.6. *If*

$$\lim_{n,m\to\infty} \frac{1}{n^q m^q} \sum_{j,k=1,1}^{n,m} s_{jk} r_{jk}\left(x\right) = 0, \quad \textit{for almost all } x \tag{2.3}$$

then we have

$$\lim_{n,m\to\infty} \frac{1}{n^{2q} m^{2q}} \sum_{j,k=1,1}^{n,m} s_{jk}^2 = 0.$$

Proof. Let $N\left[u,z\right] = \{(j,k) : u \leq j \leq n \text{ or } z \leq k \leq m\}$ and

$$T_{u,z,n,m}\left(x\right) = \sum_{(j,k)\in N[u,z]} s_{jk} r_{jk}\left(x\right).$$

Hence

$$T_{u,z,n,m}^2\left(x\right) = \sum_{(j,k)\in N[u,z]} s_{jk}^2 + 2 \sum_{\substack{(j_1,k_1),(j_2,k_2)\in N[u,z] \\ j_1\neq j_2 \text{ or } k_1\neq k_2}} s_{j_1k_1} s_{j_2k_2} r_{j_1k_1}\left(x\right) r_{j_2k_2}\left(x\right).$$

Because of the Egoroff theorem there exists a set $D \subset X$ with positive measure such that the limit in (2.3) exists uniformly on D. Therefore,

$$\int\limits_D T_{u,z,n,m}^2\left(x\right) dP\left(x\right) = P\left(D\right) \sum_{(j,k)\in N[u,z]} s_{jk}^2 + K, \tag{2.4}$$

where

$$K = 2 \sum_{\substack{(j_1, k_1), (j_2, k_2) \in N[u,z] \\ j_1 \neq j_2 \text{ or } k_1 \neq k_2}} s_{j_1 k_1} s_{j_2 k_2} \int_D r_{j_1 k_1}(x) \, r_{j_2 k_2}(x) \, dP(x).$$

By the Hölder inequality we have

$$|K| \leq 2 \left(\sum_{\substack{(j_1, k_1), (j_2, k_2) \in N[u,z] \\ j_1 \neq j_2 \text{ or } k_1 \neq k_2}} s_{j_1 k_1}^2 s_{j_2 k_2}^2 \right)^{\frac{1}{2}} \left(\sum_{\substack{(j_1, k_1), (j_2, k_2) \in N[u,z] \\ j_1 \neq j_2 \text{ or } k_1 \neq k_2}} v_{j_1 k_1 j_2 k_2}^2 \right)^{\frac{1}{2}} \tag{2.5}$$

where $v_{j_1 k_1 j_2 k_2} = \int_D r_{j_1 k_1}(x) \, r_{j_2 k_2}(x) \, dP(x)$. We know that the functions $r_{j_1 k_1}(x)$ and $r_{j_2 k_2}(x)$ are orthogonal on X (*see* [3]). So by the Bessel inequality [13] for double sequences we get

$$\sum_{\substack{1 \leq j_1 < j_2 \leq \infty \\ 1 \leq k_1 < k_2 \leq \infty}} v_{j_1 k_1 j_2 k_2}^2 \leq \int_X (\chi_D(x))^2 \, dP(x) = P(D).$$

For sufficiently large u and z, we have

$$\left(\sum_{\substack{(j_1, k_1), (j_2, k_2) \in N[u,z] \\ j_1 \neq j_2 \text{ or } k_1 \neq k_2}} v_{j_1 k_1 j_2 k_2}^2 \right)^{\frac{1}{2}} \leq \frac{P(D)}{4}.$$

It follows from (2.5) that

$$|K| \leq \left(\sum_{\substack{(j_1, k_1), (j_2, k_2) \in N[u,z] \\ j_1 \neq j_2 \text{ or } k_1 \neq k_2}} s_{j_1 k_1}^2 s_{j_2 k_2}^2 \right)^{\frac{1}{2}} \frac{P(D)}{2} \leq \frac{P(D)}{2} \sum_{(j_1,k_1) \in N[u,z]} s_{j_1 k_1}^2.$$

Combining this with (2.4) we get

$$\begin{aligned} \int_D T_{u,z,n,m}^2(x) \, dP(x) &= P(D) \sum_{(j,k) \in N[u,z]} s_{jk}^2 + K \\ &\geq \frac{P(D)}{2} \sum_{(j,k) \in N[u,z]} s_{jk}^2. \end{aligned}$$

By (2.3) we have that

$$\lim_{n,m\to\infty}\frac{1}{n^{2q}m^{2q}}\sum_{(j,k)\in N[u,z]} s_{jk}^2 = 0 \text{ and } \lim_{n,m\to\infty}\frac{1}{n^{2q}m^{2q}}\sum_{j,k=1,1}^{n,m} s_{jk}^2 = 0.$$

Hence the result follows. ☐

In the next examples we present a sequence so that it is $(C_q, 1, 1)$ summable but almost none of its subsequences are $(C_q, 1, 1)$ summable.

Example 2.7. Consider the double sequence $s_{jk} = (-1)^j (-1)^k \sqrt{j}\sqrt{k}$. Then

$$\sum_{j=1}^{\infty}\frac{(-1)^j\sqrt{j}}{j^q} = \sum_{j=1}^{\infty}\frac{(-1)^j}{j^{q-\frac{1}{2}}} \text{ is convergent in the ordinary sense for } q > \frac{1}{2},$$

and

$$\sum_{k=1}^{\infty}\frac{(-1)^k\sqrt{k}}{k^q} = \sum_{k=1}^{\infty}\frac{(-1)^k}{k^{q-\frac{1}{2}}} \text{ is convergent in the ordinary sense for } q > \frac{1}{2}.$$

On the other hand the double series $\sum_{j,k=1,1}^{\infty,\infty}\frac{(-1)^j(-1)^k}{j^{q-\frac{1}{2}}k^{q-\frac{1}{2}}}$ is convergent (see [1], page 90). Also since

$$\sum_{j=1}^{\infty}\frac{(-1)^j(-1)^k}{j^{q-\frac{1}{2}}k^{q-\frac{1}{2}}} \text{ is convergent in the ordinary sense for } k = 1, 2, ...$$

and

$$\sum_{k=1}^{\infty}\frac{(-1)^j(-1)^k}{j^{q-\frac{1}{2}}k^{q-\frac{1}{2}}} \text{ is convergent in the ordinary sense for } j = 1, 2, ...$$

then the double series $\sum_{j,k=1,1}^{\infty,\infty}\frac{(-1)^j(-1)^k}{j^{q-\frac{1}{2}}k^{q-\frac{1}{2}}}$ is convergent in the restricted sense by Theorem 1 of [5]. Since the double series $\sum_{j,k=1,1}^{\infty,\infty}\frac{(-1)^j(-1)^k\sqrt{j}\sqrt{k}}{j^qk^q}$ is convergent in the restricted sense, we get that the sequence $\left\{\frac{1}{n^qm^q}\sum_{j,k=1,1}^{n,m}(-1)^j(-1)^k\sqrt{j}\sqrt{k}\right\}$ converges to zero in the Pringsheim sense [6]. Hence the sequence $\left((-1)^j(-1)^k\sqrt{j}\sqrt{k}\right)$ is $(C_q, 1, 1)$-summable to zero. On the other hand, for the case of $q = \frac{3}{4}$,

$$\left(\frac{1}{n^{2q}m^{2q}}\sum_{j,k=1,1}^{n,m} jk\right) = \left(\frac{1}{n^{2q}m^{2q}}\frac{n(n+1)}{2}\frac{m(m+1)}{2}\right)$$

the double sequence does not converge to zero. Hence we have, by Theorem 2.6, that

$$\lim_{n,m} \frac{1}{n^q m^q} \sum_{j,k=1,1}^{n,m} (-1)^j (-1)^k \sqrt{j}\sqrt{k} r_{jk}(x) \neq 0.$$

So almost none of its subsequences are $(C_q, 1, 1)$-summable to zero.

Example 2.8. Consider the double sequence $s_{jk} = (-1)^j (-1)^k jk$. Then

$$\sum_{j=1}^{\infty} \frac{(-1)^j j}{j^q} = \sum_{j=1}^{\infty} \frac{(-1)^j}{j^{q-1}} \text{ is convergent in the ordinary sense for } q > 1,$$

and

$$\sum_{k=1}^{\infty} \frac{(-1)^k k}{k^q} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^{q-1}} \text{ is convergent in the ordinary sense for } q > 1.$$

On the other hand, the double series $\sum_{j,k=1,1}^{\infty,\infty} \frac{(-1)^j (-1)^k}{j^{q-1} k^{q-1}}$ is convergent (see [1], page 90). Also since

$$\sum_{j=1}^{\infty} \frac{(-1)^j (-1)^k}{j^{q-1} k^{q-1}} \text{ is convergent in the ordinary sense for } k = 1, 2, ...$$

and

$$\sum_{k=1}^{\infty} \frac{(-1)^j (-1)^k}{j^{q-1} k^{q-1}} \text{ is convergent in the ordinary sense for } j = 1, 2, ...$$

then the double series $\sum_{j,k=1,1}^{\infty,\infty} \frac{(-1)^j (-1)^k}{j^{q-1} k^{q-1}}$ is convergent in the restricted sense by Theorem 1 of [5]. Since the series $\sum_{j,k=1,1}^{\infty,\infty} \frac{(-1)^j (-1)^k jk}{j^q k^q}$ is convergent in the restricted sense, we get that the sequence $\left\{ \frac{1}{n^q m^q} \sum_{j,k=1,1}^{n,m} (-1)^j (-1)^k jk \right\}$ converges to 0 in the Pringsheim sense [6]. Hence the sequence $\left((-1)^j (-1)^k jk \right)$ is $(C_q, 1, 1)$-summable to 0. On the other hand, for the case of $q = \frac{3}{2}$,

$$\left(\frac{1}{n^{2q} m^{2q}} \sum_{j,k=1,1}^{n,m} j^2 k^2 \right) = \left(\frac{1}{n^{2q} m^{2q}} \frac{n(n+1)(2n+1)}{6} \frac{m(m+1)(2m+1)}{6} \right)$$

the double sequence does not converge to zero. Therefore, Theorem 2.6 implies almost none of its subsequences are $(C_q, 1, 1)$-summable to zero.

References

[1] Bromwich, M.A., *An introduction to the theory of infinite series*, Macmillan Co., 1942.

[2] Buck, R.C., Pollard, H., *Convergence and summability properties of subsequences*, Bull. Amer. Math. Soc., **49**(1943), 924-931.

[3] Crnjac, M., Čunjalo, F., Miller, H.I., *Subsequence characterizations of statistical convergence of double sequences*, Radovi Math., **12**(2004), 163-175.

[4] Garling, D.J.H., *Inequalities: A Journey into Linear Analysis*, Cambridge University Press, 2007.

[5] Móricz, F., *On the convergence in a restricted sense of multiple series*, Analysis Mathematica, **5**(1979), 135-147.

[6] Móricz, F., *The kronecker lemmas for multiple series and some applications*, Acta Math. Acad. Sci. Hungar., **37**(1981), 39-50.

[7] Orhan, C., Tas, E., Yurdakadim, T., *The Buck-Pollard property for $p-$Cesàro matrices*, Numer. Funct. Anal. Optim., **33**(2012), 1-7.

[8] Patterson, R.F., *A characterization for the limit points of double sequences*, **32**(1999), 775-780.

[9] Pringsheim, A., *On the theory of double infinite sequences of numbers. (Zur theorie der zweifach unendlichen zahlenfolgen.)*, Math. Ann., **53**(1900), 289-321.

[10] Tas, E., Orhan, C., *The Buck-Pollard property for $(C,1,1)$ method*, submitted.

[11] Tsuchikura, T., *Arithmetic means of subsequences*, Tôhoku Mathematical Journal, **2**(1950), 188-191.

[12] Unver, M., *Inclusion results for four dimensional Cesàro submethods*, Stud. Univ. Babeş-Bolyai Math., **58**(2013), 43-54.

[13] Zhizhiashvili, L.V., *Some problems in the theory of simple and multiple trigonometric and orthogonal series*, Russ. Math. Surv., **28**(1973), 65-127.

Emre Taş
Ahi Evran University, Department of Mathematics
Bağbaşı, 40100 Kırşehir, Turkey
e-mail: `emretas86@hotmail.com`

Cihan Orhan
Ankara University, Faculty of Science, Department of Mathematics
Tandoğan, 06100 Ankara, Turkey
e-mail: `orhan@science.ankara.edu.tr`

Stud. Univ. Babeş-Bolyai Math. 62(2017), No. 3, 377–394
DOI: 10.24193/subbmath.2017.3.10

Geometric properties and neighborhood results for a subclass of analytic functions involving Komatu integral

Ritu Agarwal, Gauri Shankar Paliwal and Hari S. Parihar

Abstract. In this paper, a subclass of analytic function is defined using Komatu integral. Coefficient inequalities, Fekete-Szegö inequality, extreme points, radii of starlikeness and convexity and integral means inequality for this class are obtained. Distortion theorem for the generalized fractional integration introduced by Saigo are also obtained. The inclusion relations associated with the (n,μ)-neighborhood also have been found for this class.

Mathematics Subject Classification (2010): 47A15, 46A32, 47D20.

