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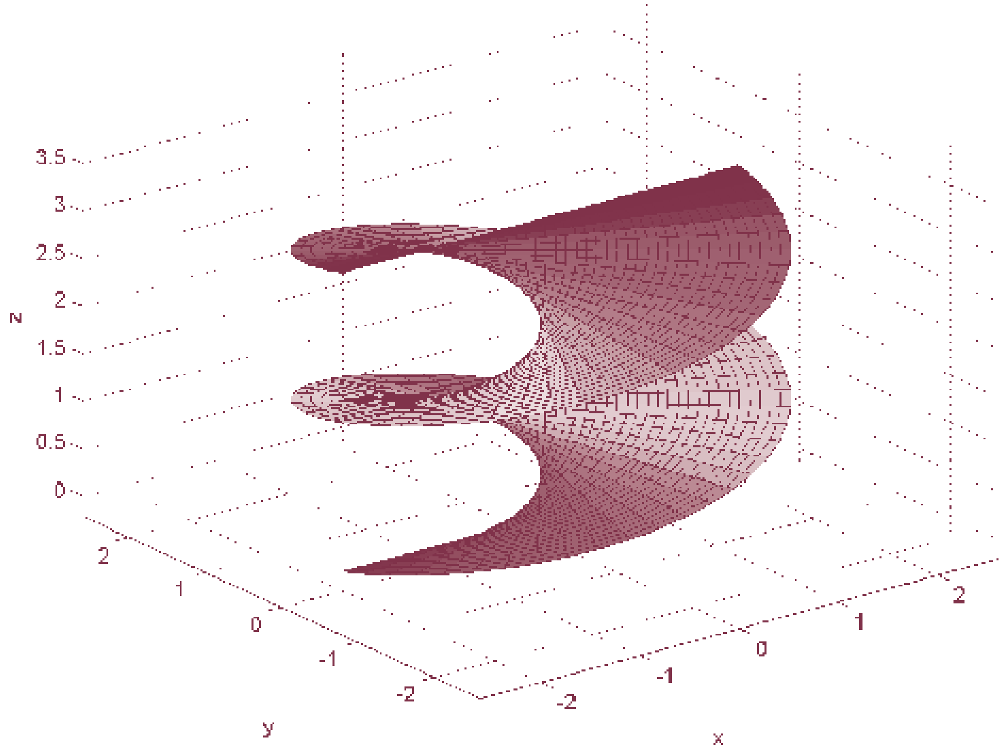

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CONTENTS

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Fekete-Szegő inequalities for certain subclass of analytic functions associated with quasi-subordination

Shashi Kant and Prem Pratap Vyas

Abstract. In this present investigation, we introduce a certain subclass $\mathcal{S}_q(\lambda, \gamma, h)$ of analytic functions which is specify in terms of a quasi-subordination. Sharp bounds of the Fekete-Szegő coefficient for functions belonging to the class $\mathcal{S}_q(\lambda, \gamma, h)$ are obtained. The results presented give improved versions for the classes involving the quasi-subordination and majorization.

Mathematics Subject Classification (2010): 30C45.

Keywords: Univalent functions, subordination, quasi-subordination, Fekete-Szegő coefficients.

1. Introduction and definitions

Let $\mathcal{A}$ denote the family of normalized functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$.
A function f in $\mathcal{A}$ is said to be univalent in $\mathbb{U}$ if f is one to one in $\mathbb{U}$. As usual, we denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of univalent functions in $\mathbb{U}$. Let g and f be two analytic functions in $\mathbb{U}$ then function g is said to be subordinate to f if there exists an analytic function w in the unit disk $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ such that

$$g(z) = f(w(z)) \quad (z \in \mathbb{U}).$$

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We denote this subordination by $g \prec f$.
In particular, if the f is univalent in $\mathbb{U}$, the above subordination is equivalent to

$$g(0) = f(0) \ \ and \ \ f(\mathbb{U}) \subset g(\mathbb{U}).$$

Further, [14] function g is said to be quasi-subordinate to f in the unit disk $\mathbb{U}$ if there exist the functions w (with constant coefficient zero) and ϕ which are analytic and bounded by one in the unit disk $\mathbb{U}$ such that

$$g(z) = \phi(z)f(w(z))$$

and this is equivalent to

$$\frac{g(z)}{\phi(z)} \prec f(z) \quad (z \in \mathbb{U}).$$

We denote this quasi-subordination by

$$g(z) \prec_q f(z) \quad (z \in \mathbb{U}).$$

It is observed that if $\phi(z) = 1 \quad (z \in \mathbb{U})$, then the quasi-subordination $\prec_q$ become the usual subordination $\prec$, and for the function $w(z) = z \quad (z \in \mathbb{U})$, the quasi-subordination $\prec_q$ become the majorization '$\ll$'. In this case:

$$g(z) \prec_q f(z) \ \Rightarrow g(z) = \phi(z)f(w(z)) \ \Rightarrow g(z) \ll f(z), \ \ (z \in \mathbb{U}).$$

The concept of majorization is due to MacGregor [8].
In geometric function theory, study a functional made up of combinations of the coefficients of the original function is a typical problem. Initially, a sharp bound of the functional $|a_3 - \nu a_2^2|$ for univalent functions $f \in \mathcal{A}$ of the form with real ν was obtained by Fekete and Szegő [3] in 1933. Since then, the problem of finding the sharp bounds for this functional $|a_3 - \nu a_2^2|$ of any compact family of functions $f \in \mathcal{A}$ with any complex number ν is generally known as the classical Fekete-Szegő problem or inequality. Fekete-Szegő problem for several subclasses of $\mathcal{A}$ have been studied by many authors (see [1], [2], [4], [12], [13], [15], [17], [18]).
Throughout this paper it is assumed that functions ϕ and h are analytic in $\mathbb{U}$.
Also let

$$\phi(z) = A_0 + A_1 z + A_2 z^2 + \cdots \quad (|\phi(z)| \leq 1, \ z \in \mathbb{U}) \tag{1.2}$$

and

$$h(z) = 1 + B_1 z + B_2 z^2 + \cdots \qquad (B_1 \in \mathbb{R}^+). \tag{1.3}$$

Motivated by earlier works in ([5],[6],[11],[16]) on quasi-subordination, we introduce here the following subclass of analytic functions:

Definition 1.1. *For $0 \leq \lambda \leq 1$ and $\gamma \in \mathbb{C}\backslash\{0\}$, a function $f \in \mathcal{A}$ given by (1.1) is said to be in the class $\mathcal{S}_q(\lambda, \gamma, h)$, if the following condition are satisfied :*

$$\frac{1}{\gamma}\Big(\frac{zf'(z)}{(1-\lambda)z + \lambda f(z)} - 1\Big) \prec_q (h(z) - 1), \tag{1.4}$$

where h is given by (1.3) and $z \in \mathbb{U}$.

It follows that a function f is in the class $\mathcal{S}_q(\lambda,\gamma,h)$ if and only if there exists an analytic function ϕ with $|\phi(z)|\le 1$, in $\mathbb{U}$ such that

$$\frac{\frac{1}{\gamma}\Big(\frac{zf'(z)}{(1-\lambda)z+\lambda f(z)}-1\Big)}{\phi(z)}\prec (h(z)-1)$$

where h is given by (1.3) and $z\in\mathbb{U}$.
If we set $\phi(z)\equiv 1\ \ (z\in\mathbb{U})$, then the class $\mathcal{S}_q(\lambda,\gamma,h)$ is denoted by $\mathcal{S}(\lambda,\gamma,h)$ satisfying the condition that

$$1+\frac{1}{\gamma}\Big(\frac{zf'(z)}{(1-\lambda)z+\lambda f(z)}-1\Big)\prec h(z)\ \ (z\in\mathbb{U}).$$

In the present paper, we find sharp bounds on the Fekete-Szegő functional for functions belonging in the class $\mathcal{S}_q(\lambda,\gamma,h)$. Several known and new consequences of these results are also pointed out. In order to derive our main results, we have to recall here the following well-known lemma:
Let Ω be class of analytic functions of the form

$$w(z)=w_1z+w_2z^2+... \tag{1.5}$$

in the unit disk $\mathbb{U}$ satisfying the condition $|w(z)|<1$.

Lemma 1.1. ([7], p. 10) *If $w(z)\in\Omega$, then for any complex number ν:*

$$|w_1|\le 1, |w_2-\nu w_1^2|\le 1+(|\nu|-1)|w_1^2|\le max\{1,|\nu|\}.$$

The result is sharp for the functions $w(z)=z$ or $w(z)=z^2$.

2. Main results

Theorem 2.1. *Let $0\le\lambda\le 1$ and $\gamma\in\mathbb{C}\backslash\{0\}$. If $f\in\mathcal{A}$ of the form (1.1) belonging to the class $\mathcal{S}_q(\lambda,\gamma,h)$, then*

$$|a_2|\le\frac{|\gamma|B_1}{2-\lambda} \tag{2.1}$$

and for any $\nu\in\mathbb{C}$

$$|a_3-\nu a_2^2|\le\frac{|\gamma|B_1}{3-\lambda}\max\{1,|\frac{B_2}{B_1}-KB_1|\}, \tag{2.2}$$

where

$$K=\gamma\Big(\frac{\nu(3-\lambda)}{(2-\lambda)^2}-\frac{\lambda}{2-\lambda}\Big). \tag{2.3}$$

The results are sharp.

Proof. Let $f\in\mathcal{S}_q(\lambda,\gamma,h)$. In view of Definition1.1, there exist then Schwarz functions w and an analytic function ϕ such that

$$\frac{1}{\gamma}\Big(\frac{zf'(z)}{(1-\lambda)z+\lambda f(z)}-1\Big)=\phi(z)(h(w(z))-1)\quad(z\in\mathbb{U}). \tag{2.4}$$

Series expansions for f and its successive derivatives from (1.1) gives us

$$\frac{1}{\gamma}\Big(\frac{zf'(z)}{(1-\lambda)z+\lambda f(z)}-1\Big)=\frac{1}{\gamma}\big[(2-\lambda)a_2z+[(3-\lambda)a_3-\lambda(2-\lambda)a_2^2]z^2+\cdots\big]. \quad (2.5)$$

Similarly from (1.2), (1.3) and (1.5), we obtain

$$h(w(z))-1=B_1w_1z+(B_1w_2+B_2w_1^2)z^2+\cdots$$

and

$$\phi(z)\big(h(w(z))-1\big)=A_0B_1w_1z+[A_1B_1w_1+A_0(B_1w_2+B_2w_1^2)]z^2+\cdots. \quad (2.6)$$

Equating (2.5) and (2.6) in view of (2.4) and comparing the coefficients of z and z^2, we get

$$a_2=\frac{\gamma A_0B_1w_1}{2-\lambda} \quad (2.7)$$

and

$$a_3=\frac{\gamma B_1}{3-\lambda}\Big[A_1w_1+A_0\{w_2+\Big(\frac{\gamma\lambda A_0B_1}{2-\lambda}+\frac{B_2}{B_1}\Big)w_1^2\}\Big]. \quad (2.8)$$

Thus, for any $\nu\in\mathbb{C}$, we have

$$\begin{aligned}a_3-\nu a_2^2&=\frac{\gamma B_1}{3-\lambda}\Big[A_1w_1+\Big(w_2+\frac{B_2}{B_1}w_1^2\Big)A_0-\Big(\frac{\nu\gamma(3-\lambda)}{(2-\lambda)^2}-\frac{\gamma\lambda}{2-\lambda}\Big)B_1A_0^2w_1^2\Big]\\&=\frac{\gamma B_1}{3-\lambda}\Big[A_1w_1+\Big(w_2+\frac{B_2}{B_1}w_1^2\Big)A_0-KB_1A_0^2w_1^2\Big], \quad (2.9)\end{aligned}$$

where K is given by (2.3).
Since $\phi(z)=A_0+A_1z+A_2z^2+\cdots$ is analytic and bounded by one in $\mathbb{U}$, therefore we have (see[10], p. 172)

$$|A_0|\le 1 \ \ and \ \ A_1=(1-A_0^2)y \quad (y\le 1). \quad (2.10)$$

From (2.9) into (2.10), we obtain

$$a_3-\nu a_2^2=\frac{\gamma B_1}{3-\lambda}\Big[yw_1+\Big(w_2+\frac{B_2}{B_1}w_1^2\Big)A_0-\Big(B_1Kw_1^2+yw_1\Big)A_0^2\Big]. \quad (2.11)$$

If A_0=0 in (2.11), we at once get

$$|a_3-\nu a_2^2|\le\frac{|\gamma|B_1}{3-\lambda}. \quad (2.12)$$

But if $A_0\neq 0$, let us then suppose that

$$G(A_0)=yw_1+\Big(w_2+\frac{B_2}{B_1}w_1^2\Big)A_0-\Big(B_1Kw_1^2+yw_1\Big)A_0^2$$

which is a quadratic polynomial in A_0 and hence analytic in $|A_0|\le 1$ and maximum value of $|G(A_0)|$ is attained at $A_0=e^{\iota\theta}$ $(0\le\theta<2\pi)$, we find that

$$\begin{aligned}\max|G(A_0)| &= \max_{0\le\theta<2\pi}|G(e^{\iota\theta})|=|G(1)|\\ &= \Big|w_2-\Big(KB_1-\frac{B_2}{B_1}\Big)w_1^2\Big|.\end{aligned}$$

Therefore, it follows from (2.11) that

$$|a_3 - \nu a_2^2| \leq \frac{|\gamma| B_1}{3-\lambda}\Big| w_2 - \Big(KB_1 - \frac{B_2}{B_1}\Big) w_1^2 \Big|, \tag{2.13}$$

which on using Lemma1.1, shows that

$$|a_3 - \nu a_2^2| \leq \frac{|\gamma| B_1}{3-\lambda} \max\{1, |\frac{B_2}{B_1} - KB_1|\},$$

and this last above inequality together with (2.12) establish the results.
The results are sharp for the function f given by

$$1 + \frac{1}{\gamma}\Big(\frac{zf'(z)}{(1-\lambda)z + \lambda f(z)} - 1\Big) = h(z),$$

$$1 + \frac{1}{\gamma}\Big(\frac{zf'(z)}{(1-\lambda)z + \lambda f(z)} - 1\Big) = h(z^2)$$

and

$$1 + \frac{1}{\gamma}\Big(\frac{zf'(z)}{(1-\lambda)z + \lambda f(z)} - 1\Big) = z(h(z) - 1).$$

This completes the proof of Theorem 2.1. □

For $\lambda = 1$ the Theorem 2.1 reduces to following corollary:

Corollary 2.2. *If $f \in \mathcal{A}$ of the form (1.1) satisfies*

$$\frac{1}{\gamma}\Big(\frac{zf'(z)}{f(z)} - 1\Big) \prec_q (h(z) - 1) \qquad (z \in \mathbb{U},\ \gamma \in \mathbb{C}\backslash\{0\}),$$

then

$$|a_2| \leq |\gamma| B_1,$$

and for some $\nu \in \mathbb{C}$

$$|a_3 - \nu a_2^2| \leq \frac{|\gamma| B_1}{2} \max\Big\{1, \Big|\frac{B_2}{B_1} + \gamma(1 - 2\nu) B_1\Big|\Big\},$$

The results are sharp.

Remark 2.3. For $\phi \equiv 1$, $\gamma = \lambda = 1$, Theorem 2.1 reduces to an improved result of given in [9].

The next theorems gives the result based on majorization.

Theorem 2.4. *Let $0 \leq \lambda \leq 1$ and $\gamma \in \mathbb{C}\backslash\{0\}$. If $f \in \mathcal{A}$ of the form (1.1) satisfies*

$$\frac{1}{\gamma}\Big(\frac{zf'(z)}{(1-\lambda)z + \lambda f(z)} - 1\Big) \ll (h(z) - 1) \quad (z \in \mathbb{U}), \tag{2.14}$$

then

$$|a_2| \leq \frac{|\gamma| B_1}{2-\lambda}$$

and for any $\nu \in \mathbb{C}$

$$|a_3 - \nu a_2^2| \leq \frac{|\gamma| B_1}{3-\lambda} \max\Big\{1, |\frac{B_2}{B_1} - KB_1|\Big\},$$

where K is given by (2.3). The results are sharp.

Proof. Assume that (2.14) holds. From the definition of majorization, there exist an analytic function ϕ such that

$$\frac{1}{\gamma}\Big(\frac{zf'(z)}{(1-\lambda)z+\lambda f(z)}-1\Big)=\phi(z)(h(z)-1)\quad (z\in\mathbb{U}).$$

Following similar steps as in the proof of Theorem 2.1, and by setting $w(z)\equiv z$, so that $w_1=1, w_n=0, n\geq 2$, we obtain

$$a_2=\frac{\gamma A_0 B_1}{2-\lambda}$$

and also we obtain that

$$a_3-\nu a_2^2=\frac{\gamma B_1}{3-\lambda}\Big[A_1+\frac{B_2}{B_1}A_0-KB_1A_0^2\Big]. \tag{2.15}$$

On putting the value of A_1 from (2.10) into (2.15), we obtain

$$a_3-\nu a_2^2=\frac{\gamma B_1}{3-\lambda}\Big[y+\frac{B_2}{B_1}A_0-\Big(B_1K+y\Big)A_0^2\Big]. \tag{2.16}$$

If $A_0=0$ in (2.16), we at once get

$$|a_3-\nu a_2^2|\leq\frac{|\gamma|B_1}{3-\lambda}. \tag{2.17}$$

But if $A_0\neq 0$, let us then suppose that

$$T(A_0)=y+\frac{B_2}{B_1}A_0-\Big(B_1K+y\Big)A_0^2$$

which is a quadratic polynomial in A_0 and hence analytic in $|A_0|\leq 1$ and maximum value of $|T(A_0)|$ is attained at $A_0=e^{\iota\theta}$ $(0\leq\theta<2\pi)$, we find that

$$\max|T(A_0)| \;=\; \max_{0\leq\theta<2\pi}|T(e^{\iota\theta})|=|T(1)|.$$

Hence, from (2.16), we obtain

$$|a_3-\nu a_2^2|\leq\frac{|\gamma|B_1}{3-\lambda}\Big|KB_1-\frac{B_2}{B_1}\Big|.$$

Thus, the assertion of Theorem 2.4 follows from this last above inequality together with (2.17). The results are sharp for the function given by

$$1+\frac{1}{\gamma}\Big(\frac{zf'(z)}{(1-\lambda)z+\lambda f(z)}-1\Big)=h(z),$$

which completes the proof of Theorem 2.4. □

Theorem 2.5. *Let $0\leq\lambda\leq 1$ and $\gamma\in\mathbb{C}\backslash\{0\}$. If $f\in\mathcal{A}$ of the form (1.1) belonging to the class $\mathcal{S}(\lambda,\gamma,h)$, then*

$$|a_2|\leq\frac{|\gamma|B_1}{2-\lambda}$$

and for any $\nu\in\mathbb{C}$

$$|a_3-\nu a_2^2|\leq\frac{|\gamma|B_1}{3-\lambda}\max\left\{1,\Big|\frac{B_2}{B_1}-KB_1\Big|\right\},$$

where K is given by (2.3), the results are sharp.

Proof. The proof is similar to Theorem 2.1, Let $f \in \mathcal{S}(\lambda, \gamma, h)$.
If $\phi(z) = 1$, then $A_0 = 1, A_n = 0$ $(n \in \mathbb{N})$. Therefore, in view of (2.7) and (2.10) and by application of Lemma 1.1, we obtain the desired assertion. The results are sharp for the function $f(z)$ given by

$$1 + \frac{1}{\gamma}\Big(\frac{zf'(z)}{(1-\lambda)z + \lambda f(z)} - 1\Big) = h(z),$$

or

$$1 + \frac{1}{\gamma}\Big(\frac{zf'(z)}{(1-\lambda)z + \lambda f(z)} - 1\Big) = h(z^2).$$

Thus, the proof of Theorem 2.5 is completed. □

Now, we determine the bounds for the functional $|a_3 - \nu a_2^2|$ for real ν.

Theorem 2.6. *Let $0 \leq \lambda \leq 1$. If $f \in \mathcal{A}$ of the form (1.1) belonging to the class $\mathcal{S}_q(\lambda, \gamma, h)$, then for real ν and γ, we have*

$$|a_3 - \nu a_2^2| \leq \begin{cases} \frac{|\gamma| B_1}{3-\lambda}\big[B_1\big(\frac{\lambda}{2-\lambda} - \frac{3-\lambda}{(2-\lambda)^2}\nu\big) + \frac{B_2}{B_1}\big] & (\nu \leq \sigma_1), \\ \frac{|\gamma| B_1}{3-\lambda} & (\sigma_1 \leq \nu \leq \sigma_1 + 2\rho), \\ -\frac{|\gamma| B_1}{3-\lambda}\big[B_1\big(\frac{\lambda}{2-\lambda} - \frac{3-\lambda}{(2-\lambda)^2}\nu\big) + \frac{B_2}{B_1}\big] & (\nu \geq \sigma_1 + 2\rho), \end{cases} \tag{2.18}$$

where

$$\sigma_1 = \frac{\lambda(2-\lambda)}{(3-\lambda)} - \frac{(2-\lambda)^2}{\gamma(3-\lambda)}\Big(\frac{1}{B_1} - \frac{B_2}{B_1^2}\Big) \tag{2.19}$$

and

$$\rho = \frac{(2-\lambda)^2}{\gamma(3-\lambda)B_1}. \tag{2.20}$$

Each of the estimates in (2.18) are sharp.

Proof. For real values of ν and γ the above bounds can be obtained from (2.2), respectively, under the following cases:

$$B_1 K - \frac{B_2}{B_1} \leq -1, \; -1 \leq B_1 K - \frac{B_2}{B_1} \leq 1 \text{ and } B_1 K - \frac{B_2}{B_1} \geq 1,$$

where K is given by (2.3). We also note the following:
(i) When $\nu < \sigma_1$ or $\nu > \sigma_1 + 2\rho$, then the equality holds if and only if $\phi(z) \equiv 1$ and $w(z) = z$ or one of its rotations.
(ii) When $\sigma_1 < \nu < \sigma_1 + 2\rho$, then the equality holds if and only if $\phi(z) \equiv 1$ and $w(z) = z^2$ or one of its rotations.
(iii) Equality holds for $\nu = \sigma_1$ if and only if $\phi(z) \equiv 1$ and $w(z) = \frac{z(z+\epsilon)}{1+\epsilon z}$ $(0 \leq \epsilon \leq 1)$, or one of its rotations, while for $\nu = \sigma_1 + 2\rho$, the equality holds if and only if $\phi(z) \equiv 1$ and $w(z) = -\frac{z(z+\epsilon)}{1+\epsilon z}$ $(0 \leq \epsilon \leq 1)$, or one of its rotations. □

The bounds of the functional $a_3 - \nu a_2^2$ for real values of ν and γ for the middle range of the parameter ν can be improved further as follows:

Theorem 2.7. *Let $0 \leq \lambda \leq 1$. If $f \in \mathcal{A}$ of the form (1.1) belonging to the class $\mathcal{S}_q(\lambda, \gamma, h)$, then for real ν and γ, we have*

$$|a_3 - \nu a_2^2| + (\nu - \sigma_1)|a_2|^2 \leq \frac{|\gamma| B_1}{3-\lambda} \quad (\sigma_1 \leq \nu \leq \sigma_1 + \rho) \tag{2.21}$$

and

$$|a_3 - \nu a_2^2| + (\sigma_1 + 2\rho - \nu)|a_2|^2 \leq \frac{|\gamma| B_1}{3-\lambda} \quad (\sigma_1 + \rho \leq \nu \leq \sigma_1 + 2\rho), \tag{2.22}$$

where σ_1 and ρ are given by (2.19) and (2.20), respectively.

Proof. Let $f \in \mathcal{S}_q(\lambda, \gamma, h)$. For real ν satisfying $\sigma_1 + \rho \leq \nu \leq \sigma_1 + 2\rho$ and using (2.7) and (2.13) we get

$$\begin{aligned} |a_3 - \nu a_2^2| + (\nu - \sigma_1)|a_2|^2 \leq \frac{|\gamma| B_1}{3-\lambda} \Bigg[|w_2| - \frac{|\gamma| B_1 (3-\lambda)}{(2-\lambda)^2} (\nu - \sigma_1 - \rho)|w_1|^2 \\ + \frac{|\gamma| B_1 (3-\lambda)}{(2-\lambda)^2} (\nu - \sigma_1)|w_1|^2 \Bigg]. \end{aligned}$$

Therefore, by virtue of Lemma 1.1, we get

$$|a_3 - \nu a_2^2| + (\nu - \sigma_1)|a_2|^2 \leq \frac{|\gamma| B_1}{3-\lambda} [1 - |w_1|^2 + |w_1|^2],$$

which yields the assertion (2.21).
If $\sigma_1 + \rho \leq \nu \leq \sigma_1 + 2\rho$, then again from (2.7), (2.13) and the application of Lemma 1.1, we have

$$\begin{aligned} |a_3 - \nu a_2^2| + (\sigma_1 + 2\rho - \nu)|a_2|^2 &\leq \frac{|\gamma| B_1}{3-\lambda} \Bigg[|w_2| + \frac{|\gamma| B_1 (3-\lambda)}{(2-\lambda)^2} (\nu - \sigma_1 - \rho)|w_1|^2 \\ &\quad + \frac{|\gamma| B_1 (3-\lambda)}{(2-\lambda)^2} (\sigma_1 + 2\rho - \nu)|w_1|^2 \Bigg] \\ &\leq \frac{|\gamma| B_1}{3-\lambda} [1 - |w_1|^2 + |w_1|^2], \end{aligned}$$

which estimates (2.22). □

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Fekete-Szegö inequality of bi-starlike and bi-convex functions of order b associated with symmetric q-derivative in conic domains

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Abstract. In this paper, two new subclasses of bi-univalent functions related to conic domains are defined by making use of symmetric q-differential operator. The initial bounds for Fekete-Szegö inequality for the functions f in these classes are estimated.

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1. Introduction

Let $\mathscr{A}$ denotes the set of all functions which are analytic in the unit disc

$$\Delta = \{z \in \mathbb{C} : |z| < 1\}$$

with Taylor's series expansion of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are normalized by $f(0) = 0, f'(0) = 1$. The subclass of $\mathscr{A}$ consisting of all univalent functions is denoted by $\mathscr{S}$. A function $f \in \mathscr{A}$ is said to be a starlike function if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \qquad (z \in \Delta).$$

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A function $f \in \mathscr{A}$ is said to be a convex function if

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right)>0 \qquad (z\in\Delta).$$

Goodman [10, 11, 12] introduced the classes uniformly starlike and uniformly convex functions as subclasses of starlike and convex functions. A starlike function (or convex function) is said to be uniformly starlike (or uniformly convex) if the image of every circular arc ζ contained in Δ, with center at ξ also in Δ is starlike (or convex) with respect to $f(\xi)$. The class of uniformly starlike functions is represented by $\mathscr{UST}$ and the class of uniformly convex functions is represented by $\mathscr{UCV}$. The class of parabolic starlike functions is represented by $\mathscr{S}_p$. Rønning [24] and Ma-Minda [18, 19] independently gave the characterization for the classes $\mathscr{S}_p$ and $\mathscr{UCV}$ as follows.
A function $f \in \mathscr{A}$ is said to be in the class $\mathscr{S}_p$ if and only if

$$\Re\left(\frac{zf'(z)}{f(z)}\right)>\left|\frac{zf'(z)}{f(z)}-1\right| \qquad (z\in\Delta).$$

A function $f \in \mathscr{A}$ is said to be in the class $\mathscr{UCV}$ if and only if

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right)>\left|\frac{zf''(z)}{f'(z)}\right| \qquad (z\in\Delta).$$

Also, it is clear that

$$f\in\mathscr{UCV}\Leftrightarrow zf'(z)\in\mathscr{S}_p.$$

Kanas and Wisniowska [16, 15], introduced k-uniformly starlike functions and k-uniformly convex functions as follows.

$$k-\mathscr{ST}=\left\{f:\ f\in\mathscr{S}\ \text{ and }\ \Re\left(\frac{zf'(z)}{f(z)}\right)>k\left|\frac{zf'(z)}{f(z)}-1\right|,\ \ z\in\Delta,\ \ k\geq 0\right\}$$
$$k-\mathscr{UCV}=\left\{f:\ f\in\mathscr{S}\ \text{ and }\ \Re\left(1+\frac{zf'(z)}{f(z)}\right)>k\left|\frac{zf''(z)}{f'(z)}\right|,\ z\in\Delta,\ k\geq 0\right\}.$$

Bharati, et al. [8], defined $k-\mathscr{ST}(\beta)$ and $k-\mathscr{UCV}(\beta)$ as follows. A function $f \in \mathscr{A}$ is said to be in the class $k-\mathscr{ST}(\beta)$ if and only if

$$\Re\left(\frac{zf'(z)}{f(z)}\right)-\beta>k\left|\frac{zf'(z)}{f(z)}-1\right| \qquad (z\in\Delta). \tag{1.2}$$

A function $f \in \mathscr{A}$ is said to be in the class $k-\mathscr{UCV}(\beta)$ if and only if

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right)-\beta>k\left|\frac{zf''(z)}{f'(z)}\right| \qquad (z\in\Delta). \tag{1.3}$$

Sim et al.[26], generalized above classes and introduced $k-\mathscr{ST}(\alpha,\ \beta)$ and $k-\mathscr{UCV}(\alpha,\ \beta)$ as below:
A function $f \in \mathscr{A}$ is said to be in the class $k-\mathscr{ST}(\alpha,\ \beta)$ if and only if

$$\Re\left(\frac{zf'(z)}{f(z)}\right)-\beta>k\left|\frac{zf'(z)}{f(z)}-\alpha\right| \qquad (z\in\Delta), \tag{1.4}$$

where $0\leq\beta<\alpha\leq 1$ and $k(1-\alpha)<1-\beta$.

A function $f \in \mathscr{A}$ is said to be in the class $k-\mathscr{UCV}(\alpha,\ \beta)$ if and only if

$$\Re\left(1+\frac{zf''(z)}{f'(z)}\right)-\beta>k\left|1+\frac{zf''(z)}{f'(z)}-\alpha\right| \qquad (z\in\Delta), \tag{1.5}$$

where $0 \leq \beta < \alpha \leq 1$ and $k(1-\alpha) < 1-\beta$.

In particular, for $\alpha = 1,\ \beta = 0$ the classes $k-\mathscr{ST}(\alpha,\ \beta)$ and $k-\mathscr{UCV}(\alpha,\ \beta)$ reduces to $k-\mathscr{ST}$ and $k-\mathscr{UCV}$ respectively. Further, for $\alpha = 1$ these classes coincides with the classes studied by Nishiwaki and Owa [20] and Shams et al. [25]. In 2017, Annamalai et al. [7], obtained second Hankel determinant of analytic functions involving conic domains.

Now we give the geometric interpretations of the classes $f \in k-\mathscr{ST}(\alpha,\ \beta)$ and $k-\mathscr{UCV}(\alpha,\ \beta)$ as follows:

A function $f \in k-\mathscr{ST}(\alpha,\ \beta)$ and $k-\mathscr{UCV}(\alpha,\ \beta)$ if and only if $\frac{zf'(z)}{f(z)}$ and $1+\frac{zf''(z)}{f'(z)}$, respectively takes all the values in the conic domain $\Omega_{k,\ \alpha,\ \beta}$

$$\Omega_{k,\ \alpha,\ \beta}=\{\omega:\ \omega\in\mathbb{C} \text{ and } k|\omega-\alpha|<\Re(\omega)-\beta\}$$

or

$$\Omega_{k,\ \alpha,\ \beta}=\left\{\omega:\ \omega\in\mathbb{C} \text{ and } k\sqrt{[\Re(\omega)-\alpha]^2+[\Im(\omega)]^2}<\Re(\omega)-\beta\right\},$$

where $0 \leq \beta < \alpha \leq 1$ and $k(1-\alpha) < 1-\beta$. Clearly $1 \in \Omega_{k,\ \alpha,\ \beta}$ and $\Omega_{k,\ \alpha,\ \beta}$ is bounded by the curve

$$\partial\Omega_{k,\ \alpha,\ \beta}=\left\{\omega:\ \omega=u+iv \text{ and } k^2(u-\alpha)^2+k^2v^2=(u-\beta)^2\right\}.$$

The Caratheodory functions $p \in \mathscr{P}$ is said to be in the class $\mathcal{P}(p_{k,\ \alpha,\ \beta})$ if and only if p takes all the values in the conic domain $\Omega_{k,\ \alpha,\ \beta}$. Analytically it is defined as follows:

$$\mathcal{P}(p_{k,\ \alpha,\ \beta})=\{p:\ p\in\mathscr{P} \text{ and } p(\Delta)\subset\Omega_{k,\ \alpha,\ \beta}\},$$
$$\mathcal{P}(p_{k,\ \alpha,\ \beta})=\{p:\ p\in\mathscr{P} \text{ and } p(z)\prec p_{k,\ \alpha,\ \beta},\ z\in\Delta\}.$$

It is interesting to note that $\partial\Omega_{k,\ \alpha,\ \beta}$ represents conic section about real axis. In particular, $\Omega_{k,\ \alpha,\ \beta}$ represents an elliptic domain for $k > 1$, parabolic domain for $k = 1$, hyperbolic domain for $0 < k < 1$. Sim et al. [26] obtained the functions $p_{k,\alpha\ \beta}(z)$ which play the role of extremal functions of $\mathcal{P}(p_{k,\ \alpha,\ \beta})$ as

$$p_{k,\alpha\ \beta}(z)=\begin{cases} \dfrac{1+(1-2\beta)z}{1-z}, & \text{for } k=0\\ \alpha+\dfrac{2(\alpha-\beta)}{\pi^2}\log^2\left(\dfrac{1+\sqrt{u_k(z)}}{1-\sqrt{u_k(z)}}\right), & \text{for } k=1\\ \dfrac{\alpha-\beta}{1-k^2}\cosh\left\{\mathfrak{u}(k)\log\left(\dfrac{1+\sqrt{u_k(z)}}{1-\sqrt{u_k(z)}}\right)\right\}+\dfrac{\beta-\alpha k^2}{1-k^2}, & \text{for } 0<k<1\\ \dfrac{\alpha-\beta}{k^2-1}\sin^2\left(\dfrac{\pi}{2K(k)}\int\limits_0^{\omega}\dfrac{dt}{\sqrt{1-t^2}\sqrt{1-t^2k^2}}\right)+\dfrac{\alpha k^2-\beta}{k^2-1}, & \text{for } k>1, \end{cases}$$

where $\mathfrak{u}(k) = \frac{2}{\pi}\cos^{-1} k$, $u_k(z) = \frac{z+\rho_k}{1+\rho_k z}$ and

$$\rho_k = \begin{cases} \left(\frac{e^A - 1}{e^A + 1}\right)^2, & \text{for } k = 1 \\ \left(\frac{\exp\left(\frac{1}{u_k(z)}\operatorname{arc\,cosh} B\right) - 1}{\exp\left(\frac{1}{u_k(z)}\operatorname{arc\,cosh} B\right) + 1}\right)^2, & \text{for } 0 < k < 1 \\ \sqrt{k}\sin\left[\frac{2K(\kappa)}{\pi}\arcsin C\right], & \text{for } k > 1 \end{cases}$$

with $A = \sqrt{\frac{1-\alpha}{2(\alpha-\beta)}}\pi$, $B = \frac{1}{\alpha-\beta}(1-k^2-\beta+\alpha k^2)$, $C = \frac{1}{\alpha-\beta}(k^2-1+\beta-\alpha k^2)$. Also

$$K(\kappa) = \int_0^\omega \frac{dt}{\sqrt{1-t^2}\sqrt{1-t^2\kappa^2}} \quad (0 < \kappa < 1),$$
$$K'(\kappa) = K(\sqrt{1-\kappa^2}) \quad (0 < \kappa < 1),$$
$$\kappa = \cosh\left(\frac{\pi K'(\kappa)}{4K(\kappa)}\right).$$

According to Koebe's $\frac{1}{4}$ theorem, every analytic and univalent function f in Δ has an inverse f^{-1} and is defined as

$$f^{-1}(f(z)) \quad = \quad z \quad (z \in \Delta) \text{ and } f(f^{-1}(w)) = w \quad \left(|w| < r_0(f); r_0(f) \geq \frac{1}{4}\right).$$

Also the function f^{-1} can be written as

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots. \tag{1.6}$$

A function $f \in \mathscr{A}$ is said to be bi-univalent if both f and analytic extension of f^{-1} in Δ are univalent in Δ. The class of all bi-univalent functions is denoted by Σ. That is a function f is said to be bi-univalent if and only if

1. f is an analytic and univalent function in Δ.
2. There exists an analytic and univalent function g in Δ such that $f(g(z)) = g(f(z)) = z$ in Δ.

The class of bi-univalent functions was introduced by Lewin [17] in 1967. Recently many researchers [1, 2, 4, 3, 14, 21, 22, 23, 28, 29, 30, 31, 33, 32, 34, 35] have introduced and investigated several interesting subclasses of the bi-univalent functions and they have found non-sharp estimates of two Taylor-Maclaurin coefficients $|a_2|$, $|a_3|$, Fekete-Szegö inequalities and second Hankel determinants. In 2017, Altinkaya and Yalçin [5, 6] estimated the coefficients and Fekete-Szegö inequalities for some subclasses of bi-univalent functions involving symmetric q-derivative operator subordinate to the generating function of Chebyshev polynomials.

Jackson [13], defined $q-$derivative operator D_q of an analytic function f of the form (1.1)as follows:

$$D_q f(z) = \begin{cases} \dfrac{f(qz) - f(z)}{(q-1)z}, & for\ z \neq 0, \\ f'(0), & for\ z = 0 \end{cases}$$

$$D_q f(0) = f'(0) \text{ and } D_q^2 = D_q(D_q f(z)).$$

If $f(z) = z^n$ for any positive integer n, the q-derivative of $f(z)$ is defined by

$$D_q z^n = \frac{(qz)^n - z^n}{qz - z} = [n]_q z^{n-1},$$

where $[n]_q = \dfrac{q^n - 1}{q - 1}$. As $q \to 1^-$ and $k \in \mathbb{N}$, we have $[n]_q \to n$ and

$$\lim_{q \to 1} (D_q f(z)) = f'(z)$$

where f' is normal derivative of f. Therefore

$$D_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}.$$

Brahim and Sidomou [9], defined the symmetric $q-$derivative operator $\widetilde{D}_q$ of an analytic function f of the form (1.1) as follows:

$$(\widetilde{D}_q f)(z) = \begin{cases} \dfrac{f(qz) - f(q^{-1}z)}{(q - q^{-1})z}, & \text{for } z \neq 0, \\ f'(0), & \text{for } z = 0 \end{cases}.$$

It is clear that $\widetilde{D}_q z^n = \widetilde{[n]}_q z^{n-1}$ and $\widetilde{D_q} f(z) = 1 + \sum_{n=2}^{\infty} \widetilde{[n]}_q a_n z^{n-1}$, where

$$\widetilde{[n]}_q = \frac{q^n - q^{-n}}{q - q^{-1}}.$$

The relation between q-derivative operator and symmetric q-derivative operator is given by

$$(\widetilde{D_q} f)(z) = D_{q^2} f(q^{-1} z).$$

If g is the inverse of f then

$$\begin{aligned} (\widetilde{D_q} g)(w) &= \frac{g(qw) - g(q^{-1}w)}{(q - q^{-1})w} \\ &= 1 - \widetilde{[2]}_q a_2 w + \widetilde{[3]}_q (2a_2^2 - a_3) w^2 - \widetilde{[4]}_q (5a_2^3 - 5a_2 a_3 + a_4) w^3 + \cdots . \end{aligned}$$

One could refer [27], for more details of $q-$ calculus and fractional $q-$calculus and their applications in Geometric Function Theory.

Motivated by the above mentioned work, in this paper, bi-starlike functions of order b and bi-convex functions of order b involving q-derivative operator subordinate to the conic domains are defined and the Fekete-Szegö inequality for the function in these classes are obtained.

Definition 1.1. A function $f \in \Sigma$ is said to be in the class $k - \mathscr{S}\mathscr{T}_{\Sigma,\, b}(\alpha,\ \beta)$, where $0 \le \beta < \alpha \le 1$ and $k(1-\alpha) < 1-\beta$ and b is a non-zero complex number, if it satisfies the following conditions:

$$1 + \frac{1}{b}\left(\frac{z\widetilde{D}_q f(z)}{f(z)} - 1\right) \prec p_{k,\,\alpha,\,\beta}(z) \qquad (z \in \Delta) \tag{1.7}$$

and for $g = f^{-1}$

$$1 + \frac{1}{b}\left(\frac{w\widetilde{D}_q g(w)}{g(w)} - 1\right) \prec p_{k,\,\alpha,\,\beta}(w) \qquad (w \in \Delta). \tag{1.8}$$

Definition 1.2. A function $f \in \Sigma$ is said to be in the class $k - \mathscr{U}\mathscr{C}\mathscr{V}_{\Sigma,\, b}(\alpha,\ \beta)$; where $0 \le \beta < \alpha \le 1$ and $k(1-\alpha) < 1-\beta$, and b is a non-zero complex number, if it satisfies the following conditions:

$$1 + \frac{1}{b}\left(\frac{\widetilde{D}_q(z\widetilde{D}_q f(z))}{\widetilde{D}_q(f(z))} - 1\right) \prec p_{k,\,\alpha,\,\beta}(z) \qquad (z \in \Delta) \tag{1.9}$$

and for $g = f^{-1}$

$$1 + \frac{1}{b}\left(\frac{\widetilde{D}_q(w\widetilde{D}_q g(w))}{\widetilde{D}_q(g(w))} - 1\right) \prec p_{k,\,\alpha,\,\beta}(w) \qquad (w \in \Delta). \tag{1.10}$$

2. Main results

In this section, initial estimates $|a_2|$, $|a_3|$ and Fekete-Szegö inequalities for the functions f in the classes $k - \mathscr{S}\mathscr{T}_{\Sigma,\, b}(\alpha,\ \beta)$ and $k - \mathscr{U}\mathscr{C}\mathscr{V}_{\Sigma,\, b}(\alpha,\ \beta)$ are obtained.

Theorem 2.1. *If $f \in k - \mathscr{S}\mathscr{T}_{\Sigma,\, b}(\alpha,\ \beta)$ and is of the form* (1.1) *then*

$$|a_2| \le \frac{|P_1|\sqrt{|P_1|b^2}}{\sqrt{|P_1^2 b\left(\widetilde{[3]}_q - \widetilde{[2]}_q\right) + 2(P_1 - P_2)\left(\widetilde{[2]}_q - 1\right)^2|}},$$

$$|a_3| \le \frac{b^2 P_1^2}{\left(\widetilde{[2]}_q - 1\right)^2} + \frac{\left|bP_1\right|}{\widetilde{[3]}_q - 1}$$

and

$$|a_3 - \mu a_2^2| \le \begin{cases} \dfrac{|P_1 b|}{\widetilde{[3]}_q - 1}, & \text{if } 0 \le |s(\mu)| \le 1 \\ \dfrac{|P_1 b|\ |s(\mu)|}{\widetilde{[3]}_q - 1} & \text{if } |s(\mu)| \ge 1, \end{cases}$$

where

$$s(\mu) = \frac{P_1^2 b(1-\mu)}{[P_1^2 b\left(\widetilde{[3]}_q - \widetilde{[2]}_q\right) + (P_1 - P_2)\left(\widetilde{[2]}_q - 1\right)^2]}.$$

Proof. Let $f \in k - \mathscr{S}\mathscr{T}_{\Sigma,\, b}(\alpha,\ \beta)$ and g be an analytic extension of f^{-1} in Δ. Then there exist two Schwarz functions $u,\ v \in \Delta$ such that

$$1 + \frac{1}{b}\left(\frac{z\widetilde{D}_q f(z)}{f(z)} - 1\right) = p_{k,\, \alpha,\, \beta}(u(z)), \tag{2.1}$$

and

$$1 + \frac{1}{b}\left(\frac{w\widetilde{D}_q g(w)}{g(w)} - 1\right) = p_{k,\, \alpha,\, \beta}(v(w)). \tag{2.2}$$

Define two functions $h,\ q \in \mathscr{P}$ such that

$$h(z) = \frac{1+u(z)}{1-u(z)} = 1 + h_1 z + h_2 z^2 + h_3 z^3 + \cdots$$

and

$$q(w) = \frac{1+v(w)}{1-v(w)} = 1 + q_1 w + q_2 w^2 + q_3 w^3 + \cdots .$$

Then

$$\begin{aligned} p_{k,\, \alpha,\, \beta}\left(\frac{h(z)-1}{h(z)+1}\right) =& 1 + \frac{P_1 h_1 z}{2} + \left(\frac{P_1}{2}(h_2 - \frac{h_1^2}{2}) + \frac{P_2 h_1^2}{4}\right) z^2 \\ &+ \left(\frac{P_1}{2}\left(\frac{h_1^3}{4} - h_1 h_2 + h_3\right) + \frac{P_2}{4}(2h_1 h_2 - h_1^3) + \frac{P_3}{8} h_1^3\right) z^3 + \cdots \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} p_{k,\, \alpha,\, \beta}\left(\frac{q(w)-1}{q(w)+1}\right) =& 1 + \frac{P_1 q_1 w}{2} + \left(\frac{P_1}{2}(q_2 - \frac{q_1^2}{2}) + \frac{P_2 q_1^2}{4}\right) w^2 \\ &+ \left(\frac{P_1}{2}\left(\frac{q_1^3}{4} - q_1 q_2 + q_3\right) + \frac{P_2}{4}(2q_1 q_2 - q_1^3) + \frac{P_3}{8} q_1^3\right) w^3 + \cdots . \end{aligned} \tag{2.4}$$

In view of (2.3) and (2.4), the equations (2.1) and (2.2) become

$$1 + \frac{1}{b}\left(\frac{z\widetilde{D}_q f(z)}{f(z)} - 1\right) = p_{k,\, \alpha,\, \beta}\left(\frac{h(z)-1}{h(z)+1}\right) \tag{2.5}$$

and

$$1 + \frac{1}{b}\left(\frac{w\widetilde{D}_q g(w)}{g(w)} - 1\right) = p_{k,\, \alpha,\, \beta}\left(\frac{v(w)-1}{v(w)+1}\right). \tag{2.6}$$

Comparing the coefficients of like powers of z in the equations (2.7) and (2.8), we get

$$\frac{1}{b}\left(\widetilde{[2]}_q - 1\right) a_2 = \frac{P_1 h_1}{2}, \tag{2.7}$$

$$\frac{1}{b}\left[\left(\widetilde{[3]}_q - 1\right) a_3 - \left(\widetilde{[2]}_q - 1\right) a_2^2\right] = \frac{P_1}{2}\left(h_2 - \frac{h_1^2}{2}\right) + \frac{P_2 h_1^2}{4}, \tag{2.8}$$

and

$$\frac{-1}{b}\left(\widetilde{[2]}_q - 1\right) a_2 = \frac{P_1 q_1}{2}, \tag{2.9}$$

$$\frac{1}{b}\left[\left(\widetilde{[3]}_q - 1\right)(2a_2^2 - a_3) - \left(\widetilde{[2]}_q - 1\right) a_2^2\right] = \frac{P_1}{2}\left(q_2 - \frac{q_1^2}{2}\right) + \frac{P_2 q_1^2}{4}. \tag{2.10}$$

From the equations (2.7) and (2.9)

$$h_1 = -q_1. \tag{2.11}$$

Now, squaring and adding the equations (2.7) from (2.9), we get

$$h_1^2 + q_1^2 = \frac{8\left(\widetilde{[2]}_q - 1\right)^2 a_2^2}{P_1^2 b^2}. \tag{2.12}$$

Next, adding (2.8) and (2.10), use the equation (2.12), one can get

$$a_2^2 = \frac{P_1^3 (h_2 + q_2) b^2}{4\left[P_1^2 b\left(\widetilde{[3]}_q - \widetilde{[2]}_q\right) + (P_1 - P_2)\left(\widetilde{[2]}_q - 1\right)^2\right]}. \tag{2.13}$$

Subtract the equation (2.10) from (2.8),

$$a_3 = a_2^2 + \frac{bP_1(h_2 - q_2)}{4\left(\widetilde{[3]}_q - 1\right)}. \tag{2.14}$$

Then using the equation (2.12), we get

$$a_3 = \frac{P_1^2 b^2 (h_1^2 + q_1^2)}{8\left(\widetilde{[2]}_q - 1\right)^2} + \frac{bP_1(h_2 - q_2)}{4\left(\widetilde{[3]}_q - 1\right)}. \tag{2.15}$$

Using the equations (2.13) and (2.14), we get

$$a_3 - \mu a_2^2 = \frac{bP_1}{4\left(\widetilde{[3]}_q - 1\right)}\left[h_2(1 + s(\mu)) + q_2(-1 + s(\mu))\right], \tag{2.16}$$

where

$$s(\mu) = \frac{P_1^2 b(1 - \mu)}{\left[P_1^2 b\left(\widetilde{[3]}_q - \widetilde{[2]}_q\right) + (P_1 - P_2)\left(\widetilde{[2]}_q - 1\right)^2\right]}.$$

By applying the modulus for the equations (2.13), (2.15) and (2.16), we get the required results. □

Theorem 2.2. *If $f \in k - \mathscr{UCV}_{\Sigma,\, b}(\alpha,\ \beta)$ and is of the form* (1.1)*, then*

$$|a_2| \leq \frac{|P_1|\,|b|\,\sqrt{|P_1|}}{\sqrt{\left|\left(\widetilde{[3]}_q\left(\widetilde{[3]}_q-1\right)-\widetilde{[2]}_q^2\left(\widetilde{[2]}_q-1\right)\right)bP_1^2+\widetilde{[2]}_q^2\left(\widetilde{[2]}_q-1\right)^2(P_1-P_2)\right|}}$$

$$|a_3| \leq \frac{P_1^2 b^2}{\widetilde{[2]}_q^2\left(\widetilde{[2]}_q-1\right)^2}+\frac{\left|bP_1\right|}{\widetilde{[3]}_q\left(\widetilde{[3]}_q-1\right)}$$

and

$$|a_3-\mu a_2^2| \leq \begin{cases} \dfrac{|P_1 b|}{\widetilde{[3]}_q\left(\widetilde{[3]}_q-1\right)}, & \text{if } 0 \leq |s(\mu)| \leq 1 \\ \dfrac{|P_1 b s(\mu)|}{\widetilde{[3]}_q\left(\widetilde{[3]}_q-1\right)} & \text{if } |s(\mu)| \geq 1, \end{cases}$$

where

$$s(\mu)=\frac{P_1^2 b(1-\mu)}{4\left[(\widetilde{[3]}_q\left(\widetilde{[3]}_q-1\right)-\widetilde{[2]}_q^2\left(\widetilde{[2]}_q-1\right))bP_1^2+\widetilde{[2]}_q^2\left(\widetilde{[2]}_q-1\right)^2(P_1-P_2)\right]}.$$

Proof. If $f \in k - \mathscr{UCV}_{\Sigma,\, b}(\alpha,\ \beta)$ and g is an analytic extension of f^{-1} in Δ, then there exist two Schwarz functions $u,\ v \in \Delta$ such that

$$1+\frac{1}{b}\left(\frac{\widetilde{D}_q(z\widetilde{D}_q f(z))}{\widetilde{D}_q(f(z))}-1\right)=p_{k,\,\alpha,\,\beta}(u(z)), \tag{2.17}$$

and

$$1+\frac{1}{b}\left(\frac{\widetilde{D}_q(w\widetilde{D}_q g(w))}{\widetilde{D}_q(g(w))}-1\right)=p_{k,\,\alpha,\,\beta}(v(w)). \tag{2.18}$$

Then in view of (2.3) and (2.4) the equations (2.17) and (2.18) reduces to

$$1+\frac{1}{b}\left(\frac{\widetilde{D}_q(z\widetilde{D}_q f(z))}{\widetilde{D}_q(f(z))}-1\right)=p_{k,\,\alpha,\,\beta}\left(\frac{h(z)-1}{h(z)+1}\right), \tag{2.19}$$

and

$$1+\frac{1}{b}\left(\frac{\widetilde{D}_q(w\widetilde{D}_q g(w))}{\widetilde{D}_q(g(w))}-1\right)=p_{k,\,\alpha,\,\beta}\left(\frac{v(w)-1}{v(w)+1}\right). \tag{2.20}$$

Comparing the coefficients of similar powers of z in equations (2.19) and (2.20)

$$\frac{1}{b}\widetilde{[2]}_q\left(\widetilde{[2]}_q-1\right)a_2=\frac{P_1 h_1}{2}, \tag{2.21}$$

$$\frac{1}{b}\left[\widetilde{[3]}_q\left(\widetilde{[3]}_q-1\right)a_3-\widetilde{[2]}_q^2\left(\widetilde{[2]}_q-1\right)a_2^2\right]=\frac{P_1}{2}(h_2-\frac{h_1^2}{2})+\frac{P_2 h_1^2}{4}, \tag{2.22}$$

and

$$\frac{-1}{b}\widetilde{[2]}_q\left(\widetilde{[2]}_q-1\right)a_2=\frac{P_1q_1}{2}, \tag{2.23}$$

$$\frac{1}{b}(\widetilde{[3]}_q\left(\widetilde{[3]}_q-1\right)(2a_2^2-a_3)-\widetilde{[2]}_q^2\left(\widetilde{[2]}_q-1\right)a_2^2)=\frac{P_1}{2}\left(q_2-\frac{q_1^2}{2}\right)+\frac{P_2q_1^2}{4}. \tag{2.24}$$

From the equations (2.21) and (2.23), we get

$$h_1=-q_1. \tag{2.25}$$

Squaring and adding the equations (2.21) from (2.23), we get

$$h_1^2+q_1^2=\frac{8(\widetilde{[2]}_q)^2\left(\widetilde{[2]}_q-1\right)^2a_2^2}{P_1^2b^2}. \tag{2.26}$$

Adding (2.22) and (2.24), and using the equation (2.26), one can get

$$a_2^2=\frac{P_1^3(h_2+q_2)b^2}{4[(\widetilde{[3]}_q\left(\widetilde{[3]}_q-1\right)-\widetilde{[2]}_q^2\left(\widetilde{[2]}_q-1\right))bP_1^2+(\widetilde{[2]}_q)^2\left(\widetilde{[2]}_q-1\right)^2(P_1-P_2)]}. \tag{2.27}$$

Subtracting the equation (2.24) from (2.22), we get

$$a_3=a_2^2+\frac{bP_1(h_2-q_2)}{4(\widetilde{[3]}_q\left(\widetilde{[3]}_q-1\right)}. \tag{2.28}$$

Using the equation (2.26), we obtain

$$a_3=\frac{P_1^2b^2(h_1^2+q_1^2)}{8\widetilde{[2]}_q^2\left(\widetilde{[3]}_q-1\right)\left(\widetilde{[2]}_q-1\right)^2}+\frac{bP_1(h_2-q_2)}{4(\widetilde{[3]}_q\left(\widetilde{[3]}_q-1\right)}. \tag{2.29}$$

Then using the equations (2.27) and (2.28), we get

$$a_3-\mu a_2^2=\frac{bP_1}{4(\widetilde{[3]}_q\left(\widetilde{[3]}_q-1\right)}\Big[h_2(1+s(\mu))+q_2(-1+s(\mu))\Big], \tag{2.30}$$

where

$$s(\mu)=\frac{bP_1^2(1-\mu)}{4[(\widetilde{[3]}_q\left(\widetilde{[3]}_q-1\right)-\widetilde{[2]}_q^2\left(\widetilde{[2]}_q-1\right)^2bP_1^2+\widetilde{[2]}_q^2\left(\widetilde{[2]}_q-1\right)^2(P_1-P_2)]}.$$

By applying modulus for the equations (2.27), (2.29) and (2.30) on both sides we get the required results. □

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Sălăgean-type harmonic multivalent functions defined by q-difference operator

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Abstract. We introduce a new subclass of Sălăgean-type harmonic multivalent functions by using $q-$difference operator. We investigate sufficient coefficient estimates, distortion bounds, extreme points, convolution properties and neighborhood for the functions belonging to this function class.

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1. Introduction

The study of harmonic functions which are multivalent in the open unit disc

$$\mathbb{D} = \{z : |z| < 1\}$$

was initiated by Duren, Hengartner and Laugesen [4]. Let $\mathcal{H}(m), (m \geq 1)$ be the class of harmonic multivalent and sense-preserving functions $f = h + \overline{g}$, where h and g have the following power series

$$h(z) = z^m + \sum_{n=2}^{\infty} a_{n+m-1} z^{n+m-1}, \; g(z) = \sum_{n=1}^{\infty} b_{n+m-1} z^{n+m-1}, \quad |b_m| < 1 \tag{1.1}$$

that are analytic and $m-$valent in $\mathbb{D}$. The class $\mathcal{H}(1)$ of harmonic univalent functions was studied by Clunie and Sheil-Small [3]. For more details of harmonic multivalent functions, one may refer to [2] and [6].

Jackson [7, 8] in 1909-1910 developed quantum calculus, popularly known as $q-$calculus. Since then it has found applications in physics, quantum mechanics, analytic number theory, Sobolev spaces, representation theory of groups, theta functions, gamma functions, operator theory, and more recently in geometric function theory. For definitions, properties and references of $q-$calculus one may refer to [1].

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In fact, $q-$calculus methodology is centered on the idea of deriving $q-$analogues results without the use of limits. Let us first recall certain notations and definitions of the $q-$calculus.

Definition 1.1. Let $q \in (0,1)$. The $q-$derivative (or $q-$difference operator) of a function f, defined on a subset Ω with $0 \in \Omega$ of $\mathbb{C}$, is given by

$$(D_q f)(z) = \begin{cases} \frac{f(z)-f(qz)}{(1-q)z}, & z \neq 0 \\ \\ f'(0), & z = 0. \end{cases}$$

We note that $\lim_{q\to 1^-}(D_q f)(z) = f'(z)$ if f is differentiable at z.

For the function $f(z) = z^n$, we observe that

$$D_q z^n = [n]_q z^{n-1},$$

where $[n]_q = \frac{1-q^n}{1-q}$. Therefore, if $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is analytic in $\mathbb{D}$, then

$$(D_q f)(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}.$$

Clearly, for $q \to 1^-$, $[n]_q \to n$. For the definitions and properties of $q-$derivative and $q-$calculus, one may refer to [1, 5, 7, 8].

The $q-$Sălăgean differential operator of a $m-$valent function h given in (1.1) is formed by

$$\begin{aligned} L_q^0 h(z) &= h(z) \\ L_q^1 h(z) &= \frac{zD_q(h(z))}{[m]_q} \\ &\vdots \\ L_q^k h(z) &= L_q(L_q^{k-1}h(z)). \end{aligned}$$

Then

$$L_q^k h(z) = z^m + \sum_{n=2}^{\infty} \left(\frac{[n+m-1]_q}{[m]_q} \right)^k a_{n+m-1} z^{n+m-1}, \tag{1.2}$$

where $[n+m-1]_q^k = (\frac{1-q^{n+m-1}}{1-q})^k$, $q \in (0,1)$, $k = 0, 1, \ldots$. Clearly, when $q \to 1^-$ and $m = 1$, the equation (1.2) reduces to Sălăgean differential operator (see [12]).

Making use of (1.1) and (1.2), we define the $q-$Sălăgean differential operator for harmonic multivalent function $L_q^k f(z) : \mathcal{H}(m) \to \mathcal{H}(m)$ by

$$L_q^k f(z) = L_q^k h(z) + (-1)^k \overline{L_q^k g(z)}, \tag{1.3}$$

where $L_q^k h(z)$ is given by (1.2) and

$$L_q^k g(z) = \sum_{n=1}^{\infty} \left(\frac{[n+m-1]_q}{[m]_q} \right)^k b_{n+m-1} z^{n+m-1}.$$

When $q \to 1^-$ the equation (1.3) reduces to Sălăgean differential operator for multivalent harmonic functions given in [11]. Motivated by definition of $q-$Sălăgean differential operator for harmonic multivalent functions, we create the following class.

Definition 1.2. For $0 \le \lambda \le 1$, $0 \le \alpha < 1$, $q \in (0,1)$ and $z \in \mathbb{D}$, a function $f \in \mathcal{H}(m)$, $(m \ge 1)$ is said to belong to the class $\mathcal{H}_q(m,k,\lambda,\alpha)$ if

$$\mathrm{Re}\left(\frac{L_q^{k+1}f(z)}{(1-\lambda)z^m+\lambda L_q^k f(z)}\right) \ge \alpha, \tag{1.4}$$

where $L_q^k f(z), (k=0,1,...)$ is defined by (1.3). A function f in this class is called $q-$Sălăgean-type harmonic multivalent function of order α.

For special values of parameters q, m, k, λ and α, we obtain several new and known subclasses as special cases; for example:

(i) If $k=0$, we get a new subclass $\mathcal{H}_q(m,\lambda,\alpha)$ as below

$$\mathrm{Re}\left(\frac{zD_qf(z)}{(1-\lambda)z^m+\lambda f(z)}\right) \ge \alpha.$$

(ii) If $k=0$, $m=1$, we get a new subclass $\mathcal{H}_q(\lambda,\alpha)$ as below

$$\mathrm{Re}\left(\frac{zD_qf(z)}{(1-\lambda)z+\lambda f(z)}\right) \ge \alpha.$$

(iii) If $k=0$, $m=1$, $q \to 1^-$, we get a known class $\mathcal{S}^*_{\mathcal{H}}(\lambda,\alpha)$ defined in [13] in the following

$$\mathrm{Re}\left(\frac{zf'(z)}{(1-\lambda)z+\lambda f(z)}\right) \ge \alpha.$$

(iv) If $k=0$, $m=1$, $\lambda=1$, $q \to 1^-$, we get a known class $\mathcal{S}^*_{\mathcal{H}}(\alpha)$ defined in [9] in the following

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) \ge \alpha.$$

(v) If $m=1$, $\lambda=1$, $q \to 1^-$, we get a known class $\mathcal{S}_{\mathcal{H}}(k,\alpha)$ defined in [10] in the following

$$\mathrm{Re}\left(\frac{L^{k+1}f(z)}{L^k f(z)}\right) \ge \alpha.$$

We also introduce a new subclass of $q-$Sălăgean-type harmonic multivalent functions using negative coefficients. Let $\mathcal{TH}_q(m,k,\lambda,\alpha)$ denote a subclass of $\mathcal{H}(m)$ that consists of harmonic functions $f = h + \overline{g}$ so that h and g are of the form

$$h(z) = z^m - \sum_{n=2}^{\infty} |a_{n+m-1}| z^{n+m-1}, \; g(z) = (-1)^k \sum_{n=1}^{\infty} |b_{n+m-1}| z^{n+m-1}, \; |b_m| < 1. \tag{1.5}$$

In Section 2, we first obtain coefficient characterization for our main class. Using this characterization, we obtain distortion and covering theorems. Finally, we obtain extreme points, convolution properties and neighborhood results for our class.

2. Main results

We first obtain two lemmas that we need for proving other results.

Lemma 2.1. *Let $0 \leq \lambda \leq 1$, $0 \leq \alpha < 1$, $q \in (0,1)$, $z \in \mathbb{D}$, and $f = h + \overline{g}$ with h and g of the form (1.1). If*

$$\sum_{n=2}^{\infty} \Omega_q(m,k,\lambda,\alpha)|a_{n+m-1}| + \sum_{n=1}^{\infty} \Psi_q(m,k,\lambda,\alpha)|b_{n+m-1}| \leq 1-\alpha, \tag{2.1}$$

where

$$\Omega_q(m,k,\lambda,\alpha) = \left(\frac{[n+m-1]_q}{[m]_q}\right)^k \left[\frac{[n+m-1]_q}{[m]_q} - \alpha\lambda\right] \tag{2.2}$$

and

$$\Psi_q(m,k,\lambda,\alpha) = \left(\frac{[n+m-1]_q}{[m]_q}\right)^k \left[\frac{[n+m-1]_q}{[m]_q} + \alpha\lambda\right], \tag{2.3}$$

then $f \in \mathcal{H}_q(m,k,\lambda,\alpha)$.

Proof. In view of (1.4), and using the fact that $Re(w) \geq \alpha$ if and only if

$$|1-\alpha+w| > |1+\alpha-w|,$$

it suffices to show that

$$\left|1-\alpha+\frac{L_q^{k+1}f(z)}{(1-\lambda)z^m+\lambda L_q^k f(z)}\right| - \left|1+\alpha-\frac{L_q^{k+1}f(z)}{(1-\lambda)z^m+\lambda L_q^k f(z)}\right| \geq 0.$$

We observe that left side of this inequality

$$\begin{aligned}
&= \left|L_q^{k+1}f(z)+(1-\alpha)[(1-\lambda)z^m+\lambda L_q^k f(z)]\right| \\
&\quad -\left|L_q^{k+1}f(z)-(1+\alpha)[(1-\lambda)z^m+\lambda L_q^k f(z)]\right|
\end{aligned}$$

$$\begin{aligned}
=& \left|(2-\alpha)z^m + \sum_{n=2}^{\infty}\left(\frac{[n+m-1]_q}{[m]_q}\right)^k \left[\frac{[n+m-1]_q}{[m]_q}+(1-\alpha)\lambda\right] a_{n+m-1}z^{n+m-1}\right. \\
&\left. -(-1)^k \sum_{n=1}^{\infty}\left(\frac{[n+m-1]_q}{[m]_q}\right)^k \left[\frac{[n+m-1]_q}{[m]_q}-(1-\alpha)\lambda\right] \overline{b_{n+m-1}z^{n+m-1}}\right| \\
&-\left| -\alpha z^m + \sum_{n=2}^{\infty}\left(\frac{[n+m-1]_q}{[m]_q}\right)^k \left[\frac{[n+m-1]_q}{[m]_q}-(1+\alpha)\lambda\right] a_{n+m-1}z^{n+m-1}\right. \\
&\left. -(-1)^k \sum_{n=1}^{\infty}\left(\frac{[n+m-1]_q}{[m]_q}\right)^k \left[\frac{[n+m-1]_q}{[m]_q}+(1+\alpha)\lambda\right] \overline{b_{n+m-1}z^{n+m-1}}\right| \\
\geq& \, 2(1-\alpha)|z|^m - \sum_{n=2}^{\infty}\left(\frac{[n+m-1]_q}{[m]_q}\right)^k \left[\frac{[n+m-1]_q}{[m]_q}+(1-\alpha)\lambda\right] |a_{n+m-1}||z|^{n+m-1} \\
&- \sum_{n=1}^{\infty}\left(\frac{[n+m-1]_q}{[m]_q}\right)^k \left[\frac{[n+m-1]_q}{[m]_q}-(1-\alpha)\lambda\right] |b_{n+m-1}||z|^{n+m-1} \\
&-\alpha|z|^m - \sum_{n=2}^{\infty}\left(\frac{[n+m-1]_q}{[m]_q}\right)^k \left[\frac{[n+m-1]_q}{[m]_q}-(1+\alpha)\lambda\right] |a_{n+m-1}||z|^{n+m-1}
\end{aligned}$$

$$-\sum_{n=1}^{\infty}\left(\frac{[n+m-1]_q}{[m]_q}\right)^k\left[\frac{[n+m-1]_q}{[m]_q}+(1+\alpha)\lambda\right]|b_{n+m-1}||z|^{n+m-1}$$

$$\geq 2(1-\alpha)|z|^m-2\sum_{n=2}^{\infty}\left(\frac{[n+m-1]_q}{[m]_q}\right)^k\left[\frac{[n+m-1]_q}{[m]_q}-\alpha\lambda\right]|a_{n+m-1}||z|^{n+m-1}$$

$$-2\sum_{n=1}^{\infty}\left(\frac{[n+m-1]_q}{[m]_q}\right)^k\left[\frac{[n+m-1]_q}{[m]_q}+\alpha\lambda\right]|b_{n+m-1}||z|^{n+m-1}$$

$$\geq 2(1-\alpha)\Bigg\{1-\sum_{n=2}^{\infty}\frac{\left(\frac{[n+m-1]_q}{[m]_q}\right)^k\left[\frac{[n+m-1]_q}{[m]_q}-\alpha\lambda\right]}{1-\alpha}|a_{n+m-1}|$$

$$-\sum_{n=1}^{\infty}\frac{\left(\frac{[n+m-1]_q}{[m]_q}\right)^k\left[\frac{[n+m-1]_q}{[m]_q}+\alpha\lambda\right]}{1-\alpha}|b_{n+m-1}|\Bigg\}\geq 0,$$

by (2.1). This completes the proof. □

The $q-$Sălăgean-type harmonic multivalent functions

$$f(z)=z^m+\sum_{n=2}^{\infty}\frac{1-\alpha}{\Omega_q(m,k,\lambda,\alpha)}x_{n+m-1}z^{n+m-1}+\sum_{n=1}^{\infty}\frac{1-\alpha}{\Psi_q(m,k,\lambda,\alpha)}\overline{y_{n+m-1}z^{n+m-1}},$$

where

$$\sum_{n=2}^{\infty}|x_{n+m-1}|+\sum_{n=1}^{\infty}|y_{n+m-1}|=1$$

shows that the coefficient bound given by (2.1) is sharp.

