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# S T U D I A
# UNIVERSITATIS BABEŞ-BOLYAI

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# 2

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**CONTENTS**

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# Professor GABRIELA KOHR – A life for research. In memoriam

Mirela Kohr and Grigore Ştefan Sălăgean

**Abstract.** We review the main contributions of Professor Gabriela Kohr in her studies in the Geometric function theory in several complex variables and complex Banach spaces with special emphasis on the theory of Loewner chains and the Loewner differential equation.

**Mathematics Subject Classification (2010):** 32A18, 32A30, 32K05, 30C45, 30C80, 30D45.

**Keywords:** Gabriela Kohr, geometric function theory of one and several complex variables, Loewner theory, mappings with parametric representation, extension operator, complex analysis.

## 1. Scientific activity of Professor Gabriela Kohr

Gabriela Kohr was born on November 20, 1967 in Teiuş (Alba), Romania.

### 1.1. Studies and degrees

- PhD in Mathematics, Babeş-Bolyai University, Cluj-Napoca, 1996.
  PhD thesis: *Contributions to the theory of univalent functions*, supervisor Professor Petru T. Mocanu, Member of the Romanian Academy.
- Licensed in Mathematics, Faculty of Mathematics, Cluj-Napoca, Romania (1986-1991).
- High School Aiud (Alba) (1982-1986).

### 1.2. Academic positions

- Teaching Assistant, Faculty of Mathematics and Computer Science, Babeş-Bolyai University, October 1991 - September 1997
- Lecturer, Faculty of Mathematics and Computer Science, Babeş-Bolyai University, October 1997 - September 2000

- Associate Professor, Faculty of Mathematics and Computer Science, Babeş-Bolyai University, October 2000-September 2006
- Professor, Faculty of Mathematics and Computer Science, Babeş-Bolyai University, October 2006-December 2020
- PhD Supervisor, Faculty of Mathematics and Computer Science, Babeş-Bolyai University, 2007-2020. PhD students: Mihai Iancu, Teodora Chirilă.

**Academic awards and distinctions**: The *Spiru Haret* award of the Romanian Academy in 2005 for the monograph: I. Graham, G. Kohr, Geometric Function Theory in One and Higher Dimensions, Marcel Dekker Inc., New York, Basel, 2003; Awards of Babeş-Bolyai University for excellence in science.
**Research interests**: Complex Analysis, Geometric Function Theory of One and Several Complex Variables.
**Research visits**: **Annual research visits in the period 1999-2020 at University of Toronto, Department of Mathematics, Toronto (Canada). Research collaboration with Professor Ian Graham**. Many other research visits at universities in Japan, SUA, Italy, Germany, England, Poland.
**Research Projects**: Coordinator of **6** national research projects (CNCSIS, UEFISCDI, Babeş-Bolyai University), member in various (international and national) research teams.
**Invited/plenary speaker** to many international conferences, or workshops, in: SUA, Canada, Japan, Italy, Germany, Finland, Norway, Sweden, Spain, Poland, Turkey, Romania.

### 1.3. Teaching activity

The lectures of Prof. Gabriela Kohr covered the following topics: Complex Analysis, Special topics in complex analysis, Complex analysis in one and higher dimensions, Geometric function theory in several complex variables, Special topics of real and complex analysis, Univalent functions and differential subordinations, Applications of complex analysis in physics, Special topics in real analysis.

The books [54] and [58] of Gabriela Kohr ([58] in collaboration) are highly appreciated for their rigorous and careful presentation and for the covered topics of real interests for all students and researchers working in Complex Analysis and related areas. Gabriela Kohr had great pedagogical skills. She captivated the students attending her courses with her clarity in teaching and passion for mathematics. The students appreciated a lot the dedication and the fairness in every single lecture, exam, or any other teaching activity of Professor Gabriela Kohr.

### 1.4. Research contributions

The list of publications of Gabriela Kohr contains an impressive number of scientific articles published in prestigious international journals, such as: Mathematische Annalen, Journal of Functional Analysis, Transactions of the American Mathematical Society, Journal d'Analyse Mathématique, Journal of Geometric Analysis, Annali della Scuola Normale Superiore di Pisa, Classe di Scienze, Canadian Journal of Mathematics, Israel Journal of Mathematics, Constructive Approximation, Journal of Mathematical Analysis and Applications, Annali di Matematica Pura ed Applicata, Proceedings of the American Mathematical Society, Journal of Approximation Theory, Annales Academiæ Scientiarum Fennicé, Analysis and Mathematical Physics.

The monograph of Ian Graham and Gabriela Kohr [32] is highly cited and recognized as one of the best books in Geometry function theory in one and several complex variables. Gabriela Kohr is also coauthor of the monograph [57], and her scientific work has more than 2600 citations. The complete list of publications of Gabriela Kohr can be found at [56].

**The main research contributions of Professor Gabriela Kohr are devoted to**

**Geometric function theory in several complex variables**

- Extensions of classical results in the theory of univalent functions to the case of several complex variables
- Theory of Loewner chains of several complex variables
- Approximation and control theory results for normalized biholomorphic mappings
- Extension operators and quasiconformal mappings

**Geometric function theory in complex Banach spaces**

- Theory of Loewner chains in complex Banach spaces
- Infinite-dimensional versions (the case of reflexive complex Banach spaces) of classical results in the theory of Loewner chains and the Loewner differential equation
- Estimation results for normalized biholomorphic mappings in complex Banach spaces

**Complex analysis on bounded symmetric domains**

- Bloch mappings and related results on bounded symmetric domains
- Harmonic and pluriharmonic mappings on complex Hilbert balls and bounded symmetric domains

**Research collaborators**

Many of the strong research contributions of Professor Gabriela Kohr have been obtained in collaboration with mathematicians from all over the world, but two of them had a very strong role in her scientific life and evolution. We refer to the outstanding research collaboration of more than 20 years, started in 1999 up to the last moment of the life of Gabi, in 2020, with Professor *Ian Graham*, University of Toronto, Canada, and Professor *Hidetaka Hamada*, Kyushu Sangyo University, Fukuoka, Japan. The remarkable collaboration and friendship of Gabi with Ian and Hidetaka highlighted her life and leaded to important and well known contributions in Geometric function theory in complex spaces. Let us note that Toronto became for Gabriela and Mirela Kohr their second home between 1999 and 2020.

Gabriela Kohr was honored to collaborate with Professor Petru T. Mocanu from Babeş-Bolyai University, Cluj-Napoca, Romania, a well known mathematician for his contributions in Geometric function theory of one complex variable. Gabi had a very important research collaboration with Professor Filippo Bracci, University

"Tor Vergata", Rome, Italy, a strong and well known mathematician for his contributions in the field. Gabi was also influenced by her collaboration with Professors Peter L. Duren, University of Michigan, Ann Arbor, USA, Ted J. Suffridge, University of Kentucky, Lexington, USA, Jerry R. Muir Jr., University of Scranton, Scranton, PA, USA, Mirela Kohr, Babeş-Bolyai University, Cluj-Napoca, Romania, Paula Curt, Babeş-Bolyai University, Mihai Iancu, Babeş-Bolyai University, Cho-Ho Chu, Queen Mary, University of London, UK, Tatsuhiro Honda, Senshu University, Japan, John A. Pfaltzgraff, University of North Carolina, USA, Piotr Liczberski, Technical University of Lodz, Poland, Martin Chuaqui, Universidad Católica de Chile, Chile, Rodrigo Hernández, Universidad Adolfo Ibánez, Vińa del Mar, Chile, and many others.

The meetings and discussions of Gabi with Professors David Shoikhet, Holon Institute of Technology and Braude Academic College, Karmiel, Israel, Oliver Roth, University of Wuerzburg, Germany, Mark Elin, Ort Braude College, Karmiel, and Grigore Sălăgean, Babeş-Bolyai University, inspired Gabi in all her research developments.

### 1.5. Main results

**A) The class $S^0(\mathbb{B}^n)$ of normalized univalent mappings on $\mathbb{B}^n$ that have parametric representation**

- The class of normalized univalent mappings in the Euclidean unit ball $\mathbb{B}^n$ of $\mathbb{C}^n$ is denoted by $S(\mathbb{B}^n)$. Therefore,

$$S(\mathbb{B}^n) := \{f : \mathbb{B}^n \to \mathbb{C}^n : f \text{ univalent}, \ f(0) = 0, \ df(0) = \mathsf{id}\}.$$

In the case of one complex variable, that is, for $n = 1$, the class $S(\mathbb{U})$ is compact, and various extremal problems have been studied by Duren, Hallenbeck and MacGregor, Pommerenke, Roth, Schaeffer and Spencer, and many others. (Recall that $\mathbb{U}$ is the open unit disc in $\mathbb{C}$.) In addition, in the case $n = 1$, every function $f \in S(\mathbb{U})$ can be embedded into a normal Loewner chain. Various results about the structure of extreme points and support points of linear problems, in particular that these points are single-slit mappings, are well known (see e.g. the first part of the book of Ian Graham and Gabriela Kohr [32]).

In higher dimensions, $n \geq 2$, the class $S(\mathbb{B}^n)$ is not compact, and there are mappings in $S(\mathbb{B}^n)$ that can not be embedded as the first element of a normalized Loewner chain (see e.g. [14] for $n = 3$), and there are no single-slit mappings.

However, in the case $n \geq 2$, the *subclass $S^0(\mathbb{B}^n)$ of $S(\mathbb{B}^n)$, consisting of mappings that have parametric representation*, is *compact*. This subclass was introduced by **I. Graham, H. Hamada** and **G. Kohr** in their remarkable paper [16]. See also the valuable book of Ian Graham and Gabriela Kohr [32] [*Geometric Function Theory in One and Higher Dimensions*, Marcel Dekker, New York, 2003]. Note that

$$S^0(\mathbb{B}^n) = \big\{f \in S(\mathbb{B}^n) : \exists\, f(z,t) \text{ Loewner chain such that } \{e^{-t} f(\cdot,t)\}_{t\geq 0} \text{ is a normal family and } f = f(\cdot,0)\big\}.$$

The class $S^0(\mathbb{B}^n)$ does not have a linear structure in the case $n = 1$, but also in higher dimensions. Therefore, it is important to analyze both linear and nonlinear extremal problems in the class $S^0(\mathbb{B}^n)$. One of the main difficulties that appears in the study of univalent mappings in higher dimensions is that the lack of an uniformization theorem does not allow to construct easily variations of a given normalized Loewner chain. **F. Bracci, I. Graham, H. Hamada** and **G. Kohr** [4] used a variational method and defined a natural class of normalized Loewner chains that they called *geräumig* Loewner chains. This method allows to construct other normalized Loewner chains with the property that from some time on, they coincide with the initial geräumig Loewner chain. The authors used their variational method in the analysis of extreme and support points of $S^0(\mathbb{B}^n)$. In particular, in the contrast to the case $n = 1$, in higher dimensions $n \geq 2$, Bracci, Graham, Hamada and Kohr [4] constructed an example of a family of mappings in $S^0(\mathbb{B}^n)$, which are bounded by a constant $M > 1$, and are not support points, nor extreme points of $S^0(\mathbb{B}^n)$. *Graham, Hamada, Kohr* and *Kohr* [26] extended these results to ***A*-normalized Loewner chains** by using their results obtained in [25] (see also [33]).

In addition, in a series of papers, [18, 26, 27, 47], Gabriela Kohr and her collaborators obtained bounded support points for various subclasses of $S(\mathbb{B}^n)$ and $S(\mathbb{U}^n)$ for $n \geq 2$, as well as for $S(\mathbb{B}_X)$, where $\mathbb{B}_X$ is a finite dimensional bounded symmetric domain with rank $r \geq 2$ in a finite dimensional complex Banach space $X$, by extending to such domains a shearing process due to F. Bracci [3] on the unit ball $\mathbb{B}^2$ of $\mathbb{C}^2$. They also generalized the extremal type results of W.E. Kirwan (1980) and R. Pell (1980) for Loewner chains in higher dimensions.

**B) Loewner chains and the Loewner PDE in several complex variables**

- Subordination chains in $\mathbb{C}^n$ were first studied by J.F. Pfaltzgraff in 1974, by extending to higher dimensions the results related to the subordination and the Loewner differential equation (the Loewner PDE) obtained by Ch. Pommerenke in 1965 and J. Becker in 1972 in the case $n = 1$ (see [32] for further details). Moreover, Pfaltzgraff showed the uniqueness of the solution to the Loewner PDE on the Euclidean unit ball of $\mathbb{C}^n$. The *existence result to the Loewner PDE on the unit ball of* $\mathbb{C}^n$ was proved by **I. Graham, H. Hamada** and **G. Kohr** [16]. The *uniqueness result to the Loewner PDE on the unit ball in A-normalized case* was proved by **P. Duren, I. Graham, H. Hamada** and **G. Kohr** [11] by using the *higher dimensional Carathéodory kernel convergence theorem* obtained by them.

The existence and uniqueness theory of Loewner chains in $\mathbb{C}^n$ has been considered by several authors, and applications of this theory have been given to characterize various subclasses of biholomorphic mappings, geometric characterizations of biholomorphic mappings that have parametric representation, as well as univalence criteria. Suggestive results in this sense have been obtained by Gabriela Kohr and her collaborators in [22], [23], [24], [28], [36], [35], [40], [41], [42], [52], among many other relevant publications. The theory of Loewner chains in higher dimensions is a research area with a strong scientific contribution of Gabriela Kohr and her collaborators.

**L. Arosio, F. Bracci, H. Hamada** and **G. Kohr** [2] proved the existence result for the solutions of the Loewner differential equation on complete hyperbolic complex manifolds by using a geometric construction of Loewner chains on complete hyperbolic complex manifolds based on a new interpretation of Loewner chains as the direct limit of evolution families. *This is a new and strong approach to the Loewner theory in complete hyperbolic complex manifolds, based on iteration and semigroup theory.*

**C) Loewner chains and nonlinear resolvents of the Carathéodory family on the unit ball in $\mathbb{C}^n$**

- Assume that $f$ is the infinitesimal generator of a one-parameter semigroup of holomorphic self-maps of the open unit disc $\mathbb{U}$ of the complex plane $\mathbb{C}$. M. Elin, D. Shoikhet and T. Sugawa [13] studied the properties of a family of non-linear resolvent functions $J_r = (I + rf)^{-1}$ of $f$, $r \geq 0$, with the additional conditions $f(0) = 0$ and $f'(0) > 0$. Note that Elin, Shoikhet, and Sugawa showed that the resolvents $J_r$ determine an inverse Loewner chain with an associated Herglotz vector field of divergence type. **I. Graham, H. Hamada** and **G. Kohr** [21] extended the result of Elin, Shoikhet, and Sugawa to the case of the Euclidean unit ball $\mathbb{B}^n$ in higher dimensions, by using their strong methods obtained during a long collaboration of 20 years, starting with [16]. Moreover, they proved that $(1 + r)J_r$ can be embedded as the first element of a normal Loewner chain, that is, $(1+r)J_r \in S^0(\mathbb{B}^n)$, and it is not an extreme point, nor a support point of $S^0(\mathbb{B}^n)$ for each $r \geq 0$, and $n \geq 1$.

**D) Approximation properties by automorphisms of $\mathbb{C}^n$ and quasiconformal diffeomorphisms in $\mathbb{C}^n$**

- If $f$ is a biholomorphic mapping on the Euclidean unit ball $\mathbb{B}^n$ such that $f(\mathbb{B}^n)$ is a Runge domain, then $f$ can be approximated locally uniformly on $\mathbb{B}^n$ by automorphisms of $\mathbb{C}^n$, whenever $n \geq 2$. This is a strong result due to E. Andersén and L. Lempert [1]. Starting from this result, it is natural to ask whether $f$ can be approximated by automorphisms of $\mathbb{C}^n$ whose restrictions to $\mathbb{B}^n$ have the same geometric property of $f$. **H. Hamada, M. Iancu, G. Kohr** and **S. Schleißinger** [42] obtained a positive answer to this question whenever $f$ is a spirallike mapping, or a convex mapping. To this end, they used the version of the Carathéodory kernel convergence theorem to the higher dimensions, a result obtained by P. Duren, I. Graham, H. Hamada and G. Kohr [11], and the property that the spirallike domains with respect to some linear operator $A \in L(\mathbb{C}^n)$ are Runge domains, a result proved by H. Hamada [36]. In particular, it follows that the first elements of $A$-normalized normal Loewner chains can be approximated locally uniformly on $\mathbb{B}^n$ by automorphisms of $\mathbb{C}^n$ whose restrictions to $\mathbb{B}^n$ are the first elements of $A$-normalized normal Loewner chains, in the case when $A$ is nonresonant. Moreover, Hamada, Iancu and Kohr [40] proved that the first elements of $A$-normalized normal Loewner chains can be approximated locally uniformly on $\mathbb{B}^n$ by automorphisms of $\mathbb{C}^n$ whose restrictions to $\mathbb{B}^n$ are the first elements of $A$-normalized normal Loewner chains including the case when $A$ is resonant, by using a geräumig Loewner chain similar to that used by G. Kohr and her collaborators in [4] and [26].

**E) Extension operators and their mapping properties**

- K. Roper and T. Suffridge (1995) introduced their extension operator which extends locally univalent functions on the unit disc $\mathbb{U}$ in $\mathbb{C}$ to locally biholomorphic mappings on the Euclidean unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$. Starting with this result, various extension operators for locally univalent functions on the unit disc $\mathbb{U}$ in $\mathbb{C}$ to higher dimensional spaces have been extensively considered. Let us mention the results of Graham and Kohr [31], Graham, Kohr, and Kohr [34] and Graham, Hamada, Kohr, and Suffridge [30] in this research direction. Related to these operators, the preservation of subfamilies of starlike mappings, spirallike mappings, the first elements of Loewner chains and Bloch mappings by extension operators have been analyzed. Graham, Hamada, Kohr and Kohr [28] obtained a unified result that shows that the *first elements of g-Loewner chains are preserved by these extension operators*, where $g$ is a convex function on $\mathbb{U}$ with $g(0) = 1$ and $\Re g(\zeta) > 0$ on $\mathbb{U}$. To show this property, they used their covering theorem for convex functions on $\mathbb{U}$. In particular, $g$-starlike mappings are preserved by these extension operators. This result implies that *various subfamilies of starlike mappings, such as starlike mappings of order $\alpha$, strongly starlike mappings of order $\alpha$ and almost starlike mappings of order $\alpha$ are preserved by these extension operators* (see also [17], [30], [32], [55], for the preservation of starlike mappings, spirallike mappings, the first elements of Loewner chains and Bloch mappings by similar extension operators).

  By following the same idea of preservation of geometric and analytic properties of various classes of mappings, J. Pfaltzgraff and T. Suffridge (1999) introduced the extension operator which maps locally biholomorphic mappings on the Euclidean unit ball $\mathbb{B}^n$ of $\mathbb{C}^n$ to locally biholomorphic mappings on the Euclidean unit ball $\mathbb{B}^{n+1}$ of $\mathbb{C}^{n+1}$. Moreover, the extension operator of Pfaltzgraff and Suffridge preserves starlikess and the first elements of normal Loewner chains. Graham, Hamada and Kohr [20] proved that the Pfaltzgraff and Suffridge type extension operator maps the *first element of normal Loewner chains on finite dimensional bounded symmetric domains to the first element of normal Loewner chains on the unit ball in a higher dimensional space.* They used a Schwarz-Pick lemma for holomorphic self mappings of bounded symmetric domains, a result obtained by them.

**F) Bloch mappings and related results on bounded symmetric domains**

- The class of Bloch functions on the unit disc $\mathbb{U}$ in $\mathbb{C}$ have been analyzed by many authors, and has various applications. Bloch functions on a bounded homogeneous domain in $\mathbb{C}^n$ have been first considered by K.T. Hahn (1975) and later by R.M. Timoney (1980). Timoney showed that many of the characterizations of Bloch functions on the unit disc apply also to Bloch functions on a bounded homogeneous domain. He defined the Bloch functions by using the Bergman metric. Such an approach is not applicable in infinite dimensions. Chu, Hamada, Honda and Kohr [6] used the infinitesimal Kobayashi metric instead of the Bergman metric, and characterized Bloch functions on bounded symmetric domains, which may be infinite dimensional, by extending several known conditions for Bloch functions on the unit disc in the complex plane.

The Bloch space of a bounded symmetric domain, which may be infinite dimensional, and properties of composition operators have been studied by Chu, Hamada, Honda and Kohr in [7]. $\alpha$-Bloch mappings on bounded symmetric domains in $\mathbb{C}^n$ have been studied by Hamada and Kohr [46].

- A distortion theorem for Bloch functions has been obtained by M. Bonk (1990). Distortion results for locally biholomorphic Bloch mappings on bounded symmetric domains in $\mathbb{C}^n$ have been obtained by Chu, Hamada, Honda and Kohr [5]. As an application, they obtained a lower bound for the Bloch constant for various classes of locally biholomorphic Bloch mappings. The distortion bound and a lower bound for the Bloch constant were given by using the constant $2c(\mathbb{B}_X)$, which they call "the diameter" and is defined by the Bergman metric.

**G) Coefficient estimates for families of univalent mappings**

- Fekete and Szegö inequality for normalized univalent functions on the unit disc $\mathbb{U}$ is an inequality involving the second and the third coefficients of univalent analytic functions. A lot of work has been done in this sense for various subclasses of $S(\mathbb{U})$. In higher dimensions, Xu and Liu (2014) obtained the Fekete and Szegö inequality for a special subclass of normalized starlike mappings on the unit ball $\mathbb{B}$ of a complex Banach space. Starting with this result, various extensions on the Fekete and Szegö inequality in higher dimensions have been obtained. However, the Fekete and Szegö inequality for the family of all starlike mappings has not been obtained. Hamada, Kohr and Kohr [50] obtained the Fekete and Szegö inequality for all starlike mappings on $\mathbb{B}$ by using a more simple proof than those provided for the previous results. As a new result (including the case on the unit disc $\mathbb{U}$), they also give the Fekete-Szegö inequality for $(1+r)J_r$, where $J_r$ is the nonlinear resolvent mapping of $f$ in the Carathéodory family $\mathcal{M}(\mathbb{B})$.

**H) Boundary behavior of families of univalent mappings**

- The Schwarz lemma at the boundary has a main role in Complex analysis due to its various applications in the geometric function theory, the theory of quasiconformal mappings, and other areas. Note that Liu, Wang and Tang (2015) obtained a variant of the Schwarz lemma for holomorphic self-mappings at the boundary of the Euclidean unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$, and Wang and Ren (2017) considered holomorphic self-mappings of strongly pseudoconvex domains in $\mathbb{C}^n$. In all these results, the assumption that the mappings are holomporphic at a smooth boundary point is essential. Hamada [37] obtained a variant of the Schwarz lemma at the boundary for holomorphic self-mappings of finite dimensional irreducible bounded symmetric domains by using the Julia-Wolff-Carathéodory type condition. His proof is based on a study of the boundary behavior of the infinitesimal Kobayashi metric near smooth boundary points, by having in view an explicit expression of the infinitesimal Kobayashi metric of the unit ball of a finite dimensional JB*-triple. Extensions and related results on bounded symmetric domains have been obtained by Graham, Hamada and Kohr [19] and Hamada and Kohr [48, 49].

**I) Harmonic and pluriharmonic mappings in $\mathbb{C}^n$**

- Extensions to higher dimensions of corresponding results valid for planar harmonic mappings have been obtained by M. Chuaqui, H. Hamada, R. Hernández and G. Kohr [8], and H. Hamada and G. Kohr [45]. P. Duren, H. Hamada, G. Kohr [12] obtained two-point distortion theorems for univalent harmonic functions on the unit disc in $\mathbb{C}$ and pluriharmonic mappings in several complex variables.

**J)** One of the main research interests of Gabriela Kohr in **Geometric function theory of several complex variables** was related to the **extensions of classical results for univalent functions to the case of (finite or infinite dimensional) complex Banach spaces**. The main results in this direction are presented below.

**1. Geometric function theory in several complex variables (the finite dimensional case)**

(a) Univalence criteria for applications of class $C^1$ on the unit ball and strictly pseudo-convex domains in $\mathbb{C}^n$ for which the Bergman kernel becomes infinite on the boundary. Extensions of Jack's, Miller's and Mocanu's lemma on the unit ball and pseudo-convex domains in $\mathbb{C}^n$. Starlikeness and convexity of order $\alpha$, strongly starlikeness of order $\alpha$ on the Euclidean unit ball in $\mathbb{C}^n$: growth and covering results, and coefficient bounds. Alpha convex mappings on the Euclidean unit ball in $\mathbb{C}^n$: necessary and sufficient conditions, geometric and analytic characterizations. Spirallike mappings of type $\alpha$ on the unit ball in $\mathbb{C}^n$: geometric and analytic characterizations. Covering, growth and distortion results (most of them being sharp), and coefficient bounds for certain compact subclasses of normalized biholomorphic mappings on the unit ball in $\mathbb{C}^n$. Higher dimensional versions of classical results in the theory of linear invariant families (L.I.F's) of one complex variable. Necessary and sufficient conditions of univalence for mappings in L.I.F's (two-point distortion results). Two-point distortion results for affine linear invariant familes of harmonic and pluriharmonic mappings.

(b) Geometric and analytic properties of certain subclasses of $S(B^n)$ generated by the (generalized) Roper-Suffridge and the Pfaltzgraff-Suffridge operators: Starlikeness and convexity properties associated with the Roper-Suffridge extension operator. Growth and covering results, L.I.F's generated by the (generalized) Roper-Suffridge extension operator. Bloch mappings and the Roper-Suffridge extension operator. Loewner chains associated with the Roper-Suffridge extension operator. Extreme points and support points associated with certain compact subsets of $S(B^n)$ generated by the Roper-Suffridge extension operator.

**2. The theory of Loewner chains in several complex variables.** We highlight the **main results of Gabriela Kohr** and her collaborators in this important research direction.

(a) *Compactness of the Carathéodory class on the unit ball in* $\mathbb{C}^n$ (*n*-dimensional version of the class of holomorphic functions on the unit disc with positive real part).

(b) *The n-dimensional version of the well known Carathéodory result, concerning the equivalence between the kernel convergence and compact convergence of univalent functions.*

(c) *Existence and uniqueness of solutions to the generalized Loewner differential equation in* $\mathbb{C}^n$.
(d) *Analytic and geometric characterizations of various subclasses of* $S(B^n)$, *by using the method of Loewner chains.*
(e) Geometric characterizations of Loewner chains on the unit ball in $\mathbb{C}^n$: *Mappings which have parametric representation on the unit ball in* $\mathbb{C}^n$. *Growth, distortion results and coefficient bounds.*
*Asymptotically starlike/spirallike maps on the Euclidean unit ball in* $\mathbb{C}^n$. *Asymptotically spirallike mappings and non-normalized univalent subordination chains in* $\mathbb{C}^n$. Geometric aspects.
(f) *Sufficient conditions for quasiregular holomorphic mappings, which can be imbedded in Loewner chains, to have quasiconformal extensions of* $\mathbb{R}^{2n}$ *onto itself.*
(g) *General and abstract constructions of Loewner chains on hyperbolic complex manifolds.*
(h) *Extreme points, support points and the Loewner variation in several complex variables.*
(i) *Approximation properties of biholomorphic mappings with parametric representation on the unit ball in* $\mathbb{C}^n$ *by automorphisms of* $\mathbb{C}^n$ *and smooth quasiconformal diffeomeomorphisms of* $\mathbb{C}^n$ *onto itself* $(n \geq 2)$.
(j) *Loewner chains and nonlinear resolvents of the Carathéodory family on the unit ball in* $\mathbb{C}^n$.

**3. Geometric function theory in complex Banach spaces (the infinite dimensional case)**

(a) The study of various subclasses of $S(B)$ (the family of normalized biholomorphic mappings on $B$, where $B$ is the unit ball in a complex Banach space): spirallikeness of type $\alpha \in (-\pi/2, \pi/2)$, convexity and $\Phi$-likeness. Geometric and analytic aspects.
(b) Sharp growth and distortion results for normalized convex (biholomorphic) mappings on the unit balls of complex Hilbert spaces.
(c) *Infinite-dimensional versions* (*the case of reflexive complex Banach spaces*) *of classical results in the theory of Loewner chains and the Loewner differential equation.*
(d) Linear invariant families on unit balls in complex Hilbert spaces.

**Conclusions**

Many of the above mentioned strong results of Gabriela Kohr receive a great scientific recognition, being cited in valuable publications, and open new important research directions. The complete list of Gabi's publications can be found at [56].

Gabriela Kohr was an outstanding professor of Babeş-Bolyai University, an excellent researcher with a strong research activity and valuable contributions in Complex Analysis and Geometric function theory in several complex variables and complex Banach spaces. The main contributions of Gabriela Kohr refer to the generalization of the class $S$ to higher dimensions via the theory of Loewner chains and to the development of this theory with her collaborators, especially with Ian Graham (Canada) and Hidetaka Hamada (Japan), in order to recover important properties, but also to point out major differences. (Note that $S = S(\mathbb{U}) = S(\mathbb{B}^1)$.)

Until the last moment of her life, Gabriela Kohr was totally dedicated to the scientific research and teaching activity. Gabi attended many international conferences, enjoying them and involving herself in her presentations. Her talent for Mathematics went hand in hand with her enormously work capacity. To these two qualities, a strong sense of responsibility and high standards in her scientific activity were added.

It was a chance and privilege for all that met and worked with Professor Gabriela Kohr. Unfortunately, her life suddenly stopped when she was in full professional ascent, when she had much more to say mathematically speaking, when her disciples needed her competent guidance in their scientific research. The loss of Gabriela Kohr left a big hole in the soul of all that knew and worked with her, especially for Mirela Kohr. Gabi remains for all of us an excellent researcher, a very rigorous person, devoted to her students. She will be always alive in our thoughts trough all her great scientific contributions, and her deep ideas will continue to inspire generations of mathematicians working in Complex Analysis and Geometric Function Theory.

With the reader's permission, the first named author (M.K.) would like to add some personal recollections about her dear twin sister, Gabriela Kohr. Gabi adored and protected Mirela until the last moment of her life. They were a team working together all the time, day in, day out. An hour without math and Mirela was wasted hour in Gabi's opinion. Gabi was the essence of Mirela's life, and she will continue her mission through Mirela. This is Mirela's promise for Gabi.

The authors of this testimonial would like to express their gratitude to all collaborators, colleagues and friends of Professor Gabriela Kohr who made this special issue possible, as a tribute and recognition to her valuable research activity.

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# $g$-Loewner chains, Bloch functions and extension operators into the family of locally biholomorphic mappings in infinite dimensional spaces

Ian Graham, Hidetaka Hamada, Gabriela Kohr and Mirela Kohr

**Abstract.** In this paper, we survey recent results obtained by the authors on the preservations of the first elements of ($g$-) Loewner chains and the Bloch mappings by the Roper-Suffridge type extension operators, the Muir type extension operators and the Pfaltzgraff-Suffridge type extension operators into the mappings on the domains in the complex Banach spaces.

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## 1. Introduction

After Roper and Suffridge [46] introduced the following extension operator

$$\Phi(f)(z) = (f(z_1), \sqrt{f'(z_1)}\tilde{z}), \quad z = (z_1, \tilde{z}) \in \mathbb{B}^n,$$

which extends locally univalent functions on the unit disc $\mathbb{U}$ in $\mathbb{C}$ to locally biholomorphic mappings on the Euclidean unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$, the preservation of starlike mappings, spirallike mappings, the first elements of Loewner chains and Bloch mappings by similar extension operators have been extensively studied (see e.g. [3], [10], [11], [14], [19], [20], [21], [22], [33], [35], [36], [40], [41], [46], [48], [49] and [50]).

The Roper-Suffridge extension operator $\Phi$ preserves the following geometric and analytic properties from the one dimensional case to higher dimensions:

(i) $\Phi(S^*(\mathbb{B}^1)) \subseteq S^*(\mathbb{B}^n)$, where $S^*(\mathbb{B}^n)$ denotes the family of normalized starlike (univalent) mappings on $\mathbb{B}^n$ ([20]).

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(ii) If $f \in S$, where $S$ denotes the family of normalized univalent functions on $\mathbb{U}$, then $\Phi(f)$ can be embedded as the first element of a Loewner chain on $\mathbb{B}^n$ ([19], [22]).

(iii) $\Phi$ maps the family of normalized univalent Bloch functions on $\mathbb{U}$ with Bloch semi-norm 1 into the family of normalized univalent Bloch mappings on $\mathbb{B}^n$ ([20]).

For further properties of the Roper-Suffridge extension operator $\Phi$, see e.g. [46].

Let $\alpha \geq 0$, $\beta \geq 0$ be given. Then the modification $\Phi_{n,\alpha,\beta}$ of the Roper-Suffridge extension operator ([19]) is given by:

$$\Phi_{n,\alpha,\beta}(f)(z) = \left( f(z_1), \left( \frac{f(z_1)}{z_1} \right)^{\alpha} (f'(z_1))^{\beta} \tilde{z} \right), \quad z = (z_1, \tilde{z}) \in \mathbb{B}^n,$$

for any $f \in \mathcal{LS}(\mathbb{U})$ such that $f(z_1) \neq 0$ for $z_1 \in \mathbb{U} \setminus \{0\}$, where $\mathcal{LS}(\mathbb{U})$ denotes the family of normalized locally univalent functions on $\mathbb{U}$. The branches of the power functions are chosen such that

$$\left( \frac{f(z_1)}{z_1} \right)^{\alpha} \bigg|_{z_1=0} = 1 \quad \text{and} \quad (f'(z_1))^{\beta} \big|_{z_1=0} = 1.$$

The extension operator $\Phi_{n,\alpha,\beta}$ has the following properties:

(i) $\Phi_{n,\alpha,\beta}(S^*(\mathbb{B}^1)) \subseteq S^*(\mathbb{B}^n)$, for $\alpha, \beta \geq 0$ with $\alpha \leq 1$, $\beta \leq 1/2$ and $\alpha + \beta \leq 1$ ([19]).