Keywords: Analytic function, Komatu integral, coefficient inequality, Fekete-Szegö inequality, extreme points, radii of starlikeness and convexity, neighborhood results.

1. Introduction

Let H denote the class of analytic function in the unit disk

$$\Delta = \{z : z \in C, |z| < 1\}$$

on the complex plane C. Let A denote the subclass of H consisting of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\Delta = \{z : z \in C, |z| < 1\}$.

Also let S be the subclass of A consisting of all univalent functions in Δ normalized by $f(0) = f'(0) - 1 = 0$.

Denote by T the subclass of S consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \;\; a_n \geq 0, \;\; z \in \Delta. \tag{1.2}$$

studied extensively by Silverman [15].

Let f and g are analytic functions defined in Δ. The function f is said to be *subordinate* to g if there exists a Schwarz function w, analytic in Δ with $w(0) = 0$, $|w(z)| < 1$, $z \in \Delta$ such that

$$f(z) = g(w(z)), (z \in \Delta). \tag{1.3}$$

We denote this subordination by $f \prec g$ or $f(z) \prec g(z), (z \in \Delta)$.

In particular, if the function g is univalent in Δ, the above subordination is equivalent to $f(0) = g(0)$ and $f(\Delta) \subset g(\Delta)$, $(z \in \Delta)$.

The convolution or Hadamard product of two functions $f(z)$ given by (1.1) and

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \tag{1.4}$$

is defined as

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n. \tag{1.5}$$

A function $f(z)$ in A is said to be in class $S^*(\alpha)$ of starlike functions of order α $(0 \leq \alpha < 1)$ in Δ, if $\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha$ for $z \in \Delta$. Let $K(\alpha)$ denote the class of all functions $f \in A$ that are convex functions of order α $(0 \leq \alpha < 1)$ in Δ, if $\text{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha$ for $z \in \Delta$. If $\alpha = 0$, the class $S^*(\alpha)$ reduces to the class S^* of starlike functions and class $K(\alpha)$ reduces to the class of convex functions K. Further, f is convex if and only if $zf'(z)$ is starlike.

Let $\phi(z)$ be an analytic function in Δ with

$$\phi(0) = 1, \;\; \phi'(0) > 0 \;\; \text{and} \;\; \text{Re}(\phi(z)) > 0, \;\; (z \in \Delta) \tag{1.6}$$

which maps the open unit disk Δ onto a region starlike with respect to 1 and is symmetric with respect to real axis. Then $S^*(\phi)$ and $K(\phi)$, respectively, be the subclasses of the normalized analytic functions f in class A, which satisfy the following subordination relations:

$$\frac{zf'(z)}{f(z)} \prec \phi(z), (z \in \Delta) \;\; \text{and} \;\; 1 + \frac{zf''(z)}{f'(z)} \prec \phi(z), (z \in \Delta)$$

These classes are introduced by Ma and Minda [8]. In their particular case when

$$\phi(z) = \frac{1 + (1 - 2\alpha)z}{1 - z} \quad (z \in \Delta; \; 0 \leq \alpha < 1), \tag{1.7}$$

these function classes would reduce, respectively, to the well known classes $S^*(\alpha)$ $(0 \leq \alpha < 1)$ of starlike function of order α in Δ and $K(\alpha)(0 \leq \alpha < 1)$ of convex functions of order α in Δ.

Definition 1.1. [4] The generalized Komatu integral operator $K_c^\delta : A \to A$ is defined for $\delta > 0$ and $c > -1$ as

$$\left(K_c^\delta f\right)(z) = \frac{(c+1)^\delta}{\Gamma(\delta) z^c} \int_0^z t^{c-1} \left(\log \frac{z}{t}\right)^{\delta - 1} f(t) dt \tag{1.8}$$

and

$$K_c^0 f(z) = f(z).$$

For $f \in A$, it can be easily verified that

$$\left(K_c^\delta f\right)(z) = z + \sum_{k=2}^{\infty} \left(\frac{c+1}{c+k}\right)^\delta a_k z^k. \tag{1.9}$$

Based on the earlier works by the authors [1], we introduce the following class.

Definition 1.2. Let $0 \le \gamma < 1$, $0 \le \rho < 1$, $\tau \in C \setminus \{0\}$, $\delta > 0$ and $c > -1$. A function $f \in S$ is in the class $R^\tau_{\delta,\gamma,\rho,c}(\phi)$ if

$$1 + \frac{1}{\tau}\left(\rho\{K_c^\delta f(z)\}' + \gamma z\{K_c^\delta f(z)\}'' - \rho\right) \prec \phi(z), \quad z \in \Delta, \tag{1.10}$$

where $\phi(z)$ is analytic function in Δ with

$$\phi(0) = 1,\ \phi'(0) > 0 \text{ and } \operatorname{Re}(\phi(z)) > 0. \tag{1.11}$$

If we set $\phi(z) = \frac{1+Az}{1+Bz}$, $(-1 \le B < A \le 1, z \in \Delta)$, in (1.10), we get

$$\begin{aligned} &R^\tau_{\delta,\gamma,\rho,c}\left(\frac{1+Az}{1+Bz}\right) = R^\tau_{\delta,\gamma,\rho,c}(A,B) \\ &= \left\{ f \in A : \left| \frac{\rho\{K_c^\delta f(z)\}' + \gamma z\{K_c^\delta f(z)\}'' - \rho}{\tau(A-B) - B\left(\rho\{K_c^\delta f(z)\}' + \gamma z\{K_c^\delta f(z)\}'' - \rho\right)} \right| < 1 \right\}, \end{aligned} \tag{1.12}$$

which is again a new class.

Some particular cases of this class discussed in the literature as:

(1) For $\delta = 0, \rho = 1$, the above class reduce to the class $R^\tau_\gamma(A,B)$ introduced by Bansal [3].

(2) For $\delta = 0, \rho = 1$, the class $R^\tau_\gamma(1-2\beta,-1) = R^\tau_\gamma(\beta)$ for $0 \le \beta < 1, \tau = C\setminus\{0\}$ was discussed recently by Swaminathan [20].

(3) $R^\tau_{0,\gamma,1,c}(1-2\beta,-1)$ with $\tau = e^{i\eta}\cos\eta$ where $-\pi/2 < \eta < \pi/2$ is considered in [11] (see also [10]).

(4) The class $R^\tau_{0,1,1,c}(0,-1)$ with $\tau = e^{i\eta}\cos\eta$ was considered in [5] with reference to the univalency of partial sums.

We denote by $P(\phi)$ the class of normalized functions defined as

$$P(\phi) = \{f \in H : f(0) = 1, f \prec \phi \in \Delta\}.$$

The problem on subordination and convolution were studied by Ruscheweyh in [12] and have found many applications in various fields. One of them is the following theorem due to Ruscheweyh and Stankiewicz [13] which will be useful in this paper.

Theorem 1.3. *Let $F, G \in A$ be any convex univalent functions in Δ. If $f \prec F$ and $g \prec G$, then $f * g \prec F * G$ in Δ.*

Observe that, in Theorem 1.3, nothing is said about the normalization of F and G.

2. Main results

Theorem 2.1. *If $f \in P(\phi) \cap S$, $n \in N$ then $(K_c^\delta)^n f(z) \prec (K_c^\delta)^n \phi(z)$, where K_c^δ is Komatu integral operator.*

Proof. If $f \in P(\phi) \cap S$, then $f(z) \prec \phi(z)$ where $\phi(z)$ is convex univalent function. It is well known that the function

$$h_1(z) = z + \sum_{n=2}^{\infty} \left(\frac{c+1}{c+n}\right)^\delta z^n, \ (\delta > 0), \tag{2.1}$$

belongs to the class K of convex univalent and normalized function and for $f \in A$

$$\begin{aligned}(f * h_1)(z) &= z + a_2 \left(\frac{c+1}{c+2}\right)^\delta z^2 + a_3 \left(\frac{c+1}{c+3}\right)^\delta z^3 + ... \\ &= \frac{(c+1)^\delta}{\Gamma(\delta) z^c} \int_0^z t^{c-1} \left(\log \frac{z}{t}\right)^{\delta-1} f(t)dt \\ &= K_c^\delta f(z).\end{aligned}$$

Therefore the function $h_2(z) = 1 + h_1(z)$ $(z \in \Delta)$ is convex univalent in Δ and for

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + ...$$

$$\begin{aligned}(p * h_2)(z) &= 1 + p_1 z + p_2 \left(\frac{c+1}{c+2}\right)^\delta z^2 + p_3 \left(\frac{c+1}{c+3}\right)^\delta z^3 + ... \\ &= 1 - \left(\frac{c+1}{c}\right)^\delta + \frac{(c+1)^\delta}{\Gamma(\delta) z^c} \left[\frac{\Gamma(\delta) z^c}{c^\delta} + \frac{p_1 z^{c+1} \Gamma(\delta)}{(c+1)^\delta} + \frac{p_2 z^{c+2} \Gamma(\delta)}{(c+2)^\delta} + ...\right] \\ &= 1 - \left(\frac{c+1}{c}\right)^\delta + \frac{(c+1)^\delta}{\Gamma(\delta) z^c} \int_0^z t^{c-1} \left(\log \frac{z}{t}\right)^{\delta-1} p(t)dt\end{aligned}$$

Thus, $f \prec \phi$. Applying Theorem 1.3, we obtain

$$\begin{aligned}& f * h_2 \prec \phi * h_2 \\ \Rightarrow & 1 - \left(\frac{c+1}{c}\right)^\delta + \frac{(c+1)^\delta}{\Gamma(\delta) z^c} \int_0^z t^{c-1} \left(\log \frac{z}{t}\right)^{\delta-1} f(t)dt \\ & \prec 1 - \left(\frac{c+1}{c}\right)^\delta + \frac{(c+1)^\delta}{\Gamma(\delta) z^c} \int_0^z t^{c-1} \left(\log \frac{z}{t}\right)^{\delta-1} \phi(t)dt \\ \Rightarrow & K_c^\delta f \prec K_c^\delta \phi, \ \delta > 0.\end{aligned}$$

Hence, the theorem is true for $n = 1$.
Again by Theorem 1.3,

$$K_c^\delta f * h_2 \prec K_c^\delta \phi * h_2$$

$$
\begin{aligned}
&\Rightarrow 1-\left(\frac{c+1}{c}\right)^{\delta}+\frac{(c+1)^{\delta}}{\Gamma(\delta)z^{c}}\int_{0}^{z}t^{c-1}\left(\log\frac{z}{t}\right)^{\delta-1}K_c^{\delta}f(t)dt\\
&\quad\prec 1-\left(\frac{c+1}{c}\right)^{\delta}+\frac{(c+1)^{\delta}}{\Gamma(\delta)z^{c}}\int_{0}^{z}t^{c-1}\left(\log\frac{z}{t}\right)^{\delta-1}K_c^{\delta}\phi(t)dt\\
&\Rightarrow K_c^{\delta}(K_c^{\delta}f)\prec K_c^{\delta}(K_c^{\delta})\phi\\
&\Rightarrow (K_c^{\delta})^{2}f\prec (K_c^{\delta})^{2}\phi.
\end{aligned}
$$

Thus, the theorem is true for $n = 2$.
Further, let the theorem is true for $n = m$ i.e.

$$(K_c^{\delta})^{m}f\prec (K_c^{\delta})^{m}\phi$$

which on application of Theorem 1.3 gives

$$(K_c^{\delta})^{m}f*h_2(z)\prec (K_c^{\delta})^{m}\phi*h_2(z)$$

$$
\begin{aligned}
&\Rightarrow 1-\left(\frac{c+1}{c}\right)^{\delta}+\frac{(c+1)^{\delta}}{\Gamma(\delta)z^{c}}\int_{0}^{z}t^{c-1}\left(\log\frac{z}{t}\right)^{\delta-1}(K_c^{\delta})^{m}f(t)dt\\
&\quad\prec 1-\left(\frac{c+1}{c}\right)^{\delta}+\frac{(c+1)^{\delta}}{\Gamma(\delta)z^{c}}\int_{0}^{z}t^{c-1}\left(\log\frac{z}{t}\right)^{\delta-1}(K_c^{\delta})^{m}\phi(t)dt\\
&\Rightarrow K_c^{\delta}[(K_c^{\delta})^{m}f](z)\prec K_c^{\delta}[(K_c^{\delta})^{m}\phi](z)\\
&\Rightarrow (K_c^{\delta})^{m+1}f(z)\prec (K_c^{\delta})^{m+1}\phi(z).
\end{aligned}
$$

The theorem follows by the principle of Mathematical induction. □

Corollary 2.2. *Let $g' \in P(\phi), \alpha < 1$. If we take $\phi(z) = \frac{1-z(2\alpha-1)}{1-z}$, $n = 1$ and*

$$h_1(z)=\sum_{n=1}^{\infty}\frac{2}{n+1}z^{n},\ \ (z\in\Delta).$$

Then

$$\frac{1}{z}\int_{0}^{z}\frac{g(t)}{t}dt\prec Q(z),\ \ (z\in\Delta),$$

where

$$Q(z)=1+2(1-2\alpha)\left[\frac{z}{2^{2}}+\frac{z^{2}}{3^{2}}+\frac{z^{3}}{4^{2}}+...\right]$$

is convex univalent function.