We now show that the condition (2.1) is also necessary for functions $f=h+\overline{g}$, where h and g are of the form (1.5)

Lemma 2.2. *Let $0\leq\lambda\leq 1$, $0\leq\alpha<1$, $q\in(0,1)$, $z\in\mathbb{D}$, and $f=h+\overline{g}$ with h and g of the form (1.5). Then $f\in\mathcal{TH}_q(m,k,\lambda,\alpha)$ if and only if*

$$\sum_{n=2}^{\infty}\Omega_q(m,k,\lambda,\alpha)|a_{n+m-1}|+\sum_{n=1}^{\infty}\Psi_q(m,k,\lambda,\alpha)|b_{n+m-1}|\leq 1-\alpha, \tag{2.4}$$

where $\Omega_q(m,k,\lambda,\alpha)$ and $\Psi_q(m,k,\lambda,\alpha)$ are, respectively, given by (2.2) and (2.3).

Proof. Since $\mathcal{TH}_q(m,k,\lambda,\alpha)\subset\mathcal{H}_q(m,k,\lambda,\alpha)$, we only need to prove the "only if" part of this theorem. Let $f\in\mathcal{TH}_q(m,k,\lambda,\alpha)$, then it satisfies (1.4) or equivalently

$$\mathrm{Re}\Bigg\{\frac{(1-\alpha)z^m-\sum_{n=2}^{\infty}\Omega_q(m,k,\lambda,\alpha)|a_{n+m-1}|z^{n+m-1}}{\Theta_q(m,k,\lambda,\alpha)}$$

$$+\frac{(-1)^{2k-1}\sum_{n=2}^{\infty}\Psi_q(m,k,\lambda,\alpha)|b_{n+m-1}|\overline{z}^{n+m-1}}{\Theta_q(m,k,\lambda,\alpha)}\Bigg\}\geq 0,$$

where

$$\Theta_q(z) = z^m - \lambda \sum_{n=2}^{\infty} \left(\frac{[n+m-1]_q}{[m]_q} \right)^k |a_{n+m-1}| z^{n+m-1} + \lambda(-1)^{2k} \sum_{n=1}^{\infty} \left(\frac{[n+m-1]_q}{[m]_q} \right)^k |b_{n+m-1}| \overline{z}^{n+m-1}.$$

The above required condition must hold for all values of $z \in \mathbb{D}$, $|z| = r < 1$. By choosing the values of z on the positive real axis where $0 \leq z = r < 1$, we must have

$$\frac{(1-\alpha) - \sum_{n=2}^{\infty} \Omega_q(m,k,\lambda,\alpha)|a_{n+m-1}|r^{n-1}}{1 - \lambda \sum_{n=2}^{\infty} \left(\frac{[n+m-1]_q}{[m]_q} \right)^k |a_{n+m-1}|r^{n-1} + \lambda \sum_{n=1}^{\infty} \left(\frac{[n+m-1]_q}{[m]_q} \right)^k |b_{n+m-1}|r^{n-1}}$$
$$- \frac{\sum_{n=2}^{\infty} \psi_q(m,k,\lambda,\alpha)|b_{n+m-1}|r^{n-1}}{1 - \lambda \sum_{n=2}^{\infty} \left(\frac{[n+m-1]_q}{[m]_q} \right)^k |a_{n+m-1}|r^{n-1} + \lambda \sum_{n=1}^{\infty} \left(\frac{[n+m-1]_q}{[m]_q} \right)^k |b_{n+m-1}|r^{n-1}} \geq 0. \tag{2.5}$$

If the condition (2.4) does not hold, then the numerator in (2.5) is negative for r sufficiently close to 1. Thus there exists a point $z_0 = r_0$ in $(0,1)$ for which the quotient in (2.5) is negative. This contradicts the required condition for $f \in \mathcal{TH}_q(m,k,\lambda,\alpha)$ and so the proof is complete. □

We now obtain distortion bounds of the class $\mathcal{TH}_q(m,k,\lambda,\alpha)$.

Theorem 2.3. *If a function f belongs to the class $\mathcal{TH}_q(m,k,\lambda,\alpha)$, then we have*

$$|f(z)| \leq (1+|b_m|)r^m + \frac{1-\alpha\lambda}{\theta_q(m,k,\lambda,\alpha)} \left(1 - \frac{1+\alpha\lambda}{1-\alpha\lambda}|b_m| \right) r^{m+1} \tag{2.6}$$

and

$$|f(z)| \geq (1-|b_m|)r^m - \frac{1-\alpha\lambda}{\theta_q(m,k,\lambda,\alpha)} \left(1 - \frac{1+\alpha\lambda}{1-\alpha\lambda}|b_m| \right) r^{m+1}, \tag{2.7}$$

where

$$\theta_q(m,k,\lambda,\alpha) = \left(\frac{[m+1]_q}{[m]_q} \right)^k \left[\frac{[m+1]_q}{[m]_q} - \alpha\lambda \right],$$

and for all $z \in \mathbb{D}$.

Proof. Let $f \in \mathcal{TH}_q(m,k,\lambda,\alpha)$. Taking the absolute value of f and using Lemma 2.2, we obtain

$$
\begin{aligned}
|f(z)| &\le (1+|b_m|)r^m + \sum_{n=2}^{\infty}(|a_{n+m-1}|+|b_{n+m-1}|)r^{n+m-1}\\
&\le (1+|b_m|)r^m + \sum_{n=2}^{\infty}(|a_{n+m-1}|+|b_{n+m-1}|)r^{m+1}\\
&\le (1+|b_m|)r^m + \frac{1-\alpha}{\theta_q(m,k,\lambda,\alpha)}\\
&\quad \times \sum_{n=2}^{\infty}\frac{\theta_q(m,k,\lambda,\alpha)}{1-\alpha}(|a_{n+m-1}|+|b_{n+m-1}|)r^{m+1}\\
&\le (1+|b_m|)r^m + \frac{1-\alpha}{\theta_q(m,k,\lambda,\alpha)}\\
&\quad \times \sum_{n=2}^{\infty}\left\{\frac{\Omega_q(m,k,\lambda,\alpha)}{1-\alpha}|a_{n+m-1}|+\frac{\Psi_q(m,k,\lambda,\alpha)}{1-\alpha}|b_{n+m-1}|\right\}r^{m+1}\\
&\le (1+|b_m|)r^m + \left\{\frac{1-\alpha\lambda}{\theta_q(m,k,\lambda,\alpha)}-\frac{1+\alpha\lambda}{\theta_q(m,k,\lambda,\alpha)}|b_m|\right\}r^{m+1}.
\end{aligned}
$$

The proof of the inequality (2.7) is similar to the proof of (2.6) and is omitted. □

The following covering result follows from the inequality (2.7) by letting r approaches to 1.

Corollary 2.4. *If $f \in \mathcal{TH}_q(m,k,\lambda,\alpha)$, then*

$$\left\{w : |w| < \left(1-\frac{1-\alpha\lambda}{\theta_q(m,k,\lambda,\alpha)}\right)-\left(1-\frac{1+\alpha\lambda}{\theta_q(m,k,\lambda,\alpha)}\right)b_m\right\} \subset f(\mathbb{D}).$$

Next, we give the extreme points of this class.

Theorem 2.5. *Let $f = h + \overline{g}$ be given by (1.5). Then $f \in clco\mathcal{TH}_q(m,k,\lambda,\alpha)$ if and only if*

$$f(z) = \sum_{n=1}^{\infty}(x_{n+m-1}h_{n+m-1}(z)+y_{n+m-1}g_{n+m-1}(z)),$$

where

$$h_m(z) = z^m, \quad h_{n+m-1}(z) = z^m - \frac{1-\alpha}{\Omega_q(m,k,\lambda,\alpha)}z^{n+m-1},\ (n \ge 2)$$

$$g_{n+m-1}(z) = z^m + (-1)^k\frac{1-\alpha}{\Psi_q(m,k,\lambda,\alpha)}\overline{z}^{n+m-1},\ (n \ge 1)$$

and

$$\sum_{n=1}^{\infty}(x_{n+m-1}+y_{n+m-1}) = 1,$$

where $x_{n+m-1} \ge 0$ and $y_{n+m-1} \ge 0$. In particular, the extreme points of $\mathcal{TH}_q(m,k,\lambda,\alpha)$ are $\{h_{n+m-1}\}$ and $\{g_{n+m-1}\}$.

Proof. For a function f of the form

$$f(z)=\sum_{n=1}^{\infty}(x_{n+m-1}h_{n+m-1}(z)+y_{n+m-1}g_{n+m-1}(z)),$$

where $\sum_{n=1}^{\infty}(x_{n+m-1}+y_{n+m-1})=1$, we have

$$f(z)=\sum_{n=1}^{\infty}(x_{n+m-1}+y_{n+m-1})z^m-\sum_{n=2}^{\infty}\frac{1-\alpha}{\Omega_q(m,k,\lambda,\alpha)}x_{n+m-1}z^{n+m-1}$$
$$+(-1)^k\sum_{n=1}^{\infty}\frac{1-\alpha}{\Psi_q(m,k,\lambda,\alpha)}y_{n+m-1}\overline{z}^{n+m-1}.$$

Then, $f\in clco\mathcal{TH}_q(m,k,\lambda,\alpha)$ because

$$\sum_{n=2}^{\infty}\frac{\Omega_q(m,k,\lambda,\alpha)}{1-\alpha}|a_{n+m-1}|+\sum_{n=1}^{\infty}\frac{\Psi_q(m,k,\lambda,\alpha)}{1-\alpha}|b_{n+m-1}|$$
$$=\sum_{n=2}^{\infty}x_{n+m-1}+\sum_{n=1}^{\infty}y_{n+m-1}=1-x_m\leq 1.$$

Conversely, suppose $f\in clco\mathcal{TH}_q(m,k,\lambda,\alpha)$. Then, by Lemma 2.2

$$|a_{n+m-1}|\leq\frac{1-\alpha}{\Omega_q(m,k,\lambda,\alpha)}$$

and

$$|b_{n+m-1}|\leq\frac{1-\alpha}{\Psi_q(m,k,\lambda,\alpha)}.$$

Putting

$$x_{n+m-1}=\frac{\Omega_q(m,k,\lambda,\alpha)}{1-\alpha}|a_{n+m-1}|,$$
$$y_{n+m-1}=\frac{\Psi_q(m,k,\lambda,\alpha)}{1-\alpha}|b_{n+m-1}|,$$

and $x_m=1-\sum_{n=2}^{\infty}x_{n+m-1}-\sum_{n=1}^{\infty}y_{n+m-1}\geq 0$, we obtain

$$\begin{aligned}f(z)&=z^m-\sum_{n=2}^{\infty}a_{n+m-1}z^{n+m-1}+(-1)^k\sum_{n=1}^{\infty}\overline{b_{n+m-1}z^{n+m-1}}\\&=z^m-\sum_{n=2}^{\infty}\frac{1-\alpha}{\Omega_q(m,k,\lambda,\alpha)}x_{n+m-1}z^{n+m-1}\\&\quad+(-1)^k\sum_{n=1}^{\infty}\frac{1-\alpha}{\Psi_q(m,k,\lambda,\alpha)}\overline{y_{n+m-1}z^{n+m-1}}\\&=z^m+\sum_{n=1}^{\infty}(h_{n+m-1}(z)-z^m)x_{n+m-1}+\sum_{n=1}^{\infty}(g_{n+m-1}(z)-z^m)y_{n+m-1}.\end{aligned}$$

Consequently, we obtain $f(z) = \sum_{n=1}^{\infty}(x_{n+m-1}h_{n+m-1}(z) + y_{n+m-1}g_{n+m-1}(z))$ as required. $\square$

Using definition of convolution and Lemma 2.2, we show that the class $\mathcal{TH}_q(m,k,\lambda,\alpha)$ is closed under convolution. Recall that the convolution of two complex-valued harmonic multivalent functions

$$f(z) = z^m - \sum_{n=2}^{\infty}|a_{n+m-1}|z^{n+m-1} + (-1)^k\sum_{n=1}^{\infty}|b_{n+m-1}|\overline{z}^{n+m-1}$$

and

$$F(z) = z^m - \sum_{n=2}^{\infty}|A_{n+m-1}|z^{n+m-1} + (-1)^k\sum_{n=1}^{\infty}|B_{n+m-1}|\overline{z}^{n+m-1}$$

is defined by

$$(f * F)(z) = z^m + \sum_{n=2}^{\infty}|a_{n+m-1}||A_{n+m-1}|z^{n+m-1}$$
$$+(-1)^k\sum_{n=1}^{\infty}|b_{n+m-1}||B_{n+m-1}|\overline{z}^{n+m-1}.$$

Theorem 2.6. *For $0 \le \beta \le \alpha < 1$, suppose*

$$f \in \mathcal{TH}_q(m,k,\lambda,\alpha) \text{ and } F \in \mathcal{TH}_q(m,k,\lambda,\beta).$$

Then

$$f * F \in \mathcal{TH}_q(m,k,\lambda,\alpha) \subset \mathcal{TH}_q(m,k,\lambda,\beta).$$

Proof. Let

$$f(z) = z^m - \sum_{n=2}^{\infty}|a_{n+m-1}|z^{n+m-1} + (-1)^k\sum_{n=1}^{\infty}|b_{n+m-1}|\overline{z}^{n+m-1}$$

be in $\mathcal{TH}_q(m,k,\lambda,\alpha)$ and

$$F(z) = z^m - \sum_{n=2}^{\infty}|A_{n+m-1}|z^{n+m-1} + (-1)^k\sum_{n=1}^{\infty}|B_{n+m-1}|\overline{z}^{n+m-1},$$

be in $\mathcal{TH}_q(m,k,\lambda,\beta)$. Since $F \in \mathcal{TH}_q(m,k,\lambda,\beta)$, we note that $|A_{n+m-1}| \le 1$ and $|B_{n+m-1}| \le 1$. We want to show that if $f * F$ satisfy the condition given in Lemma 2.2, then

$$\sum_{n=2}^{\infty}\frac{\Omega_q(m,k,\lambda,\alpha)}{1-\alpha}|a_{n+m-1}||A_{n+m-1}| + \sum_{n=1}^{\infty}\frac{\Psi_q(m,k,\lambda,\alpha)}{1-\alpha}|b_{n+m-1}||B_{n+m-1}|$$
$$\le \sum_{n=2}^{\infty}\frac{\Omega_q(m,k,\lambda,\alpha)}{1-\alpha}|a_{n+m-1}| + \sum_{n=1}^{\infty}\frac{\Psi_q(m,k,\lambda,\alpha)}{1-\alpha}|b_{n+m-1}| \le 1.$$

In view of Lemma 2.2, it follows that $f * F \in \mathcal{TH}_q(m,k,\lambda,\alpha) \subset \mathcal{TH}_q(m,k,\lambda,\beta)$. $\square$

Finally, we define $q-\delta-$neighborhood and then investigate a containment property. The $q-\delta-$neighborhood of a function $f = h + \overline{g}$ in $\mathcal{H}_q(m, k, \lambda, \alpha)$ is defined as the set:

$$N_{q,\delta}(f) = \Big\{F(z) = z^m + B_m\overline{z}^m + \sum_{n=2}^{\infty}(A_{n+m-1}z^{n+m-1} + B_{n+m-1}\overline{z}^{n+m-1}) : \sum_{n=2}^{\infty}\left(\frac{[n+m-1]_q}{[m]_q}\right)^k\Big\{\left[\frac{[n+m-1]_q}{[m]_q} - \alpha\lambda\right]|a_{n+m-1} - A_{n+m-1}| + \left[\frac{[n+m-1]_q}{[m]_q} + \alpha\lambda\right]|b_{n+m-1} - B_{n+m-1}|\Big\} + (1+\alpha\lambda)|b_m - B_m| \le (1-\alpha)\delta,\ \delta > 0\Big\}.$$

Theorem 2.7. *If f given by (1.1) satisfies the condition (2.1) and*

$$\delta \le \left[1 - \frac{1}{\frac{[m+1]_q}{[m]_q} - \alpha\lambda}\right]\left(1 - \frac{1+\alpha\lambda}{1-\alpha}|b_m|\right), \tag{2.8}$$

then $N_{q,\delta}(f) \subset \mathcal{H}_q(m, k, \lambda, \alpha)$.

Proof. For any $f \in \mathcal{H}_q(m, k, \lambda, \alpha)$, suppose

$$F(z) = z^m + B_m\overline{z}^m + \sum_{n=2}^{\infty}(A_{n+m-1}z^{n+m-1} + B_{n+m-1}\overline{z}^{n+m-1})$$

belongs to $N_{q,\delta}(f)$. Then we have

$$\begin{aligned}
&(1+\alpha\lambda)|B_m| + \sum_{n=2}^{\infty}\left(\frac{[n+m-1]_q}{[m]_q}\right)^k\Big\{\left[\frac{[n+m-1]_q}{[m]_q} - \alpha\lambda\right]|A_{n+m-1}| + \\
&\quad\left[\frac{[n+m-1]_q}{[m]_q} + \alpha\lambda\right]|B_{n+m-1}|\Big\} \\
&\le (1+\alpha\lambda)|B_m - b_m| + (1+\alpha\lambda)|b_m| + \sum_{n=2}^{\infty}\left(\frac{[n+m-1]_q}{[m]_q}\right)^k \\
&\quad\Big\{\left[\frac{[n+m-1]_q}{[m]_q} - \alpha\lambda\right]|A_{n+m-1} - a_{n+m-1}| \\
&\quad + \left[\frac{[n+m-1]_q}{[m]_q} + \alpha\lambda\right]|B_{n+m-1} - b_{n+m-1}|\Big\} \\
&\quad + \sum_{n=2}^{\infty}\left(\frac{[n+m-1]_q}{[m]_q}\right)^k\Big\{\left[\frac{[n+m-1]_q}{[m]_q} - \alpha\lambda\right]|a_{n+m-1}| \\
&\quad + \left[\frac{[n+m-1]_q}{[m]_q} + \alpha\lambda\right]|b_{n+m-1}|\Big\} \\
&\le (1-\alpha)\delta + (1+\alpha\lambda)|b_m| + \frac{1}{\frac{[m+1]_q}{[m]_q} - \alpha\lambda}[(1-\alpha) - (1+\alpha\lambda)b_m] \le 1-\alpha,
\end{aligned} \tag{2.9}$$

by given condition (2.8). Therefore, it follows that $F \in \mathcal{H}_q(m, k, \lambda, \alpha)$. This completes the proof this theorem. □

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On the existence of positive solutions of a class of parabolic reaction diffusion systems

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Abstract. In this paper, we show the existence of continuous positive solutions of a class of nonlinear parabolic reaction diffusion systems with initial conditions using techniques of functional analysis and potential analysis.

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Keywords: Reaction diffusion systems, parabolic systems, positive solutions, Green function.

1. Introduction

The modeling and the mathematical analysis of parabolic systems, in particular, reaction diffusion systems, has been the subject of in-depth studies of several mathematicians in recent years, as they appear in the modeling of a large variety of phenomena, not only in biology and chemistry, but also in engineering, economics and ecology, such as gas dynamics, fusion processes, cellular processes, disease propagation, industrial processes , catalytic transport of contaminants in the environment, population dynamics, flame spread and others.

For the mathematical analysis of this type of problem, various methods and elaborate techniques have been proposed, see for example Mesbahi et *al.* [1], [2], [16], [15], Gontara [9], Lions [10], Maâgli et *al.* [13], [12], Pierre [17] and Zhang [20], [19]. We refer the reader to Arakelian and Gauthier [3], Armitage and Gardiner [4] and Port [18] for more details on the potential arguments of the theory that interest us mainly in this work.

The subject of this paper is in this context, we will take care to study the existence of positive solutions of the following nonlinear parabolic reaction diffusion

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system

$$\begin{cases} -\dfrac{\partial u}{\partial t} + \Delta u = \lambda p(x,t) f(v) \\ -\dfrac{\partial v}{\partial t} + \Delta v = \mu q(x,t) g(w) \\ -\dfrac{\partial w}{\partial t} + \Delta w = \eta r(x,t) h(z) \\ -\dfrac{\partial z}{\partial t} + \Delta z = \varrho e(x,t) k(u) \end{cases} \tag{1.1}$$

with $(x,t) \in \mathbb{R}^n \times (0,\infty)$ and the initial conditions

$$\begin{cases} u(x,0) = \varphi(x) \quad , \quad v(x,0) = \psi(x) \\ w(x,0) = \gamma(x) \quad , \quad z(x,0) = \zeta(x) \end{cases} , \ \forall x \in \mathbb{R}^n \tag{1.2}$$

where $n \geq 3$, φ, ψ, γ and $\zeta : \mathbb{R}^n \rightarrow [0,\infty)$ are continuous, the constants λ, μ, η and ϱ are nonnegative, f, g, h and $k : (0,\infty) \rightarrow [0,\infty)$ are nondecreasing and continuous. p, q, r and $e : \mathbb{R}^n \times (0,\infty) \rightarrow [0,\infty)$ are measurable functions and satisfy an appropriate hypotheses related to the parabolic Kato class $P^\infty(\mathbb{R}^n)$ introduced in Zhang [19].

Before stating the main result of this work, it is worth mentioning that several mathematicians have dealt with this type of problem using various analytical and numerical techniques and methods, under different hypotheses as appropriate, see for example, Bachar et *al.* [5], Maâgli et *al.* [7], [6], [13]-[14], Ghergu and Radulescu [8], Gontara [9], Ma [11], Zhang [20], [19] and Zhao [21].

Concerning the problem $(1.1)-(1.2)$ in the case of a single equation of the form

$$\begin{cases} \Delta u - \dfrac{\partial u}{\partial t} = q(x,t) u^{p+1} \quad , \quad \mathbb{R}^n \times (0,\infty) \\ u(x,0) = u_0(x) \quad , \quad x \in \mathbb{R}^n, \ n \geq 3 \end{cases} \tag{1.3}$$

Zhang in [20] discussed the existence and the asymptotic behavior of solutions to this problem, he proved the following result:

Theorem 1.1. *Suppose* $p > 0$, $q \in P^\infty(\mathbb{R}^n)$. *For any* $M > 1$, *there is a constant* $b_0 > 0$ *such that for each nonnegative* $u_0 \in C^2(\mathbb{R}^n)$ *satisfying* $\|u_0\|_{L^\infty(\mathbb{R}^n)} \leq b_0$, *there exists a positive and continuous solution* u *of* (1.3) *such that*

$$M^{-1} \int_{\mathbb{R}^n} G(x,t,y,0) u_0(y) dy \leq u(x,t) \leq M \int_{\mathbb{R}^n} G(x,t,y,0) u_0(y) dy$$

for all $(x,t) \in \mathbb{R}^n \times (0,\infty)$.

G denotes the fundamental solution of the heat equation $\Delta u - \dfrac{\partial u}{\partial t} = 0$ in $\mathbb{R}^n \times (0,\infty)$ given for $t > s$ and $x, \ y \in \mathbb{R}^n$ by

$$G(x,t,y,s) = \frac{1}{[4\pi(t-s)]^{\frac{n}{2}}} \exp\left(-\frac{|x-y|^2}{4(t-s)}\right)$$

In [12], the authors considered the problem (1.3) with boundary condition u_0, not necessarily bounded function. The nonlinearity $u\varphi(.,u)$ is required to satisfy some conditions related to the parabolic Kato class $P^{\infty}(\mathbb{R}^n)$. They gave existence results and similar estimates on the solutions as in [20].

In [13], a similar problem as (1.3) has been treated in the half space $\mathbb{R}^n_+$. The elliptical version of (1.3) was studied in [7]. Dans [9], the authors examined the problem (1.1) – (1.2) in the case of a system with two equations.

2. Statement of the main result

2.1. Assumptions

To study problem (1.1) – (1.2), we consider the following definition and hypotheses:

Definition 2.1. We say that a nonnegative superharmonic function ω satisfies condition (H_0) if ω is locally bounded in $\mathbb{R}^n$ $(n \geq 3)$ and the map $(x,t) \mapsto P\omega(x,t)$ is continuous in $\mathbb{R}^n \times (0,\infty)$, where P is defined below.

$(P_t)_{t>0}$ on $\mathbb{R}^n$ denotes the Gauss semigroup defined for each nonnegative measurable function Φ on $\mathbb{R}^n$ by

$$P_t\Phi(x) = P\Phi(x,t) = \int_{\mathbb{R}^n} G(x,t,y,0)\Phi(y)dy \ \ , \ t>0, \ x \in \mathbb{R}^n$$

The family $(P_t)_{t>0}$ is a markovian semigroup. Moreover, a nonnegative superharmonic function ω on $\mathbb{R}^n$ satisfies for every $t>0$, $P_t\omega \leq \omega$, and consequently the mapping $t \mapsto P_t\omega$ is nonincreasing. We remark that for each nonnegative measurable function Φ on $\mathbb{R}^n$, the map $(x,t) \to P_t\Phi(x)$ is lower semicontinuous on $\mathbb{R}^n \times (0,\infty)$ and becomes continuous if Φ is further bounded.

Remark 2.2. We note that every bounded superharmonic function in $\mathbb{R}^n$ satisfies (H_0), see Gontara and Turki [9] and Mâagli et *al.* [12].

We fix four nonnegative superharmonic functions ω, θ, δ and ϕ satisfying condition (H_0). Let us introduce the required hypotheses on the initial values φ, ψ, γ and ζ the nonlinear terms:

(H_1) There exist four constants $c_i > 1$, $1 \leq i \leq 4$, such that

$$\frac{1}{c_1}\omega(x) \leq \psi(x) \leq c_1\omega(x) \quad , \quad \frac{1}{c_2}\theta(x) \leq \varphi(x) \leq c_2\theta(x)$$
$$\frac{1}{c_3}\delta(x) \leq \gamma(x) \leq c_3\delta(x) \quad , \quad \frac{1}{c_4}\phi(x) \leq \zeta(x) \leq c_4\phi(x)$$

and

$$\lim_{t\to 0} P_t\psi(x) = \psi(x) \quad , \quad \lim_{t\to 0} P_t\varphi(x) = \varphi(x)$$
$$\lim_{t\to 0} P_t\gamma(x) = \gamma(x) \quad , \quad \lim_{t\to 0} P_t\zeta(x) = \zeta(x)$$

(H_2) $f,\ g,\ h,\ k : (0,\infty) \to [0,\infty)$ are nondecreasing and continuous.

(H_3) The functions $p,\ q,\ r$ and e are measurable nonnegative and for each $c>0$, the functions

$$\tilde{p}_c = \frac{pf(cP\omega)}{P\theta} \quad , \quad \tilde{q}_c = \frac{qg(cP\delta)}{P\omega} \quad , \quad \tilde{r}_c = \frac{rh(cP\phi)}{P\delta} \quad , \quad \tilde{e}_c = \frac{ek(cP\theta)}{P\phi}$$

belong to the parabolic Kato class $P^{\infty}(\mathbb{R}^n)$.

To study $(1.1)-(1.2)$, a basic assumptions on p, q, r and e requires to fix four superharmonic functions ω, θ, δ and ϕ on $\mathbb{R}^n$ satisfying condition (H_0).

2.2. The main result

Now, we can state the main result of this work:

Theorem 2.3. *Assume $(H_1)-(H_3)$. Then there exist four constants λ_0, μ_0, η_0 and ϱ_0 such that for each $\lambda \in [0,\lambda_0)$, $\mu \in [0,\mu_0)$, $\eta \in [0,\eta_0)$ and $\varrho \in [0,\varrho_0)$, the problem $(1.1)-(1.2)$ has a positive continuous solution (u,v,w,z) in $(\mathbb{R}^n \times (0,\infty))^4$ satisfying for each $t>0$ and $x \in \mathbb{R}^n$*

$$\begin{cases} (1-\frac{\lambda}{\lambda_0})P\varphi(x,t) \le u(x,t) \le P\varphi(x,t) \\ (1-\frac{\mu}{\mu_0})P\psi(x,t) \le v(x,t) \le P\psi(x,t) \\ (1-\frac{\eta}{\eta_0})P\gamma(x,t) \le w(x,t) \le P\gamma(x,t) \\ (1-\frac{\varrho}{\varrho_0})P\zeta(x,t) \le z(x,t) \le P\zeta(x,t) \end{cases} \tag{2.1}$$

This document is organized as follows: In the next section, we give some technical results and to recall some theoretical tools that are essential to prove our main result. The last section is devoted to the proof of the main result, Theorem 2.3. The difficulties in this section are similar to those in [5]-[9], [13]-[14] and [20]-[21], and the techniques are of the same spirit, but specific new difficulties due to the nature of the system must be handled.

3. Preliminary results

We give here some essential results proved in [12], we can also see [19], [21], which were retained for the proof of our result. Now, we recall the definition of the Kato class $P^{\infty}(\mathbb{R}^n)$.

Definition 3.1. A Borel measurable function q in $\mathbb{R}^{n+1}$ belongs to the Kato class $P^{\infty}(\mathbb{R}^n)$ if for all $c>0$,

$$\lim_{\epsilon\to 0} \sup_{(x,t)\in\mathbb{R}^n\times\mathbb{R}} \int_{t-\epsilon}^{t+\epsilon}\int_{B(x,\sqrt{\epsilon})} G_c(x,|t-s|,y,0)\,|q(y,s)|\,dyds = 0$$

and

$$\sup_{(x,t)\in\mathbb{R}^n\times\mathbb{R}} \int_{-\infty}^{+\infty}\int_{\mathbb{R}^n} G_c(x,|t-s|,y,0)\,|q(y,s)|\,dyds < \infty$$

where

$$G_c(x,t,y,s) = \frac{1}{(t-s)^{\frac{n}{2}}}\exp(-c\frac{|x-y|^2}{t-s}) \quad \text{for } t>s \text{ and } x,\ y \in \mathbb{R}^n$$

In the following, we give a class of functions belonging to $P^{\infty}(\mathbb{R}^n)$.

Proposition 3.2. *(i)* $L^\infty(\mathbb{R}^n) \otimes L^1(\mathbb{R}) \subset P^\infty(\mathbb{R}^n)$.

(ii) Let $1 \leq p < \infty$ *and* $q \geq 1$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$. *Then for* $\sigma > \frac{np}{2}$ *and* $\tau < \frac{2}{p} - \frac{n}{\sigma} < \upsilon$, *we have*

$$\frac{L^\sigma(\mathbb{R}^n)}{|\cdot|^\tau (1+|\cdot|)^{\upsilon-\tau}} \otimes L^q(\mathbb{R}) \subset P^\infty(\mathbb{R}^n)$$

(iii) $P^\infty(\mathbb{R}^n) \subset L^1_{loc}(\mathbb{R}^{n+1})$.

We denote for any measurable function Φ on $\mathbb{R}^n \times (0,\infty)$, the potential

$$V\Phi(x,t) = \int_0^t \int_{\mathbb{R}^n} G(x,t,y,s)\Phi(y,s)dyds = \int_0^t P_{t-s}(\Phi(.,s))(x)ds$$

Proposition 3.3. *Let* q *be a nonnegative function in* $P^\infty(\mathbb{R}^n)$, *then there exists a positive constant* α_q *such that for each superharmonic function* υ *in* $\mathbb{R}^n$, *we have*

$$V(qP\upsilon)(x,t) \leq \alpha_q P\upsilon(x,t) \quad , \quad \textit{for} \;\; (x,t) \in \mathbb{R}^n \times (0,\infty)$$

Proposition 3.4. *Let* υ *be a superharmonic function in* $\mathbb{R}^n$ *satisfying* (H_0) *and* q *be a nonnegative function in* $P^\infty(\mathbb{R}^n)$. *Then the family of functions*

$$\left\{(x,t) \to Vf(x,t) = \int_0^t \int_{\mathbb{R}^n} G(x,t,y,s)f(y,s)dyds,\ |f| \leq qP\upsilon\right\}$$

is equicontinuous in $\mathbb{R}^n \times [0,\infty)$.

Moreover, for each $x \in \mathbb{R}^n$ *we have* $\lim_{t\to 0} Vf(x,t) = 0$, *uniformly on* f.

We therefore conclude the following result on the continuity needed to obtain the proof of Theorem 2.3.

Proposition 3.5. *Assuming the hypothesis* (H_1). *Then the functions* $P\varphi$, $P\psi$, $P\gamma$ *and* $P\zeta$ *are continuous in* $\mathbb{R}^n \times (0,\infty)$.

Proof. We prove that $P\varphi$ is continuous in $\mathbb{R}^n \times (0,\infty)$.
Let c_2 be the constant given in (H_1). We write for each $t > 0$ and $x \in \mathbb{R}^n$

$$c_2 P_t\theta(x) = P_t(c_2\theta - \varphi)(x) + P_t\varphi(x)$$

So, from (H_0) we have $(x,t) \mapsto P\theta(x,t)$ is continuous in $\mathbb{R}^n \times (0,\infty)$ and from the fact that $(x,t) \mapsto P_t(c_2\theta - \varphi)(x)$ and $(x,t) \mapsto P_t\varphi(x)$ are lower semicontinuous in $\mathbb{R}^n \times (0,\infty)$, we deduce that $(x,t) \mapsto P_t\varphi(x)$ is continuous in $\mathbb{R}^n \times (0,\infty)$.
Similarly, we can prove the continuity of $P\psi$, $P\gamma$ and $P\zeta$ in $\mathbb{R}^n \times (0,\infty)$. □

4. Proof of the main result

Let

$$\begin{aligned}\lambda_0 &= \inf_{(x,t)\in\mathbb{R}^n\times(0,\infty)} \frac{P\varphi(x,t)}{V(pf(P\psi))(x,t)}\\ \mu_0 &= \inf_{(x,t)\in\mathbb{R}^n\times(0,\infty)} \frac{P\psi(x,t)}{V(qg(P\gamma))(x,t)}\\ \eta_0 &= \inf_{(x,t)\in\mathbb{R}^n\times(0,\infty)} \frac{P\gamma(x,t)}{V(rh(P\zeta))(x,t)}\\ \varrho_0 &= \inf_{(x,t)\in\mathbb{R}^n\times(0,\infty)} \frac{P\zeta(x,t)}{V(ek(P\varphi))(x,t)}\end{aligned}$$

Proposition 4.1. *Suppose that the hypotheses $(H_1)-(H_3)$ are satisfied, then the constants λ_0, μ_0, η_0 and ϱ_0 are positive.*

Proof. The hypothesis (H_1) leads to $\psi \leq c_1\omega$.
From the fact that f is nondecreasing and p is nonnegative, we have

$$V(pf(P\psi)) \leq V(pf(c_1 P\omega))$$

Hence, by hypothesis (H_3) and Proposition 3.3, there exist $\tilde{p}_{c_1} \in P^\infty(\mathbb{R}^n)$ and a positive constant $\alpha_{\tilde{p}_{c_1}}$ such that for each $(x,t)\in\mathbb{R}^n\times(0,\infty)$, we have

$$V(pf(P\psi))(x,t) \leq V(\tilde{p}_{c_1}P\theta)(x,t) \leq \alpha_{\tilde{p}_{c_1}} P\theta(x,t)$$

So, using again (H_1) we find for each $(x,t)\in\mathbb{R}^n\times(0,\infty)$

$$\frac{P\varphi(x,t)}{V(pf(P\psi))(x,t)} \geq \frac{\frac{1}{c_1}P\theta(x,t)}{\alpha_{\tilde{p}_{c_1}}P\theta(x,t)} = \frac{1}{c_1\alpha_{\tilde{p}_{c_1}}} > 0$$

In the same way, we prove that

$$\frac{P\psi(x,t)}{V(qg(P\gamma))(x,t)} > 0 \quad , \quad \frac{P\gamma(x,t)}{V(rh(P\zeta))(x,t)} > 0 \quad , \quad \frac{P\zeta(x,t)}{V(ek(P\varphi))(x,t)} > 0$$

which implies that $\lambda_0 > 0$, $\mu_0 > 0$, $\eta_0 > 0$, $\varrho_0 > 0$. □

Proof. (**of Theorem 2.3).** Let $\lambda\in[0,\lambda_0)$, $\mu\in[0,\mu_0)$, $\eta\in[0,\eta_0)$ and $\varrho\in[0,\varrho_0)$. We define the sequences $(u_j)_{j\geq 0}$, $(v_j)_{j\geq 0}$, $(w_j)_{j\geq 0}$, and $(z_j)_{j\geq 0}$ by

$$\begin{cases} v_0 = P\psi \quad , \quad z_0 = P\zeta \\ u_j = P\varphi - \lambda V(pf(v_j)) \\ w_j = P\gamma - \eta V(rh(z_j)) \\ z_{j+1} = P\zeta - \varrho V(ek(u_j)) \\ v_{j+1} = P\psi - \mu V(qg(w_j)) \end{cases}$$

We are determined to prove for all $j\in\mathbb{N}$,

$$0 < (1-\frac{\lambda}{\lambda_0})P\varphi \leq u_j \leq u_{j+1} \leq P\varphi \tag{4.1}$$

$$0 < (1 - \frac{\eta}{\eta_0})P\gamma \le w_j \le w_{j+1} \le P\gamma \tag{4.2}$$

$$0 < (1 - \frac{\mu}{\mu_0})P\psi \le v_{j+1} \le v_j \le P\psi \tag{4.3}$$

$$0 < (1 - \frac{\varrho}{\varrho_0})P\zeta \le z_{j+1} \le z_j \le P\zeta \tag{4.4}$$

We note that according to the definition of λ_0, μ_0, η_0 and ϱ_0 that, for each $(x,t) \in \mathbb{R}^n \times (0,\infty)$

$$\lambda_0 V(pf(P\psi))(x,t) \le P\varphi(x,t) \tag{4.5}$$

$$\mu_0 V(qg(P\gamma))(x,t) \le P\psi(x,t) \tag{4.6}$$

$$\eta_0 V(rh(P\zeta))(x,t) \le P\gamma(x,t) \tag{4.7}$$

$$\varrho_0 V(ek(P\varphi))(x,t) \le P\zeta(x,t) \tag{4.8}$$

From (4.5) and (4.7), we have

$$\begin{aligned} u_0 &= P\varphi - \lambda V(pf(P\psi)) \ge P\varphi - \frac{\lambda}{\lambda_0}P\varphi = (1 - \frac{\lambda}{\lambda_0})P\varphi > 0 \\ w_0 &= P\gamma - \eta V(rh(P\zeta)) \ge P\gamma - \frac{\eta}{\eta_0}P\gamma = (1 - \frac{\eta}{\eta_0})P\gamma > 0 \end{aligned}$$

Then

$$\begin{aligned} z_1 - z_0 &= -\varrho V(ek(u_0)) \le 0 \\ v_1 - v_0 &= -\mu V(qg(w_0)) \le 0 \end{aligned}$$

Since f and h are nondecreasing, we obtain

$$\begin{aligned} u_1 - u_0 &= \lambda V(p(f(v_0) - f(v_1))) \ge 0 \\ w_1 - w_0 &= \eta V(r(h(z_0) - h(z_1))) \ge 0 \end{aligned}$$

Now, since v_0, z_0 are nonnegatives ($v_0 > 0 \Rightarrow u_0 \le P\varphi,\ z_0 > 0 \Rightarrow w_0 \le P\gamma$) and g, k are nondecreasing, we deduce from (4.6) and (4.8) that

$$\begin{aligned} z_1 &= P\zeta - \varrho V(ek(u_0)) \ge (1 - \frac{\varrho}{\varrho_0})P\zeta > 0 \\ v_1 &= P\psi - \mu V(qg(w_0)) \ge (1 - \frac{\mu}{\mu_0})P\psi > 0 \end{aligned}$$

which gives us

$$u_1 \le P\varphi \quad \text{and} \quad w_1 \le P\gamma$$

Finally, we find

$$\begin{cases} 0 < (1 - \frac{\lambda}{\lambda_0})P\varphi \le u_0 \le u_1 \le P\varphi \\ 0 < (1 - \frac{\eta}{\eta_0})P\gamma \le w_0 \le w_1 \le P\gamma \\ 0 < (1 - \frac{\mu}{\mu_0})P\psi \le v_1 \le v_0 \le P\psi \\ 0 < (1 - \frac{\varrho}{\varrho_0})P\zeta \le z_1 \le z_0 \le P\zeta \end{cases}$$

By induction, we suppose that (4.1), (4.2), (4.3)and (4, 4) hold for j. Since g, k are nondecreasing and $u_{j+1} \leq P\varphi,\ w_{j+1} \leq P\gamma$, we have

$$\begin{aligned} z_{j+2} - z_{j+1} &= \varrho V(e(k(u_j) - k(u_{j+1}))) \leq 0 \\ v_{j+2} - v_{j+1} &= \mu V(q(g(w_j) - g(w_{j+1}))) \leq 0 \end{aligned}$$

and

$$\begin{aligned} z_{k+2} &= P\zeta - \varrho V(ek(u_{k+1})) \geq P\zeta - \varrho V(ek(P\varphi)) \geq (1 - \frac{\varrho}{\varrho_0})P\zeta \\ v_{k+2} &= P\psi - \mu V(qg(w_{k+1})) \geq P\psi - \mu V(qg(P\gamma)) \geq (1 - \frac{\mu}{\mu_0})P\psi \end{aligned}$$

Using the two relations (4.6) and (4.8), we have

$$\begin{aligned} 0 &< (1 - \frac{\varrho}{\varrho_0})P\zeta \leq z_{j+2} \leq z_{j+1} \leq P\zeta \\ 0 &< (1 - \frac{\mu}{\mu_0})P\psi \leq v_{j+2} \leq v_{j+1} \leq P\psi \end{aligned}$$

Now, using that f, h are nondecreasing, we have

$$\begin{aligned} u_{j+2} - u_{j+1} &= \lambda V(p(f(v_{j+1}) - f(v_{j+2}))) \geq 0 \\ w_{j+2} - w_{j+1} &= \eta V(r(h(z_{j+1}) - h(z_{j+2}))) \geq 0 \end{aligned}$$

Since $z_{j+2} > 0$, $v_{j+2} > 0$, we obtain

$$\begin{aligned} 0 &< (1 - \frac{\lambda}{\lambda_0})P\varphi \leq u_{j+1} \leq u_{j+2} \leq P\varphi \\ 0 &< (1 - \frac{\eta}{\eta_0})P\gamma \leq w_{j+1} \leq w_{j+2} \leq P\gamma \end{aligned}$$

Therefore, the sequences $(u_j)_{j\geq 0}$, $(v_j)_{j\geq 0}$, $(w_j)_{j\geq 0}$ and $(z_j)_{j\geq 0}$ converge respectively to u, v, w and z satisfying (2.1). We claim that

$$u = P\varphi - \lambda V(pf(v)) \tag{4.9}$$

$$w = P\gamma - \eta V(rh(z)) \tag{4.10}$$

$$z = P\zeta - \varrho V(ek(u)) \tag{4.11}$$

$$v = P\psi - \mu V(qg(w)) \tag{4.12}$$

Since $v_j \leq P\psi$ and $z_j \leq P\zeta$ for all $j \in \mathbb{N}$, using hypotheses $(H_1), (H_3)$ and the fact that f, h are nondecreasing, there exist $\tilde{p}_{c_1}, \tilde{r}_{c_4} \in P^\infty(\mathbb{R}^n)$ such that

$$pf(v) \leq pf(c_1 P\omega) \leq \tilde{p}_{c_1} P\theta \tag{4.13}$$

$$rh(z) \leq rh(c_4 P\phi) \leq \tilde{r}_{c_4} P\delta \tag{4.14}$$

and so

$$\begin{aligned} p\,|f(v_j) - f(v)| &\leq 2\tilde{p}_{c_1} P\theta\ ,\ \text{for all } j \in \mathbb{N} \\ r\,|h(z_j) - h(z)| &\leq 2\tilde{r}_{c_4} P\delta\ ,\ \text{for all } j \in \mathbb{N} \end{aligned}$$

Now, from Proposition 3.4 and by Lebesgue's theorem, we can deduce

$$\begin{aligned} \lim_{k\to\infty} V(pf(v_k)) &= V(pf(v)) \\ \lim_{k\to\infty} V(rh(z_k)) &= V(rh(z)) \end{aligned}$$

So, letting $j \to \infty$ in equations

$$u_j = P\varphi - \lambda V(pf(v_j)), \qquad w_j = P\gamma - \eta V\left(rh\left(z_j\right)\right)$$

we have (4.9) and (4.10). Similarly, we obtain (4.11) and (4.12).
Next, we affirm that (u, v, w, z) satisfies

$$\begin{cases} \Delta u - \dfrac{\partial u}{\partial t} = \lambda pf(v) \\ \Delta v - \dfrac{\partial v}{\partial t} = \mu qg(w) \\ \Delta w - \dfrac{\partial w}{\partial t} = \eta rh(z) \\ \Delta z - \dfrac{\partial z}{\partial t} = \varrho ek(u) \end{cases} \tag{4.15}$$

Since θ, δ satisfies (H_0) and $\tilde{p}_{c_1}, \tilde{r}_{c_4} \in P^\infty(\mathbb{R}^n)$, using Proposition 3.2, we have

$$\tilde{p}_{c_1} P\theta,\ \tilde{r}_{c_4} P\delta \in L^1_{loc}(\mathbb{R}^n \times (0,\infty))$$

Moreover (4.13), (4.14) and Proposition 3.4 imply that

$$pf(v),\ rh(z) \in L^1_{loc}(\mathbb{R}^n \times (0,\infty))$$

and

$$V(pf(v)),\ V(rh(z)) \in C(\mathbb{R}^n \times (0,\infty)) \subset L^1_{loc}(\mathbb{R}^n \times (0,\infty))$$

Similarly

$$qg(w),\ V(qg(w)),\ ek(u),\ V(ek(u)) \in L^1_{loc}(\mathbb{R}^n \times (0,\infty))$$

Now, applying the heat operator $\Delta - \dfrac{\partial}{\partial t}$ in (4.9), (4.10), (4.11) and (4.12), (u, v, w, z) is clearly a positive solution (in the sense of distributions) of (4.15).
Furthermore since $V(pf(v))$, $V(qg(w))$, $V(rh(z))$ and $V(ek(u))$ are continuous in $\mathbb{R}^n \times (0,\infty)$ and using Proposition 3.5, we deduce from (4.9), (4.10), (4.11) and (4.12) that

$$(u, v, w, z) \in (C(\mathbb{R}^n \times (0,\infty)))^4$$

which implies according to hypothesis (H_1) and proposition 3.4 that

$$\begin{aligned}
\lim_{t\to 0} u(x,t) &= \lim_{t\to 0} P\varphi(x,t) = \varphi(x)\ , \quad x \in \mathbb{R}^n \\
\lim_{t\to 0} v(x,t) &= \lim_{t\to 0} P\psi(x,t) = \psi(x)\ , \quad x \in \mathbb{R}^n \\
\lim_{t\to 0} w(x,t) &= \lim_{t\to 0} P\gamma(x,t) = \gamma(x)\ , \quad x \in \mathbb{R}^n \\
\lim_{t\to 0} z(x,t) &= \lim_{t\to 0} P\zeta(x,t) = \zeta(x)\ , \quad x \in \mathbb{R}^n
\end{aligned}$$

This completes the proof of our theorem. □

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Existence of solutions for an impulsive boundary value problem with nonlinear derivative dependence on unbounded intervals via variational methods

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Abstract. In this paper, we employ the critical point theory and iterative methods to establish the existence of solutions for an impulsive boundary value problem with nonlinear derivative dependence on the half-line.

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Keywords: Impulsive BVPs, unbounded interval, nonlinear derivative dependence, iterative methods, variational methods.

1. Introduction

In this paper, we consider the solvability of an impulsive boundary value problem with nonlinear derivative dependence on the half-line. More precisely, we consider the problem

$$\begin{cases} -(p(t)u'(t))' &=& f(t,u(t),u'(t)), \quad \text{a.e. } t\geq 0,\ t\neq t_j, \\ u(0)=u(+\infty) &=& 0, \\ \triangle(p(t_j)u'(t_j)) &=& g(t_j)I_j(u(t_j)), \quad\ \ j\in\{1,2,\ldots\}, \end{cases} \tag{1.1}$$

where $f:[0,+\infty)\times\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$ is measurable in $t\in[0,+\infty)$ for each $(x,\xi)\in\mathbb{R}\times\mathbb{R}$, and continuous in $(x,\xi)\in\mathbb{R}\times\mathbb{R}$ for a.e. $t\in[0,+\infty)$. We assume that the impulsive functions $I_j:\mathbb{R}\longrightarrow\mathbb{R}$ are continuous where $t_0=0<t_1<t_2<\ldots<t_j<\ldots<t_m\to+\infty$, as $m\to\infty$, are the impulse points.
The coefficient $p:[0,+\infty)\longrightarrow(0,+\infty)$ satisfies $\frac{1}{p}\in L^1(0,+\infty)$, and

$$M=\int_0^{+\infty}\left(\int_t^{+\infty}\frac{1}{p(s)}ds\right)dt<+\infty.$$

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We define the jump

$$\triangle(p(t_j)u'(t_j)) = p(t_j^+)u'(t_j^+) - p(t_j^-)u'(t_j^-),$$

where $u'(t_j^+) = \lim_{t\to t_j^+} u'(t)$ and $u'(t_j^-) = \lim_{t\to t_j^-} u'(t)$ stand for the right and the left limits of u' at t_j, respectively. Finally $g : [0,+\infty) \longrightarrow [0,+\infty)$ is a continuous function that satisfies

$$\sum_{j=1}^{+\infty} g(t_j) < +\infty.$$

Recently, in [2, 3], the authors obtained the existence of solutions for BVPs associated to impulsive equations on unbounded domains by using variational methods. In [4], de Figueiredo, Girardi and Matzeu proved the existence of solution for semilinear elliptic equations with dependence on the gradient through an iterative technique. However, there are few papers that have studied the existence of solutions for impulsive boundary value problems similar to the problem (1.1) by using variational methods coupled with the iterative methods.

In order to use variational methods, we consider a family of boundary value problems with no dependence on the derivative. Namely, for each $w \in H^1_{0,p}(0,+\infty)$, we consider the problem

$$\begin{cases} -(p(t)u'(t))' &=& f(t,u(t),w'(t)), \quad \text{a.e. } t \geq 0,\ t \neq t_j, \\ u(0) = u(+\infty) &=& 0, \\ \triangle(p(t_j)u'(t_j)) &=& g(t_j)I_j(u(t_j)), \quad j \in \{1,2,\ldots\}. \end{cases} \tag{1.2}$$

The class of problems (1.2) is of variational type and we can resolve them by variational methods and the existence of a solution for the initial problem is obtained by iterative methods.

Now we need to define the following Banach space and this before giving the variational formulation of (1.2).

$$H^1_{0,p}(0,+\infty) = \{u \in AC[0,+\infty),\mathbb{R}) \mid u(0) = u(+\infty) = 0,\ \sqrt{p}u' \in L^2(0,+\infty)\},$$

equipped with the norm

$$\|u\|_{0,p} = \sqrt{\int_0^{+\infty} p(t)u'^2(t)dt + \int_0^{+\infty} u^2(t)dt},$$

or the equivalent norm

$$\|u\|_p = \|u\|_{L^2} + \|\sqrt{p}u'\|_{L^2}.$$

Moreover the space $H^1_{0,p}(0,+\infty)$ is reflexive (see [2]).

Lemma 1.1. *On $H^1_{0,p}(0,+\infty)$, the quantity $\|u\| = \sqrt{\int_0^{+\infty} p(t)u'^2(t)dt}$ is a norm which is equivalent to the $H^1_{0,p}(0,+\infty)$-norm.*

Now let us recall the following essential embeddings (see [2]).

Lemma 1.2. *$(H^1_{0,p}(0,+\infty), \|\cdot\|)$ embeds in $(C_0[0,+\infty), \|u\|_\infty)$, where*

$$C_0[0,+\infty) = \{u \in C([0,+\infty),\mathbb{R}) \mid \lim_{t\to+\infty} u(t) = 0\} \text{ and } \|u\|_\infty = \sup_{t\in[0,+\infty)} |u(t)|.$$

Lemma 1.3. *$H^1_{0,p}(0,+\infty)$ embeds continuously in $C_0[0,+\infty)$ and in $L^2(0,+\infty)$.*

Lemma 1.4. *The embedding $H^1_{0,p}(0,+\infty) \hookrightarrow C_0[0,+\infty)$ is compact with*

$$\|u\|_\infty \leq M_1\|u\|,$$

where

$$M_1 = \sqrt{\|\frac{1}{p}\|_{L^1}}.$$

2. Preliminaries

First we recall some basic definitions and lemmas which are used in this paper.

Lemma 2.1. *(Minimization Principle*[1]*) Let X be a reflexive Banach space and J a functional defined on X such that*
(1) $\lim\limits_{\|u\|\to+\infty} J(u) = +\infty$ (coercivity condition),
(2) J is sequentially weakly lower semi-continuous.
Then J is lower bounded on X and achieves its lower bound at some point u_0.

Definition 2.2. Let X be a real Banach space, $J \in C^1(X,\mathbb{R})$. If any sequence $(u_n) \subset X$ for which $(J(u_n))$ is bounded in $\mathbb{R}$ and $J'(u_n) \longrightarrow 0$ as $n \to +\infty$ in X' possesses a convergent subsequence, then we say that J satisfies the Palais-Smale condition (PS condition for brevity).

Lemma 2.3. ([5, Theorem 2.2], [6, Theorem 3.1]) [Mountain Pass Theorem] *Let X be a real Banach space and $J \in C^1(X,\mathbb{R})$ satisfying the (PS) condition. Suppose that $J(0) = 0$ and*
(1) there are constants $\rho, \alpha > 0$ such that $J(u) \geq \alpha$ for all $u \in X$ with $\|u\| = \rho$,
(2) there exists $u_0 \in X$ such that $\|u_0\| > \rho$ and $J(u_0) < \alpha$.
Then J possesses a critical value such that $c \geq \alpha$. Moreover, c can be characterized as

$$c = \inf_{\gamma\in\Gamma} \max_{u\in\gamma([0,1])} J(u),$$

where

$$\Gamma = \big\{\gamma \in C([0,1],X) : \gamma(0) = 0,\ \gamma(1) = u_0\big\}.$$

3. Variational setting

Take $v \in H^1_{0,p}(0,+\infty)$, multiply the equation in problem (1.1) by v and integrate over $(0,+\infty)$, we obtain

$$-\int_0^{+\infty} (p(t)u'(t))'v(t)dt = \int_0^{+\infty} f(t,u(t),u'(t))v(t)dt.$$

The first term is

$$
\begin{aligned}
-\int_0^{+\infty}(p(t)u'(t))'v(t)dt &= -\sum_{j=0}^{+\infty}\int_{t_j}^{t_{j+1}}(p(t)u'(t))'v(t)dt \\
&= \sum_{j=1}^{+\infty} g(t_j)I_j(u(t_j))v(t_j) + \int_0^{+\infty} p(t)u'(t)v'(t)dt.
\end{aligned}
$$

Hence

$$\int_0^{+\infty} p(t)u'(t)v'(t)dt = -\sum_{j=1}^{+\infty} g(t_j)I_j(u(t_j))v(t_j) + \int_0^{+\infty} f(t,u(t),u'(t))v(t)dt.$$

Definition 3.1. We say that a function $u \in H^1_{0,p}(0,+\infty)$ is a weak solution of Problem (1.1) if

$$\int_0^{+\infty} p(t)u'(t)v'(t)dt + \sum_{j=1}^{+\infty} g(t_j)I_j(u(t_j))v(t_j) - \int_0^{+\infty} f(t,u(t),u'(t))v(t)dt = 0,$$

for every $v \in H^1_{0,p}(0,+\infty)$.¸

Proposition 3.2. *Suppose that the following conditions hold:*
(H_1) *There exists constant* $\sigma > 2$ *and two positive functions* φ, ψ *such that* $\varphi \in L^1(0,+\infty), \psi \in L^\infty(0,+\infty)$ *with*

$$|f(t,x,\xi)| \leq \varphi(t)|x|^\sigma \psi(\xi), \quad \textit{for a.e. } t \in [0,+\infty), x \in \mathbb{R}, \xi \in \mathbb{R}.$$

(I_0) *There exist positive constants* c_0 *and* ν *such that*

$$|I_j(x)| \leq c_0|x|^\nu, \quad \forall x \in \mathbb{R},\ j \in \{1,2,\ldots\}.$$

Then, for each $w \in H^1_{0,p}(0,+\infty)$ *fixed, the functional* $J_w : H^1_{0,p}(0,+\infty) \longrightarrow \mathbb{R}$ *defined by*

$$J_w(u) = \frac{1}{2}\|u\|^2 + \sum_{j=1}^{+\infty} g(t_j)\int_0^{u(t_j)} I_j(\tau)d\tau - \int_0^{+\infty} F(t,u(t),w'(t))dt,$$

where $F(t,u,\xi) = \int_0^u f(t,s,\xi)ds$, *is continuous, differentiable and*

$$
\begin{aligned}
(J'_w(u),v) \quad = \quad & \int_0^{+\infty} p(t)u'(t)v'(t)dt + \sum_{j=1}^{+\infty} g(t_j)I_j(u(t_j))v(t_j) \\
& - \int_0^{+\infty} f(t,u(t),w'(t))v(t)dt,
\end{aligned}
\tag{3.1}
$$

for all $v \in H^1_{0,p}(0,+\infty)$.

Proof. Claim 1. Let $w \in H^1_{0,p}(0,+\infty)$ fixed. Then J_w is Gâteaux-differentiable. Indeed, for all $v \in H^1_{0,p}(0,+\infty)$, we have

$$\begin{aligned}
J_w(u+hv) - J_w(u) &= \frac{1}{2}\int_0^{+\infty} p(t)(u'(t)+hv'(t))^2 dt \\
&+ \sum_{j=1}^{+\infty} g(t_j)\int_0^{u(t_j)+hv(t_j)} I_j(\tau)d\tau \\
&- \int_0^{+\infty} F(t,u(t)+hv(t),w'(t))dt \\
&- \frac{1}{2}\int_0^{+\infty} p(t)u'^2(t)dt - \sum_{j=1}^{+\infty} g(t_j)\int_0^{u(t_j)} I_j(\tau)d\tau \\
&+ \int_0^{+\infty} F(t,u(t),w'(t))dt \\
&= h\int_0^{+\infty} p(t)u'(t)v'(t)dt + \frac{h^2}{2}\int_0^{+\infty} p(t)v'^2(t)dt \\
&+ \sum_{j=1}^{+\infty} g(t_j)\left[\int_0^{u(t_j)+hv(t_j)} I_j(\tau)d\tau - \int_0^{u(t_j)} I_j(\tau)d\tau\right] \\
&- \int_0^{+\infty} \Big[F(t,u(t)+hv(t),w'(t)) - F(t,u(t),w'(t))\Big]dt
\end{aligned}$$

$$\begin{aligned}
J_w(u+hv) - J_w(u) &= h\int_0^{+\infty} p(t)u'(t)v'(t)dt + \frac{h^2}{2}\int_0^{+\infty} p(t)v'^2(t)dt \\
&+ h\sum_{j=1}^{+\infty} g(t_j)I_j(u(t_j)+c_h v(t_j))v(t_j) \\
&- h\int_0^{+\infty} f(t,u(t)+\theta_h v(t),w'(t))v(t)dt,
\end{aligned}$$

where $0<\theta_h<1$ and $0<c_h<1$ from the Mean Value Theorem. Thus

$$\begin{aligned}
\frac{J_w(u+hv) - J_w(u)}{h} &= \int_0^{+\infty} p(t)u'(t)v'(t)dt + \frac{h}{2}\int_0^{+\infty} p(t)v'^2(t)dt \\
&+ \sum_{j=1}^{+\infty} g(t_j)I_j(u(t_j)+c_h v(t_j))v(t_j) \\
&- \int_0^{+\infty} f(t,u(t)+\theta_h v(t),w'(t))v(t)dt.
\end{aligned}$$

By (H_1), (I_0) and the Lebesgue Dominated Convergence Theorem, we obtain

$$\lim_{h\to 0}\frac{J_w(u+hv)-J_w(u)}{h} = \int_0^{+\infty} p(t)u'(t)v'(t)dt + \sum_{j=1}^{+\infty} g(t_j)I_j(u(t_j))v(t_j) - \int_0^{+\infty} f(t,u(t),w'(t))v(t)dt,$$

so that, J_w is Gâteaux-differentiable and

$$(J'_w(u),v) = \int_0^{+\infty} p(t)u'(t)v'(t)dt + \sum_{j=1}^{+\infty} g(t_j)I_j(u(t_j))v(t_j) - \int_0^{+\infty} f(t,u(t),w'(t))v(t)dt,$$

for all $v \in H^1_{0,p}(0,+\infty)$. Therefore a critical point of J_w is a weak solution of Problem (1.2).

Claim 2. J'_w is continuous.

Indeed, let (u_n) be a sequence in $H^1_{0,p}(0,+\infty)$ such that $u_n \longrightarrow u$ as $n \longrightarrow +\infty$. From Lemma 1.4, we have (u_n) converges uniformly to u on $[0,+\infty)$ as $n \longrightarrow +\infty$. Since f and I_j are continuous, then

$$f(t,u_n(t),w'(t)) \longrightarrow f(t,u(t),w'(t)), \quad I_j(u_n(t_j)) \longrightarrow I_j(u(t_j))$$

as $n \longrightarrow +\infty$ and it follows from (H_1) that

$$\begin{aligned} |f(t,u_n(t),w'(t))| &\leq \varphi(t)|u_n(t)|^\sigma|\psi(w'(t))| \\ &\leq \varphi(t)\|u_n\|_\infty^\sigma|\psi(w'(t))| \\ &\leq M_1^\sigma\varphi(t)\|u_n\|^\sigma|\psi(w'(t))|. \end{aligned}$$

And by (I_0), we have

$$\begin{aligned} |I_j(u_n(t_j))| &\leq c_0|u_n(t_j)|^\nu \\ &\leq c_0\|u_n\|_\infty^\nu \\ &\leq M_1^\nu c_0\|u_n\|^\nu. \end{aligned}$$

Then from the Lebesgue Dominated Convergence Theorem, we obtain

$$\lim_{n\to+\infty}\int_0^{+\infty} f(t,u_n(t),w'(t))dt = \int_0^{+\infty} f(t,u(t),w'(t))dt,$$

and

$$\lim_{n\to+\infty}\sum_{j=1}^{+\infty} g(t_j)I_j(u_n(t_j)) = \sum_{j=1}^{+\infty} g(t_j)I_j(u(t_j)).$$

So

$$\begin{aligned}(J'_w(u_n) - J'_w(u), v) &= \int_0^{+\infty} p(t)(u'_n(t) - u'(t))v'(t)dt \\ &+ \sum_{j=1}^{+\infty} g(t_j)\Big[I_j(u_n(t_j)) - I_j(u(t_j))\Big]v(t_j) \\ &- \int_0^{+\infty} \Big[f(t, u_n(t), w'(t)) - f(t, u(t), w'(t))\Big]v(t)dt.\end{aligned}$$

Passing to the limit in $(J'_w(u_n) - J'_w(u), v)$ when $n \longrightarrow +\infty$, using assumptions (H_1), (I_0) and the Lebesgue Dominated Convergence Theorem, we obtain that $J'_w(u_n) \longrightarrow J'_w(u)$, as $n \longrightarrow +\infty$.
Consequently, $J_w \in C^1(H^1_{0,p}(0, +\infty), \mathbb{R})$. □

4. Main results

4.1. Nontrivial weak solution

Theorem 4.1. *Assume that f satisfies (H_1), I_j satisfies (I_0) and the following hypotheses:*

(H_2) $\lim\limits_{x\to 0} \frac{f(t,x,\xi)}{x} = 0$, *uniformly in $t \in [0, +\infty)$ and $\xi \in \mathbb{R}$.*

(H_3) *There exist positive functions $c_1, c_2 \in L^1(0, +\infty)$, and $\mu > 2$ such that*

(a) $F(t, x, \xi) \geq c_1(t)|x|^\mu - c_2(t)$, for a.e. $t \geq 0$, and all $x \in \mathbb{R}$, $\xi \in \mathbb{R}$,

(b) $\mu F(t, x, \xi) \leq x f(t, x, \xi)$, for a.e. $t \geq 0$, and all $x \in \mathbb{R}$, $\xi \in \mathbb{R}$.