(ii) If $f \in S$, then $\Phi_{n,\alpha,\beta}(f)$ can be embedded as the first element of a Loewner chain on $\mathbb{B}^n$, for $\alpha, \beta \geq 0$ with $\alpha \leq 1$, $\beta \leq 1/2$ and $\alpha + \beta \leq 1$ ([19]).

(iii) $\Phi_{n,0,\beta}$ maps the family of normalized univalent Bloch functions on $\mathbb{U}$ with Bloch semi-norm 1 into the family of normalized univalent Bloch mappings on $\mathbb{B}^n$, for all $\beta \in [0, 1/2]$ ([22]).

The Muir extension operator $\Phi_{n,Q}$, which is another modification of the Roper-Suffridge extension operator, is given by ([40])

$$\Phi_{n,Q}(f)(z) = \left( f(z_1) + Q(\tilde{z}) f'(z_1), \sqrt{f'(z_1)}\, \tilde{z} \right), \quad z = (z_1, \tilde{z}) \in \mathbb{B}^n,$$

where $f \in \mathcal{LS}(\mathbb{U})$ and $Q : \mathbb{C}^{n-1} \to \mathbb{C}$ is a homogeneous polynomial mapping of degree 2. The branch of the power function is chosen such that $\sqrt{f'(z_1)}\big|_{z_1=0} = 1$.

One of the properties of the Muir extension operator is as follows:

(i) $\Phi_{n,Q}(S^*(\mathbb{B}^1)) \subseteq S^*(\mathbb{B}^n)$ if and only if $\|Q\| \leq 1/4$ ([40]).

Muir [41] also studied the extension operator $\Phi_G : S \to S(\mathbb{B}^n)$ given by

$$\Phi_G(f)(z) = \left( f(z_1) + G\big(\sqrt{f'(z_1)}\, \tilde{z}\big), \sqrt{f'(z_1)}\, \tilde{z} \right), \quad z = (z_1, \tilde{z}) \in \mathbb{B}^n,$$

where $G : \mathbb{C}^{n-1} \to \mathbb{C}$ is a holomorphic function such that $G(0) = 0$ and $DG(0) = 0$, and the branch of the power function is chosen such that

$$\sqrt{f'(z_1)}\big|_{z_1=0} = 1.$$

Note that $DG(0)$ is the Frechét derivative of $G$ at 0. One of the properties of the extension operator $\Phi_G$ is as follows:

(i) If $\alpha \in [0, 1)$ and $\Phi_G(S^*(\alpha)) \subseteq S^*(\mathbb{B}^n)$, where $S^*(\alpha)$ denotes the family of all normalized starlike functions of order $\alpha$ on $\mathbb{U}$, then $G$ is a homogeneous polynomial of degree 2 from $\mathbb{C}^{n-1}$ into $\mathbb{C}$ and $\|G\| \leq 1/4$ ([41]).

Further study of the above operator has been given in [41], [50] (cf. [11]).

On the other hand, $g$-Loewner chains have been extensively studied in [13], [15], [17], [31]. Chirilă ([3], [4]) studied the preservation of the first elements of $g$-Loewner chains by the extension operators $\Phi_{n,\alpha,\beta}$ and $\Phi_{n,Q}$ on $\mathbb{B}^n$, in the case that $g(\zeta) = \frac{1-\zeta}{1+(1-2\gamma)\zeta}$ for $\zeta \in \mathbb{U}$ and $\gamma \in (0,1)$.

Let $\Phi_{n,r} : \mathcal{LS}(\mathbb{B}^n) \to \mathcal{LS}(\mathbb{B}^{n+r})$ be the Pfaltzgraff-Suffridge type extension operator, where $\mathcal{LS}(\mathbb{B}^n)$ denotes the family of normalized locally univalent mappings from $\mathbb{B}^n$ to $\mathbb{C}^n$, given by (see [23] and [43], in the case $r = 1$)

$$\Phi_{n,r}(f)(z) = \left(f(x), [J_f(x)]^{\frac{1}{n+1}} y\right), \quad z = (x,y) \in \mathbb{B}^{n+r}, \tag{1.1}$$

where $J_f(x)$ is the Jacobian determinant of $f$ at $x$, and $r \geq 1$ is an integer. The branch of the power function is chosen such that $[J_f(x)]^{1/(n+1)}\big|_{x=0} = 1$. We note that the operator $\Phi_{1,r}$ reduces to the Roper-Suffridge extension operator. The Pfaltzgraff-Suffridge type extension operator $\Phi_{n,r}$ has the following properties (see [23] in the case $r = 1$):

(i) $\Phi_{n,r}(S^*(\mathbb{B}^n)) \subseteq S^*(\mathbb{B}^{n+r})$.
(ii) If $f \in S(\mathbb{B}^n)$ can be embedded as the first element of a Loewner chain on $\mathbb{B}^n$, then $F = \Phi_{n,r}(f)$ can be embedded as the first element of a Loewner chain on $\mathbb{B}^{n+r}$.

Let $Y$ be a complex Banach space and let $r \geq 1$. Recently, the authors [18] studied the Roper-Suffridge type extension operator $\Phi_{\alpha,\beta}$ that provides a way of extending a locally univalent function $f$ on $\mathbb{U}$ to a locally biholomorphic mapping $F \in H(\Omega_r)$, where $\Omega_r = \{(z_1, w) \in \mathbb{C} \times Y : |z_1|^2 + \|w\|_Y^r < 1\}$ and proved the preservation result of the first element of a $g$-Loewner chain and the Bloch mappings by the Roper-Suffridge type extension operator $\Phi_{\alpha,\beta}$. They also studied the Muir type extension operator $\Phi_{P_k}$ that provides a way of extending a locally univalent function $f$ on $\mathbb{U}$ to a locally biholomorphic mapping $F \in H(\Omega_k)$, where $k \geq 2$ is an integer and $P_k : Y \to \mathbb{C}$ is a homogeneous polynomial mapping of degree $k$, and proved the preservation result of the first element of a Loewner chain and the Bloch mappings by the Muir type extension operator $\Phi_{P_k}$.

In [16], Graham, Hamada and Kohr have considered a generalization of the Pfaltzgraff-Suffridge extension operator on bounded symmetric domains in $\mathbb{C}^n$, and proved that if $\mathbb{B}_X$ is a bounded symmetric domain in $X = \mathbb{C}^n$, and $\mathfrak{F}_{n,\alpha}$ is an extension operator which maps normalized locally biholomorphic mappings on $\mathbb{B}_X$ to locally biholomorphic mappings on $\mathbb{D}_\alpha$, where $\mathbb{D}_\alpha \subseteq \mathbb{B}_X \times \mathbb{B}_Y$ is a certain domain with $\mathbb{B}_X \times \{0\} \subset \mathbb{D}_\alpha$, then $\mathfrak{F}_{n,\alpha}$ extends the first elements of Loewner chains from $\mathbb{B}_X$ to the first elements of Loewner chains on $\mathbb{D}_\alpha$, when $\alpha \geq n/(2c(\mathbb{B}_X))$, where $c(\mathbb{B}_X)$ is a constant defined by the Bergman metric on $X$ (see (5.1)). Also, they proved that normalized locally univalent I-Bloch mappings, which have finite trace order on $\mathbb{B}_X$, are mapped into R-Bloch mappings on $\Omega_\alpha$ by the operator $\mathfrak{F}_{n,\alpha}$ when $\alpha \geq 1/2$, where $\Omega_\alpha \subset X \times Y$ is a bounded balanced convex domain such that $\mathbb{B}_X \times \{0\} \subset \Omega_\alpha \subseteq \mathbb{D}_\alpha$.

In this paper, we survey the above results obtained in [16] and [18].

## 2. Preliminaries

Let $X$ and $Y$ be complex Banach spaces. Let $L(X,Y)$ denote the family of continuous linear operators from $X$ to $Y$. The family $L(X,X)$ is denoted by $L(X)$, and the identity in $L(X)$ is denoted by $I_X$. Let $\Omega \subset X$ be a domain which contains the origin and let $H(\Omega)$ be the family of holomorphic mappings from $\Omega$ into $X$. If a mapping $f \in H(\Omega)$ satisfies $f(0) = 0$, $Df(0) = I_X$, we say that $f$ is normalized, where $Df(z)$ is the Fréchet derivative of $f$ at $z$. Let $\mathcal{L}S(\Omega)$ denote the family of normalized locally biholomorphic mappings on $\Omega$ and let $S(\Omega)$ denote the family of normalized biholomorphic mappings on $\Omega$. Also, let $S^*(\Omega)$ (respectively, $K(\Omega)$) be the subset of $S(\Omega)$ consisting of starlike (respectively, convex) mappings on $\Omega$, where a mapping $f \in S(\Omega)$ is said to be starlike (respectively, convex) if $f(\Omega)$ is a starlike (respectively, convex) domain in $X$. The family $S(\mathbb{U})$ is denoted by $S$, where $\mathbb{U}$ is the unit disc in $\mathbb{C}$. The family $S^*(\mathbb{U})$ (respectively, $K(\mathbb{U})$) is denoted by $S^*$ (respectively, $K$).

**Definition 2.1 (cf. [29]).** *Let $X$ be a complex Banach space and let $\Omega \subseteq X$ be a bounded balanced domain. Also, let $\gamma \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and let $f \in H(\Omega)$. We say that $f$ is spirallike of type $\gamma$ on $\Omega$ if $f \in S(\Omega)$ and $\exp(-e^{-i\gamma}t)f(\Omega) \subseteq f(\Omega)$, for all $t \geq 0$.*

*In the case $\gamma = 0$, a spirallike mapping $f$ of type 0 is starlike in the usual sense.*

*Let $\widehat{S}_\gamma(\Omega)$ denote the family of spirallike mappings of type $\gamma$ on $\Omega$.*

**Assumption 2.1.** Let $g : \mathbb{U} \to \mathbb{C}$ be a univalent holomorphic function such that $g(0) = 1$ and $\Re g(\zeta) > 0$ on $\mathbb{U}$.

Next we recall the notions of subordination and Loewner chain on a complex Banach space $X$ (see e.g. [16], [18], [21] and [45]).

**Definition 2.2.** *Let $X$ be a complex Banach space and let $\Omega \subseteq X$ be a domain which contains the origin.*

(i) *If $f, g \in H(\Omega)$, we say that $f$ is subordinate to $g$ (denoted by $f \prec g$ ) if there exists a Schwarz mapping $v$ (i.e. $v \in H(\Omega)$, $v(0) = 0$ and $v(\Omega) \subseteq \Omega$ ) such that $f = g \circ v$.*

(ii) *A mapping $f : \Omega \times [0, \infty) \to X$ is called a univalent subordination chain if $f(\cdot, t)$ is univalent on $\Omega$, $f(0, t) = 0$ for $t \geq 0$, and $f(\cdot, s) \prec f(\cdot, t)$, $0 \leq s \leq t < \infty$. A univalent subordination chain $f : \Omega \times [0, \infty) \to X$ is called a Loewner chain if $f(\cdot, t)$ is biholomorphic on $\Omega$ and $Df(0, t) = e^t I_X$, for all $t \geq 0$.*

**Remark 2.3.** Note that if $f : \Omega \times [0, \infty) \to X$ is a Loewner chain, then the subordination condition is equivalent to the existence of a unique biholomorphic Schwarz mapping $v = v(\cdot, s, t)$, called the transition mapping associated with $f(x, t)$, such that $f(x, s) = f(v(x, s, t), t)$ for $x \in \Omega$ and $t \geq s \geq 0$. Also, $Dv(0, s, t) = e^{s-t} I_X$ for $t \geq s \geq 0$ (see e.g. [21]).

For various applications of the Loewner theory in the study of univalent mappings in higher dimensions, see e.g. [21, Chapter 8].

For $x \in X \setminus \{0\}$, we define

$$T(x) = \big\{ l_x \in L(X, \mathbb{C}) : \ l_x(x) = \|x\|_X, \ \|l_x\| = 1 \big\}.$$

Then $T(x) \neq \emptyset$ in view of the Hahn-Banach theorem.

Let $\mathbb{B}_X$ be the unit ball of a complex Banach space $X$. Next, we recall the definition of the Carathéodory family $\mathcal{M} = \mathcal{M}(\mathbb{B}_X)$ in $H(\mathbb{B}_X)$ (see [47]):

$$\mathcal{M}(\mathbb{B}_X) = \big\{h \in H(\mathbb{B}_X) : h(0) = 0, Dh(0) = I_X,$$

$$\Re l_x(h(x)) > 0, \forall x \in \mathbb{B}_X \setminus \{0\}, \forall l_x \in T(x)\big\}.$$

If $X = \mathbb{C}$, then $f \in \mathcal{M}(\mathbb{U})$ if and only if $f(x)/x \in \mathcal{P}$, where

$$\mathcal{P} = \big\{p \in H(\mathbb{U}) : p(0) = 1, \Re p(z_1) > 0, \forall z_1 \in \mathbb{U}\big\}$$

is the Carathéodory family on $\mathbb{U}$.

**Definition 2.4 (cf.** [1], [9]**).** *Let $X$ be a complex Banach space. A mapping $h = h(x,t)$ : $\mathbb{B}_X \times [0,\infty) \to X$ is called a generating vector field (Herglotz vector field) if the following conditions hold:*

(i) *$h(\cdot,t) \in \mathcal{M}(\mathbb{B}_X)$, for a.e. $t \geq 0$;*
(ii) *$h(x,\cdot)$ is strongly measurable on $[0,\infty)$, for all $x \in \mathbb{B}_X$.*

**Definition 2.5 (see e.g.** [13] **and** [15]**).** *Let $g : \mathbb{U} \to \mathbb{C}$ satisfy Assumption* 2.1. *Also, let $h \in H(\mathbb{B}_X)$ be normalized. We say that $h$ belongs to the family $\mathcal{M}_g = \mathcal{M}_g(\mathbb{B}_X)$ if*

$$\frac{1}{\|x\|_X} l_x(h(x)) \in g(\mathbb{U}), \quad \forall x \in \mathbb{B}_X \setminus \{0\}, \quad \forall l_x \in T(x).$$

Further, we define the notion of a $g$-Loewner chain in the case of complex Banach spaces (not necessarily reflexive), where $g : \mathbb{U} \to \mathbb{C}$ satisfies Assumption 2.1. In the case $X = \mathbb{C}^n$, see [13], [15].

**Definition 2.6.** *Let $g : \mathbb{U} \to \mathbb{C}$ satisfy Assumption* 2.1. *We say that a mapping $f = f(x,t) : \mathbb{B}_X \times [0,\infty) \to X$ is a $g$-Loewner chain if the following conditions hold:*

(i) *$f(x,t)$ is a Loewner chain such that $\{e^{-t}f(\cdot,t)\}_{t\geq 0}$ is uniformly bounded on each ball $\rho\mathbb{B}_X$ $(0 < \rho < 1)$;*
(ii) *there exists a null set $E \subset [0,\infty)$ such that $\frac{\partial f}{\partial t}(x,t)$ exists for $t \in [0,\infty) \setminus E$ and for all $x \in \mathbb{B}_X$, and there exists a generating vector field $h = h(x,t)$ : $\mathbb{B}_X \times [0,\infty) \to X$ with $h(\cdot,t) \in \mathcal{M}_g(\mathbb{B}_X)$ for $t \in [0,\infty) \setminus E$, such that*

$$\frac{\partial f}{\partial t}(x,t) = Df(x,t)h(x,t), \quad t \in [0,\infty) \setminus E, \forall x \in \mathbb{B}_X. \tag{2.1}$$

**Remark 2.7.** In general, if $X$ is a complex Banach space and if $f(x,t)$ satisfies condition (i) of Definition 2.6, it is not known whether $\frac{\partial f}{\partial t}(x,t)$ exists for $x \in \mathbb{B}_X$ and $t \in [0,\infty) \setminus E$, where $E \subset [0,\infty)$ is a null set. Also, if $\frac{\partial f}{\partial t}(x,t)$ exists for $x \in \mathbb{B}_X$ and $t \in [0,\infty) \setminus E$, it is not known whether there exists a generating vector field $h(x,t)$ such that the Loewner differential equation (2.1) holds. However, positive answers to these questions may be obtained in the case of separable reflexive complex Banach spaces. A discussion of Loewner chains and the associated Loewner differential equation in the case of separable reflexive complex Banach spaces may be found in [32]. In the finite dimensional case $X = \mathbb{C}^n$, see [44, Chapter 6] for $n = 1$; see [1], [9], and [13], in the case $n \geq 2$.

**Definition 2.8 (see [26]).** *Let $g : \mathbb{U} \to \mathbb{C}$ satisfy the conditions of Assumption 2.1. A mapping $f \in \mathcal{L}S(\mathbb{B}_X)$ is said to be $g$-starlike if $h \in \mathcal{M}_g(\mathbb{B}_X)$, where*

$$h(x) = [Df(x)]^{-1} f(x), \quad x \in \mathbb{B}_X.$$

*Let $S_g^*(\mathbb{B}_X)$ denote the class of $g$-starlike mappings on $\mathbb{B}_X$.*

**Definition 2.9 (see e.g. [5], and [37]).** *A complex Banach space $X$ is called a JB*-triple if $X$ is a complex Banach space equipped with a continuous Jordan triple product*

$$X \times X \times X \to X \qquad (x, y, z) \mapsto \{x, y, z\}$$

*satisfying*

($J_1$) *$\{x, y, z\}$ is symmetric bilinear in the outer variables, but conjugate linear in the middle variable,*

($J_2$) $\{a, b, \{x, y, z\}\} = \{\{a, b, x\}, y, z\} - \{x, \{b, a, y\}, z\} + \{x, y, \{a, b, z\}\}$,

($J_3$) *$x\Box x \in L(X, X)$ is a hermitian operator with spectrum $\geqq 0$,*

($J_4$) $\|\{x, x, x\}\| = \|x\|^3$

*for $a, b, x, y, z \in X$, where the box operator $x\Box y : X \to X$ is defined by*

$$x\Box y(\cdot) = \{x, y, \cdot\},$$

*and $\|\cdot\|$ is the norm on $X$.*

A complex Banach space $X$ is a JB*-triple if and only if the open unit ball of $X$ is homogeneous (see e.g. [5, Section 3.3]).

Next we recall the notion of R-Bloch mappings on the unit ball of a complex Banach space $X$ and also that of I-Bloch mappings on the unit ball of a JB*-triple.

**Definition 2.10.** (i) (*cf.* [25]) *Let $\mathbb{B}_X$ be the unit ball of a complex Banach space $X$ and let $f : \mathbb{B}_X \to Y$ be a holomorphic mapping. We say that $f$ is an R-Bloch mapping on $\mathbb{B}_X$ if*

$$\sup_{x \in \mathbb{B}_X} (1 - \|x\|^2)\|Df(x)x\| < \infty. \tag{2.2}$$

(ii) (*cf.* [6], [7], [24]) *Let $\mathbb{B}_X$ be the unit ball of a JB*-triple $X$ and let $f : \mathbb{B}_X \to Y$ be a holomorphic mapping. We say that $f$ is an I-Bloch mapping on $\mathbb{B}_X$ if*

$$\sup_{g \in \mathrm{Aut}(\mathbb{B}_X)} \|D(f \circ g)(0)\| < \infty, \tag{2.3}$$

*where $\mathrm{Aut}(\mathbb{B}_X)$ denotes the family of biholomorphic automorphisms of $\mathbb{B}_X$.*

**Remark 2.11.** (i) When $\mathbb{B}_X$ is the unit ball of a JB*-triple $X$, I-Bloch mappings are R-Bloch mappings by [34, Corollary 3.6] (cf. [7, Corollary 3.5], [24]). Chu, Hamada, Honda and Kohr [8, Example 2.9] and Miralles [39, Proposition 2.5] independently gave an example such that the converse is not true for $\mathbb{B}_X = \mathbb{U}^2$.

(ii) When $\mathbb{B}_X$ is a Hilbert ball and $Y = \mathbb{C}$, then conditions (2.2) and (2.3) are equivalent to the following relation:

$$\sup_{x \in \mathbb{B}_X} (1 - \|x\|^2)\|Df(x)\| < \infty, \tag{2.4}$$

by [2, Proposition 2.4, Theorems 2.6 and 3.8] (cf, [25, Theorem 2.8]). Moreover, (2.2), (2.3) and (2.4) give equivalent semi-norms for a holomorphic function $f : \mathbb{B}_X \to \mathbb{C}$ which satisfies one of the relations (2.2), (2.3) and (2.4). Then for $f \in H(\mathbb{B}_X)$, by considering the function $f_a = \langle f, a\rangle$ with $\|a\| = 1$, we obtain that conditions (2.2), (2.3) and (2.4) are equivalent. Namely, the notions of R-Bloch mappings and I-Bloch mappings are equivalent to the usual notion of Bloch mappings on the Hilbert ball. In particular, $f \in H(\mathbb{U})$ is a Bloch function if and only if

$$\sup_{\zeta\in\mathbb{U}}(1-|\zeta|^2)|f'(\zeta)| < \infty.$$

Next, we recall the notion of a linearly invariant family (L.I.F.) and the trace-order of a L.I.F. on the unit ball $\mathbb{B}_X$ of a finite-dimensional complex Banach space $X$ ([28]; cf. [42], [21, Chapter 10]).

**Definition 2.12.** *Let $X$ be a complex Banach space and let $\mathbb{B}_X$ be the open unit ball of $X$. A family $\mathcal{F} \subseteq H(\mathbb{B}_X)$ is called a linearly invariant family (L.I.F.) if the following conditions hold:*

(i) $\mathcal{F} \subseteq \mathcal{L}S(\mathbb{B}_X)$;
(ii) $\Lambda_\phi(f) \in \mathcal{F}$, *for all* $f \in \mathcal{F}$ *and* $\phi \in \mathrm{Aut}(\mathbb{B}_X)$,

*where $\Lambda_\phi(f)$ is the Koebe transform given by*

$$\Lambda_\phi(f)(x) = [D\phi(0)]^{-1}[Df(\phi(0))]^{-1}(f(\phi(x)) - f(\phi(0))), \quad \forall x \in \mathbb{B}_X.$$

**Definition 2.13 ([28]; cf. [42]).** *If $\mathcal{F}$ is a linearly invariant family on the unit ball of a finite dimensional complex Banach space $X$, we define the trace order of $\mathcal{F}$, by*

$$\operatorname{ord}\mathcal{F} = \sup_{f\in\mathcal{F}}\sup_{\|y\|=1}\left\{\frac{1}{2}\left|\operatorname{trace}\left[D^2f(0)(y,\cdot)\right]\right|\right\}.$$

Since the trace is a similarity invariant, the above definition is well-defined. When $X = \mathbb{C}$ and $\mathbb{B}_X = \mathbb{U}$, the trace order is the usual order of a linearly invariant family on $\mathbb{U}$.

Let $\Lambda[\{f\}]$ be the linearly invariant family generated by $f \in \mathcal{L}S(\mathbb{B}_X)$ (see [28]; cf. [42]). In this case, $\operatorname{ord}\Lambda[\{f\}]$ is called the trace order of $f$.

## 3. Roper-Suffridge type extension operators

Let $Y$ be a complex Banach space and let $r \geq 1$. Also, let

$$\Omega_r = \left\{(z_1, w) \in Z = \mathbb{C}\times Y : |z_1|^2 + \|w\|_Y^r < 1\right\}. \tag{3.1}$$

Then, the Minkowski function of $\Omega_r$ is a complete norm $\|\cdot\|_Z$ on $Z$ and $\Omega_r$ is the unit ball of $Z$ with respect to this norm. Let $\alpha, \beta \geq 0$ and let $\Phi_{\alpha,\beta} : S \to S(\Omega_r)$ be the Roper-Suffridge type extension operator given by

$$\Phi_{\alpha,\beta}(f)(z_1, w) = \left(f(z_1), \left(\frac{f(z_1)}{z_1}\right)^\alpha (f'(z_1))^\beta w\right), \quad (z_1, w) \in \Omega_r. \tag{3.2}$$

The branches of the power functions are chosen such that

$$\left(\frac{f(z_1)}{z_1}\right)^\alpha\bigg|_{z_1=0} = 1 \quad \text{and} \quad (f'(z_1))^\beta|_{z_1=0} = 1.$$

### 3.1. $g$-Loewner chains and Roper-Suffridge type extension operators

Let $g : \mathbb{U} \to \mathbb{C}$ be a convex (univalent) function which satisfies Assumption 2.1. In the first part of this section, we are concerned with preservation of the first elements of $g$-Loewner chains from $\mathbb{U}$ into $\Omega_r$ under the Roper-Suffridge type extension operators $\Phi_{\alpha,\beta}$, where $r \geq 1$ (cf. [1, Theorem 7.1], [3, Theorem 2.1], [14, Corollary 2.9], [19, Theorem 2.1], [22, Theorem 2.1]).

**Theorem 3.1.** *Let $g : \mathbb{U} \to \mathbb{C}$ be a convex (univalent) function which satisfies Assumption 2.1. Let $Y$ be a complex Banach space and let $\Omega_r$ be the unit ball of $Z = \mathbb{C} \times Y$ given by (3.1), where $r \geq 1$. Let $\Phi_{\alpha,\beta}$ be the Roper-Suffridge type extension operator given by (3.2). Assume that $f \in S$ can be embedded as the first element of a $g$-Loewner chain on $\mathbb{U}$. Then $F = \Phi_{\alpha,\beta}(f) \in S(\Omega_r)$ can be embedded as the first element of a $g$-Loewner chain on $\Omega_r$ for $\alpha \in [0,1]$, $\beta \in [0,1/r]$, $\alpha + \beta \leq 1$.*

As a corollary of Theorem 3.1, we obtain the following preservation of the first elements of Loewner chains from $\mathbb{U}$ into the unit ball $\Omega_r$ under the Roper-Suffridge type extension operators $\Phi_{\alpha,\beta}$ (cf. [14, Corollary 2.9], [19, Theorem 2.1], [22, Theorem 2.1], [36]).

**Corollary 3.2.** *Let $\Omega_r$ and $\Phi_{\alpha,\beta}$ be as in Theorem 3.1. If $f \in S$, then $F = \Phi_{\alpha,\beta}(f) \in S(\Omega_r)$ can be embedded as the first element of a Loewner chain on $\Omega_r$ for $\alpha \in [0,1]$, $\beta \in [0,1/r]$, $\alpha + \beta \leq 1$.*

As another consequence of Theorem 3.1, we obtain that the Roper-Suffridge type extension operators $\Phi_{\alpha,\beta}$ preserve $g$-starlike mappings. This result is a generalization of [20, Theorem 2.2], in the case $Y = \mathbb{C}^{n-1}$, $r = 2$ and $g(\zeta) = \frac{1-\zeta}{1+\zeta}$, $\zeta \in \mathbb{U}$ (cf. [3, Corollary 2.2], [4, Corollary 2.3]).

**Corollary 3.3.** *Let $\Omega_r$, $\Phi_{\alpha,\beta}$ and $g$ be as in Theorem 3.1. If $f$ is a $g$-starlike mapping on $\mathbb{U}$, then $F = \Phi_{\alpha,\beta}(f) \in S(\Omega_r)$ is also a $g$-starlike mapping on $\Omega_r$ for $\alpha \in [0,1]$, $\beta \in [0,1/r]$, $\alpha + \beta \leq 1$.*

As particular cases of Corollary 3.3, we obtain that strongly starlike mappings of order $d \in (0,1]$ and almost starlike mappings of order $d \in [0,1)$ (see e.g. [21]) are preserved by the Roper-Suffridge type extension operators $\Phi_{\alpha,\beta}$ for $\alpha \in [0,1]$, $\beta \in [0,1/r]$, $\alpha + \beta \leq 1$.

In the case $\beta = 0$, [26, Theorem 5.1] can be generalized as follows.

**Theorem 3.4.** *Let $\Omega_r$ and $\Phi_{\alpha,\beta}$ be as in Theorem 3.1. Let $g$ be a univalent holomorphic function on $\mathbb{U}$ which satisfies Assumption 2.1 such that $g(\mathbb{U})$ is a starlike domain with respect to 1. Assume that $f \in S$ can be embedded as the first element of a $g$-Loewner chain on $\mathbb{U}$. Then $F = \Phi_{\alpha,0}(f) \in S(\Omega_r)$ can be embedded as the first element of a $g$-Loewner chain on $\Omega_r$ for $\alpha \in [0,1]$.*

As a corollary of Theorem 3.4, we obtain the following generalization of [27, Theorem 5.3] to certain complex Banach spaces.

**Corollary 3.5.** *Let $\Omega_r$ and $\Phi_{\alpha,\beta}$ be as in Theorem 3.1. If $f$ is a parabolic starlike mapping of order $d \in [0,1)$ on $\mathbb{U}$, then $F = \Phi_{\alpha,0}(f) \in S(\Omega_r)$ is also a parabolic starlike mapping of order $d$ on $\Omega_r$ for $\alpha \in [0,1]$.*

### 3.2. Bloch mappings and Roper-Suffridge type extension operators

In the second part of this section, we show that normalized univalent Bloch functions on $\mathbb{U}$ (respectively normalized uniformly locally univalent Bloch functions on $\mathbb{U}$) are extended to $R$-Bloch mappings on $\Omega_r$ by the Roper-Suffridge type extension operators $\Phi_{\alpha,\beta}$, for $\alpha > 0$ and $\beta \in [0, 1/r)$ (respectively for $\alpha = 0$ and $\beta \in [0, 1/r]$).

The following theorem is a generalization of [20, Theorem 2.6] and [22, Theorem 4.1] to certain complex Banach spaces (cf. [10, Proposition 6.1]).

**Theorem 3.6.** *Let $\Omega_r$ and $\Phi_{\alpha,\beta}$ be as in Theorem* 3.1. *If $f \in S$ is a Bloch function on $\mathbb{U}$, then $F = \Phi_{\alpha,\beta}(f) \in S(\Omega_r)$ is an $R$-Bloch mapping on $\Omega_r$ for $\alpha > 0$ and $\beta \in [0, 1/r)$.*

In the case $\alpha = 0$ and $\beta \in [0, 1/r]$, we obtain that uniformly locally univalent Bloch functions on $\mathbb{U}$ are extended to $R$-Bloch mappings on $\Omega_r$ by the extension operator $\Phi_{0,\beta}$. This result is a generalization of [20, Theorem 2.6] and [22, Theorem 4.1] to certain complex Banach spaces and also is an improvement of Theorem 3.6.

**Theorem 3.7.** *Let $\Omega_r$ and $\Phi_{\alpha,\beta}$ be as in Theorem* 3.1. *If $f \in \mathcal{LS}(\mathbb{U})$ is a uniformly locally univalent Bloch function on $\mathbb{U}$, then $F = \Phi_{0,\beta}(f) \in \mathcal{LS}(\Omega_r)$ is an $R$-Bloch mapping on $\Omega_r$ for $\beta \in [0, 1/r]$.*

## 4. Muir type extension operators

Let $k \geq 2$ be an integer and let $Y$ be a complex Banach space and let $\Omega_k$ be the unit ball of $Z = \mathbb{C} \times Y$ given by (3.1). Let $P_k : Y \to \mathbb{C}$ be a homogeneous polynomial mapping of degree $k$. The Muir type extension operator $\Phi_{P_k}$ is defined by (cf. [40])

$$\Phi_{P_k}(f)(z) = \Big(f(z_1) + P_k(w)f'(z_1), (f'(z_1))^{\frac{1}{k}} w\Big), \quad z = (z_1, w) \in \Omega_k, \tag{4.1}$$

where $f$ is a locally univalent function on $\mathbb{U}$, normalized by $f(0) = f'(0) - 1 = 0$. The branch of the power function is chosen such that $(f'(z_1))^{\frac{1}{k}}|_{z_1=0} = 1$.

### 4.1. $g$-Loewner chains and Muir type extension operators

We begin this section with the following preservation result of the first elements of $g$-Loewner chains by the Muir type extension operators $\Phi_{P_k}$, where $g$ is a convex function on $\mathbb{U}$ which satisfies Assumption 2.1. In the case $Y = \mathbb{C}^{n-1}$, $k = 2$ and $g(\zeta) = \frac{1-\zeta}{1+(1-2\gamma)\zeta}$, $\zeta \in \mathbb{U}$, where $\gamma \in (0, 1)$, see [4, Theorem 3.1] (cf. [33, Theorem 5.6], [35, Theorem 2.1 and Corollary 2.2] ).

**Theorem 4.1.** *Let $k \geq 2$ be an integer. Let $Y$ be a complex Banach space and let $\Omega_k$ be the unit ball of $Z = \mathbb{C} \times Y$ given by* (3.1). *Let $P_k : Y \to \mathbb{C}$ be a homogeneous polynomial mapping of degree $k$ and let $\Phi_{P_k}$ be the Muir type extension operator given by* (4.1). *Let $g$ be a convex function on $\mathbb{U}$ which satisfies Assumption* 2.1. *Assume that $f \in S$ can be embedded as the first element of a $g$-Loewner chain on $\mathbb{U}$ and that $\|P_k\| \leq d(1, \partial g(\mathbb{U}))/4$, where*

$$d(1, \partial g(\mathbb{U})) = \inf_{\zeta \in \partial g(\mathbb{U})} |\zeta - 1|.$$

*Then $F = \Phi_{P_k}(f) \in S(\Omega_k)$ can be embedded as the first element of a $g$-Loewner chain on $\Omega_k$.*

As a corollary of Theorem 4.1, we obtain the following result. This result is a generalization of [35, Theorem 2.1 and Corollary 2.2], in the case $Y = \mathbb{C}^{n-1}$, $k = 2$ and $g(\zeta) = \frac{1-\zeta}{1+\zeta}$, $\zeta \in \mathbb{U}$ to certain complex Banach spaces (cf. [33, Theorem 5.6]).