This particular result is given by Janusz Sokol [18].

3. Coefficient inequality

Theorem 3.1. *Let $f \in R_{\gamma}^{\tau}(A, B)$ [3]. Then f is in the class $R^{\tau}_{\delta,\gamma,\rho,c}$ if and only if*

$$\sum_{k=2}^{\infty}(1+B)k\{\rho+\gamma(k-1)\}\left(\frac{c+1}{c+k}\right)^{\delta}a_k\le|\tau(A-B)|. \tag{3.1}$$

The result is sharp for the function $f(z)$ given by the following form

$$f(z) = z + \frac{|\tau(A-B)|}{2(1+B)(\rho+\gamma)\left(\frac{c+1}{c+2}\right)^{\delta}} z^2. \tag{3.2}$$

Proof. For $|z| = 1$, we have

$$\begin{aligned}
&\left|\rho\{K_c^{\delta} f(z)\}' + \gamma z\{K_c^{\delta} f(z)\}'' - \rho\right| \\
&- \left|\tau(A-B) - B[\rho\{K_c^{\delta} f(z)\}' + \gamma z\{K_c^{\delta} f(z)\}'' - \rho]\right| \\
=&\left|\rho\left[1 + \sum_{k=2}^{\infty} k\left(\frac{c+1}{c+k}\right)^{\delta} a_k z^{k-1}\right] + \gamma z\left[\sum_{k=2}^{\infty}\left(\frac{c+1}{c+k}\right)^{\delta} k(k-1)a_k z^{k-2}\right] - \rho\right| \\
&- \left|\tau(A-B) - B\left[\rho\left\{1 + \sum_{k=2}^{\infty} k\left(\frac{c+1}{c+k}\right)^{\delta} a_k z^{k-1}\right\}\right.\right. \\
&\qquad\qquad\left.\left. + \gamma z\sum_{k=2}^{\infty} k(k-1)\left(\frac{c+1}{c+k}\right)^{\delta} a_k z^{k-2} - \rho\right]\right| \\
\leq&\rho\sum_{k=2}^{\infty}\left(\frac{c+1}{c+k}\right)^{\delta} k a_k + \gamma\sum_{k=2}^{\infty}\left(\frac{c+1}{c+k}\right)^{\delta} k(k-1)a_k - |\tau(A-B)| \\
&+ B\left|\rho\sum_{k=2}^{\infty} k\left(\frac{c+1}{c+k}\right)^{\delta} a_k z^{k-1} + \gamma z\sum_{k=2}^{\infty} k(k-1)\left(\frac{c+1}{c+k}\right)^{\delta} a_k z^{k-2}\right| \\
\leq&\rho\sum_{k=2}^{\infty} k\left(\frac{c+1}{c+k}\right)^{\delta} a_k + \gamma\sum_{k=2}^{\infty} k(k-1)\left(\frac{c+1}{c+k}\right)^{\delta} a_k - |\tau(A-B)| \\
&+ B\rho\sum_{k=2}^{\infty} k\left(\frac{c+1}{c+k}\right)^{\delta} a_k + B\gamma\sum_{k=2}^{\infty} k(k-1)\left(\frac{c+1}{c+k}\right)^{\delta} a_k \\
\leq&(1+B)\rho\sum_{k=2}^{\infty} k\left(\frac{c+1}{c+k}\right)^{\delta} a_k + (1+B)\gamma\sum_{k=2}^{\infty} k(k-1)\left(\frac{c+1}{c+k}\right)^{\delta} a_k - |\tau(A-B)| \\
&\leq 0. \qquad\qquad \text{(By hypothesis)}
\end{aligned}$$

Thus, by maximum modulus theorem, $f \in R^{\tau}_{\delta,\gamma,\rho,c}(A,B)$.
Conversely, assume that

$$\left|\frac{\rho\{K_c^{\delta} f(z)\}' + \gamma z\{K_c^{\delta} f(z)\}'' - \rho}{\tau(A-B) - B\{\rho\{K_c^{\delta} f(z)\}' + \gamma z\{K_c^{\delta} f(z)\}'' - \rho\}}\right| < 1$$

$$\Rightarrow\left|\frac{\rho\sum_{k=2}^{\infty} k\left(\frac{c+1}{c+k}\right)^{\delta} a_k z^{k-1} + \gamma\sum_{k=2}^{\infty} k(k-1)\left(\frac{c+1}{c+k}\right)^{\delta} a_k z^{k-1}}{\tau(A-B) - B\left\{\rho\sum_{k=2}^{\infty} k\left(\frac{c+1}{c+k}\right)^{\delta} a_k z^{k-1} + \gamma\sum_{k=2}^{\infty} k(k-1)\left(\frac{c+1}{c+k}\right)^{\delta} a_k z^{k-1}\right\}}\right| < 1.$$

Since $|Re(z)| < |z|$,

$$Re\left[\frac{\sum_{k=2}^{\infty} k\{\rho+\gamma(k-1)\}\left(\frac{c+1}{c+k}\right)^{\delta} a_k z^{k-1}}{|\tau(A-B)| - B\sum_{k=2}^{\infty} k\{\rho+\gamma(k-1)\}\left(\frac{c+1}{c+k}\right)^{\delta} a_k z^{k-1}}\right] < 1.$$

By choosing the value of z on the real axis so that $K_c^{\delta} f(z)$ is real. Let $z \to 1^-$ through real values. So we can write as

$$\sum_{k=2}^{\infty} k\{\rho+\gamma(k-1)\}\left(\frac{c+1}{c+k}\right)^{\delta} a_k \le |\tau(A-B)| - B\sum_{k=2}^{\infty} k\{\rho+\gamma(k-1)\}\left(\frac{c+1}{c+k}\right)^{\delta} a_k \le |\tau(A-B)|$$

□

Corollary 3.2. *Let $f(z) \in R^{\tau}_{\delta,\gamma,\rho,c}(A,B)$, then*

$$a_k \le \frac{|\tau(A-B)|}{(1+B)k\{\rho+\gamma(k-1)\}\left(\frac{c+1}{c+k}\right)^{\delta}}; \quad k \ge 2.$$

4. Fekete-Szegö inequality

We recall the following lemma to prove our results:

Lemma 4.1. [6] *If $p_1(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + ...(z \in \Delta)$ is a function with positive real part, then for any complex number ε,*

$$|c_3 - \varepsilon c_2^2| \le 2\ \max\{1, |2\varepsilon - 1|\}$$

and the result is sharp for the functions given by

$$p_1(z) = \frac{1+z^2}{1-z^2} \quad or \quad p_1(z) = \frac{1+z}{1-z}.$$

Theorem 4.2. *Let*

$$\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + ... \tag{4.1}$$

where $\phi(z) \in A$ with $\phi'(0) > 0$.
If $f(z)$ given by (1.1) belongs to $R^{\tau}_{\delta,\gamma,\rho,c}(\phi)$ $(\gamma, \rho \in [0,1)$; $\tau \in C \setminus \{0\}$; $\delta > 0$; $c > -1)$, $z \in \Delta$, then for any complex number ν

$$|a_3 - \nu a_2^2| \le \frac{|\tau|\, B_1}{3(\rho+2\gamma)}\left(\frac{c+3}{c+1}\right)^{\delta} \max\left\{1, \left|\frac{B_2}{B_1} - \frac{3\nu B_1 \tau(\rho+2\gamma)(c+2)^{2\delta}}{(\rho+\gamma)^2(c+3)^{\delta}(c+1)^{\delta}}\right|\right\}. \tag{4.2}$$

The result is sharp for the functions $\frac{1+z^2}{1-z^2}$ or $\frac{1+z}{1-z}$.

Proof. If $f(z) \in R^{\tau}_{\delta,\gamma,\rho,c}(\phi)$, then there exists a Schwarz function w analytic in Δ with $w(0) = 0$ and $|w(z)| < 1$, $(z \in \Delta)$ such that

$$1 + \frac{1}{\tau}(\rho\{K_c^{\delta} f(z)\}' + \gamma z\{K_c^{\delta} f(z)\}'' - \rho) = \phi(w(z)). \tag{4.3}$$

Define the function $p_1(z)$ by

$$p_1(z) = \frac{1+w(z)}{1-w(z)} = 1 + c_1 z + c_2 z^2 + c_3 z^3 + ... \tag{4.4}$$

Since $w(z)$ is a Schwarz function, we see that $Re(p_1(z)) > 0$ and $p_1(0) = 1$.
Define the function $p(z)$ by

$$p(z) = 1 + \frac{1}{\tau}[\rho\{K_c^\delta f(z)\}' + \gamma z\{K_c^\delta f(z)\}'' - \rho] = 1 + b_1 z + b_2 z^2 + b_3 z^3 + ... \tag{4.5}$$

In view of (4.3), (4.4), (4.5)

$$\begin{aligned} p(z) = \phi\left[\frac{p_1(z)-1}{p_1(z)+1}\right] &= \phi\left[\frac{c_1 z + c_2 z^2 + c_3 z^3 + ...}{2 + c_1 z + c_2 z^2 + c_3 z^3 + ...}\right] \\ &= \phi\left[\frac{c_1 z}{2} + \frac{1}{2}\left(c_2 - \frac{c_1^2}{2}\right) z^2 + ...\right]. \end{aligned}$$

From equation (4.1)

$$p(z) = 1 + \frac{B_1 c_1 z}{2} + \frac{B_1}{2}\left(c_2 - \frac{c_1^2}{2}\right) z^2 + \frac{B_2 c_1^2}{4} z^2 + \dots. \tag{4.6}$$

Now, from (1.9)

$$K_c^\delta f(z) = z + \left(\frac{c+1}{c+2}\right)^\delta a_2 z^2 + \left(\frac{c+1}{c+3}\right)^\delta a_3 z^3 + ...,$$

$$\{K_c^\delta f(z)\}' = 1 + 2\left(\frac{c+1}{c+2}\right)^\delta a_2 z + 3\left(\frac{c+1}{c+3}\right)^\delta a_3 z^2 + ...,$$

and

$$\{K_c^\delta f(z)\}'' = 2\left(\frac{c+1}{c+2}\right)^\delta a_2 + 6\left(\frac{c+1}{c+3}\right)^\delta a_3 z + \dots.$$

From equation (4.5)

$$\begin{aligned} p(z) = 1 + \frac{1}{\tau}\Bigg[&\left\{2\rho\left(\frac{c+1}{c+2}\right)^\delta a_2 + 2\gamma\left(\frac{c+1}{c+2}\right)^\delta a_2\right\} z \\ &+ \left\{3\rho\left(\frac{c+1}{c+3}\right)^\delta a_3 + 6\gamma\left(\frac{c+1}{c+3}\right)^\delta a_3\right\} z^2 + ...\Bigg] \end{aligned} \tag{4.7}$$

Thus from (4.6) and (4.7)

$$\frac{B_1 c_1}{2} = \frac{2(\rho+\gamma)}{\tau}\left(\frac{c+1}{c+2}\right)^\delta a_2 \;\Rightarrow\; a_2 = \frac{B_1 c_1 \tau}{4(\rho+\gamma)}\left(\frac{c+2}{c+1}\right)^\delta$$

$$\frac{B_1}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{B_2 c_1^2}{4} = \frac{3a_3(\rho+2\gamma)}{\tau}\left(\frac{c+1}{c+3}\right)^\delta$$

$$\Rightarrow a_3 = \frac{\tau}{3(\rho+2\gamma)}\left(\frac{c+3}{c+1}\right)^\delta\left[\frac{B_1}{2}\left(c_2 - \frac{c_1^2}{2}\right) + \frac{B_2 c_1^2}{4}\right]$$

Therefore, we have

$$a_3-\nu a_2^2=\frac{\tau}{3(\rho+2\gamma)}\left(\frac{c+3}{c+1}\right)^{\delta}\left[\frac{B_1}{2}\left(c_2-\frac{c_1^2}{2}\right)+\frac{B_2c_1^2}{4}\right]-\nu\frac{B_1^2c_1^2\tau^2}{4(\rho+\gamma)^2}\left(\frac{c+2}{c+1}\right)^{2\delta}$$

Simplifying, we get

$$a_3-\nu a_2^2=\frac{\tau B_1}{6(\rho+2\gamma)}\left(\frac{c+3}{c+1}\right)^{\delta}(c_2-\varepsilon c_1^2),$$

where

$$\varepsilon=\frac{1}{2}\left\{1-\frac{B_2}{B_1}+\frac{3\nu B_1\tau(\rho+2\gamma)(c+2)^{2\delta}}{(\rho+\gamma)^2(c+3)^{\delta}(c+1)^{\delta}}\right\}.$$

Thus

$$|a_3-\nu a_2^2|=\frac{|\tau|B_1}{6(\rho+2\gamma)}\left(\frac{c+3}{c+1}\right)^{\delta}|c_2-\varepsilon c_1^2|$$

By application of the Lemma (4.1), we obtain

$$|a_3-\nu a_2^2|\le\frac{2|\tau|B_1}{6(\rho+2\gamma)}\left(\frac{c+3}{c+1}\right)^{\delta}\max\{1,|2\varepsilon-1|\}$$

$$|a_3-\nu a_2^2|\le\frac{|\tau|B_1}{3(\rho+2\gamma)}\left(\frac{c+3}{c+1}\right)^{\delta}\max\left\{1,\left|\frac{B_2}{B_1}+\frac{3\nu B_1\tau(\rho+2\gamma)(c+2)^{2\delta}}{(\rho+\gamma)^2(c+3)^{\delta}(c+1)^{\delta}}\right|\right\}$$