(I_1) *There exists $0 < \gamma \leq 2$ such that*

$$\gamma \int_0^x I_j(s)ds \geq x I_j(x) > 0,\ \forall\, x \in \mathbb{R} \setminus \{0\},\ \forall\, j \in \{1, 2, \ldots\}.$$

Then there exist positive constants d_1, d_2 such that, for each $w \in H^1_{0,p}(0, +\infty)$, Problem (1.2) has at least one nontrivial weak solution u_w satisfying

$$d_1 \leq \|u_w\| \leq d_2.$$

Proof. Claim 1. Let $w \in H^1_{0,p}(0, +\infty)$ fixed. Then J_w satisfies the (PS) condition.
Indeed, let $(u_n) \subset H^1_{0,p}(0, +\infty)$ such that $(J_w(u_n))$ is bounded and $J'_w(u_n) \longrightarrow 0$ as $n \longrightarrow +\infty$. Using $(H_3)(b)$ and (I_1), there exists some $d > 0$ such that

$$\begin{aligned}d &\geq \mu J_w(u_n) - (J'_w(u_n), u_n) \\ &\geq \left(\frac{\mu}{2} - 1\right)\|u_n\|^2 \\ &- \int_0^{+\infty} \Big(\mu F(t, u_n(t), w'(t)) - f(t, u_n(t), w'(t))u_n(t)\Big)dt \\ &+ \sum_{j=1}^{+\infty} g(t_j)\left(\mu \int_0^{u_n(t_j)} I_j(\tau)d\tau - I_j(u_n(t_j))u_n(t_j)\right) \\ &\geq \left(\frac{\mu}{2} - 1\right)\|u_n\|^2.\end{aligned}$$

Since $\mu > 2$, it follows that (u_n) is bounded in $H^1_{0,p}(0,+\infty)$.
Then there exists a subsequence of (u_n) still denoted (u_n) such that (u_n) converges weakly to some u in $H^1_{0,p}(0,+\infty)$ because (u_n) is bounded in the reflexive Banach space $H^1_{0,p}(0,+\infty)$. Lemma 1.4 implies that (u_n) converges uniformly to u on $[0,+\infty)$. Thus

$$\lim_{n\to+\infty}\sum_{j=1}^{+\infty} g(t_j)\Big(I_j(u_n(t_j)) - I_j(u(t_j))\Big)(u_n(t_j) - u(t_j)) = 0$$

and

$$\lim_{n\to+\infty}\int_0^{+\infty}\Big(f(t,u_n(t),w'(t)) - f(t,u(t),w'(t))\Big)(u_n(t) - u(t))\,dt = 0.$$

Since $\lim_{n\to+\infty} J'(u_n) = 0$ and (u_n) converges weakly to some u, we get

$$\lim_{n\to+\infty}(J'_w(u_n) - J'_w(u), u_n - u) = 0.$$

From (3.1), we have

$$\begin{aligned}(J'_w(u_n) - J'_w(u), u_n - u) &= \|u_n - u\|^2\\ &+\sum_{j=1}^{+\infty} g(t_j)\big(I_j(u_n(t_j)) - I_j(u(t_j))\big)(u_n(t_j) - u(t_j))\\ &-\int_0^{+\infty}\big(f(t,u_n(t),w'(t)) - f(t,u(t),w'(t))\big)(u_n(t) - u(t))\,dt.\end{aligned}$$

Hence $\lim\limits_{n\to+\infty}\|u_n - u\| = 0$. Thus (u_n) converges strongly to u in $H^1_{0,p}(0,+\infty)$.
Consequently J_w satisfies the (PS) condition.
Claim 2. Let $w \in H^1_{0,p}(0,+\infty)$ fixed. Then there exist $\rho > 0$ and $\alpha > 0$, independent of w, such that $J_w(u) \geq \alpha, \quad \forall u \in H^1_{0,p}(0,+\infty), \|u\| = \rho$.
Indeed, let $0 < \varepsilon < \frac{1}{M}$. By (H_2), there exists $\delta > 0$ such that

$$|x| \leq \delta \Longrightarrow |f(t,x,\xi)| \leq \varepsilon|x|, \quad \forall t \in [0,+\infty), \xi \in \mathbb{R}.$$

We have $\|u\|^2_{L^2} \leq M\|u\|^2$ (see [2]) , so we deduce that

$$\int_0^{+\infty}|F(t,u(t),w'(t))dt| \leq \frac{\varepsilon}{2}\|u\|^2_{L^2} \leq \frac{\varepsilon}{2}M\|u\|^2, \ \text{ for a.e. } t \geq 0,$$

whenever $\|u\|_\infty \leq \delta$.
By choosing $0 < \rho \leq \frac{\delta}{M_1}$ and $\alpha = \frac{1}{2}(1-\varepsilon M)\rho^2$, hence for $\|u\| = \rho$ (note $\|u\|_\infty \leq \delta$), we get

$$\begin{aligned}J_w(u) &= \frac{1}{2}\|u\|^2 + \sum_{j=1}^{+\infty} g(t_j)\int_0^{u(t_j)} I_j(\tau)d\tau - \int_0^{+\infty} F(t,u(t),w'(t))dt\\ &\geq \frac{1}{2}\|u\|^2 - \int_0^{+\infty} F(t,u(t),w'(t))dt\\ &\geq \frac{1}{2}(1-\varepsilon M)\|u\|^2 = \alpha.\end{aligned}$$

So there are $\rho > 0$ and $\alpha > 0$ such that $J_w(u) \geq \alpha, \ \ \forall u \in H^1_{0,p}(0,+\infty)$ with $\|u\| = \rho$.
Claim 3. Let $w \in H^1_{0,p}(0,+\infty)$ *fixed. Then there exists* $T_0 > 0$*, independent of* w*, such that*

$$J_w(\vartheta u^*) \leq 0, \ \forall \vartheta \geq T_0,$$

where $u^* \in H^1_{0,p}(0,+\infty)$ *with* $\|u^*\| = 1$*.*
Indeed, from (I_1), there exists $c_3 > 0$ such that

$$\int_0^x I_j(s)ds \leq c_3|x|^\gamma, \ \text{ for every } \ x \in \mathbb{R}.$$

Take an arbitrary $u^* \in H^1_{0,p}(0,+\infty)$ with $\|u^*\| = 1$ and using Lemma 1.4, $(H_3)(a)$, we obtain

$$\begin{aligned}
J_w(\vartheta u^*) &= \frac{1}{2}\vartheta^2\|u^*\|^2 + \sum_{j=1}^{+\infty} g(t_j)\int_0^{\vartheta u^*(t_j)} I_j(\tau)d\tau \\
&- \int_0^{+\infty} F(t,\vartheta u^*(t), w'(t))dt \\
&\leq \frac{1}{2}\vartheta^2 + c_3|\vartheta|^\gamma \|u^*\|_\infty^\vartheta \sum_{j=1}^{+\infty} g(t_j) \\
&- |\vartheta|^\mu \int_0^{+\infty} c_1(t)|u^*(t)|^\mu dt + \int_0^{+\infty} c_2(t)dt \\
&\leq \frac{1}{2}\vartheta^2 + c_3|\vartheta|^\gamma M_1^\gamma \sum_{j=1}^{+\infty} g(t_j) \\
&- |\vartheta|^\mu \int_0^{+\infty} c_1(t)|u^*(t)|^\mu dt + \int_0^{+\infty} c_2(t)dt \leq 0,
\end{aligned}$$

when $\vartheta \geq T_0$ for some T_0 large, since $\mu > 2 \geq \gamma$.
By Proposition 3.2, the functional j_w is in $C^1(H^1_{0,p}(0,+\infty),\mathbb{R})$. Lemma 2.3 guarantees that J_w possesses a critical point which is a weak solution of Problem (1.2).
Claim 4. Let $w \in H^1_{0,p}(0,+\infty)$ *fixed. Then there is a constant* $d_1 > 0$*, independent of* w*, such that* $\|u_w\| \geq d_1$*, for all solution* u_w *obtained above.*
Indeed, let u_w be a solution of Problem (1.2). Then

$$\|u_w\|^2 + \sum_{j=1}^{+\infty} g(t_j)I_j(u_w(t_j))u_w(t_j) = \int_0^{+\infty} f(t,u_w(t),w'(t))u_w(t)dt.$$

It follows from (H_1) and (H_2) that,

$$|f(t,x,\xi)| \leq \varepsilon|x| + \varphi(t)|x|^\sigma\psi(\xi), \quad \text{for } t \in [0,+\infty), x \in \mathbb{R}, \xi \in \mathbb{R}.$$

Then

$$\begin{aligned}\|u_w\|^2 &\leq \|u_w\|^2 + \sum_{j=1}^{+\infty} g(t_j) I_j(u_w(t_j)) u_w(t_j) \\ &= \int_0^{+\infty} f(t, u_w(t), w'(t)) u_w(t) dt \\ &\leq \varepsilon \int_0^{+\infty} |u_w(t)|^2 dt + \int_0^{+\infty} \varphi(t) |u_w(t)|^{\sigma+1} \psi(w'(t)) dt \\ &\leq \varepsilon M \|u_w\|^2 + \|\varphi\|_{L^1} \; \|\psi\|_{L^\infty} \|u_w\|_\infty^{\sigma+1} \\ &\leq \varepsilon M \|u_w\|^2 + M_1^{\sigma+1} \|\varphi\|_{L^1} \; \|\psi\|_{L^\infty} \|u_w\|^{\sigma+1},\end{aligned}$$

which implies that

$$(1 - \varepsilon M) \|u_w\|^2 \leq M_1^{\sigma+1} \|\varphi\|_{L^1} \; \|\psi\|_{L^\infty} \|u_w\|^{\sigma+1}.$$

Hence

$$\|u_w\| \geq d_1, \quad \text{for some } d_1 > 0.$$

Claim 5. Let $w \in H^1_{0,p}(0, +\infty)$ fixed. Then there is a constant $d_2 > 0$, independent of w, such that $\|u_w\| \leq d_2$, for all solution u_w obtained above.
Indeed, by the characterization of the critical point and (H_3), it follows that

$$|J_w(u_w)| \leq \max_{\vartheta \in [0, +\infty)} J_w(\vartheta u^*),$$

where u^* is given in Claim 3.
From $(H_3)(a)$, we get

$$\begin{aligned}|J_w(u_w)| \leq & \max_{\vartheta \in [0,+\infty)} \left\{ \frac{1}{2}\vartheta^2 + c_3 |\vartheta|^\gamma M_1^\gamma \sum_{j=1}^{+\infty} g(t_j) - |\vartheta|^\mu \int_0^{+\infty} c_1(t) |u^*(t)|^\mu dt \right. \\ & \left. + \int_0^{+\infty} c_2(t) dt \right\}.\end{aligned}$$

We define K on $[0, +\infty)$ such that

$$K(\vartheta) = \frac{1}{2}\vartheta^2 + c_3 |\vartheta|^\gamma M_1^\gamma \sum_{j=1}^{+\infty} g(t_j) - |\vartheta|^\mu \int_0^{+\infty} c_1(t) |u^*(t)|^\mu dt + \int_0^{+\infty} c_2(t) dt,$$

and since $\mu > 2$, $K(\vartheta)$ can achieve its maximum at some ϑ_0.
Hence

$$|J_w(u_w)| \leq K(\vartheta_0).$$

On the other hand, we have

$$\begin{aligned}\left(1-\frac{2}{\mu}\right)\|u_w\|^2 &= 2J_w(u_w)-\frac{2}{\mu}(J'_w(u_w),u_w)\\ &+ 2\int_0^{+\infty}\Big[F(t,u_w(t),w'(t))-\frac{u_w(t)}{\mu}f(t,u_w(t),w'(t))\Big]dt\\ &+ 2\sum_{j=1}^{+\infty}g(t_j)\Big[\frac{u_w(t_j)}{\mu}I_j(u_w(t_j))-\int_0^{u_w(t_j)}I_j(\tau)d\tau\Big].\end{aligned}$$

Using $(H_3)(b),(I_1)$ and $(J'_w(u_w),u_w)=0$, we obtain

$$\left(1-\frac{2}{\mu}\right)\|u_w\|^2\le K(\vartheta_0).$$

Hence

$$\begin{aligned}\|u_w\| &\le \left(\frac{K(\vartheta_0)}{1-\frac{2}{\mu}}\right)^{\frac{1}{2}}\\ &\le d_2,\end{aligned}\tag{4.1}$$

we can choose $d_2=\left(\frac{K(\vartheta_0)}{1-\frac{2}{\mu}}\right)^{\frac{1}{2}}$, which is independent of w. □

Theorem 4.2. *Assume hypotheses* $(H_1)-(H_3),(I_0),(I_1)$ *hold and*
(H_4) *there exist positive constants* L_1 *and* L_2 *such that*

$$\begin{aligned}|f(t,x,\xi)-f(t,y,\xi)|\le L_1|x-y|,\quad &\forall t\in[0,+\infty),\ x,y\in[0;M_1d_2],\ \xi\in\mathbb{R},\\ |f(t,x,\xi)-f(t,x,\xi')|\le L_2|\xi-\xi'|,\quad &\forall t\in[0,+\infty),\ x\in[0;M_1d_2],\ \xi,\xi'\in\mathbb{R},\end{aligned}$$

(I_2) *there exist positive constants* α_j *such that*

$$|I_j(x)-I_j(y)|\le\alpha_j|x-y|,\quad\forall x,y\in[0;M_1d_2],\ j\in\{1,2,\ldots\}.$$

Then Problem (1.1) *has at least one nontrivial weak solution provided that*

$$0<\frac{L_2M}{1-L_1M-M_1^2\sum_{j=1}^{+\infty}g(t_j)\alpha_j}<1.$$

Proof. We construct a sequence $(u_n)\subset H^1_{0,p}(0,+\infty)$ as solutions of the problem

$$(P_n)\begin{cases}-(p(t)u'_n(t))' = f(t,u_n(t),u'_{n-1}(t)), & \text{a.e. } t\ge0,\ t\ne t_j,\\ u_n(0)=u_n(+\infty) = 0,\\ \triangle(p(t_j)u'_n(t_j)) = g(t_j)I_j(u_n(t_j)), & j\in\{1,2,\ldots\},\end{cases}$$

given in Theorem 4.1, starting with an arbitrary $u_0\in H^1_{0,p}(0,+\infty)$.
It follows from (4.1) and Lemma 1.4 that

$$\|u_n\|_\infty\le M_1d_2.$$

Using (P_{n+1}) and (P_n), we obtain

$$\int_0^{+\infty} p(t)u'_{n+1}(t)(u'_{n+1}(t)-u'_n(t))dt = -\sum_{j=1}^{+\infty} g(t_j)I_j(u_{n+1}(t_j))(u_{n+1}(t_j)-u_n(t_j)) + \int_0^{+\infty} f(t,u_{n+1}(t),u'_n(t))(u_{n+1}(t)-u_n(t))dt,$$

and

$$\int_0^{+\infty} p(t)u'_n(t)(u'_{n+1}(t)-u'_n(t))dt = -\sum_{j=1}^{+\infty} g(t_j)I_j(u_n(t_j))(u_{n+1}(t_j)-u_n(t_j)) + \int_0^{+\infty} f(t,u_n(t),u'_{n-1}(t))(u_{n+1}(t)-u_n(t))dt.$$

By subtracting, we obtain

$$\|u_{n+1}-u_n\|^2 = -\sum_{j=1}^{+\infty} g(t_j)\Big[I_j(u_{n+1}(t_j))-I_j(u_n(t_j))\Big](u_{n+1}(t_j)-u_n(t_j)) + \int_0^{+\infty} \Big[f(t,u_{n+1}(t),u'_n(t))-f(t,u_n(t),u'_{n-1}(t))\Big](u_{n+1}(t)-u_n(t))dt,$$

then

$$\|u_{n+1}-u_n\|^2 = -\sum_{j=1}^{+\infty} g(t_j)\Big[I_j(u_{n+1}(t_j))-I_j(u_n(t_j))\Big](u_{n+1}(t_j)-u_n(t_j)) + \int_0^{+\infty} \Big[f(t,u_{n+1}(t),u'_n(t))-f(t,u_n(t),u'_n(t))\Big](u_{n+1}(t)-u_n(t))dt + \int_0^{+\infty} \Big[f(t,u_n(t),u'_n(t))-f(t,u_n(t),u'_{n-1}(t))\Big](u_{n+1}(t)-u_n(t))dt.$$

By (H_4) and (I_2), we get

$$\|u_{n+1}-u_n\|^2 \le \sum_{j=1}^{+\infty} g(t_j)\alpha_j|u_{n+1}(t_j)-u_n(t_j)|^2 + L_1\int_0^{+\infty} |u_{n+1}(t)-u_n(t)|^2dt + L_2\int_0^{+\infty} |u'_n(t)-u'_{n-1}(t)|\,|u_{n+1}(t)-u_n(t)|dt.$$

Using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\|u_{n+1}-u_n\|^2 &\leq \|u_{n+1}-u_n\|_\infty^2 \sum_{j=1}^{+\infty} g(t_j)\alpha_j + L_1\|u_{n+1}-u_n\|_{L^2}^2 \\ &\quad + L_2\|u_n'-u_{n-1}'\|_{L^2}\,\|u_{n+1}-u_n\|_{L^2} \\ &\leq M_1^2\|u_{n+1}-u_n\|^2 \sum_{j=1}^{+\infty} g(t_j)\alpha_j + L_1 M\|u_{n+1}-u_n\|^2 \\ &\quad + L_2 M\|u_n-u_{n-1}\|\,\|u_{n+1}-u_n\|,\end{aligned}$$

which implies that

$$\|u_{n+1}-u_n\| \leq \frac{L_2 M}{1-L_1M-M_1^2\sum_{j=1}^{+\infty} g(t_j)\alpha_j}\|u_n-u_{n-1}\|.$$

Since

$$0<\frac{L_2 M}{1-L_1M-M_1^2\sum_{j=1}^{+\infty} g(t_j)\alpha_j}<1,$$

it follows that (u_n) is a Cauchy sequence in the reflexive Banach space $H^1_{0,p}(0,+\infty)$. Then the sequence (u_n) strongly converges in $H^1_{0,p}(0,+\infty)$ to some $u\in H^1_{0,p}(0,+\infty)$. Since $\|u_n\|\geq d_1,\ \forall n\in\mathbb{N}$, it follows that $u\neq 0$.
Consequently, we obtain a nontrivial solution for Problem (1.1). □

Now we prove the existence of a solution for the problem (1.1) by using the Minimization principle.

4.2. The sublinear case

Theorem 4.3. *Suppose that the following conditions hold:*
(H_5) *There exist a constant* $\alpha\in[0,1)$ *and positive functions* $a_1,b_1\in L^1(0,+\infty)$ *such that*

$$|f(t,x,\xi)|\leq a_1(t)|x|^\alpha+b_1(t),\ \text{ for a.e. } t\in[0,+\infty) \text{ and all } x\in\mathbb{R},\xi\in\mathbb{R}.$$

(I_3) *There exist constants* $c_4>0$ *and* $\beta\in[0,1)$ *such that*

$$|I_j(s)|\leq c_4|s|^\beta,\ \forall\, s\in\mathbb{R},\ j\in\{1,2,\ldots\}.$$

Then there exists positive constant d_3 *such that, for each* $w\in H^1_{0,p}(0,+\infty)$, *Problem (*1.2*) has at least one weak solution* u_w *satisfying*

$$\|u_w\|\leq d_3.$$

Proof. Claim 1. Let $w\in H^1_{0,p}(0,+\infty)$ *fixed. The functional* J_w *is well defined.*
Indeed, take u in $H^1_{0,p}(0,+\infty)$. From (H_5), we deduce that

$$|F(t,u(t),w'(t))|\leq\frac{a_1(t)}{\alpha+1}|u(t)|^{\alpha+1}+b_1(t)|u(t)|.$$

Thus, by using Lemma 1.4

$$\begin{aligned}\left|\int_0^{+\infty} F(t,u(t),w'(t))dt\right| &\leq \|u\|_\infty^{\alpha+1}\int_0^{+\infty} a_1(t)dt + \|u\|_\infty \int_0^{+\infty} b_1(t)dt \\ &\leq \frac{M_1^{\alpha+1}}{\alpha+1}\|u\|^{\alpha+1}\int_0^{+\infty} a_1(t)dt + M_1\|u\|\int_0^{+\infty} b_1(t)dt \\ &\leq \frac{M_1^{\alpha+1}}{\alpha+1}\|u\|^{\alpha+1}\|a_1\|_{L^1} + M_1\|u\|\|b_1\|_{L^1}.\end{aligned}$$

It follows from (I_3) that

$$\begin{aligned}\left|\sum_{j=1}^{+\infty} g(t_j)\int_0^{u(t_j)} I_j(\tau)d\tau\right| &\leq \frac{c_4}{\beta+1}\|u\|_\infty^{\beta+1}\sum_{j=1}^{+\infty} g(t_j) \\ &\leq \frac{c_4 M_1^{\beta+1}}{\beta+1}\|u\|^{\beta+1}\sum_{j=1}^{+\infty} g(t_j).\end{aligned}$$

Hence

$$\begin{aligned}|J_w(u)| \leq \frac{1}{2}\|u\|^2 &+ \frac{c_4 M_1^{\beta+1}}{\beta+1}\|u\|^{\beta+1}\sum_{j=1}^{+\infty} g(t_j) \\ &+ \frac{M_1^{\alpha+1}}{\alpha+1}\|u\|^{\alpha+1}\|a_1\|_{L^1} + M_1\|u\|\|b_1\|_{L^1} \\ &< \infty.\end{aligned}$$

Claim 2. Let $w \in H^1_{0,p}(0,+\infty)$ fixed. J_w is sequentially weakly lower semicontinuous. Indeed, let (u_n) be a sequence in $H^1_{0,p}(0,+\infty)$ such that $u_n \rightharpoonup u$ in $H^1_{0,p}(0,+\infty)$, as $n \to \infty$. Lemma 1.4 implies that (u_n) converges uniformly to u on $[0,+\infty)$ and by the fact that the norm is weakly lower semicontinuous, we have

$$\liminf_{n\to+\infty}\|u_n\| \geq \|u\|.$$

Using the Lebesgue Dominated Convergence Theorem and the continuity of the functions f and $I_j, j \in \{1,2,\ldots\}$, we obtain

$$\begin{aligned}\liminf_{n\to+\infty} J_w(u_n) &= \liminf_{n\to+\infty}\left(\frac{1}{2}\|u_n\|^2 + \sum_{j=1}^{+\infty} g(t_j)\int_0^{u_n(t_j)} I_j(\tau)d\tau \right. \\ &\qquad \left. - \int_0^{+\infty} F(t,u_n(t),w'(t))dt\right) \\ &\geq \frac{1}{2}\|u\|^2 + \sum_{j=1}^{+\infty} g(t_j)\int_0^{u(t_j)} I_j(\tau)d\tau - \int_0^{+\infty} F(t,u(t),w'(t))dt \\ &= J(u).\end{aligned}$$

Consequently, J_w is sequentially weakly lower semicontinuous.

Claim 3. Let $w \in H^1_{0,p}(0,+\infty)$ fixed. J_w is coercive.
Indeed, From $(H_5),(I_3)$ and Lemma 1.4, we have

$$\begin{aligned} J_w(u) \geq \frac{1}{2}\|u\|^2 \;-\; \frac{c_4 M_1^{\beta+1}}{\beta+1}\|u\|^{\beta+1}\sum_{j=1}^{+\infty} g(t_j) \\ -\frac{M_1^{\alpha+1}}{\alpha+1}\|u\|^{\alpha+1}\|a_1\|_{L^1} \;-\; M_1\|u\|\|b_1\|_{L^1}. \end{aligned} \tag{4.2}$$

Since $\alpha < 1$ and $\beta < 1$, then (4.2) implies that

$$\lim_{\|u\|\longrightarrow+\infty} J_w(u) = +\infty.$$

So, by Lemma 2.1, J_w has a minimum point u_w. Under hypothesis (H_5) and using the same ideas as in Proposition 3.2, we get, J_w is Gâteaux differentiable. Thus u_w is a critical point of J_w.
Claim 4. Let $w \in H^1_{0,p}(0,+\infty)$ fixed. Then $\|u_w\| \leq d_3$, for some $d_3 > 0$, for all solutions u_w obtained above.
Indeed, let u_w be a solution of Problem (1.2). Then

$$\|u_w\|^2 = \int_0^{+\infty} f(t,u_w(t),w'(t))u_w(t)dt - \sum_{j=1}^{+\infty} g(t_j)I_j(u_w(t_j))u_w(t_j).$$

By (H_5) and (I_3), we get

$$\begin{aligned} \|u_w\|^2 &\leq \int_0^{+\infty} a_1(t)|u_w(t)|^{\alpha+1}dt + \int_0^{+\infty} b_1(t)|u_w(t)|dt \\ &\quad +c_4\sum_{j=1}^{+\infty} g(t_j)|u_w(t_j)|^{\beta+1} \\ &\leq \|u_w\|_\infty^{\alpha+1}\int_0^{+\infty} a_1(t)dt + \|u_w\|_\infty\int_0^{+\infty} b_1(t)dt + c_4\|u_w\|_\infty^{\beta+1}\sum_{j=1}^{+\infty} g(t_j) \\ &\leq M_1^{\alpha+1}\|u_w\|^{\alpha+1}\|a_1\|_{L^1} + M_1\|u_w\|\|b_1\|_{L^1} + c_4 M_1^{\beta+1}\|u_w\|^{\beta+1}\sum_{j=1}^{+\infty} g(t_j). \end{aligned}$$

Hence

$$\|u_w\| \leq d_3, \quad \text{for some } d_3 > 0.$$

Therefor u_w is a weak solution of Problem (1.2). □

Remark 4.4. In addition, if $u_w \in H^2_p(t_j,t_{j+1})$, for all $j \in \{1,2,\ldots\}$, where

$$H^2_p(t_j,t_{j+1}) = \{u \in AC[0,+\infty),\mathbb{R}) : \sqrt{p}u' \in L^2(t_j,t_{j+1}),\ (pu')' \in L^2(t_j,t_{j+1})\},$$

then u_w will be called a strong solution of Problem (1.2).

Proposition 4.5. *In (H_5), assume that $a_1, b_1 \in L^2(0,+\infty)$. Then every weak solution is a strong solution of Problem (1.2).*

Proof. We know that $u_w \in H^1_{0,p}(0,+\infty)$ is a critical point of J_w. Then, for any $v \in H^1_{0,p}(0,+\infty)$, we have

$$\int_0^{+\infty} p(t)u_w'(t)v'(t)dt + \sum_{j=1}^{+\infty} g(t_j)I_j(u_w(t_j))v(t_j) - \int_0^{+\infty} f(t,u_w(t),w'(t))v(t)dt = 0. \tag{4.3}$$

For $j \in \{1,2,\ldots\}$, if $v \in H^1_{0,p}(t_j,t_{j+1})$ $(v = v_j)$, then

$$\int_{t_j}^{t_{j+1}} p(t)u_w'(t)v'(t)dt = \int_{t_j}^{t_{j+1}} f(t,u_w(t),w'(t))v(t)dt.$$

So $u_{w,j} \in H^1_{0,p}(t_j,t_{j+1})$ is a solution of the equation:

$$-(p(t)u_w')' = f(t,u_w(t),w'(t)),\ t \in (t_j,t_{j+1}), \tag{4.4}$$

Since, $u_w \in C_0[0,+\infty)$, and by (H_5), we get

$$|f(t,u_w(t),w'(t))|^2 \leq 2\left(a_1(t)^2\|u_w\|_\infty^{2\alpha} + b_1(t)^2\right),$$

thus $u_{w,j} \in H^2_p\,(t_j,t_{j+1})$. Then (4.4), implies that the limits
$u'(t_j^+), u'(t_j^-)$, $\quad j \in \{1,2,\ldots\}$ exist.
Using the integration by parts in (4.3), we obtain

$$\begin{aligned} 0 &= -\sum_{j=0}^{j=+\infty} \int_{t_j}^{t_{j+1}} (p(t)u_w'(t))'v(t)dt - \sum_{j=1}^{+\infty} \triangle(p(t_j)u_w'(t_j))v(t_j) \\ &\quad + \sum_{j=1}^{+\infty} g(t_j)I_j(u_w(t_j))v(t_j) - \int_0^{+\infty} f(t,u_w(t),w'(t))v(t)dt. \end{aligned}$$

Since u_w satisfies the equation in problem (1.2) a.e. on $[0,+\infty)$, we deduce that

$$\sum_{j=1}^{+\infty} g(t_j)I_j(u_w(t_j))v(t_j) = \sum_{j=1}^{+\infty} \triangle(p(t_j)u_w'(t_j))v(t_j), \quad \text{for all}\, v \in H^1_{0,p}(0,+\infty).$$

Thus

$$\triangle(p(t_j)u_w'(t_j)) = g(t_j)I_j(u_w(t_j)), \ \text{ for every } \ j \in \{1,2,\ldots\}.$$

Actually, u_w is even a classical solution, i.e., $u \in C^2(t_j,t_{j+1})$, for all $j \in \{1,2,\ldots\}$, when $f : [0,+\infty) \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous. □

Theorem 4.6. *Assume that* $(H_4),(H_5),(I_2)$ *and* (I_3) *hold.*
Then Problem (1.1) *has at least one classical solution provided that*

$$0 < \frac{L_2 M}{1 - L_1 M - M_1^2 \sum_{j=1}^{+\infty} g(t_j)\alpha_j} < 1.$$

Proof. The proof is similar to the proof of Theorem 4.2. □

Example 4.7. Consider the impulsive boundary value problem

$$\begin{cases} -(e^t u'(t))' &= \frac{\sqrt{|u|}}{(1+t)^2}\cos u' + \frac{1}{(1+t)^3}, \quad \text{a.e. } t \geq 0,\ t \neq t_j, \\ u(0) = u(+\infty) &= 0, \\ \triangle(e^j u'(j)) &= \frac{\sqrt[3]{u(j)}}{1+j^2}, \qquad j \in \{1, 2, \ldots\}. \end{cases} \tag{4.5}$$

We know that all hypotheses of Theorem 4.3 are satisfied with

$$f(t, x, \xi) = \frac{\sqrt{|x|}}{(1+t)^2}\cos\xi + \frac{1}{(1+t)^3},$$

$$\alpha = 1/2,\ a_1(t) = \frac{1}{(1+t)^2},\ b_1(t) = \frac{1}{(1+t)^3},$$

$$I_j(s) = s^{1/3},\ \beta = \frac{1}{3},\ c_4 = 1,$$

$$g(t) = \frac{1}{1+t^2} \text{ and } \sum_{j=1}^{\infty} g(j) = \frac{\pi}{4}.$$

Consequently, problem (4.5) has at least one solution.

4.3. The limit case $\alpha = 1$

Theorem 4.8. *Suppose that* (I_3) *holds and*
(H_6) *there exist positive functions* $a_2, b_2 \in L^1(0, +\infty)$ *with* $||a_2||_{L^1} < \frac{1}{M_1^2}$ *and*

$$|f(t, x, \xi)| \leq a_2(t)|x| + b_2(t), \quad \text{for a.e. } t \in [0, +\infty) \text{ and } \forall x \in \mathbb{R}, \xi \in \mathbb{R}.$$

Then there exists positive constant d_4 *such that, for each* $w \in H^1_{0,p}(0, +\infty)$*, Problem* (1.2) *has at least one weak solution* u_w *satisfying*

$$\|u_w\| \leq d_4.$$

Proof. Claim 1. Let $w \in H^1_{0,p}(0, +\infty)$ *fixed.* J_w *is sequentially weakly lower semicontinuous.*
Indeed, we use the same technique as in the proof of Theorem 4.3.
Claim 2. Let $w \in H^1_{0,p}(0, +\infty)$ *fixed.* J_w *is coercive.*
Indeed, by (H_6), we obtain

$$|F(t, u(t), w'(t))| \leq \frac{a_2(t)}{2}|u(t)|^2 + b_2(t)|u(t)|,$$

hence

$$\begin{aligned} \left|\int_0^{+\infty} F(t, u(t), w'(t))dt\right| &\leq \int_0^{+\infty} \left(\frac{a_2(t)}{2}|u(t)|^2 + b_2(t)|u(t)|\right) dt \\ &\leq \frac{M_1^2}{2}\|u\|^2\|a_2\|_{L^1} + M_1\|u\|\|b_2\|_{L^1}. \end{aligned}$$

Thus

$$\begin{aligned} J_w(u) \quad &\geq \quad \frac{1}{2}\left(1 - M_1^{\,2}\|a_{\,2}\|_{L^1}\right)\|u\|^2 - M_1\|u\|\|b_2\|_{L^1} \\ &\quad - \frac{c_4 M_1^{\beta+1}}{\beta+1}\|u\|^{\,\beta+1}\sum_{j=1}^{+\infty} g(t_j). \end{aligned} \tag{4.6}$$

Since $\|a_2\|_{L^1} < \frac{1}{M_1^2}$ and $\beta < 1$, we pass to the limit in (4.6) when $n \to +\infty$, we get

$$\lim_{\|u\| \longrightarrow +\infty} J_w(u) = +\infty.$$

Therefore, J_w is coercive.
By applying Lemma 2.1, we find that J_w has a minimum point u_w. Under hypothesis (H_6) and using the same ideas as in Proposition 3.2, we get, J_w is Gâteaux differentiable. Then u_w is a critical point of J_w which is a weak solution of Problem (1.2).
Claim 3. Let $w \in H^1_{0,p}(0,+\infty)$ fixed. Then $\|u_w\| \leq d_4$, for some $d_4 > 0$, for all solutions u_w obtained above.
Indeed, let u_w be a solution of Problem (1.2). Then

$$\|u_w\|^2 = \int_0^{+\infty} f(t, u_w(t), w'(t))u_w(t)dt - \sum_{j=1}^{+\infty} g(t_j)I_j(u_w(t_j))u_w(t_j).$$

It follows from (H_6) and (I_3) that

$$\begin{aligned} \|u_w\|^2 \quad &\leq \quad \int_0^{+\infty} a_2(t)|u_w(t)|^2 dt + \int_0^{+\infty} b_2(t)|u_w(t)|dt \\ &\quad + c_4\sum_{j=1}^{+\infty} g(t_j)|u_w(t_j)|^{\beta+1} \\ &\leq \quad \|u_w\|_\infty^2 \int_0^{+\infty} a_2(t)dt + \|u_w\|_\infty \int_0^{+\infty} b_2(t)dt + c_4\|u_w\|_\infty^{\beta+1}\sum_{j=1}^{+\infty} g(t_j) \\ &\leq \quad M_1^2\|a_2\|_{L^1}\|u_w\|^2 + M_1\|u_w\|\|b_2\|_{L^1} + c_4 M_1^{\beta+1}\|u_w\|^{\beta+1}\sum_{j=1}^{+\infty} g(t_j). \end{aligned}$$

Thus

$$(1 - M_1^2\|a_2\|_{L^1})\|u_w\|^2 \quad \leq \quad M_1\|u_w\|\|b_2\|_{L^1} + c_4 M_1^{\beta+1}\|u_w\|^{\beta+1}\sum_{j=1}^{+\infty} g(t_j).$$

Hence

$$\|u_w\| \leq d_4, \quad \text{for some } d_4 > 0.$$

□

Theorem 4.9. *Assume that $(H_4), (H_6), (I_2)$ and (I_3) hold.*
Then Problem (1.1) has at least one weak solution provided that

$$0 < \frac{L_2 M}{1 - L_1 M - M_1^2\sum_{j=1}^{+\infty} g(t_j)\alpha_j} < 1.$$

Proof. Reasoning like in the proof of Theorem 4.2, we can prove that Problem (1.1) has at least one weak solution. □

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General stabilization of a thermoelastic systems with a boundary control of a memory type

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Abstract. In this paper we consider an n-dimensional thermoelastic system, in a bounded domain, where the memory-type damping is acting on a part of the boundary and where the resolvent kernel k of $-g'(t)/g(0)$ satisfies $k''(t) \geq \gamma(t)(-k'(t))^p$, $t \geq 0$, $1 < p < \frac{3}{2}$. We establish a general decay result, from which the usual exponential and polynomial decay rates are only special cases. This work generalizes and improves earlier results in the literature.

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Keywords: Thermoelasticity, general decay, memory type, boundary damping, resolvent kernel.

1. Introduction

In [4], Messaoudi and Al-Khulaifi studied the following problem

$$\begin{cases} |u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-s)\,\Delta u(s)\,ds = 0, & \text{in } \Omega \times (0, +\infty) \\ u(x,t) = 0, & x \in \partial\Omega,\ t \geq 0 \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^n$ $(n \geq 1)$ with a smooth boundary $\partial\Omega$, ρ is a positive real number such that $0 < \rho \leq 2/(n-2)$ if $n \geq 3$ and $\rho > 0$ if $n = 1, 2$, and g is a positive nonincreasing function. They obtained a general decay rate where the relaxation functions satisfies

$$g'(t) \leq -\xi(t)\, g^p(t), \quad \forall t \geq 0, \quad 1 \leq p < \frac{3}{2}.$$

Stabilization of thermoelastic systems has been studied by many researchers. Different mechanisms have been utilized to stabilize such systems and several decay and stability results have been obtained. In this regard we mention, among many

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others, the work of Dafermos [2], Messaoudi and Al-Shehri [3], Muñoz Rivera [7], Rivera and Barreto [8], Rivera and Racke [9], Racke and Shibata [11].

In the present work, we are concerned with

$$\begin{cases} u_{tt} - \mu \triangle u - (\mu + \lambda) \nabla (divu) + \beta \nabla \theta = 0, & \text{in } \Omega \times (0, +\infty) \\ c\theta_t - \kappa \triangle \theta + \beta divu_t = 0, & \text{in } \Omega \times (0, +\infty) \\ u(.,0) = u_0, \quad u_t(.,0) = u_1, \quad \theta(.,0) = \theta_0, & \text{in } \Omega \\ u = 0, & \text{on } \Gamma_0 \times [0, +\infty) \\ u(x,t) = -\int_0^t g(t-s) \left(\mu \frac{\partial u}{\partial \nu} + (\mu + \lambda)(divu)\nu \right)(s)\, ds, & \text{on } \Gamma_1 \times [0, +\infty) \\ \theta = 0, & \text{on } \Gamma \times [0, +\infty), \end{cases} \tag{1.2}$$

which is a thermoelastic system subjected to the effect of a viscoelastic damping acting on a part of the boundary. Here Ω is a bounded domain of $\mathbb{R}^n$ $(n \geq 2)$ with a smooth boundary $\Gamma = \Gamma_0 \cup \Gamma_1$, ν is the unit outward normal vector to Γ, $u = u(x,t) \in \mathbb{R}^n$ is the displacement vector, $\theta = \theta(x,t)$ is the difference temperature. The relaxation function g is positive and differentiable function and the boundary condition on Γ_1 is the nonlocal condition responsible for the memory effect. The coefficients c, κ, μ, λ are positive constants, where μ, λ are Lame moduli and $\beta \neq 0$ is a real number. By considering the resolvent kernel of $-g'/g(0)$, the boundary condition takes the form

$$\frac{\partial u}{\partial \nu} = -\frac{1}{g(0)} (u_t + k * u_t), \quad \text{on } \Gamma_1 \times [0, +\infty),$$

where k is the resolvent kernel of $-g'/g(0)$.
Messaoudi and Al-Shehri [3] considered (1.2) for a wider class of kernels k satisfying

$$k(0) > 0, \quad k(t) \geq 0, \quad k'(t) \leq 0, \quad k''(t) \geq \gamma(t)(-k'(t)),$$

where $\gamma : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a function satisfying the following conditions

$$\gamma(t) \geq 0, \quad \gamma'(t) \leq 0, \quad \text{and} \int_0^\infty \gamma(t)dt = +\infty, \tag{1.3}$$

they proved a more general energy decay result.
Recently, Mustafa [10] treated system (1.2), for k satisfying

$$k(0) > 0, \quad \lim_{t \to \infty} k(t) = 0, \quad k'(t) \leq 0, \tag{1.4}$$

$$k''(t) \geq H(-k'(t)), \quad \forall t > 0, \tag{1.5}$$

where H is a positive function, which is linear or strictly increasing, strictly convex of class C^2 on $(0, r]$, $r < 1$, and $H(0) = 0$ and proved for $u_0 = 0$ on Γ_1, an explicit energy decay formula, from which the usual exponential and polynomial decay rates are only special cases.
The aim of this work is to study problem (1.2) for k satisfies

$$k(0) > 0,\ k(t) \geq 0,\ k'(t) \leq 0,\ k''(t) \geq \gamma(t)(-k'(t))^p,\ t \geq 0,\ 1 < p < \frac{3}{2}, \tag{1.6}$$

where γ satisfies (1.3).

2. Notation and transformation

In this section we introduce our problem, as well as some notation and lemmas. The partition Γ_0 and Γ_1 of boundary are closed, disjoint, with $meas\,(\Gamma_0) > 0$ and satisfying

$$\Gamma_1 = \{x \in \Gamma : \ m\,(x)\,.\nu \geq \delta > 0\}\,, \quad \Gamma_0 = \{x \in \Gamma : \ m\,(x)\,.\nu \leq 0\}\,, \tag{2.1}$$

where $m\,(x) = x - x_0$, for some $x_0 \in \mathbb{R}^n$.

Similarly to [5, 3, 6], applying Volterra's inverse operator, the boundary condition

$$u\,(x,t) = -\int_0^t g\,(t-s)\left(\mu \frac{\partial u}{\partial \nu} + (\mu+\lambda)\,(divu)\,\nu\right)(s)\,ds, \quad \text{on } \ \Gamma_1 \times [0,+\infty),$$

can be transformed into

$$\mu \frac{\partial u}{\partial \nu} + (\mu+\lambda)\,(divu)\,\nu = -\frac{1}{g\,(0)}\,(u_t + k * u_t)\,, \quad \text{on } \ \Gamma_1 \times [0,+\infty),$$

where $*$ denotes the convolution product

$$(\varphi * \psi)(t) = \int_0^t \varphi(t-s)\psi(s)ds,$$

and k is the resolvent kernel of $-g'/g(0)$ which satisfies

$$k + \frac{1}{g\,(0)}\,(g' * k) = -\frac{1}{g\,(0)} g'.$$

Taking $\eta = 1/g\,(0)$ and assuming throughout the paper that $u_0 = 0$ on Γ_1, we arrive at

$$\mu \frac{\partial u}{\partial \nu} + (\mu+\lambda)\,(divu)\,\nu = -\eta\,(u_t + k(0)u + k' * u)\,, \quad \text{on } \ \Gamma_1 \times [0,+\infty). \tag{2.2}$$

Therefore, we will use the boundary relation (2.2) instead of the third equation in (1.2).

Since we are interested in relaxation functions of more general decay, we would like to know if the resolvent kernel k, involved in (2.2), inherits some properties of the relaxation function involved in $(1.2)_3$. The following Lemma answers this question.

Let $h : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be continuous. Let k be its resolvent, i.e.

$$k(t) = h(t) + (k * h)(t), \tag{2.3}$$

It is well known that k is continuous and positive (see [1, 9]).

Lemma 2.1. *Let $p > 1$, $\gamma : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be a nonincreasing function satisfying $\gamma(0) > 0$, and*

$$\begin{aligned} C_p \ &= \ \sup_{t\geq 0} \int_0^t \left(1 + \int_0^t \gamma^{2p-1}(\zeta)\,d\zeta\right)^{\frac{1}{2p-2}} \left(1 + \int_0^{t-s} \gamma^{2p-1}(\zeta)\,d\zeta\right)^{-\frac{1}{2p-2}} \\ &\quad \times \left(1 + \int_0^s \gamma^{2p-1}(\zeta)\,d\zeta\right)^{-\frac{1}{2p-2}} ds. \end{aligned}$$

Assume that there exists C *and* $1 - CC_p > 0$ *such that*

$$h(t) \leq \frac{C}{\left(1 + \int_0^t \gamma^{2p-1}(\zeta)\, d\zeta\right)^{\frac{1}{2p-2}}}.$$

Then there exists $\widetilde{C}$ *such that*

$$k(t) \leq \frac{\widetilde{C}}{\left(1 + \int_0^t \gamma^{2p-1}(\zeta)\, d\zeta\right)^{\frac{1}{2p-2}}}.$$

Proof. We set

$$k_p(t) = k(t)\left(1 + \int_0^t \gamma^{2p-1}(\zeta)\, d\zeta\right)^{\frac{1}{2p-2}},$$

and

$$h_p(t) = h(t)\left(1 + \int_0^t \gamma^{2p-1}(\zeta)\, d\zeta\right)^{\frac{1}{2p-2}}.$$

By multiplying (2.3) by $\left(1 + \int_0^t \gamma^{2p-1}(\zeta)\, d\zeta\right)^{\frac{1}{2p-2}}$, we obtain

$$\begin{aligned}
k_p(t) &= h_p(t) + \int_0^t \left(1 + \int_0^t \gamma^{2p-1}(\zeta)\, d\zeta\right)^{\frac{1}{2p-2}} k(t-s)\, h(s)\, ds \\
&= h_p(t) + \int_0^t \left(1 + \int_0^t \gamma^{2p-1}(\zeta)\, d\zeta\right)^{\frac{1}{2p-2}} \left(1 + \int_0^{t-s} \gamma^{2p-1}(\zeta)\, d\zeta\right)^{-\frac{1}{2p-2}} \\
&\times k_p(t-s)\, h(s)\, ds \\
&= h_p(t) + \int_0^t \left(1 + \int_0^t \gamma^{2p-1}(\zeta)\, d\zeta\right)^{\frac{1}{2p-2}} \left(1 + \int_0^{t-s} \gamma^{2p-1}(\zeta)\, d\zeta\right)^{-\frac{1}{2p-2}} \\
&\times \left(1 + \int_0^s \gamma^{2p-1}(\zeta)\, d\zeta\right)^{-\frac{1}{2p-2}} k_p(t-s) \left(1 + \int_0^s \gamma^{2p-1}(\zeta)\, d\zeta\right)^{\frac{1}{2p-2}} h(s)\, ds \\
&= h_p(t) + \int_0^t \left(1 + \int_0^t \gamma^{2p-1}(\zeta)\, d\zeta\right)^{\frac{1}{2p-2}} \left(1 + \int_0^{t-s} \gamma^{2p-1}(\zeta)\, d\zeta\right)^{-\frac{1}{2p-2}} \\
&\times \left(1 + \int_0^s \gamma^{2p-1}(\zeta)\, d\zeta\right)^{-\frac{1}{2p-2}} k_p(t-s)\, h_p(s)\, ds,
\end{aligned}$$

Consequently,

$$\sup_{0 \leq s \leq t} k_p(s) \leq \sup_{0 \leq s \leq t} h_p(s) + CC_p \sup_{0 \leq s \leq t} k_p(s) \leq C + CC_p \sup_{0 \leq s \leq t} k_p(s),$$

which implies

$$\sup_{0 \leq s \leq t} k_p(s) \leq \frac{C}{1 - CC_p}, \ \forall t > 0.$$

Hence

$$k_p(t) \leq \frac{C}{1 - CC_p}.$$

Therefore

$$k(t) \leq \frac{C}{1-CC_p}\left(1+\int_0^t \gamma^{2p-1}(\zeta)\,d\zeta\right)^{-\frac{1}{2p-2}}.$$

Finally, we obtain the result of the lemma

$$k(t) \leq \frac{\widetilde{C}}{\left(1+\int_0^t \gamma^{2p-1}(\zeta)\,d\zeta\right)^{\frac{1}{2p-2}}}. \qquad \square$$

Let us define

$$(\varphi \circ \psi)(t) = \int_0^t \varphi(t-s)\left|\psi(t)-\psi(s)\right|^2 ds,$$

$$(\varphi \Diamond \psi)(t) = \int_0^t \varphi(t-s)\left(\psi(t)-\psi(s)\right) ds.$$

By using Hölder's inequality, we have

$$\left|(\varphi \Diamond \psi)(t)\right|^2 \leq \left(\int_0^t |\varphi(s)|\,ds\right)(|\varphi| \circ \psi)(t). \tag{2.4}$$

Lemma 2.2 ([9]). *If* $\varphi,\ \psi \in C^1(\mathbb{R}^+)$, *then*

$$(\varphi * \psi)\psi_t = -\frac{1}{2}\varphi(t)\left|\psi(t)\right|^2 + \frac{1}{2}\varphi' \circ \psi - \frac{1}{2}\frac{d}{dt}\left(\varphi \circ \psi - \left(\int_0^t \varphi(s)ds\right)\left|\psi(t)\right|^2\right). \tag{2.5}$$

Let us define

$$V = \left\{u \in H^1(\Omega) : \ u = 0 \quad \text{on} \quad \Gamma_0\right\}.$$

The well-posedness of system (1.2) is presented in the following theorem, which can be proved, using the Galerkin method as in [9].

Theorem 2.3. *Let* $k \in W^{2,1}(\mathbb{R}^+) \cap W^{1,\infty}(\mathbb{R}^+)$, $u_0 \in \left(H^2(\Omega) \cap V\right)^n$, $\theta_0 \in H^2(\Omega) \cap H_0^1(\Omega)$, *and* $u_1 \in V^n$, *with*

$$\frac{\partial u_0}{\partial \nu} + \eta u_0 = 0 \quad \text{on } \Gamma_1.$$

Then there exists a unique strong solution u *of system* (1.2), *such that*

$$\begin{aligned} u &\in C\left(\mathbb{R}^+; \left(H^2(\Omega) \cap V\right)^n\right) \cap C^1\left(\mathbb{R}^+; V^n\right) \cap C^2\left(\mathbb{R}^+; L^2(\Omega)^n\right), \\ \theta &\in C\left(\mathbb{R}^+; H^2(\Omega) \cap H_0^1(\Omega)\right) \cap C^1\left(\mathbb{R}^+; H_0^1(\Omega)\right). \end{aligned}$$

3. Decay of solutions

In this section we study the asymptotic behavior of the solutions of system (1.2) when the resolvent kernel k satisfies the assumption

$$k(0) > 0, \quad k(t) \geq 0, \quad k'(t) \leq 0, \quad k''(t) \geq \gamma(t)\left(-k'(t)\right)^p, \tag{3.1}$$

where $t \geq 0$, $1 < p < \frac{3}{2}$ and $\gamma : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a function satisfying

$$\gamma(t) > 0, \quad \gamma'(t) \leq 0. \tag{3.2}$$

By multiplying the first equation in (1.2) by u_t and the second equation in (1.2) by θ and integrating over Ω, using integration by parts and boundary conditions (2.2) and (2.5), one can easily find that the first order energy of system (1.2) is given by (see Lemma 3.1 below).

$$\begin{aligned} E(t) &= \frac{1}{2}\int_{\Omega}\left[|u_t|^2+\mu|\nabla u|^2+(\mu+\lambda)(divu)^2+c\theta^2\right]dx \\ &-\frac{\eta}{2}\int_{\Gamma_1}(k'\circ u)(t)d\Gamma_1+\frac{\eta}{2}\int_{\Gamma_1}k(t)|u|^2 d\Gamma_1. \end{aligned} \tag{3.3}$$

Lemma 3.1. *The energy of the solution of* (1.2) *satisfies*

$$\begin{aligned} E'(t) &= -\kappa\int_{\Omega}|\nabla\theta|^2dx-\eta\int_{\Gamma_1}|u_t|^2d\Gamma_1+\frac{\eta}{2}k'(t)\int_{\Gamma_1}|u|^2d\Gamma_1 \\ &-\frac{\eta}{2}\int_{\Gamma_1}(k''\circ u)(t)\,d\Gamma_1\leq 0. \end{aligned} \tag{3.4}$$

Proof. Direct differentiation, using Eqs. (1.2) and (2.2), gives

$$\begin{aligned} E'(t) &= -\kappa\int_{\Omega}|\nabla\theta|^2dx-\eta\int_{\Gamma_1}|u_t|^2d\Gamma_1+\frac{\eta}{2}k'(t)\int_{\Gamma_1}|u|^2d\Gamma_1 \\ &-\frac{\eta}{2}\int_{\Gamma_1}(k''\circ u)(t)\,d\Gamma_1. \end{aligned}$$

and consequently, we obtain (3.4) for strong solutions. This result and all estimates below remain valid for weak solutions by a simple density argument. □

The following crucial lemmas will be used in the proof of our result.

Lemma 3.2. *The solution u of* (1.2) *satisfies*

$$\|u(t)-u(s)\|^2_{L^2(\Gamma_1)}\leq CE(0), \quad \forall s\in[0,t].$$

Proof. Using the trace theorem and (3.3), we obtain, for all $s\in[0,t]$,

$$\begin{aligned} \|u(t)-u(s)\|^2_{L^2(\Gamma_1)} &\leq c\|\nabla u(t)-\nabla u(s)\|_2^2 \\ &\leq c\left(\|\nabla u(t)\|_2^2+\|\nabla u(s)\|_2^2\right) \\ &\leq c'(E(t)+E(s)) \\ &\leq C(E(0)). \end{aligned}$$

□

Lemma 3.3. *Assume that k satisfies (*3.1*). Then*

$$\int_0^{+\infty}\gamma(t)\left[-k'(t)\right]^{1-\sigma}dt<+\infty,\ \forall\sigma<2-p.$$

Proof. Recalling (3.1), we easily see that

$$\begin{aligned}\gamma(t)\left[-k'(t)\right]^{1-\sigma} &= \gamma(t)\left(-k'(t)\right)^p\left[-k'(t)\right]^{1-\sigma-p} \\ &\leq k''(t)\left[-k'(t)\right]^{1-\sigma-p}.\end{aligned}$$

Then, integration gives

$$\begin{aligned}\int_0^{+\infty}\gamma(t)\left[-k'(t)\right]^{1-\sigma}dt &\leq \int_0^{+\infty}k''(t)\left[-k'(t)\right]^{1-\sigma-p}dt \\ &= -\left.\frac{\left[-k'(t)\right]^{2-p-\sigma}}{2-p-\sigma}\right|_0^{+\infty} < +\infty, \end{aligned} \tag{3.5}$$

since $\sigma < 2-p$ and $-k'$ is nonnegative and nonincreasing. □

Lemma 3.4. *Assume that k satisfies* (3.1)*. Then the solution u of* (1.2) *satisfies*

$$\left[\int_{\Gamma_1}\left(\gamma(t)\left(-k'\right)^p\circ u\right)d\Gamma_1\right]^{\frac{1}{2p-1}} \leq \left[\int_{\Gamma_1}\left(k''\circ u\right)d\Gamma_1\right]^{\frac{1}{2p-1}}.$$

Proof. Using the fact that γ is nonincreasing, we get

$$\left(-k'(t-s)\right)^p\gamma(t-s) \geq \left(-k'(t-s)\right)^p\gamma(t).$$

Multiplication by $|u(t)-u(s)|^2$ and integration over $(0,t)\times\Gamma_1$, we obtain

$$\int_{\Gamma_1}\int_0^t\left(-k'(t-s)\right)^p\gamma(t-s)\left|u(t)-u(s)\right|^2 ds d\Gamma_1$$

$$\geq \int_{\Gamma_1}\int_0^t\left(-k'(t-s)\right)^p\gamma(t)\left|u(t)-u(s)\right|^2 ds d\Gamma_1,$$

then, by using (3.1) , we find

$$\int_{\Gamma_1}\left(\gamma(t)\left(-k'\right)^p\circ u\right)d\Gamma_1 \leq \int_{\Gamma_1}k''\circ u d\Gamma_1,$$

hence

$$\left[\int_{\Gamma_1}\left(\gamma(t)\left(-k'\right)^p\circ u\right)d\Gamma_1\right]^{\frac{1}{2p-1}} \leq \left[\int_{\Gamma_1}\left(k''\circ u\right)d\Gamma_1\right]^{\frac{1}{2p-1}}.$$ □

Lemma 3.5. *Assume that k satisfies* (3.1)*. Then there exists $C>0$ such that the solution u of* (1.2) *satisfies*

$$\int_{\Gamma_1}\gamma(t)\left(-k'\circ u\right)d\Gamma_1 \leq C\left[-E'(t)\right]^{\frac{1}{2p-1}}.$$

Proof. It easy to see that

$$\begin{aligned}\int_{\Gamma_1}(-k'\circ u)\,d\Gamma_1 &= \int_{\Gamma_1}\int_0^t -k'(t-s)\left|u(t)-u(s)\right|^2 dsd\Gamma_1\\ &= \int_{\Gamma_1}\int_0^t \left[-k'(t-s)\right]^{(1-\sigma)\frac{p-1}{p-1+\sigma}}\left(\left|u(t)-u(s)\right|^2\right)^{\frac{p-1}{p-1+\sigma}}\\ &\quad\times\left[-k'(t-s)\right]^{1-(1-\sigma)\frac{p-1}{p-1+\sigma}}\left(\left|u(t)-u(s)\right|^2\right)^{\frac{\sigma}{p-1+\sigma}} dsd\Gamma_1\\ &= \int_{\Gamma_1}\int_0^t \left[-k'(t-s)\right]^{(1-\sigma)\frac{p-1}{p-1+\sigma}}\left(\left|u(t)-u(s)\right|^2\right)^{\frac{p-1}{p-1+\sigma}}\\ &\quad\times\left[-k'(t-s)\right]^{\frac{\sigma p}{p-1+\sigma}}\left(\left|u(t)-u(s)\right|^2\right)^{\frac{\sigma}{p-1+\sigma}} dsd\Gamma_1.\end{aligned}$$

Using Hölder's inequality, for

$$s=\frac{p-1+\sigma}{p-1} \text{ and } s'=\frac{p-1+\sigma}{\sigma},$$

and Lemma 3.2, we arrive at

$$\begin{aligned}\int_{\Gamma_1}(-k'\circ u)\,d\Gamma_1 &\le \left[\int_{\Gamma_1}\int_0^t\left[-k'(t-s)\right]^{1-\sigma}\left|u(t)-u(s)\right|^2 dsd\Gamma_1\right]^{\frac{p-1}{p-1+\sigma}}\\ &\quad\times\left[\int_{\Gamma_1}\int_0^t\left[-k'(t-s)\right]^{p}\left|u(t)-u(s)\right|^2 dsd\Gamma_1\right]^{\frac{\sigma}{p-1+\sigma}}\\ &\le \left[\int_{\Gamma_1}\int_0^t\left[-k'(t-s)\right]^{1-\sigma}\left|u(t)-u(s)\right|^2 dsd\Gamma_1\right]^{\frac{p-1}{p-1+\sigma}}\\ &\quad\times\left[\int_{\Gamma_1}\left((-k')^p\circ u\right)d\Gamma_1\right]^{\frac{\sigma}{p-1+\sigma}}\\ &\le C\left[\int_0^t\left[-k'(t-s)\right]^{1-\sigma} dsd\Gamma_1\right]^{\frac{p-1}{p-1+\sigma}}\\ &\quad\times\left[\int_{\Gamma_1}\left((-k')^p\circ u\right)d\Gamma_1\right]^{\frac{\sigma}{p-1+\sigma}}.\end{aligned}$$

By taking $\sigma=\frac{1}{2}$, we have

$$\int_{\Gamma_1}(-k'\circ u)\,d\Gamma_1 \le C\left[\int_0^t\left[-k'(s)\right]^{\frac{1}{2}} dsd\Gamma_1\right]^{\frac{2p-2}{2p-1}}\left[\int_{\Gamma_1}\left((-k')^p\circ u\right)d\Gamma_1\right]^{\frac{1}{2p-1}}. \quad (3.6)$$

Multiply both sides of (3.6) by $\gamma(t)$, recall Lemma 3.3 and Lemma 3.4 and use Lemma 3.1 to get

$$\begin{aligned}
&\gamma(t)\int_{\Gamma_1}(-k'\circ u)\,d\Gamma_1\\
\leq\;& C\gamma(t)\left[\int_0^t[-k'(s)]^{\frac{1}{2}}\,dsd\Gamma_1\right]^{\frac{2p-2}{2p-1}}\left[\int_{\Gamma_1}\left((-k')^p\circ u\right)d\Gamma_1\right]^{\frac{1}{2p-1}}\\
\leq\;& C\gamma(t)^{\frac{2p-2}{2p-1}}\left[\int_0^t[-k'(s)]^{\frac{1}{2}}\,dsd\Gamma_1\right]^{\frac{2p-2}{2p-1}}\gamma(t)^{\frac{1}{2p-1}}\left[\int_{\Gamma_1}\left((-k')^p\circ u\right)d\Gamma_1\right]^{\frac{1}{2p-1}}\\
\leq\;& C\left[\int_0^t\gamma(s)[-k'(s)]^{\frac{1}{2}}\,dsd\Gamma_1\right]^{\frac{2p-2}{2p-1}}\left[\int_{\Gamma_1}\left(\gamma(t)(-k')^p\circ u\right)d\Gamma_1\right]^{\frac{1}{2p-1}}\\
\leq\;& C\left[\int_0^{+\infty}\gamma(s)[-k'(s)]^{\frac{1}{2}}\,dsd\Gamma_1\right]^{\frac{2p-2}{2p-1}}\left[\int_{\Gamma_1}(k''\circ u)\,d\Gamma_1\right]^{\frac{1}{2p-1}}\\
\leq\;& C[-E'(t)]^{\frac{1}{2p-1}}.
\end{aligned}$$

□

For completeness, we adopt without proof the following result from [3].

Lemma 3.6 ([3])**.** *There exist positive constants N, M, m, c, and t_0 such that the functional*

$$L(t)=NE(t)+\int_{\Omega}u_t.\,[M+(n-1)\,u]\,dx,$$

is equivalent to $E(t)$ and satisfies

$$L'(t)\leq -mE(t)-c\int_{\Gamma_1}(k'\circ u)(t)\,d\Gamma_1,\ \forall t\geq t_0. \tag{3.7}$$

Theorem 3.7. *Given $(u_0,u_1,\theta_0)\in\left(V^n,\left(L^2(\Omega)\right)^n,H_0^1(\Omega)\right)$. Assume that (2.1) and (3.1) − (3.2) hold, with $\lim_{t\to\infty}k(t)=0$. Then for each $t_0>0$, there exists a strictly positive constant C' such that the solution u of (1.2) satisfies, for all $t\geq t_0$,*

$$E(t)\leq C'\left[\frac{1}{\int_0^t\gamma^{2p-1}(s)\,ds+1}\right]^{\frac{1}{2p-2}} \tag{3.8}$$

Moreover,

$$\textit{If}\int_0^{+\infty}E(t)<+\infty, \tag{3.9}$$

then

$$E(t)\leq C'\left[\frac{1}{\int_0^t\gamma^{p}(s)\,ds+1}\right]^{\frac{1}{p-1}} \tag{3.10}$$

4. Proof of the main result

Multiplying (3.7) by $\gamma(t)$, and recall lemma 3.5, we obtain

$$\gamma(t)L'(t) \leq -m\gamma(t)E(t) + C\left[-E'(t)\right]^{\frac{1}{2p-1}}.$$

Multiplication of the last inequality by $\gamma^{\alpha}(t)E^{\alpha}(t)$, where $\alpha = 2p-2$, gives

$$\gamma^{\alpha+1}(t)E^{\alpha}(t)L'(t) \leq -m\gamma^{\alpha+1}(t)E^{\alpha+1}(t) + C\gamma^{\alpha}(t)E^{\alpha}(t)\left[-E'(t)\right]^{\frac{1}{\alpha+1}}.$$

Use of Young's inequality, with $q = \alpha+1$ and $q^* = \frac{\alpha+1}{\alpha}$, yields, for any $\varepsilon > 0$,

$$\begin{aligned}\gamma^{\alpha+1}(t)E^{\alpha}(t)L'(t) &\leq -m\gamma^{\alpha+1}(t)E^{\alpha+1}(t) + C\left[\varepsilon\gamma^{\alpha+1}(t)E^{\alpha+1}(t) - C_{\varepsilon}E'(t)\right] \\ &= -\left(m - \varepsilon C\right)\gamma^{\alpha+1}(t)E^{\alpha+1}(t) - C'E'(t)\end{aligned}$$

We then choose $\varepsilon < \dfrac{m}{C}$ and recall that $\gamma' \leq 0$ and $E' \leq 0$, to get

$$\left(\gamma^{\alpha+1}E^{\alpha}L\right)'(t) \leq \gamma^{\alpha+1}(t)E^{\alpha}(t)L'(t) \leq -c\gamma^{\alpha+1}(t)E^{\alpha+1}(t) - C'E'(t),$$

which implies that

$$\left(\gamma^{\alpha+1}(t)E^{\alpha}(t)L(t) + C'E(t)\right)' \leq -c\gamma^{\alpha+1}(t)E^{\alpha+1}(t) \tag{4.1}$$

Let

$$F(t) = \gamma^{\alpha+1}(t)E^{\alpha}(t)L(t) + C'E(t), \tag{4.2}$$

where $F(t) \sim E(t)$. Then

$$F'(t) \leq -c\gamma^{\alpha+1}(t)F^{\alpha+1}(t) = -c\gamma^{2p-1}(t)F^{2p-1}(t). \tag{4.3}$$

Integrating over $(0,t)$ and using the fact that $F \sim E$, we obtain

$$E(t) \leq C'\left[\frac{1}{\int_0^t \gamma^{2p-1}(s)\,ds + 1}\right]^{\frac{1}{2p-2}}.$$

To establish (3.10), we consider (3.9). Let

$$\eta(t) = \int_0^t \|u(t) - u(t-s)\|_2^2 ds.$$

Assume that $\eta(t) > 0$. Then multiplying (3.7) by $\gamma(t)$, we obtain

$$\begin{aligned}\gamma(t)L'(t) &\leq -m\gamma(t)E(t) - c\gamma(t)\int_{\Gamma_1}(k'\circ u)(t)d\Gamma_1 \\ &= -m\gamma(t)E(t) \\ &+ c\frac{\eta(t)}{\eta(t)}\int_{\Gamma_1}\int_0^t\left[\gamma^p(s)(-k')^p(s)\right]^{\frac{1}{p}}\|u(t) - u(t-s)\|_2^2\,dsd\Gamma_1,\end{aligned} \tag{4.4}$$

where

$$\begin{aligned}
\eta(t) = \int_0^t \|u(t) - u(t-s)\|_2^2\, ds &\leq 2\int_0^t \|u(t)\|_2^2 + \|u(t-s)\|_2^2\, ds \\
&\leq 2C_\Omega \int_0^t \|\nabla u(t)\|_2^2 + \|\nabla u(t-s)\|_2^2\, ds \\
&\leq 2C_\Omega \int_0^t [E(t) + E(t-s)]\, ds \\
&\leq 4C_\Omega \int_0^t E(t-s)\, ds = 4C_\Omega \int_0^t E(s)\, ds \\
&< 4C_\Omega \int_0^{+\infty} E(s)\, ds < +\infty.
\end{aligned}$$

Applying Jensen's inequality for the third term of (4.4), with $G(y) = y^{\frac{1}{p}}$, $y > 0$, $f(s) = \gamma^p(s)(-k')^p(s)$ and $h(s) = \|u(t) - u(t-s)\|_2^2$, for $y > 0$ and $s > 0$, we get

$$\begin{aligned}
&\gamma(t) L'(t) \leq -m\gamma(t) E(t) \\
&\qquad + c\eta(t)\left[\frac{1}{\eta(t)} \int_{\Gamma_1}\int_0^t \left[\gamma^p(s)(-k')^p(s)\right] \|u(t) - u(t-s)\|_2^2\, ds d\Gamma_1\right]^{\frac{1}{p}}.
\end{aligned}$$

If $\eta(t) = 0$, then previous inequality still has a sense because $p > 1$. By using the fact that γ is nonincreasing, to see that

$$\begin{aligned}
&\gamma(t) L'(t) \leq -m\gamma(t) E(t) \\
&\qquad + c\eta^{\frac{p-1}{p}}(t)\left[\gamma^{p-1}(0) \int_{\Gamma_1}\int_0^t \gamma(s)(-k')^p(s)\|u(t) - u(t-s)\|_2^2\, ds d\Gamma_1\right]^{\frac{1}{p}} \\
&\qquad \leq -m\gamma(t) E(t) + C'\left(\int_{\Gamma_1} (k'' \circ u)\, d\Gamma_1\right)^{\frac{1}{p}} \\
&\qquad \leq -m\gamma(t) E(t) + C'(-E'(t))^{\frac{1}{p}}.
\end{aligned}$$

Multiplying by $\gamma^\alpha(t) E^\alpha(t)$, for $\alpha = p - 1$, and repeating the same computations as in above, we arrive at

$$E(t) \leq C'\left[\frac{1}{\int_0^t \gamma^p(s)\, ds + 1}\right]^{\frac{1}{p-1}}.$$

This completes the proof of our main result.

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Well-posedness and exponential decay for a laminated beam with distributed delay term

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Abstract. In this paper, we study the well-posedness and the asymptotic behavior of a one-dimensional laminated beam system with a distributed delay term in the first equation, where the heat conduction is given by Fourier's law effective in the rotation angle displacements. We first give the well-posedness of the system by using the semigroup method. Then, we show that the system is exponentially stable under the assumption of equal wave speeds.

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Keywords: Laminated beam, Fourier's law, distributed delay term, well-posedness, exponential decay.

1. Introduction

Recent advances in smart laminated composite structures in the past two decades resulted in the application of these new generation of structures in modern industries, including automotive, robot arms, aerospace and civil engineering. Such structures are mainly work in harsh dynamic conditions, particularly the design of their piezoelectric materials can be used as both actuators and sensors. Hansen and Spies in [8, 9] derived the mathematical model for two-layered beams with structural damping due to the interfacial slip, the system is given by the following equations:

$$\begin{cases} \rho_1\varphi_{tt} + G\left(\psi - \varphi_x\right)_x = 0, \\ \rho_2\left(3w - \psi\right)_{tt} - G\left(\psi - \varphi_x\right) - D\left(3w - \psi\right)_{xx} = 0, \\ \rho_2 w_{tt} + G\left(\psi - \varphi_x\right) + \frac{4}{3}\gamma w + \frac{4}{3}\beta w_t - D w_{xx} = 0, \end{cases} \tag{1.1}$$

where $(x,t) \in (0,1) \times (0,+\infty)$, and $\varphi = \varphi(x,t)$ is the transversal displacement, $\psi = \psi(x,t)$ denotes the rotational displacement, and $w = w(x,t)$ is proportional to the amount of slip along the interface at time t and longitudinal spatial variable x. The coefficients $\rho_1, G, \rho_2, D, \gamma, \beta > 0$ are the density of the beams, the shear stiffness, mass

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moment of inertia, flexural rigidity, adhesive stiffness of the beams and the adhesive damping parameter, respectively.