**Corollary 4.2.** *Let $\Omega_k$ and $\Phi_{P_k}$ be as in Theorem 4.1, where $\|P_k\| \leq 1/4$. If $f \in S$, then $F = \Phi_{P_k}(f)$ can be embedded as the first element of a Loewner chain on $\Omega_k$.*

In view of Theorem 4.1, it would be interesting to give an answer to the following questions:

**Question 4.3.** *Under the assumptions of Theorem 4.1, is the coefficient bound $\|P_k\| \leq d(1, \partial g(\mathbb{U}))/4$ also necessary for the preservation of the first elements of g-Loewner chains under the Muir type extension operator $\Phi_{P_k}$?*

**Question 4.4.** *Under the assumptions of Theorem 4.1, is the coefficient bound $\|P_k\| \leq d(1, \partial g(\mathbb{U}))/4$ sharp for the preservation of the first elements of g-Loewner chains under the Muir type extension operator $\Phi_{P_k}$?*

In the case that $f \in K$ can be embedded as the first element of a $g$-Loewner chain $f(z_1, t)$ on $\mathbb{U}$ such that $f(\cdot, t)$ is convex on $\mathbb{U}$ for $t \geq 0$, then Theorem 4.1 may be refined as follows (cf. [4], [35], [40]).

**Proposition 4.5.** *Let $\Omega_k$ and $\Phi_{P_k}$ be as in Theorem 4.1. Let $g$ be a convex function on $\mathbb{U}$ which satisfies Assumption 2.1. Assume that $f \in K$ can be embedded as the first element of a g-Loewner chain $f(z_1, t)$ on $\mathbb{U}$, such that $e^{-t}f(\cdot, t) \in K$, for all $t \geq 0$. If $\|P_k\| \leq d(1, \partial g(\mathbb{U}))/2$, then $F = \Phi_{P_k}(f) \in S(\Omega_k)$ can be embedded as the first element of a g-Loewner chain on $\Omega_k$.*

Let $g$ be a linear fractional transformation with real coefficients, which satisfies Assumption 2.1. Then the image $g(\mathbb{U})$ is one of the following sets:

$$g(\mathbb{U}) = \left\{\zeta \in \mathbb{C} : \left|\zeta - \frac{1}{2\gamma}\right| < \frac{\delta}{2\gamma}\right\},\ \gamma > 0,\ \delta \in (0, 1],\ |2\gamma - 1| < \delta,$$
$$g(\mathbb{U}) = \{\zeta \in \mathbb{C} : \Re\zeta > \delta\},\ \delta \in [0, 1).$$

As a corollary of Theorem 4.1, we obtain the following results.

**Corollary 4.6.** *Let $\Omega_k$ and $\Phi_{P_k}$ be as in Theorem 4.1. Let $g$ be a linear fractional transformation with real coefficients which satisfies Assumption 2.1. Assume that $f \in S$ can be embedded as the first element of a g-Loewner chain on $\mathbb{U}$.*

*(i) If $g(\mathbb{U}) = \left\{\zeta \in \mathbb{C} : \left|\zeta - \frac{1}{2\gamma}\right| < \frac{\delta}{2\gamma}\right\}$, where $\gamma > 0$, $\delta \in (0, 1]$, and $|2\gamma - 1| < \delta$, and if $\|P_k\| \leq (\delta - |2\gamma - 1|)/(8\gamma)$, then $F = \Phi_{P_k}(f) \in S(\Omega_k)$ can be embedded as the first element of a g-Loewner chain on $\Omega_k$.*

*(ii) If $g(\mathbb{U}) = \{\zeta \in \mathbb{C} : \Re\zeta > \delta\}$, where $\delta \in [0, 1)$, and if $\|P_k\| \leq (1 - \delta)/4$, then $F = \Phi_{P_k}(f) \in S(\Omega_k)$ can be embedded as the first element of a g-Loewner chain on $\Omega_k$.*

As in Corollary 3.3, we obtain the following result (cf. [4, Corollary 3.3], [35, Corollary 2.3] [40, Theorem 4.1]).

**Corollary 4.7.** *Let $\Omega_k$, $\Phi_{P_k}$ and $g$ be as in Theorem 4.1. If $f$ is a g-starlike mapping on $\mathbb{U}$ and if $\|P_k\| \leq d(1, \partial g(\mathbb{U}))/4$, then $F = \Phi_{P_k}(f) \in S(\Omega_k)$ is also a g-starlike mapping on $\Omega_k$.*

In particular, we have the following corollary.

**Corollary 4.8.** *Let* $\Omega_k$ *and* $\Phi_{P_k}$ *be as in Theorem* 4.1.

(*i*) *If* $f : \mathbb{U} \to \mathbb{C}$ *is a strongly starlike mapping of order* $d \in (0,1]$ *on* $\mathbb{U}$ *and if* $\|P_k\| \le \sin\left(\frac{\pi}{2}d\right)/4$, *then* $F = \Phi_{P_k}(f) \in S(\Omega_k)$ *is also a strongly starlike mapping of order* $d$ *on* $\Omega_k$.

(*ii*) *If* $f : \mathbb{U} \to \mathbb{C}$ *is an almost starlike mapping of order* $d \in [0,1)$ *on* $\mathbb{U}$ *and if* $\|P_k\| \le (1-d)/4$, *then* $F = \Phi_{P_k}(f) \in S(\Omega_k)$ *is also an almost starlike mapping of order* $d$ *on* $\Omega_k$.

Taking into account Corollary 4.7, it would be interesting to give an answer to the following question.

**Question 4.9.** *Under the same assumptions of Corollary* 4.7, *is the condition* $\|P_k\| \le d(1, \partial g(\mathbb{U}))/4$ *necessary for the preservation of* $g$*-starlikeness under the Muir type extension operator* $\Phi_{P_k}$*?*

Note that if $g(\zeta) = \frac{1-\zeta}{1+\zeta}$, $\zeta \in \mathbb{U}$, $k = 2$ and $Y = \mathbb{C}^{n-1}$, the answer is positive, in view of [40, Theorem 4.1].

Next, let $G : Y \to \mathbb{C}$ be a holomorphic function such that $G(0) = 0$ and $DG(0) = 0$. Also, let $\Phi_{G,k} : \mathcal{LS}(\mathbb{U}) \to \mathcal{LS}(\Omega_k)$ be the following modification of the Muir extension operator (cf. [41])

$$\Phi_{G,k}(f)(z) = \Big(f(z_1) + G\big((f'(z_1))^{\frac{1}{k}} w\big), (f'(z_1))^{\frac{1}{k}} w\Big), \quad z = (z_1, w) \in \Omega_k, \qquad (4.2)$$

where $\Omega_k$ is the unit ball of $Z = \mathbb{C} \times Y$ given by (3.1). The branch of the power function is chosen such that $(f'(z_1))^{\frac{1}{k}}\big|_{z_1=0} = 1$.

It is natural to ask the following question, in connection with Corollary 4.7 (cf. [41], [50]):

**Question 4.10.** *Let* $k \ge 2$ *be an integer and let* $\Omega_k$ *be the unit ball of* $Z = \mathbb{C} \times Y$ *given by* (3.1). *Assume that* $g : \mathbb{U} \to \mathbb{C}$ *is a univalent function, which satisfies Assumption* 2.1. *Let* $G : Y \to \mathbb{C}$ *be a holomorphic function such that* $G(0) = 0$ *and* $DG(0) = 0$. *If* $\Phi_{G,k}(S_g^*(\mathbb{U})) \subseteq S^*(\Omega_k)$, *what conditions for* $G$ *must be satisfied ?*

The following result provides an answer to the above question (cf. [41, Theorem 5.1], [50, Theorem 3.1]).

**Theorem 4.11.** *Let* $\Omega_k$ *be as in Theorem* 4.1. *Let* $g$ *be a univalent function with real coefficients on* $\mathbb{U}$, *which satisfies Assumption* 2.1. *Assume that there exists the limit*

$$a := \liminf_{r \to 1^-} \frac{g(r)}{1-r} < +\infty. \qquad (4.3)$$

*Let* $G : Y \to \mathbb{C}$ *be a holomorphic function such that* $G(0) = 0$ *and* $DG(0) = 0$ *and* $\Phi_{G,k}$ *be the extension operator given in* (4.2). *Let* $f$ *be a* $g$*-starlike function on* $\mathbb{U}$ *such that* $\frac{f(\zeta)}{\zeta f'(\zeta)} = g(\zeta)$ *for* $\zeta \in \mathbb{U}$. *If* $\Phi_{G,k}(f)$ *is a starlike mapping on* $\Omega_k$, *then* $G$ *is a polynomial of degree at most* $k$.

As a corollary of Theorem 4.11, we obtain the following result (cf. [41, Corollary 5.2], [50, Corollary 3.2]).

**Corollary 4.12.** *Let* $\Omega_k$, $\Phi_{G,k}$ *and* $g$ *be as in Theorem* 4.11. *If* $\Phi_{G,k}(S_g^*(\mathbb{U})) \subseteq S^*(\Omega_k)$, *then* $G$ *is a polynomial of degree at most* $k$.

### 4.2. Bloch mappings and Muir type extension operators

The next result shows that normalized uniformly locally univalent Bloch functions on $\mathbb{U}$ are extended to normalized locally univalent $R$-Bloch mappings on $\Omega_k$ by the Muir type extension operators $\Phi_{P_k}$ given by (4.1).

**Theorem 4.13.** *Let $\Omega_k$ and $\Phi_{P_k}$ be as in Theorem 4.1. If $f \in \mathcal{LS}(\mathbb{U})$ is a uniformly locally univalent Bloch function on $\mathbb{U}$, then $F = \Phi_{P_k}(f) \in \mathcal{LS}(\Omega_k)$ is an R-Bloch mapping on $\Omega_k$.*

As a corollary of Theorem 4.13, we obtain the following result.

**Corollary 4.14.** *Let $\Omega_k$ and $\Phi_{P_k}$ be as in Theorem 4.1. If $f \in S$ is a Bloch function on $\mathbb{U}$, then $F = \Phi_{P_k}(f) \in S(\Omega_k)$ is an R-Bloch mapping on $\Omega_k$.*

## 5. Pfaltzgraff-Suffridge type extension operators

In this section, let $X$ be an $n$-dimensional JB*-triple. Also, let $\mathbb{B}_X$ be the open unit ball of $X$ with respect to the norm $\|\cdot\|_X$ and for every $x, y \in X$, let $B(x,y) \in L(X)$ be the Bergman operator defined by

$$B(x,y)(z) = z - 2(x\square y)(z) + \{x, \{y, z, y\}, x\}, \quad z \in X.$$

If $f \in H(\mathbb{B}_X)$, let $J_f(x) = \det Df(x)$, $x \in \mathbb{B}_X$. Also, let $h_0$ be the Bergman metric on $X$ at 0, and let $c(\mathbb{B}_X)$ be the constant given by (see [28])

$$c(\mathbb{B}_X) = \frac{1}{2} \sup_{x,y\in\mathbb{B}_X} |h_0(x,y)|. \tag{5.1}$$

In view of [24, Lemma 2.4] (cf. [30, Lemma 2.2]), the following distortion result holds:

$$\det B(x,x) \geq (1 - \|x\|_X^2)^{2c(\mathbb{B}_X)}, \quad x \in \mathbb{B}_X. \tag{5.2}$$

Equality holds for every $x \in X$ such that $x/\|x\|_X$ is a maximal tripotent in $X$.

Next, let $Y$ be a complex Banach space with the norm $\|\cdot\|_Y$, and let $\mathbb{B}_Y$ be the unit ball of $Y$. For $\alpha > 0$, let

$$\mathbb{D}_\alpha = \Big\{(x,y) \in \mathbb{B}_X \times Y : \|y\|_Y < [\det B(x,x)]^{1/(4\alpha c(\mathbb{B}_X))}\Big\} \tag{5.3}$$

and

$$\Omega_\alpha = \Big\{(x,y) \in X \times Y : \|x\|_X^2 + \|y\|_Y^{2\alpha} < 1\Big\}. \tag{5.4}$$

Also, for $\alpha > 0$, let $\mathfrak{F}_{n,\alpha} : \mathcal{LS}(\mathbb{B}_X) \to \mathcal{LS}(\mathbb{D}_\alpha)$ be the Pfaltzgraff-Suffridge type extension operator given by

$$\mathfrak{F}_{n,\alpha}(f)(z) = \Big(f(x), [J_f(x)]^{1/(2\alpha c(\mathbb{B}_X))} y\Big), \quad z = (x,y) \in \mathbb{D}_\alpha. \tag{5.5}$$

The branch of the power function is chosen such that $[J_f(x)]^{1/(2\alpha c(\mathbb{B}_X))}|_{x=0} = 1$. Note that this branch is well defined on $\mathbb{B}_X$, since $\mathbb{B}_X$ is a starlike domain with respect to the origin in $X = \mathbb{C}^n$. It is not difficult to deduce that if $f \in \mathcal{LS}(\mathbb{B}_X)$ and $F = \mathfrak{F}_{n,\alpha}(f)$, then $F \in H(\mathbb{D}_\alpha)$ and the Frechét derivative $DF(z)$ has a bounded inverse at each point $z \in \mathbb{D}_\alpha$, i.e. $F$ is locally biholomorphic on $\mathbb{D}_\alpha$. Hence the Pfaltzgraff-Suffridge type extension operator $\mathfrak{F}_{n,\alpha}$ is well defined and extends normalized locally biholomorphic mappings on $\mathbb{B}_X$ into normalized locally biholomorphic mappings on the domain $\mathbb{D}_\alpha$.

**Example 5.1.** (i) If $X$ is the space $\mathbb{C}^n$ with the Euclidean norm $\|\cdot\|_e$, then $\mathbb{B}_X = \mathbb{B}^n$, $\det B(x,x) = (1-\|x\|_e^2)^{n+1}$, and $c(\mathbb{B}^n) = \frac{n+1}{2}$ (see e.g. [28]). Therefore, we have $\mathbb{D}_\alpha = \Omega_\alpha$ for $\alpha > 0$, that is

$$\mathbb{D}_\alpha = \Big\{(x,y) \in \mathbb{C}^n \times Y : \|x\|_e^2 + \|y\|_Y^{2\alpha} < 1\Big\}.$$

In this case, the operator $\mathfrak{F}_{n,\alpha}$ will be denoted by $\Gamma_{n,\alpha}$. Thus, we obtain the extension operator $\Gamma_{n,\alpha} : \mathcal{L}S(\mathbb{B}^n) \to \mathcal{L}S(\Omega_\alpha)$ given by (see [14, Definition 2.7]):

$$\Gamma_{n,\alpha}(f)(z) = \Big(f(x), [J_f(x)]^{1/(\alpha(n+1))}y\Big),\ \forall\, f \in \mathcal{L}S(\mathbb{B}^n),\ z = (x,y) \in \Omega_\alpha. \tag{5.6}$$

If $\alpha = 1$, $\mathbb{B}_X = \mathbb{B}^n$ and $\mathbb{B}_Y = \mathbb{B}^r$, then $\Omega_1 = \mathbb{B}^{n+r}$ and the operator $\Gamma_{n,1}$ reduces to the Pfaltzgraff-Suffridge type extension operator $\Phi_{n,r}$. On the other hand, if $n = 1$ and $\alpha = 1$, then the operator $\Gamma_{1,1}$ reduces to the Roper-Suffridge extension operator $\Psi : \mathcal{L}S(\mathbb{B}^1) \to \mathcal{L}S(\mathbb{B})$ given by (cf. [46]; see also [14])

$$\Psi(f)(z) = \Big(f(x), \sqrt{f'(x)}y\Big),\ z = (x,y) \in \mathbb{B},$$

where $\mathbb{B} = \big\{(x,y) \in \mathbb{C} \times Y : |x|^2 + \|y\|_Y^2 < 1\big\}$.

(ii) If $X = \mathbb{C}^n$ with respect to the maximum norm $\|\cdot\|_\infty$, then $c(\mathbb{U}^n) = n$ (see [28]), and $\det B(x,x) = \prod_{j=1}^n (1-|x_j|^2)^2$, $x = (x_1,\dots,x_n) \in \mathbb{U}^n$. Denoting the domain $\mathbb{D}_\alpha$ by $\Delta_\alpha$ for $\alpha > 0$, we obtain that

$$\Delta_\alpha = \Big\{(x,y) \in \mathbb{U}^n \times \mathbb{B}_Y : \|y\|_Y < \prod_{j=1}^n (1-|x_j|^2)^{1/(2n\alpha)}\Big\}. \tag{5.7}$$

In this case, we denote the operator $\mathfrak{F}_{n,\alpha}$ by $\Theta_{n,\alpha}$. Thus, we obtain the extension operator $\Theta_{n,\alpha} : \mathcal{L}S(\mathbb{U}^n) \to \mathcal{L}S(\Delta_\alpha)$ given by (cf. [14])

$$\Theta_{n,\alpha}(f)(z) = \Big(f(x), [J_f(x)]^{1/(2n\alpha)}y\Big), \quad z = (x,y) \in \Delta_\alpha. \tag{5.8}$$

### 5.1. Loewner chains and Pfaltzgraff-Suffridge type extension operators

We begin this section with the preservation of Loewner chains from the open unit ball $\mathbb{B}_X$ of an $n$-dimensional JB*-triple $X$ into the domain $\mathbb{D}_\alpha$ given by (5.3) by the Pfaltzgraff-Suffridge type extension operator $\mathfrak{F}_{n,\alpha}$. This result is a generalization of [23, Theorem 2.1] (cf. [14, Theorem 2.1]).

**Theorem 5.2.** *Let $\mathbb{B}_X$ be the unit ball of an $n$-dimensional JB*-triple $X$, and let $\alpha \geq \frac{n}{2c(\mathbb{B}_X)}$. Also, let $\mathbb{D}_\alpha \subset Z = X \times Y$ be the domain given by* (5.3) *and $\mathfrak{F}_{n,\alpha}$ be the Pfaltzgraff-Suffridge type extension operator given by* (5.5)*. Assume that $f \in S(\mathbb{B}_X)$ can be embedded as the first element of a Loewner chain on $\mathbb{B}_X$. Then $\mathfrak{F}_{n,\alpha}(f) \in S(\mathbb{D}_\alpha)$ can be embedded as the first element of a Loewner chain on $\mathbb{D}_\alpha$.*

As corollaries of Theorem 5.2, we obtain the following results (cf. [10], [14], [20], [21, Chapter 11] ).

**Corollary 5.3.** *Let $\mathbb{B}_X$, $\mathbb{D}_\alpha$ and $\mathfrak{F}_{n,\alpha}$ be as in Theorem* 5.2*. If $\gamma \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $f \in \widehat{S}_\gamma(\mathbb{B}_X)$, then $\mathfrak{F}_{n,\alpha}(f) \in \widehat{S}_\gamma(\mathbb{D}_\alpha)$. In particular, if $f \in S^*(\mathbb{B}_X)$, then $\mathfrak{F}_{n,\alpha}(f) \in S^*(\mathbb{D}_\alpha)$.*

Let $\mathbb{B}_X = \mathbb{B}^n$ be the Euclidean unit ball in $\mathbb{C}^n$. Since $c(\mathbb{B}^n) = \frac{n+1}{2}$, in view of Theorem 5.2 and Corollary 5.3, we obtain the following consequence (cf. [14, Corollary 2.8], [23, Theorem 2.1]).

**Corollary 5.4.** *Let $\Gamma_{n,\alpha}$ be the extension operator given by* (5.6)*, and let $\Omega_\alpha$ be the domain given by* (5.4)*, where $\alpha \geq \frac{n}{n+1}$. Then the following statements hold:*

(i) *If $f \in S(\mathbb{B}^n)$ can be embedded as the first element of a Loewner chain on $\mathbb{B}^n$, then $\Gamma_{n,\alpha}(f)$ can be embedded as the first element of a Loewner chain on $\Omega_\alpha$.*

(ii) *If $\gamma \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $f \in \widehat{S}_\gamma(\mathbb{B}^n)$, then $\Gamma_{n,\alpha}(f) \in \widehat{S}_\gamma(\Omega_\alpha)$. In particular, if $f \in S^*(\mathbb{B}^n)$, then $\Gamma_{n,\alpha}(f) \in S^*(\Omega_\alpha)$.*

*(iii) If $d \in [0, 1)$ and $f \in S(\mathbb{B}^n)$ is an almost starlike mapping of order $d$ on $\mathbb{B}^n$, then $\Gamma_{n,\alpha}(f)$ is almost starlike of order $d$ on $\Omega_\alpha$.*

If $\mathbb{B}_X = \mathbb{U}^n$, then $c(\mathbb{U}^n) = n$, and we obtain the following result from Theorem 5.2 and Corollary 5.3.

**Corollary 5.5.** *Let $\Theta_{n,\alpha}$ be the extension operator given by* (5.8)*, and let $\Delta_\alpha$ be the domain given by* (5.7)*, where $\alpha \geq 1/2$. Then the following statements hold:*

(i) *If $f \in S(\mathbb{U}^n)$ can be embedded as the first element of a Loewner chain on $\mathbb{U}^n$, then $\Theta_{n,\alpha}(f)$ can be embedded as the first element of a Loewner chain on $\Delta_\alpha$.*

(ii) *If $\gamma \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $f \in \widehat{S}_\gamma(\mathbb{U}^n)$, then $\Theta_{n,\alpha}(f) \in \widehat{S}_\gamma(\Delta_\alpha)$. In particular, if $f \in S^*(\mathbb{U}^n)$, then $\Theta_{n,\alpha}(f) \in S^*(\Delta_\alpha)$.*

Next, we mention the following suggestive examples. If we combine Examples 5.6 and 5.7 with Theorem 5.2 and Corollary 5.3, we obtain concrete examples of starlike, spirallike of type $\gamma$, and mappings which can be embedded as the first elements of Loewner chains on the domain $\mathbb{D}_\alpha$, where $\alpha \geq \frac{n}{2c(\mathbb{B}_X)}$. If we combine Examples 5.6 and 5.7 with Corollary 5.4, we also obtain concrete examples of almost starlike mappings of order $d$ on the domain $\Omega_\alpha$, where $\alpha \geq \frac{n}{n+1}$.

**Example 5.6.** Let $f \in \mathcal{L}S(\mathbb{U})$. Let $u \in X \setminus \{0\}$ be fixed and let $l_u \in T(u)$. Also, let $F_u \in H(\mathbb{B}_X)$ be given by

$$F_u(z) = \frac{f(l_u(z))}{l_u(z)} z, \quad z \in \mathbb{B}_X. \tag{5.9}$$

Then we have

$$[DF_u(z)]^{-1} F_u(z) = \frac{f(l_u(z))}{f'(l_u(z)) l_u(z)} z, \quad z \in \mathbb{B}_X.$$

Consequently, we deduce the following statements:

(i) $F_u \in S^*(\mathbb{B}_X)$ if and only if $f \in S^*$.

(ii) $F_u \in \widehat{S}_\gamma(\mathbb{B}_X)$, $\gamma \in (-\frac{\pi}{2}, \frac{\pi}{2})$ if and only if $f \in \widehat{S}_\gamma(\mathbb{U})$.

(iii) $F_u$ is almost starlike of order $d \in [0, 1)$ on $\mathbb{B}_X$ if and only if $f$ is almost starlike of order $d$ on $\mathbb{U}$.

We recall that a Loewner chain $(F_t)_{t \geq 0}$ on $\mathbb{B}_X$ is said to be normal if the family $\{e^{-t} F_t\}_{t \geq 0}$ is a normal family on $\mathbb{B}_X$.

**Example 5.7.** Let $f \in \mathcal{L}S(\mathbb{U})$. Let $u \in X \setminus \{0\}$ be fixed and let $l_u \in T(u)$. Also, let $F_u \in H(\mathbb{B}_X)$ be given by (5.9). Then $F_u$ may be embedded in a normal Loewner chain on $\mathbb{B}_X$ if and only if $f \in S$.

### 5.2. Bloch mappings and Pfaltzgraff-Suffridge type extension operators

Next, we prove that locally univalent I-Bloch mappings on $\mathbb{B}_X$ of finite trace order are extended to R-Bloch mappings on $\Omega_\alpha$ by the Pfaltzgraff-Suffridge type extension operator $\mathfrak{F}_{n,\alpha}$, for $\alpha \geq \frac{1}{2}$. In the case $n = 1$, $f \in \mathcal{LS}(\mathbb{U})$ is uniformly locally univalent on $\mathbb{U}$ if and only if $f$ has a finite order (see [12, Theorem 2.1], [38]). Therefore, the following results are generalizations of Theorem 3.7.

**Theorem 5.8.** *Let $\mathbb{B}_X$ be the open unit ball of an $n$-dimensional JB*-triple $X$. Let $\mathfrak{F}_{n,\alpha}$ be the Pfaltzgraff-Suffridge type extension operator given by* (5.5)*, and let $\Omega_\alpha$ be the domain given by* (5.4)*, where $\alpha \geq \frac{1}{2}$. If $f \in \mathcal{LS}(\mathbb{B}_X)$ is an I-Bloch mapping on $\mathbb{B}_X$ which has finite trace order, then $F = \mathfrak{F}_{n,\alpha}(f) \in \mathcal{LS}(\Omega_\alpha)$ is an R-Bloch mapping on $\Omega_\alpha$.*

Next, we obtain the following consequences of Theorem 5.8.

**Corollary 5.9.** *Let $\mathbb{B}_X$, $\mathfrak{F}_{n,\alpha}$ and $\Omega_\alpha$ be as in Theorem* 5.8*. If $f \in \mathcal{LS}(\mathbb{B}_X)$ is a bounded mapping on $\mathbb{B}_X$ which has finite trace order, then $F = \mathfrak{F}_{n,\alpha}(f) \in \mathcal{LS}(\Omega_\alpha)$ is an R-Bloch mapping on $\Omega_\alpha$.*

**Corollary 5.10.** *Let $\mathbb{B}_X$, $\mathfrak{F}_{n,\alpha}$ and $\Omega_\alpha$ be as in Theorem* 5.8*. Then the following statements hold:*

(i) *If $f \in K(\mathbb{B}_X)$ is an I-Bloch mapping on $\mathbb{B}_X$, then $F = \mathfrak{F}_{n,\alpha}(f) \in S(\Omega_\alpha)$ is an R-Bloch mapping on $\Omega_\alpha$.*

(ii) *If $f \in K(\mathbb{B}_X)$ is a bounded mapping on $\mathbb{B}_X$, then $F = \mathfrak{F}_{n,\alpha}(f) \in S(\Omega_\alpha)$ is an R-Bloch mapping on $\Omega_\alpha$.*

As a corollary of Theorem 5.8, we obtain that the Pfaltzgraff-Suffridge type extension operator $\Gamma_{n,1}$ given by (5.6) maps locally univalent Bloch mappings of finite trace order from the Euclidean unit ball $\mathbb{B}^n$ into locally univalent Bloch mappings on the unit ball $\mathbb{B}_H$ of a complex Hilbert space $H$ with $\dim H \geq n+1$. Note that $\mathbb{B}_H$ can be regarded as the domain

$$\Omega_1 = \{(x, y) \in \mathbb{C}^n \times H_1 : \|x\|_e^2 + \|y\|_{H_1}^2 < 1\},$$

where $H_1$ is a complex Hilbert space with $\dim H_1 \geq 1$.

**Corollary 5.11.** *Let $\mathbb{B}_H$ be the unit ball of a complex Hilbert space $H$ with $\dim H \geq n+1$. Then the following statements hold:*

(i) *If $f \in \mathcal{LS}(\mathbb{B}^n)$ is a Bloch mapping, which has finite trace order, then $F = \Gamma_{n,1}(f) \in \mathcal{LS}(\mathbb{B}_H)$ is a Bloch mapping on $\mathbb{B}_H$.*

(ii) *If $f \in K(\mathbb{B}^n)$ is a bounded mapping on $\mathbb{B}^n$, then $F = \Gamma_{n,1}(f) \in S(\mathbb{B}_H)$ is a Bloch mapping on $\mathbb{B}_H$.*

In view of Corollary 5.11, we obtain the following result related to the preservation of normalized locally univalent Bloch functions under the Roper-Suffridge extension operator (cf. Theorem 3.7). This result is an improvement of [20, Theorem 2.6].

**Corollary 5.12.** *Let $\mathbb{B}_H$ be the unit ball of a complex Hilbert space $H$ with* $\dim H \geq 2$*, and let $f \in \mathcal{LS}(\mathbb{U})$. Then the following statements hold:*

(i) *If $f$ is a uniformly locally univalent Bloch function on $\mathbb{U}$, then $F = \Gamma_{1,1}(f) \in \mathcal{LS}(\mathbb{B}_H)$ is a Bloch mapping on $\mathbb{B}_H$.*

(ii) *If $f$ is a bounded convex function on $\mathbb{U}$, then $F = \Gamma_{1,1}(f) \in S(\mathbb{B}_H)$ is a Bloch mapping on $\mathbb{B}_H$.*

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# Runge pairs of $\Phi$-like domains

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**Abstract.** We prove that if $E \subseteq \mathbb{C}^n$ is a $\Phi$-like domain and $D \subseteq E$ is a $\Phi\big|_D$-like domain, then $(D, E)$ is a Runge pair. Certain applications, examples and questions are also provided.

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## 1. Introduction

Every starlike domain in $\mathbb{C}^n$ is a Runge domain. According to [27, p. 410], this observation goes back at least to Almer [3] and it has been rediscovered several times (cf. [7], [9]). Some of the proofs use the envelopes of holomorphy and/or a result due to Docquier and Grauert [10] (see e.g. [7], [29]). A simple proof has been given by El Kasimi [11]. Hamada [17] has adapted this proof to prove that every spirallike domain in $\mathbb{C}^n$ is a Runge domain.

In this paper, we want to exploit further El Kasimi's ideas [11], in order to develop a criterion for two domains in $\mathbb{C}^n$ to form a Runge pair, in terms of $\Phi$-likeness, a notion introduced by Brickman [8], for one complex variable, and later extended by Gurganus [16], for several complex variables. Certain general results that give sufficient conditions for two pseudoconvex domains in $\mathbb{C}^n$ to form a Runge pair are given in e.g. [10], [23, Theorem 4.25] and [29] (cf. [19, Proposition 3.1.22]). However, in our case, the domains are not necessarily pseudoconvex. The following is our main result.

**Theorem 1.1.** *Let $E \subseteq \mathbb{C}^n$ be a $\Phi$-like domain. If $D \subseteq E$ is a $\Phi\big|_D$-like domain, then $(D, E)$ is a Runge pair.*

For the proof, we combine the ideas from the proof of [11, Proposition 1] with some results from the theory of semigroups of holomorphic self-mappings, extended by Abate [2] for domains in $\mathbb{C}^n$.

We shall provide some examples that point out various aspects of our main result. For example, the domains in Theorem 1.1 are not necessarily Runge domains,

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even though they form a Runge pair. As an application to Theorem 1.1, we prove that every $\Phi$-like domain admits a Runge exhaustion - see [12]. Also, we obtain a result related to [4, Proposition 5.1]. Further, we shall take a look at the invariant domains of a semigroup of holomorphic mappings on a $\Phi$-like domain. Furthermore, we deduce the Runge property of spirallike domains after generalizing [17, Theorem 3.1]. Finally, we point out a version of our main result for taut domains.

## 2. Preliminaries

For every open sets $D \subseteq \mathbb{C}^n$ and $E \subseteq \mathbb{C}^m$ we denote by $\mathcal{H}(D, E)$ the space of holomorphic mappings from $D$ into $E$. For every open set $D \subseteq \mathbb{C}^n$, we denote by $\mathcal{O}(D)$ the space of holomorphic functions from $D$ into $\mathbb{C}$. We consider the topology of locally uniform convergence on these spaces. Also, we denote by $L(\mathbb{C}^n, \mathbb{C}^m)$ the space of complex linear mappings from $\mathbb{C}^n$ into $\mathbb{C}^m$.

We present the definition of Runge pairs (see e.g. [24]).

**Definition 2.1.** *Let $D \subseteq E \subseteq \mathbb{C}^n$ be open sets. We say that $(D, E)$ is a Runge pair, if, for every $f \in \mathcal{O}(D)$ and every compact set $K \subset D$, there exists a sequence in $\mathcal{O}(E)$ that converges uniformly on $K$ to $f$. We say that an open set $D \subseteq \mathbb{C}^n$ is Runge, if $(D, \mathbb{C}^n)$ is a Runge pair.*

**Remark 2.2.** Let $D \subseteq E \subseteq \mathbb{C}^n$ be open sets. We note that $(D, E)$ is a Runge pair if and only if the family of functions in $\mathcal{O}(E)$ restricted to $D$ is dense in $\mathcal{O}(D)$. In particular, $D$ is Runge if and only if the family of complex polynomial functions on $\mathbb{C}^n$ is dense in $\mathcal{O}(D)$.

In the case $n = 1$, $(D, E)$ is a Runge pair if and only if $E \setminus D$ has no nonempty relatively open, compact subsets (see e.g. [21, Theorem 4.9]), i.e., each connected component of $E \setminus D$ is not compact. In particular, we have the well known Runge theorem: $D \subseteq \mathbb{C}$ is a Runge domain if and only if $D$ is simply connected.

Next, we consider the definition of a $\Phi$-like domain. It was introduced by Brickman [8], in dimension one, as a generalization of starlike and spirallike domains in $\mathbb{C}$. Later, Gurganus [16] extended the definition to higher dimensions.

In the following, we use the notation $m(A) = \min\{\Re\langle A(z), z\rangle : \|z\| = 1\}$, for $A \in L(\mathbb{C}^n, \mathbb{C}^n)$, where $\|\cdot\|$ is the Euclidean norm.