Equality in (4.2) is obtained when

$$p_1(z)=\frac{1+z^2}{1-z^2}\quad or\quad p_1(z)=\frac{1+z}{1-z}.\qquad\square$$

For class $R^{\tau}_{\delta,\gamma,\rho,c}(A,B)$

$$\phi(z)=\frac{1+Az}{1+Bz}=1+(A-B)z-(AB-B^2)z^2+...$$

Thus writing $B_1=A-B$ and $B_2=-B(A-B)$ in the Theorem 3.1, we get the following corollary:

Corollary 4.3. *If $f(z)$ given by (1.1) belongs to $R^{\tau}_{\delta,\gamma,\rho,c}(A,B)$, then*

$$|a_3-\nu a_2^2|\le\frac{|\tau|(A-B)}{3(\rho+2\gamma)}\left(\frac{c+3}{c+1}\right)^{\delta}\max\left\{1,\left|B-\frac{3\nu(A-B)\tau(\rho+2\gamma)(c+2)^{2\delta}}{(\rho+\gamma)^2(c+3)^{\delta}(c+1)^{\delta}}\right|\right\}.$$

5. Distortion theorem

Saigo's fractional calculus operator $I^{\alpha,\beta,\eta}_{0,z}f(z)$ of $f(z)\in A$ is defined by Srivastava et al. [19] (see also, Saigo [14]) as follows:

Definition 5.1. For real numbers $\alpha>0,\beta$ and η , the fractional integral operator $I^{\alpha\beta,\eta}_{0,z}f(z)$ of $f(z)$ is defined by

$$I^{\alpha\beta,\eta}_{0,z}f(z)=\frac{z^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^z(z-\zeta)^{\alpha-1}\,{}_2F_1\left[\alpha+\beta,-\eta;\alpha;1-\frac{\zeta}{z}\right]f(\zeta)d\zeta,$$

where $f(z)$ is an analytic function in a simply-connected region of the z-plane containing the origin with the order $f(z) = O(|z|^\epsilon)$ $(z \to 0)$, $\epsilon > \max\{0, \beta - \eta\} - 1$, and the multiplicity of $(z-\zeta)^{\alpha-1}$ is removed by requiring $\log(z-\zeta)$ to be real when $z - \zeta > 0$.

In order to derive the inequalities involving Saigo's fractional operators, we need the following lemma due to Srivastava, Saigo and Owa [19].

Lemma 5.2. *Let $\alpha > 0, \beta$ and η be real. Then, for $k > \max\{0, \beta - \eta\} - 1$,*

$$I_{0,z}^{\alpha,\beta,\eta} z^k = \frac{\Gamma(k+1)\Gamma(k-\beta+\eta+1)}{\Gamma(k-\beta+1)\Gamma(k+\alpha+\eta+1)} z^{k-\beta}. \tag{5.1}$$

Theorem 5.3. *Let $f \in R^{\tau}_{\delta,\gamma,\rho,c}(A,B)$, then*

$$|I_{0,z}^{\alpha,\beta,\eta} f(z)| \le \frac{\Gamma(2-\beta+\eta)|z|^{1-\beta}}{\Gamma(2-\beta)\Gamma(2+\alpha+\eta)} \left[1 + \frac{(2-\beta+\eta)|\tau(A-B)||z|}{(2-\beta)(2+\alpha+\eta)(1+B)(\rho+\gamma)\left(\frac{c+1}{c+2}\right)^{\delta}} \right] \tag{5.2}$$

and

$$|I_{0,z}^{\alpha,\beta,\eta} f(z)| \ge \frac{\Gamma(2-\beta+\eta)|z|^{1-\beta}}{\Gamma(2-\beta)\Gamma(2+\alpha+\eta)} \left[1 - \frac{(2-\beta+\eta)|\tau(A-B)||z|}{(2-\beta)(2+\alpha+\eta)(1+B)(\rho+\gamma)\left(\frac{c+1}{c+2}\right)^{\delta}} \right] \tag{5.3}$$

The equalities in (5.2) and (5.3) are attained for the function $f(z)$ given by (3.2)

Proof. The generalized Saigo [19] fractional integration of $f \in A$ for real numbers $\alpha > 0$, β and η, is given by

$$I_{0,z}^{\alpha,\beta,\eta} f(z) = \sum_{k=1}^{\infty} \frac{\Gamma(k+1)\Gamma(k-\beta+\eta+1)}{\Gamma(k-\beta+1)\Gamma(k+\alpha+\eta+1)} a_k z^{k-\beta}, \qquad (a_1 = 1)$$

$$\Rightarrow \frac{\Gamma(2-\beta)\Gamma(2+\alpha+\eta)}{\Gamma(2-\beta+\eta)} z^{\beta} I_{0,z}^{\alpha,\beta,\eta} f(z) = z + \sum_{k=2}^{\infty} B^{\alpha,\beta,\eta}(k) a_k z^k,$$

where

$$B^{\alpha,\beta,\eta}(k) = \frac{\Gamma(k+1)\Gamma(k-\beta+\eta+1)\Gamma(2-\beta)\Gamma(2+\alpha+\eta)}{\Gamma(k-\beta+1)\Gamma(k+\alpha+\eta+1)\Gamma(2-\beta+\eta)}.$$

Therefore,

$$\frac{B^{\alpha,\beta,\eta}(k)}{B^{\alpha,\beta,\eta}(k+1)} = \frac{(k-\beta+1)(k+\alpha+\eta+1)}{(k+1)(k-\beta+\eta+1)} = \frac{1 + \left(\frac{\alpha+\eta}{k+1}\right)}{1 + \left(\frac{\eta}{k-\beta+1}\right)}.$$

Now, $(\alpha+\eta) > \eta$ and $\frac{1}{k+1} > \frac{1}{k-\beta+1}$ for $\beta < 0$. Therefore,

$$\frac{\alpha+\eta}{k+1} > \frac{\eta}{k-\beta+1},$$

and hence

$$B^{\alpha,\beta,\eta}(k) > B^{\alpha,\beta,\eta}(k+1)$$

Therefore, $B^{\alpha,\beta,\eta}(k)$, $\beta < 0$ is decreasing for k, Then

$$B^{\alpha,\beta,\eta}(k) \leq B^{\alpha,\beta,\eta}(2) = \frac{2(2-\beta+\eta)}{(2-\beta)(2+\alpha+\eta)}.$$

By using Theorem 3.1, we have

$$\sum_{k=2}^{\infty} a_k \leq \frac{|\tau(A-B)|}{2(1+B)(\rho+\gamma)\left(\frac{c+1}{c+2}\right)^{\delta}}; \quad k \geq 2.$$

Thus

$$\left|\frac{\Gamma(2-\beta)\Gamma(2+\alpha+\eta)}{\Gamma(2-\beta+\eta)} z^{\beta} I_{0,z}^{\alpha,\beta,\eta} f(z)\right| \leq |z| + B^{\alpha,\beta,\eta}(2)|z|^2 \sum_{k=2}^{\infty} a_k$$

$$\Rightarrow \left|I_{0,z}^{\alpha,\beta,\eta} f(z)\right| \leq \frac{\Gamma(2-\beta+\eta)|z|^{1-\beta}}{\Gamma(2-\beta)\Gamma(2+\alpha+\eta)}\left[1+\frac{(2-\beta+\eta)|\tau(A-B)||z|}{(2-\beta)(2+\alpha+\eta)(1+B)(\rho+\gamma)(\frac{c+1}{c+2})^{\delta}}\right].$$

Following the similar steps as above, we obtain

$$\left|I_{0,z}^{\alpha,\beta,\eta} f(z)\right| \geq \frac{\Gamma(2-\beta+\eta)|z|^{1-\beta}}{\Gamma(2-\beta)\Gamma(2+\alpha+\eta)}\left[1-\frac{(2-\beta+\eta)|\tau(A-B)||z|}{(2-\beta)(2+\alpha+\eta)(1+B)(\rho+\gamma)(\frac{c+1}{c+2})^{\delta}}\right].$$

□

6. Extreme points

Theorem 6.1. *Let* $f_1(z) = z$ *and*

$$f_k(z) = z + \frac{|\tau(A-B)|}{k(1+B)\{\rho+\gamma(k-1)\}\left(\frac{c+1}{c+k}\right)^{\delta}} z^k.$$

Then $f \in R^{\tau}_{\delta,\gamma,\rho,c}(A,B)$ *if and only if* $f(z)$ *can be expressed in the form*

$$f(z) = \lambda_1 f_1(z) + \sum_{k=2}^{\infty} \lambda_k f_k(z) \tag{6.1}$$

where

$$\lambda_1 + \sum_{k=2}^{\infty} \lambda_k = 1, \quad (\lambda_1 \geq 0,\ \lambda_k \geq 0).$$

Proof. Let $f(z)$ is given by (6.1). Then

$$f(z) = \lambda_1 z + \sum_{k=2}^{\infty} \lambda_k z + \frac{|\tau(A-B)|}{k(1+B)\{\rho+\gamma(k-1)\}\left(\frac{c+1}{c+k}\right)^{\delta}} \lambda_k z^k = z + \sum_{k=2}^{\infty} t_k z^k,$$

where

$$t_k = \frac{|\tau(A-B)|\lambda_k}{k(1+B)\{\rho+\gamma(k-1)\}\left(\frac{c+1}{c+k}\right)^{\delta}}.$$

Now,

$$\sum_{k=2}^{\infty} \frac{k(1+B)\{\rho+\gamma(k-1)\}\left(\frac{c+1}{c+k}\right)^{\delta}}{|\tau(A-B)|} t_k = \sum_{k=2}^{\infty} \lambda_k = 1-\lambda_1 < 1.$$

Therefore, $f \in R^{\tau}_{\delta,\gamma,\rho,c}(A,B)$.

Conversely, suppose that, $f \in R^{\tau}_{\delta,\gamma,\rho,c}(A,B)$, then by (3.1)

$$a_k < \frac{|\tau(A-B)|}{k(1+B)\{\rho+\gamma(k-1)\}\left(\frac{c+1}{c+k}\right)^{\delta}}, \quad k \geq 2.$$

So, if we set

$$\lambda_k = \frac{k(1+B)\{\rho+\gamma(k-1)\}\left(\frac{c+1}{c+k}\right)^{\delta} a_k}{|\tau(A-B)|} < 1, \quad k \geq 2$$

and $\qquad \lambda_1 = 1 - \sum_{k=2}^{\infty} \lambda_k$, then,

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k = z + \sum_{k=2}^{\infty} \frac{|\tau(A-B)|}{k(1+B)\{\rho+\gamma(k-1)\}\left(\frac{c+1}{c+k}\right)^{\delta}} z^k,$$

$$\Rightarrow f(z) = \lambda_1 f_1(z) + \sum_{k=2}^{\infty} \lambda_k f_k(z)$$

which leads to (6.1). □

From the Theorem 6.1, it follows that:

Corollary 6.2. *The extreme points of the class $R^{\tau}_{\delta,\gamma,\rho,c}(A,B)$ are the functions $f_1(z)$ and $f_k(z)$, $(k \geq 2)$.*

7. Radii of starlikeness and convexity

Theorem 7.1. *Let $f \in R^{\tau}_{\delta,\gamma,\rho,c}(A,B)$. Then $f(z)$ is starlike of order α $(0 \leq \alpha < 1)$ in $|z| < r_1$ where*

$$r_1 = \inf_k \left[\frac{(1-\alpha)k(1+B)\{\rho+\gamma(k-1)\}(\frac{c+1}{c+k})^{\delta}}{(k-\alpha)|\tau(A-B)|}\right]^{\frac{1}{k-1}}.$$

Proof. For $0 \leq \alpha < 1$, we require to show that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1-\alpha,$$

that is, for $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$,

$$\frac{\sum_{k=2}^{\infty} a_k (k-1)|z|^{k-1}}{1 - \sum_{k=2}^{\infty} a_k |z|^{k-1}} < 1 - \alpha$$

or, alternatively $\sum_{k=2}^{\infty} a_k \left(\frac{k-\alpha}{1-\alpha}\right) |z|^{k-1} < 1$, which holds if

$$|z|^{k-1} < \left[\frac{(1-\alpha)k(1+B)\{\rho+\gamma(k-1)\}(\frac{c+1}{c+k})^{\delta}}{(k-\alpha)|\tau(A-B)|}\right].$$

$$\Rightarrow r_1 = \inf_k \left[\frac{(1-\alpha)k(1+B)\{\rho+\gamma(k-1)\}(\frac{c+1}{c+k})^{\delta}}{(k-\alpha)|\tau(A-B)|}\right]^{\frac{1}{k-1}} \qquad \square$$

Noting the fact that $f(z)$ is convex iff $zf'(z)$ is starlike, we have

Theorem 7.2. *Let $f \in R^{\tau}_{\delta,\gamma,\rho,c}(A,B)$. Then f is convex of order α $(0 \le \alpha < 1)$ in $|z| < r_2$ where*

$$r_2 = \inf_k \left[\frac{(1-\alpha)(1+B)\{\rho+\gamma(k-1)\}(\frac{c+1}{c+k})^{\delta}}{(k-\alpha)|\tau(A-B)|}\right]^{\frac{1}{k-1}}.$$

8. Neighborhood results

Definition 8.1. For $f \in A$ of the form (1.1) and $\mu \ge 0$. We define a $(n,\mu)-$neighborhood of a function f by

$$N_{n,\mu}(f) = \left\{ g : g \in A, g(z) = z + \sum_{k=n+1}^{\infty} b_k z^k \textit{ and } \sum_{k=n+1}^{\infty} k|a_k - b_k| \le \mu \right\}. \qquad (8.1)$$

In particular, for the identity function $e(z) = z$, we immediately have

$$N_{n,\mu}(e) = \left\{ g : g \in A, g(z) = z + \sum_{k=n+1}^{\infty} b_k z^k \textit{and} \sum_{k=n+1}^{\infty} k|b_k| \le \mu \right\} \qquad (8.2)$$

where $n \in N \setminus \{1\}$.