In recent years, an increasing interest has been developed to determine the asymptotic behavior of the solution of several laminated beam problems, we refer the reader to [3, 12, 13, 14, 15, 23, 24] and the references therein. In [23], Raposo considered system (1.1) with two frictional dampings of the form:

$$\begin{cases} \rho_1\varphi_{tt} + G(\psi - \varphi_x)_x + k_1\varphi_t = 0, \\ \rho_2(3w - \psi)_{tt} - G(\psi - \varphi_x) - D(3w - \psi)_{xx} + k_2(3w - \psi)_t = 0, \\ \rho_2 w_{tt} + G(\psi - \varphi_x) + \frac{4}{3}\gamma w + \frac{4}{3}\beta w_t - Dw_{xx} = 0, \end{cases}$$

where $(x,t) \in (0,1) \times (0,+\infty)$, and obtained the exponential decay result under appropriate initial and boundary conditions. In [24], Wang, Xu and Yung considered system (1.1) with the cantilever boundary conditions and two different wave speeds ($\sqrt{\frac{G}{\rho_1}}$ and $\sqrt{\frac{D}{\rho_2}}$). W. Liu and W. Zhao [14] considered a coupled system of a laminated beam with Fourier's type heat conduction, which has the form:

$$\begin{cases} \rho\varphi_{tt} + G(\psi - \varphi_x)_x = 0, \\ I_\rho(3w - \psi)_{tt} - G(\psi - \varphi_x) - D(3w - \psi)_{xx} + \sigma\theta_x = 0, \\ I_\rho w_{tt} + G(\psi - \varphi_x) + \frac{4}{3}\gamma w + \frac{4}{3}\beta w_t - Dw_{xx} = 0, \\ k\theta_t - \tau\theta_{xx} + \sigma(3w - \psi)_{tx} = 0, \end{cases}$$

where $(x,t) \in (0,1) \times (0,+\infty)$, they used the energy method to prove an exponential decay result for the case of equal wave speeds. (See also [1, 5, 11, 16, 17]).

Time delays arise in many applications of most phenomena naturally modulate by partial differential equations problems, depending not only on the present state but also on some past occurrences. The presence of delay may be a source of instability. It may turn a well-behaved system into a wild one. For example, it was shown in [6, 7, 10, 20, 21, 25] that an arbitrarily small delay may destabilize a system that is uniformly asymptotically stable in the absence of delay unless additional control terms have been used. In [21], Nicaise and Pignotti considered wave equation with linear frictional damping and internal distributed delay

$$u_{tt} - \triangle u + \mu_1 u_t + a(x)\int_{\tau_1}^{\tau_2} \mu_2(s)\, u_t(t-s)\, ds = 0, \text{ in } \Omega \times (0,\infty),$$

with initial and mixed Dirichlet-Neumann boundary conditions and a is a function chosen in an appropriate space. They established exponential stability of the solution under the assumption that

$$\|a\|_\infty \int_{\tau_1}^{\tau_2} \mu_2(s)\, ds < \mu_1.$$

Regarding the similar result concerning boundary distributed delay see [2, 18, 19]. Moreover, Nicaise, Pignotti and Valein [22] replaced the constant delay term in the boundary condition of [20] by a time-varying delay term and obtained an exponential decay result under an appropriate assumption on the weights of the damping and delay.

In this work, we consider the laminated beam system where the heat flux is given by Fourier's law with distributed delay term. The system is written as

$$\begin{cases} \rho_1\varphi_{tt} + G(\psi-\varphi_x)_x + \mu_0\varphi_t + \int_{\tau_1}^{\tau_2}\mu(s)\varphi_t(x,t-s)\,ds = 0, \\ \rho_2(3w-\psi)_{tt} - G(\psi-\varphi_x) - D(3w-\psi)_{xx} + \sigma\theta_x = 0, \\ \rho_2 w_{tt} + G(\psi-\varphi_x) + \frac{4}{3}\gamma w + \frac{4}{3}\beta w_t - Dw_{xx} = 0, \\ k\theta_t - \tau\theta_{xx} + \sigma(3w-\psi)_{tx} = 0, \end{cases} \tag{1.2}$$

where $(x,t)\in(0,1)\times(0,+\infty)$, and ρ_1, G, ρ_2, D, σ, γ, β, k, τ are positive constant coefficients, with the Dirichlet-Neumann boundary conditions:

$$\begin{cases} \varphi(0,t) = \psi_x(0,t) = w_x(0,t) = \theta(0,t) = 0, & t\in[0,+\infty), \\ \varphi_x(1,t) = \psi(1,t) = w(1,t) = \theta_x(1,t) = 0, & t\in[0,+\infty), \end{cases} \tag{1.3}$$

and the initial conditions:

$$\begin{cases} \varphi(x,0) = \varphi_0(x), \varphi_t(x,0) = \varphi_1(x), & x\in[0,1], \\ \psi(x,0) = \psi_0(x), \psi_t(x,0) = \psi_1(x), & x\in[0,1], \\ w(x,0) = w_0(x), w_t(x,0) = w_1(x), & x\in[0,1], \\ \theta(x,0) = \theta_0(x), & x\in[0,1], \\ \varphi_t(x,-t) = f_0(x,t), & (x;t)\in(0,1)\times(0,\tau_2), \end{cases} \tag{1.4}$$

where τ_1 and τ_2 are two real numbers with $0\le\tau_1<\tau_2$, μ_0 is a positive constant, and $\mu:[\tau_1,\tau_2]\longrightarrow\mathbb{R}$ is an L^∞ function, $\mu\ge 0$ almost everywhere, and the initial data $(\varphi_0,\varphi_1,3w_0-\psi_0,3w_1-\psi_1,w_0,w_1,\theta_0,f_0)$ belong to a suitable Sobolev space.

Here, we prove the well-posedness and stability results for problem on the following parameter, under the assumption

$$\mu_0 \ge \int_{\tau_1}^{\tau_2}\mu(s)\,ds. \tag{1.5}$$

The rest of our paper is organized as follows. In Section 2, by using Hille-Yosida theorem, we state and prove the well posedness of problem (1.2)-(1.4). In Section 3, by using the perturbed energy method, we then establish the exponential result if and only if $\frac{\rho_1}{G}=\frac{\rho_2}{D}$.

2. Well-posedness of the problem

In this section, we will prove that system (1.2)-(1.4) are well posed using semigroup theory by introducing the following new variable as in [21].

$$z(x,\rho,t,s) = \varphi_t(x,t-\rho s), \quad x\in(0,1), \ \rho\in(0,1), \ t>0, \ s\in(\tau_1,\tau_2). \tag{2.1}$$

Then, we have

$$sz_t(x,\rho,t,s) + z_\rho(x,\rho,t,s) = 0, \quad x\in(0,1), \ \rho\in(0,1), \ t>0, \ s\in(\tau_1,\tau_2). \tag{2.2}$$

Therefore, problem (1.2) takes the form:

$$\begin{cases} \rho_1\varphi_{tt} + G(\psi-\varphi_x)_x + \mu_0\varphi_t + \int_{\tau_1}^{\tau_2}\mu(s)z(x,1,t,s)ds = 0, \\ \rho_2(3w-\psi)_{tt} - G(\psi-\varphi_x) - D(3w-\psi)_{xx} + \sigma\theta_x = 0, \\ \rho_2 w_{tt} + G(\psi-\varphi_x) + \frac{4}{3}\gamma w + \frac{4}{3}\beta w_t - Dw_{xx} = 0, \\ k\theta_t - \tau\theta_{xx} + \sigma(3w-\psi)_{tx} = 0, \end{cases} \tag{2.3}$$

with the Dirichlet-Neumann boundary conditions:

$$\begin{cases} \varphi(0,t) = \psi_x(0,t) = w_x(0,t) = \theta(0,t) = 0, & t \in [0,+\infty), \\ \varphi_x(1,t) = \psi(1,t) = w(1,t) = \theta_x(1,t) = 0, & t \in [0,+\infty), \end{cases} \tag{2.4}$$

and the initial conditions:

$$\begin{cases} \varphi(x,0) = \varphi_0(x), \ \varphi_t(x,0) = \varphi_1(x), \ x \in [0,1], \\ \psi(x,0) = \psi_0(x), \ \psi_t(x,0) = \psi_1(x), \ x \in [0,1], \\ w(x,0) = w_0(x), \ w_t(x,0) = w_1(x), \ x \in [0,1], \\ \theta(x,0) = \theta_0(x), \ x \in [0,1], \\ \varphi_t(x,-t) = f_0(x,t), \ (x,t) \in (0,1) \times (0,\tau_2) \\ z(x,0,t,s) = \varphi_t(x,t) \ \text{ on } \ (0,1) \times (0,\infty) \times (\tau_1,\tau_2), \\ z(x,\rho,0,s) = f_0(x,\rho,s) \ \text{ on } \ (0,1) \times (0,1) \times (\tau_1,\tau_2). \end{cases} \tag{2.5}$$

Introducing the vector function

$$U = (\varphi, \varphi_t, 3w-\psi, (3w-\psi)_t, w, w_t, \theta, z)^T,$$

problem (2.3)-(2.5) can be written as

$$\begin{cases} \partial_t U = AU, \\ U(x,0) = U^0(x) = (\varphi_0, \varphi_1, 3w_0-\psi_0, 3w_1-\psi_1, w_0, w_1, \theta_0, f_0)^T. \end{cases} \tag{2.6}$$

Where the operator A is defined by

$$AU = \begin{pmatrix} \varphi_t \\ -\frac{G}{\rho_1}(\psi-\varphi_x)_x - \frac{\mu_0}{\rho_1}\varphi_t - \frac{1}{\rho_1}\int_{\tau_1}^{\tau_2} \mu(s) z(x,1,t,s)\, ds \\ (3w-\psi)_t \\ \frac{G}{\rho_2}(\psi-\varphi_x) + \frac{D}{\rho_2}(3w-\psi)_{xx} - \frac{\sigma}{\rho_2}\theta_x \\ w_t \\ -\frac{G}{\rho_2}(\psi-\varphi_x) - \frac{4\gamma}{3\rho_2}w - \frac{4\beta}{3\rho_2}w_t + \frac{D}{\rho_2}w_{xx} \\ \frac{\tau}{\kappa}\theta_{xx} - \frac{\sigma}{\kappa}(3w-\psi)_{tx} \\ -s^{-1}z_\rho \end{pmatrix}$$

We consider the following spaces

$$\begin{aligned} H_*^1(0,1) &= \left\{\chi/\chi \in H^1(0,1) : \chi(0) = 0\right\}, \\ \widetilde{H}_*^1(0,1) &= \left\{\chi/\chi \in H^1(0,1) : \chi(1) = 0\right\}. \end{aligned}$$

Let

$$\begin{aligned} \mathcal{H} = {} & H_*^1(0,1) \times L^2(0,1) \times \widetilde{H}_*^1(0,1) \times L^2(0,1) \times \widetilde{H}_*^1(0,1) \\ & \times L^2(0,1) \times L^2(0,1) \times L^2\left((0,1)\times(\tau_1,\tau_2), H^1(0,1)\right), \end{aligned}$$

be the Hilbert space equipped with the inner product

$$\begin{aligned}\left\langle U,\widetilde{U}\right\rangle_{\mathcal{H}} &= \int_0^1 [\rho_1\varphi_t\widetilde{\varphi}_t+\rho_2(3w-\psi)_t\left(3\widetilde{w}-\widetilde{\psi}\right)_t+3\rho_2 w_t\widetilde{w}_t dx+k\theta\widetilde{\theta}\\ &\quad+4\gamma w\widetilde{w}+G(\psi-\varphi_x)\left(\widetilde{\psi}-\widetilde{\varphi}_x\right)+D(3w-\psi)_x\left(3\widetilde{w}-\widetilde{\psi}\right)_x\\ &\quad+3Dw_x\widetilde{w}_x]dx+\int_0^1\int_{\tau_1}^{\tau_2}s\mu(s)\int_0^1 z(x,\rho,s)\,\widetilde{z}(x,\rho,s)\,d\rho ds dx.\end{aligned}$$

The domain of A is

$$D(A)=\left\{\begin{array}{c} U\in\mathcal{H}\mid\varphi\in H^2(0,1)\cap H_*^1(0,1),\ \theta\in H_*^1(0,1),\\ 3w-\psi,w\in H^2(0,1)\cap\widetilde{H}_*^1(0,1),\\ \varphi_t\in H_*^1(0,1),(3w-\psi)_t,w_t\in\widetilde{H}_*^1(0,1),\\ \varphi_x(1,t)=\psi_x(0,t)=w_x(0,t)=0,\varphi_t(x)=z(x,0,s)\,in(0,1)\end{array}\right\},\quad (2.7)$$

and it is dense in $\mathcal{H}$. The well-posedness of problem (2.6) is ensured by

Theorem 2.1. *Assume that $U^0\in\mathcal{H}$ and (1.5) holds, then problem (2.6) exists a unique weak solution $U\in C(\mathbb{R}^+;\mathcal{H})$. Moreover, if $U^0\in D(A)$, then*

$$U\in C\left(\mathbb{R}^+;D(A)\cap C^1\left(\mathbb{R}^+;\mathcal{H}\right)\right). \quad (2.8)$$

Proof. To prove the well-posedness result, it suffices to show that $A:D(A)\to\mathcal{H}$ is a maximal monotone operator, which means A is dissipative and $Id-A$ is surjective.
First, we prove that A is dissipative.
For any $U=(\varphi,\varphi_t,3w-\psi,(3w-\psi)_t,w,w_t,\theta,z)^T\in D(A)$, by using the inner product and integrating by parts, we have

$$\begin{aligned}\langle AU,U\rangle_{\mathcal{H}} &= -\mu_0\int_0^1\varphi_t^2(x)\,dx-\int_0^1\varphi_t(x)\left(\int_{\tau_1}^{\tau_2}\mu(s)\,z(x,1,s)\,ds\right)dx\\ &\quad-4\beta\int_0^1 w_t^2dx-\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}\mu(s)\,z^2(x,1,s)\,dsdx\\ &\quad-\tau\int_0^1\theta_x^2dx+\frac{1}{2}\int_{\tau_1}^{\tau_2}\mu(s)\,ds\int_0^1\varphi_t^2(x)\,dx.\end{aligned}$$

Now, using Young's and Cauchy–Schwarz' inequalities, we can estimate,

$$\begin{aligned}&-\int_0^1\varphi_t(x)\left(\int_{\tau_1}^{\tau_2}\mu(s)\,z(x,1,s)\,ds\right)dx\\ &\le\frac{1}{2}\left(\int_{\tau_1}^{\tau_2}\mu(s)\,ds\right)\int_0^1\varphi_t^2(x)\,dx+\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}\mu(s)\,z^2(x,1,s)\,dsdx.\end{aligned}$$

Therefore, from the assumption (1.5) we have

$$\begin{aligned}&\langle AU,U\rangle_{\mathcal{H}}\\ &\le -\tau\int_0^1\theta_x^2dx-4\beta\int_0^1 w_t^2dx+\left(-\mu_0+\int_{\tau_1}^{\tau_2}\mu(s)\,ds\right)\int_0^1\varphi_t^2(x)\,dx\le 0.\end{aligned}$$

Consequently, A is a dissipative operator.
Next, we prove that the operator $Id-A$ is surjective.

Given $F = (f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8)^T \in \mathcal{H}$, we prove that there exists a unique $U = (\varphi, \varphi_t, 3w - \psi, (3w-\psi)_t, w, w_t, \theta, z)^T \in D(A)$ such that

$$(Id - A) U = F, \tag{2.9}$$

that is,

$$\begin{cases} \varphi - \varphi_t = f_1, \\ (\rho_1 + \mu_0)\varphi_t - G\varphi_{xx} - G(3w-\psi)_x + 3Gw_x + \int_{\tau_1}^{\tau_2} \mu(s) z(x,1,t,s)\, ds \\ \qquad = \rho_1 f_2, \\ (3w-\psi) - (3w-\psi)_t = f_3, \\ \rho_2 (3w-\psi)_t + G\varphi_x + G(3w-\psi) - D(3w-\psi)_{xx} - 3Gw + \sigma\theta_x \\ \qquad = \rho_2 f_4, \\ w - w_t = f_5, \\ \left(\rho_2 + \frac{4\beta}{3}\right) w_t - G\varphi_x - G(3w-\psi) + \left(3G + \frac{4\gamma}{3}\right) w - Dw_{xx} = \rho_2 f_6, \\ k\theta - \tau\theta_{xx} + \sigma(3w-\psi)_{tx} = kf_7, \\ z + s^{-1} z_\rho = f_8. \end{cases} \tag{2.10}$$

From $(2.10)_1$,$(2.10)_3$ and $(2.10)_5$ we have

$$\begin{cases} \varphi_t = \varphi - f_1, \\ (3w-\psi)_t = (3w-\psi) - f_3, \\ w_t = w - f_5. \end{cases} \tag{2.11}$$

Inserting (2.11) into $(2.10)_2$, $(2.10)_4$, $(2.10)_6$ and $(2.10)_7$, we get

$$\begin{cases} (\mu_0 + \rho_1)\varphi - G\varphi_{xx} - G(3w-\psi)_x + 3Gw_x + \int_{\tau_1}^{\tau_2} \mu(s) z(x,1,t,s)\, ds \\ \qquad = \rho_1 (f_1 + f_2) + \mu_0 f_1, \\ \rho_2 (3w-\psi) + G\varphi_x + G(3w-\psi) - D(3w-\psi)_{xx} - 3Gw + \sigma\theta_x \\ \qquad = \rho_2 (f_3 + f_4), \\ \left(\rho_2 + \frac{4\beta}{3}\right) w - G\varphi_x - G(3w-\psi) + \left(3G + \frac{4\gamma}{3}\right) w - Dw_{xx} \\ \qquad = \rho_2 (f_5 + f_6) + \frac{4\beta}{3} f_5, \\ k\theta - \tau\theta_{xx} + \sigma(3w-\psi)_x = \sigma(f_3)_x + kf_7, \\ z + s^{-1} z_\rho = f_8. \end{cases} \tag{2.12}$$

Using (2.11) and the fact that $z(x,0,s) = \varphi_t(x)$, we get

$$z(x,\rho,s) = \varphi(x) e^{-\rho s} - f_1 e^{-\rho s} + s e^{-\rho s} \int_0^\rho f_8(x,\delta,s) e^{\delta s} d\delta, \tag{2.13}$$

In order to solve (2.10), we consider the following variational formulation

$$B\left((\varphi, 3w-\psi, w, \theta)^T, \left(\widetilde{\varphi}, 3\widetilde{w} - \widetilde{\psi}, \widetilde{w}, \widetilde{\theta}\right)^T \right) = L\left(\widetilde{\varphi}, 3\widetilde{w} - \widetilde{\psi}, \widetilde{w}, \widetilde{\theta}\right)^T, \tag{2.14}$$

where $B : \left[H_*^1(0,1) \times \widetilde{H}_*^1(0,1) \times \widetilde{H}_*^1(0,1) \times L^2(0,1)\right]^2 \longrightarrow \mathbb{R}$ is the bilinear form defined by

$$
\begin{aligned}
& B\left((\varphi, 3w-\psi, w, \theta)^T, \left(\widetilde{\varphi}, 3\widetilde{w}-\widetilde{\psi}, \widetilde{w}, \widetilde{\theta}\right)^T\right) \\
= & \int_0^1 G(-\varphi_x+\psi)\left(-\widetilde{\varphi}_x+\widetilde{\psi}\right) dx + \int_0^1 (\mu_0+\rho_1)\varphi\widetilde{\varphi} dx + \int_0^1 k\theta\widetilde{\theta} dx \\
& + \int_0^1 \rho_2 (3w-\psi)\left(3\widetilde{w}-\widetilde{\psi}\right) dx + \int_0^1 (3\rho_2+4\beta+4\gamma) w\widetilde{w} dx \\
& + \int_0^1 D(3w-\psi)_x \left(3\widetilde{w}-\widetilde{\psi}\right)_x dx + \int_0^1 3Dw_x\widetilde{w}_x dx + \tau\int_0^1 \theta_x\widetilde{\theta}_x dx \\
& + \sigma\int_0^1 \theta_x\left(3\widetilde{w}-\widetilde{\psi}\right) dx + \sigma\int_0^1 (3w-\psi)_x \widetilde{\theta} dx \\
& + \int_0^1 \varphi\widetilde{\varphi}\int_{\tau_1}^{\tau_2} \mu(s) e^{-s} ds dx,
\end{aligned}
$$

and $L : \left[H_*^1(0,1) \times \widetilde{H}_*^1(0,1) \times \widetilde{H}_*^1(0,1) \times L^2(0,1)\right] \longrightarrow \mathbb{R}$ is the linear form defined by

$$
\begin{aligned}
& L\left(\widetilde{\varphi}, 3\widetilde{w}-\widetilde{\psi}, \widetilde{w}, \widetilde{\theta}\right)^T \\
= & \int_0^1 \rho_1 (f_1+f_2)\widetilde{\varphi} dx + \int_0^1 \mu_0 f_1 \widetilde{\varphi} dx + \int_0^1 \rho_2 (f_3+f_4)\left(3\widetilde{w}-\widetilde{\psi}\right) dx \\
& + \int_0^1 3\rho_2 (f_5+f_6)\widetilde{w} dx + \int_0^1 4\beta f_5 \widetilde{w} dx + \int_0^1 \sigma (f_3)_x \widetilde{\theta} dx + \int_0^1 k f_7 \widetilde{\theta} dx \\
& - \int_0^1 \widetilde{\varphi}\int_{\tau_1}^{\tau_2} \mu(s) z_0(x,s) ds dx.
\end{aligned}
$$

Now, for $V = H_*^1(0,1) \times \widetilde{H}_*^1(0,1) \times \widetilde{H}_*^1(0,1) \times L^2(0,1)$ equipped with the norm

$$\|(\varphi, 3w-\psi, w, \theta)\|_V^2 = \|-\varphi_x+\psi\|_2^2 + \|\varphi\|_2^2 + \|3w_x-\psi_x\|_2^2 + \|w_x\|_2^2 + \|\theta_x\|_2^2 .$$

It is easy to verify that $B(.,.)$ is continuous and coercive, and $L(.)$ is continuous. So applying the Lax-Milgram theorem, problem (2.14) admits a unique solution

$$\varphi \in H_*^1(0,1), \quad (3w-\psi) \in \widetilde{H}_*^1(0,1), \quad w \in \widetilde{H}_*^1(0,1), \quad \theta \in L^2(0,1).$$

The substitution of $\varphi, 3w-\psi$ and w into (2.11), we obtain

$$\varphi_t \in H_*^1(0,1), \quad (3w-\psi)_t \in \widetilde{H}_*^1(0,1), \quad w_t \in \widetilde{H}_*^1(0,1).$$

Applying the classical elliptic regularity, it follows from (2.12) that

$$
\begin{aligned}
\varphi & \in H^2(0,1)\cap H_*^1(0,1), (3w-\psi) \in H^2(0,1)\cap \widetilde{H}_*^1(0,1), \theta \in H_*^1(0,1), \\
w & \in H^2(0,1)\cap \widetilde{H}_*^1(0,1), \varphi_x(1) = (3w-\psi)_x(0) = w_x(0) = 0.
\end{aligned}
$$

Therefore, the operator $Id - A$ is surjective. Consequently, the well-posedness result stated in Theorem 2.1 follows from the Hille–Yosida theorem (see [4]). □

3. Exponential stability of solution

In this section, we show that, under the assumption $\mu_0 \geq \int_{\tau_1}^{\tau_2} \mu(s)\,ds$ and for $\frac{\rho_1}{G} = \frac{\rho_2}{D}$, the solution of problem (2.3)-(2.5) decays exponentially to the study state. To achieve our goal we use the energy method to produce a suitable Lyapunov functional. We define the energy functional $E(t)$ as

$$\begin{aligned} E(t) := &\frac{1}{2}\int_0^1 \Big[\rho_1\varphi_t^2 + \rho_2 (3w_t - \psi_t)^2 + 3\rho_2 w_t^2 + G(\psi - \varphi_x)^2 + 4\gamma w^2 + k\theta^2 \\ &+ D(3w_x - \psi_x)^2 + 3Dw_x^2\Big]\,dx + \frac{1}{2}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\mu(s)\, z^2(x,\rho,s,t)\,ds\,d\rho\,dx. \end{aligned} \tag{3.1}$$

Theorem 3.1. *Assume that $\frac{\rho_1}{G} = \frac{\rho_2}{D}$ and (1.5) holds. Let $U^0 \in \mathcal{H}$, then there exists positive constants c_0, c_1 such that the energy $E(t)$ associated with problem (2.3)-(2.5) satisfies,*

$$E(t) \leq c_0 e^{-c_1 t}, t \geq 0. \tag{3.2}$$

In order to prove this result, we need the following lemmas.

Lemma 3.2. *Let $(\varphi, \psi, w, \theta, z)$ be the solution of (2.3)-(2.5) and assume (1.5) holds. Then the energy functional, defined by (3.1) satisfies*

$$\frac{d}{dt}E(t) \leq -4\beta\int_0^1 w_t^2 dx - \tau\int_0^1 \theta_x^2 dx - \left(\mu_0 - \int_{\tau_1}^{\tau_2}\mu(s)\,ds\right)\int_0^1 \varphi_t^2 dx \leq 0. \tag{3.3}$$

Proof. Multiplying $(2.3)_1$, $(2.3)_2$, $(2.3)_3$ and $(2.3)_4$ by $\varphi_t, 3(w-\psi)_t$, $3w_t$ and θ, respectively, and integrating over $(0,1)$, using integration by parts and the boundary conditions in (2.4), we get

$$\begin{aligned} &\frac{d}{dt}\left[\frac{1}{2}\left(\rho_1\int_0^1 \varphi_t^2 dx + G\int_0^1 (\psi - \varphi_x)^2 dx\right)\right] \\ &= G\int_0^1 (\psi - \varphi_x)\psi_t dx - \mu_0\int_0^1 \varphi_t^2 dx - \int_0^1 \varphi_t \int_{\tau_1}^{\tau_2}\mu(s)\varphi_t(x,t-s)\,ds\,dx, \end{aligned} \tag{3.4}$$

$$\begin{aligned} &\frac{d}{dt}\left[\frac{1}{2}\left(\rho_2\int_0^1 (3w_t - \psi_t)^2 dx + D\int_0^1 (3w_x - \psi_x)^2 dx\right)\right] \\ = \;& G\int_0^1 (\psi - \varphi_x)(3w - \psi)_t dx - \sigma\int_0^1 \theta_x (3w - \psi)_t dx, \end{aligned} \tag{3.5}$$

$$\begin{aligned} &\frac{d}{dt}\left[\frac{1}{2}\left(3\rho_2\int_0^1 w_t^2 dx + 4\gamma\int_0^1 w^2 dx + 3D\int_0^1 w_x^2 dx\right)\right] \\ = \;& -3G\int_0^1 (\psi - \varphi_x) w_t dx - 4\beta\int_0^1 w_t^2 dx, \end{aligned} \tag{3.6}$$

and

$$\frac{d}{dt}\left[\frac{1}{2}k\int_0^1 \theta^2 dx\right] = \sigma\int_0^1 (3w - \psi)_t\theta_x dx - \tau\int_0^1 \theta_x^2 dx. \tag{3.7}$$

On the other hand, multiplying (2.2) by $\mu(s)\, z(x,\rho,s,t)$ and integrating over $(0,1)\times(0,1)\times(\tau_1,\tau_2)$, we obtain

$$\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\mu(s)\, z(x,\rho,s,t)\, z_t(x,\rho,s,t)\, ds d\rho dx$$

$$+\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} \mu(s)\, z(x,\rho,s,t)\, z_\rho(x,\rho,s,t)\, ds d\rho dx = 0,$$

thus, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\mu(s)\, z^2(x,\rho,s,t)\, ds d\rho dx \\ =\ & -\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}\mu(s)\, z^2(x,1,s,t)\, ds dx + \frac{1}{2}\int_{\tau_1}^{\tau_2}\mu(s)\, ds \int_0^1 \varphi_t^2 dx. \end{aligned} \tag{3.8}$$

Summing up (3.4)-(3.8), we arrive at

$$\begin{aligned} \frac{d}{dt}E(t) =\ & -4\beta\int_0^1 w_t^2 dx - \left(\mu_0 - \frac{1}{2}\left(\int_{\tau_1}^{\tau_2}\mu(s)\, ds\right)\right)\int_0^1 \varphi_t^2 dx \\ & -\tau\int_0^1 \theta_x^2 dx - \int_0^1 \varphi_t \int_{\tau_1}^{\tau_2}\mu(s)\, z(x,1,s,t)\, ds dx \\ & -\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}\mu(s)\, z^2(x,1,s,t)\, ds dx. \end{aligned} \tag{3.9}$$

Young's and Cauchy–Schwarz' inequalities applied to the fourth term on the right-hand side yield

$$\begin{aligned} -\int_0^1 \varphi_t \int_{\tau_1}^{\tau_2}\mu(s)\, z(x,1,s,t)\, ds dx \ \leq\ & \frac{1}{2}\left(\int_{\tau_1}^{\tau_2}\mu(s)\, ds\right)\int_0^1 \varphi_t^2 dx \\ & +\frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}\mu(s)\, z^2(x,1,s,t)\, ds dx. \end{aligned} \tag{3.10}$$

Simple substitution of (3.10) into (3.9) and using (1.5) give (3.3), which concludes the proof. □

Now, we are going to construct a Lyapunov functional equivalent to the energy. For this, we will prove several lemmas with the purpose of creating negative counterparts of the terms that appear in the energy.

Lemma 3.3. *Let $(\varphi,\psi,w,\theta,z)$ be the solution of (2.3)-(2.5). Then the functional*

$$F_1(t) := -\rho_1\int_0^1 \varphi\varphi_t dx \tag{3.11}$$

satisfies the estimate

$$\begin{aligned} F_1'(t) &\leq -\frac{\rho_1}{2}\int_0^1 \varphi_t^2 dx + C_1\int_0^1 (\psi-\varphi_x)^2 dx + C_2\int_0^1 (3w_x-\psi_x)^2 dx \\ &\quad + C_3\int_0^1 w_x^2 dx + \frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}\mu(s) z^2(x,1,s,t) dsdx. \end{aligned} \tag{3.12}$$

where

$$\begin{aligned} C_1 &= \frac{3G}{2}+\frac{\mu_0^2}{\rho_1}+\int_{\tau_1}^{\tau_2}\mu(s) ds, \quad C_2 = G+\frac{2\mu_0^2}{\rho_1}+2\int_{\tau_1}^{\tau_2}\mu(s) ds, \\ C_3 &= 9G+\frac{18\mu_0^2}{\rho_1}+18\int_{\tau_1}^{\tau_2}\mu(s) ds. \end{aligned}$$

Proof. Taking the derivative of $F_1(t)$ with respect to t, using the first equation in (2.3), and integrating by parts, gives

$$\begin{aligned} F_1'(t) &= -\rho_1\int_0^1 \varphi_t^2 dx - G\int_0^1 (\psi-\varphi_x)\varphi_x dx + \mu_0\int_0^1 \varphi_t\varphi dx \\ &\quad + \int_0^1 \varphi\int_{\tau_1}^{\tau_2}\mu(s) z(x,1,t,s) dsdx. \end{aligned}$$

Note that

$$-G\int_0^1 (\psi-\varphi_x)\varphi_x dx = G\int_0^1 (\psi-\varphi_x)^2 dx - G\int_0^1 \psi(\psi-\varphi_x) dx.$$

Then, we deduce that

$$\begin{aligned} F_1'(t) &= -\rho_1\int_0^1 \varphi_t^2 dx + G\int_0^1 (\psi-\varphi_x)^2 dx - G\int_0^1 \psi(\psi-\varphi_x) dx \\ &\quad + \mu_0\int_0^1 \varphi_t\varphi dx + \int_0^1 \varphi\int_{\tau_1}^{\tau_2}\mu(s) z(x,1,t,s) dsdx. \end{aligned}$$

We then use Young's inequality, we obtain

$$\begin{aligned} F_1'(t) &\leq -\frac{\rho_1}{2}\int_0^1 \varphi_t^2 dx + \frac{3G}{2}\int_0^1 (\psi-\varphi_x)^2 dx + \frac{G}{2}\int_0^1 \psi_x^2 dx \\ &\quad + \left(\frac{\mu_0^2}{2\rho_1}+\frac{1}{2}\int_{\tau_1}^{\tau_2}\mu(s) ds\right)\int_0^1 \varphi^2 dx \\ &\quad + \frac{1}{2}\int_0^1\int_{\tau_1}^{\tau_2}\mu(s) z^2(x,1,s,t) dsdx. \end{aligned}$$

By using (1.5) and the trivial relation

$$\int_0^1 \varphi^2 dx \leq 2\int_0^1 (\psi-\varphi_x)^2 dx + 2\int_0^1 \psi_x^2 dx,$$

we obtain

$$F_1'(t) \leq -\frac{\rho_1}{2}\int_0^1 \varphi_t^2 dx + \left(\frac{3G}{2} + \frac{\mu_0^2}{\rho_1} + \int_{\tau_1}^{\tau_2} \mu(s)\, ds\right)\int_0^1 (\psi - \varphi_x)^2 dx + \left(\frac{G}{2} + \frac{\mu_0^2}{\rho_1} + \int_{\tau_1}^{\tau_2} \mu(s)\, ds\right)\int_0^1 \psi_x^2 dx + \frac{1}{2}\int_0^1 \int_{\tau_1}^{\tau_2} \mu(s)\, z^2(x,1,s,t)\, dsdx.$$

Note that

$$\int_0^1 \psi_x^2 dx = \int_0^1 (\psi_x - 3w_x + 3w_x)^2 dx \leq 2\int_0^1 (3w_x - \psi_x)^2 dx + 18\int_0^1 w_x^2 dx.$$

Then the estimate (3.12) is established. □

Lemma 3.4. *Let $(\varphi, \psi, w, \theta, z)$ be the solution of (2.3)-(2.5). Then the functional*

$$F_2(t) := \rho_2 \int_0^1 (3w - \psi)(3w - \psi)_t\, dx \tag{3.13}$$

satisfies the estimate

$$F_2'(t) \leq -\frac{D}{2}\int_0^1 (3w_x - \psi_x)^2 dx + \rho_2 \int_0^1 (3w_t - \psi_t)^2 dx + \frac{G^2}{D}\int_0^1 (\psi - \varphi_x)^2 dx + \frac{\sigma^2}{D}\int_0^1 \theta^2 dx. \tag{3.14}$$

Proof. By differentiating $F_2(t)$ with respect to t, then exploiting the second equation in (2.3), and integrating by parts, we obtain

$$F_2'(t) = -D\int_0^1 (3w_x - \psi_x)^2 dx + \rho_2 \int_0^1 (3w_t - \psi_t)^2 dx + G\int_0^1 (\psi - \varphi_x)(3w - \psi)\, dx + \sigma \int_0^1 (3w - \psi)_x\, \theta dx. \tag{3.15}$$

Using Young's inequality, we obtain estimate (3.14). □

Lemma 3.5. *Let $(\varphi, \psi, w, \theta, z)$ be the solution of (2.3)-(2.5). Then the functional*

$$F_3(t) := \rho_2 \int_0^1 w w_t dx \tag{3.16}$$

satisfies, for any $\varepsilon_1 > 0$, the estimate

$$F_3'(t) \leq -\left(\frac{4\gamma}{3} - \varepsilon_1\right)\int_0^1 w^2 dx - D\int_0^1 w_x^2 dx + C_4(\varepsilon_1)\int_0^1 w_t^2 dx + \frac{G^2}{2\varepsilon_1}\int_0^1 (\psi - \varphi_x)^2 dx. \tag{3.17}$$

where

$$C_4(\varepsilon_1) = \rho_2 + \frac{8\beta^2}{9\varepsilon_1}.$$

Proof. By differentiating $F_3(t)$ with respect to t, then exploiting the third equation in (2.3), and integrating by parts, we obtain

$$\begin{aligned} F_3'(t) &= \rho_2 \int_0^1 w_t^2 dx - G \int_0^1 w(\psi - \varphi_x) dx - \frac{4}{3}\gamma \int_0^1 w^2 dx \\ &\quad -\frac{4}{3}\beta \int_0^1 w w_t dx - D \int_0^1 w_x^2 dx. \end{aligned}$$

Using Young's inequality with $\varepsilon_1 > 0$, we obtain estimate (3.17). □

Lemma 3.6. *Let $(\varphi, \psi, w, \theta, z)$ be the solution of (2.3)-(2.5). Then the functional*

$$F_4(t) := \frac{k\rho_2}{\sigma} \int_0^1 (3w - \psi)_t \int_0^x \theta dy dx \tag{3.18}$$

satisfies, for any $\varepsilon_2 > 0$, the estimate

$$\begin{aligned} F_4'(t) &\le -\frac{\rho_2}{2} \int_0^1 (3w_t - \psi_t)^2 dx + C_5(\varepsilon_2) \int_0^1 \theta^2 dx + \varepsilon_2 \int_0^1 (\psi - \varphi_x)^2 dx \\ &\quad +\varepsilon_2 \int_0^1 (3w_x - \psi_x)^2 dx + \frac{\tau\rho_2}{2\sigma^2} \int_0^1 \theta_x^2 dx, \end{aligned} \tag{3.19}$$

where

$$C_5(\varepsilon_2) = k + \frac{k^2 D^2}{4\varepsilon_2 \sigma^2} + \frac{k^2 G^2}{4\varepsilon_2 \sigma^2}.$$

Proof. By differentiating $F_4(t)$ with respect to t, using the second and the fourth equations in (2.3), and integrating by parts, we obtain

$$\begin{aligned} F_4'(t) &= -\rho_2 \int_0^1 (3w_t - \psi_t)^2 dx + \frac{kG}{\sigma} \int_0^1 (\psi - \varphi_x) \int_0^x \theta dy dx \\ &\quad -\frac{kD}{\sigma} \int_0^1 (3w - \psi)_x \theta dx + k \int_0^1 \theta^2 dx + \frac{\tau\rho_2}{\sigma} \int_0^1 (3w - \psi)_t \theta_x dx. \end{aligned} \tag{3.20}$$

Then, using Young's and Poincaré inequalities with $\varepsilon_2 > 0$, we arrive at (3.19). □

Lemma 3.7. *Let $(\varphi, \psi, w, \theta, z)$ be the solution of (2.3)-(2.5). Then the functional*

$$F_5(t) := \rho_2 \int_0^1 w_t (\psi - \varphi_x) dx + \rho_2 \int_0^1 w_t \varphi_x dx - \frac{D\rho_1}{G} \int_0^1 (w_x \varphi_t - w_{xt} \varphi) dx \tag{3.21}$$

satisfies, for any $\varepsilon_3 > 0$, the estimate

$$\begin{aligned} F_5'(t) &\le -\frac{G}{2} \int_0^1 (\psi - \varphi_x)^2 dx + \varepsilon_3 \int_0^1 (3w_t - \psi_t)^2 dx + \frac{16\gamma^2}{9G} \int_0^1 w^2 dx \\ &\quad +C_6 \int_0^1 w_x^2 dx + C_7(\varepsilon_3) \int_0^1 w_t^2 dx + \frac{D\mu_0}{2G} \int_0^1 \varphi_t^2 dx \\ &\quad +\frac{D}{2G} \int_0^1 \int_{\tau_1}^{\tau_2} \mu(s) z^2(x, 1, s, t) ds dx, \end{aligned} \tag{3.22}$$

where $C_6 = \frac{D\mu_0}{2G} + \frac{D}{2G} \int_{\tau_1}^{\tau_2} \mu(s) ds$, $C_7(\varepsilon_3) = \frac{16\beta^2}{9G} + \frac{\rho_2^2}{2\varepsilon_3} + 9\varepsilon_3$.

Proof. Using the first and the third equations in (2.3), and integrating by parts, we obtain

$$\begin{aligned}
&\frac{d}{dt}\left\{\rho_2\int_0^1 w_t\left(\psi-\varphi_x\right)dx\right\}\\
=\;&\frac{D\rho_1}{G}\left\{\frac{d}{dt}\int_0^1\left(w_x\varphi_t-w_{xt}\varphi\right)dx-\int_0^1 w_{tt}\varphi_x dx\right\}+\frac{\mu_0 D}{G}\int_0^1 w_x\varphi_t dx\\
&+\frac{D}{G}\int_0^1 w_x\int_{\tau_1}^{\tau_2}\mu\left(s\right)\varphi_t\left(x,t-s\right)dsdx-G\int_0^1\left(\psi-\varphi_x\right)^2dx\\
&-\frac{4\gamma}{3}\int_0^1 w\left(\psi-\varphi_x\right)dx-\frac{4\beta}{3}\int_0^1 w_t\left(\psi-\varphi_x\right)dx+\rho_2\int_0^1 w_t\psi_t dx\\
&-\frac{d}{dt}\left\{\rho_2\int_0^1 w_t\varphi_x dx\right\}+\rho_2\int_0^1 w_{tt}\varphi_x dx
\end{aligned}$$

We conclude for

$$\begin{aligned}
F_5'\left(t\right)\;=\;&D\left(\frac{\rho_2}{D}-\frac{\rho_1}{G}\right)\int_0^1 w_{tt}\varphi_x dx+\frac{\mu_0 D}{G}\int_0^1 w_x\varphi_t dx\\
&+\frac{D}{G}\int_0^1 w_x\int_{\tau_1}^{\tau_2}\mu\left(s\right)z\left(x,1,s,t\right)dsdx-G\int_0^1\left(\psi-\varphi_x\right)^2dx\\
&-\frac{4\gamma}{3}\int_0^1 w\left(\psi-\varphi_x\right)dx-\frac{4\beta}{3}\int_0^1 w_t\left(\psi-\varphi_x\right)dx+\rho_2\int_0^1 w_t\psi_t dx.
\end{aligned}$$

Using Young's inequality and $\frac{\rho_2}{D}=\frac{\rho_1}{G}$, we obtain (3.22). □

Lemma 3.8. *Let* $(\varphi,\psi,w,\theta,z)$ *be the solution of* (2.3)*-*(2.5) *and* (2.2)*. Then the functional*

$$F_6(t):=\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} se^{-s\rho}\mu\left(s\right)z^2\left(x,\rho,s,t\right)dsd\rho dx \tag{3.23}$$

satisfies, for some positive constant n*, the following estimate*

$$\begin{aligned}
F_6^{'}(t)\;\leq\;&-n\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} s\mu\left(s\right)z^2\left(x,\rho,s,t\right)dsd\rho dx\\
&-n\int_0^1\int_{\tau_1}^{\tau_2}\mu\left(s\right)z^2\left(x,1,s,t\right)dsdx+\mu_0\int_0^1\varphi_t^2 dx.
\end{aligned} \tag{3.24}$$

Proof. By differentiating $F_6\left(t\right)$ with respect to t, and using the equation (2.2), we obtain

$$\begin{aligned}
F_6^{'}(t)\;=\;&-2\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} e^{-s\rho}\mu\left(s\right)z\left(x,\rho,s,t\right)z_\rho\left(x,\rho,s,t\right)dsd\rho dx\\
=\;&-\int_0^1\int_{\tau_1}^{\tau_2}\mu\left(s\right)\left[e^{-s}z^2\left(x,1,s,t\right)-z^2\left(x,0,s,t\right)\right]dsdx\\
&-\int_0^1\int_0^1\int_{\tau_1}^{\tau_2} se^{-s\rho}\mu\left(s\right)z^2\left(x,\rho,s,t\right)dsd\rho dx.
\end{aligned} \tag{3.25}$$

Using the fact that $z(x,0,s,t)=\varphi_t$ and $e^{-s}\leq e^{-s\rho}\leq 1$, for all $0<\rho<1$, we obtain

$$\begin{aligned} F_6'(t) \leq & -\int_0^1\int_{\tau_1}^{\tau_2} e^{-s}\mu(s)\,z^2(x,1,s,t)\,dsdx+\int_{\tau_1}^{\tau_2}\mu(s)\,ds\int_0^1\varphi_t^2dx \\ & -n_1\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}\mu(s)\,z^2(x,\rho,s,t)\,dsd\rho dx. \end{aligned} \tag{3.26}$$

Because $-e^{-s}$ is an increasing function, we have $-e^{-s}\leq -e^{-\tau_2}$, for all $s\in[\tau_1,\tau_2]$. Finally, setting $n=e^{-\tau_2}$ and recalling (1.5), we obtain (3.24). □

Next, we define a Lyapunov functional $L(t)$ and show that it is equivalent to the energy functional $E(t)$.

Lemma 3.9. *Let $N,N_2,N_3,N_4,N_5,N_6>0$ and $\frac{\rho_1}{G}=\frac{\rho_2}{D}$, we define*

$$L(t):=NE(t)+F_1(t)+\sum_{i=2}^{i=6}N_iF_i(t) \tag{3.27}$$

For two positive constants β_1 and β_2, we have

$$\beta_1E(t)\leq L(t)\leq\beta_2E(t),\forall t\geq 0. \tag{3.28}$$

Proof. Now, let

$$\mathcal{L}(t)=F_1(t)+\sum_{i=2}^{i=6}N_iF_i(t)$$

$$\begin{aligned} |\mathcal{L}(t)| \leq & \rho_1\int_0^1|\varphi\varphi_t|\,dx+N_2\rho_2\int_0^1|(3w-\psi)(3w-\psi)_t|\,dx \\ & +N_3\rho_2\int_0^1|ww_t|\,dx+N_4\frac{k\rho_2}{\sigma}\int_0^1\left|(3w-\psi)_t\int_0^x\theta dy\right|dx \\ & +N_5\rho_2\int_0^1|w_t(\psi-\varphi_x)|\,dx+N_5\frac{D\rho_1}{G}\int_0^1|(w_x\varphi_t-w_{xt}\varphi)|\,dx \\ & +N_5\rho_2\int_0^1|w_t\varphi_x|\,dx \\ & +N_6\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}\left|se^{-s\rho}\mu(s)\,z^2(x,\rho,s,t)\right|dsd\rho dx. \end{aligned}$$

Exploiting Young's, Poincaré, Cauchy-Schwarz inequalities, (3.1), and the fact that $e^{-s\rho}\leq 1$ for all $\rho\in[0,1]$, we obtain

$$\begin{aligned} |\mathcal{L}(t)|\leq c\int_0^1\Big[\varphi_t^2+(3w_t-\psi_t)^2+w_t^2+(\psi-\varphi_x)^2+(3w_x-\psi_x)^2+w_x^2+w^2 \\ +\theta^2\Big]\,dx+c\int_0^1\int_0^1\int_{\tau_1}^{\tau_2}s\mu(s)\,z^2(x,\rho,z,t)\,dsd\rho dx\leq cE(t). \end{aligned}$$

Consequently, $|L(t)-NE(t)|\leq cE(t)$, which yields

$$(N-c)\,E(t)\leq L(t)\leq(N+c)\,E(t).$$

Choosing such that $(N-c)>0$, we obtain estimate (3.28). □

Now, we are ready to state and prove the main result of this section.

Proof. (of Theorem 3.1). By differentiating (3.27) and recalling (3.12), (3.14), (3.17), (3.19), (3.22) and (3.24), we obtain

$$\begin{aligned}
L'(t) \leq\ & -\left[\left(\mu_0 - \int_{\tau_1}^{\tau_2} \mu(s)\,ds\right) N + \frac{\rho_1}{2} - \frac{D\mu_0}{2G} N_5 - \mu_0 N_6\right] \int_0^1 \varphi_t^2 dx \\
& - \left[\frac{4\gamma}{3} N_3 - \varepsilon_1 N_3 - \frac{16\gamma^2}{9G} N_5\right] \int_0^1 w^2 dx \\
& - \left[\tau N - \frac{\tau\rho_2}{2\sigma^2} N_4\right] \int_0^1 \theta_x^2 dx \\
& - \left[D N_3 - C_3 - C_6 N_5\right] \int_0^1 w_x^2 dx + \left[\frac{\sigma^2}{D} N_2 + C_5(\varepsilon_2) N_4\right] \int_0^1 \theta^2 dx \\
& - \left[\frac{G}{2} N_5 - C_1 - \frac{G^2}{D} N_2 - \frac{G^2}{2\varepsilon_1} N_3 - \varepsilon_2 N_4\right] \int_0^1 (\psi - \varphi_x)^2 dx \\
& - \left[\frac{\rho_2}{2} N_4 - \rho_2 N_2 - \varepsilon_3 N_5\right] \int_0^1 (3w_t - \psi_t)^2 dx \\
& - \left[4\beta N - C_4(\varepsilon_1) N_3 - C_7(\varepsilon_3) N_5\right] \int_0^1 w_t^2 dx \\
& - \left[\frac{D}{2} N_2 - C_2 - \varepsilon_2 N_4\right] \int_0^1 (3w_x - \psi_x)^2 dx \\
& - \left[n N_6\right] \int_0^1 \int_0^1 \int_{\tau_1}^{\tau_2} s\mu(s) z^2(x,\rho,s,t)\,ds\,d\rho\,dx \\
& - \left[n N_6 - \frac{1}{2} - \frac{D}{2G} N_5\right] \int_0^1 \int_{\tau_1}^{\tau_2} \mu(s) z^2(x,1,s,t)\,ds\,dx. \qquad (3.29)
\end{aligned}$$

At this point, we need to choose our constants very carefully. First, we take N_2 large enough, such that

$$\frac{D}{2} N_2 - C_2 \geq 0.$$

Then, we choose N_4 and N_5 large enough, so that

$$\frac{\rho_2}{2} N_4 - \rho_2 N_2 \geq 0, \quad \frac{G}{2} N_5 - C_1 - \frac{G^2}{D} N_2 \geq 0.$$

Next, we pick ε_1 small and choose N_3 large enough, such that

$$D N_3 - C_3 - C_6 N_5 \geq 0, \ \frac{4\gamma}{3} N_3 - \varepsilon_1 N_3 - \frac{16\gamma^2}{9G} N_5 \geq 0.$$

Then, we select N_3 even smaller (if needed) and ε_2, ε_3 small enough, so that

$$\frac{D}{2} N_2 - C_2 - \varepsilon_2 N_4 \geq 0, \ \frac{\rho_2}{2} N_4 - \rho_2 N_2 - \varepsilon_3 N_5 \geq 0,$$

$$\frac{G}{2} N_5 - C_1 - \frac{G^2}{D} N_2 - \frac{G^2}{2\varepsilon_1} N_3 - \varepsilon_2 N_4 \ \geq 0.$$

Furthermore, we choose N_6 large enough, so that

$$nN_6 - \frac{D}{2G}N_5 - \frac{1}{2} \geq 0.$$

Finally, we choose N so large such that

$$\begin{aligned} \left(\mu_0 - \int_{\tau_1}^{\tau_2} \mu(s)\, ds\right) N + \frac{\rho_1}{2} - \frac{D\mu_0}{2G} N_5 - \mu_0 N_6 &\geq 0, \\ 4\beta N - C_4(\varepsilon_1) N_3 - C_7(\varepsilon_3) N_5 &\geq 0. \end{aligned}$$

Thus, we deduce that there exist positive constants α_1 and α_2 such that (3.29) becomes

$$\begin{aligned} L'(t) &\leq -\alpha_1 E(t) - \left[\tau N - \frac{\tau\rho_2}{2\sigma^2} N_4\right] \int_0^1 \theta_x^2 dx + \alpha_2 \int_0^1 \theta^2 dx \\ &\quad - \left[nN_6 - \frac{1}{2} - \frac{D}{2G} N_5\right] \int_0^1 \int_{\tau_1}^{\tau_2} \mu(s)\, z^2(x,1,s,t) ds dx \\ &\leq -\alpha_1 E(t) + \alpha_2 \int_0^1 \theta_x^2 dx. \end{aligned}$$

By (3.3), we obtain

$$L'(t) \leq -\alpha_1 E(t) - \alpha_3 E'(t), \tag{3.30}$$

for some $\alpha_3 > 0$. It is obvious that

$$\mathfrak{L}(t) = L(t) + \alpha_3 E(t) \sim E(t).$$

Next, exploiting (3.30), we get

$$\mathfrak{L}'(t) = L'(t) + \alpha_3 E'(t) \leq -\alpha_1 E(t) \leq -c_1 \mathfrak{L}(t), \tag{3.31}$$

for some $c_1 > 0$. Integration (3.31) over $(0,t)$, leads to

$$\mathfrak{L}(t) \leq \mathfrak{L}(0)\, e^{-c_1 t}, \ \forall\, t \geq 0. \tag{3.32}$$

It gives the desired result theorem 3.1 when combined with the equivalence of $L(t)$ and $E(t)$. □

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Global nonexistence of solutions to system of Klein-Gordon equations with degenerate damping and strong source terms in viscoelasticity

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Abstract. In this paper, we consider a system of nonlinear viscoelastic wave equations with degenerate damping and source terms. We prove, with positive initial energy, the global nonexistence of solution by concavity method.

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Keywords: Global nonexistence, nonlinear viscoelastic wave equations, positive initial energy, concavity method.

1. Introduction

In this paper, we consider a system of viscoelastic wave equations with degenerate damping and strong nonlinear source terms

$$\begin{cases} u_{tt} - \Delta u + m_1^2 \cdot u + \displaystyle\int_0^t g\,(t-s)\,\Delta u\,(x,s)\,ds + \left(a\,|u|^k + b\,|v|^l\right)|u_t|^{m-1}u_t = f_1(u,v), \\ v_{tt} - \Delta v + m_2^2 \cdot v + \displaystyle\int_0^t h\,(t-s)\,\Delta v\,(x,s)\,ds + \left(c\,|v|^\theta + d\,|u|^\varrho\right)|v_t|^{r-1}v_t = f_2(u,v), \end{cases} \tag{1.1}$$

where $m, r > 0, k, l, \theta, \varrho \geq 1$ and the functions $f_1\,(u,v)\,, f_2\,(u,v)$ are defined by

$$\begin{aligned} f_1(\xi_1,\xi_2) &= a_1|\xi_1+\xi_2|^{2(\rho+1)}(\xi_1+\xi_2) + b_1|\xi_1|^{\rho}\xi_1|\xi_2|^{(\rho+2)} \\ f_2(\xi_1,\xi_2) &= a_1|\xi_1+\xi_2|^{2(\rho+1)}(\xi_1+\xi_2) + b_1|\xi_1|^{(\rho+2)}|\xi_2|^{\rho}\xi_2, a_1, b_1 > 0, \end{aligned} \tag{1.2}$$

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where $\rho > -1$. In (1.1), $u = u(x,t), v = v(x,t)$, where $x \in \Omega$ is a bounded domain of $\mathbb{R}^N$ $(N \geq 1)$ with a smooth boundary $\partial\Omega$ and $t > 0$, $a, b, c, d, m_1, m_2 > 0$.
To above system (1.1), we add the initial conditions given by

$$(u(0), v(0)) = (u_0, v_0), (u_t(0), v_t(0)) = (u_1, v_1), x \in \Omega \tag{1.3}$$

and boundary conditions given by

$$u(x) = v(x) = 0, x \in \partial\Omega. \tag{1.4}$$

This kind of problems arise in viscoelasticity. Dafermos was the first who study this type in [9], where the general decay was treated. In the last decades, problems related to system (1.1) had a lot of attention and many results appeared on the existence and long time behavior of solutions. See in this directions ([6, 3, 2, 4, 5, 8, 7, 11, 14, 17, 20, 19, 21, 27, 26]) and references therein.
In the absence of viscoelastic term, some special cases of the single wave equations with nonlinear damping and nonlinear source terms in the form

$$u_{tt} - \Delta u + a|u_t|^{m-1}u_t = b|u|^{p-1}u. \tag{1.5}$$

With nonlinear damping and source terms, it arises in the quantum-field and used to describe the movement of charged electromagnetic fields. Equation (1.5) equipped with initial and bounded conditions of Dirichlet type has been extensively studied and many results regarding existence, blow up and asymptotic behavior of solutions have been obtained. Many authors have studied the single wave equations in the presence of various mechanisms of dissipation, damping and non-linear sources. See ([1, 15, 18, 10, 12, 13, 24, 25, 28]) and references therein.
In [16], authors considered the nonlinear viscoelastic system

$$\begin{cases} u_{tt} - \Delta u + \displaystyle\int_0^t g(t-s)\Delta u(x,s)ds + |u_t|^{m-1} u_t = f_1(u,v), \\ v_{tt} - \Delta v + \displaystyle\int_0^t h(t-s)\Delta v(x,s)ds + |v_t|^{r-1} v_t = f_2(u,v), \end{cases} \tag{1.6}$$

where

$$\begin{aligned} f_1(u,v) &= a|u+v|^{2(\rho+1)}(u+v) + b|u|^{\rho}u|v|^{(\rho+2)} \\ f_2(u,v) &= a|u+v|^{2(\rho+1)}(u+v) + b|u|^{(\rho+2)}|v|^{\rho}v, \end{aligned} \tag{1.7}$$

The global nonexistence theorem for some solutions with positive energy was proved using a method applied in [22].
In [23], the authors studied the nonlinear viscoelastic system in (1.6), where they obtained the decay of solutions for system. Under some restrictions on the nonlinearities of damping and source terms, they proved that, for some class of relaxation functions and some restrictions on the initial data, the rate of decay of relaxation functions affects the rate of decay of solution for system.
In this paper, we consider system (1.1)-(1.4) and proved a global nonexistence result of solutions. We extended to result in [16] and [27] to more general cases.

2. Preliminaries

In this section, we present some notations and Lemmas.
We assume that the relaxation functions $g, h \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ satisfying

$$\begin{cases} 1-\int_0^\infty g(s)ds = l' > 0, \quad g(t) \geq 0, \quad g'(t) \leq 0, \\ 1-\int_0^\infty h(s)ds = k' > 0, \quad h(t) \geq 0, \quad h'(t) \leq 0, \end{cases} \quad t \geq 0. \tag{2.1}$$

We introduce the "*modified*" energy functional E associated to our system

$$2E(t) = \|u_t\|_2^2 + \|v_t\|_2^2 + 2\left(m_1^2\|u\|_2^2 + m_2^2\|v\|_2^2\right) + J(u,v) - 2\int_\Omega F(u,v)\,dx, \tag{2.2}$$

where $F(u,v)$ is defined for all $(u,v) \in \mathbb{R}^2$,

$$\begin{aligned} F(u,v) &= \frac{1}{2(\rho+2)}\left[uf_1(u,v) + vf_2(u,v)\right], \\ &= \frac{1}{2(\rho+2)}\left[|u+v|^{2(\rho+2)} + 2|uv|^{\rho+2}\right] \geq 0, \end{aligned}$$

where

$$\frac{\partial F}{\partial u} = f_1(u,v), \frac{\partial F}{\partial v} = f_2(u,v),$$

and

$$\begin{aligned} J(u,v) &= \left(1-\int_0^t g(s)\,ds\right)\|\nabla u\|_2^2 + \left(1-\int_0^t h(s)\,ds\right)\|\nabla v\|_2^2 \\ &+ (g \circ \nabla u) + (h \circ \nabla v). \end{aligned} \tag{2.3}$$

Noting by

$$\begin{cases} (g \circ u)(t) = \int_0^t g(t-\tau)\|u(t) - u(\tau)\|_2^2\,d\tau, \\ (h \circ v)(t) = \int_0^t h(t-\tau)\|v(t) - v(\tau)\|_2^2\,d\tau. \end{cases} \tag{2.4}$$

We suppose that ρ satisfies

$$\begin{cases} -1 < \rho, & \text{if} \quad N = 1, 2, \\ -1 < \rho \leq \frac{4-N}{N-2} & \text{if} \quad N \geq 3. \end{cases} \tag{2.5}$$

Lemma 2.1. [22] *There exist two positive constants c_0 and c_1 with the end goal that*

$$\frac{c_0}{2(\rho+2)}\left(|u|^{2(\rho+2)} + |v|^{2(\rho+2)}\right) \leq F(u,v) \leq \frac{c_1}{2(\rho+2)}\left(|u|^{2(\rho+2)} + |v|^{2(\rho+2)}\right).$$

Lemma 2.2. *Assume that (2.5) holds. There exists $\eta > 0$, such that for any $(u,v) \in H_0^1(\Omega) \times H_0^1(\Omega)$, the inequality*

$$2(\rho+2)\int_\Omega F(u,v)\,dx \leq \eta\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right)^{\rho+2} \tag{2.6}$$

holds.

Lemma 2.3. *Let $\nu > 0$, be a real positive number and let $L(t)$ be a solution of the ordinary differential inequality*

$$\frac{dL(t)}{dt} \geq \xi L^{1+\nu}(t) \tag{2.7}$$

defined in $[0, \infty)$.
If $L(0) > 0$, then the solution does not exist for $t \geq L(0)^{-\nu} \xi^{-\nu} \nu^{-1}$.

Proof. By simple integration of (2.7), we have

$$L^{-\nu}(0) - L^{-\nu}(t) \geq \xi \nu t.$$

Then, we obtain the following estimate

$$L^{\nu}(t) \geq \left[L^{-\nu}(0) - \xi \nu t\right]^{-1}. \tag{2.8}$$

Then, the RHS of (2.8) is unbounded for

$$\xi \nu t = L^{-\nu}(0).$$

The proof is completed. □

3. Blow up result

Lemma 3.1. *Assume that (2.5) holds. Let (u, v) be the solution of the system (1.1)–(1.4) then the energy functional is a non-increasing function, that is, for all $t \geq 0$,*

$$\begin{aligned} E'(t) &= -\int_\Omega \left(|u(t)|^k + |v(t)|^l\right) |u_t(t)|^{m+1} dx \\ &- \int_\Omega \left(|v(t)|^\theta + |u(t)|^\varrho\right) |v_t(t)|^{r+1} dx \\ &+ \frac{1}{2}(g' \circ \nabla u) + \frac{1}{2}(h' \circ \nabla v) - \frac{1}{2} g(s) \|\nabla u\|_2^2 - \frac{1}{2} h(s) \|\nabla v\|_2^2 . \end{aligned} \tag{3.1}$$

Lemma 3.2. *Suppose that (2.5) holds. Let (u, v) be the solution of the system (1.1)-(1.4), then the energy functional is a non-increasing function, that is, for all $t > 0$,*

$$\begin{aligned} \frac{dE(t)}{dt} &= -\int_\Omega \left(|u(t)|^k + |v(t)|^l\right) |u_t(t)|^{m+1} dx \\ &\quad -\int_\Omega \left(|v(t)|^\theta + |u(t)|^\vartheta\right) |v_t(t)|^{r+1} dx. \end{aligned} \tag{3.2}$$

The proof of Lemma 3.1 can be done by using a classical calculations.
Our main result reads as follows

Theorem 3.3. *Suppose that (2.5) holds. Assume further that*

$$\rho > \max\left(\frac{k+m-3}{2}, \frac{l+m-3}{2}, \frac{\theta+r-3}{2}, \frac{\varrho+r-3}{2}\right), \tag{3.3}$$

and that there exists p *such that* $2 < p < 2(\rho+2)$, *for which*

$$\max\left(\int_0^\infty g(s)\,ds, \int_0^\infty h(s)\,ds\right) < \frac{(p/2)-1}{(p/2)-1+1/(2p)}, \tag{3.4}$$

holds. Then any solution of problem (1.1)*–*(1.4)*, with initial data satisfying*

$$\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2 + m_1^2\|u_0\|_2^2 + m_2^2\|v_0\|_2^2 > \alpha_1^2, \text{ and } E(0) < E_2 \tag{3.5}$$

blows up in finite time, where the constants α_1 *and* E_2 *are defined in* (3.6)*.*

We take $a = b = c = d = 1, a_1 = b_1 = 1$ for convenience. We introduce the following constants

$$B = \eta^{\frac{1}{2(\rho+2)}}, \qquad \alpha_1 = B^{-\frac{\rho+2}{\rho+1}}, \qquad E_1 = \left(\frac{1}{2} - \frac{1}{2(\rho+2)}\right)\alpha_1^2, \tag{3.6}$$

$$E_2 = \left(\frac{1}{p} - \frac{1}{2(\rho+2)}\right)\alpha_1^2,$$

where η is the optimal constant in (2.6).