**Definition 2.3.** *Let $\Omega \subseteq \mathbb{C}^n$ be a domain. If $0 \in \Omega$ and there exists $\Phi \in \mathcal{H}(\Omega, \mathbb{C}^n)$ such that $\Phi(0) = 0$, $m(D\Phi(0)) > 0$ and, for every $z \in \Omega$, the initial value problem*

$$\frac{\partial w}{\partial t}(z, t) = -\Phi(w(z, t)), \quad t \geq 0, \qquad w(z, 0) = z, \tag{2.1}$$

*has a solution $w(z, \cdot)$ on $[0, \infty)$ such that $w(z, t) \in \Omega$, $t \geq 0$, and $w(z, t) \to 0$, as $t \to \infty$, then we say that $\Omega$ is a $\Phi$-like domain.*

The initial value problem (2.1) is related to the study of one-parameter semigroups of holomorphic self-mappings (see e.g. [1], [25]). We consider below the definition of a one-parameter semigroup on a domain in $\mathbb{C}^n$ (see e.g. [2]).

**Definition 2.4.** *Let $\Omega \subseteq \mathbb{C}^n$ be a domain. We say that $w : \Omega \times [0, \infty) \to \Omega$ is a one-parameter semigroup (or, simply, a semigroup) on $\Omega$ if $t \mapsto w_t = w(\cdot, t)$ is a* *continuous map from* $[0,\infty)$ *into* $\mathcal{H}(\Omega,\Omega)$, $w_0 = \mathrm{id}_\Omega$ *and* $w_{s+t} = w_s \circ w_t$, *for all* $s,t \geq 0$.

**Remark 2.5.** For every one-parameter semigroup $w$ on a domain $\Omega$, $w_t$ is univalent on $\Omega$, for all $t \geq 0$ (see [2, Proposition 1]).

It is well known that there is a one-to-one correspondence between one-parameter semigroups and infinitesimal generators. To be more precise, we consider first the following definition (see e.g. [1], [25]).

**Definition 2.6.** *Let* $\Omega \subseteq \mathbb{C}^n$ *be a domain. We say that* $\Phi \in \mathcal{H}(\Omega,\mathbb{C}^n)$ *is an infinitesimal generator on* $\Omega$ *if, for every* $z \in \Omega$, *the initial value problem* (2.1) *associated to* $\Phi$ *has a solution* $w(z,\cdot)$ *on* $[0,\infty)$. *We mention that, if these solutions exist, then they are necessarily unique* (*see e.g.* [2]).

**Remark 2.7.** For every semigroup $w$ on a domain $\Omega \subseteq \mathbb{C}^n$, there is a unique infinitesimal generator $\Phi$, which is given by $\Phi(z) = \lim_{t\searrow 0} \frac{1}{t}(z - w(z,t))$, locally uniformly with respect to $z \in \Omega$, such that $w(z,\cdot)$ is the solution on $[0,\infty)$ of the initial value problem (2.1) associated to $\Phi$, for every $z \in \Omega$ (see [2, Theorem 5]). Conversely, let $\Phi$ be an infinitesimal generator on a domain $\Omega \subseteq \mathbb{C}^n$ and let $w : \Omega \times [0,\infty) \to \Omega$ be such that, for every $z \in \Omega$, $w(z,\cdot)$ is the solution of the initial value problem (2.1) associated to $\Phi$. Then $w$ is a one-parameter semigroup on $\Omega$ (see [2, p. 169]). In particular, $w_t = w(\cdot,t) \in \mathcal{H}(\Omega,\Omega)$, for all $t \geq 0$.

**Remark 2.8.** Taking into account Definitions 2.3 and 2.4, it is clear that if $\Omega \subseteq \mathbb{C}^n$ is a $\Phi$-like domain, then $\Phi$ is an infinitesimal generator on $\Omega$.

The following family of infinitesimal generators on the Euclidean unit ball $\mathbb{B}^n$ plays an important role in the geometric function theory in several complex variables (see [15], [25]).

**Definition 2.9.** *Let*

$$\mathcal{N} = \{h \in \mathcal{H}(\mathbb{B}^n,\mathbb{C}^n) : h(0) = 0, \Re\langle h(z), z\rangle > 0, z \in \mathbb{B}^n \setminus \{0\}\}.$$

By [16, Lemma 2], $\mathbb{B}^n$ is a $\Phi$-like domain with respect to every $\Phi \in \mathcal{N}$. In particular, every mapping in $\mathcal{N}$ is an infinitesimal generator on $\mathbb{B}^n$.

## 3. Main result

The following lemmas are useful in our proof of the main result.

The first lemma points out that in every $\Phi$-like domain there is a sufficiently small ball which is invariant with respect to the semigroup generated by $\Phi$ (see the discussion in [16, p. 393]).

**Lemma 3.1.** *Let* $\Omega \subseteq \mathbb{C}^n$ *be a domain with* $0 \in \Omega$ *and let* $\Phi \in \mathcal{H}(\Omega,\mathbb{C}^n)$ *be an infinitesimal generator on* $\Omega$ *with* $\Phi(0) = 0$ *and* $m(D\Phi(0)) > 0$. *Also, let* $w$ *be the semigroup on* $\Omega$ *generated by* $\Phi$. *Then there exists* $\delta > 0$ *such that* $\delta\mathbb{B}^n \subset \Omega$ *and* $w_t(\delta\mathbb{B}^n) \subseteq \delta\mathbb{B}^n$, *for all* $t \geq 0$.

*Proof.* Let $A = D\Phi(0)$ and let $\omega(z) = \Phi(z) - Az$, for $z \in \Omega$. In view of the definition of the Fréchet differential of $\Phi$ at 0, we have $\lim_{z\to 0} \frac{\|\omega(z)\|}{\|z\|} = 0$. Let $\delta > 0$ be such that $\delta\mathbb{B}^n \subset \Omega$ and $\|\omega(z)\| \le \frac{m(A)}{2}\|z\|$, for all $z \in \delta\mathbb{B}^n$. Then

$$\Re\langle \Phi(z), z\rangle = \Re\langle Az, z\rangle + \Re\langle \omega(z), z\rangle \ge m(A)\|z\|^2 - \|\omega(z)\|\|z\| \ge \frac{m(A)}{2}\|z\|^2 > 0,$$

for all $z \in \delta\mathbb{B}^n \setminus \{0\}$.

Let $h(z) = \frac{1}{\delta}\Phi(\delta z)$, for $z \in \mathbb{B}^n$. In view of the above inequalities, $h \in \mathcal{N}$. For every $z \in \mathbb{B}^n$, let $v(z,\cdot)$ be the solution on $[0,\infty)$ of the initial value problem

$$\frac{\partial v}{\partial t}(z,t) = -h(v(z,t)), \quad t \ge 0, \qquad v(z,0) = z,$$

which is given by [16, Lemma 2]. We easily deduce that

$$\frac{\partial}{\partial t}\big(\delta v(\frac{1}{\delta}\zeta, t)\big) = -\Phi(\delta v(\frac{1}{\delta}\zeta, t)), \quad t \ge 0,\ \zeta \in \delta\mathbb{B}^n.$$

For every $\zeta \in \delta\mathbb{B}^n$, $w(\zeta,\cdot)$ is the unique solution on $[0,\infty)$ of the initial value problem (2.1) associated to $\Phi$, and thus $w(\zeta,t) = \delta v(\frac{1}{\delta}\zeta, t) \in \delta\mathbb{B}^n$, for all $t \ge 0$. □

In Definition 2.3, we have that every trajectory in a $\Phi$-like domain, given by the initial value problem (2.1), converges to 0, as time goes to infinity. In the following we prove that the convergence is actually uniform on compact subsets of $\Omega$.

**Lemma 3.2.** *Let $\Omega \subseteq \mathbb{C}^n$ be a $\Phi$-like domain. Also, let $w$ be the semigroup on $\Omega$ generated by $\Phi$. Then $w_t \to 0$, as $t \to \infty$, locally uniformly on $\Omega$.*

*Proof.* In view of Vitali's theorem, it suffices to show that $\{w_t\}_{t\ge 0}$ is a normal family. Let $K \subset \Omega$ be a compact set. Also, let $\delta > 0$ be such that $w_t(\delta\mathbb{B}^n) \subseteq \delta\mathbb{B}^n$, for all $t \ge 0$, whose existence is ensured by Lemma 3.1. By Definition 2.3, for every $z \in \Omega$, $w_t(z) \to 0$, as $t \to \infty$. For every $z \in \Omega$, let $t_z \ge 0$ be such that $w_{t_z}(z) \in \delta\mathbb{B}^n$, and then let $V_z \subset \Omega$ be an open set with $z \in V_z$ such that $w_{t_z}(V_z) \subseteq \delta\mathbb{B}^n$. Since $K$ is compact, there exist $z_1, \dots, z_m \in K$ such that $K \subset V_{z_1} \cup \dots \cup V_{z_m}$. Let $T = \max\{t_{z_1}, \dots, t_{z_m}\}$. Then, for every $j \in \{1,\dots,m\}$ and $t \ge T$, we have

$$w_t(V_{z_j}) = w_{t-t_{z_j}}(w_{t_{z_j}}(V_{z_j})) \subseteq w_{t-t_{z_j}}(\delta\mathbb{B}^n) \subseteq \delta\mathbb{B}^n.$$

Hence $w_t(K) \subseteq \delta\mathbb{B}^n$, for all $t \ge T$. Since $t \mapsto w(\cdot,t)$ is a continuous map from $[0,\infty)$ into $\mathcal{H}(\Omega,\Omega)$, we deduce that $w$ is continuous on $\Omega \times [0,\infty)$, and thus $w(K \times [0,T])$ is compact. Therefore we conclude that the family $\{w_t\}_{t\ge 0}$ is bounded on $K$. □

The next lemma is a consequence of [18, Theorem 1.1], which tells us that every semigroup extends holomorphically in a neighborhood of any nonnegative time.

**Lemma 3.3.** *Let $w$ be a semigroup on a domain $\Omega \subseteq \mathbb{C}^n$. Then, for every compact $K \subset \Omega$, there exist an open set $V \subseteq \Omega$ that contains $K$ and a domain $U \subseteq \mathbb{C}$ that contains the interval $[0,\infty)$ such that $w\big|_{V\times[0,\infty)}$ has a holomorphic extension to $V \times U$ which takes values in $\Omega$.*

*Proof.* *Step* 1. Let $\Phi$ be the infinitesimal generator of $w$ and let $z_0 \in \Omega$. By [18, Theorem 1.1] (cf. [22, Theorem 1.8.10]), there exists $\varepsilon > 0$ such that

$$V_{z_0} = \left\{ z \in \mathbb{C}^n : |z_j - z_{0,j}| < \varepsilon,\ j = \overline{1,n} \right\} \subset \Omega$$

and, for every $z \in V_{z_0}$, the initial value problem

$$\frac{\partial v}{\partial \zeta}(z, \zeta) = -\Phi(v(z, \zeta)), \quad |\zeta| < \varepsilon, \qquad v(z, 0) = z, \tag{3.1}$$

has a unique holomorphic solution $v(z, \cdot)$ on $U_{z_0,0} = \{\zeta \in \mathbb{C} : |\zeta| < \varepsilon\}$ which takes values in $\Omega$ and depends holomorphically on $z \in V_{z_0}$. Let $z \in V_{z_0}$. In view of (3.1), we have that $v(z, \cdot)$ is a solution on $[0, \varepsilon)$ of the initial value problem

$$\frac{\partial v}{\partial t}(z, t) = -\Phi(v(z, t)), \quad t \in [0, \varepsilon), \qquad v(z, 0) = z.$$

Taking into account the uniqueness of this solution, we deduce that $w(z, t) = v(z, t)$, for all $t \in [0, \varepsilon)$. Hence $w$ has a holomorphic extension to $V_{z_0} \times U_{z_0,0}$ which takes values in $\Omega$. Since $w_{t_0}$ is holomorphic on $\Omega$ and $w_t = w_{t_0} \circ w_{t-t_0}$, for all $t \geq t_0 \geq 0$, we deduce that $w$ has a holomorphic extension to $V_{z_0} \times U_{z_0,t_0}$, where $U_{z_0,t_0} = \left\{ \zeta : \zeta - t_0 \in U_{z_0,0} \right\}$, for all $t_0 \geq 0$, which takes values in $\Omega$.
*Step* 2. Using the notations from the previous step, let $z_1, z_2 \in \Omega$ and $t_1, t_2 \in [0, \infty)$ be such that $V_{z_1} \cap V_{z_2} \neq \emptyset$ and $U_{z_1,t_1} \cap U_{z_2,t_2} \neq \emptyset$ and let $v_j$ be the holomorphic extension of $w$ to $V_{z_j} \times U_{z_j,t_j}$, $j \in \{1, 2\}$. If $(z, \zeta) \in (V_{z_1} \times U_{z_1,t_1}) \cap (V_{z_2} \times U_{z_2,t_2})$, then $v_1(z, \zeta) = v_2(z, \zeta)$. Indeed, since $v_1(z, t) = v_2(z, t) = w(z, t)$, for all $z \in V_{z_1} \cap V_{z_2}$ and $t \in U_{z_1,t_1} \cap U_{z_2,t_2} \cap [0, \infty) \neq \emptyset$, we deduce, by the Identity Principle, that $v_1(z, \zeta) = v_2(z, \zeta)$, for all $z \in V_{z_1} \cap V_{z_2}$ and $\zeta \in U_{z_1,t_1} \cap U_{z_2,t_2}$.
*Step* 3. For every $z_0 \in \Omega$, let $U_{z_0} = \bigcup_{t_0 \geq 0} U_{z_0,t_0}$ and note that $U_{z_0}$ is a domain in $\mathbb{C}$ that contains $[0, \infty)$ and $w$ has a well defined holomorphic extension to $V_{z_0} \times U_{z_0}$, in view of the previous step, which takes values in $\Omega$.
*Step* 4. Since $K$ is compact, there exist $z_1, \ldots, z_m \in K$ such that $K$ is a subset of the open set $V = V_{z_1} \cup \ldots \cup V_{z_m}$. $U = U_{z_1} \cap \ldots \cap U_{z_m}$ is a domain that contains $[0, \infty)$. $w$ has a holomorphic extension to $V \times U$ (this extension is well defined in view of the second step) which takes values in $\Omega$. □

The next lemma is a consequence of a result of Laufer [21, Theorem 4.11].

**Lemma 3.4.** *Let $E \subseteq \mathbb{C}^n$ be a domain and let $K \subset E$ be a compact set. Then there exists an open set $V \subset E$ relatively compact in $E$ (i.e., $\overline{V}$ is a compact subset of $E$) such that $K \subset V$ and, for every $f \in \mathcal{O}(V)$, there exists a sequence in $\mathcal{O}(E)$ that converges uniformly on $K$ to $f$.*

*Proof.* Taking into account that every domain admits a normal exhaustion (see e.g. [24, p. 17, E.1.2]), we deduce that there exists a sequence of open sets $\{V_k\}_{k \in \mathbb{N}}$ relatively compact in $E$ such that $K \subset V_1 \subset V_2 \subset \ldots$ and $E = \bigcup_{k=1}^{\infty} V_k$. By [21, Theorem 4.11], there exists $m \in \mathbb{N}$ such that, for every $f \in \mathcal{O}(V_m)$, there exists a sequence in $\mathcal{O}(E)$ that converges uniformly on $K$ to $f$. □

**Remark 3.5.** We mention that the above lemma cannot, in general, be strengthened, in the sense that: if $E$ is a domain and $K \subset E$ is a compact set, then there exists an open set $V \subset E$ relatively compact in $E$ such that $K \subset V$ and $(V, E)$ is a Runge pair (see the example of Fornaess and Zame [12]).

Now, we are ready to prove the main result.

**Theorem 3.6.** *Let $E \subseteq \mathbb{C}^n$ be a $\Phi$-like domain. If $D \subseteq E$ is a $\Phi\big|_D$-like domain, then $(D, E)$ is a Runge pair.*

*Proof.* We follow some basic ideas in the proof of [11, Proposition 1] (cf. the proof of [17, Theorem 3.1]). Let $f \in \mathcal{O}(D)$ and let $K \subset D$ be a compact set. We note that $\mathcal{O}(E)$ is a linear subspace of the space $C(K)$ of continuous functions on $K$ with the supremum norm. Hence, by a consequence of the Hahn-Banach theorem (see [26, Theorem 5.19]), $f$ can be approximated uniformly on $K$ by functions in $\mathcal{O}(E)$ if and only if every continuous linear functional on $C(K)$ which vanishes on $\mathcal{O}(E)$ also vanishes at $f$. Taking into account the Riesz-Markov-Kakutani representation theorem for $C(K)$ (see [26, Theorem 6.19]), it suffices to prove that: if $\mu$ is a complex Borel measure on $K$ such that $\int_K g(z)\mathrm{d}\mu(z) = 0$, for all $g \in \mathcal{O}(E)$, then $\int_K f(z)\mathrm{d}\mu(z) = 0$. This strategy of proof has been used in [11] and [17] (cf. the proof of [26, Theorem 13.6]).

Fix a measure $\mu$ that satisfies the above assumptions. Let $w$ be the semigroup on $E$ generated by $\Phi$. Let $V \subset E$ be an open set relatively compact in $E$ such that $K \subset V$ and every function in $\mathcal{O}(V)$ can be approximated uniformly on $K$ by functions in $\mathcal{O}(E)$, whose existence is ensured by Lemma 3.4. Since $D$ is a domain that contains the origin, Lemma 3.2 implies that there exists $T \geq 0$ such that $w_t(V) \subseteq D$, for all $t \geq T$. Hence, for every $t \geq T$, $f \circ w_t$ is well-defined and holomorphic on $V$, and thus, there exists a sequence $(g_k)_{k\in\mathbb{N}}$ in $\mathcal{O}(E)$ such that $g_k \to f \circ w_t$, as $k \to \infty$, uniformly on $K$. Therefore, we have

$$\int_K f(w(z,t))\mathrm{d}\mu(z) = 0, \qquad \text{for all } t \geq T. \tag{3.2}$$

Since $D$ is a $\Phi\big|_D$-like domain, we have that $w$ restricted to $D\times[0,\infty)$ is a semigroup on $D$ (in particular, $w_t(D) \subseteq D$, for all $t \geq 0$). In the following, we use the same notation $w$ for this restriction. By Lemma 3.3, there exist an open set $W \subseteq D$ that contains $K$ and a domain $U \subseteq \mathbb{C}$ that contains the interval $[0,\infty)$ such that $w\big|_{W\times[0,\infty)}$ has a holomorphic extension to $W \times U$ which takes values in $D$. We use the same notation $w$ for this extension. The function $\varphi : U \to \mathbb{C}$ given by $\varphi(\zeta) = \int_K f(w(z,\zeta))\mathrm{d}\mu(z)$, $\zeta \in U$, is well-defined and holomorphic on $U$. In view of (3.2), we have $\varphi(t) = 0$, for all $t \in [T,\infty)$. By the Identity Principle, we deduce that $\varphi(\zeta) = 0$, for all $\zeta \in U$. In particular, $\varphi(0) = 0$, and thus $\int_K f(z)\mathrm{d}\mu(z) = 0$. Taking into account the discussion at the beginning, the proof is complete. □

## 4. Applications, examples and questions

**Definition 4.1.** *We say that $D \subseteq \mathbb{C}^n$ is a starlike domain with respect to 0 if $rz \in D$, for all $z \in D$ and $r \in [0,1]$. We say that $D \subseteq \mathbb{C}^n$ is a starlike domain if there exists $z_0 \in D$ such that $-z_0 + D$ is starlike with respect to 0.*

**Remark 4.2.** Theorem 3.6 implies that every starlike domain is Runge (cf. [7, p. 666], [11, Proposition 1], [19, Corollary 3.1.23]).

*Proof.* Let $D \subseteq \mathbb{C}^n$ be a starlike domain with respect to 0 and $I$ be the identity mapping on $\mathbb{C}^n$. $D$ is an $I\big|_D$-like domain and $\mathbb{C}^n$ is an $I$-like domain (see [16, Section 4]). Theorem 3.6 implies that $D$ is a Runge domain. The general case follows easily from Definitions 2.1 and 4.1. □

The following proposition, related to [16, Theorem 1, Corollary 1], is useful in providing some examples of Φ-like domains.

**Proposition 4.3.** *Let $\Omega \subseteq \mathbb{C}^n$ be a $\Phi_1$-like domain and let $F : \Omega \to \mathbb{C}^n$ be a univalent mapping with $F(0) = 0$ such that $DF(0)$ and $D\Phi_1(0)$ commute. Then $F(\Omega)$ is a $\Phi_2$-like domain, where $\Phi_2 \in \mathcal{H}(F(\Omega), \mathbb{C}^n)$ is given by*

$$\Phi_2(z) = DF(F^{-1}(z))\Phi_1(F^{-1}(z)), \quad z \in F(\Omega).$$

*In particular, if $\Omega$ is a starlike domain with respect to 0 and $F : \Omega \to \mathbb{C}^n$ is a univalent mapping with $F(0) = 0$, then $F(\Omega)$ is a Φ-like domain, where $\Phi \in \mathcal{H}(F(\Omega), \mathbb{C}^n)$ is given by*

$$\Phi(z) = DF(F^{-1}(z))F^{-1}(z), \quad z \in F(\Omega).$$

*Proof.* Let $v$ be the semigroup on $\Omega$ generated by $\Phi_1$. Let $w : F(\Omega) \times [0, \infty) \to \mathbb{C}^n$ be given by $w(z, t) = F(v_t(F^{-1}(z)))$, for $z \in F(\Omega)$. Then it is not difficult to show that $w$ is a semigroup on $F(\Omega)$, which is generated by $\Phi_2$. Moreover, since $v_t(\zeta) \to 0$, as $t \to \infty$, for all $\zeta \in \Omega$, then $w_t(z) \to 0$, as $t \to \infty$, for all $z \in F(\Omega)$. Also, it is easy to prove that $\Phi_2(0) = 0$ and $D\Phi_2(0) = D\Phi_1(0)$. Hence $F(\Omega)$ is a $\Phi_2$-like domain.

The particular result for a starlike domain with respect to 0 follows by taking $\Phi_1 = \mathrm{id}_\Omega$. □

**Question 4.4.** *Is every Φ-like domain $E \subseteq \mathbb{C}^n$ biholomorphic to a starlike domain $D \subseteq \mathbb{C}^n$ for $n \geq 2$?*

**Remark 4.5.** Every Φ-like domain $E \subseteq \mathbb{C}^n$ is simply connected.

*Proof.* Let $w$ be the semigroup on $E$ generated by Φ. Let $f : \partial\mathbb{U} \to E$ be a continuous closed curve. Let $F : \overline{\mathbb{U}} \to E$ be given by $F(r\zeta) = w(f(\zeta), -\log r)$, for $r \in (0, 1]$, $\zeta \in \partial\mathbb{U}$, and $F(0) = 0$. Since $w(\cdot, 0) = \mathrm{id}_E$, we have $F\big|_{\partial\mathbb{U}} = f$. Since $w$ is jointly continuous on $E \times [0, \infty)$ (it has continuous partial derivatives) and $w_t \to 0$, as $t \to \infty$, uniformly on the compact set $f(\partial\mathbb{U})$ (see Lemma 3.2), we deduce that $F$ is continuous on $\overline{\mathbb{U}}$. Hence $f$ can be continuously contracted inside $E$ to 0. □

The following example points out that the domains $D \subseteq E \subseteq \mathbb{C}^n$ in Theorem 3.6 are not necessarily Runge domains, for $n \geq 2$ (in the case $n = 1$, $D$ and $E$ are always Runge domains, because they are simply connected, see Remarks 2.2 and 4.5). We shall use the example of non-Runge domain biholomorphic to a polydisc due to Wermer [30], as it is presented in [20, Example 6.8].

**Example 4.6.** Let $\varphi : \mathbb{C}^3 \to \mathbb{C}^3$ be given by $\varphi(z) = (z_1, z_1z_2 + z_3, z_1z_2^2 - z_2 + 2z_2z_3)$, for $z = (z_1, z_2, z_3) \in \mathbb{C}^3$. Let $\varepsilon_1, \varepsilon_2 \in (0, \frac{1}{2})$, $\varepsilon_1 \leq \varepsilon_2$, be sufficiently small such that $\varphi$ is biholomorphic on the polydiscs

$$P_j = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : |z_1| < 1 + \varepsilon_j, |z_2| < 1 + \varepsilon_j, |z_3| < \varepsilon_j\}, \ j \in \{1, 2\}.$$

For the existence of such $\varepsilon_j$, $j \in \{1,2\}$, see [20, Example 8.4].

Let $E = \varphi(P_2)$ and $D = \varphi(P_1)$. Then $E$ and $D$ satisfy the assumptions of Theorem 3.6 (i.e., $E$ is a $\Phi$-like domain and $D$ is a $\Phi\big|_D$-like domain, for some $\Phi \in \mathcal{H}(E, \mathbb{C}^n)$), but neither of these domains is Runge.

*Proof.* Let $\Phi : E \to \mathbb{C}^3$ be given by $\Phi(\zeta) = D\varphi(\varphi^{-1}(\zeta))\varphi^{-1}(\zeta)$, for $\zeta \in E$. Since $P_1$ and $P_2$ are starlike domains with respect to 0, we deduce, by Proposition 4.3, that $E$ is a $\Phi$-like domain and $D$ is a $\Phi\big|_D$-like domain. In view of [20, Example 6.8], neither of these domains is Runge. □

The next example is related to the previous one.

**Example 4.7.** Let $\phi : \mathbb{C}^2 \to \mathbb{C}^2$ be a univalent mapping with $\phi(0) = 0$ such that $\phi(\mathbb{C}^2)$ is not a Runge domain (the existence of such a mapping is due to Wold [31]). Then there exists $r > 0$ such that $E = \phi(\mathbb{C}^2)$ and $D = \phi(r\mathbb{B}^2)$ are not Runge domains, but they satisfy the assumptions of Theorem 3.6 with $\Phi(z) = D\phi(\phi^{-1}(z))\phi^{-1}(z)$, for $z \in E$.

*Proof.* The result follows in view of [5, Example 2.2] (cf. [24, Theorem VI.1.12]) and Proposition 4.3. □

Regarding Theorem 3.6, we consider the following useful proposition.

**Proposition 4.8.** *Let $E \subseteq \mathbb{C}^n$ be a $\Phi$-like domain and let $D \subseteq E$. Also, let $w$ be the semigroup on $E$ generated by $\Phi$. Then the following are equivalent:*
i) *$D$ is a $\Phi|_D$-like domain.*
ii) *$D$ is a domain with $0 \in D$ and $w_t(D) \subseteq D$, for all $t \geq 0$.*
iii) *there exists an open set $U \subseteq E$ with $0 \in U$ such that $D = w(U, [0, \infty))$.*

*Proof.* The equivalence i)⇔ii) is straightforward, in view of Definition 2.3. For the implication ii)⇒iii), we just take $U = D$. For the implication iii)⇒ii), we observe that: $D = \bigcup_{t \geq 0} w_t(U)$ is open, $w_t(D) = w(U, [t, \infty)) \subseteq D$, for all $t \geq 0$, and every $z \in D$ is connected with 0 through the path

$$\gamma(s) = \begin{cases} w(z, -\log s), & s \in (0, 1], \\ 0, & s = 0. \end{cases}$$

□

Related to [12] (see also Remark 3.5), we have the following corollary, which tells us that every $\Phi$-like domain has a *Runge exhaustion* (see [12, p. 1]).

**Corollary 4.9.** *Let $E$ be a $\Phi$-like domain. Then for every compact set $K \subset E$, there exists an open set $V \subset E$ relatively compact in $E$ such that $K \subset V$ and $(V, E)$ is a Runge pair. Moreover, $V$ can be chosen to be a $\Phi\big|_V$-like domain.*

*Proof.* Let $w$ be the semigroup on $E$ generated by $\Phi$. Since $K \subset E$ is compact, there exists an open set $U \subset E$ relatively compact in $E$ such that $0 \in U$ and $K \subset U$ (one can take $U$ to be a finite union of small balls that cover $K \cup \{0\}$). Let $V = w(U, [0, \infty))$. In view of Lemma 3.2, we have that $V$ is relatively compact in $E$. By Proposition 4.8, $V$ is a $\Phi\big|_V$-like domain. By Theorem 3.6, $(V, E)$ is a Runge pair. □

**Remark 4.10.** In [12, p. 1] it is pointed out that every pseudoconvex domain has a Runge exhaustion (see e.g. the proof of [24, Theorem VI.1.17]; see also [23, Theorem 4.25]). Regarding Corollary 4.9, we mention that, in general, not every Φ-like domain is pseudoconvex. For example, let

$$E = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < 1, |z_2| < 1\} \setminus \big\{(\zeta, 0) : \frac{1}{2} \le |\zeta| < 1\big\}.$$

Then $E$ is a starlike domain with respect to 0. To prove that it is not pseudoconvex, it suffices to observe that every holomorphic function on $E$ is holomorphic on the Hartogs domain

$$\{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < 1, \frac{1}{2} < |z_2| < 1\} \cup \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < \frac{1}{2}, |z_2| < 1\} \subset E$$

and thus it extends to the whole unit polydisc centered at 0 (see e.g. [24, Theorem II.1.1]).

In view of the above example, we also note that the domains in Theorem 3.6 are not necessarily pseudoconvex.

**Question 4.11.** *Taking a look at the arguments used in* [7, p. 666] (*see also* [29]) *to prove that every starlike domain is Runge, we ask the following: is it possible to prove Theorem* 3.6 *using envelopes of holomorphy and the result of Docquier and Grauert* [10]*?*

The following corollary of Theorem 3.6, is related to [4, Proposition 5.1].

**Corollary 4.12.** *Let $E \subseteq \mathbb{C}^n$ be a Φ-like domain. Let $w$ be the semigroup on $E$ generated by Φ. Then $(w_t(E), E)$ is a Runge pair, for all $t \ge 0$.*

*Proof.* Let $t \ge 0$ be fixed and let $D = w_t(E)$. Then

$$w_\tau(D) = w_\tau(w_t(E)) = w_{\tau+t}(E) = w_t(w_\tau(E)) \subseteq w_t(E) = D, \text{ for all } \tau \ge 0.$$

Since $\Phi(0) = 0$, we have $w_t(0) = 0$, by [1, Proposition 2.5.23], and thus $0 \in D$. By Remark 2.5 and Proposition 4.8, $D$ is a $\Phi\big|_D$-like domain. Hence, by Theorem 3.6, we deduce that $(D, E)$ is a Runge pair. □

Looking at Theorem 3.6 and Corollary 4.12, one might suspect that: if $E \subseteq \mathbb{C}^n$ is a Φ-like domain and $D \subseteq E$ is a $\Phi\big|_D$-like domain, then there exists $t \ge 0$ such that $D = w_t(E)$, where $w$ is the semigroup on $E$ generated by Φ. The following example shows that this is, in general, false.

**Example 4.13.** Let $E = \mathbb{B}^n$ and let $D = \mathbb{B}^n \setminus \big\{te_1 : t \in [\frac{1}{2}, 1]\big\}$, where $e_1 = (1, 0, \ldots, 0)$. Let $\Phi = \mathrm{id}_E$. Then $E$ is a Φ-like domain and $D$ is a $\Phi\big|_D$-like domain, but $D \ne w_\tau(E)$, for all $\tau \ge 0$, where $w$ is the semigroup on $E$ generated by Φ.

*Proof.* Since $w_t(z) = e^{-t}z$, for $z \in \mathbb{B}^n$, $t \ge 0$, we easily deduce that $E$ is a Φ-like domain. We observe that

$$w_\tau(D) = \big(e^{-\tau}\mathbb{B}^n\big) \setminus \left\{te_1 : t \in \left[\frac{e^{-\tau}}{2}, e^{-\tau}\right]\right\} \subset \mathbb{B}^n \setminus \left\{te_1 : t \in \left[\frac{e^{-\tau}}{2}, 1\right]\right\} \subset D,$$

for all $\tau \ge 0$. Since $0 \in D$, we have that $D$ is a $\Phi\big|_D$-like domain, by Proposition 4.8. Since $w_\tau(E) = e^{-\tau}\mathbb{B}^n$, we deduce that $D \ne w_\tau(E)$, for all $\tau \ge 0$. □

Taking into account Proposition 4.8, we shall use the following definition.

**Definition 4.14.** *Let $w$ be a semigroup on a domain $E \subseteq \mathbb{C}^n$. We say that $D \subseteq E$ is an invariant domain of $w$, if $w_t(D) \subseteq D$, for all $t \geq 0$.*

We consider the following question (cf. [25, Chapter 7]).

**Question 4.15.** *Let $w$ be a semigroup on a domain $E \subseteq \mathbb{C}^n$ and $D \subset E$ be a domain. Under what conditions is $D$ an invariant domain of $w$?*

**Remark 4.16.** Let $E \subseteq \mathbb{C}^n$ be a $\Phi$-like domain and let $w$ be the semigroup on $E$ generated by $\Phi$. By Theorem 3.6 and Proposition 4.8, a necessary condition for a domain $D \subseteq E$ with $0 \in D$ to be an invariant domain of $w$ is $(D, E)$ to be a Runge pair. In the next example, we shall see that this condition is not sufficient.

**Example 4.17.** Let $E$ be the unit disc centered at the origin of $\mathbb{C}$ and let

$$D = E \setminus \left\{ \frac{t+i}{\sqrt{2}} : t \in [0,1] \right\}.$$

Let $\Phi = \mathrm{id}_E$. Then $E$ is a $\Phi$-like domain and $D \subset E$ is a domain with $0 \in D$ such that $(D, E)$ is a Runge pair, but $D$ is not an invariant domain of $w$, where $w$ is the semigroup on $E$ generated by $\Phi$.

*Proof.* Since $w_t(z) = e^{-t}z$, for $z \in E$, $t \geq 0$, it is easy to prove that $D$ is not an invariant domain of $w$. However, $D$ is a Runge domain, because it is simply connected (see e.g. Remark 2.2), and thus $(D, E)$ is a Runge pair. □

We mention that the previous example can be easily extended to higher dimensions, in view of [22, Corollary 1.7.8].