Theorem 8.2. *If*

$$\mu = \frac{|\tau(A-B)|}{(1+B)(\rho+n\gamma)\left(\frac{c+1}{c+n+1}\right)^{\delta}}$$

then,

$$R^{\tau}_{\delta,\gamma,\rho,c}(A,B) \subset N_{n,\mu}(e)$$

Proof. For a function $f \in R^{\tau}_{\delta,\gamma,\rho,c}(A,B)$ of the form (1.1), Theorem 3.1 immediately yields,

$$\sum_{k=n+1}^{\infty}(1+B)k\{\rho+\gamma(k-1)\}\left(\frac{c+1}{c+k}\right)^{\delta}a_k \le |\tau(A-B)|,$$

where, $n \in N \setminus \{1\}$.

$$\Rightarrow (1+B)(\rho+n\gamma)\left(\frac{c+1}{c+n+1}\right)^{\delta}\sum_{k=n+1}^{\infty}ka_k \le |\tau(A-B)|$$

$$\Rightarrow \sum_{k=n+1}^{\infty}ka_k \le \frac{|\tau(A-B)|}{(1+B)(\rho+n\gamma)\left(\frac{c+1}{c+n+1}\right)^{\delta}} = \mu. \qquad \square$$

A function, $f \in A$ is said to be in the class $R^{\tau,\alpha}_{\delta,\gamma,\rho,c}(A,B)$, if there exists a function $g \in R^{\tau}_{\delta,\gamma,\rho,c}(A,B)$, such that

$$\left|\frac{f(z)}{g(z)}-1\right| < 1-\alpha, \;\; (z \in U, \; 0<\alpha<1). \tag{8.3}$$

Now, we determine the neighborhood for the class $R^{\tau,\alpha}_{\delta,\gamma,\rho,c}(A,B)$.

Theorem 8.3. *If $g \in R^{\tau}_{\delta,\gamma,\rho,c}(A,B)$ and*

$$\alpha = 1-\frac{\mu(1+B)(\rho+n\gamma)\left(\frac{c+1}{c+n+1}\right)^{\delta}}{n(1+B)(\rho+n\gamma)\left(\frac{c+1}{c+n+1}\right)^{\delta}-|\tau(A-B)|}. \tag{8.4}$$

Then,

$$N_{n,\mu}(g) \subset R^{\tau,\alpha}_{\delta,\gamma,\rho,c}(A,B).$$

Proof. Suppose that, $f \in N_{\mu}(g)$ we then find from the definition (8.1) that,

$$\sum_{k=n+1}^{\infty}k|a_k-b_k| \le \mu,$$

which implies that the coefficient inequality:

$$\sum_{k=n+1}^{\infty}|a_k-b_k| \le \frac{\mu}{n+1} \quad (n \in N).$$

Next since, $g \in R^{\tau}_{\delta,\gamma,\rho,c}(A,B)$, we have

$$\sum_{k=n+1}^{\infty}b_k \le \frac{|\tau(A-B)|}{(n+1)(1+B)(\rho+n\gamma)\left(\frac{c+1}{c+n+1}\right)^{\delta}},$$

so that,

$$\left|\frac{f(z)}{g(z)}-1\right| \leq \frac{\sum_{k=n+1}^{\infty}|a_k-b_k|}{1-\sum_{k=n+1}^{\infty}b_k}$$

$$\leq \frac{\mu}{(n+1)\left[1-\frac{|\tau(A-B)|}{(n+1)(1+B)(\rho+n\gamma)\left(\frac{c+1}{c+n+1}\right)^{\delta}}\right]}$$

$$\leq \frac{\mu(1+B)(\rho+n\gamma)\left(\frac{c+1}{c+n+1}\right)^{\delta}}{(n+1)(1+B)(\rho+n\gamma)\left(\frac{c+1}{c+n+1}\right)^{\delta}-|\tau(A-B)|} \leq 1-\alpha \tag{8.5}$$

provided that α is given precisely by (8.4). Thus by definition $f \in R^{\tau,\alpha}_{\delta,\gamma,\rho,c}(A,B)$ for α given by (8.4). This completes the proof. □

9. Integral means inequality

In 1975, Silverman[15] (see, e.g., [17]) found that the function $f_2(z) = z - \frac{z^2}{2}$ is often extremal over the family T and applied this function to resolve his integral means inequality, conjectured in [16] that

$$\int_0^{2\pi}|f(re^{i\theta})|^{\eta}d\theta \leq \int_0^{2\pi}|f_2(re^{i\theta})|^{\eta}d\theta, \tag{9.1}$$

for all $f \in T, \eta > 0$ and $0 < r < 1$ and settled in 1997. He also proved his conjecture for the subclasses $S^*(\alpha)$ and $K(\alpha)$ of T.

Lemma 9.1. [7] *If $f(z)$ and $g(z)$ are analytic in Δ with $f(z) \prec g(z)$, then*

$$\int_0^{2\pi}|f(re^{i\theta})|^{\eta}d\theta \leq \int_0^{2\pi}|g(re^{i\theta})|^{\eta}d\theta, \tag{9.2}$$

$$\eta \geq 0,\ z = re^{i\theta} \text{ and } 0 < r < 1.$$

Application of Lemma (9.1) to function of f in the class $R^{\tau}_{\delta,\gamma,\rho,c}(A,B)$, gives the following result.

Theorem 9.2. *Let $\eta > 0$. If $f \in R^{\tau}_{\delta,\gamma,\rho,c}(A,B)$ is given by (1.1) and $f_2(z)$ is defined by*

$$f_2(z) = z + \frac{|\tau(A-B)|}{2(1+B)(\rho+\gamma)\left(\frac{c+1}{c+2}\right)^{\delta}}z^2 \tag{9.3}$$

$$= z + \frac{1}{\phi_B^A(2,\delta,\gamma,\rho,c,\tau)}z^2,$$

where,

$$\phi_B^A(2,\delta,\gamma,\rho,c,\tau) = \frac{2(1+B)(\rho+\gamma)\left(\frac{c+1}{c+2}\right)^{\delta}}{|\tau(A-B)|}.$$

then, for $z = re^{i\theta}, 0 < r < 1$, we have

$$\int_0^{2\pi} |f(z)|^\eta d\theta \leq \int_0^{2\pi} |f_2(z)|^\eta d\theta. \tag{9.4}$$

Proof. For function f of the form (1.1) is equivalent to proving that

$$\int_0^{2\pi} \left|1 + \sum_{k=2}^{\infty} a_k z^{k-1}\right|^\eta d\theta \leq \int_0^{2\pi} \left|1 + \frac{1}{\phi_B^A(2, \delta, \gamma, \rho, c, \tau)} z\right|^\eta d\theta$$

By Lemma (9.1), it suffices to show that

$$1 + \sum_{k=2}^{\infty} a_k z^{k-1} \prec 1 + \frac{1}{\phi_B^A(2, \delta, \gamma, \rho, c, \tau)} z.$$

Setting

$$1 + \sum_{k=2}^{\infty} a_k z^{k-1} = 1 + \frac{1}{\phi_B^A(2, \delta, \gamma, \rho, c, \tau)} w(z)$$

and using Theorem 3.1, we obtain

$$|w(z)| \leq \left|\sum_{k=2}^{\infty} \phi_B^A(2, \delta, \gamma, \rho, c, \tau) a_k z^{k-1}\right| \leq |z| \sum_{k=2}^{\infty} \phi_B^A(2, \delta, \gamma, \rho, c, \tau) a_k \leq |z|$$

which completes the proof. □

10. Conclusion

We conclude this paper in view of the function class $R^\tau_{\delta,\gamma,\rho,c}(\phi)$ defined by the subordination relation involving arbitrary coefficients and Komatu integral operator $K_c^\delta : A \to A$ defined for $\delta > 0$ and $c > -1$. The classes defined earlier by Bansal [3], Swaminathan [20], Ponnusamy [11] (see also [10]) and Li [5] follow as special cases of this class defined by the authors. The main result gives sufficient condition for coefficient inequalities. Some particular results in this paper leads to the results given earlier by Sokol [18]. A few geometric properties are obtained for this class.

References

[1] Agarwal, R., Paliwal, G.S., *On the Fekete-Szego problem for certain subclasses of analytic function*, Proceedings of the Second International Conference on Soft Computing for Problem Solving (SocProS 2012), December 28-30, 2012, Advances in Intelligent Systems and Computing, **236**(2014), 353-361.

[2] Atshan, W.G., *Fractional Calculus on a Subclass of Spiral-Like Functions Defined by Komatu Operator*, International Mathematical Forum, **3**(32)(2008), 1587-1594.

[3] Bansal, D., *Fekete-Szegö problem for a new class of analytic function*, International Journal of Mathematics and Mathematical Science, 2011, article ID 143096, 5 pp.

[4] Komatu, Y., *On analytic prolongation of a family of integral operators*, Mathematica (Cluj), **32**(55)(1990), 141-145.

[5] Li, J.L., *On some classes of analytic functions*, Mathematica Japonica, **40**(3)(1994), 523-529.

[6] Libera, R.J., Zlotkiewicz, E.J., *Coefficient bounds for the inverse of a function with derivative in ρ,* Proc. Amer. Math. Soc., **87**(2)(1983), 251-257.

[7] Littlewood, J.E., *On inequalities in theory of functions*, Proc. London Math. Soc., **23**(1925), 481-519.

[8] Ma, W., Minda, D., *A unified treatment of some special classes of univalent functions*, in: Proceedings of the Conference on Complex Analysis (Li, Z., Ren, F., Yang, L. and Zhang, S. eds.) Conference Proceeding and Lecture Notes in Analysis, International Press, Cambridge, Massachusetts, **1**(1994), 157-169.

[9] Murugusundaramoorthy, G., Srivastava, H.M., *Neighborhood of certain classes of analytic functions of complex order*, Journal of Inequalities in Pure and Applied Mathematics, **5**(2)(2004), art. 24.

[10] Ponnusamy, S., *Neighborhoods and Carathéodory functions,* J. Anal., **4**(1996), 41-51.

[11] Ponnusamy, S., Ronning, F., *Integral transforms of a class of analytic functions*, Complex Var. Elliptic Equ., **53**(5)(2008), 423-434.

[12] Rusheweyh, St., *Convolution in Geometric Function Theory,* Les Presses de l'Univ. de Montreal, 1982.

[13] Rusheweyh, St., Stankiewicz, J., *Subordination under convex univalent function*, Bull. Pol. Acad. Sci. Math., **33**(1985), 499-502.

[14] Saigo, M., *A remark on integral operators involving the Gauss hypergeometric functions*, Math. Rep. College General Ed. Kyushu Univ., **11**(1978), 135-143.

[15] Silverman, H., *Univalent functions with negative coefficients*, Proc. Amer. Math. Soc., **51**(1975), 109-116.

[16] Silverman, H., *A survey with open problems on univalent functions whose coefficients are negative*, Rocky Mountain J. Math., **21**(1991), 1099-1125.

[17] Silverman, H., *Integral means for univalent functions with negative coefficients*, Houston J. Math., **23**(1997), 169-174.

[18] Sokol, J., *On sufficient condition for starlikeness of certain integral of analytic function*, J. Math. Appl., **28**(2006), 127-130.

[19] Srivastava, H.M., Saigo, M., Owa, S., *A class of distortion theorems involving certain operators of fractional calculus of starlike functions*, J. Math. Anal. Appl., **131**(1988), 412-420.

[20] Swaminathan, A., *Sufficient conditions for hypergeometric functions to be in a certain class of analytic functions,* Computers and Mathematics with Applications, **59**(4)(2010), 1578-1583.

[21] Vijaya, K., Deepa, K., *Neighborhood and partial sums results on the class of starlike functions involving Dziok-Srivastava operator*, Stud. Univ. Babeş-Bolyai Math., **58**(2)(2013), 171180.