Lemma 3.4. [22] *Suppose that* (2.5)*,* (3.3) *and* (3.4) *hold. Let* (u, v) *be a solutions of* (1.1)*–*(1.4)*. Assume further that* $E(0) < E_2$ *and*

$$\|\nabla u_0\|_2^2 + \|\nabla v_0\|_2^2 + m_1^2\|u_0\|_2^2 + m_2^2\|v_0\|_2^2 > \alpha_1^2. \tag{3.7}$$

Then, there exists a constant $\alpha_2 > \alpha_1$ *such that*

$$J(t) > \alpha_2^2, \tag{3.8}$$

and

$$2(\rho+2)\int_\Omega F(u,v)\,dx \ge (B\alpha_2)^{2(\rho+2)}, \ \forall\, t \ge 0. \tag{3.9}$$

Proof of Theorem 3.3*.* The proof is similar to one given in [14] with the necessary modification imposed by the nature of our problem. We assume that the solutions exists for all t and we get a contradiction. We set

$$H(t) = E_2 - E(t). \tag{3.10}$$

By using the definition of $H(t)$, we obtain

$$\begin{aligned} H'(t) &= -E'(t) \\ &= \int_\Omega \left(|u(t)|^k + |v(t)|^l\right)|u_t(t)|^{m+1}\,dx \\ &+ \int_\Omega \left(|v(t)|^\theta + |u(t)|^\varrho\right)|v_t(t)|^{r+1}\,dx \\ &- \frac{1}{2}(g' \circ \nabla u) - \frac{1}{2}(h' \circ \nabla v) + \frac{1}{2}g(s)\|\nabla u\|_2^2 + \frac{1}{2}h(s)\|\nabla v\|_2^2 \\ &\ge 0, \ \forall\, t \ge 0. \end{aligned} \tag{3.11}$$

Therefore,

$$H(0) = E_2 - E(0) > 0. \tag{3.12}$$

Then,

$$
\begin{aligned}
0 &< H(0) \leq H(t) \\
&= E_2 - \frac{1}{2}\left(\|u_t\|_2^2 + \|v_t\|_2^2 + m_1^2\|u\|_2^2 + m_2^2\|v\|_2^2\right) - \frac{J(t)}{2} \\
&+ \frac{1}{2(\rho+2)}\left[\|u+v\|_{2(\rho+2)}^{2(\rho+2)} + 2\|uv\|_{\rho+2}^{\rho+2}\right].
\end{aligned} \tag{3.13}
$$

Note that from (2.1) and (3.8), we get

$$
\begin{aligned}
E_2 - \frac{1}{2}\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) - \frac{J(t)}{2} &< E_2 - \frac{1}{2}\alpha_2^2 \\
&< E_2 - \frac{1}{2}\alpha_1^2 \\
&< E_1 - \frac{1}{2}\alpha_1^2 \\
&= -\frac{1}{2(\rho+2)}\alpha_1^2 < 0, \ \forall\, t \geq 0.
\end{aligned} \tag{3.14}
$$

Thus, by using (3.14) and Lemma 2.1, we get

$$
\begin{aligned}
0 &< H(0) \leq H(t) \leq \frac{1}{2(\rho+2)}\left[\|u+v\|_{2(\rho+2)}^{2(\rho+2)} + 2\|uv\|_{\rho+2}^{\rho+2}\right] \\
&\leq \frac{c_1}{2(\rho+2)}\left(\|u\|_{2(\rho+2)}^{2(\rho+2)} + \|v\|_{2(\rho+2)}^{2(\rho+2)}\right), \ \forall\, t \geq 0.
\end{aligned} \tag{3.15}
$$

We define the function M as

$$
M(t) = \frac{1}{2}\int_\Omega \left(u^2 + v^2\right)(x,t)\,dx, \tag{3.16}
$$

and let

$$
L(t) = H^{1-\sigma}(t) + \varepsilon M'(t), \tag{3.17}
$$

for ε small to be chosen later and

$$
\begin{aligned}
0 < \sigma \leq \min &\left\{\frac{1}{2}, \frac{2\rho+3-(k+m)}{2(m+1)(\rho+2)}, \frac{2\rho+3-(l+m)}{2(m+1)(\rho+2)},\right. \\
&\left.\frac{2\rho+3-(\varrho+r)}{2(r+1)(\rho+2)}, \frac{2\rho+3-(\theta+r)}{2(r+1)(\rho+2)}, \frac{2\rho+2}{4(\rho+2)}\right\}.
\end{aligned} \tag{3.18}
$$

By differentiation of (3.17) with respect to time and using (1.1), we get

$$\begin{aligned}
L'(t) &= (1-\sigma) H^{-\sigma}(t) H'(t) + \varepsilon \left(\|u_t\|_2^2 + \|v_t\|_2^2\right) \\
&- \varepsilon \left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right) \\
&- \varepsilon \int_\Omega u \left(|u(t)|^k + |v(t)|^l\right) |u_t|^{m-1} u_t dx \\
&- \varepsilon \int_\Omega v \left(|v(t)|^\theta + |u(t)|^\varrho\right) |v_t|^{r-1} v_t dx \\
&+ \varepsilon \int_\Omega (u f_1(u,v) + v f_2(u,v)) dx \\
&+ \varepsilon \int_\Omega \nabla u(t) \int_0^t g(t-s) \nabla u(\tau) dxds \\
&+ \varepsilon \int_\Omega \nabla v(t) \int_0^t h(t-s) \nabla v(\tau) dxds. \qquad (3.19)
\end{aligned}$$

Then,

$$\begin{aligned}
L'(t) &= (1-\sigma) H^{-\sigma}(t) H'(t) + \varepsilon \left(\|u_t\|_2^2 + \|v_t\|_2^2 + m_1^2 \|u\|_2^2 + m_2^2 \|v\|_2^2\right) \\
&- \varepsilon \left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right) \\
&- \varepsilon \int_\Omega u \left(|u(t)|^k + |v(t)|^l\right) |u_t|^{m-1} u_t dx \\
&- \varepsilon \int_\Omega v \left(|v(t)|^\theta + |u(t)|^\varrho\right) |v_t|^{r-1} v_t dx \\
&+ \varepsilon \left(\|u+v\|_{2(\rho+2)}^{2(\rho+2)} + 2\|uv\|_{\rho+2}^{\rho+2}\right) \\
&+ \varepsilon \left(\int_0^t g(s) ds\right) \|\nabla u\|_2^2 + \left(\int_0^t h(s) ds\right) \|\nabla v\|_2^2 \\
&+ \varepsilon \int_0^t g(t-s) \int_\Omega \nabla u(t) . [\nabla u(\tau) - \nabla u(t)] dxds \\
&+ \varepsilon \int_0^t h(t-s) \int_\Omega \nabla v(t) . [\nabla v(\tau) - \nabla v(t)] dxds. \qquad (3.20)
\end{aligned}$$

By using Cauchy-Schwartz and Young's inequalities, we obtain the following estimate

$$\begin{aligned}
&\int_0^t g(t-s) \int_\Omega \nabla u(t) . [\nabla u(\tau) - \nabla u(t)] dxds \\
\leq\ & \int_0^t g(t-s) \|\nabla u\|_2 \|\nabla u(\tau) - \nabla u(t)\|_2 d\tau \\
\leq\ & \lambda (g \circ \nabla u) + \frac{1}{4\lambda} \left(\int_0^t g(s) ds\right) \|\nabla u\|_2^2, \qquad \lambda > 0 \qquad (3.21)
\end{aligned}$$

and

$$\begin{aligned} & \int_0^t h(t-s) \int_\Omega \nabla v(t) \,.\, [\nabla v(\tau) - \nabla v(t)]\, dxds \\ \leq \; & \lambda (h \circ \nabla v) + \frac{1}{4\lambda} \left(\int_0^t h(s)\, ds \right) \|\nabla v\|_2^2, \qquad \lambda > 0. \end{aligned} \tag{3.22}$$

Adding $pE(t)$ and using the definition of $H(t), E_2$ leads to

$$\begin{aligned} L'(t) \; \geq \; & (1-\sigma) H^{-\sigma}(t) H'(t) \\ + \; & \varepsilon \left(1 + \frac{p}{2}\right) \left(\|u_t\|_2^2 + \|v_t\|_2^2 + m_1^2 \|u\|_2^2 + m_2^2 \|v\|_2^2 \right) \\ + \; & \varepsilon \left(\frac{p}{2} - \lambda\right) [(g \circ \nabla u) + (h \circ \nabla v)] + p\varepsilon H(t) - p\varepsilon E_2 \\ - \; & \varepsilon \int_\Omega u \left(|u(t)|^k + |v(t)|^l \right) |u_t|^{m-1} u_t dx \\ - \; & \varepsilon \int_\Omega v \left(|v(t)|^\theta + |u(t)|^\varrho \right) |v_t|^{r-1} v_t dx \\ + \; & \varepsilon \left(1 - \frac{p}{2(\rho+2)}\right) \left(\|u+v\|_{2(\rho+2)}^{2(\rho+2)} + 2\|uv\|_{\rho+2}^{\rho+2} \right) \\ + \; & \varepsilon \left[\left(\frac{p}{2} - 1\right) - \left(\frac{p}{2} - 1 + \frac{1}{4\lambda}\right) \int_0^\infty g(s)\, ds \right] \|\nabla u\|_2^2 \\ + \; & \varepsilon \left[\left(\frac{p}{2} - 1\right) - \left(\frac{p}{2} - 1 + \frac{1}{4\lambda}\right) \int_0^\infty h(s)\, ds \right] \|\nabla v\|_2^2, \end{aligned} \tag{3.23}$$

for some λ such that

$$a_1 = \frac{p}{2} - \lambda > 0,$$

and

$$a_2 = \left[\left(\frac{p}{2} - 1\right) - \left(\frac{p}{2} - 1 + \frac{1}{4\lambda}\right) \max \left(\int_0^\infty g(s)\, ds, \; \int_0^\infty h(s)\, ds \right) \right] > 0.$$

Then, (3.23) can be estimated as follows

$$\begin{aligned} L'(t) \; \geq \; & (1-\sigma) H^{-\sigma}(t) H'(t) + \varepsilon \left(1 + \frac{p}{2}\right) \left(\|u_t\|_2^2 + \|v_t\|_2^2 \right) \\ + \; & \varepsilon a_1 [(g \circ \nabla u) + (h \circ \nabla v)] + p\varepsilon H(t) - p\varepsilon E_2 \\ - \; & \varepsilon \int_\Omega u \left(|u(t)|^k + |v(t)|^l \right) |u_t|^{m-1} u_t dx \\ - \; & \varepsilon \int_\Omega v \left(|v(t)|^\theta + |u(t)|^\varrho \right) |v_t|^{r-1} v_t dx \\ + \; & \varepsilon \left(1 - \frac{p}{2(\rho+2)}\right) \left(\|u+v\|_{2(\rho+2)}^{2(\rho+2)} + 2\|uv\|_{\rho+2}^{\rho+2} \right) \\ + \; & \varepsilon a_2 \left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2 \right). \end{aligned} \tag{3.24}$$

By taking $c_3 = 1 - \dfrac{p}{\rho+2} - 2E_2 (B\alpha_2)^{-2(\rho+2)} > 0$, since $\alpha_2 > B^{-\frac{2(\rho+2)}{\rho+1}}$. Consequently, (3.24) takes the form

$$\begin{aligned}
L'(t) &\geq (1-\sigma) H^{-\sigma}(t) H'(t) \\
&+ \varepsilon \left(1+\frac{p}{2}\right)\left(\|u_t\|_2^2 + \|v_t\|_2^2 + m_1^2 \|u\|_2^2 + m_2^2 \|v\|_2^2\right) \\
&+ \varepsilon a_1 [(g \circ \nabla u) + (h \circ \nabla v)] \\
&+ \varepsilon a_2 \left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right) + p\varepsilon H(t) \\
&+ \varepsilon c_3 \left(\|u+v\|_{2(\rho+2)}^{2(\rho+2)} + 2\|uv\|_{\rho+2}^{\rho+2}\right) \\
&- \varepsilon \int_\Omega u \left(|u(t)|^k + |v(t)|^l\right) |u_t|^{m-1} u_t dx \\
&- \varepsilon \int_\Omega v \left(|v(t)|^\theta + |u(t)|^\varrho\right) |v_t|^{r-1} v_t dx. \qquad (3.25)
\end{aligned}$$

By using Young's inequality, we have

$$XY \leq \frac{\delta^\alpha X^\alpha}{\alpha} + \frac{\delta^{-\beta} Y^\beta}{\beta}, \qquad (3.26)$$

where $X, Y \geq 0$, $\delta > 0$ and $\alpha, \beta > 0$ such that $1/\alpha + 1/\beta = 1$, we obtain

$$\left|u |u_t|^{m-1} u_t\right| \leq \frac{\delta_1^{m+1}}{m+1} |u|^{m+1} + \frac{m}{m+1} \delta_1^{-(m+1)/m} |u_t|^{m+1}, \ \forall\ \delta_1 \geq 0 \qquad (3.27)$$

and

$$\begin{aligned}
&\int_\Omega \left(|u(t)|^k + |v(t)|^l\right) \left|u |u_t|^{m-1} u_t\right| dx \\
\leq\ & \frac{\delta_1^{m+1}}{m+1} \int_\Omega \left(|u(t)|^k + |v(t)|^l\right) |u|^{m+1} dx \\
&+ \frac{m}{m+1} \delta_1^{-(m+1)/m} \int_\Omega \left(|u(t)|^k + |v(t)|^l\right) |u_t|^{m+1} dx. \qquad (3.28)
\end{aligned}$$

Similarly, for any $\delta_2 > 0$,

$$\left|v |v_t|^{r-1} v_t\right| \leq \frac{\delta_2^{r+1}}{r+1} |v|^{r+1} + \frac{r}{r+1} \delta_2^{-(r+1)/r} |v_t|^{r+1}, \qquad (3.29)$$

which gives

$$\begin{aligned}
&\int_\Omega \left(|v(t)|^\theta + |u(t)|^\varrho\right) \left|v |v_t|^{r-1} v_t\right| dx \\
\leq\ & \frac{\delta_2^{r+1}}{r+1} \int_\Omega \left(|v(t)|^\theta + |u(t)|^\varrho\right) |v|^{r+1} dx \\
+\ & \frac{r}{r+1} \delta_2^{-(r+1)/r} \int_\Omega \left(|v(t)|^\theta + |u(t)|^\varrho\right) |v_t|^{r+1} dx. \qquad (3.30)
\end{aligned}$$

Then, we obtain

$$
\begin{aligned}
L'(t) &\geq (1-\sigma) H^{-\sigma}(t) H'(t) \\
&+ \varepsilon \left(1+\frac{p}{2}\right)\left(\|u_t\|_2^2 + \|v_t\|_2^2 + m_1^2 \|u\|_2^2 + m_2^2 \|v\|_2^2\right) \\
&+ \varepsilon a_1 \left[(g \circ \nabla u) + (h \circ \nabla v)\right] \\
&+ \varepsilon a_2 \left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right) + p\varepsilon H(t) \\
&+ \varepsilon c_3 \left(\|u+v\|_{2(\rho+2)}^{2(\rho+2)} + 2\|uv\|_{\rho+2}^{\rho+2}\right) \\
&- \varepsilon \frac{\delta_1^{m+1}}{m+1} \int_\Omega \left(|u(t)|^k + |v(t)|^l\right) |u|^{m+1} dx \\
&- \varepsilon \frac{m}{m+1} \delta_1^{-\frac{(m+1)}{m}} \int_\Omega \left(|u(t)|^k + |v(t)|^l\right) |u_t|^{m+1} dx \\
&- \varepsilon \frac{\delta_2^{r+1}}{r+1} \int_\Omega \left(|v(t)|^\theta + |u(t)|^\varrho\right) |v|^{r+1} dx \\
&- \varepsilon \frac{r}{r+1} \delta_2^{-\frac{(r+1)}{r}} \int_\Omega \left(|v(t)|^\theta + |u(t)|^\varrho\right) |v_t|^{r+1} dx. \qquad (3.31)
\end{aligned}
$$

Choosing δ_1 and δ_2 such that

$$
\delta_1^{-\frac{(m+1)}{m}} = M_1 H(t)^{-\sigma}, \delta_2^{-\frac{(r+1)}{r}} = M_2 H(t)^{-\sigma}, \qquad (3.32)
$$

for M_1 and M_2 large constants to be fixed later. Thus, by using (3.32), we obtain

$$
\begin{aligned}
L'(t) &\geq ((1-\sigma) - M\varepsilon) H^{-\sigma}(t) H'(t) \\
&+ \varepsilon \left(1+\frac{p}{2}\right)\left(\|u_t\|_2^2 + \|v_t\|_2^2 + m_1^2 \|u\|_2^2 + m_2^2 \|v\|_2^2\right) \\
&+ \varepsilon a_1 \left[(g \circ \nabla u) + (h \circ \nabla v)\right] \\
&+ \varepsilon a_2 \left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right) + p\varepsilon H(t) \\
&+ \varepsilon c_3 \left(\|u+v\|_{2(\rho+2)}^{2(\rho+2)} + 2\|uv\|_{\rho+2}^{\rho+2}\right) \\
&- \varepsilon M_1^{-m} H^{\sigma m}(t) \int_\Omega \left(|u(t)|^k + |v(t)|^l\right) |u|^{m+1} dx \\
&- \varepsilon \frac{m}{m+1} \delta_1^{-\frac{(m+1)}{m}} \int_\Omega \left(|u(t)|^k + |v(t)|^l\right) |u_t|^{m+1} dx \\
&- \varepsilon M_2^{-r} H^{\sigma r}(t) \int_\Omega \left(|v(t)|^\theta + |u(t)|^\varrho\right) |v|^{r+1} dx \\
&- \varepsilon \frac{r}{r+1} \delta_2^{-\frac{(r+1)}{r}} \int_\Omega \left(|v(t)|^\theta + |u(t)|^\varrho\right) |v_t|^{r+1} dx, \qquad (3.33)
\end{aligned}
$$

where $M = m/(m+1) M_1 + r/(r+1) M_2$. Therefore, we have

$$
\int_\Omega \left(|u(t)|^k + |v(t)|^l\right) |u|^{m+1} dx = \|u\|_{k+m+1}^{k+m+1} + \int_\Omega |v|^l |u|^{m+1} dx, \qquad (3.34)
$$

and

$$\int_{\Omega}\left(|v(t)|^{\theta}+|u(t)|^{\varrho}\right)|v|^{r+1}dx=\|v\|_{\theta+r+1}^{\theta+r+1}+\int_{\Omega}|u|^{\varrho}|v|^{r+1}dx. \tag{3.35}$$

Also by using Young's inequality, we obtain

$$\begin{aligned}\int_{\Omega}|v|^{l}|u|^{m+1} &\leq \frac{l}{l+m+1}\delta_1^{(l+m+1)/l}\|v\|_{l+m+1}^{l+m+1}\\ &+ \frac{m+1}{l+m+1}\delta_1^{-(l+m+1)/(m+1)}\|u\|_{l+m+1}^{l+m+1},\\ \int_{\Omega}|u|^{\varrho}|v|^{r+1} &\leq \frac{\varrho}{\varrho+r+1}\delta_2^{(\varrho+r+1)/\varrho}\|u\|_{\varrho+r+1}^{\varrho+r+1}\\ &+ +\frac{r+1}{\varrho+r+1}\delta_2^{-(\varrho+r+1)/(r+1)}\|v\|_{\varrho+r+1}^{\varrho+r+1}.\end{aligned}$$

Therefore,

$$\begin{aligned}&H^{\sigma m}(t)\int_{\Omega}\left(|u(t)|^{k}+|v(t)|^{l}\right)|u|^{m+1}dx\\ =\ &H^{\sigma m}(t)\|u\|_{k+m+1}^{k+m+1}+\frac{l}{l+m+1}\delta_1^{(l+m+1)/l}H^{\sigma m}(t)\|v\|_{l+m+1}^{l+m+1}\\ +\ &\frac{m+1}{l+m+1}\delta_1^{-(l+m+1)/(m+1)}H^{\sigma m}(t)\|u\|_{l+m+1}^{l+m+1},\end{aligned} \tag{3.36}$$

and

$$\begin{aligned}&H^{\sigma r}(t)\int_{\Omega}\left(|v(t)|^{\theta}+|u(t)|^{\varrho}\right)|v|^{r+1}dx\\ =\ &H^{\sigma r}(t)\|v\|_{\theta+r+1}^{\theta+r+1}+\frac{\varrho}{\varrho+r+1}\delta_2^{\frac{\varrho+r+1}{\varrho}}H^{\sigma r}(t)\|u\|_{\varrho+r+1}^{\varrho+r+1}\\ +\ &\frac{r+1}{\varrho+r+1}\delta_2^{-\frac{(\varrho+r+1)}{r+1}}H^{\sigma r}(t)\|v\|_{\varrho+r+1}^{\varrho+r+1}.\end{aligned} \tag{3.37}$$

Since (3.3) holds, we get by using (3.18)

$$\begin{cases} H^{\sigma m}(t)\|u\|_{k+m+1}^{k+m+1}\leq c_5\left(\|u\|_{2(\rho+2)}^{2\sigma m(\rho+2)+k+m+1}+\|v\|_{2(\rho+2)}^{2\sigma m(\rho+2)}\|u\|_{k+m+1}^{k+m+1}\right),\\ H^{\sigma r}(t)\|v\|_{\theta+r+1}^{\theta+r+1}\leq c_6\left(\|v\|_{2(\rho+2)}^{2\sigma r(\rho+2)+\theta+r+1}+\|u\|_{2(\rho+2)}^{2\sigma r(\rho+2)}\|v\|_{\theta+r+1}^{\theta+r+1}\right).\end{cases} \tag{3.38}$$

This implies

$$\begin{aligned}&\tfrac{l}{l+m+1}\delta_1^{\frac{l+m+1}{l}}H^{\sigma m}(t)\|v\|_{l+m+1}^{l+m+1}\\ &\leq c_7\tfrac{l}{l+m+1}\delta_1^{\frac{l+m+1}{l}}\left(\|v\|_{2(\rho+2)}^{2\sigma m(\rho+2)+l+m+1}+\|u\|_{2(\rho+2)}^{2\sigma m(\rho+2)}\|v\|_{l+m+1}^{l+m+1}\right),\end{aligned} \tag{3.39}$$

and

$$\begin{aligned}&\tfrac{\varrho}{\varrho+r+1}\delta_2^{\frac{\varrho+r+1}{\varrho}}H^{\sigma r}(t)\|u\|_{\varrho+r+1}^{\varrho+r+1}\\&\le c_8\tfrac{\varrho}{\varrho+r+1}\delta_2^{\frac{\varrho+r+1}{\varrho}}\left(\|u\|_{2(\rho+2)}^{2\sigma r(\rho+2)+\varrho+r+1}+\|v\|_{2(\rho+2)}^{2\sigma r(\rho+2)}\|u\|_{\varrho+r+1}^{\varrho+r+1}\right).\end{aligned} \tag{3.40}$$

Using (3.18) and the algebraic inequality

$$z^{\nu}\le(z+1)\le\left(1+\frac{1}{a}\right)(z+a),\ \forall\ z\ge 0,0<\nu\le 1,a>0, \tag{3.41}$$

we get, for all $t\ge 0$,

$$\begin{cases}\|u\|_{2(\rho+2)}^{2\sigma m(\rho+2)+k+m+1}\le d\left(\|u\|_{2(\rho+2)}^{2(\rho+2)}+H(0)\right)\le d\left(\|u\|_{2(\rho+2)}^{2(\rho+2)}+H(t)\right),\\ \|v\|_{2(\rho+2)}^{2\sigma r(\rho+2)+\theta+r+1}\le d\left(\|v\|_{2(\rho+2)}^{2(\rho+2)}+H(t)\right),\ \forall\ t\ge 0,\end{cases} \tag{3.42}$$

where $d=1+1/H(0)$. Similarly

$$\begin{cases}\|v\|_{2(\rho+2)}^{2\sigma m(\rho+2)+l(m+1)}\le d\left(\|v\|_{2(\rho+2)}^{2(\rho+2)}+H(0)\right)\le d\left(\|v\|_{2(\rho+2)}^{2(\rho+2)}+H(t)\right),\\ \|u\|_{2(\rho+2)}^{2\sigma r(\rho+2)+\varrho(r+1)}\le d\left(\|u\|_{2(\rho+2)}^{2(\rho+2)}+H(t)\right),\ \forall\ t\ge 0.\end{cases} \tag{3.43}$$

Also, since

$$(X+Y)^s\le C(X^s+Y^s),\qquad X,Y\ge 0,s>0, \tag{3.44}$$

by using (3.18) and (3.41) we have

$$\begin{aligned}\|v\|_{2(\rho+2)}^{2\sigma m(\rho+2)}\|u\|_{k+m+1}^{k+m+1}&\le c_9\left(\|v\|_{2(\rho+2)}^{2(\rho+2)}+\|u\|_{k+m+1}^{2(\rho+2)}\right)\\&\le c_{10}\left(\|v\|_{2(\rho+2)}^{2(\rho+2)}+\|u\|_{2(\rho+2)}^{2(\rho+2)}\right),\end{aligned} \tag{3.45}$$

similarly

$$\|u\|_{2(\rho+2)}^{2\sigma r(\rho+2)}\|v\|_{\theta+r+1}^{\theta+r+1}\le c_{11}\left(\|u\|_{2(\rho+2)}^{2(\rho+2)}+\|v\|_{2(\rho+2)}^{2(\rho+2)}\right), \tag{3.46}$$

$$\|u\|_{2(\rho+2)}^{2\sigma m(\rho+2)}\|v\|_{l+m+1}^{l+m+1}\le c_{12}\left(\|u\|_{2(\rho+2)}^{2(\rho+2)}+\|v\|_{2(\rho+2)}^{2(\rho+2)}\right) \tag{3.47}$$

and

$$\|v\|_{2(\rho+2)}^{2\sigma r(\rho+2)}\|u\|_{\varrho+r+1}^{\varrho+r+1}\le c_{13}\left(\|v\|_{2(\rho+2)}^{2(\rho+2)}+\|u\|_{2(\rho+2)}^{2(\rho+2)}\right). \tag{3.48}$$

Taking into account (3.36)-(3.48), then (3.33) written as

$$\begin{aligned}
L'(t) &\geq ((1-\sigma)-M\varepsilon) H^{-\sigma}(t) H'(t) \\
&+ 2\varepsilon\left(\|u_t\|_2^2+\|v_t\|_2^2+m_1^2\|u\|_2^2+m_2^2\|v\|_2^2\right) \\
&+ \varepsilon\left[2-CM_1^{-m}\left(1+\frac{l}{l+m+1}\delta_1^{\frac{l+m+1}{l}}+\frac{m+1}{l+m+1}\delta_1^{-\frac{(l+m+1)}{m+1}}\right)\right. \\
&- \left. CM_2^{-r}\left(1+\frac{\varrho}{\varrho+r+1}\delta_2^{\frac{\varrho+r+1}{\varrho}}+\frac{r+1}{\varrho+r+1}\delta_2^{-\frac{(\varrho+r+1)}{r+1}}\right)\right] H(t) \\
&+ \varepsilon\left[c_4-CM_1^{-m}\left(1+\frac{l}{l+m+1}\delta_1^{\frac{l+m+1}{l}}+\frac{m+1}{l+m+1}\delta_1^{-\frac{(l+m+1)}{m+1}}\right)\right. \\
&- \left. CM_2^{-r}\left(1+\frac{\varrho}{\varrho+r+1}\delta_2^{\frac{\varrho+r+1}{\varrho}}+\frac{r+1}{\varrho+r+1}\delta_2^{-\frac{(\varrho+r+1)}{r+1}}\right)\right] \\
&\times \left(\|u\|_{2(\rho+2)}^{2(\rho+2)}+\|v\|_{2(\rho+2)}^{2(\rho+2)}\right). \qquad (3.49)
\end{aligned}$$

At this point and for large values of M_1 and M_2, we can find positive constants Λ_1 and Λ_2 such that (3.49) becomes

$$\begin{aligned}
L'(t) &\geq ((1-\sigma)-M\varepsilon) H^{-\sigma}(t) H'(t) \\
&+ 2\varepsilon\left(\|u_t\|_2^2+\|v_t\|_2^2+m_1^2\|u\|_2^2+m_2^2\|v\|_2^2\right) \\
&+ \varepsilon\Lambda_1\left(\|u(t)\|_{2(\rho+2)}^{2(\rho+2)}+\|v(t)\|_{2(\rho+2)}^{2(\rho+2)}\right)+\varepsilon\Lambda_2 H(t). \qquad (3.50)
\end{aligned}$$

Once M_1 and M_2 are fixed (hence Λ_1 and Λ_2), we choose ε small enough so that $((1-\sigma)-M\varepsilon)\geq 0$ and

$$L(0)=H^{1-\sigma}(0)+\varepsilon\int_\Omega [u_0.u_1+v_0.v_1]\,dx>0. \qquad (3.51)$$

Therefore, there exists $\Gamma>0$ such that (3.50) can be written as

$$L'(t)\geq \varepsilon\Gamma\left(H(t)+\|u_t\|_2^2+\|v_t\|_2^2+\|u\|_{2(\rho+2)}^{2(\rho+2)}+\|v\|_{2(\rho+2)}^{2(\rho+2)}\right). \qquad (3.52)$$

Then, we have $L(t)\geq L(0)>0$, for all $t\geq 0$. Next, by using Holder's and Young's inequalities, we have the estimate

$$\begin{aligned}
&\left(\int_\Omega u.u_t(x,t)\,dx+\int_\Omega v.v_t(x,t)\,dx\right)^{\frac{1}{1-\sigma}} \\
&\leq C\left(\|u\|_{2(\rho+2)}^{\frac{\tau}{1-\sigma}}+\|u_t\|_2^{\frac{s}{1-\sigma}}+\|v\|_{2(\rho+2)}^{\frac{\tau}{1-\sigma}}+\|v_t\|_2^{\frac{s}{1-\sigma}}\right), \qquad (3.53)
\end{aligned}$$

for $1/\tau+1/s=1$. We takes $s=2(1-\sigma)$, to get $\dfrac{\tau}{1-\sigma}=\dfrac{2}{1-2\sigma}$. From (3.10) and (3.41), we have

$$\|u\|_{2(\rho+2)}^{\frac{2}{1-2\sigma}}\leq d\left(\|u\|_{2(\rho+2)}^{2(\rho+2)}+H(t)\right), \qquad (3.54)$$

and

$$\|v\|_{2(\rho+2)}^{\frac{2}{1-2\sigma}}\leq d\left(\|v\|_{2(\rho+2)}^{2(\rho+2)}+H(t)\right),\ \forall\, t\geq 0. \qquad (3.55)$$

Consequently, (3.53) can be written as

$$\begin{aligned} &\left(\int_\Omega uu_t\left(x,t\right)dx+\int_\Omega vv_t\left(x,t\right)dx\right)^{\frac{1}{1-\sigma}}\\ \leq\ & c_{14}\left(\|u\|_{2(\rho+2)}^{2(\rho+2)}+\|v\|_{2(\rho+2)}^{2(\rho+2)}+\|u_t\|_2^2+\|v_t\|_2^2\right)\\ +\ & c_{14}\left(m_1^2\|u\|_2^2+m_2^2\|v\|_2^2+H\left(t\right)\right),\ \forall\, t\geq 0. \end{aligned}$$

Also, we have

$$\begin{aligned} L^{\frac{1}{1-\sigma}}\left(t\right) &= \left(H^{1-\sigma}\left(t\right)+\varepsilon\int_\Omega\left(u.u_t+v.v_t\right)\left(x,t\right)dx\right)^{\frac{1}{(1-\sigma)}}\\ &\leq c_{15}\left(H\left(t\right)+\left|\int_\Omega\left(u.u_t\left(x,t\right)+v.v_t\left(x,t\right)\right)dx\right|^{\frac{1}{(1-\sigma)}}\right)\\ &\leq c_{16}\left[H\left(t\right)+\|u\|_{2(\rho+2)}^{2(\rho+2)}+\|v\|_{2(\rho+2)}^{2(\rho+2)}+\|u_t\|_2^2\right]\\ &+ c_{16}\left[\|v_t\|_2^2+m_1^2\|u\|_2^2+m_2^2\|v\|_2^2\right],\ \forall\, t\geq 0, \end{aligned} \tag{3.56}$$

from (3.56) and (3.52), we get

$$L'\left(t\right)\geq a_0L^{\frac{1}{1-\sigma}}\left(t\right),\ \forall\, t\geq 0. \tag{3.57}$$

Finally, a simple integration of (3.57) gives the desired result. □

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The combined Shepard operator of inverse quadratic and inverse multiquadric type

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Abstract. Starting with the classical, the modified and the iterative Shepard methods, we construct some new Shepard type operators, using the inverse quadratic and the inverse multiquadric radial basis functions. Given some sets of points, we compute some representative subsets of knot points following an algorithm described by J.R. McMahon in 1986.

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Keywords: Shepard operator, inverse quadratic, inverse multiquadric, knot points.

1. Preliminaries

Over the time Shepard method, introduced in 1968 in [21], has been improved in order to get better reproduction qualities, higher accuracy and lower computational cost (see, e.g.,[2]-[9], [22], [23]).

Let f be a real-valued function defined on $X \subset \mathbb{R}^2$, and $(x_i, y_i) \in X$, $i = 1, ..., N$ some distinct points. The bivariate Shepard operator is defined by

$$(S_\mu f)(x, y) = \sum_{i=1}^{N} A_{i,\mu}(x, y) f(x_i, y_i), \tag{1.1}$$

where

$$A_{i,\mu}(x, y) = \frac{\prod\limits_{\substack{j=1 \\ j \neq i}}^{N} r_j^\mu(x, y)}{\sum\limits_{k=1}^{N} \prod\limits_{\substack{j=1 \\ j \neq k}}^{N} r_j^\mu(x, y)}, \tag{1.2}$$

with the parameter $\mu > 0$ and $r_i(x, y)$ denoting the distances between a given point $(x, y) \in X$ and the points (x_i, y_i), $i = 1, ..., N$.

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In [11], Franke and Nielson introduced a method for improving the accuracy in reproducing a surface with the bivariate Shepard approximation. This method has been further improved in [10], [20], [19], and it is given by:

$$(Sf)(x,y) = \frac{\sum\limits_{i=1}^{N} W_i(x,y) f(x_i,y_i)}{\sum\limits_{i=1}^{N} W_i(x,y)}, \tag{1.3}$$

with

$$W_i(x,y) = \left[\frac{(R_w - r_i(x,y))_+}{R_w r_i(x,y)}\right]^2, \tag{1.4}$$

where R_w is a radius of influence about the node (x_i, y_i) and it is varying with i. R_w is taken as the distance from node i to the jth closest node to (x_i, y_i) for $j > N_w$ (N_w is a fixed value) and j as small as possible within the constraint that the jth closest node is significantly more distant than the $(j-1)$st closest node (see, e.g. [19]). As it is mentioned in [14], this modified Shepard method is one of the most powerful software tools for the multivariate approximation of large scattered data sets.

A.V. Masjukov and V.V. Masjukov introduced in [15] an iterative modification for the Shepard operator that requires no artificial parameter, such as a radius of influence or number of nodes. So, they defined the iterative Shepard operator as

$$u(x,y) = \sum_{k=0}^{K}\sum_{j=1}^{N}\left[u_j^{(k)} w\left((x-x_j, y-y_j)/\tau_k\right) / \sum_{p=1}^{N} w\left((x_p - x_j, y_p - y_j)/\tau_k\right)\right], \tag{1.5}$$

where w is the weight function, continuously differentiable, with the properties that

$$w(x,y) \geq 0,\ \forall (x,y) \in \mathbb{R}^2,\ w(0,0) > 0 \text{ and } w(x,y) = 0 \text{ if } \|(x,y)\| > 1,$$

and $u_j^{(k)}$ denotes the interpolation residuals at the kth step, with $u_j^{(0)} \equiv u_j$.

2. The Shepard operators combined with the inverse quadratic and inverse multiquadric radial basis functions

Let f be a real-valued function defined on $X \subset \mathbb{R}^2$. We denote by $\mathbf{x}$ the point $(x,y) \in X$ and we assume that $\mathbf{x_i} = (x_i, y_i) \in X$, $i = 1, ..., N'$, are some given interpolation nodes.

The radial basis functions (RBF) are some modern and very efficient tools for interpolating scattered data, thus they are intensively used (see, e.g., [1], [12] – [14], [18]). In the sequel we use two radial basis functions that are positive definite, the inverse quadratic RBF and the inverse multiquadric RBF.

Consider the two radial basis functions as

$$\phi_i^\beta(x,y) = \sum_{j=1}^{i} \alpha_j \left[1+(\epsilon r_j)^2\right]^\beta + ax + by + c, \quad i=1,...,N', \tag{2.1}$$

with ϵ being a shape parameter and $r_j(x,y) = \sqrt{(x-x_j)^2+(y-y_j)^2}$.

For $\beta = -1$, $\phi_{\mathbf{i}}^{\mathbf{-1}}$ is the *inverse quadratic RBF* and for $\beta = -1/2$, $\phi_{\mathbf{i}}^{\mathbf{-1/2}}$ is the *inverse multiquadric RBF*.

The coefficients $\alpha_j,\ a,\ b,\ c$ are obtained as solutions of systems of the form

$$\begin{pmatrix} 1 & [1+(\epsilon r_{12})^2]^\beta & \cdots & [1+(\epsilon r_{1N'})^2]^\beta & x_1 & y_1 & 1 \\ [1+(\epsilon r_{21})^2]^\beta & 1 & \cdots & [1+(\epsilon r_{2N'})^2]^\beta & x_2 & y_2 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ [1+(\epsilon r_{N'1})^2]^\beta & [1+(\epsilon r_{N'2})^2]^\beta & \cdots & 1 & x_{N'} & y_{N'} & 1 \\ x_1 & x_2 & \cdots & x_{N'} & 0 & 0 & 0 \\ y_1 & y_2 & \cdots & y_{N'} & 0 & 0 & 0 \\ 1 & 1 & \cdots & 1 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{N'} \\ a \\ b \\ c \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_{N'} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

with $r_{ij} = \sqrt{(x_i-x_j)^2+(y_i-y_j)^2}$ and $f_i = f(\mathbf{x}_i)$.

Shortly, this system can be written as

$$\begin{pmatrix} A & X^T \\ X & O_3 \end{pmatrix} \cdot \begin{pmatrix} \boldsymbol{\alpha} \\ \mathbf{u} \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ \mathbf{0} \end{pmatrix},$$

considering the following notations:

- $A \in \mathcal{M}_{N'\times N'}(\mathbb{R})$, with the element on the entry (i,j) being $a_{ij} = \left[1+(\epsilon r_{ij})^2\right]^\beta$, where $r_{ij} = \sqrt{(x_i-x_j)^2+(y_i-y_j)^2}$, $i,j = 1,...,N'$ and $\beta \in \{-1,\ -1/2\}$;
- $X \in \mathcal{M}_{3\times N'}(\mathbb{R})$, $X = \begin{pmatrix} x_1 & ... & x_{N'} \\ y_1 & ... & y_{N'} \\ 1 & ... & 1 \end{pmatrix}$, O_3 is the zero square matrix of order 3;
- $\mathbf{u} = (a,\ b,\ c)^T$, $\boldsymbol{\alpha} = (\alpha_1,...,\alpha_{N'})^T$, $\mathbf{0} = (0,\ 0,\ 0)^T$;
- $\mathbf{f} = (f_1,...,f_{N'})^T$, with $f_i = f(\mathbf{x}_i)$.

First, consider the classical Shepard operator given in (1.1).

Definition 2.1. The classical Shepard operator combined with the inverse quadratic and inverse multiquadric RBF is defined as

$$(S_\mu^\beta f)(\mathbf{x}) = \sum_{i=1}^{N'} A_{i,\mu}(\mathbf{x})\phi_i^\beta(\mathbf{x}), \tag{2.2}$$

where $A_{i,\mu}$, $i = 1,...,N'$, are defined by (1.2), for a given parameter $\mu > 0$ and ϕ_i^β are given in (2.1), for $\beta \in \{-1, -1/2\}$ and $i = 1,...,N'$.

Furthermore, we consider the improved form of the Shepard operator, given in (1.3).

Definition 2.2. We define the modified Shepard operator combined with the inverse quadratic and inverse multiquadric RBF as:

$$(S_W^{\beta} f)(\mathbf{x}) = \frac{\sum\limits_{i=1}^{N'} W_i\left(\mathbf{x}\right)\phi_i^{\beta}(\mathbf{x})}{\sum\limits_{i=1}^{N'} W_i\left(\mathbf{x}\right)}, \tag{2.3}$$

with W_i, $i = 1, ..., N'$, given by (1.4) and ϕ_i^{β} defined in (2.1), for $\beta \in \{-1, -1/2\}$ and $i = 1, ..., N'$.

Finally, we follow the idea proposed in [15], which consists of using an iterative procedure that requires no artificial parameters.

Definition 2.3. The iterative Shepard operator combined with the inverse quadratic and inverse multiquadric RBF is defined as

$$u_{\phi^{\beta}}(\mathbf{x}) = \sum_{k=0}^{K}\sum_{j=1}^{N'}\left[u_{\phi_j^{\beta}}^{(k)} w\left((\mathbf{x}-\mathbf{x_j})/\tau_k\right) / \sum_{p=1}^{N'} w\left((\mathbf{x_p}-\mathbf{x_j})/\tau_k\right)\right], \tag{2.4}$$

with $\beta \in \{-1, -1/2\}$, where $u_{\phi_j^{\beta}}^{(k)}$ are the interpolation residuals at the kth step given by

$$u_{\phi_j^{\beta}}^{(0)} = \phi_j(\mathbf{x_j}),\ \mathbf{x_j} \in X,\ j = 1, ..., N'$$

and

$$u_{\phi_j^{\beta}}^{(k+1)} = u_{\phi_j^{\beta}}^{(k)} - \sum_{q=1}^{N'}\left[u_{\phi_q^{\beta}}^{(k)} w\left((\mathbf{x_j}-\mathbf{x_q})/\tau_k\right) / \sum_{p=1}^{N'} w\left((\mathbf{x_p}-\mathbf{x_q})/\tau_k\right)\right].$$

The functions ϕ_i^{β} are given in (2.1). We follow ideas from [15] for the parameters' choice. As an example, the sequence $\{\tau_k\}$ of scale factors is defined as

$$\tau_k = \tau_0\gamma^k, \qquad 0 < \gamma < 1.$$

The setup parameter τ_k can be chosen such that it decreases from an initial value τ_0, which is given for instance as

$$\tau_0 > \sup_{(x,y)\in X}\ \max_{1\le j\le N'} \|(\mathbf{x}-\mathbf{x_j})\|$$

to the final value τ_K such that

$$\tau_K < \min_{i\neq j} \|(\mathbf{x_i}-\mathbf{x_j})\|.$$

The behaviour of u_{ϕ}^{β} does not change very much for γ between 0.6 and 0.95, as shown in [15]. One can also choose smaller values for γ if the nodes are sparse and a decreased computational time is desired.

Finally, the weight function w is given by

$$w(\mathbf{x}) = w(x)w(y),$$

with

$$w(x) = \begin{cases} 5(1-|x|)^4 - 4(1-|x|)^5, & |x| < 1 \\ 0, & |x| \geq 1 \end{cases}.$$

We apply the three operators on two sets of points. For the first way, we consider a set of N initial interpolation nodes $\mathbf{x_i},\ i = 1, ..., N$, and for the second way, we consider a smaller set of $k \in \mathbb{N}^*$ knot points $\hat{\mathbf{x}}_\mathbf{j},\ j = 1, ..., k$, that will be representative for the original set. This set is obtained following the next steps (see, e.g., [16] and [17]):

Algorithm 2.4.
1. Consider the first subset of k knot points, $k < N$, randomly generated;
2. Using the Euclidean distance between two points, find the closest knot point for every point;
3. For the knot points with no point assigned, replace the knot by the nearest point;
4. Compute the arithmetic mean of all the points that are closest to the same knot and compute in this way the new subset of knot points;
5. Repeat steps 2-4 until the subset of knot points has not change for two consecutive iterations.

3. Numerical examples

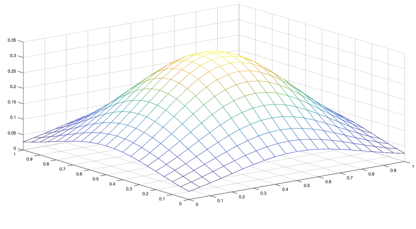
We consider the following test functions (see, e.g., [10], [20], [19]):

$$\begin{aligned} &\text{Gentle:} \quad f_1(x,y) = \exp[-\tfrac{81}{16}((x-0.5)^2 + (y-0.5)^2)]/3, \\ &\text{Saddle:} \quad f_2(x,y) = \frac{(1.25 + \cos 5.4y)}{6 + 6(3x-1)^2}, \\ &\text{Sphere:} \quad f_3(x,y) = \sqrt{64 - 81((x-0.5)^2 + (y-0.5)^2)}/9 - 0.5. \end{aligned} \tag{3.1}$$

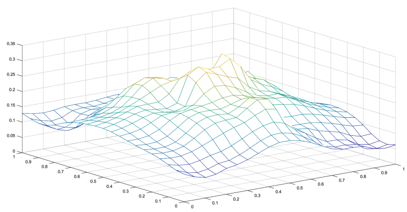
Tables 1 - 3 contain the maximum errors for approximating the functions (3.1) by the classical, the modified and the iterative Shepard operators given, respectively, by (1.1), (1.3) and (1.5), and the errors of approximating by the operators introduced in (2.2), (2.3) and (2.4). We construct the operators for both radial basis functions - the inverse quadratic and the inverse multiquadric. For each function we consider a set of $N = 100$ random points in $[0,1] \times [0,1]$, a subset of $k = 25$ representative knots, $\mu = 3$, $N_w = 19$, $K = 20$, $\tau_0 = 3$ and $\gamma = 0.66,\ 0.84,\ 0.91$.

In Figures 1 - 4 we plot the graphs of $f_1,\ f_2,\ f_3$ and of the corresponding Shepard operators $S_\mu^\beta f$, $S_W^\beta f$ and u_{ϕ^β}, combined with the inverse quadratic ($\beta = -1$) and the inverse multiquadric ($\beta = -1/2$) radial basis functions. We consider the sets of the $k = 25$ representative knot points.

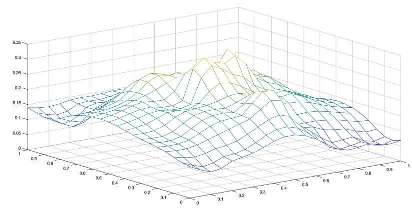
We remark that $S_W^\beta f$ and u_{ϕ^β} have better approximation properties than the classical Shepard operator $S_\mu^\beta f$, the results for u_{ϕ^β} depending on the values of γ. Also, we notice better approximation errors for the lower number of knots obtained using the Algorithm 2.4.



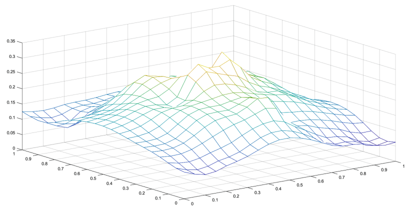
Function f_1.



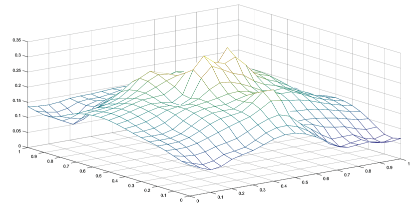
$S_\mu^{-1} f_1,\ \epsilon = 5.5.$



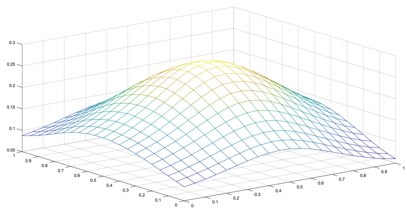
$S_\mu^{-1/2} f_1,\ \epsilon = 10.$



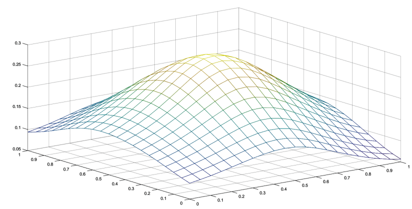
$S_W^{-1} f_1,\ \epsilon = 5.5.$



$S_W^{-1/2} f_1,\ \epsilon = 10.$

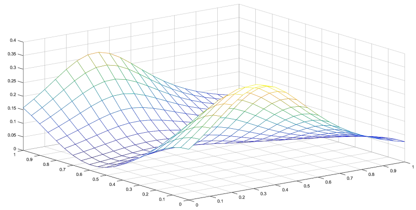


$u_{\phi^{-1}},\ \epsilon = 5.5,\ \gamma = 0.91.$

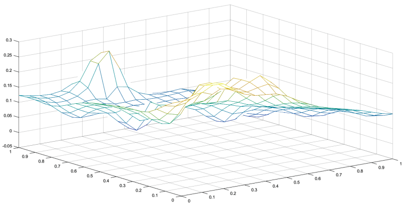


$u_{\phi^{-1/2}},\ \epsilon = 10,\ \gamma = 0.91.$

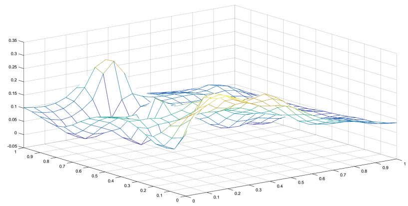
Figure 1. Graphs for f_1.



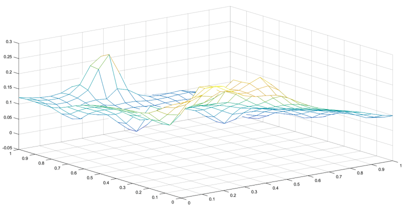
Function f_2.



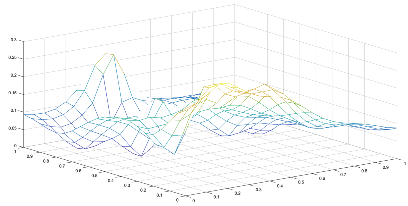
$S_\mu^{-1} f_2,\ \epsilon = 10.$



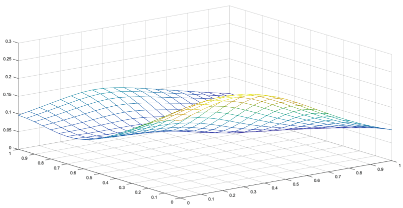
$S_\mu^{-1/2} f_2,\ \epsilon = 10.$



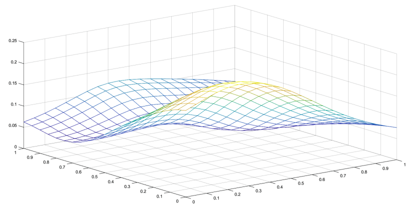
$S_W^{-1} f_2,\ \epsilon = 10.$



$S_W^{-1/2} f_2,\ \epsilon = 10.$

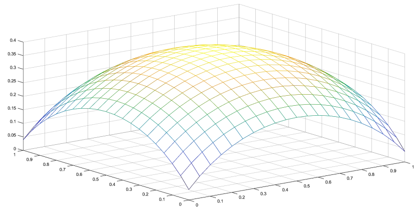


$u_{\phi^{-1}},\ \epsilon = 10,\ \gamma = 0.91.$

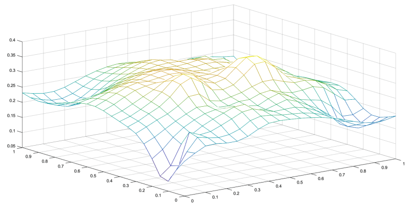


$u_{\phi^{-1/2}},\ \epsilon = 10,\ \gamma = 0.91.$

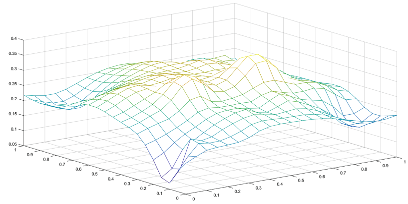
FIGURE 2. Graphs for f_2.



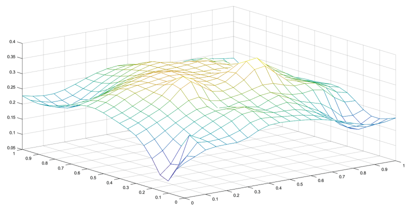
Function f_3.



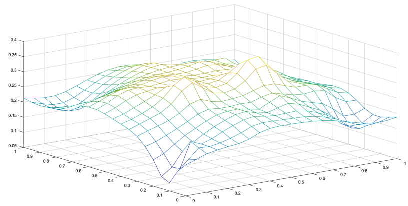
$S_\mu^{-1} f_3,\ \epsilon = 5.5.$



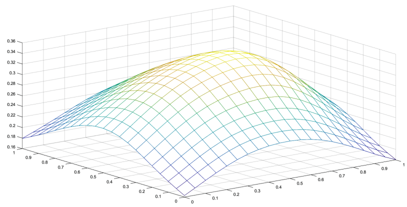
$S_\mu^{-1/2} f_3,\ \epsilon = 9.$



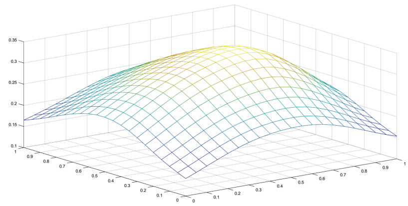
$S_W^{-1} f_3,\ \epsilon = 5.5.$



$S_W^{-1/2} f_3,\ \epsilon = 9.$



$u_{\phi^{-1}},\ \epsilon = 5.5,\ \gamma = 0.91.$



$u_{\phi^{-1/2}},\ \epsilon = 9,\ \gamma = 0.91.$

FIGURE 4. Graphs for f_3.

TABLE 1. Maximum approximation errors for the **Gentle** function.

	ϵ	Classical S_μ		Modified S_W		Iterative u_ϕ		
		k=25	N=100	k=25	N=100	γ (input)	k=25	N=100
f_1	–	0.0864	0.0855	0.0725	0.0644	0.66	0.0967	0.1158
						0.84	0.0757	0.1159
						0.91	0.0528	0.1105
ϕ^{-1}	5.5	0.1023	0.5564	0.0994	0.5543	0.66	0.1061	0.2866
						0.84	0.0847	0.2644
						0.91	0.0627	0.2396
	10	0.1313	0.1876	0.1293	0.1681	0.66	0.1026	0.1488
						0.84	0.0772	0.1251
						0.91	0.0579	0.1123
$\phi^{-1/2}$	9	0.1098	0.2402	0.1063	0.2219	0.66	0.1002	0.2155
						0.84	0.0866	0.1985
						0.91	0.0686	0.1887
	10	0.1129	0.2292	0.1096	0.2094	0.66	0.0994	0.1936
						0.84	0.0854	0.1750
						0.91	0.0673	0.1653

TABLE 2. Maximum approximation errors for the **Saddle** function.

	ϵ	Classical S_μ		Modified S_W		Iterative u_ϕ		
		k=25	N=100	k=25	N=100	γ (input)	k=25	N=100
f_2	–	0.1096	0.1152	0.0970	0.1033	0.66	0.2083	0.2051
						0.84	0.1902	0.1828
						0.91	0.1633	0.1567
ϕ^{-1}	7	0.1669	0.9372	0.1575	0.8615	0.66	0.2198	0.3754
						0.84	0.2103	0.4007
						0.91	0.1938	0.4456
	10	0.1813	0.1693	0.1828	0.1697	0.66	0.2175	0.1909
						0.84	0.2045	0.1797
						0.91	0.1825	0.1626
$\phi^{-1/2}$	9	0.1677	0.5409	0.1639	0.4933	0.66	0.2301	0.3125
						0.84	0.2222	0.3202
						0.91	0.2077	0.3344
	10	0.1582	0.2952	0.1630	0.2659	0.66	0.2292	0.2000
						0.84	0.2195	0.2020
						0.91	0.2029	0.2028

Table 3. Maximum approximation errors for the **Sphere** function.

	ϵ	Classical S_μ		Modified S_W		Iterative u_ϕ		
		k=25	N=100	k=25	N=100	γ (input)	k=25	N=100
f_3	–	0.2011	0.2156	0.1934	0.1744	0.66	0.1837	0.1850
						0.84	0.1730	0.1743
						0.91	0.1593	0.1645
ϕ^{-1}	5	0.1849	1.3107	0.1806	1.1997	0.66	0.1576	0.2703
						0.84	0.1488	0.4361
						0.91	0.1390	0.5255
	5.5	0.1926	0.9074	0.1898	0.8297	0.66	0.1637	0.1925
						0.84	0.1533	0.2901
						0.91	0.1456	0.3494
$\phi^{-1/2}$	7	0.1584	0.8948	0.1526	0.8150	0.66	0.1401	0.2258
						0.84	0.1291	0.3072
						0.91	0.1183	0.3464
	9	0.1796	0.3682	0.1779	0.3341	0.66	0.1537	0.1772
						0.84	0.1417	0.2091
						0.91	0.1344	0.2216

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A strong converse inequality for the iterated Boolean sums of the Bernstein operator

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Abstract. We establish a two-term strong converse estimate of the rate of approximation by the iterated Boolean sums of the Bernstein operator. The characterization is stated in terms of appropriate moduli of smoothness or K-functionals.

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1. Main results

The Bernstein operator is defined for $f \in C[0,1]$ and $x \in [0,1]$ by

$$B_n f(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{n,k}(x), \quad p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}.$$

Its iterated Boolean sum $\mathcal{B}_{r,n} : C[0,1] \to C[0,1]$ is then defined by

$$\mathcal{B}_{r,n} = I - (I - B_n)^r,$$

where I stands for the identity and $r \in \mathbb{N}$.

Gonska and Zhou [9] estimated the uniform norm of the approximation error for $\mathcal{B}_{r,n}$. They proved a neat direct inequality and a Stechkin-type converse inequality. The former states

$$\|\mathcal{B}_{r,n} f - f\| \le c\left(\omega_\varphi^{2r}(f, n^{-1/2}) + \frac{1}{n^r}\|f\|\right). \tag{1.1}$$

Above $\|\circ\|$ denotes the uniform norm on the interval $[0,1]$, c is a constant independent of the approximated function and the order of the operator (not necessarily the same

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at each occurrence), and $\omega_\varphi^r(f,t)$ denotes the Ditzian-Totik modulus of smoothness with $\varphi(x) = \sqrt{x(1-x)}$, which is given by (see [5, Chapter 1])

$$\omega_\varphi^r(f,t) = \sup_{0<h\le t} \|\Delta_{h\varphi}^r f\|,$$

where

$$\Delta_{h\varphi(x)}^r f(x) = \begin{cases} \displaystyle\sum_{k=0}^{r} (-1)^k \binom{r}{k} f\left(x + \left(\frac{r}{2} - k\right) h\varphi(x)\right), & x \pm rh\varphi(x)/2 \in [0,1], \\ 0, & \text{otherwise.} \end{cases}$$

Let us recall that the modulus $\omega_\varphi^r(f,t)$ is equivalent to the K-functional

$$K_{r,\varphi}(f,t^r) = \inf_{g\in AC_{loc}^{r-1}(0,1)} \left\{ \|f-g\| + t^r \|\varphi^r g^{(r)}\| \right\}.$$

More precisely, we say that $\Phi(f,t)$ and $\Psi(f,t)$ are equivalent and write

$$\Phi(f,t) \sim \Psi(f,t)$$

if there exists a constant c such that $c^{-1}\Phi(f,t) \le \Psi(f,t) \le c\,\Phi(f,t)$ for all f and t under consideration. Thus there holds (see [5, Theorem 2.1.1])

$$K_{r,\varphi}(f,t^r) \sim \omega_\varphi^r(f,t), \quad 0 < t \le t_0 \tag{1.2}$$

with some fixed $t_0 > 0$. It was shown in [11, Theorem 2.7] that we can take $t_0 = 2/r$. A smaller value of t_0 was given in [2, Chapter 6, Theorem 6.2].

Since the operator $\mathcal{B}_{r,n}$ preserves the algebraic polynomials of degree 1 and the modulus $\omega_\varphi^{2r}(f,n^{-1/2})$ is invariant to translation of f by such polynomials, we immediately deduce from (1.1) the estimate

$$\|\mathcal{B}_{r,n}f - f\| \le c\left(\omega_\varphi^{2r}(f,n^{-1/2}) + \frac{1}{n^r}E_1(f)\right), \tag{1.3}$$

where $E_1(f)$ is the best approximation of f by algebraic polynomials of degree 1 in the uniform norm on $[0,1]$.

Later on Ding and Cao [3] characterized the error of the multivariate generalization of $\mathcal{B}_{r,n}$ on the simplex. In the univariate case, the direct inequality they proved is of the form

$$\|\mathcal{B}_{r,n}f - f\| \le c\,K_r(f,n^{-r}), \tag{1.4}$$

where

$$K_r(f,t) = \inf_{g\in C^{2r}[0,1]} \{\|f-g\| + t\|D^r g\|\}, \quad Dg = \varphi^2 g''.$$

They also proved a strong converse inequality of type D (in the terminology introduced in [4]), that is

$$K_r(f,n^{-r}) \le c \max_{k\ge n} \|\mathcal{B}_{r,k}f - f\|. \tag{1.5}$$

As it was shown in [6, Theorem 5.1],

$$K_r(f,t) \sim K_{2r,\varphi}(f,t) + tE_1(f), \quad 0 < t \le 1. \tag{1.6}$$

Therefore, taking also into account (1.2), we see that the function characteristics on the right side of (1.3) and (1.4) are equivalent.

Quite recently, Cheng and Zhou [1] derived another converse inequality from the Stechkin-type converse inequality in [9]. It is similar to (1.5), though weaker than it.

Our main result improves (1.5). We will prove the following strong converse inequality of type B according to [4].

Theorem 1.1. *Let $r \in \mathbb{N}$. There exists $R \in \mathbb{N}$ such that for all $f \in C[0,1]$ and $k, n \in \mathbb{N}$ with $k \geq Rn$ there holds*

$$K_r(f, n^{-r}) \leq c \left(\frac{k}{n}\right)^r (\|\mathcal{B}_{r,n}f - f\| + \|\mathcal{B}_{r,k}f - f\|) .$$

In particular,

$$K_r(f, n^{-r}) \leq c \,(\|\mathcal{B}_{r,n}f - f\| + \|\mathcal{B}_{r,Rn}f - f\|) .$$

Let us recall that the assertion of the theorem for $r = 1$ was established in [4, Theorem 8.1] and then improved to a one-term converse inequality (i.e. $R = 1$) in [10, 12].

As we mentioned earlier in (1.6), the more complicated K-functional $K_r(f,t)$ can be replaced with the simpler function characteristics $K_{2r,\varphi}(f,t) + tE_1(f)$. In addition to this, we will establish also the following equivalence relation.

Theorem 1.2. *Let $r \in \mathbb{N}$. For all $f \in C[0,1]$ and $0 < t \leq 1$ we have*

$$K_r(f,t) \sim K_{2r,\varphi}(f,t) + K_{2,\varphi}(f,t).$$

Taking into account (1.2), we arrive at the following relation between $K_r(f,t)$ and the Ditzian-Totik modulus.

Corollary 1.3. *Let $r \in \mathbb{N}$. For all $f \in C[0,1]$ and $n \in \mathbb{N}$ such that $n \geq r^2$ we have*

$$K_r(f, n^{-r}) \sim \omega_\varphi^{2r}(f, n^{-1/2}) + \omega_\varphi^2(f, n^{-r/2}).$$

We establish Theorem 1.1 by means of the method given in [4]. To this end, we need a Voronovskaya-type inequality and several Bernstein-type inequalities, which relate the approximation operator $\mathcal{B}_{r,n}$ to the differential operator D^r. They are given in Section 2. Then, in the next section we prove Theorem 1.1. We present the short argument that verifies Theorem 1.2 in the last section.

2. Voronovskaya- and Bernstein-type inequalities for $\mathcal{B}_{r,n}$

We will use the following inequalities, which were obtained by Gonska and Zhou [9, (2) and (4)] for algebraic polynomials, but, as it is easy to see, the same considerations verify them for all functions in $C^{2r}[0,1]$.

Proposition 2.1. *For $g \in C^{2r}[0,1]$ there hold:*

(a) $\|\varphi^{2r} g^{(2r)}\| \leq c\,\|D^r g\|$;
(b) $\|D^j g\| \leq c\,\|D^r g\|, \quad j = 1, \dots, r.$

We proceed to two Voronovskaya-type estimates (cf. [9, Lemma 4]).

Proposition 2.2. *Let $r \in \mathbb{N}$. For all $g \in C^{2r+2}[0,1]$ and all $n \in \mathbb{N}$ there hold*

$$\left\|\mathcal{B}_{r,n}g - g - \frac{(-1)^{r-1}}{(2n)^r}\, D^r g\right\| \le \frac{c}{n^{r+1}} \left(\|\varphi^2 g^{(3)}\| + \|\varphi^{2r+2} g^{(2r+2)}\|\right) \tag{2.1}$$

and

$$\left\|\mathcal{B}_{r,n}g - g - \frac{(-1)^{r-1}}{(2n)^r}\, D^r g\right\| \le \frac{c}{n^{r+1}}\, \|D^{r+1} g\|. \tag{2.2}$$

Proof. Assertion (2.1) for $r = 1$ follows from [8, Proposition 2.3].
Next, we set $J_{r,n}g = (I - B_n)^r g$ and

$$V_{r,n}g = \mathcal{B}_{r,n}g - g - \frac{(-1)^{r-1}}{(2n)^r}\, D^r g.$$

For $r \ge 2$ we use the relation

$$V_{r,n}g = V_{1,n}J_{r-1,n}g - \frac{1}{2n} D V_{r-1,n}g.$$

It implies

$$\|V_{r,n}g\| \le \|V_{1,n}J_{r-1,n}g\| + \frac{1}{n}\, \|\varphi^2 (V_{r-1,n}g)''\|. \tag{2.3}$$

By virtue of (2.1) with $r = 1$,

$$\|V_{1,n}J_{r-1,n}g\| \le \frac{c}{n^2}\left(\|\varphi^2 (J_{r-1,n}g)^{(3)}\| + \|\varphi^4 (J_{r-1,n}g)^{(4)}\|\right). \tag{2.4}$$

Further, we estimate the first term on the right above by means of [7, Corollary 4.7] with $p = \infty$, $r - 1$ in place of r, $s = 3$ and $w = \varphi^2$ (i.e. $\gamma_0 = \gamma_1 = 1$). Thus we get

$$\|\varphi^2 (J_{r-1,n}g)^{(3)}\| \le \frac{c}{n^{r-1}} \left(\|\varphi^2 g^{(3)}\| + \|\varphi^{2r} g^{(2r+1)}\|\right). \tag{2.5}$$

Similarly, again by [7, Corollary 4.7] with $p = \infty$ and $r - 1$ in place of r, but $s = 4$ and $w = \varphi^4$ (i.e. $\gamma_0 = \gamma_1 = 2$) we have for the other term

$$\|\varphi^4 (J_{r-1,n}g)^{(4)}\| \le \frac{c}{n^{r-1}} \left(\|\varphi^4 g^{(4)}\| + \|\varphi^{2r+2} g^{(2r+2)}\|\right). \tag{2.6}$$

Next, by virtue of [7, Proposition 2.1] with $p = \infty$, $j = 1$, $m = 2r - 1$, $w_1 = \varphi^4$ (i.e. $\gamma_{1,0} = \gamma_{1,1} = 2$), $w_2 = \varphi^{2r+2}$ (i.e. $\gamma_{2,0} = \gamma_{2,1} = r + 1$) and $g^{(3)}$ in place of g, we get

$$\|\varphi^4 g^{(4)}\| \le c \left(\|\varphi^2 g^{(3)}\| + \|\varphi^{2r+2} g^{(2r+2)}\|\right). \tag{2.7}$$

Likewise, by means of the same proposition with $p = \infty$, $m = 2r - 1$, $w_2 = \varphi^{2r+2}$ and $g^{(3)}$ in place of g, but with $j = 2r - 2$ and $w_1 = \varphi^{2r}$ (i.e. $\gamma_{1,0} = \gamma_{1,1} = r$), we get

$$\|\varphi^{2r} g^{(2r+1)}\| \le c \left(\|\varphi^2 g^{(3)}\| + \|\varphi^{2r+2} g^{(2r+2)}\|\right). \tag{2.8}$$

Combining, (2.4)-(2.8), we get

$$\|V_{1,n}J_{r-1,n}g\| \le \frac{c}{n^{r+1}}\left(\|\varphi^2 g^{(3)}\| + \|\varphi^{2r+2} g^{(2r+2)}\|\right). \tag{2.9}$$

It remains to estimate the second term on the right side of (2.3). To this end, we apply [7, Corollary 4.11] with $p = \infty$, $r-1$ in place of r, $s = 2$, and $w = \varphi^2$ (i.e. $\gamma_0 = \gamma_1 = 1$) and get

$$\|\varphi^2 (V_{r-1,n} g)''\| \le \frac{c}{n^r} \left(\|\varphi^2 g^{(3)}\| + \|\varphi^{2r+2} g^{(2r+2)}\| \right). \tag{2.10}$$

Now, (2.3), (2.9) and (2.10) imply (2.1) for $r \ge 2$.
To prove the second assertion of the proposition, we observe that Proposition 2.1(a) with $r+1$ in place of r yields

$$\|\varphi^{2r+2} g^{(2r+2)}\| \le c\, \|D^{r+1} g\|. \tag{2.11}$$

Also, by virtue of [7, Proposition 2.1] with $p = \infty$, $j = 1$, $m = 2r$, $w_1 = \varphi^2$ (i.e. $\gamma_{1,0} = \gamma_{1,1} = 1$), $w_2 = \varphi^{2r+2}$ (i.e. $\gamma_{2,0} = \gamma_{2,1} = r+1$) and $g^{(2)}$ in place of g, we get

$$\|\varphi^2 g^{(3)}\| \le c \left(\|\varphi^2 g^{(2)}\| + \|\varphi^{2r+2} g^{(2r+2)}\| \right).$$

Taking into account (2.11) and Proposition 2.1(b) with $j = 1$ and $r+1$ in place of r, we arrive at

$$\|\varphi^2 g^{(3)}\| \le c\, \|D^{r+1} g\|. \tag{2.12}$$

Now, (2.2) follows from (2.1), (2.11) and (2.12). □

Next we shall establish several Bernstein-type inequalities.

Proposition 2.3. *Let $r \in \mathbb{N}$. Then for all $f \in C[0,1]$ and $n \in \mathbb{N}$ there holds*

$$\|D^r \mathcal{B}_{r,n} f\| \le c\, n^r \|f\|.$$

Proof. It is established by induction on r that (cf. [9, p. 24])

$$D^r g = \varphi^2 \sum_{i=2}^{r+1} q_{r,i-2}\, g^{(i)} + \sum_{i=2}^{r} \varphi^{2i}\, \tilde{q}_{r,r-i}\, g^{(i+r)},$$

where $q_{r,j}$ and $\tilde{q}_{r,j}$ are algebraic polynomials of degree at most j.
Therefore

$$\|D^r g\| \le c \left(\sum_{i=2}^{r+1} \|\varphi^2 g^{(i)}\| + \sum_{i=2}^{r} \|\varphi^{2i} g^{(i+r)}\| \right). \tag{2.13}$$

Let $r \ge 2$. We apply [7, Proposition 2.1] with $p = \infty$, $j = i-2$, where $i \in \{2, \dots, r+1\}$, $m = 2r-2$, $w_1 = \varphi^2$ (i.e. $\gamma_{1,0} = \gamma_{1,1} = 1$), $w_2 = \varphi^{2r}$ (i.e. $\gamma_{2,0} = \gamma_{2,1} = r$) and $g^{(2)}$ in place of g to get

$$\|\varphi^2 g^{(i)}\| \le c \left(\|\varphi^2 g^{(2)}\| + \|\varphi^{2r} g^{(2r)}\| \right), \quad i = 2, \dots, r+1. \tag{2.14}$$

Also, this trivially holds for $r = 1$.