Next, we ask the following question.

**Question 4.18.** *Let $w$ be a semigroup on a domain $E \subseteq \mathbb{C}^n$ and let $D \subseteq E$ be an invariant domain of $w$. Under what conditions is $(D, E)$ a Runge pair?*

By Theorem 3.6 and Proposition 4.8, we have that: if $E \subseteq \mathbb{C}^n$ is a $\Phi$-like domain and $D \subseteq E$ is an invariant domain of the semigroup $w$ generated by $\Phi$ such that $0 \in D$, then $(D, E)$ is a Runge pair. In the following example, we show that the condition $0 \in D$ is essential.

**Example 4.19.** Let $E$ be the unit disc centered at the origin of $\mathbb{C}$ and let $D = E \setminus \{0\}$. Let $\Phi = \mathrm{id}_E$. Then $E$ is a $\Phi$-like domain and $D \subseteq E$ is an invariant domain of $w$, where $w$ is the semigroup on $E$ generated by $\Phi$, but $(D, E)$ is not a Runge pair.

*Proof.* Since $w_t(z) = e^{-t}z$, for $z \in E$, $t \geq 0$, it is easy to show that $D$ is an invariant domain of $w$. Since $D$ is not simply connected, $(D, E)$ is not a Runge pair (see Remark 2.2). □

Next, we consider an example in higher dimension, related to the previous one.

**Example 4.20.** Let $E \subset \mathbb{C}^2$ be the unit polydisc centered at the origin of $\mathbb{C}^2$. Let $D = \{(z, w) \in \mathbb{C}^2 : |z| < |w| < 1\}$ (this domain is known as the Hartogs triangle, see e.g. [19]). Let $\Phi = \mathrm{id}_E$, and let $w$ be the semigroup on $E$ generated by $\Phi$. Then $D$ is an invariant domain of $w$, but $(D, E)$ is not a Runge pair.

*Proof.* We have $w_t(z,w) = (e^{-t}z, e^{-t}w)$, for $(z,w) \in E$, $t \geq 0$. Clearly, we have that $D$ is an invariant domain of $w$. However, since $E$ is a Runge domain and $D$ is not a Runge domain, we have that $(D,E)$ is not a Runge pair. For the sake of clarity, we give here an elementary argument for the fact that $D$ is not Runge. Let $f \in \mathcal{O}(D)$ be given by $f(z,w) = \frac{1}{w}$, for $(z,w) \in D$. Suppose that there is a sequence $(p_k)_{k\in\mathbb{N}}$ of polynomial functions such that $p_k \to f$, as $k \to \infty$, locally uniformly on $D$. In particular, $p_k(0,\cdot) \to f(0,\cdot)$, as $k \to \infty$, uniformly on the circle

$$\Gamma = \left\{(0,\zeta) : |\zeta| = \frac{1}{2}\right\}.$$

Hence $0 = \displaystyle\int_\Gamma p_k(0,\zeta)\mathrm{d}\zeta \to \int_\Gamma f(0,\zeta)\mathrm{d}\zeta = 2\pi i$, as $k \to \infty$, which is a contradiction. □

We note that in both of the previous two examples the invariant domain (of the corresponding semigroup) is not simply connected, hence we consider the following question.

**Question 4.21.** *Let $n \geq 2$. Does there exist simply connected domains $D \subset E \subseteq \mathbb{C}^n$ such that $D$ is an invariant domain of a semigroup on $E$ and $(D,E)$ is not a Runge pair?*

From now on, we consider a slight modification of the definition of a Φ-like domain in $\mathbb{C}^n$, by dropping the condition $m(D\Phi(0)) > 0$ in Definition 2.3.

**Definition 4.22.** *Let $\Omega \subseteq \mathbb{C}^n$ be a domain. If $0 \in \Omega$ and there exists $\Phi \in \mathcal{H}(\Omega,\mathbb{C}^n)$ such that $\Phi(0) = 0$, and, for every $z \in \Omega$, the initial value problem*

$$\frac{\partial w}{\partial t}(z,t) = -\Phi(w(z,t)), \quad t \geq 0, \qquad w(z,0) = z, \tag{4.1}$$

*has a solution $w(z,\cdot)$ on $[0,\infty)$ such that $w(z,t) \in \Omega$, $t \geq 0$, and $w(z,t) \to 0$, as $t \to \infty$, then we say that $\Omega$ is a Φ-like domain.*

Let $\|\cdot\|$ be an arbitrary norm on $\mathbb{C}^n$ and let $B$ be the unit ball of $\mathbb{C}^n$ with respect to this norm. Then, $\mathbb{C}^n$ may be regarded as a finite dimensional complex Banach space with respect to this norm. For each $z \in \mathbb{C}^n \setminus \{0\}$, let

$$T(z) = \big\{l_z \in L(\mathbb{C}^n,\mathbb{C}) : \ l_z(z) = \|z\|, \ \|l_z\| = 1\big\}.$$

This set is nonempty by the Hahn-Banach theorem.

For a domain $\Omega \subseteq \mathbb{C}^n$ with $0 \in \Omega$, let

$$\mathcal{N}(\Omega) = \{h \in \mathcal{H}(\Omega,\mathbb{C}^n) : \ h(0) = 0, \Re l_z(h(z)) > 0, z \in \Omega \setminus \{0\}, l_z \in T(z)\}.$$

If $\Phi \in \mathcal{N}(rB)$ for some $r \in (0,\infty)$, then we have $\Re l_z(D\Phi(0)z) > 0$ for $z \in rB\setminus\{0\}$ and $l_z \in T(z)$ by [28, Lemma 3]. Therefore, in view of [14, Theorem 3.1] and considering the map $\Psi(z) = r^{-1}\Phi(rz)$ for $z \in B$, we obtain that $B$ is a Ψ-like domain, so $rB$ is a Φ-like domain (use Proposition 4.3 with $F = rI$) and Lemma 3.2 holds for every mapping $\Phi \in \mathcal{N}(rB)$ for arbitrary $r \in (0,\infty)$.

In the case of $\infty B = \mathbb{C}^n$, if $\Phi \in \mathcal{N}(\mathbb{C}^n)$ and $w_r$ is the semigroup generated by $\Phi\big|_{rB}$ for $r \in (0,\infty)$, then, in view of the uniqueness of the solution of the initial value problem (4.1), there exists a semigroup generated by $\Phi$ such that $w(z,\cdot) = w_r(z,\cdot)$, for all $z \in rB$ and $r \in (0, \infty)$, and, in view of the above, $\mathbb{C}^n$ is a $\Phi$-like domain and Lemma 3.2 holds also for every $\Phi \in \mathcal{N}(\mathbb{C}^n)$.

Thus, as in the proof of Theorem 3.6, we obtain the following theorem.

**Theorem 4.23.** *Let $B$ be the unit ball of $\mathbb{C}^n$ with respect to an arbitrary norm on $\mathbb{C}^n$ and let $D \subseteq \mathbb{C}^n$ be a domain. If $D \subseteq rB$ for some $r \in (0, \infty]$ and there exists a $\Phi \in \mathcal{N}(rB)$ such that $D$ is a $\Phi\big|_D$-like domain, then $D$ is a Runge domain.*

Next, we consider the definition of a spirallike domain (see [13]).

**Definition 4.24.** *Let $A \in L(\mathbb{C}^n, \mathbb{C}^n)$ be such that $m(A) > 0$. We say that a domain $D \subseteq \mathbb{C}^n$ is $A$-spirallike if $0 \in D$ and $e^{-tA}z \in D$, for all $z \in D$ and $t \geq 0$. In the case $A = I$, we obtain the definition of a starlike domain with respect to 0.*

We obtain the following corollary from Theorem 4.23.

**Corollary 4.25** ([17, Theorem 3.1])**.** *Let $A \in L(\mathbb{C}^n, \mathbb{C}^n)$ be such that $m(A) > 0$ and let $D \subseteq \mathbb{C}^n$ be an $A$-spirallike domain. Then $D$ is a Runge domain.*

*Proof.* Let $\Phi = A$. Then the semigroup $w$ on $\mathbb{C}^n$ generated by $\Phi$ is given by $w(z, t) = e^{-tA}z$, for $z \in \mathbb{C}^n$, $t \geq 0$. By Proposition 4.8, $D$ is a $\Phi\big|_D$-like domain. $m(A) > 0$ implies $A \in \mathcal{N}(\mathbb{C}^n)$, with respect to the Euclidean norm. So, Theorem 4.23 implies that $D$ is a Runge domain. □

In view of [6, Theorem 2.1], we ask the following question.

**Question 4.26.** *Let $E \subset \mathbb{C}^n$ be a $\Phi$-like domain. Under what conditions on $\Phi$ do we have the following extension of the Andersén-Lempert theorem: if $f : E \to \mathbb{C}^n$ is a biholomorphic mapping whose image $f(E)$ is a Runge domain, then $f$ can be approximated by automorphisms of $\mathbb{C}^n$ locally uniformly on $E$?*

In the case $\Phi = A \in L(\mathbb{C}^n, \mathbb{C}^n)$ with $k_+(A) < 2m(A)$, where $k_+(A)$ is the largest real part of the eigenvalues of $A$, Hamada [17, Theorem 4.2] proved that the theorem holds.

Finally, we give a version of Theorem 3.6 for taut domains. For various properties of the taut domains, see e.g. [1].

**Definition 4.27.** *Let $\mathbb{U} \subset \mathbb{C}$ be the unit disc. A domain $\Omega \subset \mathbb{C}^n$ is said to be taut, if for every sequence $(f_j)_{j \in \mathbb{N}}$ in $\mathcal{H}(\mathbb{U}, \Omega)$ there exists a subsequence $(f_{j_k})_{k \in \mathbb{N}}$ that either converges locally uniformly on $\mathbb{U}$ to a map $f \in \mathcal{H}(\mathbb{U}, \Omega)$ or diverges locally uniformly on $\mathbb{U}$ (i.e., for any two compact sets $K \subset \mathbb{U}$ and $L \subset \Omega$ there exists $k_0 \in \mathbb{N}$ such that $f_{j_k}(K) \cap L = \emptyset$ for $k \geq k_0$).*

**Theorem 4.28.** *Let $E \subseteq \mathbb{C}^n$ be a taut domain with $0 \in E$ that has an infinitesimal generator $\Phi$ such that $\Phi(0) = 0$ and the spectrum of $D\Phi(0)$ lies in the right half-plane $\{\zeta \in \mathbb{C} : \Re\zeta > 0\}$. If $D \subseteq E$ is a $\Phi\big|_D$-like domain, then $(D, E)$ is a Runge pair.*

*Proof.* The proof follows from the proof of Theorem 3.6, once we have proved that $w_t \to 0$, as $t \to \infty$, locally uniformly on $E$, where $w$ is the semigroup on $E$ generated by $\Phi$. By [1, Theorem 2.5.21, Proposition 2.5.23] (see also [1, Corollary 2.1.17]), we deduce that there exists $\rho \in \mathcal{H}(E, E)$ such that $w_t \to \rho$, as $t \to \infty$, locally uniformly on $E$. By [1, Corollary 2.4.2, Proposition 2.5.23 (ii)], we have that $\rho \equiv 0$. □

**Remark 4.29.** i) If $E \subseteq \mathbb{C}^n$ is a domain that satisfies the assumptions in Theorem 4.28, then, using the same arguments, we deduce that Corollaries 4.9 and 4.12 hold also in this case.

ii) Let $B$ be the unit ball of $\mathbb{C}^n$ with respect to an arbitrary norm on $\mathbb{C}^n$. If $r \in (0, \infty)$, then $rB$ is a taut domain (see [1, Corollary 2.1.11]), and, if $\Phi \in \mathcal{N}(rB)$, then $\Phi$ is an infinitesimal generator and the real part of each eigenvalue of $D\Phi(0)$ is positive, because $\Re l_z(D\Phi(0)z) > 0$ for $z \in rB \setminus \{0\}$ and $l_z \in T(z)$ (see the discussion before Theorem 4.23). So, Theorem 4.28 implies Theorem 4.23 in the case $r \in (0, \infty)$ and the case $r = \infty$ follows from the finite cases, in view of Definition 2.1.

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# Shearing maps and a Runge map of the unit ball which does not embed into a Loewner chain with range $\mathbb{C}^n$

Filippo Bracci and Pavel Gumenyuk

*Dedicated to the memory of our friend Professor Gabriela Kohr*

**Abstract.** In this paper we study the class of "shearing" holomorphic maps of the unit ball of the form $(z, w) \mapsto (z + g(w), w)$. Besides general properties, we use such maps to construct an example of a normalized univalent map of the ball onto a Runge domain in $\mathbb{C}^n$ which however cannot be embedded into a Loewner chain whose range is $\mathbb{C}^n$.

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**Keywords:** Loewner chains, geometric function theory, embedding problem.

## 1. Introduction

Let $\mathbb{B}^n := \{z \in \mathbb{C}^n : \|z\| < 1\}$ be the Euclidean unit ball in $\mathbb{C}^n$ and $\mathbb{D} := \mathbb{B}^1$. Let

$$H(\mathbb{B}^n) := \big\{f : \mathbb{B}^n \to \mathbb{C}^n : f \text{ is holomorphic}\big\}.$$

As usual, we endow $H(\mathbb{B}^n)$ with the topology of uniform convergence on compacta. We say that $f \in H(\mathbb{B}^n)$ is normalized if $f(0) = 0$ and $df(0) = \mathsf{Id}$. Let

$$S(\mathbb{B}^n) := \big\{f \in H(\mathbb{B}^n) : f \text{ is normalized univalent on } \mathbb{B}^n\big\}.$$

Also, let

$$U_0(\mathbb{C}^n) := \{f : \mathbb{C}^n \to \mathbb{C}^n \text{ univalent}, f(0) = 0, df(0) = \mathsf{Id}\},$$

and

$$A_0(\mathbb{C}^n) := \{f \in U_0(\mathbb{C}^n) : f \text{ is surjective}\}.$$

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Note that $A_0(\mathbb{C}^n)$ consists of all the automorphisms of $\mathbb{C}^n$ tangent to the identity at the origin, and that for $n > 1$ such a group is huge: in fact, for every jet of the form

$$\mathsf{Id} + \sum_{j=2}^{k} P_j,$$

where $k \in \mathbb{N}$, $k \geq 2$ and $P_j$ is a homogeneous polynomial vector of degree $j$, there exists $F \in A_0(\mathbb{C}^n)$ such that

$$F = \mathsf{Id} + \sum_{j=2}^{k} P_j + o(k)$$

(see [8]). Also, clearly, $A_0(\mathbb{C}^n) \circ S(\mathbb{B}^n) = S(\mathbb{B}^n)$.

Clearly, $A_0(\mathbb{C}) = U_0(\mathbb{C}) = \{\mathsf{Id}\}$, while, for $n > 1$, $A_0(\mathbb{C}^n) \subsetneq U_0(\mathbb{C}^n)$, due to the existence of the so-called Fatou-Bieberbach domains.

As it is well known, $S(\mathbb{D})$ is compact in $H(\mathbb{D})$. The compactness of a family of holomorphic maps is a strong property, because, for instance, it allows growth estimates, estimates on the differentials, existence of support points, Koebe's type theorem, Bieberbach's conjecture, and so on.

In higher dimension the family $S(\mathbb{B}^n)$ is not compact. (For instance, for every $A > 0$, the restriction to $\mathbb{B}^n$ of the automorphism $(z, w) \mapsto (z + Aw^2, w)$ belongs to $S(\mathbb{B}^n)$ but, for $A \to +\infty$ there is no limit point.) It is then natural to ask:

**Question 1.1.** Is there a "natural" compact set $\mathcal{K} \subseteq S(\mathbb{B}^n)$ such that

$$A_0(\mathbb{C}^n) \circ \mathcal{K} = S(\mathbb{B}^n)?$$

For $n = 1$, clearly $\mathcal{K} = S(\mathbb{D})$. For $n > 1$, several natural compact subclasses of $S(\mathbb{B}^n)$ has been introduced. We need some preliminaries in order to define one of the most natural.

**Definition 1.2.** A family $(f_t)_{t\geq 0}$ of holomorphic mappings on $\mathbb{B}^n$ is called a *Loewner chain* if $\{e^{-t} f_t\}_{t\geq 0} \subseteq S(\mathbb{B}^n)$ and $f_s(\mathbb{B}^n) \subseteq f_t(\mathbb{B}^n)$, $0 \leq s \leq t < \infty$.

If, in addition, $\{e^{-t} f_t\}_{t\geq 0}$ is a normal family, then $(f_t)_{t\geq 0}$ is called a *normal Loewner chain*.

For a Loewner chain $(f_t)_{t\geq 0}$ the set

$$R(f_t) := \bigcup_{t\geq 0} f_t(\mathbb{B}^n)$$

is called the *Loewner range* of $(f_t)_{t\geq 0}$.

We say that a mapping $f \in S(\mathbb{B}^n)$ *embeds into a Loewner chain* $(f_t)_{t\geq 0}$ if $f_0 = f$. Let

$S^0(\mathbb{B}^n) := \big\{f \in S(\mathbb{B}^n) : f$ embeds into a normal Loewner chain $(f_t)_{t\geq 0}\big\}$,
$S^1(\mathbb{B}^n) := \big\{f \in S(\mathbb{B}^n) : f$ embeds into a Loewner chain $(f_t)_{t\geq 0}\big\}$,
$S_R(\mathbb{B}^n) := \big\{f \in S(\mathbb{B}^n) : f(\mathbb{B}^n)$ is a Runge domain$\big\}$.

For $n = 1$, by the so-called Pommerenke's embedding theorem and Runge's theorem, $S(\mathbb{D}) = S^0(\mathbb{D}) = S^1(\mathbb{D}) = S_R(\mathbb{D})$ (see [15]). This fact suggested to Gabriela Kohr [14] and I. Graham, H. Hamada and G. Kohr [9] (see also [11]) to define the remarkable class $S^0(\mathbb{B}^n)$. Among many other properties, they showed that $S^0(\mathbb{B}^n)$ is compact, and every normalized convex or starlike mapping of $\mathbb{B}^n$ is contained in $S^0(\mathbb{B}^n)$. Moreover, we have

**Theorem 1.3 (Graham, Kohr, Pfaltzgraff** [12]**).** *Let $(f_t)_{t\geq 0}$ be a Loewner chain. Then there exist $\phi \in U_0(\mathbb{C}^n)$ and a normal Loewner chain $(g_t)_{t\geq 0}$ such that $f_t = \phi \circ g_t$, for all $t \geq 0$. In particular,*

$$S^1(\mathbb{B}^n) = U_0(\mathbb{C}^n) \circ S^0(\mathbb{B}^n).$$

However, although the above theorem is very tempting to make guess that one can take $\mathcal{K} = S^0(\mathbb{B}^n)$, it turns out that

$$A_0(\mathbb{C}^n) \circ S^0(\mathbb{B}^n) \subsetneq S^1(\mathbb{B}^n).$$

Indeed, let $(g_t)$ be a normal Loewner chain. By [2, Prop. 2.1], the Loewner range of $(g_t)$ is $\mathbb{C}^n$ and thus, by a theorem of Docquier-Grauert [6], every $f \in S^0(\mathbb{B}^n)$ has Runge image in $\mathbb{C}^n$. Therefore, also every $h \in A_0(\mathbb{C}^n) \circ S^0(\mathbb{B}^n)$ has Runge image in $\mathbb{C}^n$. However, there exists $g \in S^1(\mathbb{B}^n)$ such that $g(\mathbb{B}^n)$ is not Runge (see [2, Example 2.2]).

Constructing an example of a biholomorphic image of the ball $\mathbb{B}^3$ whose image is not Runge in any bigger domain, J. E. Fornæss and E. Wold [7] recently proved that

$$S(\mathbb{B}^3) \neq S^1(\mathbb{B}^3),$$

which shows, in particular, that, at least for $n \geq 3$, the compact class $\mathcal{K}$ has to be rather exotic.

As the above discussion shows, the main issues are due to Runge-ness property. Thus one might ask what happens if we restrict to the class $S_R(\mathbb{B}^n)$.

Since $S^0(\mathbb{B}^n) \subset S_R(\mathbb{B}^n)$, it is thus natural to ask (cfr. Question Q1) in [2]):

**Question 1.4.** Is it true that $S_R(\mathbb{B}^n) = A_0(\mathbb{C}^n) \circ S^0(\mathbb{B}^n)$?

The main result of this note is the following:

**Theorem 1.5.** *There exists a map $f \in S_R(\mathbb{B}^2)$ that cannot be embedded into a Loewner chain whose range is $\mathbb{C}^2$. In particular, $A_0(\mathbb{C}^2) \circ S^0(\mathbb{B}^2) \subsetneq S_R(\mathbb{B}^2)$.*

This result can be easily generalized to any $n \geq 3$; hence, for $n > 1$,

$$A_0(\mathbb{C}^n) \circ S^0(\mathbb{B}^n) \subsetneq S_R(\mathbb{B}^n).$$

The proof relies on the construction of an example of the form

$$\mathbb{B}^2 \ni (z, w) \mapsto (z + g(w), w),$$

where $g : \mathbb{D} \to \mathbb{C}$ is holomorphic and $g(0) = g'(0) = 0$. Thus, in Section 2 we study general properties of the class of such maps, which we call *shearing maps*. In Section 3, we prove first the following new growth estimate for the differential of maps in $S^0(\mathbb{B}^n)$. For a linear map $L : \mathbb{C}^n \to \mathbb{C}^n$, we denote by $\|L\|$ the operator norm of $L$.

**Proposition 1.6.** *Let $f \in S^0(\mathbb{B}^n)$, $n \geq 1$. Then for every $r \in (0, 1)$ the following inequality holds:*

$$\|df(z)\| \leq \frac{(1 + \sqrt{r})^2}{(1 - r)^3}, \quad \|z\| \leq r.$$

Then, choosing a (non-normal, in the sense of geometric function theory) function $g$, we obtain an example of a function in $S_R(\mathbb{B}^2)$ such that the growth of the differential is faster than predicted for maps in $A_0(\mathbb{B}^2) \circ S^0(\mathbb{B}^2)$.

The following question remains open:

**Question 1.7.** Is it true that $S_R(\mathbb{B}^n) \subset U_0(\mathbb{C}^n) \circ S^0(\mathbb{B}^n)$? Equivalently, due to Theorem 1.3, is it true that every $f \in S_R(\mathbb{B}^n)$ embeds into some Loewner chain, possibly with range different from $\mathbb{C}^n$?

The thoughts on which this work is based, germinated in Cluj in 2015 when the first author was visiting Gabriela Kohr. We separated with the promise to continue to work on this subject together. As often happens, years passed by, the material left in a drawer, and, when the sad news of the premature departure of Gabi arrived, we could only realize that the time was over and we could never benefit again of the amazing ideas of Gabi. We can just thank Gabi for sharing with us her enthusiasm, her deep intuitions, her constant support and many years of friendship.

The authors also warmly thank Mihai Iancu for very fruitful discussions on the subject.

## 2. The class of shearing maps in $S_R(\mathbb{B}^2)$

Fix a holomorphic function $g : \mathbb{D} \to \mathbb{C}$ with $g(0) = g'(0) = 0$. Let $f : \mathbb{B}^2 \to \mathbb{C}^2$ be given by

$$f(z) := (z_1 + g(z_2), z_2),\ z := (z_1, z_2) \in \mathbb{B}^2.$$

Then $f \in S(\mathbb{B}^2)$. If $g$ is an entire function, then $f$ extends to an automorphism of $\mathbb{C}^2$.

**Proposition 2.1.** $f \in S_R(\mathbb{B}^2)$.

*Proof.* Consider the Taylor expansion for $g$:

$$g(\zeta) = \sum_{k=2}^{\infty} a_k \zeta^k,\ \zeta \in \mathbb{D}.$$

For every $m \in \mathbb{N}$, $m \geq 2$, let

$$g_m(\zeta) := \sum_{k=2}^{m} a_k \zeta^k,\ \zeta \in \mathbb{C},$$

and $f_m(z) := (z_1 + g_m(z_2), z_2),\ z = (z_1, z_2) \in \mathbb{C}^2$. Since, for every $m \in \mathbb{N}$, $f_m$ is an automorphism of $\mathbb{C}^2$, we deduce that $f_m\big|_{\mathbb{B}^2} \in S_R(\mathbb{B}^2)$ (see [2]). Thus $f \in S_R(\mathbb{B}^2)$, because $S_R(\mathbb{B}^2)$ is closed in $H(\mathbb{B}^2)$ (see [2]). □

**Proposition 2.2.** *Assume that*

$$g(\zeta) = \sum_{k=2}^{\infty} a_k \zeta^k,\ \zeta \in \mathbb{D},\ \textit{and} \sum_{k=2}^{\infty} k|a_k| < \infty.$$

*Then $f$ embeds into a Loewner chain with range $\mathbb{C}^2$. In particular, $f \in S^1(\mathbb{B}^2)$.*

*Proof.* As before, let $g_m(\zeta) := \sum_{k=2}^{m} a_k\zeta^k$, $\zeta \in \mathbb{C}$, and $f_m(z) := (z_1 + g_m(z_2), z_2)$, $z = (z_1, z_2) \in \mathbb{C}^2$. Then $f_m$ extends to an automorphism of $\mathbb{C}^2$ and

$$f_m^{-1}(z) = (z_1 - g_m(z_2), z_2),\ z = (z_1, z_2) \in \mathbb{C}^2,$$

for all $m \in \mathbb{N}$. Hence

$$(f_m^{-1} \circ f)(z) = (z_1 + \sum_{k=m+1}^{\infty} a_k z_2^k, z_2),\ z = (z_1, z_2) \in \mathbb{C}^2,$$

for all $m \in \mathbb{N}$. Let $N \in \mathbb{N}$ be sufficiently large such that

$$\sum_{k=N+1}^{\infty} k|a_k| \leq 1.$$

Then, in view of [10, Lemma 2.2], we have that $f_N^{-1} \circ f \in S^0(\mathbb{B}^2)$. Since $f_N$ is an automorphism of $\mathbb{C}^2$, we conclude that $f$ embeds into a Loewner chain with range $\mathbb{C}^2$, by Theorem 1.3. □

Denote by $S^*(\mathbb{B}^n)$ the subset of $S(\mathbb{B}^n)$ consisting of all starlike mappings of $\mathbb{B}^n$.

**Proposition 2.3.** *Assume that*

$$g(\zeta) = \sum_{k=2}^{\infty} a_k\zeta^k,\ \zeta \in \mathbb{D},\ \textit{and}\ \sum_{k=2}^{\infty}(k-1)|a_k| \leq \frac{3\sqrt{3}}{2}.$$

*Then $f \in S^*(\mathbb{B}^2)$. In particular, $f \in S^0(\mathbb{B}^2)$. The constant $\frac{3\sqrt{3}}{2}$ is sharp.*

*Proof.* Since we have

$$\begin{aligned}\Re\langle[df(z)]^{-1}f(z), z\rangle &= \|z\|^2 + \Re\sum_{k=2}^{\infty}(1-k)a_k z_2^k\overline{z}_1 \\ &\geq \|z\|^2 - \frac{2}{3\sqrt{3}}\|z\|^2\sum_{k=2}^{\infty}(k-1)|a_k| \\ &\geq 0\end{aligned}$$

for all $z \in \mathbb{B}^2 \setminus \{0\}$, $f$ is starlike.
By [4], if $f \in S^0(\mathbb{B}^2)$, then $|a_2| \leq \frac{3\sqrt{3}}{2}$. In particular, the map $(z_1, z_2) \mapsto (z_1 + a_2 z_2^2, z_2)$ does not belong to $S^*(\mathbb{B}^2)$ if $|a_2| > \frac{3\sqrt{3}}{2}$. Hence, $\frac{3\sqrt{3}}{2}$ is sharp. □

**Proposition 2.4.** *If $f \in S^*(\mathbb{B}^2)$, then $g$ is bounded.*

*Proof.* First, we prove the following: if $f \in S^*(\mathbb{B}^2)$, then, for every $\alpha \in (0, 1]$ and $z = (z_1, z_2) \in \mathbb{B}^2$, we have

$$\left|z_1 + g(z_2) - \frac{1}{\alpha}g(\alpha z_2)\right|^2 + |z_2|^2 < \frac{1}{\alpha^2}. \tag{2.1}$$

Indeed, if $f(\mathbb{B}^2)$ is a starlike domain with respect to the origin, then, for every $\alpha \in (0, 1]$ and $z = (z_1, z_2) \in \mathbb{B}^2$, there exists $z' = (z_1', z_2') \in \mathbb{B}^2$ such that $f(z') = \alpha f(z)$, i.e. $z_2' = \alpha z_2$ and $z_1' = \alpha z_1 + \alpha g(z_2) - g(\alpha z_2)$. Rewriting the condition $|z_1'|^2 + |z_2'|^2 < 1$, we deduce (2.1).

Now, suppose that $g$ is not bounded and fix $\alpha_0 \in (0,1)$. We deduce that

$$\sup_{\zeta\in\mathbb{D}} \left|g(\zeta) - \frac{1}{\alpha_0} g(\alpha_0\zeta)\right| = \infty.$$

Hence (2.1) does not hold for $z = (0,\zeta)$, $\zeta \in \mathbb{D}$, and thus $f$ is not starlike. □

**Definition 2.5 (see** [1, 3]**).** A domain $D \subset \mathbb{C}^n$ is called a starshapelike domain if there is an automorphism $\psi : \mathbb{C}^n \to \mathbb{C}^n$ such that $\psi(D)$ is a starlike domain with respect to the origin. A mapping $f \in S(\mathbb{B}^n)$ is called starshapelike if $f(\mathbb{B}^n)$ is a starshapelike domain.

Using similar arguments as in the case of Propositions 2.2 and 2.3, we deduce the following result.

**Proposition 2.6.** *If* $\sum_{k=2}^{\infty} k|a_k| < \infty$, *then $f$ is starshapelike.*

## 3. The proof of Theorem 1.5

We start by proving the new growth estimate on the differential of functions in $S^0(\mathbb{B}^n)$, which we stated in the Introduction.

*Proof of Proposition* 1.6. Let $\rho \in (0,1)$ be arbitrary. Since $\|f(\zeta)\| \le \frac{\|\zeta\|}{(1-\|\zeta\|)^2}$ for all $\zeta \in \mathbb{B}^n$ (see e.g. [9]), we see that $F(\zeta) := \rho^{-1}(1-\rho)^2 f(\rho\zeta)$, $\zeta \in \mathbb{B}^n$, is a holomorphic self-map of $\mathbb{B}^n$. In view of [13, Theorem 4.6] (cf. [5, Lemma 3]), we deduce that:

$$\left\|dF(\zeta)\right\| \le \frac{1}{1-\|\zeta\|^2}, \quad \zeta \in \mathbb{B}^n.$$

After elementary computations, we get:

$$\left\|df(\rho\zeta)\right\| \le \frac{1}{(1-\rho)^2(1-\|\zeta\|^2)}, \quad \zeta \in \mathbb{B}^n. \tag{3.1}$$

Let $r \in (0,1)$. Substituting $\sqrt{r}$ for $\rho$ in (3.1), we obtain:

$$\left\|df(\sqrt{r}\zeta)\right\| \le \frac{1}{(1-\sqrt{r})^2(1-r)}, \quad \|\zeta\| \le \sqrt{r},$$

and thus, replacing $\sqrt{r}\zeta$ with $z$, we have:

$$\left\|df(z)\right\| \le \frac{(1+\sqrt{r})^2}{(1-r)^3}, \quad \|z\| \le r, \tag{3.2}$$

and we are done. □

With the aid of Propositions 1.6 and 2.1, we are now ready to prove Theorem 1.5.

*Proof of Theorem* 1.5. As mentioned in the Introduction, $A_0(\mathbb{C}^2) \circ S^0(\mathbb{B}^2) \subset S_R(\mathbb{B}^2)$. In order to see that the inclusion is strict, consider the shearing map $f : \mathbb{B}^2 \to \mathbb{C}^2$ defined by

$$f(z) := (z_1 + z_2^2 h(z_2), z_2),\ z = (z_1, z_2) \in \mathbb{B}^2, \quad \text{where}\ \ h(\zeta) := \exp(i/(1-\zeta)^3),\ \zeta \in \mathbb{D}.$$

According to Proposition 2.1, $f \in S_R(\mathbb{B}^2)$. Let us show that nevertheless $f$ does not embed into a Loewner chain with range $\mathbb{C}^2$. By Theorem 1.3, the latter is equivalent to $f \notin A_0(\mathbb{C}^2) \circ S^0(\mathbb{B}^2)$.
Suppose on the contrary that there exists a normalized automorphism $\Phi : \mathbb{C}^2 \to \mathbb{C}^2$ such that $\Phi^{-1} \circ f \in S^0(\mathbb{B}^2)$.
By elementary computations, we obtain:

$$df(z) = \frac{1}{(1-z_2)^4}\begin{bmatrix} 0 & 3iz_2^2 h(z_2) \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 2z_2 h(z_2) \\ 0 & 0 \end{bmatrix}, \quad z = (z_1, z_2) \in \mathbb{B}^2.$$

Observe that:

$$|h(r)| = 1, \quad r \in (0,1). \tag{3.3}$$

Therefore, for all $r \in (1/2, 1)$,

$$\|df(0,r)\| \geq \frac{3r^2}{(1-r)^4} - (1+2r) \geq \frac{2r^2}{(1-r)^4}.$$

By (3.3), we have that $r \mapsto f(0,r)$ is bounded on $(0,1)$. Taking into account that $\Phi$ is an automorphism of $\mathbb{C}^2$, it follows that there exists a compact set $K \subset \mathbb{C}^2$ such that $\Phi^{-1}(f(0,r)) \in K$ for all $r \in (0,1)$. Let

$$C := \max_{w \in K} \|d\Phi(w)\|.$$

On the other hand, we have

$$df(z) = d\Phi(\Phi^{-1}(f(z)))\, d(\Phi^{-1} \circ f)(z), \quad z \in \mathbb{B}^2,$$

and hence, we deduce that:

$$\|d(\Phi^{-1} \circ f)(0,r)\| \geq \frac{1}{\|d\Phi(\Phi^{-1}(f(0,r)))\|}\|df(0,r)\| \geq \frac{2r^2}{C(1-r)^4}, \quad r \in (0, 1/2).$$

On the other hand, $\Phi^{-1} \circ f \in S^0(\mathbb{B}^2)$ and hence, by (3.2),

$$\|d(\Phi^{-1} \circ f)(0,r)\| \leq \frac{4}{(1-r)^3}, \quad r \in (0,1),$$

and thus we have arrived to an obvious contradiction, which completes the proof. □

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# Advanced versions of the inverse function theorem

David Shoikhet

*Dedicated to the memory of Professor Gabriela Kohr*

**Abstract.** This short opus is dedicated to the bright memory of the distinguished mathematician *Gabriela Kohr* and her mathematical heritage. *Gabriela Kohr*'s contribution to analysis of one and several complex variables brought new knowledge into the modern theory as well as new colors to the subject. During our meetings with Gabriela at various conferences she always proposed some interesting and often nonstandard questions related to classical issues as well as new directions. It is worth to be mentioned her excellent book [50] together with Ian Graham on classical and modern problems in Geometric Function Theory in complex spaces (see also, [46], [56], [27], [22], [24], [49] and [48]).