Ritu Agarwal
Malaviya National Institute of Technology, Jaipur, India
e-mail: `ragarwal.maths@mnit.ac.in`

Gauri Shankar Paliwal
JECRC University, Jaipur, India
e-mail: gaurishankarpaliwal@gmail.com

Hari S. Parihar
Central University of Rajasthan, Kishangarh, Ajmer, India
e-mail: hsparihar@rediffmail.com

Stud. Univ. Babeş-Bolyai Math. 62(2017), No. 3, 395–405
DOI: 10.24193/subbmath.2017.3.11

Best proximity problems for Ćirić type multivalued operators satisfying a cyclic condition

Adrian Magdaş

Abstract. The aim of this paper is to present some best proximity results for multivalued cyclic operators satisfying a Ćirić type condition. Our results extend to the multivalued case some recent results in the literature.

Mathematics Subject Classification (2010): 47H04, 47H09, 54H25.

Keywords: Best proximity point, multivalued operator, proximinal set, property UC, comparison function, multivalued Ćirić type cyclic operator, Ulam-Hyers stability.

1. Introduction and preliminaries

The standard notations and terminologies in nonlinear analysis will be used throughout this paper.

Let (X, d) be a metric space. We denote:

$P(X) := \{Y \subset X \mid Y \text{ is nonempty}\}$; $P_b(X) := \{Y \in P(X) \mid Y \text{ is bounded}\}$;

$P_{cl}(X) := \{Y \in P(X) \mid Y \text{ is closed}\}$; $P_{cv}(X) := \{Y \in P(X) \mid Y \text{ is convex}\}$;

$P_{cp}(X) := \{Y \in P(X) \mid Y \text{ is compact}\}$; $P_{cl,cv}(X) := P_{cl}(X) \cap P_{cv}(X)$.

If $T : Y \subset X \to P(X)$ is a multivalued operator, then

$$Graph(T) := \{(x, y) \in Y \times X \mid y \in T(x)\}$$

denotes the graph of T.

Let us define the following (generalized) functionals used in this paper:

- the diameter functional

$$\delta : P(X) \times P(X) \to \mathbb{R}_+,\ \delta(A, B) = \sup\{d(a, b) \mid a \in A,\ b \in B\};$$

- the gap functional

$$D : P(X) \times P(X) \to \mathbb{R}_+,\ D(A, B) = \inf\{d(a, b) \mid a \in A,\ b \in B\};$$

• the generalized excess functional

$$\rho : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\},\ \rho(A, B) = \sup\{D(a, B) \mid a \in A\};$$

• the generalized Pompeiu-Hausdorff functional

$$H : P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\},\ H(A, B) = \max\{\rho(A, B), \rho(B, A)\}.$$

In 2003, Kirk, Srinivasan and Veeramani generalized Banach's contraction principle introducing the concept of cyclic contraction.

Theorem 1.1. [3] *Let A and B be non-empty closed subsets of a complete metric space (X, d). Suppose that $T : A \cup B \to A \cup B$ is an operator satisfying:*
(i) $T(A) \subset B$, $T(B) \subset A$;
(ii) there exists $k \in (0, 1)$ such that for any $x \in A$ and $y \in B$,

$$d(T(x), T(y)) \leq kd(x, y).$$

Then, T has a unique fixed point in $A \cap B$.

The best proximity problem for a cyclic multivalued operator is as follows:
If (X, d) is a metric space, $A, B \in P(X)$, $T : A \cup B \to P(X)$ is a multivalued operator satisfying the cyclic condition $T(A) \subset B, T(B) \subset A$, then we are interested to find

$$x^* \in A \cup B \text{ such that } D(x^*, Tx^*) = D(A, B). \tag{1.1}$$

x^* is said to be a best proximity point of T.

Eldred and Veeramani proved in 2006 a theorem (see [1]) which ensures the existence of a best proximity point of cyclic contractions in the framework of uniformly convex Banach spaces.

In 2009, Suzuki, Kikkawa and Vetro introduced the property UC and extended Eldred and Veeramani theorem to metric spaces with the property UC.

Theorem 1.2. [12] *Let (X, d) be a metric space and let A and B be nonempty subsets of X such that (A, B) satisfies the property UC. Assume that A is complete. Let $T : A \cup B \to X$ be a cyclic mapping, that is $T(A) \subset B$ and $T(B) \subset A$. Assume that there exists $k \in (0, 1)$ such that*

$$d(T(x), T(y)) \leq k \max\{d(x, y), d(x, T(x)), d(y, T(y))\} + (1 - k)D(A, B)$$

for all $x \in A$ and $y \in B$. Then the following hold:
(i) T has a unique best proximity point $z \in A$.
(ii) z is a unique fixed point of T^2 in A.
(iii) $\left(T^{2n}(x)\right)$ converges to z for every $x \in A$.
(iv) T has at least one best proximity point in B.
(v) If (B, A) satisfies the property UC, then $T(z)$ is a unique best proximity point in B and $\left(T^{2n}(y)\right)$ converges to $T(z)$ for every $y \in B$.

The purpose of this paper is to extend Suzuki, Kikkawa and Vetro theorem to multivalued Ćirić type cyclic operator in the framework of metric spaces with the property UC.

We recall now the following notions and results.

Lemma 1.3. *Let (X, d) be a metric space, $A, B \in P(X)$. Then for any $\varepsilon > 0$ and for any $a \in A$ there exists $b \in B$ such that*

$$d(a, b) \leq H(A, B) + \varepsilon.$$

Definition 1.4. Let (X, d) be a metric space, $Y \in P(X)$. We denote

$$P_Y(x) = \{y \in Y \mid d(x, y) = D(x, Y)\} \text{ for } x \in X.$$

The set Y is called proximinal if for any $x \in X$, $P_Y(x)$ is nonempty. If for any $x \in X$, $P_Y(x)$ is singleton, then Y is called Chebyshev set.

Obviously, any Chebysev set is proximinal.

We denote $P_{prox}(X) = \{Y \in P(X) \mid Y \text{ is proximinal}\}$.

Remark 1.5. Let (X, d) be a metric space. Then

$$P_{cp}(X) \subset P_{prox}(X) \subset P_{cl}(X).$$

Remark 1.6. [2] Every closed convex subset of a uniformly convex Banach space is a Chebyshev set.

For details concerning the above notions see [7], [9] and [11].

Several types of comparison functions have been considered in literature. In this paper we shall refer only to the following one:

Definition 1.7. [10] A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a comparison function if it satisfies:

(i) φ is increasing;

(ii) $(\varphi^n(t))_{n \in \mathbb{N}}$ converges to 0 as $n \to \infty$, for all $t \in \mathbb{R}_+$.

If the condition (ii) is replaced by:

(iii) $\sum_{k=0}^{\infty} \varphi^k(t) < \infty$, for any $t > 0$,

then φ is called a strong comparison function.

It is evident that a strong comparison function is comparison function.

Lemma 1.8. [9] *If $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a comparison function, then $\varphi(t) < t$, for any $t > 0$, $\varphi(0) = 0$ and φ is continuous at* 0.

Example 1.9. The following functions $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ are comparison functions:

(1) $\varphi(t) = at$, where $a \in [0, 1[$.

(2) $\varphi(t) = \begin{cases} \frac{1}{2}t, & \text{for } t \in [0, 1] \\ t - \frac{1}{2}, & \text{for } t > 1 \end{cases}$.

(3) $\varphi(t) = at + \frac{1}{2}[t]$, where $a \in]0, \frac{1}{2}[$.

(4) $\varphi(t) = \frac{t}{1+t}$.

The first three examples are strong comparison functions, and the forth example is a comparison function which is not a strong comparison function. For more examples and considerations on comparison functions see [8], [9].

Definition 1.10. [12]. Let A and B be nonempty subsets of a metric space (X, d). Then (A, B) is said to satisfy the property UC if for $(x_n)_{n\in\mathbf{N}}$ and $(z_n)_{n\in\mathbf{N}}$ sequences in A and $(y_n)_{n\in\mathbf{N}}$ a sequence in B such that $d(x_n, y_n) \to D(A, B)$ and $d(z_n, y_n) \to D(A, B)$ as $n \to \infty$, then $d(x_n, z_n) \to 0$ as $n \to \infty$.

The following are examples of pairs of nonempty subsets of a metric space satisfying the property UC.

Proposition 1.11. *Any pair of nonempty subsets (A, B) of a metric space (X, d) with $D(A, B) = 0$ enjoy the property UC.*

Proposition 1.12. [1]. *Any pair of nonempty subsets (A, B) of a uniformly convex Banach space with A convex enjoy the property UC.*

2. Main results

We start this section by presenting the concept of multivalued Ćirić type cyclic operator.

Definition 2.1. Let (X, d) be a metric space, $A, B \in P(X)$, and $T : A \cup B \to P(X)$ a multivalued operator. If:

(i) $T(A) \subset B, T(B) \subset A$;

(ii) there exists a comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that for any $x \in A$, $y \in B$,

$$H(T(x), T(y)) \leq \varphi(M(x, y) - D(A, B)) + D(A, B),$$

where

$$M(x, y) = \max\left\{d(x, y), D(x, T(x)), D(y, T(y)), \frac{1}{2}[D(x, T(y)) + D(y, T(x))]\right\},$$

then T is called a multivalued Ćirić type cyclic operator.

Example 2.2. The following operators are multivalued Ćirić type cyclic operators:

(1) A multivalued cyclic contraction (see [5]) i.e. a multivalued cyclic operator $T : A \cup B \to P(X)$ satisfying the condition:

there exists $k \in]0, 1[$ such that for any $x \in A$, $y \in B$,

$$H(T(x), T(y)) \leq kd(x, y) + (1 - k)D(A, B).$$

(2) A multivalued cyclic operator $T : A \cup B \to P(X)$ satisfying a Chatterjea type condition:

there exists $k \in]0, \frac{1}{2}[$ such that for any $x \in A$, $y \in B$,

$$H(T(x), T(y)) \leq k(D(x, T(y)) + D(y, T(x))) + (1 - 2k)D(A, B).$$

(3) A multivalued cyclic operator $T : A \cup B \to P(X)$ satisfying a Reich type condition:

there exists $a, b, c \in \mathbb{R}_+, s = a + b + c < 1$, such that for any $x \in A$, $y \in B$,

$$H(T(x), T(y)) \leq ad(x, y) + bD(x, T(x)) + cD(y, T(y)) + (1 - s)D(A, B).$$

Our first main result extends the following theorem to the case of multivalued Ćirić type cyclic operator in the setting of proximinal values.

Theorem 2.3. [5] *Let A and B be nonempty subsets of a metric space (X, d) such that (A, B) satisfies the property UC and A is complete. Let $T : A \cup B \to P(X)$ be a multivalued cyclic contraction with closed bounded valued. Then T has a best proximity point in A.*

The following lemma will be used in the proof of our results.

Lemma 2.4. [5]. *Let be (A, B) a pair of nonempty subsets of a metric space (X, d), satisfying the property UC, and let be a sequence $(x_n)_{n\in\mathbf{N}}$ in A. If there exists a sequence $(y_n)_{n\in\mathbf{N}}$ in B such that $d(x_n, y_n) \to D(A, B)$ and $d(x_{n+1}, y_n) \to D(A, B)$, then $(x_n)_{n\in\mathbf{N}}$ is a Cauchy sequence.*

The first main result of this paper is the following

Theorem 2.5. *(X, d) be a complete metric space, $A \in P_{cl}(X), B \in P(X)$, such that (A, B) satisfies the property UC. If $T : A \cup B \to P_{prox}(X)$ is a multivalued Ćirić type cyclic operator, then the following statements hold:*

(i) T has a best proximity point $x_A^* \in A$;

(ii) there exists a sequence $(x_n)_{n\in\mathbf{N}}$ with $x_0 \in A$ and $x_{n+1} \in T(x_n)$, such that $(x_{2n})_{n\in\mathbf{N}}$ converges to x_A^*.

Proof. (i)+(ii) We construct a sequence of successive approximations of T starting from an arbitrary $x \in A$ in the following way:

$$x_0 = x \in A;$$
$$x_{n+1} \in T(x_n) \text{ such that } d(x_n, x_{n+1}) = D(x_n, T(x_n)), \text{ for } n \geq 0,$$

the existence of x_{n+1} being assured by the proximinality of $T(x_n)$.
Then, for $n \geq 1$,

$$\begin{aligned} d(x_n, x_{n+1}) = D(x_n, T(x_n)) &\leq H(T(x_{n-1}), T(x_n)) \\ &\leq \varphi(M(x_{n-1}, x_n) - D(A, B)) + D(A, B), \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} M(x_{n-1}, x_n) \quad = \quad & \max\Big\{d(x_{n-1}, x_n), D(x_{n-1}, T(x_{n-1})), D(x_n, T(x_n)), \\ & \frac{1}{2}[D(x_{n-1}, T(x_n)) + D(x_n, T(x_{n-1}))]\Big\}\Big). \end{aligned}$$

Notice that

$$D(x_{n-1}, T(x_{n-1})) = d(x_{n-1}, x_n) \text{ and } D(x_n, T(x_{n-1})) = 0.$$

Using the triangle inequality,

$$\begin{aligned} D(x_{n-1}, T(x_n)) &\leq d(x_{n-1}, x_n) + D(x_n, T(x_n)) \\ &= d(x_{n-1}, x_n) + d(x_n, x_{n+1}), \; n \geq 1. \end{aligned}$$

So

$$\frac{1}{2}[D(x_{n-1}, T(x_n)) + D(x_n, T(x_{n-1}))] \leq \frac{1}{2}[d(x_{n-1}, x_n) + d(x_n, x_{n+1})],$$

and

$$M(x_{n-1}, x_n) \le \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\},\ n \ge 1.$$

Denoting $z_n = d(x_n, x_{n+1}) - D(A, B)$ and using the monotonicity of φ, (2.1) becomes

$$z_n \le \varphi(\max\{z_{n-1}, z_n\}),\ \text{for } n \ge 1.$$

Because $\varphi(t) < t$, for any $t > 0$, we get

$$z_n \le \varphi(z_{n-1}),\ \text{for any } n \ge 1.$$

Thus

$$z_n \le \varphi^{n-1}(z_1) \to 0,\ \text{so } d(x_n, x_{n+1}) \to D(A, B) \text{ when } n \to \infty.$$

Since

$$(x_{2n})_{n\in\mathbf{N}} \subset A, (x_{2n+2})_{n\in\mathbf{N}} \subset A,\ \text{and } (x_{2n+1})_{n\in\mathbf{N}} \subset B,$$

by Lemma 2.4, $(x_{2n})_{n\in\mathbf{N}}$ is a Cauchy sequence in the complete metric space X. Hence, the Cauchy sequence $(x_{2n})_{n\in\mathbf{N}}$ converges to a point x_A^* which lies in A because $(x_{2n})_{n\ge 0} \subset A$ and A is closed.