Let $r \ge 3$. Similarly, [7, Proposition 2.1] with $p = \infty$, $j = i + r - 2$, where $i \in \{2, \dots, r-1\}$, $m = 2r-2$, $w_1 = \varphi^{2i}$ (i.e. $\gamma_{1,0} = \gamma_{1,1} = i$), where $i \in \{2, \dots, r\}$, $w_2 = \varphi^{2r}$ (i.e. $\gamma_{2,0} = \gamma_{2,1} = r$) and $g^{(2)}$ in place of g to get

$$\|\varphi^{2i} g^{(i+r)}\| \le c \left(\|\varphi^2 g^{(2)}\| + \|\varphi^{2r} g^{(2r)}\| \right), \quad i = 2, \dots, r-1. \tag{2.15}$$

The above estimate trivially holds for $i = r$, $r \ge 2$, as well.

The inequalities (2.13)-(2.15) yield

$$\|D^r g\| \le c\left(\|\varphi^2 g^{(2)}\| + \|\varphi^{2r} g^{(2r)}\|\right), \quad r \in \mathbb{N}. \tag{2.16}$$

Setting $g = \mathcal{B}_{r,n} f$ we get

$$\|D^r \mathcal{B}_{r,n} f\| \le c\left(\|\varphi^2 (\mathcal{B}_{r,n} f)^{(2)}\| + \|\varphi^{2r} (\mathcal{B}_{r,n} f)^{(2r)}\|\right). \tag{2.17}$$

Then we take into account that the operator $\mathcal{B}_{r,n}$ is a linear combination of iterates of B_n and also that (see [5, (9.3.7)])

$$\|\varphi^{2\ell} (B_n g)^{(2\ell)}\| \le c\,\|\varphi^{2\ell} g^{(2\ell)}\|, \quad g \in C^{2\ell}[0,1], \tag{2.18}$$

to derive from (2.17) the estimate

$$\|D^r \mathcal{B}_{r,n} f\| \le c\left(\|\varphi^2 (B_n f)^{(2)}\| + \|\varphi^{2r} (B_n f)^{(2r)}\|\right).$$

Now, the assertion of the proposition follows from

$$\|\varphi^{2\ell} (B_n f)^{(2\ell)}\| \le c\, n^{\ell} \|f\|, \quad \ell \in \mathbb{N},$$

which was established in [5, Theorem 9.4.1]. □

Proposition 2.4. *Let $r \in \mathbb{N}$. Then for all $g \in C^{2r}[0,1]$ and $n \in \mathbb{N}$ there holds*

$$\|D^{r+1} \mathcal{B}_{r,n} g\| \le c\, n \|D^r g\|.$$

Proof. We make use of (2.16) with $r+1$ in place of r and $\mathcal{B}_{r,n} g$ in place of g, then apply (2.18), [7, Proposition 4.13(a)] with $p = \infty$, $w = \varphi^{2r}$ (i.e. $\gamma_0 = \gamma_1 = r$), $\ell = 1$, $s = 2r$, and, finally, Proposition 2.1 with $j = 1$, to arrive at

$$\begin{aligned}
\|D^{r+1} \mathcal{B}_{r,n} g\| &\le c\left(\|\varphi^2 (\mathcal{B}_{r,n} g)^{(2)}\| + \|\varphi^{2r+2} (\mathcal{B}_{r,n} g)^{(2r+2)}\|\right) \\
&\le c\left(\|\varphi^2 g^{(2)}\| + \|\varphi^{2r+2} (B_n g)^{(2r+2)}\|\right) \\
&\le c\left(\|\varphi^2 g^{(2)}\| + n\,\|\varphi^{2r} g^{(2r)}\|\right) \\
&\le c\, n \|D^r g\|.
\end{aligned}$$

Thus the proposition is verified. □

3. A proof of the converse inequalities

Equipped with the estimates established in the previous section, we are now ready to verify Theorem 1.1.

Proof of Theorem 1.1. We apply [4, Theorem 3.2] with the operator $Q_n = \mathcal{B}_{r,n}$ and the spaces $X = C[0,1]$ (with the uniform norm on $[0,1]$), $Y = C^{2r}[0,1]$ and $Z = C^{2r+2}[0,1]$.
As is known,

$$\|B_n f\| \le \|f\|.$$

Therefore, since $\mathcal{B}_{r,n}$ is linear combination of iterates of B_n, we have

$$\|\mathcal{B}_{r,n} f\| \le c\,\|f\|, \quad f \in C[0,1],\ n \in \mathbb{N}.$$

Thus [4, (3.3)] is satisfied.

By virtue of the Voronovskaya-type inequality (2.2), we have [4, (3.4)] with $(-1)^{r-1}D^r$ in place of D, $\Phi(f) = \|D^{r+1}f\|$, $\lambda(n) = (2n)^{-r}$ and $\lambda_1(\alpha) = c\,n^{-r-1}$, where the constant c is the one in (2.2).

Next, Proposition 2.4 with $g = \mathcal{B}_{r,n}f$ implies [4, (3.5)] with $\ell = 1$ and $m = 2$.

Finally, Proposition 2.3 yields [4, (3.6)]. □

Let us note that [4, Theorems 10.4 and 10.5] are not applicable because condition (c) there is not satisfied.

4. Relations between K-functionals

Proof of Theorem 1.2. In view of (1.6), it is sufficient to show that

$$K_{2r,\varphi}(f,t) + tE_1(f) \sim K_{2r,\varphi}(f,t) + K_{2,\varphi}(f,t), \quad 0 < t \le 1. \tag{4.1}$$

Trivially, for any $g \in C[0,1]$ such that $g \in AC^1_{loc}(0,1)$ and $\varphi^2 g'' \in L_\infty[0,1]$, and any $t \in (0,1]$ we have the estimates

$$tE_1(f) \le \|f-g\| + t\,\|g - B_1 g\| \le \|f-g\| + ct\|\varphi^2 g''\|;$$

hence

$$tE_1(f) \le c\,K_{2,\varphi}(f,t), \quad 0 < t \le 1.$$

Above we used the inequality

$$\|g - B_1 g\| \le \|\varphi^2 g''\|,$$

which is directly established by Taylor's formula (see e.g. [4, p. 87]).
To complete the proof of (4.1), it remains to show that

$$K_{2,\varphi}(f,t) \le c\left(K_{2r,\varphi}(f,t) + tE_1(f)\right), \quad 0 < t \le 1. \tag{4.2}$$

Let $g \in C[0,1]$ be such that $g \in AC^{2r-1}_{loc}(0,1)$ and $\varphi^{2r}g^{(2r)} \in L_\infty[0,1]$. Then, by e.g. [7, (2.9)] with $p = \infty$, $w = 1$, $j = 1$ and $m = r$, we deduce that $\varphi^2 g^{(2)} \in L_\infty[0,1]$ too, as, moreover,

$$\|\varphi^2 g^{(2)}\| \le c\left(\|g\| + \|\varphi^{2r}g^{(2r)}\|\right).$$

Consequently, we have for $t \in (0,1]$

$$\begin{aligned} K_{2,\varphi}(f,t) &\le \|f-g\| + t\|\varphi^2 g^{(2)}\| \\ &\le c\left(\|f-g\| + t\|\varphi^{2r}g^{(2r)}\|\right) + ct\|f\|. \end{aligned}$$

Taking the infimum on g, we arrive at

$$K_{2,\varphi}(f,t) \le c\left(K_{2r,\varphi}(f,t) + t\|f\|\right).$$

Finally, we replace f with $f - p_1$, where p_1 is the algebraic polynomial of degree 1 of best approximation in $C[0,1]$ to f, to get (4.2). □

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A modified Post Widder operators preserving e^{Ax}

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Abstract. In the present paper, we discuss the approximation properties of modified Post-Widder operators, which preserve the test function e^{Ax}. We establish weighted approximation and a direct quantitative estimate for the modified operators.

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Keywords: Post-Widder operators, weighted approximation, quantitative estimate.

1. Post-Widder operators

In the recent years some sequences of linear positive operators and the operators of integral type have been studied in [2], [3] and [4] etc. Also the moments of several operators have been provided in [8]. In the present article, we discuss the vatiant of an integral operators viz. Post-Widder operators. Post-Widder operators are defined for $f \in C[0, \infty)$ as (see [13]):

$$P_n(f, x) \quad := \quad \frac{1}{n!}\left(\frac{n}{x}\right)^{n+1} \int_0^\infty t^n \, e^{-\frac{nt}{x}} \, f(t) \; dt.$$

Following [7], we have

$$P_n(e^{\theta t}, x) \quad = \quad \left(1 - \frac{x\theta}{n}\right)^{-(n+1)}. \tag{1.1}$$

Very recently Gupta-Agrawal in [6] and Gupta-Tachev in [11] considered different forms of modified Post-Widder operators preserving the test functions $e_r, r \in N$. Gupta-Singh in [9] estimated some quantitative convergence results of Post-Widder operators preserving e^{ax}, e^{bx}.

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Let us consider that the Post-Widder operators preserve the test function e^{Ax}, then we start with the following form

$$\widetilde{P}_n(f,x) := \frac{1}{n!}\left(\frac{n}{a_n(x)}\right)^{n+1}\int_0^\infty t^n\, e^{-\frac{nt}{a_n(x)}}\, f(t)\, dt.$$

Then using (1.1), we have

$$\widetilde{P}_n(e^{At},x) = e^{Ax} = \left(1-\frac{a_n(x)A}{n}\right)^{-(n+1)},$$

implying

$$a_n(x) = \frac{n}{A}(1-e^{-Ax/(n+1)}).$$

Thus our modified operators $\widetilde{P}_n$ take the following form

$$\widetilde{P}_n(f,x) := \frac{1}{n!}\left[\frac{A}{(1-e^{-Ax/(n+1)})}\right]^{(n+1)} \int_0^\infty t^n\, e^{-\frac{At}{(1-e^{-Ax/(n+1)})}}\, f(t)\, dt, \tag{1.2}$$

with $x\in(0,\infty)$ and $\widetilde{P}_n(f,0)=f(0)$, which preserve constant and the test function e^{Ax}.

2. Lemmas

Lemma 2.1. *We have for $\theta>0$ that*

$$\widetilde{P}_n(e^{\theta t},x) = \left(1-\frac{(1-e^{-Ax/(n+1)})\theta}{A}\right)^{-(n+1)}.$$

It may be observed that $\widetilde{P}_n(e^{\theta t},x)$ may be treated as m.g.f. of the operators $\widetilde{P}_n$, which may be utilized to obtain the moments of (1.2). Let $\mu_r^{\widetilde{P}_n}(x)=\widetilde{P}_n(e_r,x)$, where $e_r(t)=t^r$, $r\in N\cup\{0\}$. The moments are given by

$$\begin{aligned}\mu_r^{\widetilde{P}_n}(x) &= \left[\frac{\partial^r}{\partial\theta^r}\widetilde{P}_n(e^{\theta t},x)\right]_{\theta=0}\\ &= \left[\frac{\partial^r}{\partial\theta^r}\left\{\left(1-\frac{(1-e^{-Ax/(n+1)})\theta}{A}\right)^{-(n+1)}\right\}\right]_{\theta=0}.\end{aligned}$$

Few moments are given below:

$$\begin{aligned}\mu_0^{\widetilde{P}_n}(x) &= 1,\\ \mu_1^{\widetilde{P}_n}(x) &= \frac{(n+1)}{A}(1-e^{-Ax/(n+1)}),\\ \mu_2^{\widetilde{P}_n}(x) &= \frac{(n+1)(n+2)}{A^2}(1-e^{-Ax/(n+1)})^2.\end{aligned}$$

Lemma 2.2. *The moments of arbitrary order, satisfy the following*

$$\mu_k^{\widetilde{P}_n}(x) \quad = \quad \frac{(n+1)_k}{A^k}(1-e^{-Ax/(n+1)})^k, k=0,1,....,$$

where the Pochhammer symbol is defined by

$$(c)_0 = 1, \quad (c)_k = c(c+1)\cdots(c+k-1).$$

Further by linearity property and using Lemma 2.2, we have the following lemma:

Lemma 2.3. *The central moments* $U_r^{\widetilde{P}_n}(x) = \widetilde{P}_n((t-x)^r, x)$ *are given below:*

$$U_k^{\widetilde{P}_n}(x) = \sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}x^{k-j}(1-e^{-Ax/(n+1)})^j\frac{(n+1)_j}{A^j}, \quad k=0,1,\ldots .$$

Also, for each $n \in N$, *we have*

$$U_1^{\widetilde{P}_n}(x) \quad = \quad \frac{(n+1)}{A}(1-e^{-Ax/(n+1)}-1)-x,$$
$$U_2^{\widetilde{P}_n}(x) \quad = \quad \frac{(n+1)(n+2)}{A^2}(1-e^{-Ax/(n+1)})^2+x^2-2x\frac{(n+1)}{A}(1-e^{-Ax/(n+1)}).$$

Lemma 2.4. *For the central moments* $U_{2k}^{\widetilde{P}_n}(x) = \widetilde{P}_n((t-x)^{2k}, x)$, *we have*

$$U_{2k}^{\widetilde{P}_n}(x) = O(n^{-k}), n \to \infty, k = 1,2,3,\cdots$$

Proof. We observe that

$$\widetilde{P}_n(f,x) = P_n(f, \alpha_n(x)),$$

where

$$a_n(x) \quad = \quad \frac{n}{A}(1-e^{-Ax/(n+1)}).$$

It is easy to verify $y > 1-e^{-y} > y - \frac{y^2}{2}$ for $y \in [0,\infty)$. We set $y = Ax/(n+1)$ and get

$$x\left(\frac{n}{n+1}\right) > \alpha_n(x) > x\left(\frac{n}{n+1}\right) - \left(\frac{Ax}{n+1}\right)^2 \cdot \frac{n}{2A}.$$

Hence

$$\frac{x}{n+1} < x - \alpha_n(x) < \frac{x}{n+1} + \frac{Ax^2 n}{2(n+1)^2} = O(n^{-1}),$$

by fixed $x \in [0,\infty)$. Therefore

$$\begin{aligned}
\widetilde{P}_n((t-x)^{2k}, x) &= P_n((t-x)^{2k}, \alpha_n(x)) \\
&= P_n((t-\alpha_n(x)+\alpha_n(x)-x)^{2k}, \alpha_n(x)) \\
&\leq C(k)P_n((t-\alpha_n(x))^{2k}, \alpha_n(x)) + P_n((x-\alpha_n(x)^{2k}, \alpha_n(x)) \\
&\leq C(k).\frac{1}{n^k} + (x-\alpha_n(x))^{2k} = O(n^{-k}).
\end{aligned}$$

This completes the proof of Lemma 2.4. □

3. Weighted approximation

We also analyse the behaviour of the operators on some weighted spaces. Set $\phi(x) = 1 + e^{Ax},\ x \in R^+$ and consider the following weighted spaces:

$$\begin{aligned} B_\phi(R^+) &= \{f : R^+ \to R : |f(x) \le C_1(1 + e^{Ax})\}, \\ C_\phi(R^+) &= B_\phi(R^+) \cap C(R^+), \\ C_\phi^k(R^+) &= \left\{ f \in C_\phi(R^+) : \lim_{x\to\infty} \frac{f(x)}{1+e^{Ax}} = C_2 < \infty \right\}, \end{aligned}$$

where C_1, C_2 are constants depending on f. The norm is defined as

$$||f||_\phi = \sup_{x\in R^+} \frac{|f(x)|}{1+e^{Ax}}.$$

Theorem 3.1. *For each $f \in C_\phi^k(R^+)$, we have*

$$\lim_{n\to\infty} ||\widetilde{P}_n f - f||_\phi = 0.$$

Proof. Following [1, Th. 1] in order to prove the result we have to prove

$$\lim_{n\to\infty} ||\widetilde{P}_n(e^{iAt/2}) - e^{iAx/2}||_\phi = 0, i = 0, 1, 2.$$

The result is true for $i = 0, i = 2$. It remains to verify it for $i = 1$. By Lemma 2.1 we have

$$\begin{aligned} & ||\widetilde{P}_n(e^{At/2}) - e^{Ax/2}||_\phi \\ = & \sup_{x\in R^+} \frac{\left| \left(1 - \frac{(1-e^{-Ax/(n+1)})}{2}\right)^{-(n+1)} - e^{Ax/2} \right|}{1+e^{Ax}} \\ = & \sup_{x\in R^+} \frac{\left| \left(1+e^{-Ax/(n+1)}\right)^{-(n+1)} 2^{n+1} - e^{Ax/2} \right|}{1+e^{Ax}} \\ = & \sup_{x\in R^+} \frac{\left| e^{Ax}\left(1+e^{Ax/(n+1)}\right)^{-(n+1)} 2^{n+1} - e^{Ax/2} \right|}{1+e^{Ax}} \\ = & \sup_{x\in R^+} \left[\frac{e^{Ax}}{1+e^{Ax}}\right] \cdot \left| \left(\frac{2}{1+e^{Ax/(n+1)}}\right)^{n+1} - e^{-Ax/2} \right|. \end{aligned} \tag{3.1}$$

Obviously $\frac{e^{Ax}}{1+e^{Ax}} \in \left[\frac{1}{2}, 1\right),\ A > 0,\ x > 0$. We set $t = e^{Ax/2}, t \in [1, \infty)$ for $x \in (0, \infty)$. Then (3.1) implies

$$\left| \left(\frac{2}{1+t^{2/(n+1)}}\right)^{n+1} - t^{-1} \right| = t^{-1} \left| \left(\frac{2t^{1/(n+1)}}{1+t^{2/(n+1)}}\right)^{n+1} - 1 \right| = g(t). \tag{3.2}$$

In (3.2), we set $t^{1/(n+1)} = y \in [1,\infty)$. Hence

$$\begin{aligned} g(t) = h(y) &= y^{-(n+1)} \left| \left(\frac{2y}{1+y^2}\right)^{n+1} - 1 \right| \\ &= \left| \left(\frac{2}{1+y^2}\right)^{n+1} - y^{-(n+1)} \right| \\ &= y^{-(n+1)} - \left(\frac{2}{1+y^2}\right)^{n+1}. \end{aligned} \tag{3.3}$$

We have $h(1) = 0$, $h(+\infty) = \lim_{y\to\infty} h(y) = 0$. To find the global maxima of $h(y)$ we solve the equation $h'(y) = 0$. Simple calculations imply that $h'(y_0) = 0$ for y_0 satisfying the equation

$$\frac{2}{1+y_0^2} = y_0^{-(n+3)/(n+2)}, y_0 \in (1,\infty). \tag{3.4}$$

The equations (3.3) and (3.4) imply

$$h(y) \le h(y_0) = y_0^{-(n+1)} - y_0^{-(n+3)(n+1)/(n+2)}. \tag{3.5}$$

The proof will be completed if we show

$$h(y_0) < \frac{1}{2(n+3)}, n \to \infty. \tag{3.6}$$

We set in (3.5) $y_0^{n+1} = z_0 \in (1,+\infty)$. Then $h(y_0) = z_0^{-1} - z_0^{-(n+3)/(n+2)} < \max p(z)$ with $p(z) = z^{-1} - z^{-(n+3)/(n+2)}$. We compute that $p'(z_1)$ for $z_1 = \left(\frac{n+3}{n+2}\right)^{n+2}$. Therefore

$$\begin{aligned} p(z_1) &= \left(\frac{n+3}{n+2}\right)^{-(n+2)} - \left(\frac{n+3}{n+2}\right)^{-(n+3)} \\ &= \left(\frac{n+3}{n+2}\right)^{-(n+2)} \left[1 - \left(\frac{n+3}{n+2}\right)^{-1}\right] \\ &= \left(1 + \frac{1}{n+2}\right)^{-(n+2)} \frac{1}{n+3} < \frac{1}{2(n+3)}, \end{aligned}$$

due to $\lim\limits_{n\to\infty} \left(1 + \frac{1}{n+2}\right)^{-(n+2)} = e^{-1} < 1/2$. □

4. A direct quantitative estimate

Our goal in this section is to obtain a quantitative form of the statement in Theorem 3.1. For the sake of simplicity we slightly modify the weight function and instead of $\phi(x) = 1 + e^{Ax}, x \in R^+$ we consider $\phi(x) = e^{Ax}, x \in R^+$, For continuous functions on $[0,\infty)$ with exponential growth i.e.

$$||f||_A := \sup_{x\in[0,\infty)} |f(x) \cdot e^{-Ax}| < \infty, A > 0, \tag{4.1}$$

it is easy to observe that

$$||\widetilde{P}_n f||_A \leq ||f||_A. \tag{4.2}$$

Consequently if the following function series is uniformly convergent on $[0,\infty)$

$$S(x) = \sum_{k=0}^{\infty} u_k(x), x \in [0,\infty),$$

then

$$\widetilde{P}_n(S(t),x) = \sum_{k=0}^{\infty} \widetilde{P}_n(u_k(t),x), x \in [0,\infty), \tag{4.3}$$

where the last series is also uniformly convergent. For our goals in this section we need the first order exponential modulus of continuity, studied by Ditzian in [5] and defined as

$$\omega_1(f,\delta,A) := \sup_{h\leq\delta, 0\leq x<\infty} |f(x)-f(x+h)|e^{-Ax}.$$

We consider the sequence of operators $\widetilde{P}_n : E \to C[0,\infty)$, where the domain of the operator $\widetilde{P}_n$ contains the space of functions f with exponential growth, i.e. $||f||_A < \infty$. Our main result states the following:

Theorem 4.1. *Let $\widetilde{P}_n : E \to C[0,\infty)$ be sequence of linear positive operators of Post-Widder type defined in (1.2). Then*

$$|\widetilde{P}_n(f,x) - f(x)| \leq e^{Ax}[3 + C(n,x)]\omega_1(f, \sqrt{U_2^{\widetilde{P}_n}(x)}, A),$$

where

$$C(n,x) = 2\sum_{k=1}^{\infty} \frac{A^k}{k!}\sqrt{U_{2k}^{\widetilde{P}_n}(x)},\ n \to \infty \text{ for fixed } x \in [0,\infty).$$

Proof. We observe that

$$|f(t)-f(x)| \leq \begin{cases} e^{Ax}\omega_1(f,\delta,A), |t-x| \leq \delta \\ e^{Ax}\omega_1(f,k\delta,A), \delta \leq |t-x| \leq k\delta, \end{cases} \tag{4.4}$$

where k is the smallest natural number in the above upper bound. Now [12, Lemma 2.2] (also see [10]) implies

$$\begin{aligned} \omega_1(f,k\delta,A) &\leq ke^{A(k-1)\delta}\omega_1(f,\delta,A) \\ &\leq \omega_1(f,\delta,A)\left[\frac{|t-x|}{\delta}+1\right]e^{A.|t-x|}. \end{aligned} \tag{4.5}$$

Now (4.4) and (4.5) imply

$$|f(t)-f(x)| \leq \left[1 + \left(\frac{|t-x|}{\delta}+1\right)e^{A.|t-x|}\right]e^{Ax}\omega_1(f,\delta,A). \tag{4.6}$$

For fixed $x \in [0,\infty)$ the following series is uniformly convergent for $t \in [0,\infty)$

$$\begin{aligned} S_1(t,x) &= e^{A.|t-x|} = \sum_{k=0}^{\infty} \frac{(A|t-x|)^k}{k!} \\ \frac{|t-x|}{\delta} S_1(t,x) &= \frac{|t-x|}{\delta} + \frac{1}{\delta}\sum_{k=1}^{\infty} \frac{A^k|t-x|^{k+1}}{k!}. \end{aligned} \tag{4.7}$$

Obviously for linear positive operators $\widetilde{P}_n$ using (4.4), (4.6) and (4.7), we obtain

$$\begin{aligned} |\widetilde{P}_n(f(t)-f(x)| &\leq \widetilde{P}_n(|f(t)-f(x)|,x) \\ &\leq e^{Ax}\Big\{1+\widetilde{P}_n(S_1(t,x),x)+\frac{1}{\delta}\widetilde{P}_n(|t-x|,x) \\ &\quad +\frac{1}{\delta}\sum_{k=1}^{\infty}\frac{A^k\widetilde{P}_n(|t-x|^{k+1},x)}{k!}\Big\}\omega_1(f,\delta,A). \end{aligned} \tag{4.8}$$

From Cauchy Schwarz inequality, we have

$$\begin{aligned} \widetilde{P}_n(|t-x|^{k+1},x) &\leq \sqrt{\widetilde{P}_n((t-x)^2,x)}\sqrt{\widetilde{P}_n((t-x)^{2k},x)} \\ &= \sqrt{U_2^{\widetilde{P}_n}(x)}\sqrt{U_{2k}^{\widetilde{P}_n}(x)}. \end{aligned} \tag{4.9}$$

Further

$$S_1(t,x) = 1 + A|t-x| + \sum_{k=2}^{\infty}\frac{(A|t-x|)^k}{k!}.$$

Hence

$$\widetilde{P}_n(S_1(t,x),x) \leq 1 + A\sqrt{U_2^{\widetilde{P}_n}(x)} + \sum_{k=2}^{\infty}\frac{A^k\sqrt{U_{2k}^{\widetilde{P}_n}(x)}}{k!}. \tag{4.10}$$

From Lemma 2.4, for fixed $x \in [0,\infty)$, we have

$$U_{2k}^{\widetilde{P}_n}(x) = O(n^{-k}), n \to \infty. \tag{4.11}$$

We set in (4.8) that

$$\delta = \sqrt{U_2^{\widetilde{P}_n}(x)} = O(n^{-1/2}), n \to \infty. \tag{4.12}$$

Therefore estimates (4.8)-(4.12) imply

$$|\widetilde{P}_n(f,x)-f(x)| \leq e^{Ax}[3+C(n,x)]\omega_1(f,\sqrt{U_2^{\widetilde{P}_n}(x)},A),$$

where

$$C(n,x) = A\sqrt{U_2^{\widetilde{P}_n}(x)} + \sum_{k=2}^{\infty}\frac{A^k\sqrt{U_{2k}^{\widetilde{P}_n}(x)}}{k!} + \sum_{k=1}^{\infty}\frac{A^k\sqrt{U_{2k}^{\widetilde{P}_n}(x)}}{k!} = O(n^{-1/2}), n \to \infty,$$

by fixed $x \in [0,\infty)$. This completes the proof of theorem. □

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On a new family of generalized Bernstein operators

Maria Talpău Dimitriu

Abstract. In this paper we remark that α-Bernstein operators, introduced by X. Y. Chen et al., are combinations of two known operators (Stancu and Bernstein operators) and we establish the preservation of global smoothness properties by these linear operators, the global smoothness being expressed by a Lipschitz condition with a certain second order modulus of continuity.

Mathematics Subject Classification (2010): 41A36, 41A17.

Keywords: Bernstein-type operators, global smoothness preservation, second order modulus of continuity.

1. Introduction

X.Y. Chen et al. [5] introduced and studied a family of operators as follows. For a function $f : [0,1] \longrightarrow \mathbb{R}$, the α-Bernstein operators $T_{n,\alpha}$, $n \in \mathbb{N}$, $\alpha \in \mathbb{R}$ fixed, are defined by

$$T_{n,\alpha}(f,x) = \sum_{j=0}^{n} p_{n,j}^{(\alpha)}(x) f\left(\frac{j}{n}\right),\ x \in [0,1], \tag{1.1}$$

where $p_{1,0}^{(\alpha)}(x) = 1-x$, $p_{1,1}^{(\alpha)}(x) = x$ and for $n \geq 2$,

$$p_{n,j}^{(\alpha)}(x) = \left[\binom{n-2}{j}(1-\alpha)x + \binom{n-2}{j-2}(1-\alpha)(1-x) + \binom{n}{j}\alpha x(1-x)\right] \cdot x^{j-1}(1-x)^{n-j-1}$$

with the convention

$$\binom{k}{l} = \begin{cases} \dfrac{k!}{l!(k-l)!}, & \text{if } 0 \leq l \leq k, \\ 0, & \text{else.} \end{cases}$$

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It is obvious that for $\alpha = 1$ the class of Bernstein operators is obtained

$$T_{n,1}(f,x) = \sum_{j=0}^{n} p_{n,j}(x) f\left(\frac{j}{n}\right) = B_n(f,x),\ p_{n,j}(x) = \binom{n}{j} x^j (1-x)^{n-j}.$$

We note that for the α-Bernstein operator another representation can be obtained as follows.

$$\begin{aligned} T_{n,\alpha}(f,x) &= (1-\alpha)\left[\sum_{j=0}^{n-2} p_{n-2,j}(x)(1-x) f\left(\frac{j}{n}\right) + \sum_{j=2}^{n} p_{n-2,j-2}(x) x f\left(\frac{j}{n}\right)\right] \\ &\quad +\alpha \sum_{j=0}^{n} p_{n,j}(x) f\left(\frac{j}{n}\right) \\ &= (1-\alpha)\left[\sum_{j=0}^{n-2} p_{n-2,j}(x)(1-x) f\left(\frac{j}{n}\right) + \sum_{j=0}^{n-2} p_{n-2,j}(x) x f\left(\frac{j+2}{n}\right)\right] \\ &\quad +\alpha B_n(f,x) \\ &= (1-\alpha)\sum_{j=0}^{n-2} p_{n-2,j}(x) \sum_{i=0}^{1} p_{1,i}(x) f\left(\frac{j+2i}{n}\right) + \alpha B_n(f,x) \end{aligned}$$

The following generalized Bernstein operators was introduced by D. D. Stancu (see [11])

$$S_{n,r,s}(f,x) = \sum_{j=0}^{n-rs} p_{n-rs,j}(x) \sum_{i=0}^{s} p_{s,i}(x) f\left(\frac{j+ir}{n}\right), \tag{1.2}$$

$f \in C[0,1]$, $x \in [0,1]$, where $n \in \mathbb{N}$, $r, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ such that $rs < n$. Bernstein's operators are obtained for $s = 0$ or $s = 1$, $r = 0$ or $s = 1$, $r = 1$.

So the α-Bernstein operator can be expressed as

$$T_{n,\alpha}(f,x) = (1-\alpha) S_{n,2,1}(f,x) + \alpha B_n(f,x). \tag{1.3}$$

In [13] we introduced a two dimensional generalization of the Stancu operators (1.2) and established certain results related to the global smoothness preservation with respect to a second order modulus of continuity for functions defined on the 2-dimensional simplex. The corresponding results in the one-dimensional case are presented in the following section.

The preservation of global smoothness properties by the Bernstein operators was studied in [7], [8], [4], [2], [6], [3]. In [14], D.-X. Zhou showed that the Lipschitz classes with respect to the second order modulus

$$\omega_2(f,t) = \sup\{|f(x-h) - 2f(x) + f(x+h)| : x \pm h \in [0,1],\ 0 < h \le t\}$$

are not preserved by the Bernstein operators. He introduced the following second order modulus of smoothness

$$\begin{aligned} \widetilde{\omega}_2(f,t) &= \sup\{|f(x+h_1+h_2) - f(x+h_1) - f(x+h_2) + f(x)| : \\ &\qquad x, x+h_1+h_2 \in [0,1],\ h_1, h_2 > 0,\ h_1+h_2 \le 2t\} \end{aligned}$$

and proved the following result:
Theorem A. *Let $f \in C[0,1]$, $n \in \mathbb{N}$, $M > 0$ and $0 < \mu \leq 1$.*
If $\widetilde{\omega}_2(f,t) \leq Mt^\mu$, $0 < t \leq \frac{1}{2}$, then $\widetilde{\omega}_2(B_n f,t) \leq Mt^\mu$, $0 < t \leq \frac{1}{2}$.
For the Bernstein-type operators

$$L_n(f,x) = \sum_{j=0}^{n} p_{n,j}(x) F_{n,j}(f),\ f \in C[0,1],\ x \in [0,1], \tag{1.4}$$

where $F_{n,j} : C[0,1] \longrightarrow \mathbb{R}$, $j = \overline{1,n}$, are linear positive functionals, in [12] we studied simultaneous global smoothness preservation in terms of modulus of continuity ω_2^* introduced by R. Păltănea [9], [10] and independently by J. Adell and J. de la Cal [1], defined for $f \in \mathbf{C}[0,1]$ and $t > 0$ by

$$\begin{aligned} \omega_2^*(f,t) &= \sup\{|(1-\lambda)f(x) + \lambda f(y) - f((1-\lambda)x + \lambda y, y))| : \\ &\qquad x, y \in [0,1],\ x < y,\ y - x \leq 2t,\ \lambda \in [0,1]\}. \end{aligned}$$

The preservation of global smoothness properties by α-Bernstein operators is obtained as a consequence of global smoothness preservation by Stancu operators.

2. Global smoothness preservation

Lemma 2.1. *For $f \in C[0,1]$, $0 \leq x < y \leq 1$, $\lambda \in [0,1]$ we have*

$$\begin{aligned} &S_{n,r,s}(f,(1-\lambda)x + \lambda y) \\ &= \sum_{k_1+l_1=0}^{s} \sum_{k_2+l_2=0}^{n-rs} p_{s,k_1,l_1}(x, y-x) p_{n-rs,k_2,l_2}(x, y-x) \cdot \\ &\cdot \sum_{m_1=0}^{l_1} \sum_{m_2=0}^{l_2} p_{l_1,m_1}(\lambda) p_{l_2,m_2}(\lambda) f\left(\frac{k_2 + m_2 + r(k_1 + m_1)}{n}\right), \end{aligned} \tag{2.1}$$

where

$$p_{n,k,l}(x,y) = \frac{n!}{k!l!\,(n-k-l)!} x^k y^l (1-x-y)^{n-k-l}.$$

Proof. Let $f \in C[0,1]$, $0 \leq x < y \leq 1$, $\lambda \in [0,1]$.
For the Bernstein type operator (1.4) in [12], proceeding similarly as in [14] (see also [4]), we obtained:

$$\begin{aligned} &L_n(f,(1-\lambda)x + \lambda y) \\ &= \sum_{k+l=0}^{n} p_{n,k,l}(x, y-x) \sum_{m=0}^{l} p_{l,m}(\lambda) F_{n,k+m}(f). \end{aligned} \tag{2.2}$$

Repeating the application of an adapted version of relation (2.2) yields

$$S_{n,r,s}(f,(1-\lambda)x + \lambda y) = \sum_{j=0}^{n-rs} p_{n-rs,j}((1-\lambda)x+\lambda y) \sum_{i=0}^{s} p_{s,i}((1-\lambda)x+\lambda y) f\left(\frac{j+ir}{n}\right)$$

$$= \sum_{j=0}^{n-rs} p_{n-rs,j}((1-\lambda)x+\lambda y) \sum_{k_1+l_1=0}^{s} p_{s,k_1,l_1}(x,y-x) \sum_{m_1=0}^{l_1} p_{l_1,m_1}(\lambda) f\left(\frac{j+r(k_1+m_1)}{n}\right)$$

$$= \sum_{k_1+l_1=0}^{s} p_{s,k_1,l_1}(x,y-x) \sum_{m_1=0}^{l_1} p_{l_1,m_1}(\lambda) \sum_{j=0}^{n-rs} p_{n-rs,j}((1-\lambda)x+\lambda y) f\left(\frac{j+r(k_1+m_1)}{n}\right)$$

$$= \sum_{k_1+l_1=0}^{s} p_{s,k_1,l_1}(x,y-x) \sum_{m_1=0}^{l_1} p_{l_1,m_1}(\lambda)\cdot$$

$$\cdot \sum_{k_2+l_2=0}^{n-rs} p_{n-rs,k_2,l_2}(x,y-x) \sum_{m_2=0}^{l_2} p_{l_2,m_2}(\lambda) f\left(\frac{k_2+m_2+r(k_1+m_1)}{n}\right).$$

□

Theorem 2.2. *Let $f \in C[0,1]$, $M>0$ and $\mu \in (0,1]$. If*

$$\omega_1(f,t) \le Mt^\mu,\ t\in(0,1],$$

then

$$\omega_1(S_{n,r,s}f,t) \le Mt^\mu,\ t\in(0,1].$$

Proof. Let $x,y\in[0,1]$ be such that $|x-y|\le t$. We can assume that $x<y$.

$$\begin{aligned}
&|S_{n,r,s}(f,x) - S_{n,r,s}(f,y)| \\
&\le \sum_{k_1+l_1=0}^{s} \sum_{k_2+l_2=0}^{n-rs} p_{s,k_1,l_1}(x,y-x) p_{n-rs,k_2,l_2}(x,y-x)\cdot \\
&\cdot \left| f\left(\frac{k_2+rk_1}{n}\right) - f\left(\frac{k_2+l_2+r(k_1+l_1)}{n}\right)\right| \\
&\le \sum_{k_1+l_1=0}^{s} \sum_{k_2+l_2=0}^{n-rs} p_{s,k_1,l_1}(x,y-x) p_{n-rs,k_2,l_2}(x,y-x)\omega_1\left(f,\frac{l_2+rl_1}{n}\right) \\
&\le \sum_{k_1+l_1=0}^{s} \sum_{k_2+l_2=0}^{n-rs} p_{s,k_1,l_1}(x,y-x) p_{n-rs,k_2,l_2}(x,y-x) M\left(\frac{l_2+rl_1}{n}\right)^\mu \\
&\le M\left(\sum_{k_1+l_1=0}^{s} \sum_{k_2+l_2=0}^{n-rs} p_{s,k_1,l_1}(x,y-x) p_{n-rs,k_2,l_2}(x,y-x)\frac{l_2+rl_1}{n}\right)^\mu \\
&= M\left(\sum_{k_2+l_2=0}^{n-rs} p_{n-rs,k_2,l_2}(x,y-x)\frac{l_2}{n} + \sum_{k_1+l_1=0}^{s} p_{s,k_1,l_1}(x,y-x)\frac{rl_1}{n}\right)^\mu \\
&= M\left(\frac{n-rs}{n}(y-x) + \frac{rs}{n}(y-x)\right)^\mu \le Mt^\mu.
\end{aligned}$$

□

The next result relates to the global smoothness preservation by Stancu operators in terms of modulus of continuity ω_2^*.

Theorem 2.3. *Let $f \in C[0,1]$, $M > 0$ and $\mu \in (0,1]$. If*

$$\omega_2^*(f,t) \le Mt^\mu,\ t \in \left(0, \frac{1}{2}\right],$$

then

$$\omega_2^*(S_{n,r,s}f,t) \le Mt^\mu,\ t \in \left(0, \frac{1}{2}\right].$$

Proof. Let $t \in \left(0, \frac{1}{2}\right]$, $x, y \in [0,1]$, $x < y$, $y - x \le 2t$, $\lambda \in [0,1]$. By using the representation (2.1), we obtain:

$$\begin{aligned}
&|(1-\lambda)S_{n,r,s}f(x) + \lambda S_{n,r,s}f(y) - S_{n,r,s}f((1-\lambda)x - \lambda y)| \\
&\le \sum_{k_1+l_1=0}^{s} \sum_{k_2+l_2=0}^{n-rs} p_{s,k_1,l_1}(x, y-x) p_{n-rs,k_2,l_2}(x, y-x) \cdot \\
&\cdot \Bigg|(1-\lambda) f\left(\frac{k_2 + rk_1}{n}\right) + \lambda f\left(\frac{k_2 + l_2 + r(k_1 + l_1)}{n}\right) \\
&- \sum_{m_1=0}^{l_1} \sum_{m_2=0}^{l_2} p_{l_1,m_1}(\lambda) p_{l_2,m_2}(\lambda) f\left(\frac{k_2 + m_2 + r(k_1 + m_1)}{n}\right)\Bigg| \\
&\le \sum_{\substack{k_1+l_1=0 \\ l_2+rl_1 \ne 0}}^{s} \sum_{k_2+l_2=0}^{n-rs} p_{s,k_1,l_1}(x, y-x) p_{n-rs,k_2,l_2}(x, y-x) \cdot \\
&\cdot \sum_{m_1=0}^{l_1} \sum_{m_2=0}^{l_2} p_{l_1,m_1}(\lambda) p_{l_2,m_2}(\lambda) \cdot \Bigg| \left(1 - \frac{m_2 + rm_1}{l_2 + rl_1}\right) f\left(\frac{k_2 + rk_1}{n}\right) \\
&+ \frac{m_2 + rm_1}{l_2 + rl_1} f\left(\frac{k_2 + rk_1}{n} + \frac{l_2 + rl_1}{n}\right) - f\left(\frac{k_2 + rk_1}{n} + \frac{m_2 + rm_1}{n}\right)\Bigg| \\
&\le \sum_{\substack{k_1+l_1=0 \\ l_2+rl_1 \ne 0}}^{s} \sum_{k_2+l_2=0}^{n-rs} p_{s,k_1,l_1}(x, y-x) p_{n-rs,k_2,l_2}(x, y-x) \cdot \\
&\cdot \sum_{m_1=0}^{l_1} \sum_{m_2=0}^{l_2} p_{l_1,m_1}(\lambda) p_{l_2,m_2}(\lambda) \omega_2^*\left(f, \frac{l_2 + rl_1}{2n}\right) \\
&\le M \sum_{k_1+l_1=0}^{s} \sum_{k_2+l_2=0}^{n-rs} p_{s,k_1,l_1}(x, y-x) p_{n-rs,k_2,l_2}(x, y-x) \left(\frac{l_2 + rl_1}{2n}\right)^\mu \\
&\le M \left(\sum_{k_1+l_1=0}^{s} \sum_{k_2+l_2=0}^{n-rs} p_{s,k_1,l_1}(x, y-x) p_{n-rs,k_2,l_2}(x, y-x) \frac{l_2 + rl_1}{2n}\right)^\mu \\
&= M \left(\frac{n - rs}{2n}(y - x) + \frac{rs}{2n}(y - x)\right)^\mu \\
&= M \left(\frac{y - x}{2}\right)^\mu \le Mt^\mu.
\end{aligned}$$

Hence $\omega_2^*\left(S_{n,r,s}f,t\right) \le Mt^\mu$. □

For $n \in \mathbb{N}$, $\alpha \in [0,1]$, $r_1, s_1, r_2, s_2 \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}$ such that $r_1s_1, r_2s_2 < n$, we consider the operators

$$T_{n,\alpha}^{r_1,s_1,r_2,s_2}(f,x) = (1-\alpha)S_{n,r_1,s_1}(f,x) + \alpha S_{n,r_2,s_2}(f,x), \tag{2.3}$$

$f \in C[0,1]$, $x \in [0,1]$.
From Theorem 2.2, Theorem 2.3 and the inequalities

$$\omega_1\left(T_{n,\alpha}^{r_1,s_1,r_2,s_2}f,t\right) \le (1-\alpha)\omega_1\left(S_{n,r_1,s_1}f,t\right) + \alpha\omega_1\left(S_{n,r_2,s_2}f,t\right),$$
$$\omega_2^*\left(T_{n,\alpha}^{r_1,s_1,r_2,s_2}f,t\right) \le (1-\alpha)\omega_2^*\left(S_{n,r_1,s_1}f,t\right) + \alpha\omega_2^*\left(S_{n,r_2,s_2}f,t\right),$$

we obtain the final result:

Theorem 2.4. *Let $f \in C[0,1]$, $M > 0$ and $\mu \in (0,1]$.*

1. *If $\omega_1(f,t) \le Mt^\mu$, $t \in (0,1]$, then $\omega_1\left(T_{n,\alpha}^{r_1,s_1,r_2,s_2}f,t\right) \le Mt^\mu$, $t \in (0,1]$.*
2. *If $\omega_2^*(f,t) \le Mt^\mu$, $t \in \left(0,\frac{1}{2}\right]$, then $\omega_2^*\left(T_{n,\alpha}^{r_1,s_1,r_2,s_2}f,t\right) \le Mt^\mu$, $t \in \left(0,\frac{1}{2}\right]$.*

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On possible generalisations of quasi-contractions

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Abstract. This paper investigates whether some fixed point theorems for quasi-contractions on metric spaces introduced by Ćirić in [1] and generalised by Kumam et al. in [2] can be improved further. It turns out that the answer is negative. We provide two examples of complete metric spaces and two operators without fixed points. We prove that for any possible straightforward relaxation of generalised quasi-contractive conditions, one of these operators satisfies the condition.

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Keywords: Quasi-contractions, fixed point theorems, metric spaces.

1. Introduction and preliminary results

Banach's contraction principle is a fundamental result in the study of fixed points of operators defined on complete metric spaces. This principle can be stated as follows.

Let (X, d) be a complete metric space and $T\colon X \to X$ be a self-map. If there exists $q \in [0, 1)$ such that for all $x, y \in X$

$$d(Tx, Ty) \le q \cdot d(x, y), \tag{C1}$$

then T has a unique fixed point $x^* \in X$. Furthermore, for any $x_0 \in X$ the sequence $x_{n+1} = Tx_n$ converges to x^* in X.

Due to its wide range of applicability in different fields of mathematics, several generalisations have appeared. This paper focuses on one possible "branch" of these improvements, that of quasi-contractions, that we present below.

1.1. Quasi-contractions

The notion was first introduced by Ćirić in [1], hence it is sometimes referred to as Ćirić-type contractions. It consists of two separate improvements of Banach's original principle.

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On the one hand, the requirement of completeness of (X, d) is relaxed to T-orbitally completeness. We recall that the *orbit* of $T\colon X \to X$ is defined as

$$O_T(x) = \{x, Tx, \dots, T^n x, \dots\},$$

and a metric space is T*-orbitally complete* if every Cauchy sequence in $O_T(x)$ is convergent in X.

On the other hand, the contractive condition (C1) is relaxed as well and it is replaced with the following:

$$d(Tx, Ty) \le q \cdot \max\{d(x, y), d(Tx, x), d(Tx, y), d(Ty, x), d(Ty, y)\}. \tag{C2}$$

With these improvements, the operator T still has a unique fixed point $x^* \in X$ and for any $x_0 \in X$ the sequence $x_{n+1} = Tx_n$ converges to x^*.

1.2. Generalised quasi-contractions

Ćirić's idea was developed further by Kumam et al. in [2]. The authors introduced the notion of generalised quasi-contraction which uses the condition

$$\begin{aligned} d(Tx, Ty) \le q \cdot \max\{&d(x, y), d(Tx, x), d(Tx, y), d(Ty, x), d(Ty, y), \\ &d(T^2x, x), d(T^2x, y), d(T^2x, Tx), d(T^2x, Ty)\}, \end{aligned} \tag{C3}$$

and proved a fixed point theorem similar to Ćirić's.

Focusing on conditions (C1-C3), the following questions arise naturally:

- Why is (C3) not symmetric in x and y, i.e. why are the $d(T^2y, \cdot)$ terms excluded? More generally, can one include terms of the form $d(T^k y, \cdot)$, where $k \ge 2$?
- Can one introduce additional terms of the form $d(T^k x, \cdot)$, where $k \ge 3$, in the set on the right-hand side?
- What is the "most general" version of these types of conditions that guarantees the existence and uniqueness of the fixed point of the operator in concern?

In the next section we answer all the questions above: the conclusion is that condition (C3) cannot be relaxed further.

2. Main result

Theorem 2.1. *There exists a complete metric space (X, d) and an operator $T\colon X \to X$ such that T has no fixed points, while for some $q \in (0, 1)$ and for every $x, y \in X$ we have*

$$\begin{aligned} d(Tx, Ty) \le q \cdot \max\{&d(x, y), d(Tx, x), d(Tx, y), d(Ty, x), d(Ty, y), \\ &d(T^2x, x), d(T^2x, y), d(T^2x, Tx), d(T^2x, Ty), D\}, \end{aligned} \tag{C}$$

where D is one of the distances

$$\begin{cases} d(T^a x, T^b y), & \text{for some } a \ge 3, b \ge 0, \\ d(T^a x, T^b y), & \text{for some } a \ge 0, b \ge 2, \\ d(T^a x, T^b x), & \text{for some } a \ge 3, b \ge 0, a \ne b, \\ d(T^a y, T^b y), & \text{for some } a \ge 2, b \ge 0, a \ne b. \end{cases} \tag{$\mathcal{D}$}$$

The proof is obtained by constructing two different examples, depending on the type of the distance D. First, we construct a space and an operator that deals with distances of type $D = d(T^{k+1}x, T^k y)$, for some $k \geq 2$. Then we give another construction that discusses the remaining cases for D. We claim that both examples satisfy condition (C) with the respective D.

Claim 2.2. *Let $X = \{2^n : n \in \mathbb{N}\}$, $d(x,y) = |x-y|$ and $T(x) = 2x$. Obviously (X,d) is a complete metric space and T does not have fixed points. Furthermore, let $D = d(T^{k+1}x, T^k y)$, with $k \geq 2$ arbitrary. Then condition* (C) *holds for all $x, y \in X$.*

Proof. For every $x, y \in X$, there exists $m, n \in \mathbb{N}$, such that $x = 2^m$ and $y = 2^n$.
We have three cases.

- If $m > n$, then
$$d(Tx,Ty) = 2^{m+1} - 2^{n+1} \leq 2^{m+1} - 2^n = \frac{1}{2}(2^{m+2} - 2^{n+1}) = \frac{1}{2}d(T^2x, Ty).$$
- If $m = n-1$, then
$$d(Tx,Ty) = 2^{n+1} - 2^n = \frac{2}{3}(2^2-1)2^{n-1} = \frac{2}{3}(2^{n+1} - 2^{n-1}) = \frac{2}{3}d(Ty, x).$$
- If $m < n-1$, then $0 \leq 2^{n-m-1} - 2$. Adding $2 \cdot 2^{n-m-1} - 1$ to both sides, we obtain
$$(2^{n-m} - 1) \leq 3(2^{n-m-1} - 1).$$
Now using that $k \geq 2$, we can write
$$\begin{aligned} d(Tx,Ty) &= 2^{n+1} - 2^{m+1} = 2^{m+1}(2^{n-m} - 1) \leq 2^{m+1} \cdot 3(2^{n-m-1} - 1) \\ &= \frac{3}{2^k}(2^{n+k} - 2^{m+1+k}) \leq \frac{3}{4}(2^{n+k} - 2^{m+1+k}) = \frac{3}{4}d(T^k y, T^{k+1}x). \end{aligned}$$

In conclusion, (C) holds with $q = \frac{3}{4}$. □

Claim 2.3. *Let $X = \{z^n \mid n \in \mathbb{N}\}$, where $z = -1 + i\sqrt{3}$, $d(x,y) = |x-y|$ and $T(x) = zx$. Obviously (X,d) is a complete metric space and T does not have fixed points. Furthermore, let D be one of the distances from $(\mathcal{D})$, which is not included in Claim* 2.2. *Then condition* (C) *holds for all $x, y \in X$.*

We present two lemmas that we use in the proof of Claim 2.3. In the forthcoming proofs, we use the following facts without mention: $|z| = 2$, $|z-1| = \sqrt{7}$, $|z^2-1| = \sqrt{21}$, $|z^3-1| = 7$ and $|z+1| = \sqrt{3}$.

Lemma 2.4. *If $z = -1 + i\sqrt{3}$, then $D = |z^{u+2} - z^v| \geq \sqrt{21}$, for all $u, v \geq 0$, with $u + 2 \neq v$.*

Proof. We have the following cases.

- If $u = 0$ and $v = 0$, then $D = |z^2 - 1| = \sqrt{21}$.
- If $u = 0$ and $v = 1$, then $D = |z^2 - z| = 2|z-1| = 2\sqrt{7} > \sqrt{21}$.
- If $u = 0$ and $v = 3$, then $D = |z^2 - z^3| = 4|z-1| = 4\sqrt{7} > \sqrt{21}$.
- If $u = 1$ and $v = 0$, then $D = |z^3 - 1| = 7 > \sqrt{21}$.
- If $u = 1$ and $v = 1$, then $D = |z^3 - z| = 2\sqrt{21} > \sqrt{21}$.

- If $u = 1$ and $v = 2$, then $D = |z^3 - z^2| = 4\sqrt{7} > \sqrt{21}$.
- If $u > 1$ or $v > 3$, then

$$D = |z^{u+2} - z^v| \geq ||z^{u+2} - |z^v|| = |2^{u+2} - 2^v| \geq 8 > \sqrt{21}.$$

□

Lemma 2.5. *If* $z = -1+i\sqrt{3}$, *then* $D = |z^{u+3} - z^v| \geq 7$, *for all* $u, v \geq 0$, *with* $u+3 \neq v$.

Proof. We have the following cases.

- If $u = 0$ and $v = 0$, then $D = |z^3 - 1| = 7$.
- If $u = 0$ and $v = 1$, then $D = |z^3 - z| = 2|z^2 - 1| = 2\sqrt{21} > 7$.
- If $u = 0$ and $v = 2$, then $D = |z^3 - z^2| = 4|z - 1| = 4\sqrt{7} > 7$.
- If $u > 0$ or $v > 3$, then $D = |z^{u+3} - z^v| \geq ||z^{u+3} - |z^v|| = |2^{u+3} - 2^v| \geq 8$.

□

Proof of Claim 2.3. We have four cases.

- If $m = n + s$, with $s \geq 2$ then $d(Tx, Ty) \leq q_1 d(Tx, x)$, where $q_1 = \frac{5}{2\sqrt{7}} < 1$. Indeed, we have

$$\begin{aligned} d(Tx, Ty) &= |z^{n+s+1} - z^{n+1}| = 2^{n+1}|z^s - 1| \leq 2^{n+1}(|z^s| + 1) = 2^{n+1}(2^s + 1) \\ &\leq 2^{n+1}(2^s + 2^{s-2}) = 5 \cdot 2^{n-1+s} = \frac{5}{2\sqrt{7}} \cdot \sqrt{7} \cdot 2^{n+s} \\ &= \frac{5}{2\sqrt{7}}|z - 1||z^{n+s}| = \frac{5}{2\sqrt{7}} d(Tx, x). \end{aligned}$$

- If $n = m + s$, with $s \geq 2$ one can similarly prove that $d(Tx, Ty) \leq \frac{5}{2\sqrt{7}} d(Ty, y)$.
- If $m = n + 1$, then $d(Tx, Ty) \leq q_2 d(T^2x, Ty)$, where $q_2 = \frac{\sqrt{3}}{3} < 1$. Indeed, we have

$$\begin{aligned} d(Tx, Ty) &= |z^{n+2} - z^{n+1}| = 2^{n+1}|z - 1| = 2^{n+1}\sqrt{7} \\ &= \frac{\sqrt{3}}{3} \cdot \sqrt{21} \cdot 2^{n+1} = \frac{\sqrt{3}}{3}|z^2 - 1||z^{n+1}| = \frac{\sqrt{3}}{3} d(T^2x, Ty). \end{aligned}$$

- If $m = n - 1$, then there exists $q_3 \in (0, 1)$, such that $d(Tx, Ty) \leq q_1 D$ and D is any distance from $(\mathcal{D})$ that was not considered in Claim 2.2.

  To prove this statement, we observe that D can have the following forms.

  – If $D = d(T^a x, T^b y)$ for some $a \geq 3, b \geq 0$, $a \neq b + 1$, then

  $$D = |z^{n-1+a} - z^{n+b}|.$$

  – If $D = d(T^a x, T^b y)$ for some $a \geq 0$, $b \geq 2$, $a \neq b + 1$, then

  $$D = |z^{n-1+a} - z^{n+b}|.$$

  – If $D = d(T^a x, T^b x)$ for some $a \geq 3, b \geq 0$, $a \neq b$, then

  $$D = |z^{n-1+a} - z^{n-1+b}|.$$

  – If $D = d(T^a y, T^b y)$ for some $a \geq 2, b \geq 0$, $a \neq b$, then

  $$D = |z^{n+a} - z^{n+b}|.$$

This implies that

- either $D = |z^{n+a} - z^{n+b}|$ for some $a \geq 2, b \geq 0$, $a \neq b$,
- or $D = |z^{n+a} - z^{n-1+b}|$ for some $a \geq 2, b \geq 0$, $a + 1 \neq b$.

On the one hand, using Lemma 2.4 we have

$$d(Tx, Ty) = |z^n - z^{n+1}| = 2^n\sqrt{7} = \frac{\sqrt{3}}{3}2^n\sqrt{21}$$
$$\leq \frac{\sqrt{3}}{3}|z^n||z^a - z^b| = \frac{\sqrt{3}}{3}|z^{n+a} - z^{n+b}|.$$

On the other hand, using Lemma 2.5 we have

$$d(Tx, Ty) = |z^n - z^{n+1}| = 2^n|1 - z| = 2^n\sqrt{7} = \frac{2}{\sqrt{7}}2^{n-1} \cdot 7$$
$$\leq \frac{2}{\sqrt{7}}|z^{n-1}||z^{a+1} - z^b| = \frac{2}{\sqrt{7}}|z^{n+a} - z^{n-1+b}|.$$

The above two cases conclude the proof. □

Remark 2.6. The proofs of Lemma 2.4, 2.5 and Claim 2.3 can be carried out with fewer steps than presented (some cases can be merged). However, we think that these shortenings detriment the readability of the paper.

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Triangle angle sums related to translation curves in Sol geometry

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Abstract. After having investigated the geodesic and translation triangles and their angle sums in **Nil** and $\widetilde{\mathbf{SL_2R}}$ geometries we consider the analogous problem in **Sol** space that is one of the eight 3-dimensional Thurston geometries. We analyse the interior angle sums of translation triangles in **Sol** geometry and prove that it can be larger or equal than π. In our work we will use the projective model of **Sol** described by E. Molnár in [9].

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1. Introduction

In the Thurston spaces can be introduced in a natural way (see [9]) translations mapping each point to any point. Consider a unit vector at the origin. Translations, postulated at the beginning carry this vector to any point by its tangent mapping. If a curve $t \to (x(t), y(t), z(t))$ has just the translated vector as tangent vector in each point, then the curve is called a *translation curve*. This assumption leads to a system of first order differential equations, thus translation curves are simpler than geodesics and differ from them in **Nil**, $\widetilde{\mathbf{SL_2R}}$ and **Sol** geometries. In $\mathbf{E}^3$, $\mathbf{S}^3$, $\mathbf{H}^3$, $\mathbf{S}^2\times\mathbf{R}$ and $\mathbf{H}^2\times\mathbf{R}$ geometries the mentioned curves coincide with each other.

Therefore, the translation curves also play an important role in **Nil**, $\widetilde{\mathbf{SL_2R}}$ and **Sol** geometries and often seem to be more natural in these geometries, than their geodesic lines.

A translation triangle in Riemannian geometry and more generally in metric geometry a figure consisting of three different points together with the pairwise-connecting translation curves. The points are known as the vertices, while the translation curve segments are known as the sides of the triangle.

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In the geometries of constant curvature $\mathbf{E}^3$, $\mathbf{H}^3$, $\mathbf{S}^3$ the well-known sums of the interior angles of geodesic (or translation) triangles characterize the space. It is related to the Gauss-Bonnet theorem which states that the integral of the Gauss curvature on a compact 2-dimensional Riemannian manifold M is equal to $2\pi\chi(M)$ where $\chi(M)$ denotes the Euler characteristic of M. This theorem has a generalization to any compact even-dimensional Riemannian manifold (see e.g. [4], [6], [8]).

In [5] we investigated the angle sum of translation and geodesic triangles in $\widetilde{\mathbf{SL_2R}}$ geometry and proved that the possible sum of the interior angles in a translation triangle must be greater or equal than π. However, in geodesic triangles this sum is less, greater or equal to π.

In [19] we considered the analogous problem for geodesic triangles in **Nil** geometry and proved that the sum of the interior angles of geodesic triangles in **Nil** space is larger, less or equal than π.

In [2] K. Brodaczewska showed, that sum of the interior angles of translation triangles of the **Nil** space is larger or equal than π.

However, in $\mathbf{S}^2\times\mathbf{R}$, $\mathbf{H}^2\times\mathbf{R}$ and **Sol** Thurston geometries there are no result concerning the angle sums of translation or geodesic triangles. Therefore, it is interesting to study similar question in the above three geometries. Now, we are interested in *translation triangles* in **Sol** space [15, 20].

In Section 2 we describe the projective model and the isometry group of **Sol**, moreover, we give an overview about its translation curves.

Remark 1.1. We note here, that nowadays the **Sol** geometry is a widely investigated space concerning its manifolds, tilings, geodesic and translation ball packings and probability theory (see e.g. [1], [3], [7], [11], [12], [13], [17], [18] and the references given there).

In Section 3 we study the **Sol** *translation triangles and prove that their interior angle sums can be larger or equal than π.*

2. On Sol geometry

In this Section we summarize the significant notions and notations of real **Sol** geometry (see [9], [15]).

Sol is defined as a 3-dimensional real Lie group with multiplication

$$(a,b,c)(x,y,z) = (x+ae^{-z}, y+be^{z}, z+c). \tag{2.1}$$

We note that the conjugacy by (x,y,z) leaves invariant the plane (a,b,c) with fixed c:

$$(x,y,z)^{-1}(a,b,c)(x,y,z) = (x(1-e^{-c})+ae^{-z}, y(1-e^{c})+be^{z}, c). \tag{2.2}$$

Moreover, for $c=0$, the action of (x,y,z) is only by its z-component, where $(x,y,z)^{-1}=(-xe^{z},-ye^{-z},-z)$. Thus the $(a,b,0)$ plane is distinguished as a *base plane* in **Sol**, or by other words, $(x,y,0)$ is normal subgroup of **Sol**. **Sol** multiplication can also be affinely (projectively) interpreted by "right translations" on its points as the following matrix formula shows, according to (2.1):

$$(1;a,b,c) \to (1;a,b,c)\begin{pmatrix}1 & x & y & z\\ 0 & e^{-z} & 0 & 0\\ 0 & 0 & e^{z} & 0\\ 0 & 0 & 0 & 1\end{pmatrix} = (1; x + ae^{-z}, y + be^{z}, z + c) \tag{2.3}$$

by row-column multiplication.

This defines "translations" $\mathbf{L}(\mathbf{R}) = \{(x,y,z) : x,\ y,\ z \in \mathbf{R}\}$ on the points of space $\mathbf{Sol} = \{(a,b,c) : a,\ b,\ c \in \mathbf{R}\}$. These translations are not commutative, in general. Here we can consider $\mathbf{L}$ as projective collineation group with right actions in homogeneous coordinates as usual in classical affine-projective geometry. We will use the Cartesian homogeneous coordinate simplex $E_0(\mathbf{e}_0)$, $E_1^\infty(\mathbf{e}_1)$, $E_2^\infty(\mathbf{e}_2)$, $E_3^\infty(\mathbf{e}_3)$, $(\{\mathbf{e}_i\} \subset \mathbf{V}^4$ with the unit point $E(\mathbf{e} = \mathbf{e}_0 + \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3))$ which is distinguished by an origin E_0 and by the ideal points of coordinate axes, respectively. Thus **Sol** can be visualized in the affine 3-space $\mathbf{A}^3$ (so in Euclidean space $\mathbf{E}^3$) as well [9].

In this affine-projective context E. Molnár has derived in [9] the usual infinitesimal arc-length square at any point of **Sol**, by pull back translation, as follows

$$(ds)^2 := e^{2z}(dx)^2 + e^{-2z}(dy)^2 + (dz)^2. \tag{2.4}$$

Hence we get infinitesimal Riemann metric invariant under translations, by the symmetric metric tensor field g on **Sol** by components as usual.

It will be important for us that the full isometry group Isom(**Sol**) has eight components, since the stabilizer of the origin is isomorphic to the dihedral group $\mathbf{D_4}$, generated by two involutive (involutory) transformations, preserving (2.4):

$$(1)\ \ y \leftrightarrow -y;\quad (2)\ x \leftrightarrow y;\quad z \leftrightarrow -z;\quad \text{i.e. first by } 3\times 3 \text{ matrices :}$$

$$(1)\ \begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{pmatrix};\quad (2)\ \begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}; \tag{2.5}$$

with its product, generating a cyclic group $\mathbf{C_4}$ of order 4

$$\begin{pmatrix}0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & -1\end{pmatrix};\ \begin{pmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{pmatrix};\ \begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & -1\end{pmatrix};\ \mathbf{Id} = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}.$$

Or we write by collineations fixing the origin $O(1,0,0,0)$:

$$(1)\ \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix},\ (2)\ \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}\ \text{of form (2.3).} \tag{2.6}$$

A general isometry of **Sol** to the origin O is defined by a product $\gamma_O\tau_X$, first γ_O of form (2.6) then τ_X of (2.3). To a general point $A(1,a,b,c)$, this will be a product $\tau_A^{-1}\gamma_O\tau_X$, mapping A into $X(1,x,y,z)$.

Conjugacy of translation τ by an above isometry γ, as $\tau^\gamma = \gamma^{-1}\tau\gamma$ also denotes it, will also be used by (2.3) and (2.6) or also by coordinates with above conventions. We note here that the **Sol**-space is translation-complete, i.e. every two points of the

Sol-space can be connected through one translation arc and every three points form a triangle, when they are not on the same translation curve.

We remark only that the role of x and y can be exchanged throughout the paper, but this leads to the mirror interpretation of **Sol**. As formula (2.4) fixes the metric of **Sol**, the change above is not an isometry of a fixed **Sol** interpretation. Other conventions are also accepted and used in the literature.

Sol *is an affine metric space (affine-projective one in the sense of the unified formulation of* [9]*). Therefore its linear, affine, unimodular, etc. transformations are defined as those of the embedding affine space.*

2.1. Translation curves

We consider a **Sol** curve $(1, x(t), y(t), z(t))$ with a given starting tangent vector at the origin $O(1, 0, 0, 0)$

$$u = \dot{x}(0), \ v = \dot{y}(0), \ w = \dot{z}(0). \tag{2.7}$$

For a translation curve let its tangent vector at the point $(1, x(t), y(t), z(t))$ be defined by the matrix (2.3) with the following equation:

$$(0, u, v, w) \begin{pmatrix} 1 & x(t) & y(t) & z(t) \\ 0 & e^{-z(t)} & 0 & 0 \\ 0 & 0 & e^{z(t)} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = (0, \dot{x}(t), \dot{y}(t), \dot{z}(t)). \tag{2.8}$$

Thus, *translation curves* in **Sol** geometry (see [10] and [11]) are defined by the first order differential equation system $\dot{x}(t) = ue^{-z(t)}$, $\dot{y}(t) = ve^{z(t)}$, $\dot{z}(t) = w$, whose solution is the following:

$$\begin{gathered} x(t) = -\frac{u}{w}(e^{-wt} - 1), \ y(t) = \frac{v}{w}(e^{wt} - 1), \ z(t) = wt, \ \text{if } w \neq 0 \text{ and} \\ x(t) = ut, \ y(t) = vt, \ z(t) = z(0) = 0 \ \text{ if } w = 0. \end{gathered} \tag{2.9}$$

We assume that the starting point of a translation curve is the origin, because we can transform a curve into an arbitrary starting point by translation (2.3), moreover, unit velocity translation can be assumed :

$$\begin{gathered} x(0) = y(0) = z(0) = 0; \\ u = \dot{x}(0) = \cos\theta\cos\phi, \quad v = \dot{y}(0) = \cos\theta\sin\phi, \quad w = \dot{z}(0) = \sin\theta; \\ -\pi \leq \phi \leq \pi, \ -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}. \end{gathered} \tag{2.10}$$

Definition 2.1. The translation distance $d^t(P_1, P_2)$ between the points P_1 and P_2 is defined by the arc length of the above translation curve from P_1 to P_2.