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**Keywords:** Univalent starlike, spirallike functions.

## 1. Introduction

A number of the results presented in this manuscript is based mostly on the joint and works with Filippo Bracci, Mark Elin, Victor Khatskevich, Marina Levenstein, Simeon Reich and Toshiyuki Sugawa [58], [14], [19], [35], [33], [68], [95] as well as addendum from Vladimir Mazi'ya and Gregory Kresin [64], [63], [66] who had many joint mathematical interests with *Gabriela Kohr*. Also we would like to mentioned a great contribution to the theory of semigroups of holomorphic mappings and complex dynamical systems developed by Leonardo Arosio, Filippo Bracci, Manuel D. Contreras and Santiago Diaz-Madrigal and Hidetaka Hamada (see [7], [12], [14], [13] and references therein).

A deep understanding and knowledge of *Gabriela Kohr* in various topics related to generalizations of the Loewner chains to higher dimensions is presented in her joint book in with Ian Graham [50]. In a parallel way Filippo Bracci, Manuel D.

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Contreras, Santiago Diaz-Madrigal and Hidetaka Hamada [15], [13], [20], [17] and [7] have developed geometrical aspects of this theory; some of them were probably waft by Gabrela's work. They have produced very interesting questions and problems (as well as their solutions) which in our opinion will give a push for further investment the Loewner Theory to general complex analysis. In particular, Leandro Arosio, Filippo Bracci, Hidetaka Hamada and *Gabriela Kohr* [7] have presented a new geometric construction of Loewner chains in one and several complex variables which holds on a complete hyperbolic complex manifold M and proved that there is essentially a one-to-one correspondence between evolution families of order d and Loewner chains of the same order. As a consequence they obtained a solution for any Loewner-Kufarev PDE, given by univalent mappings.

Finally, we would like to highlight some questions and problems inspired by *Gabriela Kohr* which for the one-dimensional case have discussed and developed independently by Mark Elin and Fiana Jacobson [31] and [30].

As far as we will see below that actually the inverse function is an element of the so-called *resolvent family* of a discrete (or continuous) semigroup of holomorphic mappings. By using this fact and previous investigations in [39] and [40] in order to answer some *Gabriela Kohr's* questions one can employ the results in [31] to establish new features of nonlinear resolvents of holomorphic generators of one-parameter semigroups acting in the open unit disk. Since the class of nonlinear resolvents consists of univalent functions, it can be studied in the frameworks of classical and modern geometric function theories. In this way in works [31] and [30] the authors establish some distortion and covering results as well as pointed out the order of starlikeness and strong starlikeness of resolvents. It is shown that any resolvent admits quasiconformal extension to the complex plane $\mathbb{C}$. Also, they obtain some characteristics of semigroups generated by these resolvents.

Also, we have to mention a recent work of Xiu-Shuang Ma, Saminathan Ponnusamy and Toshiyuki Sugawa on spirallikeness and strongly starlikeness of harmonic functions.

## 2. Preliminary notions and results

It often happens in mathematics, in examinations of classical issues, that one can discover (sometimes surprisingly) a number of renewed problems and questions.

In this short survey we trace some traits and relationships between invertibility and the numerical range of holomorphic mappings in the one dimensional and (partially) higher dimensional cases.

It is well known that for holomorphic mappings in Banach spaces the *Inverse Function Theorem, the Implicit Function Theorem and the Fixed Point Theorem* are closely related each other.

The classical *Inverse Function Theorem* says.

**Theorem 2.1.** *Let $\mathbb{X}$ be a complex Banach space and let $F$ be a holomorphic mapping in a neighborhood of the origin such that*

$$F(0) = 0 \text{ and } F'(0) \text{ is the invertible linear operator on } \mathbb{X}.$$

*Then there are positive numbers* $r$ *and* $\rho$ *such that* $F(\mathcal{B}_r) \supset \mathcal{B}_\rho$ *and* $F^{-1} : \mathcal{B}_\rho \to \mathcal{B}_r$ *is a well defined holomorphic mapping on* $D_\rho$ *.These numbers* $r$ *and* $\rho$ *are often called the Bloch radii for* $F$ *(cf. for example,* [52], [53], [55], [82], [50], [35] *and* [37]*).*

Note in passing that the pair $(r, \rho)$ is not uniquely defined. See details in [59], [55], [50], [82], [35] and [37]. This manuscript in a sense can be considered an additional chapter to the book [37].

Let $\mathbb{X}^*$ denote the dual of the Banach space $\mathbb{X}$ and let $\langle z, z^* \rangle$ denote the duality pairing of $z^* \in \mathbb{X}^*$ and $z \in \mathbb{X}$. For each $z \in \mathbb{X}$ the set $J(z)$ defined by

$$J(z) = \left\{ z^* \in \mathbb{X}^* : \ \langle z, z^* \rangle = \|z\|^2 = \|z^*\|^2 \right\}$$

is not empty by virtue of the Hanh-Banach theorem and is a closed and convex bounded subset of $\mathbb{X}^*$. The mapping $J : z \mapsto z^*$ is in general multi-valued, however it is single-valued if $\mathbb{X}^*$ is strictly convex. For a Hilbert space $\mathbb{X} = \mathbb{H}$ the semi-scalar product $\langle \cdot, \cdot \rangle$ in $\mathbb{X} * \mathbb{X}$, can be just identify with the standard inner product in $\mathbb{H}$.

Let $D$ and $\Omega$ be domains in $\mathbb{X}$ and let $Hol(D, \Omega)$ be the set of all holomorphic mappings on $D$ with values in $\Omega$. If $D = \Omega$, then we list write $Hol(D)$ for the set $Hol(D, D)$ of holomorphic self-mappings of $D$.

### 2.1. Holomorphically accretive and dissipative mappings

Let $\mathbb{X}$ be a complex Banach space with its dual $\mathbb{X}^*$. By $\langle \cdot, \cdot \rangle$ we denote the semi-scalar product in $\mathbb{X} * \mathbb{X}^*$, so that $\langle z, z^* \rangle = \|z\|^2$ and $|\langle z, w^* \rangle| \le \|z\| \, \|w\|$.

Let $\mathcal{B}$ be the open unit ball in $\mathbb{X}$ and let $f : \mathcal{B} \to \mathbb{X}$ be a *holomorphic* mapping on $\mathcal{B}$.

**Definition 2.2.** (cf. [34]) Let $f \in \mathrm{Hol}(\mathcal{B}, \mathbb{X})$. We say that $f$ is (holomorphically) accretive on $\mathcal{B}$ if

$$\overline{\lim_{s \to 1^-}} \inf \mathrm{Re} \, \langle f_s(z), z^* \rangle \ge \varepsilon \ge 0,$$

where $f_s(z) = f(sz)$, $0 \le s < 1$, $\|z\| = 1$. It is called to be *strongly (holomorphically) accretive* on $\mathcal{B}$ if $\varepsilon > 0$. Respectively, a holomorphic mapping $g : \mathcal{B} \to \mathbb{X}$ is called *(holomorphically) dissipative if* $f = -g$ is *(holomorphically) accretive* on $\mathcal{B}$.

We call these conditions *one side estimates* (see, for example, [3]).

Let $f \in Hol(\mathcal{B}, \mathbb{X})$ admit a continuous extension onto $\overline{\mathcal{B}}$-the closure of $\mathcal{B}$ and be such that

$$\mathrm{Re} \, \langle f(z), z^* \rangle \ge 0$$

for all $z \in \partial \mathcal{B}$ -the boundary of $\mathcal{B}$. Then $f : \mathcal{B} \to \mathbb{X}$ is obviously *(holomoprphically) accretive* on $\mathcal{B}$. *It is strongly holomorphically accretive if*

$$\mathrm{Re} \, \langle f(z), z^* \rangle \ge \varepsilon > 0, \ z \in \partial \mathcal{B}.$$

In this connection we recall the Bohl - Poincare´- Krasnoselskii fixed point theorem.

**Theorem 2.3.** [62] *Let* $\mathcal{B}$ *be the open unit ball of a real Hilbert space* $\mathbb{H}$ *with the inner product* $\langle \cdot, \cdot \rangle$ *and let* $\Phi : \mathcal{B} \to \mathcal{B}$ *be a* completely continuous (compact) *mapping on* $\overline{\mathcal{B}}$ *(not necessarily holomorphic). If condition*

$$\langle \Phi(z), z \rangle \le 1, \ z \in \partial \mathcal{B},$$

*holds, then* $\Phi$ *has at least one fixed point in* $\overline{\mathcal{B}}$. *If* $\langle \Phi(z), z\rangle < 1$, $z \in \partial\mathcal{B}$, *then* $\Phi$ *has a unique fixed point in* $\mathcal{B}$.

Analogously, if $\mathcal{B}$ is the open unit ball in a complex Hilbert space $\mathbb{H}$ and $f : \mathcal{B} \to \mathbb{H}$ is holomorphically accretive (respectively, dissipative) *completely continuous vector field which does not vanish on* $\partial\mathcal{B}$ then it has at least one null point in $\mathcal{B}$. With some additional restrictions a similar result holds also for Banach spaces.

Theorem 2.3 has many applications to the solvability of nonlinear equations. One-sided estimates of such type have been systematically used in many fields. For example, in [62] it is mentioned Galerkin's approximation methods, the theory of equations with potential operators, monotone operator theory and nonlinear integral and partial differential equations. One of the main points in Theorem 2.3 is, of course, the compactness of the mapping $\Phi$ (or more generally the complete continuity of the vector field defined by $I - \Phi$) which allows us to use the methods of the rotation theory of vector fields or degree theory [62]. Since we are interested in the class of holomorphic vector fields, we note that in *infinite dimensional spaces* this class is not contained in the class of completely continuous vector fields. Moreover, in this case the intersection of these classes is quite narrow.

Despite this lack of compactness, there exists a well-developed fixed point theory for holomorphic mappings in Hilbert spaces and Banach spaces (see, for example, [3], [13], [37], [35], [44], [45], [66] and [50]). In particular, for a complex Hilbert space one can reach more information.

**Theorem 2.4.** [4] *Let* $\mathbb{H}$ *be a complex Hilbert space and let* $\mathcal{B}$ *be the open unit ball in* $\mathbb{H}$. *Suppose that* $f$ *is a holomorphic mapping in* $\mathcal{B}$ *which has a uniformly continuous extension onto* $\overline{\mathcal{B}}$ *and satisfies the boundary condition*

$$\operatorname{Re}\langle f(z), z\rangle \geq 0, \quad \textit{respectively,} \quad \operatorname{Re}\langle f(z), z\rangle \leq 0,$$

*for all* $z \in \partial\mathcal{B}$. *The following assertions hold:*

1. Null $f/\overline{\mathcal{B}} \neq \varnothing$;
2. If Null $f/\mathcal{B} \neq \varnothing$, *then it is an affine sub-manifold of* $\mathcal{B}$.

**Corollary 2.5.** *If* $f$ *satisfies one of the above boundary conditions and has no null point on* $\partial\mathcal{B}$, *then it has a unique null point in* $\mathcal{B}$. *In particular, if* $\operatorname{Re}\langle f(z), z\rangle > 0$, *(respectively,* $\operatorname{Re}\langle f(z), z\rangle < 0$ *),* $z \in \partial\mathcal{B}$, *then* $f$ *has a unique null point in* $\mathcal{B}$.

### 2.2. One sided estimates in Banach spaces

Let $\mathcal{B}$ be the open unit ball in a complex Banach space $\mathbb{X}$. The following result can be easily obtained from [4] (Theorem 3).

**Theorem 2.6.** *Let* $f : \mathcal{B} \to \mathbb{X}$ *be a holomorphic mapping on* $\mathcal{B}$ *which admits a uniformly continuous extension to the boundary* $\partial\mathcal{B}$. *Assume also that* $f$ *is* strongly holomorphically accretive on $\mathcal{B}$. *Then* $f$ *has a unique null point in* $\mathcal{B}$.

Clearly this result can be rephrased in the terms of fixed points. To do this we first recall the following version of the famous Earle-Hamilton Theorem [29]: for the unit ball in a complex Banach space.

*If* $F \in Hol(\mathcal{B})$ *is such that* $F(\mathcal{B})$ *lies strictly inside* $\mathcal{B}$, *that is*

$$\inf\left(\|F(x) - y\| \geq \sigma > 0,\ x \in \mathcal{B},\ y \in \partial\mathcal{B}\right),$$

*then* $F$ *has a unique fixed point in* $\mathcal{B}$.

The standard proof of this theorem is based on the construction of pseudo-metric $\rho$ on $\mathcal{B}$ such that $F$ is a strict contraction with respect to $\rho$, i.e.,

$$\rho(F(x), F(y)) \leq k\rho(x, y)$$

for some $k \in (0,1)$. For some generalized versions of the Earl-Hamilton Theorem and additional information see [54] and references therein.

**Theorem 2.7.** [91] *(cf.* [85]*) Let* $G$ *be a holomorphic self-mapping of* $\mathcal{B}$ *which admits a uniformly continuous extension onto the boundary* $\partial\mathcal{B}$ *and satisfies the following boundary condition*

$$\operatorname{Re}\langle G(z), z^*\rangle \leq 1 - \delta,$$

*for some* $\delta > 0$ *and all* $z \in \partial\mathcal{B}$. *Then* $G$ *has a unique fixed point in* $\mathcal{B}$.

It can be easily seen that for the unit ball in a complex Banach space theorem 2.7 is a generalization of the Earle-Hamilton Theorem. For more details see also [55]. The latter theorem can be also extended to a wider class of pseudo-contractive mappings [68], [19] and [37].

The results in Theorem 2.6 and Corollary 2.5 can be completed as follows.

**Theorem 2.8.** [91] *Let* $\mathcal{B}$ *be the open unit ball in a complex Banach space* $\mathbb{X}$ *and let* $F : \mathcal{B} \to \mathbb{X}$ *be a holomorphic mapping on* $\mathcal{B}$. *Assume that* $F$ *admits a continuous extension onto* $\overline{\mathcal{B}}$ *and for some* $\varepsilon > 0$ *the condition of* **strong accretivity** *holds:*

$$\operatorname{Re}\langle F(z), z^*\rangle \geq \varepsilon, \quad z \in \partial\mathcal{B}.$$

*Then the inverse mapping* $z(w) = F^{-1}(w)$ *is well-defined and holomorphic on the ball* $\|w\| < \varepsilon$. *In other words, the numbers* $R = 1$ *and* $r = \varepsilon$ *are the Bloch radii for* $F$. *Moreover, for each* $w : \|w\| < \varepsilon$ *and* $z_0 \in \mathcal{B}$ *the sequence* $z_{n+1} = z_n - F(z_n) + w$, $n = 0, 1, ...,$*converges locally uniformly to this solution* $z(w)$ *on* $\mathcal{B}$.

Clearly this result implies the classical inverse function theorem mentioned above. Earlier results in this theme see also in [16].

In the next part we get down to the one-dimensional case which itself has non-standard particular qualities and has been developed in various directions. We start this part quoting the estimates suggested in [65]. Actually those evaluations can be also obtained by employing the results below.

Assume that $F : \Delta \to \mathbb{C}$ is a holomorphic mapping on the open unit disk $\Delta$ normalized by the conditions $F(0) = 0$ and $F'(0) = 1$ and admits a continuous extension onto $\partial\Delta$ the boundary of $\Delta$.

Define a holomorphic mapping $f$ on $\Delta$ by

$$f(z) = z - F(z),$$

so that $f(0) = 0$ and $f'(0) = 0$.

We also suppose that for some real numbers $\theta$ and $N$ the following condition is satisfied

$$\max_{z\in\partial\Delta} \operatorname{Re}\left(e^{i\theta} f(z)\,\overline{z}\right) \le N.$$

**Theorem 2.9.** *Under the above conditions the inverse mapping mapping $F^{-1}$ is a well-defined holomorphic mapping in the disk*

$$\Omega = \left\{ w \in \mathbb{C}\colon\ |w| < \left((1+2N)^{\frac{1}{2}} - (2N)^{\frac{1}{2}}\right)^2 \right\}$$

*and satisfies the following modulus estimate*

$$\left|F^{-1}(w)\right| \le 1 - \left(\frac{2N}{1+2N}\right)^{\frac{1}{2}}.$$

*Thus,* $\Phi(N) = \left((1+2N)^{\frac{1}{2}} - (2N)^{\frac{1}{2}}\right)^2$ *is a lower estimate for the Bloch radius $R_B$.*

**Remark 2.10.** Recently G. Kresin by using some more delicate calculations has shown that the latter estimate can be improved as follows.

$$R_B \ge \pi^{-1}\left((4N+\pi)^{\frac{1}{2}} - 2N^{\frac{1}{2}}\right)^2 > \left((1+2N)^{\frac{1}{2}} - (2N)^{\frac{1}{2}}\right)^2.$$

Further we discuss some geometrical aspects of the inverse functions.

**Definition 2.11.** Let $\mathcal{D}$ be a circular domain in $\mathbb{X}$. A locally biholomorphic mapping $G : \mathcal{D} \to \mathbb{X}$, $G(0) = 0$, is called star-like if for each $z \in \mathcal{D}$ and $t \in [0,1]$ the line $tG(z) \in G(\mathcal{D})$.

In the one dimensional case where $\mathcal{D} = \cdot$ -the open unit disk in $\mathbb{C}$, a locally univalent mapping $G : \cdot \to \mathbb{C}$, $G(0) = 0$, is star-like if and only if it satisfies the inequality

$$\frac{zG'(z)}{G(z)} \ge \alpha \ge 0.$$

If $\alpha > 0$ the mapping $G$ is characterized as *star-like of order* $\alpha$ (see additional details and certain sources in [57] and [50], [51], [94], [97], [98] and [35].

**Theorem 2.12.** *(cf.* [40]*) Let $f \in \operatorname{Hol}(\Delta, \mathbb{C})$ with $f'(0) \ne 0$. Then there exist numbers $r$ $(0 < r < 1)$ such that $f^{-1}$ is a star-like self-mapping of $\Delta_r$.*

In a parallel way for a domain $D$ in $\mathbb{C}$ we consider the *resolvent equation*

$$z - \lambda f(z) = w \tag{2.1}$$

where, in general, $\lambda$ and $w$ are elements in $\mathbb{C}$.

**Definition 2.13.** One says that $f \in \operatorname{Hol}(\Delta, \mathbb{C})$ is a *locally semi-complete vector field* on $\Delta$ if there is $r$ $(0 < r \le 1)$ such that for each $w : |w| < r$ and each $\lambda > 0$ the equation $z - \lambda f(z) = w$ has a unique solution $z = \Phi(\lambda, w) \in \Delta_r$. If $r = 1$, then $f$ is just said to be *semi-complete* on $\Delta$.

**Remark 2.14.** Another way to define a semi-complete vector field is through ordinary differential equations and the Cauchy problem (see, the next Section). As a matter of fact, for bounded convex domains both definitions are equivalent.

**Remark 2.15.** Sometimes it is more convenient in place of equation (2.1) to define the resolvent family $\Psi_\lambda\,(=\Psi\,(\lambda,w))$ by solving equation $z+\lambda f\,(z)=w$, $\lambda>0$, $|w|<r\leq 1$ (see, for example, [82], [80] and [35]).

Clearly, if $h\in \mathrm{Hol}(\Delta,\mathbb{C})$ with $h\,(0)=0$ and $h'\,(0)\neq -1$ is a locally semi-complete vector field on $\Delta$, then setting $f\,(z)=z+h\,(z)$ we get that $f^{-1}$ exists and is an element of the resolvent family $\{\mathcal{J}_\lambda\}_{\lambda\geq 0}$ with $\lambda=1$.

We conclude these observations with a simple consequence of the classical maximum principle (or, alternatively, the classical Schwarz Lemma) to deduce the following relations.

**Lemma 2.16.** *Let $\Delta$ be the open unit disk in $\mathbb{C}$ and let $h\in \mathrm{Hol}(\Delta,\mathbb{C})$ with $h\,(0)=0$. Assume that*

$$\sup_{z\in\Delta} \operatorname{Re} h\,(z)\,\overline{z}=N<\infty. \tag{2.2}$$

*Then*

$$L=\operatorname{Re} h'\,(0)\quad \leq N.$$

**Theorem 2.17.** [91] *Let $h$ be holomorphic in the open unit disk with $h\,(0)=0$, and let*

$$N=\sup_{z\in\Delta} h\,(z)\,\overline{z}<\infty.$$

*Then $h$ is a locally semi-complete vector field if and only if the following condition holds:*

$$L=h'\,(0)<\min\{0,N\}\,.$$

*In this case $h$ is semi-complete on each disk of radius $r\in\left(0,\dfrac{-L}{2N-L}\right)$.*

## 3. Semi-complete vector fields and semigroups

Let $\mathbb{X}$ be a complex Banach space and let $\mathbb{X}^*$ denote the dual of the Banach space $\mathbb{X}$ and let $\langle z,z^*\rangle$ denote the duality pairing of $z^*\in\mathbb{X}^*$ and $z\in\mathbb{X}$. For each $z\in\mathbb{X}$ the set $J(z)$ defined by

$$J(z)=\left\{z^*\in\mathbb{X}^*:\ \langle z,z^*\rangle=\|z\|^2=\|z^*\|^2\right\} \tag{3.1}$$

is not empty by virtue of the Hahn-Banach theorem and is a closed and convex bounded subset of $\mathbb{X}^*$.

The mapping $J:z\mapsto z^*$ is in general multi-valued, however it is single-valued if $\mathbb{X}^*$ is strictly convex.

Let $D$ be a domain in $\mathbb{X}$ and let $\mathrm{Hol}(D,\mathbb{X})$ be the set of all holomorphic mappings on $D$ with values in $\mathbb{X}$.

**Definition 3.1.** A mapping $f\in Hol(D,\mathbb{X})$ is said to be a semi-complete vector field on $D$ if the Cauchy problem

$$\begin{cases} \frac{\partial u(t,z)}{\partial t}+f(u(t,z))=0\\ u(0,z)=z\end{cases} \tag{3.2}$$

has a unique solution $u=u(t,z)\in D$ for all $z\in D$ and $t\geq 0$.

Furthermore, one can show (see [82] and [86]) that the function $u(t,z)$ satisfies the following partial differential equation

$$\frac{\partial u(t,z)}{\partial t}+\frac{\partial u(t,z)}{\partial z}f(z)=0,\quad z\in D.$$

**Definition 3.2.** A mapping $f$ is a *complete* vector field if the solution of (3.2) exists for all $t\in\mathbb{R}$ and $z\in D$.

In other words, $f$ is complete if both $f$ and $-f$ are semi-complete.

Note also that $f$ is complete if and only if this solution $\{u(t,\cdot)\}$ of (3.2) is a group (with respect to the parameter $t\in(-\infty,\infty)$) of automorphisms of $D$.

The set of semi-complete vector fields on $D$ will be denoted by $\mathcal{G}(D)$. The set of complete vector fields is denoted by $\mathcal{G}_{aut}(D)$.

Various presentations of semi-complete and complete vector fields on the open unit ball $\mathcal{B}$ in $\mathbb{C}^n$, general Hilbert and Banach spaces can be found in [1], [2], [3], [86], [82], [35] and [37].

It is well known that in the case where $D=\Delta=\{z\in\mathbb{C}:|z|<1\}$, a semi-complete vector field is complete if and only if it admits the representation

$$f(z)=a-\overline{a}z^2+ibz$$

for some complex number $a$ and real $b$ (see, for example, [86] and [35]).

If a family $\{F_t=u(t,\cdot)\}$, $t\geq 0$, $(t\in R)$ forms a semigroup ( group) of holomorphic self-mappings of $\Delta$, it follows from the remarkable result of E. Berkson and H. Porta [9] that the limit

$$f(z)=\lim_{t\to 0^+}\frac{1}{t}\left(z-F_t(z)\right)$$

exists and defines a semi-complete (complete) vector field $f$ on $\Delta$.Clearly $f\in\mathrm{Hol}(\Delta,\mathbb{C})$ and determines the holomorphic generator of $\{F_t\}$ via the above formula.

In general, if $D$ is a convex domain in $\mathbb{X}$ and the latter limit exists one can identify the set $\mathcal{G}(D)$ of semi-complete vector fields with the set of all holomorphic generators on $D$. The set $\mathcal{G}(D)$ is a real cone in $\mathrm{Hol}(D,\mathbb{C})$, while the set $\mathcal{G}_{aut}(D)$ of all group generators on $D$ is a real Banach algebra (see [2] and [82]).

We observe also, that for some $z_0\in D$, the equality $F_t(z_0)=z_0$ holds for all $t\geq 0$ if and only if $f(z_0)=0$.

**Definition 3.3.** Let $D$ be a domain in $\mathbb{X}$ and let $h\in\mathrm{Hol}(D,\mathbb{X})$. One says that $h$ satisfies the **range condition** on $D$ if for each $\lambda\geq 0$ the following condition holds $(I-\lambda h)(D)\supset D$ and the equation

$$z-\lambda h(z)=w \tag{3.3}$$

*has a unique solution*

$$z=\mathcal{J}_\lambda(w)\left(=(I-\lambda h)^{-1}(w)\right) \tag{3.4}$$

*holomorphic in* $w\in D$.

In this case the family $\{\mathcal{J}_\lambda\}_{\lambda\geq 0}\in\mathrm{Hol}(D)$ is called the *resolvent family* of $h$ on $D$. Obviously, the inverse function $(I-h)^{-1}$ is an element of the resolvent family with $\lambda=1$.

**Theorem 3.4.** [82] *Let $D$ be a bounded convex domain in $\mathbb{X}$ and let $f \in \mathrm{Hol}(D, \mathbb{X})$. The mapping $f$ defines a semi-complete vector field on $D$ if and only if it satisfies the range condition of Definition* 3.3.

For the one-dimensional case we list the following geometric properties of the resolvent established in [39] and [40] among others:

- Any resolvent $J_\lambda$ is a *hyperbolically convex* self-mapping of $\Delta$ and, consequently, is a star-like function of order $\frac{1}{2}$ (see definition 2.11 and sources in [75], [69], [70], [72], [71] *and* [93]).
- Any resolvent $J_\lambda$ satisfies $\mathrm{Re}\, \frac{J_\lambda}{z} > \frac{1}{2(1+\lambda f'(0))}$. Consequently, $J_\lambda$ is a generator on $\Delta$ and, moreover, the semigroup generated by $J_\lambda$ converges to 0 uniformly on $\Delta$ with exponential squeezing coefficient $\kappa = 1/[2(1 + \lambda f'(0))]$.
- If a generator $f$ itself is a star-like function of order $\alpha > \frac{1}{2}$, then any element $J_\lambda$, $\lambda \geq 0$, of the resolvent family extends to a $(\sin \pi\alpha)$-quasiconformal mapping of $\mathbb{C}$.

Quantitative characteristics of semi-complete vector fields can be formulated as follows.

**Theorem 3.5.** *Let $\mathcal{B}$ be the open unit ball in $\mathbb{X}$ and let $f \in Hol\,(\mathcal{B}, \mathbb{X})$. Then $f \in \mathcal{G}(\mathcal{B})$ if and only if one of the following conditions hold:*

*(i) Abate's inequality* [1]*:*

$$\mathrm{Re}\left[2\left\langle f(z), z^*\right\rangle + \left\langle f'(z)z, z^*\right\rangle\left(1 - \|z\|^2\right)\right] \geq 0.$$

*(ii) Aharonov-Reich-Elin-Shoikhet's* [2] *criterion:*

$$\mathrm{Re}\left\langle f(z), z^*\right\rangle \geq \mathrm{Re}\left\langle f(0), z^*\right\rangle\left(1 - \|z\|^2\right),\ z \in \mathcal{B}.$$

Note that condition (i) originally was establish by Abate in [1] for the finite dimensional Euclidian ball. For the general Banach space it was shown in [2] (see also [82]) by using the reduction to the unit disk $\Delta$ in the complex plane $\mathbb{C}$.

Let $\mathcal{B}$ be the open unit ball in $\mathbb{X}$. Following G. Kohr let us denote by $\mathbf{D}$ the invariant differential operator on $\mathrm{Hol}(\mathcal{B}, \mathbb{X})$ defined by

$$\mathbf{D}f(z) = (1 - \|z\|^2) f'(z).$$

**Remark 3.6.** For the one-dimensional case operator $\mathbf{D}$ has the property that $\mathbf{D}f\,(z) = (f \circ T)'\,(0)$, where $T$ is the automorphism on $\Delta$ given by $T\,(w) = \frac{z+w}{1+\overline{z}w}$, $w \in \Delta$. Thus, if $\mathbf{D}$ is the invariant differential operator on $\mathrm{Hol}(\Delta, \mathbb{C})$ defined by the above formula, condition (i) of Theorem 3.5 can be written as

$$\mathrm{Re}\left[2f(z)\overline{z} + \mathbf{D}f(z)\,|z|^2\right] \geq 0.$$

So, in this case the set $\mathcal{G}(\Delta)$ can be described by the last inequality given in Remark 3.6.

Another very useful representation of the class $\mathcal{G}(\Delta)$ was obtained by Berkson and Porta in [9].

**Theorem 3.7.** *If $\Delta$ is the open unit disk in the complex plane $\mathbb{C}$, then $f \in \mathcal{G}(\Delta)$ if and only if*

$$f(z) = (z - \tau)(1 - z\overline{\tau})p(z) \tag{3.5}$$

*for some $\tau \in \overline{\Delta}$ and $p \in Hol\,(\Delta, \mathbb{C})$ with $\operatorname{Re} p(z) \geq 0$, $z \in \Delta$.*

*The equivalence of conditions (i)-(ii) and (*3.5*) by using direct complex analysis methods was shown in* [2].

In addition, we notice that since presentation (3.5) is unique it follows that $f \in \mathcal{G}(\Delta)$ *must have at most one null point in* $\Delta$.

This fact is no longer true for the higher dimensional case. If, in particular, $\mathbb{X}$ is a reflexive Banach space, then the null point set of $f \in \mathcal{G}(\mathcal{B})$ is a holomorphic retract of $\mathcal{B}$, whence a connected analytic submanifold of $D$ (see [82] and references therein). In particular, for $\mathbb{X} = \mathbb{H}$ being a complex Hilbert space, the null point set of a semi-complete vector field is an affine submanifold of $\mathbb{X}$. In any case, it follows by the uniqueness of the solution of the Cauchy problem (3.2) that the null point set of $f \in \mathcal{G}(\mathcal{B})$ coincides with the common fixed point set of the generated semigroup $S = \{u(t, \cdot)\}_{t=0}^{\infty}$. In particular, $f \in \mathcal{G}(\mathcal{B})$ has a unique null point $\tau \in \mathcal{B}$ if and only if $\tau\,(= u(t, \tau))$ is a unique fixed point of $u(t, \cdot)$ for at least one, hence, for all $t > 0$.

This point $\tau$ is referred to be the *Denjoy-Wolff point* for the semigroup $S = \{u(t, \cdot)\}_{t \geq 0}$ generated by $f$, if

$$\lim_{t \to \infty} u(t, x) = \tau, \text{ for each } x \in \mathcal{B}.$$

It is known ( see, for example, [82] and reference therein) that $\tau \in \mathcal{B}$ is the Denjoy-Wolff point of $S$ if and only if the *spectrum $\sigma\,(A)$ of the linear operator $A = f'\,(0)$ lies in the open right-half plane.*

**Proposition 3.8.** [82]*Let $D$ be a bounded convex domain in $\mathbb{X}$ and let $f \in \mathcal{G}(D)$. Then the null point set of $f$ in $D$ is a connected analytic submanifold of $D$.*

Now by $\mathcal{N}\,(\mathcal{B})$ we denote the class$\{f \in \mathcal{G}(\mathcal{B}) : f(0) = 0,\ \operatorname{Re} \sigma\,(f'\,(0)) > 0\}$. In other words, $\mathcal{N}\,(\mathcal{B})$ consists of those semi-complete vector fields which generate the semigroups with the Denjoy-Wolff point at the origin. This class of generators is closely related to the class $S^*\,(\mathcal{B})$ of star-like mappings or, more generally, the class $Sp\,(\mathcal{B})$ of spiral-like mappings on $\mathcal{B}$ (see, for example, [87] and [37]).

Namely, $h \in Sp(\mathcal{B})$ if and only if it is locally biholomorphic and satisfies the differential equation

$$Ah(x) = h'(x) \cdot f(x),$$

where $f \in \mathcal{N}\,(\mathcal{B})$ and $A = f'\,(0)$. In particular, $h$ is star-like if and only if operator $A$ can be chosen $A = I$ - the identity operator on $\mathbb{X}$ (see, for details the books [47], [86], [50], [82] and [37]). For the geometric description of the convex hull of the set $S^*\,(\cdot)$ see a pioneer work [21] (see also a recent works [26] and [42]).