For $n \ge 1$, we have

$$D(A, B) \le d(x_A^*, x_{2n-1}) \le d(x_A^*, x_{2n}) + d(x_{2n}, x_{2n-1}),$$
$$\text{so } d(x_A^*, x_{2n-1}) \to D(A, B) \text{ when } n \to \infty.$$

$$\begin{aligned} D(A, B) &\le D(x_{2n}, T(x_A^*)) \\ &\le H(T(x_{2n-1}), T(x_A^*)) \\ &\le \varphi(M(x_{2n-1}, x_A^*) - D(A, B)) + D(A, B) \\ &< M(x_{2n-1}, x_A^*) \\ &= \max\Big\{d(x_{2n-1}, x_A^*), D(x_{2n-1}, T(x_{2n-1})), D(x_{2n}, T(x_{2n})), \\ &\frac{1}{2}[D(x_{2n-1}, T(x_{2n})) + D(x_{2n}, T(x_{2n-1}))]\Big\} \end{aligned}$$

Each term from maximum's expression tends to $D(A, B)$:

$$\begin{aligned} d(x_{2n-1}, x_A^*) &\to D(A, B); \\ D(x_{2n-1}, T(x_{2n-1})) &= d(x_{2n-1}, x_{2n}) \to D(A, B); \\ D(x_{2n}, T(x_{2n})) &= d(x_{2n}, x_{2n+1}) \to D(A, B); \\ D(x_{2n}, T(x_{2n-1})) &= 0; \\ \frac{1}{2}[D(x_{2n-1}, T(x_{2n}))] &\le \frac{1}{2}[d(x_{2n-1}, x_{2n}) + D(x_{2n}, T(x_{2n}))] \to D(A, B) \end{aligned}$$

Thus

$$D(x_{2n}, T(x_A^*)) \to D(A, B).$$

Then we have

$$D(A, B) \le D(x_A^*, T(x_A^*)) \le d(x_A^*, x_{2n}) + D(x_{2n}, T(x_A^*)) \to D(A, B).$$

Therefore

$$D(x_A^*, T(x_A^*)) = D(A, B).$$

Remark 2.6. If in Theorem 2.5 $D(A,B)=0$, then we obtain a fixed point result, see Theorem 2.7 in [4].

Theorem 2.7. *Let (X,d) be a complete metric space, $A,B\in P_{cl}(X)$, such that the pairs (A,B) and (B,A) satisfy the property UC. Let $T:A\cup B\to P_{prox}(X)$ be a multivalued operator. Then the following statements hold:*

(i) If T is a multivalued Ćirić type cyclic operator, then T has at least one best proximity point in A and at least one best proximity point in B;

(ii) If T satisfies the following stronger condition:
for any $x\in A$, $y\in B$,

$$\delta(T(x),T(y))\leq\varphi(M(x,y)-D(A,B))+D(A,B),$$

then there exist a best proximity $x_A^*\in A$ and a best proximity point $x_B^*\in B$ such that:

$$d(x_A^*,x_B^*)\leq\sup\left\{t\geq 0\mid t-\varphi(t)\leq 3D(A,B)\right\}.$$

Proof. (i) It is a consequence of Theorem 2.5.

(ii)
$$\begin{aligned} d(x_A^*,x_B^*) &\leq D(x_A^*,T(x_A^*))+\delta(T(x_A^*),T(x_B^*))+D(x_B^*,T(x_B^*)) \\ &= 2D(A,B)+\delta(T(x_A^*),T(x_B^*)\leq \\ &\leq 2D(A,B)+\varphi(\max\{d(x_A^*,x_B^*),D(x_A^*,T(x_A^*)),D(x_B^*,T(x_B^*)), \\ &\quad \frac{1}{2}[D(x_A^*,T(x_B^*))+D(x_B^*,T(x_A^*))]\}-D(A,B))+D(A,B) \\ &\leq 3D(A,B)+\varphi(\max\{d(x_A^*,x_B^*),D(A,B),D(A,B), \\ &\quad \frac{1}{2}[d(x_A^*,x_B^*)+D(A,B)+d(x_B^*,x_A^*)+D(A,B)]\}-D(A,B)) \\ &= 3D(A,B)+\varphi(d(x_A^*,x_B^*)) \end{aligned}$$

Thus, $d(x_A^*,x_B^*)-\varphi(d(x_A^*,x_B^*))\leq 3D(A,B)$.

Corollary 2.8. *Let X be a uniformly convex Banach space,*

$$A,B\in P_{cl,cv}(X),\ T:A\cup B\to P_{cl,cv}(X)$$

be a multivalued operator. Then the following statements hold:

(i) If T is a multivalued Ćirić type cyclic operator, then T has at least one best proximity point in A and at least one best proximity point in B;

(ii) If T satisfies the following stronger condition:
for any $x\in A$, $y\in B$,

$$\delta(T(x),T(y))\leq\varphi(M(x,y)-D(A,B))+D(A,B),$$

then there exist a best proximity $x_A^*\in A$ and a best proximity point $x_B^*\in B$ such that:

$$\|x_A^*-x_B^*\|\leq\sup\{t\geq 0\mid t-\varphi(t)\leq 3D(A,B)\}.$$

Proof. (i) By Remark 1.6, any closed and convex set is proximinal.

Since A and B are convex, by Proposition 1.12, the pairs (A,B) and (B,A) satisfy the property UC.

Applying Theorem 2.7 we get the existence of a best proximity point $x_A^*\in A$ and a best proximity point $x_B^*\in B$.

(ii) It is an immediate consequence of Theorem 2.7.

If, in Theorem 2.7, φ is a subadditive strong comparison function, then the condition that the multivalued operator takes proximinal values can be removed. More precisely, we obtain the second main result, as follows.

Theorem 2.9. *Let (X,d) be a complete metric space, $A,B \in P_{cl}(X)$, such that (A,B) satisfies the property UC. If $T : A \cup B \to P(X)$ is a multivalued Ćirić type cyclic operator, with a subadditive strong comparison function φ, then the following statements hold:*

(i) T has a best proximity point $x_A^* \in A$;

(ii) there exists a sequence $(x_n)_{n\in\mathbf{N}}$ with $x_{n+1} \in T(x_n)$ starting from an arbitrary $(x_0,x_1) \in Graph(T)$, such that $(x_{2n})_{n\in\mathbf{N}}$ converges to x_A^*.

Proof. (i)+(ii) Let $(x,y) \in Graph(T)$ be arbitrary. We construct a sequence of successive approximations of T starting from (x,y) in the following way:

$$x_0 = x \in A \text{ and } x_1 = y \in T(x) \subseteq T(A) \subseteq B.$$

If $d(x_0,x_1) > D(A,B)$ then $\varphi(z_0) < z_0$, where $z_0 := d(x_0,x_1) - D(A,B)$.
For $\varepsilon_1 \in]0, z_0 - \varphi(z_0)[$ there exists $x_2 \in T(x_1) \subseteq T(B) \subseteq A$ such that

$$d(x_1,x_2) \leq H(T(x_0),T(x_1)) + \varepsilon_1.$$

If $d(x_1,x_2) > D(A,B)$ then $\varphi(z_1) < z_1$, where $z_1 := d(x_1,x_2) - D(A,B)$.
For $\varepsilon_2 \in]0, \min\{\varepsilon_1, z_1 - \varphi(z_1)\}[$ there exists $x_3 \in T(x_2) \subseteq T(A) \subseteq B$ such that

$$d(x_2,x_3) \leq H(T(x_1),T(x_2)) + \varepsilon_2.$$

Following this procedure in the case $z_{n-1} := d(x_{n-1},x_n) - D(A,B) > 0, n \geq 2$, we choose

$$\varepsilon_n \in]0, \min\{\varepsilon_{n-1}, z_{n-1} - \varphi(z_{n-1})\}[, \text{ for } n \geq 2. \tag{2.2}$$

There exists $x_{n+1} \in T(x_n)$ such that

$$d(x_n,x_{n+1}) \leq H(T(x_{n-1}),T(x_n)) + \varepsilon_n, n \geq 1,$$

the existence of x_{n+1} being assured by Lemma 1.3.
Since T is a multivalued Ćirić type cyclic operator, using the same reasoning as in Theorem 2.5, we have

$$z_n \leq \varphi(\max\{z_{n-1}, z_n\}) + \varepsilon_n, \text{ for } n \geq 1. \tag{2.3}$$

Using (2.2), we obtain

$$z_n < \varphi(\max\{z_{n-1}, z_n\}) + z_{n-1} - \varphi(z_{n-1}), \text{ for } n \geq 1. \tag{2.4}$$

We suppose that $z_{n-1} \leq z_n$. Using the subadditivity of φ and Lemma 1.8,

$$\varphi(z_n) = \varphi(z_n - z_{n-1} + z_{n-1}) \leq \varphi(z_n - z_{n-1}) + \varphi(z_{n-1}) \leq z_n - z_{n-1} + \varphi(z_{n-1}),$$

so $z_n \geq \varphi(z_n) + z_{n-1} - \varphi(z_{n-1})$ which contradicts (2.4).

We have $z_n \leq z_{n-1}$ and (2.3) becomes

$$\begin{aligned} z_n &\leq \varphi(z_{n-1}) + \varepsilon_n \\ &\leq \varphi\left(\varphi\left(z_{n-2}\right) + \varepsilon_{n-1}\right)) + \varepsilon_n \\ &\leq \varphi^2(z_{n-2}) + \varphi\left(\varepsilon_{n-1}\right) + \varepsilon_n \\ &\quad \ldots \\ &\leq \varphi^n(z_0) + \sum_{k=0}^{n-1} \varphi^k(\varepsilon_{n-k}) \\ &\leq \varphi^n(z_0) + \sum_{k=0}^{n-1} \varphi^k(\varepsilon_1) \to 0, \text{ when } n \to \infty. \end{aligned}$$

Then

$$d(x_n, x_{n+1}) \to D\left(A, B\right) \text{ when } n \to \infty.$$

Applying Lemma 2.4 for the sequences

$$(x_{2n})_{n\in\mathbf{N}} \subset A, (x_{2n+2})_{n\in\mathbf{N}} \subset A, \text{ and } (x_{2n+1})_{n\in\mathbf{N}} \subset B,$$

results that $(x_{2n})_{n\in\mathbf{N}}$ is a Cauchy sequence. Because the metric space X is complete and A is closed, the sequence $(x_{2n})_{n\geq 0} \subset A$ converges to a point $x_A^* \in A$. Using the same reasoning as in Theorem 2.5,

$$D(x_{2n}, T(x_A^*)) \to D(A, B), \text{ when } n \to \infty.$$

Then we have

$$D(A, B) \leq D(x_A^*, T(x_A^*)) \leq d(x_A^*, x_{2n}) + D(x_{2n}, T(x_A^*)) \to D(A, B).$$

Therefore

$$D(x_A^*, T(x_A^*)) = D(A, B).$$

If in the above construction, there exists $k \geq 1$ such that $d(x_{k-1}, x_k) = D(A, B)$, then

$$D(A, B) \leq D(x_{k-1}, T(x_{k-1})) \leq d(x_{k-1}, x_k) = D(A, B)$$

so x_{k-1} is a best proximity point of T.
We will show that, in this situation, x_k is also a best proximity point of T.

$$D(x_k, T(x_k)) \leq H(T(x_{k-1}), T(x_k)) \leq \varphi(M(x_{k-1}, x_k) - D(A, B)) + D(A, B).$$

where

$$\begin{aligned} M(x_{k-1}, x_k) &= \max\Big\{d(x_{k-1}, x_k), D(x_{k-1}, T(x_{k-1})), D(x_k, T(x_k)), \\ &\qquad \frac{1}{2}[D(x_{k-1}, T(x_k)) + D(x_k, T(x_{k-1}))]\Big\}\Big) \\ &\leq \max\Big\{D(A, B), D(x_k, T(x_k)), \\ &\qquad \frac{1}{2}[d(x_{k-1}, x_k) + D(x_k, T(x_k))]\Big\}\Big) \\ &\leq D(x_k, T(x_k)). \end{aligned}$$

Thus $D(x_k, T(x_k)) - D(A,B) \leq \varphi(D(x_k, T(x_k)) - D(A,B))$, which means

$$D(x_k, T(x_k)) = D(A,B).$$

There exists $x_{k+1} \in T(x_k)$ such that

$$d(x_k, x_{k+1}) = D(x_k, T(x_k)) = D(A,B),$$

From now on, following this procedure we construct the terms of our sequence $(x_n)_{n\in\mathbf{N}}$ with $x_{n+1} \in T(x_n)$ such that

$$d(x_n, x_{n+1}) = D(x_n, T(x_n)) = D(A,B), \text{ for any } n \geq k.$$

From this point, the proof runs in the same manner as in the case

$$d(x_n, x_{n+1}) > D(A,B), \text{ for any } n \geq 1.$$

Hereinafter we define and study the generalized Ulam-Hyers stability of the best proximity problem (1.1) for a cyclic multivalued operator.