Thus we obtain the parametric equation of the the *translation curve segment* $t(\phi, \theta, t)$ with starting point at the origin in direction

$$\mathbf{t}(\phi, \theta) = (\cos\theta\cos\phi, \cos\theta\sin\phi, \sin\theta) \tag{2.11}$$

where $t \in [0, r \in \mathbf{R}^+]$. If $\theta \neq 0$ then the system of equation is:

$$\begin{cases} x(\phi,\theta,t) = -\cot\theta\cos\phi(e^{-t\sin\theta}-1), \\ y(\phi,\theta,t) = \cot\theta\sin\phi(e^{t\sin\theta}-1), \\ z(\phi,\theta,t) = t\sin\theta. \end{cases} \tag{2.12}$$

$$\text{If } \theta = 0 \text{ then}: \; x(t) = t\cos\phi, \; y(t) = t\sin\phi, \; z(t) = 0.$$

3. Translation triangles

We consider 3 points A_1, A_2, A_3 in the projective model of **Sol** space (see Section 2). The *translation segments* a_k connecting the points A_i and A_j $(i < j,$ $i, j, k \in \{1, 2, 3\}$, $k \neq i, j)$ are called sides of the *translation triangle* with vertices A_1, A_2, A_3.

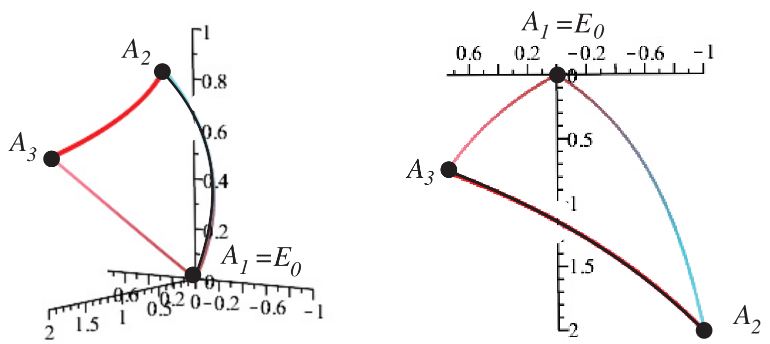

FIGURE 1. Translation triangle with vertices $A_1 = (1, 0, 0, 0)$, $A_2 = (1, -1, 2, 1)$, $A_3 = (1, 3/4, 3/4, 1/2)$.

In Riemannian geometries the metric tensor (or infinitesimal arc-length square (see (2.4)) is used to define the angle θ between two geodesic curves. If their tangent vectors in their common point are $\mathbf{u}$ and $\mathbf{v}$ and g_{ij} are the components of the metric tensor then

$$\cos(\theta) = \frac{u^i g_{ij} v^j}{\sqrt{u^i g_{ij} u^j \; v^i g_{ij} v^j}} \tag{3.1}$$

It is clear by the above definition of the angles and by the infinitesimal arc-length square (2.4), that the angles are the same as the Euclidean ones at the origin of the projective model of **Sol** geometry.

Considering a translation triangle $A_1A_2A_3$ we can assume by the homogeneity of the **Sol** geometry that one of its vertex coincide with the origin $A_1 = E_0 = (1, 0, 0, 0)$ and the other two vertices are $A_2(1, x^2, y^2, z^2)$ and $A_3(1, x^3, y^3, z^3)$.

We will consider the *interior angles* of translation triangles that are denoted at the vertex A_i by ω_i $(i \in \{1,2,3\})$. We note here that the angle of two intersecting translation curves depends on the orientation of their tangent vectors.

In order to determine the interior angles of a translation triangle $A_1A_2A_3$ and its interior angle sum $\sum_{i=1}^{3}(\omega_i)$, we define translations $\mathbf{T}_{A_i}$, $(i \in \{2,3\})$ as elements of the isometry group of **Sol**, *that maps the origin E_0 onto A_i* (see Fig. 2).

E.g. the isometry $\mathbf{T}_{A_2}$ and its inverse (up to a positive determinant factor) can be given by:

$$\mathbf{T}_{A_2} = \begin{pmatrix} 1 & x^2 & y^2 & z^2 \\ 0 & \mathrm{e}^{-z^2} & 0 & 0 \\ 0 & 0 & \mathrm{e}^{z^2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ \mathbf{T}_{A_2}^{-1} = \begin{pmatrix} 1 & -x^2\mathrm{e}^{z^2} & -y^2\mathrm{e}^{-z^2} & -z^2 \\ 0 & \mathrm{e}^{z^2} & 0 & 0 \\ 0 & 0 & \mathrm{e}^{-z^2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.2}$$

and the images $\mathbf{T}_{A_2}^{-1}(A_i)$ of the vertices A_i $(i \in \{1,2,3\})$ are the following (see also Fig. 2):

$$\begin{gathered} \mathbf{T}_{A_2}^{-1}(A_1) = A_1^2 = (1, -x^2\mathrm{e}^{z^2}, -y^2\mathrm{e}^{-z^2}, -z^2),\ \mathbf{T}_{A_2}^{-1}(A_2) = A_2^2 = E_0 = (1,0,0,0), \\ \mathbf{T}_{A_2}^{-1}(A_3) = A_3^2 = (1, (x^3-x^2)\mathrm{e}^{z^2}, (y^3-y^2)\mathrm{e}^{-z^2}, z^3-z^2). \end{gathered} \tag{3.3}$$

Similarly to the above computation we get that the images $\mathbf{T}_{A_3}^{-1}(A_i)$ of the vertices A_i $(i \in \{1,2,3\})$ are the following (see also Fig. 2):

$$\begin{gathered} \mathbf{T}_{A_3}^{-1}(A_1) = A_1^3 = (1, -x^3\mathrm{e}^{z^3}, -y^3\mathrm{e}^{-z^3}, -z^3),\ \mathbf{T}_{A_3}^{-1}(A_3) = A_2^2 = E_0 = (1,0,0,0), \\ \mathbf{T}_{A_3}^{-1}(A_2) = A_2^3 = (1, (x^2-x^3)\mathrm{e}^{z^3}, (y^2-y^3)\mathrm{e}^{-z^3}, z^2-z^3). \end{gathered} \tag{3.4}$$

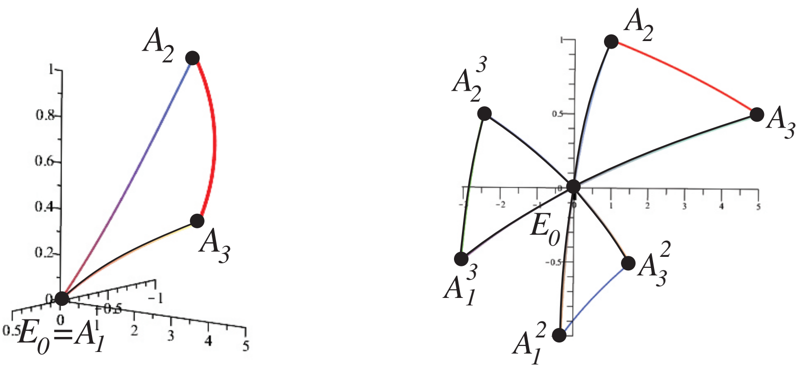

FIGURE 2. Translation triangle with vertices $A_1 = (1,0,0,0)$, $A_2 = (1,-1,1,1)$, $A_3 = (1,1/2,5,1/2)$ and its translated copies $A_1^2A_3^2E_0$ and $A_1^3A_2^3E_0$.

Our aim is to determine angle sum $\sum_{i=1}^{3}(\omega_i)$ of the interior angles of translation triangles $A_1A_2A_3$ (see Fig. 1-2). We have seen that ω_1 and the angle of translation curves with common point at the origin E_0 is the same as the Euclidean one therefore can be determined by usual Euclidean sense.

The translations $\mathbf{T}_{A_i}$ $(i = 2, 3)$ are isometries in **Sol** geometry thus ω_i is equal to the angle $(t(A^i_i, A^i_1)t(A^i_i, A^i_j))\angle$ $(i, j = 2, 3,\ i \neq j)$ (see Fig. 2) where $t(A^i_i, A^i_1)$, $t(A^i_i, A^i_j)$ are oriented translation curves ($E_0 = A^2_2 = A^3_3$) and ω_1 is equal to the angle $(t(E_0, A_2)t(E_0, A_3))\angle$ where $t(E_0, A_2)$, $t(E_0, A_3)$ are also oriented translation curves.

We denote the oriented unit tangent vectors of the oriented geodesic curves $t(E_0, A^j_i)$ with $\mathbf{t}^j_i$ where $(i, j) \in \{(1, 3), (1, 2), (2, 3), (3, 2), (3, 0), (2, 0)\}$ and $A^0_3 = A_3$, $A^0_2 = A_2$.

The Euclidean coordinates of $\mathbf{t}^j_i$ (see Section 2.1) are :

$$\mathbf{t}^j_i = (\cos(\theta^j_i)\cos(\alpha^j_i), \cos(\theta^j_i)\sin(\alpha^j_i), \sin(\theta^j_i)). \tag{3.5}$$

In order to obtain the angle of two translation curves $t_{E_0A^j_i}$ and $t_{E_0A^l_k}$ $((i, j) \neq (k, l);\ (i, j), (k, l) \in \{(1, 3), (1, 2), (2, 3), (3, 2), (3, 0), (2, 0)\})$ intersected at the origin E_0 we need to determine their tangent vectors $\mathbf{t}^r_s$ $((s, r) \in \{(1, 3), (1, 2), (2, 3), (3, 2), (3, 0), (2, 0)\})$ (see (3.5)) at their starting point E_0. From (3.5) follows that a tangent vector at the origin is given by the parameters ϕ and θ of the corresponding translation curve (see (2.12)) that can be determined from the homogeneous coordinates of the endpoint of the translation curve as the following Lemma shows:

Lemma 3.1. 1. Let $(1, x, y, z)$ $(y, z \in \mathbf{R} \setminus \{0\}, x \in \mathbf{R})$ be the homogeneous coordinates of the point $P \in$ **Sol**. The paramerters of the corresponding translation curve t_{E_0P} are the following

$$\begin{gathered}\phi = \operatorname{arccot}\Big(-\frac{x}{y}\frac{\mathrm{e}^z - 1}{\mathrm{e}^{-z} - 1}\Big),\ \theta = \operatorname{arccot}\Big(\frac{y}{\sin\phi(\mathrm{e}^z - 1)}\Big),\\ t = \frac{z}{\sin\theta},\ \text{where}\ -\pi < \phi \leq \pi,\ -\pi/2 \leq \theta \leq \pi/2,\ t \in \mathbf{R}^+.\end{gathered} \tag{3.6}$$

2. Let $(1, x, 0, z)$ $(x, z \in \mathbf{R} \setminus \{0\})$ be the homogeneous coordinates of the point $P \in$ **Sol**. The paramerters of the corresponding translation curve t_{E_0P} are the following

$$\begin{gathered}\phi = 0 \text{ or } \pi,\ \theta = \operatorname{arccot}\Big(\mp\frac{x}{(\mathrm{e}^{-z} - 1)}\Big),\\ t = \frac{z}{\sin\theta},\ \text{where}\ -\pi/2 \leq \theta \leq \pi/2,\ t \in \mathbf{R}^+.\end{gathered} \tag{3.7}$$

3. Let $(1, x, y, 0)$ $(x, y \in \mathbf{R})$ be the homogeneous coordinates of the point $P \in$ **Sol**. The paramerters of the corresponding translation curve t_{E_0P} are the following

$$\begin{gathered}\phi = \arccos\Big(\frac{x}{\sqrt{x^2 + y^2}}\Big),\ \theta = 0,\\ t = \sqrt{x^2 + y^2},\ \text{where}\ -\pi < \phi \leq \pi,\ t \in \mathbf{R}^+.\end{gathered} \tag{3.8}$$

Theorem 3.2. The sum of the interior angles of a translation triangle is greather or equal to π.

Proof. The translations $\mathbf{T}_{A_2}^{-1}$ and $\mathbf{T}_{A_3}^{-1}$ are isometries in **Sol** geometry thus ω_2 is equal to the angle $((A_2^2A_1^2),(A_2^2A_3^2))\angle$ (see Fig. 2) of the oriented translation segments $t_{A_2^2A_1^2}$, $t_{A_2^2A_3^2}$ and ω_3 is equal to the angle $((A_3^3A_1^3),(A_3^3A_2^3))\angle$ of the oriented translation segments $t_{A_3^3A_1^3}$ and $t_{A_3^3A_2^3}$ ($E_0=A_2^2=A_3^3$).

Substituting the coordinates of the points A_i^j (see (3.3) and (3.4)) $((i,j)\in\{(1,3),(1,2),\ (2,3),(3,2),(3,0),(2,0)\})$ to the appropriate equations of Lemma 3.1, it is easy to see that

$$\begin{aligned}\theta_2^0&=-\theta_1^2,\ \phi_2^0-\phi_1^2=\pm\pi\Rightarrow\mathbf{t}_2^0=-\mathbf{t}_1^2,\\ \theta_3^0&=-\theta_1^3,\ \phi_3^0-\phi_1^3=\pm\pi\Rightarrow\mathbf{t}_3^0=-\mathbf{t}_1^3,\\ \theta_3^2&=-\theta_2^3,\ \phi_3^2-\phi_2^3=\pm\pi\Rightarrow\mathbf{t}_3^2=-\mathbf{t}_2^3.\end{aligned}\tag{3.9}$$

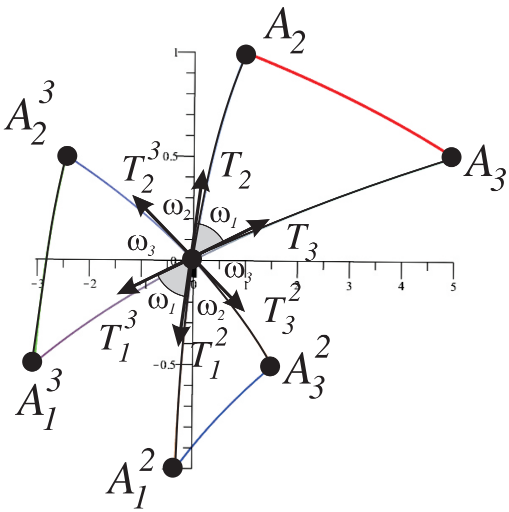

FIGURE 3. Translation triangle with vertices $A_1=(1,0,0,0)$, $A_2=(1,-1,1,1)$, $A_3=(1,1/2,5,1/2)$ and its translated copies $A_1^2A_3^2E_0$ and $A_1^3A_2^3E_0$.

The endpoints T_i^j of the position vectors $\mathbf{t}_i^j=\overrightarrow{E_0T_i^j}$ lie on the unit sphere centred at the origin. The measure of angle ω_i ($i\in\{1,2,3\}$) of the vectors $\mathbf{t}_i^j$ and $\mathbf{t}_r^s$ is equal to the spherical distance of the corresponding points T_i^j and T_r^s on the unit sphere (see Fig. 3). Moreover, a direct consequence of equations (3.9) that each point pair (T_2, T_1^2), (T_3,T_1^3), (T_2^3,T_3^2) contains antipodal points related to the unit sphere with centre E_0.

Due to the antipodality $\omega_1=T_2E_0T_3\angle=T_1^2E_0T_1^3\angle$, therefore their corresponding spherical distances are equal, as well (see Fig. 3). Now, the sum of the interior angles $\sum_{i=1}^3(\omega_i)$ can be considered as three consecutive spherical arcs $(T_3^2T_1^2)$, $(T_1^2T_1^3)$, $T_1^3T_2^3)$. Since the triangle inequality holds on the sphere, the sum of these arc lengths

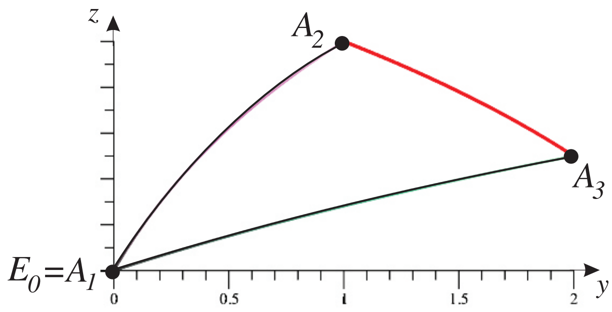

FIGURE 4. Translation triangle with vertices $A_1 = (1,0,0,0)$, $A_2 = (1,0,1,1)$, $A_3 = (1,0,2,1/2)$. The translation curve segments $t_{A_1A_2}$, $t_{A_2A_3}$, $t_{A_3A_1}$ lie on the coordinate plane $[y,z]$ and the interior angle sum of this translation triangle is $\sum_{i=1}^{3}(\omega_i) = \pi$.

is greater or equal to the half of the circumference of the main circle on the unit sphere i.e. π. □

The following lemma is an immediate consequence of the above proof:

Lemma 3.3. The angle sum $\sum_{i=1}^{3}(\omega_i)$ of a **Sol** translation triangle $A_1A_2A_3$ is π if and only if the points T_i^j $((i,j) \in \{(1,3),(1,2),\ (2,3),(3,2),(3,0),(2,0)\})$ lie in an Euclidean plane (Fig. 4).

Lemma 3.4. If the vertices of a translation triangle $A_1A_2A_3$ lie in a coordinate plane of the model of **Sol** geometry (see Section 2) or in a plane parallel to a coordinate plane then the interior angle sum $\sum_{i=1}^{3}(\omega_i) = \pi$.

Proof. We get from equation (2.12) of the translation curves that a point P lies in a coordinate plane then the corresponding translation curve t_{E_0P} also lies in the same coordinate plane.

Moreover, a direct consequence of formulas (2.3) and (2.6) than if a translation triangle $A_1A_2A_3$ lies in a coordinate plane α then its translated image by an orthogonal translation to α is in a to α parallel plane and each to α parallel plane can be derived as a tranlated copy of α. □

We can determine the interior angle sum of arbitrary translation triangle. In the following table we summarize some numerical data of interior angles of given transaltion triangles:

Table 1: $A_2(1,-1,1,1),\ A_3(1,1/2,5,z^3)$				
z^3	ω_1	ω_2	ω_3	$\sum_{i=1}^{3}(\omega_i)$
-10	1.378505	1.52957	0.39949	3.30757
-2	1.37467	1.45044	0.41389	3.23900
-1	1.36841	1.31743	0.48434	3.17018
1/100	1.35376	1.04468	0.74818	3.14661
1/10	1.35196	1.01850	0.77962	3.15008
1/2	1.34369	0.91985	0.90711	3.17066
3/4	1.33931	0.87828	0.96332	3.18092
3/2	1.34516	0.83131	0.98842	3.16489
2	1.37178	0.83021	0.94235	3.14433
5	1.46886	0.84547	0.86833	3.18265
10	1.47522	0.84678	0.86665	3.18866

Table 2: $A_2(1,-1,1,1),\ A_3(1,1/2,y^3,1/2)$				
y^3	ω_1	ω_2	ω_3	$\sum_{i=1}^{3}(\omega_i)$
-10	1.90559	0.77539	0.48862	3.16960
-2	1.99438	0.39617	0.86884	3.25939
-1	2.02152	0.38864	0.84198	3.25214
1/100	1.89224	0.42533	0.83598	3.15355
1/10	1.86415	0.43075	0.85319	3.14808
1/2	1.73149	0.45855	0.95244	3.14248
3/4	1.65752	0.47867	1.01153	3.14772
3/2	1.51011	0.54873	1.10619	3.16502
2	1.45565	0.60090	1.11440	3.17095
5	1.34369	0.91985	0.90711	3.17066
10	1.30564	1.27407	0.58095	3.16067

By the above investigation we can say that the **Sol** geometry (and the Thurston geometries) keep several interesting open questions (see e.g. [14], [16]). Detailed studies are the objective of ongoing research. Applications of the above projective method seem to be interesting in (non-Euclidean) crystallography as well.

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Optimal quadrature formulas for approximate solution of the first kind singular integral equation with Cauchy kernel

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Abstract. In the present paper in $L_2^{(m)}(-1,1)$ space the optimal quadrature formulas with derivatives are constructed for approximate solution of a singular integral equation of the first kind with Cauchy kernel. Approximate solution of the singular integral equation is obtained applying the optimal quadrature formulas. Explicit forms of coefficients for the of optimal quadrature formulas are obtained. Some numerical results are presented.

Mathematics Subject Classification (2010): 65D30, 65D32, 65R20.

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1. Introduction. Statement of the problem

The study of various problems of mathematical physics as well as specific problems from aerodynamics, electrodynamics, elasticity theory and other areas, naturally reduces to singular integral equations [5, 16]. In this case, the plane problems [5, 12, 16, 19] are reduced to solving the characteristic singular integral equation

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\varphi(x)}{x-t}\mathrm{d}x = f(t), \qquad t\in(-1,1), \tag{1.1}$$

where the singular integral is understood, here in after, in the sense of the Cauchy principal value. Equation (1.1) has four complete analytical solutions corresponding

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to the values of the parameter k (see [16],pp.49-50). In particular, for $k=-1$ the only solution of (1.1) is given by the formula

$$\varphi(t)=-\frac{\sqrt{1-t^2}}{\pi}\int_{-1}^{1}\frac{f(x)}{\sqrt{1-x^2}(x-t)}\mathtt{d}x. \tag{1.2}$$

Thus, the solution of singular integral equation of the form (1.1) can be reduced to the calculation of the weighted singular integral (1.2). Therefore, the development of effective approximate methods for calculating singular integrals are of great applied importance and one of the actual problems of computational mathematics.

Quadrature and cubature formulas are one of the methods for approximation of integrals. Many methods have been developed to construct the quadrature formulas for the singular integral (1.2). See for example, [1, 5, 6, 7, 8, 9, 12, 14, 15, 16, 19, 21, 25, 26] and literature cited therein.

Particularly, in the work [11] by Eshkuvatov, Nik Long and Abdulkawi, new quadrature formulas for evaluating the singular integral of Cauchy type with unbounded weight function on the edges is constructed. The construction of the quadrature formulas is based on the modification of the discrete vortices method and linear spline interpolation over the finite interval $[-1,1]$. It is proved that the constructed quadrature formulas converge for any singular point x not coinciding with the end points of the interval $[-1,1]$. Numerical results are given to validate the accuracy of the quadrature formulas. The error bounds are found to be of order $O(h^{\alpha}|\ln h|)$ and $O(h|\ln h|)$ in the classes of functions $H^{\alpha}([-1,1]), 0<\alpha<1$ and $C^1([-1,1])$, respectively.

In [11] the authors were used modification of the discrete vortices method and linear spline methods for approximation of the singular integrals. Constructed quadrature formulas are exact only for linear functions and these formulas are not an optimal approximation technique.

In the present paper, using the functional approach, we construct optimal quadrature formulas for approximate calculation of the integral (1.2) in the space $L_2^{(m)}(-1,1)$. We recall that $L_2^{(m)}(-1,1)$ is a Hilbert space of classes of all real functions φ defined in the interval $[-1,1]$ that differ by a polynomial of degree $(m-1)$ and square integrable with derivative of order m, and equipped with the norm

$$\|\varphi\|_{L_2^{(m)}}=\left(\int_{-1}^{1}\left(\varphi^{(m)}(x)\right)^2dx\right)^{\frac{1}{2}}.$$

It should be noted that, in particular, when $m=1$ from our numerical results we get the results of the work [11] close to each other.

We consider the following quadrature formula with derivatives

$$\int_{-1}^{1}\frac{\varphi(x)}{\sqrt{1-x^2}(x-t)}\mathtt{d}x\cong\sum_{\alpha=0}^{n}\sum_{\beta=0}^{N}C_{\alpha}[\beta]\varphi^{(\alpha)}(x_{\beta}),\quad -1<t<1, \tag{1.3}$$

in the Sobolev space $L_2^{(m)}(-1,1)$.

Here $C_\alpha[\beta]$ are the coefficients, x_β ($\in [-1,1]$) are the nodes of the quadrature formula, N is a natural number and $n = 0,1,2,...,(m-1)$.

The following difference is called *the error* of the quadrature formula (1.3):

$$(\ell,\varphi) = \int\limits_{-1}^{1} \frac{\varphi(x)}{\sqrt{1-x^2}(x-t)}\mathrm{d}x - \sum_{\alpha=0}^{n}\sum_{\beta=0}^{N} C_\alpha[\beta]\varphi^{(\alpha)}(x_\beta) = \int\limits_{-\infty}^{\infty} \ell(x)\varphi(x)\mathrm{d}x,$$

where ℓ is the error function of the formula (1.3) and has the form

$$\ell(x) = \frac{\varepsilon_{[-1,1]}(x)}{\sqrt{1-x^2}(x-t)} - \sum_{\alpha=0}^{n}\sum_{\beta=0}^{N}(-1)^\alpha C_\alpha[\beta]\delta^{(\alpha)}(x-x_\beta), \tag{1.4}$$

here $\varepsilon_{[-1,1]}(x)$ is the characteristic function of the interval $[-1,1]$, δ is the Dirac delta-function.

Since the functional ℓ of the form (1.4) is defined on the space $L_2^{(m)}(-1,1)$, it belongs to the conjugate space $L_2^{(m)*}(-1,1)$, and satisfies the following equations (see [27])

$$(\ell, x^\alpha) = 0, \quad \alpha = 0,1,2,...,(m-1).$$

The construction problem of optimal quadrature formulas of the form (1.3) in the sense of Sard [20] with the error functional (1.4) in the space $L_2^{(m)}(-1,1)$ for fixed x_β is to find the quantity

$$\|\mathring{\ell}|L_2^{(m)*}\|^2 = \inf_{C_\alpha[\beta]}(\ell,\psi_\ell),$$

where

$$\psi_\ell(x) \quad = \quad (-1)^m \ell(x) * G_m(x) + P_{m-1}(x),$$

here $G_m(x) = \frac{x^{2m-1}\mathtt{sgn}(x)}{2\cdot(2m-1)!}$, $P_{m-1}(x)$ is a polynomial of degree $(m-1)$, ψ_ℓ is the extremal function of the quadrature formula (1.3) in the space $L_2^{(m)}(-1,1)$ (see for instance, [4, 23, 24, 26]), $\mathtt{sgn}(x)$ is the signum function.

In the Hilbert spaces one can construct optimal quadrature formulas, optimal interpolation formulas, and splines using the Sobolev method which is based on using a discrete analogue of differential operator [27, 28]. Applying this method in the different Hilbert spaces optimal formulas and splines were constructed.

In the works [4, 2] for the norm of the error functional the following form was obtained

$$\begin{aligned}\|\ell|L_2^{(m)*}\|^2 &= (-1)^m\Bigg[\sum_{k=0}^{n}\sum_{\alpha=0}^{n}\sum_{\gamma=0}^{N}\sum_{\beta=0}^{N}(-1)^k \\ &\times C_k[\gamma]C_\alpha[\beta]\frac{(h\beta-h\gamma)^{2m-\alpha-k-1}\mathtt{sgn}(h\beta-h\gamma)}{2(2m-\alpha-k-1)!} \\ &-2\sum_{\alpha=0}^{n}\sum_{\beta=0}^{N}(-1)^\alpha C_\alpha[\beta]\int_{-1}^{1}\frac{(x-h\beta)^{2m-\alpha-1}\mathtt{sgn}(x-h\beta)}{2(2m-\alpha-1)!\sqrt{1-x^2}(x-t)}\mathtt{d}x \\ &+\int_{-1}^{1}\int_{-1}^{1}\frac{(x-y)^{2m-1}\mathtt{sgn}(x-y)}{2(2m-1)!\sqrt{1-x^2}\sqrt{1-y^2}(x-t)(y-t)}\mathtt{d}x\mathtt{d}y\Bigg]. \qquad (1.5)\end{aligned}$$

The rest of the paper is organized as follows. In Section 2 we give the algorithm for construction of optimal quadrature formulas of the form (1.3). Explicit formulas for coefficients of the optimal quadrature formulas of the form (1.3) are found for any natural m. In section 3 some numerical examples are provided to illustrate the validity of the algorithm.

2. The main results

Further, we suppose that $x_\beta = h\beta - 1, h = \frac{2}{N}$ and $N + 1 \geq m$.

The idea of construction of optimal quadrature formulas of the form (1.3) is as follows: First, for $m = 1$, we minimize the norm (1.5) by coefficients $C_0[\beta]$ in the space $L_2^{(1)}(0,1)$ and get the following system for finding the optimal coefficients $\mathring{C}_0[\beta]$:

$$\sum_{\gamma=0}^{N}\mathring{C}_0[\gamma]\frac{(h\beta-h\gamma)\mathtt{sgn}(h\beta-h\gamma)}{2}+\lambda_0=\int_{-1}^{1}\frac{(x-h\beta+1)\ \mathtt{sgn}(x-h\beta+1)}{2\sqrt{1-x^2}(x-t)}\mathtt{d}x,$$

$$\beta = 0, 1, ..., N,$$

$$\sum_{\gamma=0}^{N}\mathring{C}_0[\gamma]=\int_{-1}^{1}\frac{1}{\sqrt{1-x^2}(x-t)}\mathtt{d}x.$$

We note that the obtained system was solved in the work [15], i.e. there the optimal coefficients $\mathring{C}_0[\beta]$ were found in the space $L_2^{(1)}(-1,1)$.

Further, in the case $m = 2$, putting the optimal coefficients $\mathring{C}_0[\beta]$ to the expression (1.5) we minimize this norm by coefficients $C_1[\beta]$ in the space $L_2^{(2)}(-1,1)$ and find the optimal coefficients $\mathring{C}_1[\beta]$. Continuing by this manner for the cases $m = 3, 4, ..., k-1$, i.e. putting the obtained optimal coefficients $\mathring{C}_0[\beta], \mathring{C}_1[\beta], ..., \mathring{C}_{k-2}[\beta]$ to the expression of the norm (1.5) and minimizing this norm by coefficients $C_{k-1}[\beta]$ in the space $L_2^{(k)}(-1,1)$, we get the following system for finding the optimal coefficients $\mathring{C}_{k-1}[\beta]$:

$$\sum_{\gamma=0}^{N} \mathring{C}_{k-1}[\gamma] \frac{(h\beta - h\gamma)\mathtt{sgn}(h\beta - h\gamma)}{2} + (-1)^{k-1}(k-1)!\lambda_{k-1} = F_{k-1}[\beta], \tag{2.1}$$

$\beta = 0, 1, ..., N,$

$$\sum_{\gamma=0}^{N} \mathring{C}_{k-1}[\gamma] = \frac{g_{k-1}}{(k-1)!} - \sum_{i=0}^{k-2}\sum_{\gamma=0}^{N} \mathring{C}_i[\gamma] \frac{(h\gamma-1)^{k-i-1}}{(k-i-1)!}. \tag{2.2}$$

Here

$$F_{k-1}[\beta] = f_{k-1}[\beta] - \sum_{l=0}^{k-2}\sum_{\gamma=0}^{N} (-1)^{l+k-1} \mathring{C}_l[\gamma] \frac{(h\beta - h\gamma)^{k-l}\mathtt{sgn}(h\beta - h\gamma)}{2(k-l)!}, \tag{2.3}$$

where

$$\begin{aligned} f_{k-1}[\beta] &= \int_{-1}^{1} \frac{(x-h\beta+1)^k \mathtt{sgn}(x-h\beta+1)}{2\cdot k!\sqrt{1-x^2}(x-t)} \mathrm{d}x \\ &= -\frac{1}{k!}\left[\sum_{i=1}^{k} \binom{k}{i} (t-h\beta+1)^{k-i}(A_1+A_2) - \frac{(t-h\beta+1)^k}{\sqrt{1-t^2}} A_3\right], \end{aligned}$$

$$\begin{aligned} A_1 &= \sum_{j=1}^{\left[\frac{i-1}{2}\right]} \binom{i-1}{2j} (-t)^{i-2j-1} \left(-\frac{\sqrt{1-(h\beta-1)^2}}{2j} \left[(h\beta-1)^{2j-1} \right.\right. \\ &\quad \left. + \sum_{l=1}^{j-1} \frac{(2j-1)(2j-3)...(2j-2l+1)}{2^l (j-1)(j-2)...(j-l)} (h\beta-1)^{2j-2l-1} \right] \\ &\quad \left. + \frac{(2j-1)!!}{2^j j!} \arcsin(h\beta-1) \right), \\ A_2 &= \sum_{j=0}^{\left[\frac{i-2}{2}\right]} \binom{i-1}{2j+1} \sum_{l=0}^{j} \frac{(-t)^{i-2j-2}(-1)^{l+1}}{(2l+1)} \binom{j}{l} \left(\sqrt{1-(h\beta-1)^2}\right)^{2l+1} \\ &\quad + (-t)^{i-1} \arcsin(h\beta-1), \\ A_3 &= \ln\left| \frac{1-t(h\beta-1)+\sqrt{(1-t^2)(1-(h\beta-1)^2)}}{h\beta-1-t} \right|, \end{aligned}$$

$$g_{k-1} = \int_{-1}^{1} \frac{x^{k-1}}{\sqrt{1-x^2}(x-t)} \mathtt{d}x \tag{2.4}$$
$$= \pi \sum_{i=1}^{k-1} \binom{k-1}{i} t^{k-1-i} \left(\sum_{j=1}^{\left[\frac{i-1}{2}\right]} \binom{i-1}{2j} (-t)^{i-1-j} \frac{(2j-1)!!}{2^j j!} + (-t)^{i-1} \right).$$

Now we solve the system (2.1)-(2.2). The solution of the system (2.1)-(2.2) we find by the following way.
We denote

$$u(h\beta) = \sum_{\gamma=0}^{N} \mathring{C}_{k-1}[\beta] \frac{(h\beta - h\gamma)\mathtt{sgn}(h\beta - h\gamma)}{2} + (-1)^{k-1}(k-1)!\lambda_{k-1}. \tag{2.5}$$

Assume $\beta \le 0$, then from (2.5) we have

$$u(h\beta) = -\frac{h\beta}{2}\left(\frac{g_{k-1}}{(k-1)!} - \nu_{k-2}\right) - \mu_{k-1} + (-1)^{k-1}(k-1)!\lambda_{k-1},$$

where

$$\nu_{k-2} = \sum_{i=0}^{k-2}\sum_{\gamma=0}^{N} C_i[\gamma]\frac{(h\gamma)^{k-i-1}}{(k-i-1)!}, \quad \mu_{k-1} = -\frac{1}{2}\sum_{\gamma=0}^{N} C_{k-1}[\gamma](h\gamma).$$

Suppose $\beta \ge N$, then taking into account (2.5), we get

$$u(h\beta) = \frac{h\beta}{2}\left(\frac{g_{k-1}}{(k-1)!} - \nu_{k-2}\right) + \mu_{k-1} + (-1)^{k-1}(k-1)!\lambda_{k-1}.$$

We introduce the following denotations

$a^-_{k-1} = \mu_{k-1} - (k-1)!(-1)^{k-1}\lambda_{k-1}$ and $a^+_{k-1} = \mu_{k-1} + (k-1)!(-1)^{k-1}\lambda_{k-1}$.

Then we obtain that

$$u(h\beta) = \begin{cases} -\frac{h\beta}{2}\left(\frac{g_{k-1}}{(k-1)!} - \nu_{k-2}\right) - a^-_{k-1}, & \beta \le 0, \\ F_{k-1}[\beta], & 0 \le \beta \le N, \\ \frac{h\beta}{2}\left(\frac{g_{k-1}}{(k-1)!} - \nu_{k-2}\right) + a^+_{k-1}, & \beta \ge N, \end{cases} \tag{2.6}$$

where a^-_{k-1} and a^+_{k-1} are unknowns.
Hence, taking into account the values of the function $u(h\beta)$ at the points $\beta = 0$ and $\beta = N$, we get

$$a^-_{k-1} = F_{k-1}[0], \quad a^+_{k-1} = F_{k-1}[N] - \frac{1}{2}\left(\frac{g_{k-1}}{(k-1)!} - \nu_{k-2}\right).$$

Now, using the following known equality from [24]

$$h\sum_{\gamma=-\infty}^{\infty} D_1[\gamma]\frac{(h\beta - h\gamma)\mathtt{sgn}(h\beta - h\gamma)}{2} = \delta[\beta], \tag{2.7}$$

where in [22]

$$D_1[\beta] = \begin{cases} 0, & |\beta| \geq 2, \\ h^{-2}, & |\beta| = 1, \\ -2h^{-2}, & \beta = 0, \end{cases} \tag{2.8}$$

$\delta[\beta] = \begin{cases} 1, & \beta = 0, \\ 0, & \beta \neq 0, \end{cases}$ taking account of (2.6) and (2.7), for the optimal coefficients $\mathring{C}_{k-1}[\beta]$, when $0 \leq \beta \leq N$, we get the following

$$\begin{aligned} \mathring{C}_{k-1}[\beta] &= h \sum_{\gamma=-\infty}^{\infty} D_1[\beta-\gamma]u(h\gamma) = h\Bigg[\sum_{\gamma=0}^{N} D_1[\beta-\gamma]F_{k-1}[\gamma] \\ &+ \sum_{\gamma=1}^{\infty} D_1[\beta+\gamma]\left(\frac{h\gamma}{2}\left(\frac{g_{k-1}}{(k-1)!}-\nu_{k-2}\right)-a_{k-1}^{-}\right) \\ &+ \sum_{\gamma=1}^{\infty} D_1[N+\gamma-\beta]\left(\frac{1+h\gamma}{2}\left(\frac{g_{k-1}}{(k-1)!}-\nu_{k-2}\right)+a_{k-1}^{+}\right)\Bigg]. \end{aligned}$$

Hence, by virtue of (2.8), we have the following.

Theorem 2.1. *The coefficients for the optimal quadrature formulas of the form (1.3) in the Sobolev space $L_2^{(m)}(-1,1)$ are defined as follows*

$$\mathring{C}_{k-1}[0] = h^{-1}\left[F_{k-1}[1] - F_{k-1}[0] + \frac{h}{2}\left(\frac{g_{k-1}}{(k-1)!} - \nu_{k-2}\right)\right], \tag{2.9}$$

$$\mathring{C}_{k-1}[\beta] = h^{-1}\Big[F_{k-1}[\beta-1] - 2F_{k-1}[\beta] + F_{k-1}[\beta+1]\Big], \tag{2.10}$$

$$\text{for } \beta = 1, ..., N-1$$

$$\mathring{C}_{k-1}[N] = h^{-1}\left[F_{k-1}[N-1] - F_{k-1}[N] + \frac{h}{2}\left(\frac{g_{k-1}}{(k-1)!} - \nu_{k-2}\right)\right], \tag{2.11}$$

$k = 0, 1, 2, ..., m-1,$ *where for* $t \neq h\gamma - 1$, $\gamma = 0, 1, 2, ..., N$,

$$F_{k-1}[\beta] = f_{k-1}[\beta] - \sum_{l=0}^{k-2}\sum_{\gamma=0}^{N}(-1)^{l+k-1}\mathring{C}_l[\gamma]\frac{(h\beta-h\gamma)^{k-l}\,\mathrm{sgn}(h\beta-h\gamma)}{2(k-l)!},$$

$$f_{k-1}[\beta] = -\frac{1}{k!}\left[\sum_{i=1}^{k}\binom{k}{i}(t-h\beta+1)^{k-i}(A_1+A_2) - \frac{(t-h\beta+1)^k}{\sqrt{1-t^2}}A_3\right],$$

and for $t = h\gamma - 1$, $\gamma = 0, 1, 2, ..., N$,

$$F_{k-1}[\beta] = \overline{f}_{k-1}[\beta] - \sum_{l=0}^{k-2}\sum_{\gamma=0}^{N}(-1)^{l+k-1}\mathring{C}_l[\gamma]\frac{(h\beta-h\gamma)^{k-l}\,\mathrm{sgn}(h\beta-h\gamma)}{2(k-l)!},$$

$$\overline{f}_{k-1}[\beta] = -\frac{1}{k!}\left[\sum_{i=1}^{k}\binom{k}{i}(t-h\beta+1)^{k-i}(A_1+A_2)\right],$$

here

$$
\begin{aligned}
A_1 &= \sum_{j=1}^{\left[\frac{i-1}{2}\right]} \begin{pmatrix} i-1 \\ 2j \end{pmatrix} (-t)^{i-2j-1} \Bigg(-\frac{\sqrt{1-(h\beta-1)^2}}{2j} \Bigg[(h\beta-1)^{2j-1} \\
&\quad + \sum_{l=1}^{j-1} \frac{(2j-1)(2j-3)...(2j-2l+1)}{2^l (j-1)(j-2)...(j-l)} (h\beta-1)^{2j-2l-1} \Bigg] \\
&\quad + \frac{(2j-1)!!}{2^j j!} \arcsin(h\beta-1) \Bigg), \\
A_2 &= \sum_{j=0}^{\left[\frac{i-2}{2}\right]} \begin{pmatrix} i-1 \\ 2j+1 \end{pmatrix} \sum_{l=0}^{j} \frac{(-t)^{i-2j-2}(-1)^{l+1}}{(2l+1)} \begin{pmatrix} j \\ l \end{pmatrix} \left(\sqrt{1-(h\beta-1)^2}\right)^{2l+1} \\
&\quad + (-t)^{i-1} \arcsin(h\beta-1), \\
A_3 &= \ln \left| \frac{1-t(h\beta-1)+\sqrt{(1-t^2)(1-(h\beta-1)^2)}}{h\beta-1-t} \right|,
\end{aligned}
$$

$$
g_{k-1} = \pi \sum_{i=1}^{k-1} \begin{pmatrix} k-1 \\ i \end{pmatrix} t^{k-1-i} \left(\sum_{j=1}^{\left[\frac{i-1}{2}\right]} \begin{pmatrix} i-1 \\ 2j \end{pmatrix} (-t)^{i-1-j} \frac{(2j-1)!!}{2^j j!} + (-t)^{i-1} \right).
$$

From Theorem 2.1 in particular, when $m = 1$, $m = 2$ and $m = 3$. We have the following.

For the case $m = 1$.

Corollary 2.2. *For $t \neq h\gamma - 1$, coefficients of the optimal quadrature formula (1.3) with equally spaced nodes in the space $L_2^{(1)}(-1,1)$ have the following form*

$$
\begin{aligned}
\mathring{C}_0[0] &= h^{-1} \left(F_0[1] - \frac{\pi}{2} \right), \\
\mathring{C}_0[\beta] &= h^{-1} \left(F_0[\beta-1] - 2F_0[\beta] + F_0[\beta+1] \right), \quad \beta = 1,2,...,N-1, \\
\mathring{C}_0[N] &= h^{-1} \left(F_0[N-1] + \frac{\pi}{2} \right),
\end{aligned}
$$

where

$$
\begin{aligned}
F_0[\beta] &= -\arcsin(h\beta-1) \\
&\quad + \frac{t-(h\beta-1)}{\sqrt{1-t^2}} \ln \left| \frac{1-t(h\beta-1)+\sqrt{(1-t^2)(1-(h\beta-1)^2)}}{h\beta-1-t} \right|.
\end{aligned}
$$

Corollary 2.3. *For $t = h\gamma - 1$, coefficients of the optimal quadrature formula (1.3) with equally spaced nodes in the space $L_2^{(1)}(-1,1)$ have the following form* *if $\gamma=1$, i.e. for $t=h-1$:*

$$\begin{aligned}\mathring{C}_0[0] &= h^{-1}\left(\overline{F}_0[1]-\frac{\pi}{2}\right),\\ \mathring{C}_0[1] &= h^{-1}\left(F_0[0]-2\overline{F}_0[1]+F_0[2]\right),\\ \mathring{C}_0[2] &= h^{-1}\left(\overline{F}_0[1]-2F_0[2]+F_0[2]\right),\\ \mathring{C}_0[\beta] &= h^{-1}\left(F_0[\beta-1]-2F_0[\beta]+F_0[\beta+1]\right),\quad \beta=3,4,...,N-1,\\ \mathring{C}[N] &= h^{-1}\left(F_0[N-1]+\frac{\pi}{2}\right),\end{aligned}$$

if $\gamma=2,3,4,...,N-2$:

$$\begin{aligned}\mathring{C}_0[0] &= h^{-1}\left(F_0[1]-\frac{\pi}{2}\right),\\ \mathring{C}_0[\beta] &= h^{-1}\left(F_0[\beta-1]-2F_0[\beta]+F_0[\beta+1]\right),\\ &\quad \beta=1,2,...,\gamma-2 \quad \text{and} \quad \beta=\gamma+2,\gamma+3,...,N-1,\\ \mathring{C}_0[\gamma-1] &= h^{-1}\left(F_0[\gamma-2]-2F_0[\gamma-1]+\overline{F}_0[\gamma]\right),\\ \mathring{C}_0[\gamma] &= h^{-1}\left(F_0[\gamma-1]-2\overline{F}_0[\gamma]+F_0[\gamma+1]\right),\\ \mathring{C}_0[\gamma+1] &= h^{-1}\left(\overline{F}_0[\gamma]-2F_0[\gamma+1]+F_0[\gamma+2]\right),\\ \mathring{C}_0[N] &= h^{-1}\left(F_0[N-1]+\frac{\pi}{2}\right),\end{aligned}$$

if $\gamma=N-1$, i.e. for $t=1-h$:

$$\begin{aligned}\mathring{C}_0[0] &= h^{-1}\left(F_0[1]-\frac{\pi}{2}\right),\\ \mathring{C}_0[\beta] &= h^{-1}\left(F_0[\beta-1]-2F_0[\beta]+F_0[\beta+1]\right),\quad \beta=1,2,...,N-3,\\ \mathring{C}_0[N-2] &= h^{-1}\left(F_0[N-3]-2F_0[N-2]+\overline{F}_0[N-1]\right),\\ \mathring{C}_0[N-1] &= h^{-1}\left(F_0[N-2]-2\overline{F}_0[N-1]+F_0[N]\right),\\ \mathring{C}_0[N] &= h^{-1}\left(\overline{F}_0[N-1]+\frac{\pi}{2}\right),\end{aligned}$$

where $\overline{F}_0[\beta]=-\arcsin(h\beta-1)$, $F_0[\beta]$ is given in Corollary 2.2

The case $m=2$. In this case we have the following result of the work [2] as immediate corollary of Theorem 2.1.

Corollary 2.4. *For $t\neq h\gamma-1$, coefficients of the optimal quadrature formula (1.3) with equally spaced nodes in the space $L_2^{(2)}(-1,1)$ take the form*

$$\begin{aligned}\mathring{C}_1[0] &= h^{-1}\left(F_1[1]-F_1[0]+\frac{h}{2}\left(\pi-\sum_{\gamma=0}^{N}\mathring{C}_0[\gamma](h\gamma)\right)\right),\\ \mathring{C}_1[\beta] &= h^{-1}\Big(F_1[\beta-1]-2F_1[\beta]+F_1[\beta+1]\Big),\quad \beta=\overline{1,N-1}\\ \mathring{C}_1[N] &= h^{-1}\left(F_1[N-1]-F_1[N]+\frac{h}{2}\left(\pi-\sum_{\gamma=0}^{N}\mathring{C}_0[\gamma](h\gamma)\right)\right),\end{aligned}$$

where

$$F_1[\beta] \;=\; f_1[\beta]+\frac{h^2}{4}\sum_{\gamma=0}^{N}\mathring{C}_0[\gamma](h\beta-h\gamma)^2\, \mathbf{sgn}(h\beta-h\gamma),$$

$$\begin{aligned} f_1[\beta] \;=\; & -\frac{1}{2}\Bigg[-\sqrt{1-(h\beta-1)^2}+\Big(t-2h\beta+2\Big)\arcsin(h\beta-1) \\ & -\frac{(t-(h\beta-1))^2}{\sqrt{1-t^2}}\ln\left|\frac{1-t(h\beta-1)+\sqrt{(1-t^2)(1-(h\beta-1)^2)}}{h\beta-1-t}\right|\Bigg] \end{aligned}$$

and $\mathring{C}_0[\beta],\quad \beta=0,1,2,...,N$ *are defined in Corollary* 2.2.

Corollary 2.5. *For* $t=h\gamma-1$, *coefficients of the optimal quadrature formula* (1.3) *with equally spaced nodes in the space* $L_2^{(2)}(-1,1)$ *take the form*
if $\gamma=1$, *i.e. for* $t=h-1$*:*

$$\begin{aligned} \mathring{C}_1[0] &= h^{-1}\left(\overline{F}_1[1]-F_1[0]+\frac{h}{2}\left(\pi-\sum_{\gamma=0}^{N}\mathring{C}_0[\gamma](h\gamma)\right)\right), \\ \mathring{C}_1[1] &= h^{-1}\left(F_1[0]-2\overline{F}_1[1]+F_1[2]\right), \\ \mathring{C}_1[2] &= h^{-1}\left(\overline{F}_1[1]-2F_1[2]+F_1[2]\right), \\ \mathring{C}_1[\beta] &= h^{-1}\left(F_1[\beta-1]-2F_1[\beta]+F_1[\beta+1]\right), \quad \beta=3,4,...,N-1, \\ \mathring{C}_1[N] &= h^{-1}\left(F_1[N-1]-F_1[N]+\frac{h}{2}\left(\pi-\sum_{\gamma=0}^{N}\mathring{C}_0[\gamma](h\gamma)\right)\right), \end{aligned}$$

if $\gamma=2,3,4,...,N-2$*:*

$$\begin{aligned} \mathring{C}_1[0] &= h^{-1}\left(F_1[1]-F_1[0]+\frac{h}{2}\left(\pi-\sum_{\gamma=0}^{N}\mathring{C}_0[\gamma](h\gamma)\right)\right), \\ \mathring{C}_1[\beta] &= h^{-1}\left(F_1[\beta-1]-2F_1[\beta]+F_1[\beta+1]\right), \\ & \quad \beta=1,2,...,\gamma-2 \;\; \textit{and} \;\; \beta=\gamma+2,\gamma+3,...,N-1, \\ \mathring{C}_1[\gamma-1] &= h^{-1}\left(F_1[\gamma-2]-2F_1[\gamma-1]+\overline{F}_1[\gamma]\right), \\ \mathring{C}_1[\gamma] &= h^{-1}\left(F_1[\gamma-1]-2\overline{F}_1[\gamma]+F_1[\gamma+1]\right), \\ \mathring{C}_1[\gamma+1] &= h^{-1}\left(\overline{F}_1[\gamma]-2F_1[\gamma+1]+F_1[\gamma+2]\right), \\ \mathring{C}_1[N] &= h^{-1}\left(F_1[N-1]-F_1[N]+\frac{h}{2}\left(\pi-\sum_{\gamma=0}^{N}\mathring{C}_0[\gamma](h\gamma)\right)\right), \end{aligned}$$

if $\gamma = N-1$, *i.e. for* $t = 1-h$:

$$\begin{aligned}
\mathring{C}_1[0] &= h^{-1}\left(F_1[1]-F_1[0]+\frac{h}{2}\left(\pi-\sum_{\gamma=0}^{N}\mathring{C}_0[\gamma](h\gamma)\right)\right),\\
\mathring{C}_1[\beta] &= h^{-1}\left(F_1[\beta-1]-2F_1[\beta]+F_1[\beta+1]\right), \quad \beta=1,2,...,N-3,\\
\mathring{C}_1[N-2] &= h^{-1}\left(F_1[N-3]-2F_1[N-2]+\overline{F}_1[N-1]\right),\\
\mathring{C}_1[N-1] &= h^{-1}\left(F_1[N-2]-2\overline{F}_1[N-1]+F_1[N]\right),\\
\mathring{C}_1[N] &= h^{-1}\left(\overline{F}_1[N-1]-F_1[N]+\frac{h}{2}\left(\pi-\sum_{\gamma=0}^{N}\mathring{C}_0[\gamma](h\gamma)\right)\right),
\end{aligned}$$

where

$$\begin{aligned}
\overline{F}_1[\beta] &= -\frac{1}{2}\left[-\sqrt{1-(h\beta-1)^2}+\left(t-2h\beta+2\right)\arcsin(h\beta-1)\right]\\
&\quad+\frac{h^2}{4}\sum_{\gamma=0}^{N}\mathring{C}_0(h\beta-h\gamma)^2\,\mathrm{sgn}(h\beta-h\gamma).
\end{aligned}$$

$\mathring{C}_0[\beta]$, $F_1[\beta]$, $\beta = 0,1,2,...,N$ *are given in Corollaries* 2.3 *and* 2.4.

In the case $m=3$ we have the following results of the work [3] as immediate corollary of Theorem 2.1.

Corollary 2.6. *For* $t \neq h\gamma - 1$, *coefficients of the optimal quadrature formula (*1.3*) with equally spaced nodes in the space* $L_2^{(3)}(-1,1)$ *have the following form*

$$\begin{aligned}
\mathring{C}_2[0] &= h^{-1}\Big(F_2[1]-F_2[0]\\
&\quad+\frac{h}{4}\left(\pi t-\sum_{\gamma=0}^{N}\left(\mathring{C}_0[\gamma](h\gamma-1)^2+2\mathring{C}_1[\gamma](h\gamma-1)\right)\right)\Big),\\
\mathring{C}_2[\beta] &= h^{-1}\Big(F_2[\beta-1]-2F_2[\beta]+F_2[\beta+1]\Big), \quad \beta=\overline{1,N-1},\\
\mathring{C}_2[N] &= h^{-1}\Big(F_2[N-1]-F_2[N]\\
&\quad+\frac{h}{4}\left(\pi t-\sum_{\gamma=0}^{N}\left(\mathring{C}_0[\gamma](h\gamma-1)^2+2\mathring{C}_1[\gamma](h\gamma-1)\right)\right)\Big),
\end{aligned}$$

where

$$\begin{aligned}
F_2[\beta] &= f_2[\beta]-\frac{h^3}{12}\sum_{\gamma=0}^{N}\mathring{C}_0[\gamma](h\beta-h\gamma)^3\,\mathrm{sgn}(h\beta-h\gamma)\\
&\quad+\frac{h^2}{4}\sum_{\gamma=0}^{N}\mathring{C}_1[\gamma](h\beta-h\gamma)^2\,\mathrm{sgn}(h\beta-h\gamma),
\end{aligned}$$

$$
\begin{aligned}
f_2[\beta] &= \frac{1}{12}\Bigg(\Big(2t-5(h\beta-1)\Big)\sqrt{1-(h\beta-1)^2}\\
&-\Big(1+2t^2-6t(h\beta-1)+6(h\beta-1)^2\Big)\arcsin(h\beta-1)\\
&-\frac{2(t-(h\beta-1))^3}{\sqrt{1-t^2}}\ln\left|\frac{1-t(h\beta-1)+\sqrt{(1-t^2)(1-(h\beta-1)^2)}}{h\beta-1-t}\right|\Bigg).
\end{aligned}
$$

and $\mathring{C}_0[\beta], \mathring{C}_1[\beta], \quad \beta=\overline{0,N}$ *are given in Corollaries* 2.2 *and* 2.4.

Corollary 2.7. *For* $t=h\gamma-1$*, coefficients of the optimal quadrature formula* (1.3) *with equally spaced nodes in the space* $L_2^{(3)}(-1,1)$ *have the following form when* $\gamma=1$*:*

$$
\begin{aligned}
\mathring{C}_2[0] &= h^{-1}\Big(\overline{F}_2[1]-F_2[0]\\
&+\frac{h}{4}\left(\pi t-\sum_{\gamma=0}^{N}\left(\mathring{C}_0[\gamma](h\gamma-1)^2+2\mathring{C}_1[\gamma](h\gamma-1)\right)\right)\Big),\\
\mathring{C}_2[1] &= h^{-1}\left(F_2[0]-2\overline{F}_2[1]+F_2[2]\right),\\
\mathring{C}_2[2] &= h^{-1}\left(\overline{F}_2[1]-2F_2[2]+F_2[2]\right),\\
\mathring{C}_2[\beta] &= h^{-1}\left(F_2[\beta-1]-2F_2[\beta]+F_2[\beta+1]\right),\quad \beta=3,4,...,N-1,\\
\mathring{C}_2[N] &= h^{-1}\Big(F_2[N-1]-F_2[N]\\
&+\frac{h}{4}\left(\pi t-\sum_{\gamma=0}^{N}\left(\mathring{C}_0[\gamma](h\gamma-1)^2+2\mathring{C}_1[\gamma](h\gamma-1)\right)\right)\Big),
\end{aligned}
$$

when $\gamma=2,3,4,...,N-2$*:*

$$
\begin{aligned}
\mathring{C}_2[0] &= h^{-1}\Big(F_2[1]-F_2[0]\\
&+\frac{h}{4}\left(\pi t-\sum_{\gamma=0}^{N}\left(\mathring{C}_0[\gamma](h\gamma-1)^2+2\mathring{C}_1[\gamma](h\gamma-1)\right)\right)\Big),\\
\mathring{C}_2[\beta] &= h^{-1}\left(F_2[\beta-1]-2F_2[\beta]+F_2[\beta+1]\right),\quad \beta=1,2,...,\gamma-2,\\
\mathring{C}_2[\gamma-1] &= h^{-1}\left(F_2[\gamma-2]-2F_2[\gamma-1]+\overline{F}_2[\gamma]\right),\\
\mathring{C}_2[\gamma] &= h^{-1}\left(F_2[\gamma-1]-2\overline{F}_2[\gamma]+F_2[\gamma+1]\right),\\
\mathring{C}_2[\gamma+1] &= h^{-1}\left(\overline{F}_2[\gamma]-2F_2[\gamma+1]+F_2[\gamma+2]\right),\\
\mathring{C}_2[\beta] &= h^{-1}\left(F_2[\beta-1]-2F_2[\beta]+F_2[\beta+1]\right),\quad \beta=\gamma+2,\gamma+3,...,N-1,\\
\mathring{C}_2[N] &= h^{-1}\Big(F_2[N-1]-F_2[N]\\
&+\frac{h}{4}\left(\pi t-\sum_{\gamma=0}^{N}\left(\mathring{C}_0[\gamma](h\gamma-1)^2+2\mathring{C}_1[\gamma](h\gamma-1)\right)\right)\Big),
\end{aligned}
$$

when $\gamma = N-1$*:*

$$\begin{aligned}
\mathring{C}_2[0] &= h^{-1}\left(F_2[1]-F_2[0]\right.\\
&\quad\left.+\frac{h}{4}\left(\pi t-\sum_{\gamma=0}^{N}\left(\mathring{C}_0[\gamma](h\gamma-1)^2+2\mathring{C}_1[\gamma](h\gamma-1)\right)\right)\right),\\
\mathring{C}_2[\beta] &= h^{-1}\left(F_2[\beta-1]-2F_2[\beta]+F_2[\beta+1]\right),\quad \beta=1,2,...,N-3,\\
\mathring{C}_2[N-2] &= h^{-1}\left(F_2[N-3]-2F_2[N-2]+\overline{F}_2[N-1]\right),\\
\mathring{C}_2[N-1] &= h^{-1}\left(F_2[N-2]-2\overline{F}_2[N-1]+F_2[N]\right),\\
\mathring{C}_2[N] &= h^{-1}\left(\overline{F}_2[N-1]-F_2[N]\right.\\
&\quad\left.+\frac{h}{4}\left(\pi t-\sum_{\gamma=0}^{N}\left(\mathring{C}_0[\gamma](h\gamma-1)^2+2\mathring{C}_1[\gamma](h\gamma-1)\right)\right)\right),
\end{aligned}$$

where

$$\begin{aligned}
\overline{F}_2[\beta] &= \overline{f}_2[\beta]-\frac{h^3}{12}\sum_{\gamma=0}^{N}\mathring{C}_0[\gamma](h\beta-h\gamma)^3\,\mathbf{sgn}(h\beta-h\gamma)\\
&\quad+\frac{h^2}{4}\sum_{\gamma=0}^{N}\mathring{C}_1[\gamma](h\beta-h\gamma)^2\,\mathbf{sgn}(h\beta-h\gamma),
\end{aligned}$$

$$\begin{aligned}
\overline{f}_2[\beta] &= \frac{1}{12}\left(\left(2t-5(h\beta-1)\right)\sqrt{1-(h\beta-1)^2}\right.\\
&\quad\left.-\left(1+2t^2-6t(h\beta-1)+6(h\beta-1)^2\right)\arcsin(h\beta-1)\right).
\end{aligned}$$

$\mathring{C}_0[\beta], \mathring{C}_1[\beta]$, $F_2[\beta]$, $\beta = 0,1,2,...,N$ *are given in Corollaries* 2.3*,* 2.5 *and* 2.6

3. Numerical results

In this section we give some numerical results in order to show numerically convergence of the optimal quadrature formulas (1.3), with coefficients given in Theorem 2.1, in dependence on the values of N and m. Furthermore, here we compare numerical results of the quadrature formulas (1.3) with numerical results of the quadrature formula constructed in [11] in the space $L_2^{(1)}(-1,1)$.

Let us consider (1.2) and $f(t) = t^5 + t^3 + 20t$. The corresponding exact solution of (1.1) is

$$\varphi(x) = \sqrt{1-x^2}\left(x^4 + 1.5x^2 + \frac{167}{8}\right).$$

Tables 1-8 compare the exact solutions of singular integral equation in the form (1.1), the error rates of approximate solutions of quadrature formulas (16), (20), (21), (22) of the work [11], with the proposed (1.3), in which the approximate solutions of optimal quadrature formulas when $m = 1$.

These tables show that the proposed method, when $m = 1$, outperforms the results of [11], four quadrature formulas proposed in two way approaches. Our proposed theorem is applicable for arbitrary m and N. This means that the proposed optimal quadrature formulas are exact for any polynomial of degree $(m - 1)$. The error rates shown in Table 9 show that the proposed method, when $N = 20$, $m = 1$, $m = 2$, and $m = 3$ in singular integral equations, confirms the previous statement. The combination of the results illustrated in Table 9 and constructed optimal quadrature formulas by increasing N and m, allows the approximate calculations of the Fredholm singular integral equation of the first kind with high accuracy.

4. Conclusion

In the present paper, in the Sobolev space $L_2^{(m)}(-1, 1)$ we constructed the optimal quadrature formula for approximate solution of singular integral equations with Cauchy kernel. Here we found analytical forms for coefficients of the constructed optimal quadrature formulas. We applied these coefficients to approximate solution of the Fredholm singular integral equation of the first kind. We showed that singular integral equations can be solved with higher accuracy using the optimal quadrature formulas which are constructed based on Sobolev method.

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Table 1. Error terms for OQF (1.3).

N=20				
$t \neq h\gamma - 1$	Exact	Error QF(16)in [11]	OQF(1.3), m=1	Error
-0.887	10.4702332992	0.0417283096	10.4705320346	0.0002987354
-0.695	15.6980321754	0.0897051922	15.7084204293	0.0103882539
-0.495	18.5096575257	0.0818203672	18.5217620154	0.0121044897
-0.293	20.0890157504	0.0788584125	20.1010005193	0.0119847689
-0.095	20.7941454186	0.0915690660	20.8061508094	0.0120053908
0.095	20.7941454186	0.0915691610	20.8061508094	0.0120053908
0.293	20.0890157504	0.0788588154	20.1010005193	0.0119847689
0.495	18.5096575257	0.0818187549	18.5217620154	0.0121044897
0.695	15.6980321754	0.0897054275	15.7084204293	0.0103882539
0.887	10.4702332992	0.0417277253	10.4705320346	0.0002987354

Table 2. Error terms for OQF (1.3).

N=200				
$t \neq h\gamma - 1$	Exact	Error QF(16)in [11]	OQF(1.3), m=1	Error
-0.987	3.7424126959	0.0096756670	3.7423362651	0.0000764308
-0.935	8.1393802207	0.0045964403	8.1394671385	0.0000869179
-0.887	10.4702332992	0.0091676216	10.4702698056	0.0000365064
-0.695	15.6980321754	0.0026300570	15.6981798537	0.0001476783
-0.495	18.5096575257	0.0054628196	18.5098039531	0.0001464275
-0.293	20.0890157504	0.0061221725	20.0891616458	0.0001458954
-0.095	20.7941454186	0.0069000509	20.7942755288	0.0001301102
0.095	20.7941454186	0.0069001771	20.7942755288	0.0001301102
0.293	20.0890157504	0.0061240901	20.0891616457	0.0001458954
0.495	18.5096575257	0.0054671436	18.5098039531	0.0001464275
0.695	15.6980321754	0.0026330865	15.6981798537	0.0001476783
0.887	10.4702332992	0.0091552088	10.4702698056	0.0000365064
0.935	8.1393802206	0.0046156414	8.1394671385	0.0000869179
0.987	3.7424126959	0.0096747715	3.7423362651	0.0000764308

Table 3. Error terms for OQF (1.3).

N=20				
$t = h\gamma - 1$	Exact	Error QF(20)in [11]	OQF(1.3), m=1	Error
-0.9	9.9147951260	0.0434208684	9.9246302672	0.0098351412
-0.7	15.6040925306	0.0454488972	15.6182823480	0.0141898173
-0.5	18.4571664181	0.0694506386	18.4712004187	0.0140340005
-0.3	20.0499895293	0.0788768819	20.0629747128	0.0129851834
-0.1	20.7853870599	0.0814265589	20.7976199210	0.0122328611
0.1	20.7853870599	0.0814266037	20.7976199210	0.0122328611
0.3	20.0499895293	0.0788768802	20.0629747128	0.0129851834
0.5	18.4571664182	0.0694507351	18.4712004187	0.0140340005
0.7	15.6040925307	0.0454491744	15.6182823480	0.0141898173
0.9	9.9147951260	0.0434206055	9.9246302673	0.0098351412

Table 4. Error terms for OQF (1.3).

N=20				
$t = h\gamma - 1$	Exact	Error QF(20)in [11]	OQF(1.3), m=1	Error
-0.98	4.6242972766	0.0058168646	4.6243374562	0.0000401796
-0.96	6.4698890368	0.0053701073	6.4699564868	0.0000674500
-0.94	7.8405806772	0.0047221016	7.8406659308	0.0000852536
-0.90	9.9147951260	0.0032612171	9.9149040083	0.0001088823
-0.70	15.6040925306	0.0025401403	15.6042411554	0.0001486248
-0.50	18.4571664182	0.0054238405	18.4573137570	0.0001473389
-0.30	20.0499895293	0.0065655622	20.0501270539	0.0001375246
-0.10	20.7853870599	0.0068972439	20.7855174868	0.0001304269
0.10	20.7853870599	0.0068972228	20.7855174868	0.0001304269
0.30	20.0499895293	0.0065654914	20.0501270540	0.0001375247
0.50	18.4571664182	0.0054241103	18.4573137571	0.0001473389
0.70	15.6040925307	0.0025399121	15.6042411554	0.0001486248
0.90	9.9147951260	0.0032614564	9.9149040083	0.0001088823
0.94	7.8405806772	0.0047223289	7.8406659308	0.0000852536
0.96	6.4698890368	0.0053701392	6.4699564868	0.0000674500
0.98	4.6242972766	0.0058168227	4.6243374562	0.0000401796

Table 5. Error terms for OQF (1.3).

N=20				
t	Exact	Error QF(21)in [11]	OQF(1.3), m=1	Error
-0.9999	0.3305542576	0.0016697528	0.3288666955	0.0016875620
-0.9980	1.4767423383	0.0070573417	1.4696058654	0.0071364729
-0.9450	7.5265525954	0.0073648393	7.5337473788	0.0071947834
-0.9150	9.2115705777	0.0189073179	9.2305812184	0.0190106407

Table 6. Error terms for OQF (1.3).

N=200				
t	Exact	Error QF(21)in [11]	OQF(1.3), m=1	Error
-0.9999	0.3305542576	0.0001051563	0.3304914855	0.0000627721
-0.999	1.0447877703	0.0002953748	1.0446330797	0.0001546905
-0.998	1.4767423383	0.0003658651	1.4765870900	0.0001552483
-0.997	1.8076410236	0.0003931256	1.8075212773	0.0001197463
-0.995	2.3310993510	0.0004063873	2.3310939480	0.0000054030
-0.993	2.7551786054	0.0004410670	2.7552845634	0.0001059580
-0.991	3.1206688242	0.0005965699	3.1208008580	0.0001320338

Table 7. Error terms for OQF (1.3).