For the finite dimensional case $\mathbb{X} = \mathbb{C}^n$ the subclass $\mathcal{M}\,(\mathcal{B}) = \{f \in \mathcal{N}(\mathcal{B}) : f(0) = 0,\ f'\,(0) = I\}$ of $\mathcal{N}\,(\mathcal{B})$ was studied by *Gabriela Kohr* (see [60], [61], [50] and references therein). In particular, the following result was presented in [50].

**Theorem 3.9.** *If $\mathbb{X} = \mathbb{C}^n$ , then the set $\mathcal{M}\,(\mathcal{B})$ is compact.*

To motivate our further discussion we note that for the one-dimensional case characterizations of the class $\mathcal{N}(\mathcal{B})$ can be written as:

(a') Re $\left[2f(z)\overline{z}+\mathbf{D}f(z)\left|z\right|^2\right]\geq 0$ and $f(0)=0$

or

(b') Re $f(z)\overline{z}\geq 0,\ z\in\Delta\ \left(\textit{or}\ \mathrm{Re}\,\frac{f(z)}{z}\geq 0,\ z\neq 0\right).$

Surprisingly, it turns out, that formally much weaker condition than condition (a'), namely,

$$\mathrm{Re}\,f'(z)\geq 0,\quad f(0)=0, \tag{3.6}$$

also implies condition (b), that is the property of $f$ to be a semi-complete vector field on $\Delta$. The class of functions satisfying (3.6) is a well-known class consisting of univalent functions due to the Noshiro-Warshawskii Theorem (see [50], [37]). We mention *interalia* the following result.

**Theorem 3.10.** [90] *Let* $f\in Hol(\mathcal{B},\mathbb{C})$, $f(0)=f'(0)-I=0$ *satisfy the generalized Noshiro-Warshawskii condition:*

$$\mathrm{Re}\,\langle f'(x)x,x^*\rangle\geq 0,\ x\in\mathcal{B}. \tag{3.7}$$

*Then* $f$ *is a strongly semi-complete vector field satisfying*

$$\mathrm{Re}\,\langle f(x),x^*\rangle\geq(2\log 2-1)\left\|x\right\|^2>0,\ \ x\in\mathcal{B}. \tag{3.8}$$

This inspires us to consider some more general classes of holomorphic mappings defined by a convex combination of conditions (a) and (b) of Theorem 3.5.
The following question (G. Kohr) naturally rises from Theorem 3.5 and Remark 3.6.

Whether the condition,

$$\mathrm{Re}\left[\alpha\langle f(x),x^*\rangle+\langle f'(x)x,x^*\rangle\left(1-\left\|x\right\|^2\right)\right]\geq 0,$$
$$x\in\mathcal{B},\ f(0)=0,\text{and }\alpha\geq 0 \tag{3.9}$$

also characterizes the class $\mathcal{N}(\mathcal{B})$?

The answer is affirmative for all $\alpha\geq 2$. At the same time, condition (3.9) is sufficient, but is not necessary [14].

Following condition (3.9) at the end of this section we consider some special subclasses of $\mathcal{N}(\mathcal{B})$ which define the so-called parametric filtration of the class $\mathcal{N}(\mathcal{B})$ ([14] and [39]).

For $0\leq t\leq 1$ we denote $\mathcal{G}_t(\mathcal{B})$ the class which consists of functions $f\in\mathrm{Hol}(\mathcal{B},\mathbb{X})$ such that $f(0)=0$ and

$$\mathrm{Re}\left[t\langle f(x),x^*\rangle+(1-t)\langle f'(x)x,x^*\rangle\left(1-\left\|x\right\|^2\right)\right]\geq 0.$$

**Theorem 3.11.** *For each* $0\leq s\leq t\leq 1$ *the following inclusions hold*

$$\mathcal{G}_s(\mathcal{B})\subseteq\mathcal{G}_t(\mathcal{B})\subseteq\mathcal{N}(\mathcal{B})\,. \tag{3.10}$$

*Moreover,*

*(i) For all* $\frac{2}{3}\leq s\leq t\leq 1$ *the following equality holds*

$$\mathcal{G}_s(\mathcal{B})=\mathcal{G}_t(\mathcal{B})=\mathcal{N}(\mathcal{B})\,.$$

*(ii) For each* $0\leq s<t\leq\frac{2}{3}$ *the inclusion* $\mathcal{G}_s(\mathcal{B})\subset\mathcal{G}_t(\mathcal{B})$ *is strong.*

## 4. Null points of holomorphic generators in the disk

First we summarize some preliminary properties of continuous semigroups and their generators, which follow from the Berkson–Porta representation (i) of Theorem 3.5.

Consider a semigroup $S = \{F_t\}_{t\geq 0} \subset \mathrm{Hol}(\Delta)$ generated by $f \in \mathcal{G}(\Delta)$ and make the following observations.

- ⋄ If the point $\tau$ in [9] is an interior null point of $\Delta$ and $f$ does not vanish identically on $\Delta$, then $\tau$ is the unique null point of $f$ in $\Delta$, and (due to the uniqueness of the solution to the Cauchy problem (3.2)), $\tau$ is a common fixed point of $S$, i.e.,

$$F_t(\tau) = \tau \quad \text{for all} \quad t \geq 0. \tag{4.1}$$

- ⋄ If $\tau \in \partial\Delta$, then it is a fixed point of $F_t$ for each $t \geq 0$ in the sense that

$$\lim_{r\to 1^-} F_t(r\tau) = \tau. \tag{4.2}$$

In general, if $S$ is not the trivial semigroup of the identity mappings and does not contain an elliptic automorphism of $\Delta$, then the point $\tau \in \overline{\Delta}$ in (4.2) is an attractive fixed point of the semigroup $S$, i.e.,

$$\lim_{t\to\infty} F_t(z) = \tau \quad \text{for all} \quad z \in \Delta. \tag{4.3}$$

The last assertion is a continuous analog of the *Denjoy–Wolff Theorem.*

**Definition 4.1.** The point $\tau$ in (4.3) is called the **Denjoy–Wolff point** of the semigroup $S = \{F_t\}_{t\geq 0}$.

To proceed we need the following notions.

**Definition 4.2.** One says that a function $f \in \mathrm{Hol}(\Delta, \mathbb{C})$ has the **angular limit** $L$ at a point $\tau \in \partial\Delta$ denoted by $L := \angle \lim\limits_{z\to\tau} f(z)$ if $f(z) \to L$ as $z \to \tau$ in each nontangential approach region

$$\Gamma(\tau, k) = \left\{ z \in \Delta : \frac{|z-\tau|}{1-|z|} < k \right\}, \quad k > 1.$$

**Definition 4.3.** If $L$ in definition 4.2 is finite and the angular limit (finite or infinite)

$$M := \angle \lim_{z\to\tau} \frac{f(z) - L}{z - \tau}$$

exists, then $M$ is said to be the **angular derivative** of $f$ at $\tau$. We denote it by $f'(\tau)$.

It is known (see [76] p. 79) that this angular derivative exists finitely if and only if the angular limit $\angle \lim\limits_{z\to\tau} f'(z)$ exists finitely,hence $f'(\tau) = \angle \lim\limits_{z\to\tau} f'(z)$.

**Remark 4.4.** By using the Riesz–Herglotz representation (see, for example, [47]) of functions with a positive real part, one can show (see [37] and [15]) that if $\tau \in \partial\Delta$ is the boundary Denjoy–Wolff point of the semigroup $S = \{F_t\}_{t\geq 0}$ generated by $f \in \mathcal{G}(\Delta)$, then the angular derivative $f'(\tau)$ exists finitely and is a real nonnegative number. Moreover, $f'(\tau) = \lim_{r\to 1^-} \frac{f(r\tau)\overline{\tau}}{r-1}$. Some inequalities for angular derivatives related to interpolation problems are given in [11]. The second angular derivative and parabolic iteration were studied in [26].

Thus, every non-trivial semigroup $S = \{F_t\}_{t\geq 0}$ on $\Delta$ which does not contain an elliptic automorphism of $\Delta$, falls in one of three mutually exclusive classes depending on the nature of its Denjoy–Wolff point $\tau$. These classes can be described in terms of generators as follows: the Denjoy–Wolff point $\tau$ of $S$ satisfies $f(\tau) = 0$ and the semigroup $S$ must be of one of the following three types:

– **dilation type** if $\tau \in \Delta$ and $\operatorname{Re} f'(\tau) > 0$;
– **hyperbolic type** if $\tau \in \partial\Delta$ and $0 < f'(\tau) < \infty$;
– **parabolic type** if $\tau \in \partial\Delta$ and $f'(\tau) = 0$.

The real part $\operatorname{Re} f'(\tau)$ vanishes at an interior null point $\tau \in \Delta$ of a generator $f \in \mathcal{G}(\Delta)$ if and only if the semigroup $S$ generated by $f$ contains either the identity mappings or elliptic automorphisms of $\Delta$.

Without loss of generality up to appropriate Möbius transformations of the unit disk we distinguish two cases: $\tau = 0$ and $\tau = 1$.

### 4.1. Interior null points

Let $f$ be the generator of a one-parameter continuous semigroup $S = \{F_t\}_{t\geq 0}$ on $\Delta$. Suppose that $S$ is not trivial, does not contain elliptic automorphisms, and that $\tau \in \Delta$ is the interior null point of $f$. Without loss of generality we set $\tau = 0$. In this case $\tau = 0$ is the attractive fixed point of the semigroup, $\Re f'(\tau) > 0$, and the rate of convergence of the semigroup in terms of the Euclidean distance is completely determined by the following theorem (see, for example, [86]).

**Theorem 4.5.** *Let $f \in \mathcal{G}(\Delta)$ be such that $f(0) = 0$ and $\lambda := \operatorname{Re} f'(0) > 0$, and let $S = \{F_t\}_{t\geq 0}$ be the semigroup generated by $f$. Then there exists $c \in [0,1]$ such that for all $z \in \Delta$ and $t \geq 0$, the following estimates hold:*

$$(i)\ \ |z| \cdot \exp\left(-\lambda t \frac{1 + c|z|}{1 - c|z|}\right) \leq |F_t(z)| \leq |z| \cdot \exp\left(-\lambda t \frac{1 - c|z|}{1 + c|z|}\right);$$

$$(ii)\ \ \exp(-\lambda t) \frac{|z|}{(1 + c|z|)^2} \leq \frac{|F_t(z)|}{(1 - c\,|F_t(z)|)^2} \leq \exp(-\lambda t) \frac{|z|}{(1 - c|z|)^2}.$$

Inequality (ii) implies that for each $z \in \Delta$ the rate of convergence of the semigroup to its interior Denjoy–Wolff point is of exponential type.

Note that for $c = 1$ estimate (i) is due to Gurganus [51], while estimate (ii) was established by Poreda [77].

### 4.2. Boundary null points

Note that a generator $f \in \mathcal{G}(\Delta)$ may have more than one boundary null point, and for each such point $\zeta \in \partial\Delta$, the angular derivative $f'(\zeta)$ exists and is a real number or infinity (see [86], [33], [25] and [35]).

**Definition 4.6.** A point $\zeta \in \partial\Delta$ is called a **boundary regular null point** of $f \in \operatorname{Hol}(\Delta, \mathbb{C})$ if the angular (radial) derivative $f'(\zeta)$ exists finitely.

In fact, a boundary regular null point $\zeta$ of $f$ is the attractive fixed point of the semigroup $S$ generated by $f$ if and only if $f'(\zeta) \geq 0$. If for a boundary null point $\zeta \in \partial\Delta$ of $f$, $f'(\zeta) < 0$ then $\zeta$ is a **repelling (or repulsive) fixed point** of $S$ (see [36]-[32]).

For a point $\zeta \in \overline{\Delta}$, we define the class

$$\mathcal{G}[\zeta] := \{f \in \mathcal{G}(\Delta) : \ f(\zeta) = 0 \quad \text{and} \quad f'(\zeta) \quad \text{exists finitely}\}. \tag{4.4}$$

In other words, the class $\mathcal{G}[\zeta]$ is the subcone of $\mathcal{G}(\Delta)$ of all generators vanishing at the point $\zeta$, and having a finite (angular) derivative at this point.

For a boundary point $\zeta \in \partial\Delta$, each element $f$ of $\mathcal{G}[\zeta]$ has another useful parametric representation.

**Theorem 4.7.** (*see* [86] *cf. also* [87] *and* [82]) *Let* $\zeta \in \partial\Delta$. *Then* $f \in \mathcal{G}[\zeta]$ *admits the representation*

$$f(z) = (z-\zeta)(1-z\overline{\zeta})p(z) + \frac{\lambda}{2}(\overline{\zeta}z^2 - \zeta), \tag{4.5}$$

*where*

$$\lambda = f'(\zeta), \quad \operatorname{Re} p\,(z) \geq 0 \quad \textit{and} \quad \angle \lim_{z\to\zeta} \left(1 - z\overline{\zeta}\right) p\,(z) = 0. \tag{4.6}$$

It is clear that the point $\zeta$ in (4.5) is the Denjoy–Wolff point of the corresponding semigroup if and only if $\lambda \geq 0$.

In general, it turns out, that even for a boundary regular null point $\tau$ of $f$, which is not necessarily the Denjoy–Wolff point, but $f \in \mathcal{C}^3_A(\tau)$ ( i.e., $f$ has the third angular derivative at the point $\tau$, its quadratic part, say $g$, is also a generator of a semigroup of linear-fractional transformations on $\Delta$. Therefore, the natural question is: *which conditions provide* $f = g$*?*

**Theorem 4.8.** *Let* $f \in \mathcal{G}[1]$ *be of class* $\mathcal{C}^3_A(1)$ *and let* $g(z) = f'(1)(z-1) + \frac{1}{2}f''(1)(z-1)^2$ *be its quadratic part. Then*

*(i)* $g$ *is the generator of a semigroup of linear-fractional transformations on* $\Delta$*;*

*(ii)* $f'(1) - \operatorname{Re} f''(1) \geq 0$*;*

*(iii) If* $h := f - g$ *belongs to the class* $\mathcal{G}(\Delta)$ *then* $\operatorname{Re} f'''(1) \geq 0$. *Moreover,* $\operatorname{Re} f'''(1) = 0$ *if and only if* $f = g$.

*In particular,* $f(z) \equiv 0$ *if and only if* $f'(1) = f''(1) = f'''(1) = 0$.

Since for a self-mapping $F$ of $\Delta$ the mapping $I - F$ defines a semi-complete vector field (generator) on $\Delta$, the latter assertion is a generalization of the Burns-Krantz Theorem [23].

In this connection we also would like to mention that the classical Shwarz Lemma and Shawrz-Pick Lemma are the prototype of earlier rigidity results. Recently Filippo Bracci, Daniela Kraus and Oliver Roth [20] have continue the study and developments of this issue and established several versions for conformal pseudometrics on the unit disk including boundary versions of Ahlfors-Schwarz and Nehari-Schwarz Theorems, as well as for holomorphic self-mappings of strongly convex domains in $\mathbb{C}^n$ .

The so-called "slice rigidity property" of holomorphic mappings Kobayashi-isometrically preserving complex geodesics have been given by Filippo Bracci, Łukasz Kosiński, Włodzimierz Zwonek [18] More precisely.

Let $\Delta$ be the unit disc in $\mathbb{C}$ and let $F : \Delta \to \mathbb{C}$ be a Riemann mapiing such that $F(\Delta) = \Delta$. Then it was presented a necessary and sufficient condition in terms of hyperbolic distance and horocycles which assures that a compactly divergent sequence $\{z_n\} \subset \Delta$ has the property that $\{f^{-1}(z_n)\}$ converges orthogonally to a point of $\Delta$.

In this connection we also mentioned the papers [15] and [20].

In addition, verifying the proofs in [43] and [36] one shows that the point $w = 0$ is the boundary regular fixed point of the restriction of $\mathcal{J}_r$ on $\Omega$ whenever $r \in \left(-\frac{1}{f'(0)}, 0\right)$ with $\mathcal{J}_r'(0) = \frac{1}{1+rf'(0)}$.To illustrate Proposition 6.2 take Example 2 in [40] and consider the semigroup generator $f(z) = z(1-z)$. Its resolvent $\mathcal{J}_r$ is:

$$\mathcal{J}_r(w) = \frac{r+1-\sqrt{(r+1)^2-4rw}}{2r}$$

First we see that the angular limit $\angle \lim\limits_{w\to 1} \mathcal{J}_r(w) = 1$ for every $r < 0$ and $\mathcal{J}_r'(1) = \frac{1}{1-r}$ for every $r \in (-\infty, 1)$. In addition, $\mathcal{J}_r(0) = 0$ if $r \geq -1$ and $\mathcal{J}_r'(0) = \frac{1}{1+r}$ for $r > -1$. Finally, it can be seen by using the results in [43] that the maximal BFID corresponding to $\zeta = 1$ is $\Omega = \left\{z : \left|z - \frac{1}{2}\right| < \frac{1}{2}\right\}$.

## 5. Analytic extension of one-parameter semigroups

In this section we discuss the following problem.

Let $\{F_t\}_{t\geq 0}$ be the semigroup generated by an $f \in \mathcal{G}(\Delta)$. Whether there is a domain $\mathbb{Q}$ in the right half-plane such that the semigroup admits an analytic extension to $\mathbb{Q}$, preserving it's algebraic properties?

For the case when $f = A$ is a continuous linear operator on $X$ an affirmative answer was given by E. Hille [29] see also the book of Hille-Phillips [56].

**Proposition 5.1.** *Let $B$ be the infinetesimal generator of a semigroup of linear continuos operators $\{A_t\}$, $t \geqq 0$, on $X$ such that $A_t X \subset D(B)$ −the domain of definition of $B$.Assume that there is a constant $N > 0$ with $t\|BA_t\| \leq N$, $0 \leq t < 1$.Then there is a holomorphic operator function $\{A_\varsigma\} : \varsigma \in D$, where $\mathbb{Q} = \left\{\varsigma : \operatorname{Re}\varsigma > 0,\ \ |\arg\varsigma| < \frac{1}{eN}\right\}$ and $A_{\varsigma_1}A_{\varsigma_2} = A_{\varsigma_1+\varsigma_2}$ whenewer $\varsigma_1$ and $\varsigma_2$ belong to $D$. In addition, the strong limit $\lim_{\varsigma\to 0} A_\varsigma x = x$, $x \in D$, whenever $|\arg\varsigma| \leq \frac{\varepsilon}{eN}$, $\varepsilon \in (0,1)$.*

**Remark 5.2.** By using the tools and methods of the theory composition operators (see, for example, [27], [92]) one can easily establish analogs of this result for nonlinear semigroup of holomorphic mappings [37].

The following fact is a key for our considerations in the sequel.

**Theorem 5.3.** [31] *Let $\alpha, \beta \in (0, \frac{\pi}{2})$. Then the semigroup $\{F_t\}_{t\geq 0}$ generated by $f$, $f(z) = zp(z)$, can be analytically extended to the sector $\{t \in \mathbb{C} :\ \arg t \in (-\alpha, \beta)\}$ for all $z$ in the open unit disk $\Delta$ if and only if $-\frac{\pi}{2} + \alpha < \arg p(z) < \frac{\pi}{2} - \beta,\ z \in \mathbb{D}$.*

We deduce here another result from [31] which completes the material in previous sections.

**Theorem 5.4.** *Let $f$ be a semi-complete vector field and let $\mathcal{J}_r$ be its resolvent with $r \geq \frac{6}{\operatorname{Re} q}$. Denote $\gamma_r := \frac{1-A(r\operatorname{Re} q)}{1+A(r\operatorname{Re} q)}$, where $A$ is defined by*

$$A(r) := \frac{6r(1+r)}{(1+r)^3 - 3(5r-1)} \tag{5.1}$$

*Then for the semigroup $\{\Phi_{t,r}\}_{t\geq 0}$ generated by $J_r$ the following assertions hold:*

(i) *for fixed $t > 0$ the net $\{\Phi_{t,r}\}$ converges to $0$ as $r \to \infty$, uniformly on the open unit disk with the exponential squeezing coefficient*

$$\kappa(r) := \frac{\left(\operatorname{Re}(1+rq)^{\frac{1}{\gamma_r}}\right)^{\gamma_r}}{2^{1-\gamma_r}|1+rq|^2}.$$

**Theorem 5.5.** *For every fixed $z$ in the open unit disk, and $r > 0$ the mapping $\Phi_{t,r}(z)$ can be analytically extended in the parameter $t$ to the sector*

$$\left\{t \in \mathbb{C} : |\arg t - \arg(1+rq)| < \frac{\pi\gamma_r}{2}\right\}.$$

## 6. Backward flow invariant domains

To proceed we quote partially the result proved in [43] (see also [37]).

**Lemma 6.1.** *A function $f \in \mathcal{N}$ has a boundary regular null point $\zeta \in \partial\Delta$ if and only if there is a simply connected domain $\Omega \subset \Delta$ such that $f$ generates a one-parameter group $S = \{F_t\}_{-\infty<t<\infty}$ of hyperbolic automorphisms on $\Omega$ such that the points $z = 0$ and $z = \zeta$ belong to $\partial\Omega$ and are boundary regular fixed points of $S$ on $\partial\Omega$. Moreover, $f'(\zeta)$ is a real negative number.*

We call such a domain *backward flow invariant domain (or shortly BFID)*. Note that in general a BFID $\Omega$ is not unique for a point $\zeta \in \partial\Delta$, but there is a unique BFID $\Omega$ (called the maximal BFID) with the above properties such that $\Omega$ has a corner of opening $\pi$ at the point $\zeta$ (see [76]). Other characterizations of backward flow invariant domains can be found in [43], [36], [35] and [13].

An interesting phenomenon occurs when we consider the resolvent family only on BFID. Namely,

**Proposition 6.2.** *Let $f \in \mathcal{N}(\Delta)$ have a boundary regular null point $\zeta \in \partial\Delta$ and $\Omega$ is a BFID in $\Delta$ corresponding to $\zeta$. If $\Omega$ is convex, then the restriction of the resolvent family $\mathcal{J}_r$ on $\Omega$ can be continuously extended in the parameter $r \in (-\infty, 0)$ such that $\zeta$ is a boundary fixed point of $\mathcal{J}_r$ for every $r < 0$. Moreover, $\lim\limits_{r\to-\infty} \mathcal{J}_r(w) = \zeta$ whenever $w \in \Omega$.*

**Lemma 6.3.** *A function $f \in \mathcal{N}(\Delta)$ has a boundary regular null point $\zeta \in \partial\Delta$ if and only if there is a simply connected domain $\Omega \subset \Delta$ such that $f$ generates a one-parameter group $S = \{F_t\}_{-\infty<t<\infty}$ of hyperbolic automorphisms on $\Omega$ such that the points $z = 0$ and $z = \zeta$ belong to $\partial\Omega$ and are boundary regular fixed points of $S$ on $\partial\Omega$. Moreover, $f'(\zeta)$ is a real negative number.*

It follows from the Scwarz Lemma that $F_t'^{-tf'(0)} < 1$ while $F_t'^{-tf'(\zeta)} > 1$.

Thus the point $w = 0$ is a boundary regular fixed point of the restriction of $\mathcal{J}_r$ on $\Omega$ whenever $r \in \left(-\frac{1}{f'(0)}, 0\right)$ with $\mathcal{J}_r'(0) = \frac{1}{1+rf'(0)}$.

To illustrate Proposition 6.2 and the latter fact, return now to the semigroup generator $f(z) = z(1-z)$ and its resolvent

$$\mathcal{J}_r(w) = \frac{r+1-\sqrt{(r+1)^2-4rw}}{2r}$$

that were considered in [39] and [40]. First we see that the angular limit $\angle \lim_{w \to 1} \mathcal{J}_r(w) = 1$ for every $r < 0$ and $\mathcal{J}_r'(1) = \frac{1}{1-r}$ for every $r \in (-\infty, 1)$. In addition, $\mathcal{J}_r(0) = 0$ if $r \geq -1$ and $\mathcal{J}_r'(0) = \frac{1}{1+r}$ for $r > -1$. Finally, it can be shown by using the results in [43] that the maximal BFID corresponding to $\zeta = 1$ is $\Omega = \left\{ z : \left| z - \frac{1}{2} \right| < \frac{1}{2} \right\}$.

$$\operatorname{Re} \frac{w\mathcal{J}_r'(w)}{\mathcal{J}_r(w)} = \operatorname{Re} \frac{1}{1 - w\varphi'\left(\mathcal{J}_r\right)} > \frac{1}{2}\,.$$

## 7. Inverse Löwner chains

Theorem 3.11 tells us that $\Omega_r = \mathcal{J}_r(\Delta),\ 0 \leq r < \infty$, is a decreasing family of domains in the unit disk $\Delta$ (in this connection see also [20]).

One can thus introduce some aspects of Inverse Löwner theory which lead to a deeper geometric understandings of the structure of the family of nonlinear resolvents for $f \in \mathcal{N}(\Delta)$.

**Definition 7.1.** A mapping $p : \Delta \times [0, +\infty) \to \mathbb{C}$ is called a *Herglotz function* of divergence type if the following three conditions are satisfied:

(a) $p_t(z) = p(z,t)$ is analytic in $z \in \Delta$ and measurable in $t \geq 0$,
(b) $\operatorname{Re} p(z,t) > 0$ $(z \in \Delta,\ t \geq 0)$,
(c) $p(0,t)$ is locally integrable in $t \geq 0$ and

$$\int_0^\infty \operatorname{Re} p(0,t)dt = +\infty.$$

Note that the term *Herglotz function of order d* is used in [12] to mean the function $p(z,t)$ with the divergence condition being replaced by $L^d([0,\infty))$-convergence in the above definition.

The following result was proved by Becker [8].

**Theorem 7.2.** *Let $p(z,t)$ be a Herglotz function of divergence type. Then there exists a unique solution $f_t(z) = f(z,t)$, (which is analytic and univalent in $|z| < 1$ for each $t \in [0, +\infty)$ and locally absolutely continuous in $0 \leq t < \infty$ for each $z \in \Delta$) to the differential equation*

$$\dot{f}(z,t) = zf'(z,t)p(z,t) \quad (z \in \Delta,\ t \geq 0) \tag{7.1}$$

*with the normalization conditions $f_0(0) = 0$ and $f_0'(0) = 1$. Moreover, the solution satisfies $f_s \prec f_t$ for $0 \leq s \leq t$.*

Here we have written

$$\dot{f}(z,t) = \frac{\partial}{\partial t} f(z,t), \qquad f'(z,t) = \frac{\partial}{\partial z} f(z,t).$$

for the partial derivatives of $f(.,.)$. Observe that the uniqueness assertion is no longer valid if we drop the univalence condition on $f_t$. For instance, one can consider the function $\tilde{f}(z,t) = \Phi(f(z,t))$ which satisfies (7.1) as well as $\tilde{f}(0,0) = 0$ and $\tilde{f}'(0,0) = 1$ when $\Phi$ is an entire function with $\Phi(0) = 0$ and $\Phi'(0) = 1$.

We now give a definition belonging to Betker [10].

**Definition 7.3.** A family of analytic functions $g_t(z) = g(z,t)$ $(0 \le t < \infty)$ on the unit disk $\Delta$ is called an *inverse Löwner chain* if the following conditions are satisfied:

(i) $g_t : \Delta \to \mathbb{C}$ is univalent for each $t \ge 0$,
(ii) $g_t \prec g_s$ whenever $0 \le s \le t$,
(iii) $b(t) = g_t'(0)$ is locally absolutely continuous in $t \ge 0$ and $b(t) \to 0$ as $t \to \infty$.

Note that condition (ii) means that $g_t(\Delta) \subset g_s(\Delta)$ and $g_t(0) = g_s(0)$ for $0 \le s \le t$. The following lemma gives us sufficient conditions for $g(z,t)$ to be an inverse Löwner chain.

**Lemma 7.4.** *Let $g_t(z) = g(z,t)$ be a family of analytic functions on $\Delta$ for $0 \le t < \infty$ with the following properties:*

1) *$g_t$ is univalent on $\Delta$ for each $t \ge 0$,*
2) *$g(0,s) = g(0,t)$ for $0 \le s \le t$,*
3) *$g(z,0) = z$ for $z \in \Delta$,*
4) *$g(z,t)$ is locally absolutely continuous in $t \ge 0$ for each $z \in \Delta$,*
5) *the differential equation*

$$\dot{g}(z,t) = -zg'(z,t)p(z,t) \quad (z \in \Delta,\ t \ge 0) \tag{7.2}$$

*holds for a Herglotz function $p(z,t)$ of divergence type. Then $g_t(\Delta) \subset g_s(\Delta)$ for $0 \le s \le t$.*

**Corollary 7.5.** *Under the assumptions of Lemma 7.4, we suppose, in addition, that the inequality*

$$\left|\arg p(z,t)\right| < \frac{\pi\alpha}{2}, \quad z \in \Delta,\ t \ge 0, \tag{7.3}$$

*holds for a constant $0 < \alpha < 1$. Then the conformal mapping $g_t$ on $\Delta$ extends to a $k$-quasiconformal mapping of $\mathbb{C}$ for each $t \ge 0$, where $k = \sin(\pi\alpha/2)$.*

Here for a constant $0 \le k < 1$, a mapping $f : \mathbb{C} \to \mathbb{C}$ is called $k$-quasiconformal if $f$ is a homeomorphism in the Sobolev class $W^{1,2}_{\mathrm{loc}}(\mathbb{C})$ and if it satisfies $|\partial_{\bar{z}} f| \le k|\partial_z f|$ almost everywhere on $\mathbb{C}$.

**Proposition 7.6.** *The family $\mathcal{J}_r(w) = \mathcal{J}(w,r), r \ge 0$, is an inverse Löwner chain with the Herglotz function $p(w,r)$ of divergence type. In particular, $\mathcal{J}_r(\Delta) \subset \mathcal{J}_s(\Delta)$ for $0 \le s \le r$.*

**Remark 7.7.** The condition (7.3) is known to be equivalent to that the semigroup $\left\{F_t\right\}_{t\ge 0}$ in $\mathrm{Hol}(\Delta)$ generated by $f(z)$ can be analytically extended to the sector $\{t \in \mathbb{C} : |\arg t| < \pi(1-\alpha)/2\}$ in the parameter $t$ (see [41]).

By virtue of Corollary 7.5, it is enough to prove the following assertion.

**Corollary 7.8.** *Suppose that a holomorphic function $f : \Delta \to \Delta$ with $f(0) = 0$, $f'(0) > 0$ is star-like of order $\alpha$ with $\frac{1}{2} < \alpha < 1$. Then its nonlinear resolvent $\mathcal{J}_r : \Delta \to \Delta$ extends to a $k$-quasiconformal mapping of $\mathbb{C}$ for every $r \ge 0$, where $k = \sin(\pi\alpha)$.*

## 8. Rigidity properties of holomorphic generators

Let $\Delta$ be the open unit disk in the complex plane $\mathbb{C}$. By $Hol\,(\Delta, \mathbb{C})$ we denote the family of all holomorphic functions on $\Delta$. For the special case when $F \in Hol\,(\Delta, \mathbb{C})$ is a self-mapping of $\Delta$ we will simply write $F \in Hol\,(\Delta)$.

The famous rigidity theorem of D. M. Burns and S. G. Krantz [23] (which can be considered as a boundary version of the second part of the classical Schwarz Lemma) asserts:

♦ *Let $F \in Hol(\Delta, \Delta)$ be such that*

$$F\,(z) = 1 + (z-1) + O\left((z-1)^4\right),$$

*as $z \to 1$. Then $F\,(z) \equiv z$ on $\Delta$.*

It was also mentioned in [23] that the exponent 4 is sharp, and it follows from the proof of the theorem that $O\left((z-1)^4\right)$ can be replaced by $o\left((z-1)^3\right)$. There are many generalized versions of this result in different settings for one dimensional, finite dimensional and infinite dimensional situations (see, for example,[87] [88], [90], [5],[6],[11], [33],[38] and [32] and references therein). Similar results appeared earlier in the literature of conformal mappings with the additional hypothesis that $F$ is univalent (and often the function $F$ is assumed to be quite smooth - even analytic - in a neighborhood of the point $z = 1$). The theorem presented in [23] has no such hypothesis. The exponent 4 is sharp: simple geometric arguments show that the function

$$F(z) = z + \frac{1}{10}(z-1)^3$$

satisfies the conditions of the theorem with 4 replaced by 3. Note also that it follows from the proof that $O((z-1)^4)$ can be replaced by $o((z-1)^3)$.

The Burns–Krantz Theorem was improved in 1995 by Thomas L. Kriete and Barbara D. MacCluer [67], who replaced $F$ with its real part and considered the radial limit in $o((z-1)^3)$ instead of the unrestricted limit. Here is a more precise statement of their result.

**Theorem 8.1.** *Let $F \in \mathrm{Hol}(\Delta)$ with radial limit $F(1) = 1$ and angular derivative $F'(1) = 1$. If*

$$\liminf_{r\to 1^-} \frac{\mathrm{Re}\,(F(r) - r)}{(1-r)^3} = 0,$$

*then $F(z) \equiv z$ on $\Delta$.*

In [96], Roberto Tauraso and Fabio Vlacci investigated rigidity of holomorphic self-mappings of the unit disk $\Delta$ after imposing some conditions on the boundary Schwarzian derivative of $F$ defined by

$$\mathcal{S}_F(z) := \frac{F'''(z)}{F'(z)} - \frac{3}{2}\left(\frac{F''(z)}{F'(z)}\right)^2, \quad z \in \partial\Delta.$$

It is known that the Schwarzian derivative carries global information about $F$: it vanishes identically if and only if $F$ is a Möbius transformation. Initially, the original rigidity result of Burns and Krantz was extended in [96] from the identity mapping to a parabolic automorphism.