Definition 2.10. Let (X,d) be a complete metric space, $A, B \in P(X)$. Let $T : A\cup B \to P(X)$ be a multivalued operator satisfying the cyclic condition $T(A) \subset B, T(B) \subset A$. The best proximity problem (1.1) is called generalized Ulam-Hyers stable if there exists $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ increasing, continuous at 0, with $\psi(0) = 0$ and there exists $c > 0$ such that for any $\varepsilon > 0$ and $x \in B$ with

$$D(x, T(x)) \leq \varepsilon + D(A,B),$$

there exists a solution $x_A^* \in A$ of (1.1) such that

$$d(x, x_A^*) \leq \psi(\varepsilon) + c \cdot D(A,B).$$

Our stability result is the following.

Theorem 2.11. *Let (X,d) be a complete metric space, $A \in P_{cl}(X), B \in P(X)$, such that (A,B) satisfies the property UC and φ be a comparison function. Let $T : A\cup B \to P_{prox}(X)$ be a multivalued operator. Assume that:*
(i) $T(A) \subset B, T(B) \subset A$;
(ii) for any $x \in A$, $y \in B$,

$$\delta(T(x), T(y)) \leq \varphi(\max\{D(x, T(x)), D(y, T(y))\} - D(A,B)) + D(A,B).$$

*Then the best proximity problem (*1.1*) is generalized Ulam-Hyers stable.*

Proof. T is a multivalued Ćirić type cyclic operator, so the best proximity problem has at least one solution $x_A^* \in A$.

$$\begin{aligned} d(x, x_A^*) &\leq D(x, T(x)) + \delta(T(x), T(x_A^*)) + D(x_A^*, T(x_A^*)) \\ &\leq \varepsilon + D(A,B) + \varphi(\max\{D(x, T(x)), D(x_A^*, T(x_A^*))\} \\ &\quad -D(A,B)) + 2D(A,B) \\ &\leq \varepsilon + \varphi(\max\{\varepsilon + D(A,B), D(A,B)\} - D(A,B)) + 3D(A,B). \end{aligned}$$

In conclusion,

$$d(x, x_A^*) \leq \varepsilon + \varphi(\varepsilon) + 3D(A,B),$$

proving that the best proximity problem (1.1) is generalized Ulam-Hyers stable.

References

[1] Eldred, A.A., Veeramani, P., *Existence and convergence of best proximity points*, J. Math. Anal. Appl., **323**(2006), no. 2, 1001–1006.

[2] Fletcher, J., Moors, W.B., *Chebyshev sets*, J. Aust. Math. Soc., **98** (2015), 161–231.

[3] Kirk, W.A., Srinivasan, P.S., Veeramani, P., *Fixed points for mappings satisfying cyclical contractive conditions*, Fixed Point Theory, **4**(2003), no. 1, 79-89.

[4] Magdaş, A., *A fixed point theorem for Ćirić type multivalued operators satisfying a cyclical condition*, J. Nonlinear Convex Anal., **17**(2015), no. 6, 1109-1116.

[5] Neammanee, K., Kaewkhao, A., *Fixed points and best proximity points for multi-valued mapping satisfying cyclical condition*, International Journal of Mathematical Sciences and Applications, **1**(2011), 1-9.

[6] Păcurar, M., Rus, I.A., *Fixed point theory for cyclic φ-contractions*, Nonlinear Anal., **72**(2010), 1181-1187.

[7] Petruşel, G., *Cyclic representations and periodic points*, Stud. Univ. Babeş-Bolyai Math., **50**(2005), no. 3, 107–112.

[8] Rus, I.A., *Generalized Contractions and Applications*, Cluj University Press, 2001.

[9] Rus, I.A., Petruşel, A., Petruşel, G., *Fixed Point Theory*, Cluj University Press, 2008.

[10] Rus, I.A., Şerban, M.A., *Some generalizations of a Cauchy lemma and applications*, Topics in Mathematics, Computer Science and Philosophy (Şt. Cobzaş Ed.), Cluj University Press, 2008, 173-181.

[11] Singh, S.P., Watson, B., Srivastava, P., *Fixed Point Theory and Best Approximation: the KKM-map Principle*, Kluwer Academic Publishers, Dordrecht, 1997.

[12] Suzuki, T., Kikkawa, M., Vetro, C., *The existence of best proximity points in metric spaces with the property UC*, Nonlinear Anal., **71**(2009), no. 7-8, 2918–2926.

Adrian Magdaş
Babeş-Bolyai University
Faculty of Mathematics and Computer Science
Kogălniceanu Street, No. 1
400084 Cluj-Napoca, Romania
e-mail: `amagdas@yahoo.com`

Stud. Univ. Babeş-Bolyai Math. 62(2017), No. 3, 407–409

Book reviews

Boris Zacharowitsch Wulich, Geometrie der Kegel in normierten Räumen,
(Herausgegeben von Martin R. Weber) De Gruyter Studium, De Gruyter Berlin, 2017, xvi+223 p., ISBN: 978-3-11-047884-6/pbk; 978-3-11-047888-4/ebook

Boris Zakharovich Vulikh (1913-1978) was a distinguished Russian mathematician with outstanding contributions to various domains of functional analysis, mainly to the theory of ordered vector spaces. For almost thirty years he was the head of Chairs of Mathematical Analysis at Leningrad Higher Educational Institutions, from 1957 to 1963 at the Leningrad A.I. Herzen Pedagogical Institute (now A.I. Herzen State Pedagogical University of Russia, Sankt Peterburg) and from 1963 to 1978 at the Mathematics and Mechanics Faculty of the Leningrad State University (now Sankt Peterburg State University).

Besides the research papers, he wrote several well known books on analysis and functional analysis, including two on ordered vector spaces - one in 1950, jointly with L. V. Kantorovich and A. G. Pinsker, and one alone, *Introduction to the theory of partially ordered spaces* (in Russian), Leningrad 1961, an English translation being published with Wolters-Noordhoff in 1967. Less known are two booklets, *The geometry of cones in normed spaces* (72 p.), and *Special questions of the geometry of cones in normed spaces* (73 p.), published at the Kalinin (now Tver) State University in 1976 and 1977, respectively. These two booklets contain, in a condensed but complete and clear form, the basic results on cones in normed spaces, in particular, duality properties of a cone and its dual cone, and properties of the cone of positive operators between ordered normed spaces as well. Prof. Martin Weber from the Technical University of Dresden took the charge to translate into German and update them, being published as Chapters I and II in this book named *The geometry of cones in normed spaces.* This was not a simple translation, a lot of edifying footnotes are included in the text of translation. Also some interesting examples and counterexamples going back to I.I. Chuchaev (N.P. Ogarev Mordovia State University, Russia) and being only announced in the original text are included in detail into the German issue. Besides these, a consistent chapter, *Some afterthoughts by the editor of the German edition*, accompanied by a list of updated references, presents some developments in the theory of ordered vector spaces and their applications done since the publications of the Russian edition of the booklets.

It is worth to mention that Prof. Weber studied at the Leningrad State University (1963-1968) and earned a Ph.D. (Kandidat physiko-matematicheskih nauk -

Candidate in physical-mathematical sciences) in 1974 at the same University (with Prof. B. M. Makarov as supervisor). He was and remained in contact with the strong group of researchers in ordered vector spaces from Leningrad-St Petersburg University, so we have the privilege of a first hand information on the topics, people and events.

In spite of the years passed since their publication, these books by B. Z. Vulikh are still a valuable source of information for mathematicians, professionals and students as well, interested in the theory of ordered normed spaces and its applications. By translating and updating this masterpiece of mathematical exposition Prof. Martin Weber has done a wonderful (and hard) job and, at the same time, rendered a great service to the mathematical community.

S. Cobzaş

René L. Schilling; Wahrscheinlichkeit – Eine Einführung für Bachelor-Studenten. De Gruyter Studium, Walter de Gruyter GmbH, Berlin/Boston 2017, x+232 p., ISBN: 978-3-11-035065-4. Language: German; translated title: Probability – An Introduction for Bachelor Students.

Professor René L. Schilling from the Technical University in Dresden (Germany) is a well–known expert in the field of stochastic processes. This book continues the course of the author about measure and integration theory (*Maß und Integral*, published in 2015 with De Gruyter, Berlin). It is addressed to students of mathematics, natural sciences (especially physics), economics, and engineering, but also to any researcher interested in the field of probability theory and its applications.

This textbook provides a modern access to the most important results of mathematical probability theory. Prerequisites for understanding the present book are basic notions of measure and integration theory. The main topics of this book are: models of probability theory, elementary combinatorics, conditional probabilities, random variables and their independence, characteristic functions, classic limit theorems, convergence of random variables. These topics are then supplemented by the study of sums of independent random variables, laws of large numbers, zero-one laws, random walks, the central limit theorem. Conditional expectations, applications of characteristic functions, and an introduction to the theory of infinitely divisible distributions and large deviations round off the book. Lastly, the author has included an appendix at the end of the book containing a summary of the main results that are used throughout the present book, as well as, a list of discrete and continuous distributions.

The material in this book consists of definitions, properties (with proofs or with references to the literature, where the proof can be found), many examples and counterexamples, exercises, explanatory comments and helpful hints, tables and suggestive figures.

The book is clearly written and well structured. It brings together theory, practice and research topics, and can be recommended as a German textbook for probability theory courses and seminars.

Hannelore Lisei

Palle Jorgensen and Feng Tian, Non-commutative Analysis. World Scientific 2017, xxviii+533 p., ISBN: 978-981-3202-11-5 (hardcover); 978-981-3202-12-2 (softcover); 978-981-3202-14-6 (ebook)

As the authors mention in the Preface, the central themes of the book are: (i) *Operators in Hilbert space*; (ii) *Multivariable spectral theory*; (iii) *Non-commutative analysis*; (iv) *Probability theory*; (v) *Unitary representations.* The term "non-commutative analysis" is interpreted as including representations of non-Abelian groups, and non-Abelian algebras, with emphasis on Lie groups and operator algebras (C^* algebras and von Neumann algebras).

The book is oriented to applications, mainly in physics (quantum mechanics), from where the main motivation for the development of non-commutative analysis comes. According to a quotation from S. Doplicher and R. Longo (page viii), the novelty of physics of the XX century can be characterized with "a single magic word - non-commutativity". These applications are treated in two steps - in outline first and then, after developing the theoretic tools, with full details. The book is devoted to students with different backgrounds in mathematics, some of them coming from neighboring fields, so the authors tries to keep the prerequisites at a minimum. The general framework is that on Hilbert spaces, operators acting on them (with emphasis on unbounded operators) and spectral theory.

The book is divided into five parts: I. *Introduction and motivation*; II. *Topics form functional analysis and operators in Hilbert space*; III. *Applications*; IV. *Extension of operators*; V. *Appendix.*

The applications concern C^* algebras and their representations, completely positive maps, Brownian motion, Lie groups and their unitary representations. One discusses also the famous Kadison-Singer problem - Does every pure state on the von Neumann algebra of bounded diagonal operators on ℓ^2 have a unique extension to a (pure) state on the algebra $\mathcal{B}(\ell^2)$ of all bounded linear operators on ℓ^2? The authors present only in outline this problem (dating from 1959), its recent difficult solution, by N. Srivastava, A. Marcus, and D. Spielman (2013, published in Annals of Mathematics, 2015) requiring a separate book (good presentations of Kadison-Singer problem are given in the papers by P. G. Casazza et al., arXiv:math/0510024, D. Timotin, arXiv:1501.00464, M. Bownik, arXiv:1702.04578).

The book is very well written and organized. All the notions and results are motivated by examples, the Appendix contains a list of significant books in functional analysis (with telegraphic reviews) and short biographies of some relevant mathematicians and physicists who essentially contributed to the field. A lot of suggestive (and amazing) quotations are spread throughout the book.

Based on two-semester courses on functional analysis taught over the years by the first-named author, the book is highly recommended to teachers in applied functional analysis, for students in mathematics and related areas, as well as for self-study by students needing a quick access to some top research tools in mathematics and physics, paving the way to more advanced and specialized texts on non-commutative analysis, non-commutative geometry and applications.

S. Cobzaş