N=20				
t	Exact	Error QF(22)in [11]	OQF(1.3), m=1	Error
0.9999	0.3305542576	0.0016696240	0.3288666955	0.0016875620
0.9980	1.4767423383	0.0070573629	1.4696058654	0.0071364729
0.9450	7.5265525954	0.0073644315	7.5337473788	0.0071947834
0.9150	9.2115705777	0.0189067192	9.2305812184	0.0190106407

Table 8. Error terms for OQF (1.3).

N=200				
t	Exact	Error QF(22)in [11]	OQF(1.3), m=1	Error
0.9999	0.3305542576	0.0001051332	0.3304914855	0.0000627721
0.999	1.0447877703	0.0002953269	1.0446330797	0.0001546905
0.998	1.4767423383	0.0003656582	1.4765870900	0.0001552483
0.997	1.8076410236	0.0003931503	1.8075212773	0.0001197463
0.995	2.3310993510	0.0004062447	2.3310939480	0.0000054030
0.993	2.7551786054	0.0004408404	2.7552845634	0.0001059580
0.991	3.1206688242	0.0005964761	3.1208008580	0.0001320338

Table 9. Error terms for OQF (1.3).

N=20				
t	Exact	Error of OQF $m = 1$	Error of OQF $m = 2$	Error of OQF $m = 3$
-0.887	10.4702332992	0.0002987354	0.0001764502	0.0000053844
-0.695	15.6980321754	0.0103882539	0.0002530201	0.0000066140
-0.495	18.5096575257	0.0121044897	0.0001493195	0.0000104882
-0.293	20.0890157504	0.0119847689	0.0000694601	0.0000126264
-0.095	20.7941454186	0.0120053908	0.0000449824	0.0000149919
0.095	20.7941454186	0.0120053908	0.0000449824	0.0000149919
0.293	20.0890157504	0.0119847689	0.0000694601	0.0000126264
0.495	18.5096575257	0.0121044897	0.0001493195	0.0000104882
0.695	15.6980321754	0.0103882539	0.0002530201	0.0000066140
0.887	10.4702332992	0.0002987354	0.0001764502	0.0000053844

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A dynamic problem with wear involving electro-elastic-viscoplastic materials with damage

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Abstract. A dynamic contact problem is considered in the paper. The material behavior is described by electro-elastic-viscoplastic law with piezoelectric effects. The body is in contact with damage and an obstacle. The contact is frictional and bilateral with a moving rigid foundation which results in the wear of the contacting surface. The damage of the material caused by elastic deformations. The evolution of the damage is described by an inclusion of parabolic type. The problem is formulated as a coupled system of an elliptic variational inequality for the displacement, variational equation for the electric potential and a parabolic variational inequality for the damage. We establish a variational formulation for the model and we prove the existence of a unique weak solution to the problem. The proof is based on a classical existence and uniqueness result on parabolic inequalities, differential equations and fixed point arguments.

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Keywords: Damage field, piezoelectric, electro-elastic-viscoplastic, variational inequality, wear.

1. Introduction

Scientific research and recent papers in mechanics are articulated around two main components, one devoted to the laws of behavior and other devoted to boundary conditions imposed on the body. The boundary conditions reflect the binding of the body with the outside world. Recent researches use coupled laws of behavior between mechanical and electric effects or between mechanical and thermal effects(see [2]). For the case of coupled laws of behavior between mechanical and electric effects, general models can be found in (see [5]). Situations of contact between deformable bodies are very common in the industry and everyday life. Contact of braking pads with wheels, tires with roads, pistons with skirts or the complex metal forming processes

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are just a few examples. The constitutive laws with internal variables have been used in various publications in order to model the effect of internal variables in the behavior of real bodies like metals, rocks, polymers and so on, for which the rate of deformation depends on the internal variables. Some of the internal state variables considered by many authors are the spatial display of dislocation, the work-hardening of materials,

In this paper, we consider a general model for the dynamic process of frictional contact bilateral between a deformable body and an obstacle which results in the wear of the contacting surface. The material obeys an electro-elastic-viscoplastic constitutive law with piezoelectric effects. We derive a variational formulation of the problem which includes a variational second order evolution inequality. We establish the existence of a unique weak solution of the problem. The idea is to reduce the second order evolution nonlinear inequality of the system to first order evolution inequality. After this,we use classical results on first order evolution nonlinear inequalities and aquation , a parabolic variational inequality and the fixed point arguments.

The paper is structured as follows. In Section 1 we present the electro-elastic-viscoplastic contact model with friction and provide comments on the contact boundary conditions. In Section 2 we list the assumptions on the data and derive the variational formulation. In Section 3 we present our main results on existence and uniqueness which state the unique weak solvability.

2. Problem statement

Problem P: Find a displacement field $u : \Omega\times[0,T] \to \mathbb{R}^d$, a stress field $\sigma : \Omega\times[0,T] \to \mathbb{S}^d$, the an electric potentiel field $\varphi : \Omega \times [0,T] \to \mathbb{R}$, the an electric displacement field $D : \Omega \times [0,T] \to \mathbb{R}^d$, the damage field $\beta : \Omega \times [0,T] \to \mathbb{R}^d$, and the wear $\omega : \Gamma_3 \times [0,T] \to \mathbb{R}_+$ such that

$$\begin{aligned}\sigma(t) &= \mathcal{A}(\varepsilon(\dot{u}(t))) + \mathcal{B}(\varepsilon(u(t)), \beta(t)) \\ &+ \int_0^t \mathcal{G}\left(\sigma(s) - \mathcal{A}(\varepsilon(\dot{u}(s))), \varepsilon(u(s))\right) ds - \xi^* E\left(\varphi\right), \text{in } \Omega \text{ a.e. } t \in [0,T],\end{aligned} \tag{2.1}$$

$$\begin{aligned}&D = BE\left(\varphi\right) + \xi\varepsilon\left(u\right) \\ &\text{in } \Omega \times [0,T],\end{aligned} \tag{2.2}$$

$$\rho\ddot{u} = Div\ \sigma + f_0, \quad \text{in } \Omega \times [0,T], \tag{2.3}$$

$$\text{div } D = q_0 \quad \text{in } \Omega \times [0,T], \tag{2.4}$$

$$\dot{\beta} - K_1\Delta\beta + \partial\varphi_K\left(\beta\right) \ni S(\varepsilon(u), \beta), \quad \text{in } \Omega \times [0,T], \tag{2.5}$$

$$u = 0, \text{on } \Gamma_1 \times [0,T], \tag{2.6}$$

$$\sigma\nu = h, \text{on } \Gamma_2 \times [0,T], \tag{2.7}$$

$$\begin{cases}\sigma_\nu = -\alpha\left|\dot{u}_\nu\right|, \ |\sigma_\tau| = -\mu\sigma_\nu, \\ \sigma_\tau = -\lambda\left(\dot{u}_\tau - v^*\right), \ \lambda \geq 0, \ \dot{\omega} = -kv^*\sigma_\nu, \ k > 0. \text{ on } \Gamma_3 \times [0,T],\end{cases} \tag{2.8}$$

$$\frac{\partial\beta}{\partial\nu} = 0, \text{ on } \Gamma \ \times [0,T], \tag{2.9}$$

$$\varphi = 0 \text{ on } \Gamma_a \times [0,T], \tag{2.10}$$

$$D\nu = q_2 \text{on } \Gamma_b \times [0,T], \tag{2.11}$$

$$u(0) = u_0, \ v(0) = v_0, \ \beta(0) = \beta_0, \ \omega(0) = \omega_0, \text{ in } \Omega, \tag{2.12}$$

where (2.1) and (2.2) represent the electro-elastic-viscoplastic constitutive law with damage. we denote $\varepsilon(u)$ (respectively; $E(\varphi) = -\nabla\varphi$, $\mathcal{A}, \mathcal{G}$, ξ, ξ^*, B) the linearized strain tensor (respectively; electric field, the viscosity nonlinear tensor, the viscoplasticity tensor, the third order piezoelectric tensor and its transpose, the electric permittivity tensor), (2.3) represents the equation of motion where ρ represents the mass density, (2.4) represents the equilibrium equation, we mention that $Div\sigma$, $divD$ are the divergence operators. Inclusion (2.5) describes the evolution of damage field, governed by the source damage function φ, where $\partial\varphi_K(\zeta)$ is the subdifferential of indicator function of the set K of admissible damage functions.

Equalities (2.6) and (2.7) are the displacement-traction boundary conditions, respectively. (2.8) describes the frictional bilateral contact with wear described above on the potential contact surface Γ_3. (2.9) represents on Γ, a homogeneous Neumann boundary condition for the damage field. (2.10) , (2.11) represent the electric boundary conditions.The functions u_0, v_0, β_0 and ω_0 in (2.12) are the initial data.

3. Variational formulation and preliminaries

For a weak formulation of the problem, first we itroduce some notation. The indices i, j, k, l range from 1 to d and summation over repeated indices is implied. An index that follows a comma represents the partial derivative with respect to the corresponding component of the spatial variable, e. g: $u_{i.j} = \frac{\partial u_i}{\partial x_j}$. We also use the following notations

$$\begin{gathered} H = \mathbb{L}^2(\Omega)^d = \{u = (u_i)/u_i \in \mathbb{L}^2(\Omega)\}, \\ \mathcal{H} = \sigma = (\sigma_{ij})/\sigma_{ij} = \sigma_{ji} \in \mathbb{L}^2(\Omega), \\ H_1 = u = (u_i)/\varepsilon(u) \in \mathcal{H} = H^1(\Omega)^d \\ \mathcal{H}_1 = \sigma \in \mathcal{H}/Div\sigma \in H, \end{gathered}$$

The operators of deformation ε and divergence Div are defined by

$$\varepsilon(u) = (\varepsilon_{ij}(u)), \varepsilon_{ij}(u) = \tfrac{1}{2}(u_{i,j} + u_{j,i}), Div\sigma = (\sigma_{ij,j}).$$

The spaces $H, \mathcal{H}, H_1$ and $\mathcal{H}_1$ are real Hilbert spaces endowed with the canonical inner products given by

$$\begin{gathered} (u,v)_H = \textstyle\int_\Omega u_i v_i dx, \forall u, v \in H, \\ (\sigma,\tau)_{\mathcal{H}} = \textstyle\int_\Omega \sigma_{ij}\tau_{ij} dx, \forall \sigma, \tau \in \mathcal{H}, \\ (u,v)_{H_1} = (u,v)_H + (\varepsilon(u), \varepsilon(v))_{\mathcal{H}}, \forall u, v \in H_1, \\ (\sigma,\tau)_{\mathcal{H}_1} = (\sigma,\tau)_{\mathcal{H}} + (Div\sigma, Div\tau)_H, \sigma, \tau \in \mathcal{H}_1, \end{gathered}$$

We denote by $|\cdot|_H$ (respectively; $|\cdot|_{\mathcal{H}}$, $|\cdot|_{H_1}$and $|\cdot|_{\mathcal{H}1}$) the associated norm on the space H (respectively; $\mathcal{H}$, H_1 and $\mathcal{H}_1$).

Let $H_\Gamma = (H^{1/2}(\Gamma))^d$ and $\gamma : H^1(\Gamma)^d \to H_\Gamma$ be the trace map. For every element $v \in (H^1(\Gamma))^d$, we also use the notation v to denote the trace map γv of v on Γ, and we denote by v_ν and v_τ the normal and tangential components of v on Γ given by

$$v_\nu = v.\nu, v_\tau = v - v_\nu \nu$$

Similarly, for a regular (say $\mathcal{C}^1$) tensor field $\sigma : \Omega \to \mathbb{S}^d$ we define its normal and tangential components by

$$\sigma_\nu = (\sigma\nu).\nu, \sigma_\tau = \sigma\nu - \sigma_\nu \nu$$

We use standard notation for the $\mathbb{L}^p$ and the Sobolev spaces associated with Ω and Γ and, for a function $\psi \in H^1(\Omega)$ we still write ψ to denote it trace on Γ. We recall that the summation convention applies to a repeated index.
For the electric displacement field we use two Hilbert spaces

$$\mathcal{W} = \mathbb{L}^2(\Omega)^d, \mathcal{W}_1 = \left\{D \in \mathcal{W}, \operatorname{div} D \in \mathbb{L}^2(\Omega)\right\}$$

endowed with the inner products

$$(D, E)_{\mathcal{W}} = \int_\Omega D_i E_i dx, (D, E)_{\mathcal{W}_1} = (D, E)_{\mathcal{W}} + (\operatorname{div} D, \operatorname{div} E)_{\mathbb{L}^2(\Omega)}$$

and the associated norm $|.|_{\mathcal{W}}$(respectively; $|.|_{\mathcal{W}_1}$). The electric potential field is to be found in

$$W = \left\{\psi \in H^1(\Omega),\ \psi = 0 \text{ on } \Gamma_a\right\}.$$

Since *meas* $(\Gamma_a) > 0$, the following Friedrichs-Poincaré's inequality holds, thus

$$|\nabla\psi|_{\mathcal{W}} \geq c_F |\psi|_{H^1(\Omega)} \forall\psi \in W, \tag{3.1}$$

where $c_F > 0$ is a constant which depends only on Ω and Γ_a. On W, we use the inner product given by

$$(\varphi, \psi)_W = (\nabla\varphi, \nabla\psi)_{\mathcal{W}},$$

and let $|.|_W$ be the associated norm. It follows from (3.1) that $|.|_{H^1(\Omega)}$ and $|.|_W$ are equivalent norms on W and therefore $(W, |.|_W)$ is a real Hilbert space.
Moreover, by the Sobolev trace Theorem, there exists a constant $\widetilde{c}_0$, depending only on Ω, Γ_a and Γ_3 such that

$$|\psi|_{\mathbb{L}^2(\Gamma_3)} \leq \widetilde{c}_0 |\psi|_W \quad \forall\psi \in W. \tag{3.2}$$

We recall that when $D \in \mathcal{W}_1$ is a sufficiently regular function,the Green's type formula holds

$$(D, \nabla\psi)_{\mathcal{W}} + (\operatorname{div} D, \psi)_{\mathbb{L}^2(\Omega)} = \int_\Gamma D\nu.\psi da. \tag{3.3}$$

When σ is a regular function, the following Green 's type formula holds

$$(\sigma, \varepsilon(v))_{\mathcal{H}} + (Div\sigma, v)_H = \int_\Gamma \sigma\nu.vda \quad \forall v \in H_1.$$

Next, we define the space

$$V = \{u \in H_1 / \ u = 0 \ \text{ on } \Gamma_1\}.$$

Since *meas* $(\Gamma_1) > 0$, the following Korn's inequality holds

$$|\varepsilon(u)|_{\mathcal{H}} \geq c_K |v|_{H_1} \forall v \in V, \tag{3.4}$$

where $c_K > 0$ is a constant which depends only on Ω and Γ_1. On the space V we use the inner product

$$(u, v)_V = (\varepsilon(u), \varepsilon(v))_{\mathcal{H}}, \tag{3.5}$$

let $|.|_V$ be the associated norm. It follows by (3.4) that the norms $|.|_{H_1}$ and $|.|_V$ are equivalent norms on V and therefore, $(V, |.|_V)$ is a real Hilbert space. Moreover, by the Sobolev trace Theorem, there exists a constant c_0 depending only on the domain Ω, Γ_1 and Γ_3 such that

$$|v|_{\mathbb{L}^2(\Gamma_3)^d} \leq c_0 |v|_V \, \forall v \in V. \tag{3.6}$$

Finally, for a real Banach space $(X, |.|_X)$ we use the usual notation for the space $\mathbb{L}^p(0, T; X)$ and $W^{k.p}(0, T; X)$, where $1 \leq p \leq \infty, k = 1, 2......$; we also denote by $C(0, T; X)$ and $C^1(0, T; X)$ the spaces of continuous and continuously differentiable function on $[0, T]$with values in X, with the respective norms:

$$|x|_{C(0,T;X)} = \max_{t \in [0,T]} |x(t)|_X,$$

$$|x|_{C^1(0,T;X)} = \max_{t \in [0,T]} |x(t)|_X + \max_{t \in [0,T]} |\dot{x}(t)|_X.$$

In what follows, we assume the following assumptions on the problem P.
The viscosity operator $\mathcal{A} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$

$$\begin{cases} (a) \, \exists \, M_{\mathcal{A}} > 0 \text{ such that } : |\mathcal{A}(x, \varepsilon_1) - \mathcal{A}(x, \varepsilon_2)| \leq M_{\mathcal{A}} |\varepsilon_1 - \varepsilon_2| \\ \forall \, \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d, \ a.\ e.\ x \in \Omega, \\ (b) \ \exists \, m_{\mathcal{A}} > 0 \text{ such that } : |\mathcal{A}(x, \varepsilon_1) - \mathcal{A}(x, \varepsilon_2), \varepsilon_1 - \varepsilon_2| \geq m_{\mathcal{A}} |\varepsilon_1 - \varepsilon_2|^2 \\ \forall \, \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d, \ a.\ e.\ x \in \Omega, \\ (c) \ \text{The mapping } x \to \mathcal{A}(x, \varepsilon) \text{ is lebesgue measurable in } \Omega \text{ for all } \varepsilon \in \mathbb{S}^d, \\ (d) \ \text{The mapping } x \to \mathcal{A}(x, 0) \in \mathcal{H}. \end{cases} \tag{3.7}$$

The elasticity operator $\mathcal{B} : \Omega \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{S}^d$satisfies

$$\begin{cases} (a) \ \exists \, L_{\mathcal{B}} > 0 \text{ such that} \\ \ |\mathcal{B}(x, \varepsilon_1, \alpha_1) - \mathcal{B}(x, \varepsilon_2, \alpha_2)| \leq L_{\mathcal{B}} (|\varepsilon_1 - \varepsilon_2| + |\alpha_1 - \alpha_2|) \\ \ \forall \, \varepsilon_1, \varepsilon_2 \in S^d, \forall \alpha_1, \alpha_2 \in \mathbb{R}, a.\ e.\ x \in \Omega, \\ (b) \ \text{The mapping } x \to \mathcal{B}(x, \varepsilon, \alpha) \text{ is lebesgue measurable in } \Omega \\ \text{for all } \varepsilon \in S^d \text{ and } \alpha \in \mathbb{R} \\ (c) \, \text{The mapping } x \to \mathcal{B}(x, 0, 0) \in \mathcal{H}, \end{cases} \tag{3.8}$$

The viscoplasticity operator $\mathcal{G} : \Omega \times \mathbb{S}^d \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies

$$\begin{cases} (a) \ \exists \, L_{\mathcal{G}} > 0 \text{ such that} \\ \ |\mathcal{G}(x, \sigma_1, \varepsilon_1) - \mathcal{G}(x, \sigma_2, \varepsilon_2)| \leq L_{\mathcal{G}} (|\sigma_1 - \sigma_2| + |\varepsilon_1 - \varepsilon_2|) \\ \ \forall \, \sigma_1, \sigma_2 \in \mathbb{S}^d, \forall \, \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d, a.\ e.\ x \in \Omega, \\ (b) \ \text{The mapping } x \to \mathcal{G}(x, \sigma, \varepsilon) \text{ is lebesgue measurable in } \Omega \\ \text{for all } \sigma, \ \varepsilon \in \mathbb{S}^d \\ (c) \ \text{The mapping } x \to \mathcal{G}(x, 0, 0) \in \mathcal{H}, \end{cases} \tag{3.9}$$

The damage source function $S:\Omega\times\mathbb{S}^d\times\mathbb{R}\to\mathbb{R}$ satisfies

$$\begin{cases}(a)\ \exists\ M_S>0 \text{ such that}\\ \ |S(x,\varepsilon_1,\alpha_1)-S(x,\varepsilon_2,\alpha_2)|\le M_S\left(|\varepsilon_1-\varepsilon_2|+|\alpha_1-\alpha_2|\right)\\ \forall\ \varepsilon_1\ ,\varepsilon_2\in S^d,\ \forall\alpha_1,\alpha_2\in\mathbb{R}, a.\ e.\ x\in\Omega,\\ (b)\ \text{ The mapping } x\to S(x,\varepsilon,\alpha)\ \text{ is lebesgue measurable in } \Omega\\ \text{for all } \varepsilon\in S^d \text{ and } \alpha\in\mathbb{R}\\ (c)\ \text{ The mapping } x\to S(x,0,0)\in\mathbb{L}^2(\Omega),\end{cases}\tag{3.10}$$

The piezoelectric tensor $\xi=(e_{ijk}):\Omega\times S^d\to\mathbb{R}^d$ satisfies

$$\begin{cases}(a):\xi=(e_{ijk}):\Omega\times S^d\to\mathbb{R}^d,\\ (b):\xi(x,\tau)=(e_{ijk}(x)\,\tau_{jk})\ \ \forall\tau=(\tau_{ij})\in S^d, a.\ e.\ x\in\Omega,\\ (c):e_{ijk}=e_{ikj}\in\mathbb{L}^\infty(\Omega),\end{cases}\tag{3.11}$$

The electric permittivity tensor $B=(B_{ij}):\Omega\times\mathbb{R}^d\to\mathbb{R}^d$

$$\begin{cases}(a):B=(B_{ij}):\Omega\times\mathbb{R}^d\to\mathbb{R}^d,\\ (b):B(x,E)=(b_{ij}(x)\,E_j)\ \ \forall E=(E_i)\in\mathbb{R}^d,\ \text{ a.e. } x\in\Omega,\\ (c):b_{ij}=b_{ji}\in\mathbb{L}^\infty(\Omega),\\ (d):\exists\ m_B>0 \text{ such that }: b_{ij}(x)\,E_iE_j\ge m_B\,|E|^2\\ \forall E=(E_i)\in\mathbb{R}^d,\ x\in\Omega.\end{cases}\tag{3.12}$$

The mass density ρ satisfy

$$\rho\in\mathbb{L}^\infty(\Omega) \text{ there exists } \rho^*>0 \text{ such that } \rho(x)\ge\rho^*,\ \text{ a.e. } x\in\Omega\tag{3.13}$$

The body forces, surface tractions,the densities of electric charges, and the functions α and μ, satisfy

$$\begin{cases}f_0\in\mathbb{L}^2(0,T;H), h\in\mathbb{L}^2(0,T;\mathbb{L}^2(\Gamma_2)^d),\\ q_0\in L^2\left(0,T;\mathbb{L}^2(\Omega)\right), q_2\in L^2\left(0,T;\mathbb{L}^2(\Gamma_b)\right).\\ \alpha\in\mathbb{L}^\infty(\Gamma_3)\alpha(x)\ge\alpha^*>0,\ \text{ a.e. on } \Gamma_3,\\ \mu\in\mathbb{L}^\infty(\Gamma_3),\mu(x)>0,\ \text{ a.e. on } \Gamma_3,\\ K_1>0, i=0,1.\end{cases}\tag{3.14}$$

The set K of admissible damage functions defined by

$$K=\left\{\beta\in H^1(\Omega)/\ 0\le\beta\le 1\ p.p \text{ in } \Omega\right\}\tag{3.15}$$

The initial data satisfy

$$u_0\in V,\beta_0\in K,\omega_0\in\mathbb{L}^\infty(\Gamma_3).\tag{3.16}$$

We use a modified inner product on $H=\mathbb{L}^2(\Omega)^d$ given by

$$((u,v))=(\rho u,v)_{\mathbb{L}^2(\Omega)^d}, \forall u.v\in H.$$

That is, it is weighted with ρ. We let H be the associated norm

$$\|v\|_H=(\rho v,v)^{\frac{1}{2}}_{\mathbb{L}^2(\Omega)^d}, \forall v\in H.$$

We use the notation $(.,.)_{V'\times V}$ to represent the duality pairing between V' and V. Then, we have

$$(u,v)_{V'\times V}=((u,v))\,,\forall u\in H,\forall v\in V.$$

It follows from assumption (3.13) that $\|.\|_H$ and $|.|_H$ are equivalent norms on H, and also the inclusion mapping of $(V,|.|_V)$ into $(H,\|.\|_H)$ is continuous and dense. We denote by V' the dual space of V. Identifying H with its own dual, we can write the Gelfand triple $V\subset H=H'\subset V'$.

We define the function $f(t)\in V$ and $q:[0.T]\to W$ by

$$\begin{aligned}(f(t),v)_V&=\textstyle\int_\Omega f_0(t)\,vdx+\int_{\Gamma_2}h(t)\,vda\forall v\in V,t\in[0,T]\,,\\(q(t),\psi)_W&=-\textstyle\int_\Omega q_0(t)\,\psi dx+\int_{\Gamma_b}q_2(t)\,\psi da\ \forall\psi\in W,t\in[0.T]\,,\end{aligned}$$

for all u, $v\in V$, $\psi\in W$ and $t\in[0.T]$, and note that condition (3.14) imply that

$$f\in\mathbb{L}^2(0.T;V'),q\in\mathbb{L}^2(0.T;W). \tag{3.17}$$

We introduce the following bilinear

$$a_1:H^1(\Omega)\times H^1(\Omega)\to\mathbb{R},a_1(\zeta,\xi)=k_1\int_\Omega\nabla\zeta.\nabla\xi dx,\ \forall\zeta,\xi\in H^1(\Omega)\,. \tag{3.18}$$

We consider the wear functional $j:V\times V\to\mathbb{R}$,

$$j(u,v)=\int_{\Gamma_3}\alpha\,|u_\nu|\,(\mu\,|v_\tau-v^*|)\,da. \tag{3.19}$$

Finally, we consider $\phi:V\times V\to\mathbb{R}$,

$$\phi(u,v)=\int_{\Gamma_3}\alpha\,|u_\nu|\,v_\nu da,\forall v\in V. \tag{3.20}$$

We define for all $\varepsilon>0$

$$j_\varepsilon(g,v)=\textstyle\int_{\Gamma_3}\alpha\,|g_\nu|\left(\mu\sqrt{|v_\tau-v^*|^2+\varepsilon^2}\right)da,\forall v\in V.$$

Using the above notation and Green's formula, we derive the following variational formulation of mechanical problem P.

Problem PV**:** Find a displacement field $u:\Omega\times[0,T]\to V$, a stress field $\sigma:\Omega\times[0,T]\to\mathbb{S}^d$, an electric potential field $\varphi:\Omega\times[0.T]\to\mathbb{R}$, an electric displacement field $D:\Omega\times[0.T]\to\mathbb{R}^d$, the damage field $\beta:\Omega\times[0,T]\to\mathbb{R}^d$ and the wear $\omega:\Gamma_3\times[0,T]\to\mathbb{R}_+$

such that

$$\begin{aligned}&\sigma(t)=\mathcal{A}(\varepsilon(\dot{u}(t)))+\mathcal{B}(\varepsilon(u(t)),\beta(t))\\&+\int_0^t\mathcal{G}\left(\sigma(s)-\mathcal{A}(\varepsilon(\dot{u}(s))),\varepsilon(u(s))\right)ds-\xi^*E\left(\varphi\right)\ \text{, in }\Omega\text{ a.e. }t\in[0,T]\end{aligned}\tag{3.21}$$

$$\begin{aligned}&(\ddot{u}(t),w-\dot{u}(t))_{V'\times V}+(\sigma(t),\varepsilon(w-\dot{u}(t)))_{\mathcal{H}}+j(\dot{u},w)-j(\dot{u},\dot{u}(t))\\&+\phi(\dot{u},w)-\phi(\dot{u},\dot{u}(t))\geq(f(t),w-\dot{u}(t)),\ \forall u,w\in V\end{aligned}\tag{3.22}$$

$$(D(t),\nabla\psi)_{\mathbb{L}^2(\Omega)^d}+(q\left(t\right),\psi)_W=0\forall\psi\in W\tag{3.23}$$

$$\begin{aligned}&\left(\dot{\beta}\left(t\right),\zeta-\beta\left(t\right)\right)_{\mathbb{L}^2(\Omega)}+a_1\left(\beta\left(t\right),\zeta-\beta\left(t\right)\right)\geq\\&(S(\varepsilon(u\left(t\right)),\beta)\,,\zeta-\beta\left(t\right))_{\mathbb{L}^2(\Omega)},\forall\zeta\in K,\ \text{a.e. }t\in[0,T]\end{aligned}\tag{3.24}$$

$$\dot{\omega}=-kv^*\sigma_\nu,\ k>0\tag{3.25}$$

$$u\left(0\right)=u_0,v\left(0\right)=v_0,\beta\left(0\right)=\beta_0,\omega\left(0\right)=\omega_0,\ \text{in }\Omega\tag{3.26}$$

4. Existence and uniqueness result

Our main result which states the unique solvability of Problem are the following.

Theorem 4.1. *Let the assumptions* (3.7) − (3.15) *hold. Then, Problem PV has a unique solution* $(u,\sigma,\varphi,D,\beta,\omega)$ *which satisfies*

$$u\in C^1\left(0,T;H\right)\cap W^{1.2}\left(0,T;V\right)\cap W^{2.2}\left(0,T;V'\right)\tag{4.1}$$

$$\sigma\in\mathbb{L}^2(0,T;\mathcal{H}_1),Div\sigma\in\mathbb{L}^2(0,T;V')\tag{4.2}$$

$$\varphi\in W^{1.2}\left(0,T;W\right)\tag{4.3}$$

$$D\in W^{1.2}(0,T;\mathcal{W}_1)\tag{4.4}$$

$$\beta\in W^{1.2}\left(0,T;\mathbb{L}^2\left(\Omega\right)\right)\cap\mathbb{L}^2(0,T;H^1\left(\Omega\right))\tag{4.5}$$

$$\omega\in C^1(0,T;\mathbb{L}^2(\Gamma_3))\tag{4.6}$$

We conclude that under the assumptions (3.7) − (3.15), the mechanical problem (2.1) − (2.12) has a unique weak solution with the regularity (4.1) − (4.6).

The proof of this theorem will be carried out in several steps. It is based on arguments of first order evolution nonlinear inequalities, evolution equations, a parabolic variational inequality, and fixed point arguments.

First step: Let $g\in\mathbb{L}^2(0,T;V)$ and $\eta\in\mathbb{L}^2(0,T;V')$ are given, we deduce a variational formulation of Problem PV.

Problem $PV_{g\eta}$: Find a displacement field $u_{g\eta}:[0,T]\to V$ such that

$$\begin{cases}u_{g\eta}\left(t\right)\in V\quad(\ddot{u}_{g\eta}(t),w-\dot{u}_{g\eta}(t))_{V'\times V}+(\mathcal{A}\varepsilon(\dot{u}_{g\eta}(t)),\varepsilon(w-\dot{u}_{g\eta}(t))_{\mathcal{H}}+\\\left(\eta,w-\dot{u}_{g\eta}(t)\right)_{V'\times V}+j(g,w)-j(g,\dot{u}_{g\eta}(t))\geq(f(t),w-\dot{u}_{g\eta}(t)),\quad\forall w\in V.\end{cases}\tag{4.7}$$

$$\dot{u}_{g\eta}\left(0\right)=v\left(0\right)=v_0.\tag{4.8}$$

We define $f_\eta(t)\in V$ for a.e. $t\in[0,T]$ by

$$(f_\eta(t),w)_{V'\times V}=(f(t)-\eta(t),w)_{V'\times V}\,,\forall w\in V.\tag{4.9}$$

From (3.17), we deduce that

$$f_\eta\in\mathbb{L}^2(0,T;V')\tag{4.10}$$

Let now $u_{g\eta} : [0,T] \to V$ be the function defined by

$$u_{g\eta}(t) = \int_0^t v_{g\eta}(s)\,ds + u_0, \quad \forall t \in [0,T]. \tag{4.11}$$

We define the operator $A : V' \to V$ by

$$(Av, w)_{V'\times V} = (\mathcal{A}\varepsilon(v), \varepsilon(w))_{\mathcal{H}}, \forall v, w \in V. \tag{4.12}$$

Lemma 4.2. *For all $g \in \mathbb{L}^2(0,T;V)$ and $\eta \in \mathbb{L}^2(0,T;V')$, $PV_{g\eta}$ has a unique solution with the regularity*

$$v_{g\eta} \in C(0,T;H) \cap \mathbb{L}^2(0,T;V) \ and \dot{v}_{g\eta} \in \mathbb{L}^2(0,T;V'). \tag{4.13}$$

Proof. The proof from nonlinear first order evolution inequalities(see [4, 6]). □

Second step: We use the displacement field $u_{g\eta}$ to consider the following variational problem.
Let us consider now the operator $\Lambda_\eta : \mathbb{L}^2(0,T;V) \to \mathbb{L}^2(0,T;V)$, defined by

$$\Lambda_\eta g = v_{g\eta} \tag{4.14}$$

We have the following lemma.

Lemma 4.3. *The operator Λ_η has a unique fixed point $g \in \mathbb{L}^2(0,T;V)$*

Proof. Let $g_1, g_2 \in \mathbb{L}^2(0,T;V)$ and let $\eta \in \mathbb{L}^2(0.T;V')$. Using similar arguments as those in (4.7), (4.11) we find

$$\begin{aligned}&(\dot{v}_1(t) - \dot{v}_2(t), v_1(t) - v_2(t)) + (\mathcal{A}\varepsilon(v_1(t)) - \mathcal{A}\varepsilon(v_2(t)), \varepsilon(v_1(t)) - \varepsilon(v_2(t)))\\ &\quad + j(g_1, v_1(t)) - j(g_1, v_2(t)) - j(g_2, v_1(t)) + j(g_2, v_2(t)) \le 0.\end{aligned} \tag{4.15}$$

From the definition of the functional j given by (3.17), we have

$$\begin{aligned}&j(g_1, v_2(t)) - j(g_1, v_1(t)) - j(g_2, v_2(t)) + j(g_2, v_1(t))\\ &= \int_{\Gamma_3} (\alpha|g_{1\nu}| - \alpha|g_{2\nu}|)(\mu|v_{1\tau} - v^*| - \mu|v_{2\tau} - v^*|)\,da.\end{aligned} \tag{4.16}$$

From (3.6) and (3.14), we find

$$j(g_1, v_2(t)) - j(g_1, v_1(t)) - j(g_2, v_2(t)) + j(g_2, v_1(t)) \le C|g_1 - g_2|_V |v_1 - v_2|_V. \tag{4.17}$$

Integrating the (4.15) inequality with respect to time, using the initial conditions $v_2(0) = v_1(0) = v_0$, using (3.7), (4.17)and the inequality $2ab \le \frac{C}{m_{\mathcal{A}}}a^2 + \frac{m_{\mathcal{A}}}{C}b^2$ we find

$$|v_2(t) - v_1(t)|_V^2 \le C\int_0^t |g_2(s) - g_1(s)|_V^2\,ds. \tag{4.18}$$

Thus, for m sufficiently large, Λ_η^m is a contraction on $\mathbb{L}^2(0,T;V)$ and so Λ_η has a unique fixed point in this Banach space. □

Third step: We use the displacement field $u_{g\eta}$ to consider the following variational problem.
Problem $PV_{g\eta}^{\varphi}$: Find an electric potential field $\varphi_{g\eta} : \Omega \times [0.T] \to W$ such that

$$(\beta\nabla\varphi_{g\eta}(t), \nabla\psi)_{\mathbb{L}^2(\Omega)^d} - (\xi\varepsilon(u_{g\eta}(t)), \nabla\psi)_{\mathbb{L}^2(\Omega)^d} = (q(t), \psi)_W, \ \forall\psi \in W, \ t \in [0,T]. \tag{4.19}$$

We have the following result for $PV_{g\eta}^{\varphi}$:

Lemma 4.4. *There exists a unique solution* $\varphi_{g\eta} \in W^{1.2}(0.T;W)$ *satisfies* (4.19), *moreover if* φ_1 *and* φ_2 *are two solutions to* (4.19). *Then, there exists a constants* $c > 0$ *sach that*

$$|\varphi_1(t) - \varphi_2(t)|_W \le c\,|u_1(t) - u_2(t)|_V \quad \forall t \in [0,T]. \tag{4.20}$$

Proof. The proof given in Ref (see [1]). □

Fourth step: For $\phi \in C(0,T;\mathbb{L}^2(\Omega))$, we consider the following variational problem.
Problem PV_ϕ: Find the damage field $\beta_\phi : [0,T] \to K$ such that

$$\begin{aligned}&\left(\dot{\beta}_\phi(t), \zeta - \beta_\phi(t)\right)_{\mathbb{L}^2(\Omega)} + a_1(\beta_\phi(t), \zeta - \beta_\phi(t)) \ge \\ &(\phi, \zeta - \beta_\phi(t))_{\mathbb{L}^2(\Omega)}, \forall \zeta \in K, \text{ a.e. } t \in [0,T],\end{aligned} \tag{4.21}$$

$$\beta_\phi(0) = \beta_0 \tag{4.22}$$

Lemma 4.5. *There exists a unique solution* β_ϕ *to the auxiliary problem* PV_ϕ *such that*

$$\beta_\phi \in W^{1.2}\left(0,T;\mathbb{L}^2(\Omega)\right) \cap \mathbb{L}^2(0,T;H^1(\Omega))$$

Proof. The proof given in Ref (see [3]). □

By taking into account the above results and the properties of the operators $\mathcal{B}$ and $\mathcal{G}$ and of the functions ψ and S, we may consider the operator

$$\begin{aligned}&\Lambda : C(0,T;V' \times \mathbb{L}^2(\Omega)) \to C(0,T;V' \times \mathbb{L}^2(\Omega)),\\ &\Lambda(\eta,\phi)(t) = (\Lambda_1(\eta)(t), \Lambda_2(\phi)(t)),\end{aligned} \tag{4.23}$$

$$\begin{aligned}&(\Lambda_1(\eta), w)_{V'\times V} = (\mathcal{B}(\varepsilon(u_\eta(t)), \beta_\phi(t)), w)\\ &+ \left(\int_0^t \mathcal{G}\left(\sigma_\eta(s) - \mathcal{A}(\varepsilon(\dot{u}_\eta(s))), \varepsilon(u_\eta(s))\right) ds + \xi^*\nabla(\varphi), w\right)\\ &+\phi(\dot{u}_\eta, w) \quad \forall w \in V,\end{aligned} \tag{4.24}$$

$$\Lambda_2(\phi)(t) = S(\varepsilon(u_\eta(t)), \beta_\phi). \tag{4.25}$$

We have the following result.

Lemma 4.6. *The mapping* $\Lambda(\eta,\phi) : [0,T] \to V' \times \mathbb{L}^2(\Omega)$ *has a unique element* $(\eta^*,\phi^*) \in C(0,T;V' \times \mathbb{L}^2(\Omega))$ *such that* $\Lambda(\eta^*,\phi^*) = (\eta^*,\phi^*)$

Proof. Let (η_1,ϕ_1), $(\eta_2,\phi_2) \in C(0,T;V' \times \mathbb{L}^2(\Omega))$ and $t \in [0,T]$. We use the notation $u_{\eta i} = u_i, \dot{u}_{\eta i} = v_{\eta i} = v_i, \beta_{\phi i} = \beta_i$, $\varphi_{\eta i} = \varphi_i$ and $\sigma_{\eta i} = \sigma_i$, for $i = 1,2$. Using (4.24) and the relations (3.7) – (3.9), we obtain

$$\begin{aligned}&|\eta_1(t) - \eta_2(t)|_{V'}^2 \le C(|\beta_1(t) - \beta_2(t)|_{\mathbb{L}^2(\Omega)}^2\\ &+|u_1(t) - u_2(t)|_V^2 + |v_1(t) - v_2(t)|_V^2\\ &+\int_0^t (|\sigma_1(s) - \sigma_2(s)|_{\mathcal{H}_1}^2 + |v_1(s) - v_2(s)|_V^2\\ &+|u_1(s) - u_2(s)|_V^2)ds + |\varphi_1(t) - \varphi_2(t)|_W^2\\ &+\phi(v_1, v_2(t)) - \phi(v_1, v_1(t)) - \phi(v_2, v_2(t)) + \phi(v_2, v_1(t)).\end{aligned} \tag{4.26}$$

From the definition of the functional ϕ given by (3.20), and using (3.6), (3.14) we have

$$\phi(v_1, v_2(t)) - \phi(v_1, v_1(t)) - \phi(v_2, v_2(t)) + \phi(v_2, v_1(t)) \le C\left|v_1(t) - v_2(t)\right|_V^2. \quad (4.27)$$

We have

$$\left|u_2(t) - u_1(t)\right|_V \le \int_0^t \left|v_2(s) - v_1(s)\right|_V ds$$

Taking into account that

$$\sigma_i(t) = \mathcal{A}(\varepsilon(\dot{u}_i(t))) + \eta_i(t), \ \forall t \in [0,T]. \quad (4.28)$$

By (2.1), and using (3.7), we find

$$\left|\sigma_1(s) - \sigma_2(s)\right|_{\mathcal{H}_1}^2 \le C\left(\left|v_1(t) - v_2(t)\right|_V^2 + \left|\eta_1 - \eta_2\right|_{V'}^2\right). \quad (4.29)$$

It follows that

$$\begin{aligned}&\left(\dot{v}_1(t) - \dot{v}_2(t), v_1(t) - v_2(t)\right)\\ &+ \left(\mathcal{A}\varepsilon(v_1(t)) - \mathcal{A}\varepsilon(v_2(t)), \varepsilon(v_1(t)) - \varepsilon(v_2(t))\right) +\\ &+ \left(\eta_1(s) - \eta_2(s), v_1(t) - v_2(t)\right) \le j(v_1, v_2(t)) - j(v_1, v_1(t))\\ &-j(v_2, v_2(t)) + j(v_2, v_1(t)).\end{aligned} \quad (4.30)$$

From the definition of the functional j given by (3.19), and using (3.6), (3.14) we have

$$j(v_1, v_2(t)) - j(v_1, v_1(t)) - j(v_2, v_2(t)) + j(v_2, v_1(t)) \le C\left|v_1 - v_2\right|_V^2. \quad (4.31)$$

Integrating the (4.30) inequality with respect to time, using the initial conditions $v_2(0) = v_1(0) = v_0$, using (3.7), (4.31), using Cauchy-Schwartz's inequality and the inequality

$$2ab \le m_{\mathcal{A}}a^2 + \frac{1}{m_{\mathcal{A}}}b^2,$$

by Gronwall's inequality we find

$$\left|v_1(t) - v_2(t)\right|_V^2 \le C\int_0^t \left|\eta_1(s) - \eta_2(s)\right|_{V'}^2 ds. \quad (4.32)$$

Also

$$\begin{aligned}\int_0^t \left|u_1(s) - u_2(s)\right|_V^2 ds &\le C\int_0^t\int_0^s \left|\eta_1(r) - \eta_2(r)\right|_{V'}^2 drds\\ &\le C\int_0^t \left|\eta_1(s) - \eta_2(s)\right|^2 ds.\end{aligned} \quad (4.33)$$

For the damage field, from (4.21) we deduce that

$$\left(\dot{\beta}_1 - \dot{\beta}_2, \beta_1 - \beta_2\right)_{\mathbb{L}^2(\Omega)} + a_1(\beta_1 - \beta_2, \beta_1 - \beta_2) \le (\phi_1 - \phi_2, \beta_1 - \beta_2)_{\mathbb{L}^2(\Omega)},$$

$$\text{a.e. } t \in [0,T].$$

Integrating the previous inequality with respect to time, using the initial conditions $\beta_1(0) = \beta_2(0) = \beta_0$ and the inequality $a_1(\beta_2 - \beta_1, \beta_2 - \beta_1) \ge 0$, by Gronwall's inequality we find

$$\left|\beta_1(t) - \beta_2(t)\right|_{\mathbb{L}^2(\Omega)}^2 \le C\int_0^t \left|\phi_1(s) - \phi_2(s)\right|_{\mathbb{L}^2(\Omega)}^2 ds, \ \forall t \in [0,T]. \quad (4.34)$$

Applying the previous inequalities, the estimates (4.32) – (4.34), we obtain

$$|\Lambda(\eta_2,\phi_2)(t)-\Lambda(\eta_1,\phi_1)(t)|_{V'\times\mathbb{L}^2(\Omega))}\leq C\int_0^t|(\eta_2,\phi_2)(s)-(\eta_1,\phi_1)(s)|\,ds$$

Thus, for m sufficiently large, Λ^m is a contraction on $C(0,T;V'\times\mathbb{L}^2(\Omega)))$ and so Λ has a unique fixed point in this Banach space. □

We consider the operator $\mathcal{L}:C(0,T;\mathbb{L}^2(\Gamma_3))\to C(0,T;\mathbb{L}^2(\Gamma_3))$

$$\mathcal{L}\omega(t)=-kv^*\int_0^t\sigma_\nu(s)\,ds,\forall t\in[0,T]. \tag{4.35}$$

Lemma 4.7. *The operator* $\mathcal{L}:C(0,T;\mathbb{L}^2(\Gamma_3))\to C(0,T;\mathbb{L}^2(\Gamma_3))$ *has a unique element* $\omega^*\in C(0,T;\mathbb{L}^2(\Gamma_3))$, *such that* $\mathcal{L}\omega^*=\omega^*$.

Proof. Using (4.35) , we have

$$|\mathcal{L}\omega_1(t)-\mathcal{L}\omega_2(t)|^2_{\mathbb{L}^2(\Gamma_3)}\leq kv^*\int_0^t|\sigma_1(s)-\sigma_2(s)|^2\,ds, \tag{4.36}$$

From (2.1) and using (3.7) – (3.9) , we find

$$\begin{aligned}|\sigma_1(t)-\sigma_2(t)|^2_{\mathcal{H}_1}\leq C(&|\beta_1(t)-\beta_2(t)|^2_{\mathbb{L}^2(\Omega)}+|u_1(t)-u_2(t)|^2_V+|v_1(t)-v_2(t)|^2_V\\&+\int_0^t(|\sigma_1(s)-\sigma_2(s)|^2_{\mathcal{H}_1}+|v_1(s)-v_2(s)|^2_V+|u_1(s)-u_2(s)|^2_V)ds\\&+|\varphi_1(t)-\varphi_2(t)|^2_W\end{aligned} \tag{4.37}$$

By (4.26) , (4.34) , and by Gronwall's inequality we find

$$|\beta_1(t)-\beta_2(t)|^2_{\mathbb{L}^2(\Omega)}\leq C\int_0^t|u_1(s)-u_2(s)|^2_V\,ds,\ \forall t\in[0,T]. \tag{4.38}$$

And by Gronwall's inequality we find

$$|\sigma_1(t)-\sigma_2(t)|^2_{\mathcal{H}_1}\leq C\left(\int_0^t|u_1(s)-u_2(s)|^2_V\,ds+|u_1(t)-u_2(t)|^2_V+|v_1(t)-v_2(t)|^2_V\right) \tag{4.39}$$

We have

$$\int_0^t|u_1(s)-u_2(s)|^2_V\,ds+|u_1(t)-u_2(t)|^2_V+|v_1(t)-v_2(t)|^2_V$$
$$\leq C\int_0^t|v_1(s)-v_2(s)|^2_V\,ds.$$

So

$$\begin{aligned}&\textstyle\int_0^t|u_1(s)-u_2(s)|^2_V\,ds+|u_1(t)-u_2(t)|^2_V+|v_1(t)-v_2(t)|^2_V\\&\leq C\left(\int_0^t|v_1(s)-v_2(s)|^2_V\,ds++|\omega_1(t)-\omega_2(t)|^2_{\mathbb{L}^2(\Gamma_3)}\right)\end{aligned} \tag{4.40}$$

By Gronwall's inequality we find

$$\textstyle\int_0^t|u_1(s)-u_2(s)|^2_V\,ds+|u_1(t)-u_2(t)|^2_V+|v_1(t)-v_2(t)|^2_V\leq C\,|\omega_1(t)-\omega_2(t)|^2_{\mathbb{L}^2(\Gamma_3)}$$

So, we have

$$|\sigma_1(t) - \sigma_2(t)|^2_{\mathcal{H}_1} \leq C \int_0^t |\omega_1(s) - \omega_2(s)|^2_{\mathbb{L}^2(\Gamma_3)} ds \qquad (4.41)$$

Using (4.41), we find

$$|\mathcal{L}\omega_1(t) - \mathcal{L}\omega_2(t)|_{\mathbb{L}^2(\Gamma_3)} \leq C \int_0^t |\omega_1(s) - \omega_2(s)|_{\mathbb{L}^2(\Gamma_3)} ds$$

Thus, for m sufficiently large, $\mathcal{L}^m$ is a contraction on $C(0,T;\mathbb{L}^2(\Gamma_3))$ and so $\mathcal{L}$ has a unique fixed point in this Banach space. □

Now, we have all the ingredients to prove Theorem 4.1.
Existence. Let $g^* \in \mathbb{L}^2(0,T;V)$ be the fixed point of Λ_{η^*} defined by (4.14), let $(\eta^*, \phi^*) \in C(0,T;V' \times \mathbb{L}^2(\Omega)))$ be the fixed point of Λ defined by (4.23) – (4.25), let $\omega^* \in C(0,T;\mathbb{L}^2(\Gamma_3))$ be the fixed point of $\mathcal{L}\omega^*$ defined by (4.36), and let

$$(u, \varphi, \beta) = (u_{g^*\eta^*}, \varphi_{g^*\eta^*}, \beta_{\phi^*})$$

be the solutions of Problems $PV_{g^*\eta^*}$, and respectively $PV^{\varphi}_{g^*\eta^*}$, PV_{ϕ^*}. It results from (4.7), (4.8), (4.19), (4.21), (4.22) that $(u_{g^*\eta^*}, \varphi_{g^*\eta^*}, \beta_{\phi^*})$ is the solutions of Problems PV. Properties (4.1) – (4.6) follow from Lemmas 1, 3 and 4.
Uniqueness. The uniqueness of the solution is a consequence of the uniqueness of the fixed point of the operators Λ_η, $\Lambda, \mathcal{L}$ defined by (4.14), (4.23) – (4.25), (4.36) and the unique solvability of the Problem $PV_{g\eta}$ and PV_ϕ which completes the proof.

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Erratum to the paper Bogdan, M., ”Some comments on a linear programming problem”

Marcel Bogdan

Abstract. The present paper corrects an assertion of the author from [1]. The pivoting algorithms referred to, search for solving the linear programming problem.

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Keywords: Linear programming, pivoting algorithm, simplex algorithm, multiple optimal solutions.

1. Corrected assertion for the case of non-singleton solution

The standard form of a linear programming problem (LP) is $\min_{x\in S} c^T x$, where $S = \{x \in \mathbb{R}^n \,|\, A \cdot x = b, x \geq_{\mathbb{R}^n} 0_{\mathbb{R}^n}\}$, with data $c \in \mathbb{R}^n$, $A \in M_{m,n}(\mathbb{R})$, and $b \in \mathbb{R}^m$ given ([2]). Denote by $\mathcal{S}$ the set of its solutions. This paper corrects an assertion of the author from [1]. The motivation of the mentioned paper started from the clear difference between the two expressions *finding a solution to the problem* and *solving the problem*, especially when the feasible set S is not bounded. The author was not aware by the paper [5], having the same topic.

In order to correct the assertion from Proposition 3.2 in [1] into Proposition 1.3, Proposition 1.4, and Proposition 1.5, we provide the following two examples.

Example 1.1. Let $a > 0$, $b_1, b_2 > 0$ and the linear programming problem

$$\begin{cases} -x_1 - x_2 - ax_3 \longrightarrow min \\ \quad x_1 + x_2 \qquad\quad \leq b_1 \\ \qquad\quad x_2 + x_3 \leq b_1 + b_2 \\ \qquad\qquad\quad x_3 \leq b_2 \\ x_1, x_2, x_3 \geq 0. \end{cases}$$

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The four iterations are given in Figure 2 and by the classic primal simplex algorithm one may find as solutions $x^1 = (0, b_1, b_2)$ or $x^2 = (b_1, 0, b_2)$. By the extended algorithm

$$\mathcal{S} = \begin{cases} \{x^1\}, & \text{if } a > 1 \\ \{x^2\}, & \text{if } a < 1 \\ co\{x^1, x^2\}, & \text{if } a = 1. \end{cases}$$

1	A^4	A^5	A^6	
A^1	1	0	0	1
A^2	1	1	0	1
A^3	0	1	1	a
	b_1	b_1+b_2	b_2	

2	A^2	A^5	A^6	
A^1	1	-1	0	0
A^4	1	-1	0	-1
A^3	0	1	1	a
	b_1	b_2	b_2	

3	A^2	A^5	A^3	
A^1	1	-1	0	0
A^4	1	-1	0	-1
A^6	0	-1	1	-a
	b_1	0	b_2	

4	A^1	A^5	A^3	
A^2	1	1	0	0
A^4	1	0	0	-1
A^6	0	-1	1	-a
	b_1	b_1	b_2	

FIGURE 1. Optimal 0−max dual basis; bounded $\mathcal{S}$

Note that in the third tableau, for $a = 1$ we have $\max_{i\in\overline{\mathcal{B}}} \alpha_{i0} = 0 = \alpha_{10}$ and $\min_{j\in\mathcal{B}} \alpha_{0j} = 0 = \alpha_{05}$. More, it exists $j = 2 \in \mathcal{B}$ such that $\alpha_{12} = 1 > 0$.

Example 1.2. Let $a \in \mathbb{R}$ and the linear programming problem

$$\begin{cases} x_2 \longrightarrow min \\ -x_1 + x_3 = 0 \\ ax_1 + x_2 + x_4 = 1 \\ x_1, x_2, x_3, x_4 \geq 0. \end{cases}$$

From tables in Figure 3 we have

$$\mathcal{S} = \begin{cases} \{(0, 0, 0, 1)\}, & \text{if } a < 0 \\ \{(\alpha/a, 0, \alpha/a, 1-\alpha) \mid \alpha \in [0, 1]\}, & \text{if } a > 0 \\ \{(\alpha, 0, \alpha, 1) \mid \alpha \geq 0\}, & \text{if } a = 0. \end{cases}$$

1	A^3	A^4	
A^1	-1	a	0
A^2	0	1	-1
	0	1	

2	A^3	A^1	
A^4	1/a	1/a	0
A^2	1/a	1/a	-1
	1/a	1/a	

FIGURE 2. Optimal 0−max dual basis; unbounded $\mathcal{S}$

Note that in the first tableau, we have $\alpha_{10} = 0$ and $\alpha_{03} = 0$. For $a < 0$, one has $\alpha_{1j} < 0$, $j \in \mathcal{B} = \{3, 4\}$. For $a > 0$, it exists $j = 4 \in \mathcal{B}$ such that $\alpha_{14} = 1 > 0$.

An important step in the implementation of an algorithm should be a criteria that establishes the boundedness of the feasible set, weather or not it is a polytope (bounded, thus compact) or not. In its absence, the property of the solutions set regarding the boundedness is to be stated and the set itself to be obtained while the algorithm works. Related to the set of solutions for (LP), we give the following results.

Proposition 1.3. *Suppose that $x^0 = x^B$ is an optimal solution for (LP) and let B be the optimal basis. If $\max_{i \in \overline{\mathcal{B}}} \alpha_{i0} < 0$, then there is no other solution generated by B.*

Proof. Suppose $x^{B'}$ is another solution generated by B, thus $c^T x^{B'} = c^T x^0$. Let $h \in \overline{\mathcal{B}}$ and suppose that the vector A^k is replaced by A^h in B. Let $\theta = \frac{\alpha_{0k}}{\alpha_{hk}} \geq 0$ be the rate transfer. Therefore, we have

$$c^T x^0 = c^T x^{B'} = c^T x^0 + \theta \cdot (-\alpha_{h0}).$$

If $\theta > 0$ we get the contradiction since $\alpha_{h0} < 0$. If $\theta = 0 = \alpha_{0k}$, then the pivoting element α_{hk} must be strictly negative. For $j \in \mathcal{B}' = (\mathcal{B} \setminus \{k\}) \cup \{h\}$, the coordinates of $x^{B'}$ are $\alpha'_{0j} = \alpha_{0j}$ and $\alpha_{0h} = \frac{\alpha_{0k}}{\alpha_{hk}} = 0$, that is $x^{B'} = x^0$, a contradiction. □

Proposition 1.4. *Let $x^0 = x^B$ be a solution for (LP) obtained in* Step 1 *of the algorithm and B be the optimal basis. Suppose that*

$$\max_{i \in \overline{\mathcal{B}}} \alpha_{i0} = 0 = \min_{j \in \mathcal{B}} \alpha_{0j}.$$

Denote by $\overline{\mathcal{B}}_0 = \{i \in \overline{\mathcal{B}} \,|\, \alpha_{i0} = 0\}$ and $\mathcal{B}_0 = \{i \in \mathcal{B} \,|\, \alpha_{0j} = 0\}$. The following implications apply:

1. *if $\alpha_{\bar{i}j} \leq 0$, $\forall \bar{i} \in \overline{\mathcal{B}}_0$, $\forall j \in \mathcal{B}$, then $\mathcal{S}$ is unbounded.*
2. *if $\alpha_{\bar{i}j_0} > 0$, $\forall \bar{i} \in \overline{\mathcal{B}}_0$, $\forall j_0 \in \mathcal{B}_0$, then there is no other solution generated by B.*
3. *if exist $\bar{i} \in \overline{\mathcal{B}}_0$, $k \in \mathcal{B} \setminus \mathcal{B}_0$ such that $\alpha_{\bar{i}k} > 0$, then the solution is not unique.*

Proof. 1. Let $\bar{i} \in \overline{\mathcal{B}}_0$. Since $\alpha_{\bar{i}j} \leq 0$, $\forall j \in \mathcal{B}$, there is no pivoting element, consequently another solution cannot be obtained by a classic pivoting operation. There exists $\bar{c} > 0_{\mathbb{R}^n}$ such that

$$\{x^0 + \alpha \cdot c \,|\, \alpha \geq 0\} \subseteq \mathcal{S}.$$

The unboundedness direction $\bar{c} = (\bar{c}_1, ..., \bar{c}_n)$ is given by

$$\bar{c}_j = \begin{cases} -\alpha_{\bar{i}j}, & j \in \mathcal{B} \setminus \mathcal{B}_{\bar{i}0} \\ 1, & j = \bar{i} \\ 0, & \text{otherwise,} \end{cases}$$

where $\mathcal{B}_{\bar{i}0} = \{k \in \mathcal{B} \,|\, \alpha_{\bar{i}k} = 0\}$.

2. Let $\bar{i} \in \overline{\mathcal{B}}_0$ and $j_0 \in \mathcal{B}_0$. Since $\alpha_{0j_0} = 0$, by replacing vector A^{j_0} with $A^{\bar{i}}$ in B, the value of the objective function does not change

$$c^T x^0 - \alpha_{\bar{i}0} \cdot \frac{\alpha_{0j_0}}{\alpha_{\bar{i}j_0}} = c^T x^0.$$

Let $x^{B'}$ be the new optimal solution. Its coordinates $x_j^{B'}$ are $x_j^{B'} = x_j^B$, for $j \in \mathcal{B} \setminus \{j_0\}$, $\alpha'_{0\bar{i}} = \alpha_{0j_0} = 0$, and 0 in rest, (i.e. $j \notin \mathcal{B}' = (\mathcal{B} \setminus \{j_0\}) \cup \{\bar{i}\}$), therefore $x^{B'} = x^B$.

3. Let $\bar{i} \in \overline{\mathcal{B}}_0$. Consider $\alpha_{\bar{i}k}$ as pivoting element. By replacing vector A^k with $A^{\bar{i}}$, the value of the objective function does not change

$$c^T x^0 - \alpha_{\bar{i}0} \cdot \frac{\alpha_{0k}}{\alpha_{\bar{i}k}} = c^T x^0.$$

The coordinates of the new solution $x^{B'}$ are $x_j^{B'} = x_j^B - \alpha_{\bar{i}j} \cdot \frac{\alpha_{0k}}{\alpha_{\bar{i}k}}$, for $j \in \mathcal{B} \setminus \{k\}$, $\alpha'_{0\bar{i}} = \frac{\alpha_{0k}}{\alpha_{\bar{i}k}} \neq 0$, and 0 in rest, including $\alpha'_{0k} = 0$, thus $x^{B'} \neq x^B$.

□

Corresponding to item 3 above, by Example 1.1, we have $\bar{i} = 1 \in \overline{\mathcal{B}}_0$, $k = 2 \in \mathcal{B} \setminus \mathcal{B}_0$ with $\alpha_{12} > 0$. Similarly, by Example 1.2, case $a > 0$, we have $\overline{\mathcal{B}}_0 = \{1\}$ and it exists $k = 2 \in \mathcal{B} \setminus \mathcal{B}_0$ such that $\alpha_{14} > 0$.

Proposition 1.5. ([1]) *Suppose that $x^0 = x^B$ is a solution for (LP) obtained in* Step 1 *of the algorithm and that B is the optimal basis. If $\max_{i \in \overline{\mathcal{B}}} \alpha_{i0} = 0$ and $\min_{j \in \mathcal{B}} \alpha_{0j} > 0$, then*

$$\{x^0\} \subsetneq \mathit{argmin}_{x \in S} c^T x;$$

if more

α) for some $\bar{i} \in \overline{\mathcal{B}}_0$, it exists $k \in \mathcal{B}$ such that $\alpha_{\bar{i}k} > 0$, then

$$co\{x^0, x^1, ..., x^u\} \subseteq \mathcal{S},$$

with $u \leq card\,\overline{\mathcal{B}}_{0+}$, where

$$\overline{\mathcal{B}}_{0+} = \{i \in \overline{\mathcal{B}}_0 \,|\, \exists k \in \mathcal{B} \text{ such that } \alpha_{ik} > 0\};$$

β) for some $\bar{i} \in \overline{\mathcal{B}}_0$, $\alpha_{\bar{i}k} \leq 0, \forall k \in \mathcal{B}$, then the set of solutions is (convex) unbounded.

Proof. α) The proof is the same to 3. from Proposition 1.4 since $\mathcal{B}_0 = \emptyset$.

β) see [1], Proposition 3.2, 3., a3β). □

Most of the works contain background, terminology, usual notations, and basic results. Let us remind some of them. A vector $x \in \mathbb{R}^n$ is seen as a column vector and its transpose, denoted by $x^T = (x_1, ..., x_n) \in \mathbb{R}^n$, as a row vector. In particular, denote by $0^T_{\mathbb{R}^n} = (0, ..., 0) \in \mathbb{R}^n$ and by $e^T_j = (0, ..., 0, 1, 0, ..., 0)$ with 1 for the j^{th} position. The scalar product of $c \in \mathbb{R}^n$ and x is given by

$$c^T x = \sum_{i=1}^{n} c_i \cdot x_i.$$

The relation $x \geq_{\mathbb{R}^n} 0_{\mathbb{R}^n}$ means $x_i \geq 0$, for all $i \in \{1, ..., n\}$; one has $x >_{\mathbb{R}^n} 0_{\mathbb{R}^n}$ iff $x_i \geq 0$, $\forall i$ and $\exists i_0$ with $x_{i_0} > 0$. Also, $x \leq_{\mathbb{R}^n} y$ iff $y - x \geq 0_{\mathbb{R}^n}$, and $x >_{\mathbb{R}^n} y$ iff $x - y >_{\mathbb{R}^n} 0_{\mathbb{R}^n}$. The $j-$th column of a matrix $A \in M_{m,n}(\mathbb{R})$ is denoted by A^j; a matrix B consisting of m independent columns of A, $m < n$, is called *basic*. The remaining columns of A that are not in B are said to be outside the basis or nonbasic. For $A = (A^j)_{1 \leq j \leq n}$,

$$\mathcal{B} = \{j \in \{1, 2, ..., n\} \mid \exists k, A^j = B^k\},$$

$$\overline{\mathcal{B}} = \{1, 2, ..., n\} \setminus \mathcal{B} = \{i \in \{1, 2, ..., n\} \mid \nexists k, A^i = B^k\},$$

so $\{A^j\}_{1 \leq j \leq n} = \{\{A^j\}_{j \in \mathcal{B}}, \{A^i\}_{i \in \overline{\mathcal{B}}}\}$. The linear combination for A^i, $i \in \overline{\mathcal{B}}$, is given by

$$A^i = \sum_{j \in \mathcal{B}} \alpha_{ij} A^j.$$

The coordinates of b are α_{0j}, i.e.

$$b = \sum_{j \in \mathcal{B}} \alpha_{0j} A^j.$$

A basic matrix B is said to be *primal feasible* if $\alpha_{0j} \geq 0$, $\forall j \in \mathcal{B}$ and *dual feasible* if $\alpha_{i0} \leq 0$, $\forall i \in \overline{\mathcal{B}}$, respectively. If B is primal feasible and dual feasible then B is called *optimal* or *optimal basis*. Simplex algorithm 2.0, is based on the extended properties of the optimal basis. About an optimal basis B, we say that it is $0-$*max dual feasible* if $\max_{i \in \overline{\mathcal{B}}} \alpha_{i0} = 0$ and $0-$*min primal feasible* if $\min_{j \in \mathcal{B}} \alpha_{0j} = 0$, respectively.
As regards the three situations of the algorithm, when there is a unique solution, a bounded set of solutions (but not a singleton) or the unbounded set of solutions, we reformulate the following result in concordance to Proposition 1.3, Proposition 1.4, and Proposition 1.5.

Theorem 1.6. *Suppose that $x^0 = x^B$ is a solution for (LP) obtained in* Step 1 *of the algorithm and that B is the optimal basis. The following implications apply:*

1. *If B is NOT $0-$max dual feasible then the solution generated by B is unique.*
2. *If B is $0-$max dual feasible, $0-$min primal feasible, and $\alpha_{\bar{i}j} \leq 0$, $\forall \bar{i} \in \overline{\mathcal{B}}_0$, $\forall j \in \mathcal{B}$, then the set of solutions is unbounded.*
3. *If B is $0-$max dual feasible, $0-$min primal feasible, and $\alpha_{\bar{i}j} > 0$, $\forall \bar{i} \in \overline{\mathcal{B}}_0$, $\forall j \in \mathcal{B}_0$, then the solution generated by B is unique.*
4. *If B is $0-$max dual feasible, $0-$min primal feasible, and exist $\bar{i} \in \overline{\mathcal{B}}_0$, $k \in \mathcal{B} \setminus \mathcal{B}_0$ such that $\alpha_{\bar{i}k} > 0$, then the solution generated by B is not unique.*

5. *If B is $0-max$ dual feasible and it is NOT $0-min$ primal feasible, then the solution generated is not unique.*

In general, the computer algebra systems such as Octave [3], WolframAlpha [4], use the interior point algorithm implemented (the function *glpk* in Octave has the the parameter *param* that allows to use two-phase primal/dual simplex). None of these return more than one solution at one input. When the instruction/command has the parameter for the initial starting point (ex. Matlab), by changing it may be successful for returning another solution if it exists.

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Book reviews

Vijay Gupta and Michael Th. Rassias, *Computation and Approximation,*
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In the book under review, the authors have treated exponential, semi exponential, integral and hybrid operators, some of which have never been studied in the past due to their complex behavior. The book consists of three chapters.

The first chapter presents a systematic list of exponential type operators. Some of these exponential type operators associated with $x(1+x)^2$, x^3, $2x^{3/2}$ and $2x^2$ have not been studied in the past in such detail. Hence these operators may attract the interest of researchers who may wish to investigate their properties in greater depth. Furthermore, within this chapter a flavor for a possible extension of exponential-type operators to semi-exponential operators has also been indicated.

The second chapter is devoted to the treatment of several recent as well as new families of integral type operators. The authors provide here a link between original operators and their Kantorovich variants. Certain possibilities for further generalizations of such operators are also given in this chapter. Integral extensions of operators as such are not exponential type operators, but by studying such operators one may investigate many operators simultaneously, rather than studying them individually. Some original operators and their approximation properties in ordinary and simultaneous approximation are also discussed.

The third chapter deals with the investigation of the difference between two operators. Here general estimates for the difference between operators having the same but also different fundamental functions are provided. Moreover, general estimates for the difference of operators having higher-order derivatives are also discussed. In order to exemplify the theoretical results, the authors provide quantitative estimates for the differences between certain operators in ordinary and in simultaneous approximation.

Overall, the book under review is very well written and treats an active and interesting area of research. It constitutes an important contribution in the literature devoted to approximation theory and I feel it will be a very valuable source for researchers, undergraduate and postgraduate students interested to study positive linear operators. This book would also be very useful for seminar use.

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