**Theorem 8.2.** *Let* $F \in \operatorname{Hol}(\Delta) \bigcap C_A^3(1)$*. If*

$$F(1) = 1, \quad F'(1) = 1, \quad \operatorname{Re} F''(1) = 0 \quad \textit{and} \quad \operatorname{Re} \mathcal{S}_F(1) = 0,$$

*then* $F$ *is the parabolic automorphism of* $\Delta$ *defined by*

$$\frac{1 + F(z)}{1 - F(z)} = \frac{1 + z}{1 - z} + ib,$$

*where* $b = \operatorname{Im} F''(1)$*.*

In the particular case $F''(1) = F'''(1) = 0$, this reduces to the result of Burns and Krantz, i.e., $F(z) \equiv z$ on $\Delta$.

In [26] (2010), Contreras, Díaz-Madrigal and Pommerenke supplemented Theorem 8.2 as follows.

**Theorem 8.3.** *(1) A non-trivial (i.e.,* $F \neq I$*) holomorphic map* $F \in \operatorname{Hol}(\Delta)$ *is a parabolic automorphism if and only if there exists* $\zeta \in \partial\Delta$ *such that* $F \in C_A^3(\zeta)$ *and*

$$F(\zeta) = \zeta, \quad F'(\zeta) = 1, \quad \operatorname{Re}(\zeta F''(\zeta)) = 0 \quad \textit{and} \quad \mathcal{S}_F(\zeta) = 0.$$

*(2)* $F \in \operatorname{Hol}(\Delta)$ *is a hyperbolic automorphism if and only if there exist* $\zeta \in \partial\Delta$ *and* $\alpha \in (0, 1)$ *such that* $F \in C_A^3(\zeta)$ *and*

$$F(\zeta) = \zeta, \quad F'(\zeta) = \alpha, \quad \operatorname{Re}(\zeta F''(\zeta)) = \alpha(\alpha - 1) \quad \textit{and} \quad \mathcal{S}_F(\zeta) = 0.$$

The following boundary rigidity principles are given in [88]. In particular, some conditions on behavior of a holomorphic self-mapping $F$ of $\Delta$ in a neighborhood of a boundary regular fixed point (not necessarily the Denjoy–Wolff point) under which $F$ is a linear-fractional transformation have established.

It is known that if a mapping $F \in \operatorname{Hol}(\Delta)$ with the boundary regular fixed point $\tau = 1$ and $F'(1) =: \alpha$ is linear fractional, then for all $k > 0$, and for some $\beta \geq 0$ the following equality holds for all $z \in \Delta$,

$$\frac{|1 - F(z)|^2}{1 - |F(z)|^2} = \frac{\alpha\,|1 - z|^2}{(1 - |z|^2) + \alpha\beta\,|1 - z|^2}\,. \tag{8.1}$$

Moreover, $F$ is an automorphism of $\Delta$ (either hyperbolic, $\alpha \neq 1$, or parabolic, $\alpha = 1$) if and only if $\beta = 0$.

It turns out that, that under some smoothness conditions, equality (8.1) (and even some weaker condition) is also sufficient for $F \in \operatorname{Hol}(\Delta)$ to be linear fractional.

**Theorem 8.4.** *Let* $F \in \operatorname{Hol}(\Delta) \cap C_A^3(1)$*,* $F(1) = 1$ *and* $F'(1) = \alpha$*. Then* $F$ *is a linear fractional transformation if and only if the following conditions hold:*

*(i)* $\dfrac{|1 - F(z)|^2}{1 - |F(z)|^2} \leq \dfrac{1}{a}\,, \quad z \in \Delta;$

*(ii) the Schwarzian derivative* $\mathcal{S}_F(1) = 0$*.*

*So, if conditions (i) and (ii) are satisfied, then equality* (8.1) *holds for all* $z \in \Delta$*.*

In this connection we also would like to mention other two directions in generalization of the Burns-Krantz Theorem presented in [88], [89],[90] and [87].

The first one is to establish some rigidity property for those functions the third derivative of which is not necessarily zero. In other words, we assume that

$$\angle \lim_{z\to\tau} \frac{F(z)-z}{(z-1)^2} = 0, \text{ but } \angle \lim_{z\to\tau} \frac{F(z)-z}{(z-1)^3} = k.$$

It turns out that number $k$ is always nonnegative number and the value $F(0)$ lies always in the closed disk of radius $k$ centred in $k$.Moreover, $F(0)$ lies on the circle-the boundary of this disk if and only if $F$ has a special form which immediately becomes the identity mapping whenever $k = 0$.

For he second direction we have interested is to extend the mentioned above results for those functions which are not necessarily self-mappings of $\Delta$, but satisfy the so-called property to be *pseudo-contractive* on the open unit disk. Despite the latter class is much wider that the class of self mappings of $\Delta$, it preserves many properties of its fixed points as well as the rigidity property in the spirit of the Burns-Krantz Theorem.

**Theorem 8.5.** *Let $F \in Hol(\Delta, \mathbb{C})$ be pseudo-contractive on $\Delta$, with*

$$F(1) = F'(1) = 1 \text{ and } \angle \lim_{z\to\tau} \frac{F(z)-z}{(z-1)^2} = 0.$$

*Assume also that there is the angular limit*

$$\angle \lim_{z\to\tau} \frac{F(z)-z}{(z-1)^3} = \mu$$

*Then $\mu$ is a real nonnegative number with*

$$|F(0) - \mu| \leq \mu$$

*Moreover, If $F(0) = 2\mu$, then*

$$F(z) = z - \mu \frac{(z-1)^3}{1+z}.$$

*In particular, $\mu = 0$ if and only if $F(z) = z$.*

Rigidity principles related to interpolation problems can be found in [6], [5] and [11]. Boundary behavior of semigroups and rigidity at the boundary point is considered in [38].

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# The Nehari–Schwarz lemma and infinitesimal boundary rigidity of bounded holomorphic functions

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*Dedicated to the memory of Professor Gabriela Kohr*

**Abstract.** We survey a number of recent generalizations and sharpenings of Nehari's extension of Schwarz' lemma for holomorphic self–maps of the unit disk. In particular, we discuss the case of infinitely many critical points and its relation to the zero sets and invariant subspaces for Bergman spaces, as well as the case of equality at the boundary.

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## 1. Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane $\mathbb{C}$, and let $\mathcal{B}$ denote the set of holomorphic functions from $\mathbb{D}$ into $\overline{\mathbb{D}}$. A *finite Blaschke product of degree $n$* is a rational function $B \in \mathcal{B}$ of the form

$$B(z) = \eta \prod_{j=1}^{n} \frac{z_j - z}{1 - \overline{z_j} z}, \qquad |\eta| = 1, \tag{1.1}$$

with zeros $z_1, \ldots, z_n \in \mathbb{D}$, not necessarily pairwise distinct. Hence the multiplicative building blocks of finite Blaschke products are exactly the elements of the group of conformal automorphisms of $\mathbb{D}$,

$$\mathsf{Aut}(\mathbb{D}) = \left\{ \eta \frac{z_0 - z}{1 - \overline{z_0} z} \, : \, |\eta| = 1, \, z_0 \in \mathbb{D} \right\}.$$

Blaschke products are omnipresent, and occur for instance as fundamental norm-preserving factors in many important classes of holomorphic functions on $\mathbb{D}$. We refer

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to the recent monograph [19] and the references therein for an state–of–the–art account of the properties and abundant applications of FBP, the set of all finite Blaschke products. In this note, we discuss a number of recent generalizations of Nehari's celebrated extension [37] of Schwarz' lemma, a topic which is intrinsically related to FBPs, but which has not been treated in [19].

As point of departure, we note that a geometric–topological way of thinking about (non–constant) FBPs is to view them as proper holomorphic self–maps of $\mathbb{D}$ or – equivalently – as finite branched coverings of $\mathbb{D}$, see [19, Chapter 3]. From this view point, it seems natural to describe a finite Blaschke product $B$ not in terms of its zeros as in (1.1), but in terms of its *critical points*, that is, the zeros of its first derivative $B'$. That this is indeed possible is the content of the following celebrated result of M. Heins [24] (see also [19, Chapter 6], [47] and [48, 8, 53, 45, 30, 31, 41, 49]).

**Theorem 1.1 (Heins 1962).** *Let $c_1, \ldots, c_{n-1}$ be points in $\mathbb{D}$. Then there is a Blaschke product $B$ of degree $n$ with critical points $c_1, \ldots, c_{n-1}$ in $\mathbb{D}$ and no others. The Blaschke product $B$ is unique up to post–composition with an element of* $\mathsf{Aut}(\mathbb{D})$.

The Blaschke product $B$ in Theorem 1.1 can be characterized as the essentially unique extremal function in a sharpened form of the Schwarz–Pick inequality. This fundamental observation [37] is due to Nehari in 1947. In order to state Nehari's result we need to introduce some notation. We denote by $\mathcal{C}_f$ the collection of all critical points of a non–constant function $f \in \mathcal{B}$ counting multiplicities. By slight abuse of language, we call $\mathcal{C}_f$ the *critical set of* $f$ and write $\mathcal{C}_g \subseteq \mathcal{C}_f$ whenever each critical point of a function $g \in \mathcal{B}$ is also a critical point of $f \in \mathcal{B}$ of at least the same multiplicity. This is in accordance with standard practices, see [14, §4.1].

**Theorem 1.2 (The Nehari–Schwarz lemma).** *Let $f \in \mathcal{B}$ and let $B \in$ FBP such that $\mathcal{C}_B \subseteq \mathcal{C}_f$. Then:*

(i) *(Nehari–Schwarz inequality)*

$$\frac{|f'(z)|}{1-|f(z)|^2} \le \frac{|B'(z)|}{1-|B(z)|^2} \quad \text{for all } z \in \mathbb{D}; \tag{1.2}$$

(ii) *(Strong form of the Nehari–Schwarz lemma at an interior point) Equality holds in (1.2) for some point $z \in \mathbb{D} \setminus \mathcal{C}_B$ if and only if $f = T \circ B$ for some $T \in \mathsf{Aut}(\mathbb{D})$.*

**Remark 1.3 (The Schwarz–Pick lemma).** In Theorem 1.2, one can always take $B$ as a finite Blaschke product without any critical points, that is, as a conformal automorphism of $\mathbb{D}$. In this case, Theorem 1.2 reduces to the standard Schwarz–Pick lemma:

(i) (Schwarz–Pick inequality)

$$\frac{|f'(z)|}{1-|f(z)|^2} \le \frac{1}{1-|z|^2} \quad \text{for all } z \in \mathbb{D}; \tag{1.3}$$

(ii) (Strong form of the Schwarz–Pick lemma at an interior point) Equality holds in (1.3) for some point $z \in \mathbb{D}$ if and only if $f \in \mathsf{Aut}(\mathbb{D})$.

The main purpose of this note is to survey some recent sharpenings and extensions of the Nehari–Schwarz lemma. In Section 3 we discuss a generalization of the Nehari–Schwarz lemma which allows for taking into account *infinitely many* critical points instead of only finitey many as in Theorem 1.2. In Section 4 we describe the connections with the specific Bergman space $A^2_1$, in particular its zero sets and invariant subspaces. Our presentation is based on recent work of Kraus [28], Dyakonov [15, 16], and Ivrii [26, 27]. In Section 5 we discuss the so–called strong form of the Nehari–Schwarz lemma, that is, the case of equality in the Nehari–Schwarz inequality *at the boundary* which has recently been obtained in [9, Theorem 2.10] as a special case of a general boundary rigidity theorem for conformal pseudometrics. In order to make this paper self–contained we also provide a fairly concise proof of the Nehari–Schwarz inequality (1.2) in Section 2. The proof we give is slightly different from the standard proofs which can be found in [37, Corollary, p. 1037] and [24, Theorem 24.1]. The Nehari–Schwarz lemma has found many further applications, for which we refer to other works such as [6, 22, 34, 35, 45], for instance.

## 2. Proof of the Nehari–Schwarz inequality

We give a proof of the Nehari–Schwarz inequality (1.2) which is based on the observation that a finite Blaschke product $B$ has the property that

$$\lim_{|z|\to 1} \left(1-|z|^2\right) \frac{|B'(z)|}{1-|B(z)|^2} = 1 \,. \tag{2.1}$$

In fact, condition (2.1) *characterizes* finite Blaschke products (Heins [25], see also [33] and [19, Chapter 6.5]). We point out that a simple and direct proof that (2.1) holds for any finite Blaschke product

$$B(z) = \eta \prod_{j=1}^{n} \frac{z-z_j}{1-\overline{z_j} z}$$

is possible by making appeal to an identity due to Frostman [18], namely

$$\frac{1-|B(z)|^2}{1-|z|^2} = \sum_{k=1}^{n} \left( \prod_{j=1}^{k-1} \left| \frac{z-z_j}{1-\overline{z_j} z} \right|^2 \right) \frac{1-|z_k|^2}{\left|1-\overline{z_k} z\right|^2} \,, \qquad |z| \neq 1 \,, \tag{2.2}$$

and the elementary formula for the logarithmic derivative of $B$ given by

$$\frac{B'(z)}{B(z)} = \sum_{k=1}^{n} \frac{1-|z_k|^2}{(1-\overline{z_k} z)\,(z-z_k)} \,. \tag{2.3}$$

Frostman's identity (2.2) can be easily established by induction, see [19, p. 77]. Clearly, (2.2) and (2.3) immediately imply (2.1).

Using (2.1) we now give a proof of Theorem 1.2 following very closely the standard proof of Ahlfors' lemma [1] with only minor modifications.

*Proof of Theorem* 1.2 *(i).* Let $f \in \mathcal{B}$ be non–constant, so $\mathcal{C}_f$ is a discrete subset of $\mathbb{D}$. We consider the auxiliary function

$$u(z) := \log \left( \frac{|f'(z)|}{1-|f(z)|^2} \frac{1-|B(z)|^2}{|B'(z)|} \right) .$$

Since $\mathcal{C}_B \subseteq \mathcal{C}_f$ ("including multiplicities"), we see that $u$ is well–defined and real analytic on $\mathbb{D} \setminus \mathcal{C}_f$. For each $\xi \in \mathcal{C}_f$ the limit

$$\lim_{z \to \xi} u(z) \in \mathbb{R} \cup \{-\infty\}$$

exists, so $u$ extends to an upper semicontinuous function on $\mathbb{D}$ with values in $\mathbb{R} \cup \{-\infty\}$ which we continue to denote by $u$. Now, a straightforward computation reveals

$$\Delta u = 4 \left( \frac{|B'(z)|}{1-|B(z)|^2} \right)^2 \left(e^{2u}-1\right) , \qquad z \in \mathbb{D} \setminus \mathcal{C}_f .$$

In particular,

$$u^+ := \max\{u, 0\}$$

is subharmonic in $\mathbb{D}$. On the other hand, in view of the Schwarz–Pick inequality,

$$\left(1-|z|^2\right) \frac{|f'(z)|}{1-|f(z)|^2} \le 1 ,$$

we deduce from (2.1) that

$$\limsup_{|z| \to 1} u(z) \le 0 .$$

Hence $u^+ \le 0$ in $\mathbb{D}$ by the maximum principle. This implies $u \le 0$ and completes the proof of (1.2). □

**Remark 2.1 (Strong form of the Nehari–Schwarz lemma at an interior point).** The case of equality for the Nehari–Schwarz inequality (1.2) for some *interior* point $z \in \mathbb{D} \setminus \mathcal{C}_B$ can be handled in a similar way as the case of equality at some interior point for Ahlfors' lemma, which has been treated in [24, 40, 36, 12, 33]. We refer to e.g. [33, Remark 2.2 (d)] for the details.

## 3. Infinitely many critical points

We begin with an extension of the theorems of Heins' (Theorem 1.1) and Nehari–Schwarz (Theorem 1.2) essentially due to Kraus [28].

**Theorem 3.1.** *Let $\mathcal{C}$ be the critical set of a non–constant function in $\mathcal{B}$. Then there is a Blaschke product $B$ with critical set $\mathcal{C}$ such that*

$$\frac{|f'(z)|}{1-|f(z)|^2} \le \frac{|B'(z)|}{1-|B(z)|^2}$$

*for all $z \in \mathbb{D}$ and any $f \in \mathcal{B}$ such that $\mathcal{C}_f \supseteq \mathcal{C}$. If equality holds at a single point $z \notin \mathcal{C}$, then $f = T \circ B$ for some $T \in \mathsf{Aut}(\mathbb{D})$. The Blaschke product $B$ is uniquely determined by $\mathcal{C}$ up to post–composition with an element of $\mathsf{Aut}(\mathbb{D})$.*

See [28], while the case of equality has been settled in [32]. The Blaschke product $B$ in Theorem 3.1 is called *maximal Blaschke product for $\mathcal{C}$*. The set of all maximal Blaschke products will be denoted by MBP.

**Remarks 3.2 (Properties of maximal Blaschke products).**

(a) Maximal Blaschke products are indestructible: $f \in \text{MBP}$, $T \in \mathsf{Aut}(\mathbb{D}) \Longrightarrow T \circ f \in \text{MBP}$.

(b) (FBP $\subseteq$ MBP)
Maximal Blaschke products for *finite* sets $\mathcal{C}$ are *finite* Blaschke products and vice versa, see [32, Remark 1.2 (b)]. In particular, Theorem 3.1 generalizes Theorem 1.1 and Theorem 1.2.

(c) Any maximal Blaschke product is uniquely determined by its critical set up to postcomposition with an element of $\mathsf{Aut}(\mathbb{D})$. This does not hold for general infinite Blaschke products. Neat examples are the nontrivial Frostman shifts

$$\pi_a(z) := \frac{a - \pi_0(z)}{1 - \overline{a}\pi_0(z)}\,, \qquad a \in \mathbb{D} \setminus \{0\}\,,$$

of the standard singular inner function

$$\pi_0(z) := \exp\left(-\frac{1+z}{1-z}\right)$$

which are Blaschke products without critical points.

(d) The accumulation points of the critical set of a maximal Blaschke product $B$ are exactly the accumulation points of its zero set, and $B$ has an analytic continuation across any other point of the unit circle, see [32, Theorem 1.4 and Corollary 1.5].

(e) The set of maximal Blaschke products is closed with respect to composition, see [32, Theorem 1.7].

## 4. Zeros sets and invariant subspaces for Bergman spaces

**Remark 4.1 (**MBP**s and zero sets in Bergman spaces).** Theorem 3.1 shows in particular that a set $\mathcal{C} \subseteq \mathbb{D}$ is the critical set of a function in $\mathcal{B}$ if and only if it is the critical set of some maximal Blaschke product. It has been shown in [28] that this is the case if and only if $\mathcal{C}$ is the zero set of a function in the Bergman space ([14, 23])

$$A^2_1 = \left\{\varphi : \mathbb{D} \to \mathbb{C} \text{ holomorphic } : \iint_{\mathbb{D}} (1-|z|^2)\,|\varphi(z)|^2\,dxdy < \infty\right\}.$$

Hence

$$\text{MBP}/\mathsf{Aut}(\mathbb{D}) = \left\{\text{zero sets of } A^2_1\right\}. \tag{4.1}$$

This can be seen as an analogue of the classical fact that up to a rotation (= multiplication by a number $\eta \in \mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$) the zero sets of functions in the Hardy space $H^2$ are exactly the zero sets of Blaschke products,

$$\text{BP}/\mathbb{S}^1 = \left\{\text{zero sets of } H^2\right\}.$$

**Remark 4.2 (Critical sets in $\mathcal{B}$ and singly generated invariant subspaces in Bergman spaces).** Remark 4.1 has a simple operator theoretic interpretation, cf. [14, 23] for background. A closed subspace of $A^2_1$ is called *zero–based* if it is defined as the set of all $A^2_1$–functions that vanish at a prescribed set of points in $\mathbb{D}$. Each such subspace is *invariant*, that is, invariant w.r.t. to multiplication by $z$. Hence (4.1) can be trivially rewritten as

$$\text{MBP}/\mathsf{Aut}(\mathbb{D}) = \left\{\text{zero–based invariant subspaces of } A^2_1\right\} .$$

In particular, if we denote by $[H]$ the *subspace generated by a function* $H \in A^2_1$, that is, the minimal closed invariant subspace of $A^2_1$ which contains $H$, then each zero–based subspace of $A^2_1$ has the form $[B']$, meaning that it is singly generated by the derivative $B' \in A^2_1$ of some maximal Blaschke product $B$. Combining this observation with the beautiful concept of asymptotic spectral synthesis of Nikol'skii [38] and a deep result of Shimorin [43] about approximation of singly–generated invariant subspaces of Bergman spaces by zero–based subspaces, O. Ivrii [27] has recently been led to the following striking conjecture

$$\text{Inner functions}/\mathsf{Aut}(\mathbb{D}) = \left\{\text{singly generated invariant subspaces of } A^2_1\right\} ,$$

or in more explicit terms:

**Conjecture 4.3 (Ivrii [27]).** Any singly generated subspace of $A^2_1$ can be generated by the derivative of an inner function. This inner function is uniquely determined up to postcomposition with a unit disk automorphism.

This conjecture can be seen as an analogue of the celebrated result of Beurling that the invariant subspaces of $H^2$ are generated by inner functions:

$$\text{Inner functions}/\mathbb{S}^1 = \left\{\text{invariant subspaces of } H^2\right\} .$$

We refer to the original papers [26, 27] for details and a number of substantial results in support of Conjecture 4.3.

## 5. The strong form of the extended Nehari–Schwarz lemma at the boundary

We now return to Theorem 3.1 and discuss the case of equality at the boundary. For this purpose it is convenient to denote by

$$f^h(z) := \left(1-|z|^2\right) \frac{|f'(z)|}{1-|f(z)|^2}$$

the *hyperbolic derivative* of a holomorphic function $f : \mathbb{D} \to \mathbb{D}$, see [4, Definition 5.1]. If $f \in \mathcal{B}$ and $B$ is a maximal Blaschke product with $\mathcal{C}_B \subseteq \mathcal{C}_f$, then

$$\frac{f^h}{B^h} : \mathbb{D} \setminus \mathcal{C}_B \to \mathbb{R}$$

has a continuous extension to $\mathbb{D}$ which will still be denoted by $f^h/B^h$. Theorem 3.1 (see also [32, Theorem 2.2 (b)]) implies:

(i) (Extended Nehari–Schwarz inequality)

$$\frac{f^h(z)}{B^h(z)} \le 1 \qquad \text{for all } z \in \mathbb{D};$$

(ii) (Strong form of the extended Nehari–Schwarz lemma at an interior point)

$$\frac{f^h(z)}{B^h(z)} = 1 \quad \text{for some point } z \in \mathbb{D} \quad \Longleftrightarrow \quad f = T \circ B \quad \text{for some } T \in \mathsf{Aut}(\mathbb{D})\,.$$

Recently, a boundary version of this interior rigidity result for functions in $\mathcal{B}$ has been obtained in [9]:

**Theorem 5.1 (The strong form of the generalized Nehari–Schwarz lemma at the boundary).** *Let $\mathcal{C}$ be the critical set of a non–constant function in $\mathcal{B}$, $B$ a maximal Blaschke product with critical set $\mathcal{C}_B = \mathcal{C}$, and $f \in \mathcal{B}$ such that $\mathcal{C}_f \supseteq \mathcal{C}$. If*

$$\frac{f^h(z_n)}{B^h(z_n)} = 1 + o\left((1-|z_n|)^2\right)$$

*for some sequence $(z_n)$ in $\mathbb{D}$ such that $|z_n| \to 1$, then $f = T \circ B$ for some $T \in \mathsf{Aut}(\mathbb{D})$ and $f$ is a maximal Blaschke product.*

The proof of Theorem 5.1 in [9] is based on PDE methods, in particular a Harnack–type inequality for solutions of the Gauss curvature equation, see [9] for details. This approach also yields a version of the strong form of the Ahlfors–Schwarz lemma [1, 12, 24, 36, 40, 50] *at the boundary*, see [9, Theorem 2.6]. The special case $\mathcal{C} = \emptyset$ of Theorem 5.1 is the following boundary version of the strong form of the classical Schwarz–Pick lemma:

**Theorem 5.2 (The strong form of the Schwarz–Pick lemma at the boundary).** *Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic. If*

$$f^h(z_n) = 1 + o((1-|z_n|)^2)$$

*for some sequence $(z_n)$ in $\mathbb{D}$ such that $|z_n| \to 1$, then $f \in \mathsf{Aut}(\mathbb{D})$.*

The error term is sharp. For $f(z) = z^2$ we have

$$f^h(z) = \frac{2|z|}{1+|z|^2} = 1 - \frac{(1-|z|)^2}{1+|z|^2} = 1 - \frac{1}{2}\,(1-|z|)^2 + o\left((1-|z|)^2\right) \qquad (|z| \to 1)\,.$$

Hence one cannot replace "little $o$" by "big $O$" in Theorem 5.2. Theorem 5.2 can also be deduced from the inequality

$$f^h(z) \le \frac{f^h(0) + \dfrac{2|z|}{1+|z|^2}}{1 + f^h(0)\dfrac{2|z|}{1+|z|^2}} \qquad \text{for all } |z| < 1\,, \tag{5.1}$$

which has been proved by Golusin (see [20, Theorem 3] or [21, p. 335], and independently by Yamashita [51, 52], Beardon [3], and by Beardon & Minda [4, 5] as part of their elegant work on multi–point Schwarz–Pick lemmas. With hindsight, inequality (5.1) is exactly the case $w = 0$ in Corollary 3.7 of [4].

**Remark 5.3 (The boundary Schwarz–Pick lemma and the boundary Schwarz lemma of Burns and Krantz).** From Theorem 5.2 one can easily deduce the well–known boundary Schwarz lemma of Burns and Krantz [10], which asserts that if $f$ is a holomorphic selfmap of $\mathbb{D}$ such that

$$f(z) = z + o\left(|1-z|^3\right) \qquad \text{as } z \to 1\,, \tag{5.2}$$

then $f(z) \equiv z$. We refer to [9, Remark 2.2] for details.

**Remark 5.4.** Baracco, Zaitsev and Zampieri [2] have improved the boundary Schwarz lemma of Burns and Krantz by proving that if $f : \mathbb{D} \to \mathbb{D}$ is a holomorphic map such that

$$f(z_n) = z_n + o\left(|1-z_n|^3\right)$$

for some sequence $(z_n)$ in $\mathbb{D}$ converging nontangentially to 1, then $f(z) \equiv z$. Does the result of Baracco, Zaitsev and Zampieri follow from Theorem 5.2?

We refer to [7, 11, 13, 39, 44, 46] and in particular to the survey [17] by Elin et al. for more on boundary Schwarz–type lemmas in the one variable setting.

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# Linear invariance and extension operators of Pfaltzgraff–Suffridge type

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*Dedicated to the memory of Professor Gabriela Kohr*

**Abstract.** We consider the image of a linear-invariant family $\mathcal{F}$ of normalized locally biholomorphic mappings defined in the Euclidean unit ball $\mathbb{B}_n$ of $\mathbb{C}^n$ under the extension operator

$$\Phi_{n,m,\beta}[f](z,w) = \Big(f(z), [Jf(z)]^\beta w\Big), \quad (z,w) \in \mathbb{B}_{n+m} \subseteq \mathbb{C}^n \times \mathbb{C}^m,$$

where $\beta \in \mathbb{C}$, $Jf$ denotes the Jacobian determinant of $f$, and the branch of the power function taking 0 to 1 is used. When $\beta = 1/(n+1)$ and $m = 1$, this is the Pfaltzgraff–Suffridge extension operator. In particular, we determine the order of the linear-invariant family on $\mathbb{B}_{n+m}$ generated by the image in terms of the order of $\mathcal{F}$, taking note that the resulting family has minimum order if and only if either $\beta \in (-1/m, 1/(n+1)]$ and the family $\mathcal{F}$ has minimum order or $\beta = -1/m$. We will also see that order is preserved when generating a linear-invariant family from the family obtained by composing $\mathcal{F}$ with a certain type of automorphism of $\mathbb{C}^n$, leading to consequences for various extension operators including the modified Roper–Suffridge extension operator introduced by the author.

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## 1. Introduction

In this note, we generate linear-invariant families on the Euclidean unit ball $\mathbb{B}_n$ of $\mathbb{C}^n$ from other linear-invariant families defined on $\mathbb{B}_n$ or on a lower-dimensional ball (or disk) in a manner that allows us to determine the order of a new family from the order of the family from which it is generated. In many cases, the new linear-invariant families will have minimum order if the families that generate them do.

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The primary mechanism used will involve a perturbation of an extension operator originally presented by Pfaltzgraff and Suffridge [17].

In order to more thoroughly preview our work, we present some basic notation. We reserve $n, m \in \mathbb{N}$ for the dimensions of complex Euclidean spaces. If $a \in \mathbb{C}^n$ and $r > 0$, then $B_n(a; r)$ denotes the ball centered at $a$ of radius $r$. Thus $\mathbb{B}_n = B_n(0; 1)$, and we write $\mathbb{S}_n = \partial\mathbb{B}_n$ for the unit sphere. When $n = 1$, $\mathbb{D} = \mathbb{B}_1$ is the unit disk in $\mathbb{C}$. For $z \in \mathbb{C}^n$ with $n \geq 2$, we write $z = (z_1, \hat{z})$, where $z_1 \in \mathbb{C}$ and $\hat{z} = (z_2, \ldots, z_n) \in \mathbb{C}^{n-1}$. When used in matrix-algebra calculations, we treat elements of $\mathbb{C}^n$ as column vectors, although we express them as $n$-tuples. To avoid confusion, $n$-tuples are always written within parentheses $(\cdot)$ and matrices are always written within brackets $[\cdot]$. The canonical basis vectors in $\mathbb{C}^n$ are $e_1, \ldots, e_n$.

Let $L(\mathbb{C}^n, \mathbb{C}^m)$ denote the space of linear operators from $\mathbb{C}^n$ into $\mathbb{C}^m$. We write $L(\mathbb{C}^n)$ for the algebra $L(\mathbb{C}^n, \mathbb{C}^n)$ and $I_n$ for its identity. The adjoint (conjugate-transpose) of $A \in L(\mathbb{C}^n, \mathbb{C}^m)$ is $A^* \in L(\mathbb{C}^m, \mathbb{C}^n)$. If $\Omega \subseteq \mathbb{C}^n$ is open, then $H(\Omega, \mathbb{C}^m)$ is the space of all holomorphic mappings from $\Omega$ into $\mathbb{C}^m$. If $m = 1$, we use the shorthand $H(\Omega)$. For $f \in H(\Omega, \mathbb{C}^m)$, the Fréchet derivative of $f$ is $Df\colon \Omega \to L(\mathbb{C}^n, \mathbb{C}^m)$. When $m = n$, the Jacobian of $f$ is denoted $Jf = \det Df$. The second Fréchet derivative of $f$ at $z \in \Omega$ is the symmetric bilinear operator $D^2 f(z)\colon \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^m$. It is useful to note that if $a \in \mathbb{C}^n$ is fixed and $g \in H(\Omega, \mathbb{C}^m)$ is given by $g(z) = Df(z)a$ for $z \in \Omega$, then

$$Dg(z) = D^2 f(z)(a, \cdot), \qquad z \in \Omega.$$

In particular, $D^2 f(z)(a, \cdot)$ is an element of $L(\mathbb{C}^n, \mathbb{C}^m)$, and when $m = n$ we may consider its trace. We are broadly interested in the family of normalized locally biholomorphic mappings

$$\mathcal{LS}(\mathbb{B}_n) = \{f \in H(\mathbb{B}_n, \mathbb{C}^n) : f(0) = 0,\ Df(0) = I_n, \text{ and } Jf(z) \neq 0 \text{ for } z \in \mathbb{B}_n\}.$$

The subfamily of biholomorphic mappings is

$$\mathcal{S}(\mathbb{B}_n) = \{f \in \mathcal{LS}(\mathbb{B}_n) : f \text{ is biholomorphic}\}.$$

The group of unitary operators on $\mathbb{C}^n$ is $\mathcal{U}(n) \subseteq L(\mathbb{C}^n)$. The group of biholomorphic automorphisms of a domain $\Omega \subseteq \mathbb{C}^n$ (i.e., biholomorphic mappings of $\Omega$ onto $\Omega$) is $\operatorname{Aut}\Omega$. Any $\varphi \in \operatorname{Aut}\mathbb{B}_n$ has the unique decomposition $\varphi = U \circ \varphi_a$ for $U \in \mathcal{U}(n)$ and $a \in \mathbb{B}_n$, where $\varphi_a \in \operatorname{Aut}\mathbb{B}_n$ is given by

$$\varphi_a(z) = T_a\left(\frac{a - z}{1 - \langle z, a\rangle}\right) = \frac{a - P_a z - s_a Q_a z}{1 - \langle z, a\rangle}, \qquad z \in \mathbb{B}_n. \tag{1.1}$$

Here, $P_a$ is the orthogonal projection of $\mathbb{C}^n$ onto the subspace spanned by $a$, $Q_a = I_n - P_a$ is the orthogonal projection of $\mathbb{C}^n$ onto the orthogonal complement of $a$, $s_a = \sqrt{1 - \|a\|^2}$, and $T_a = P_a + s_a Q_a$. (See [8, 23].) Observe that $\varphi_a$ is an involution exchanging 0 and $a$. We also note that any $\varphi \in \operatorname{Aut}\mathbb{B}_n$ can likewise be written as $\varphi = \varphi_b \circ V|_{\mathbb{B}_n}$ for $b \in \mathbb{B}_n$ and $V \in \mathcal{U}(n)$. (See [19].)

To prove the classical Koebe distortion theorem for $\mathcal{S}(\mathbb{D})$, a standard technique is to apply the bound on the second coefficient of the Taylor series expansion of a function in $\mathcal{S}(\mathbb{D})$ to the function formed by composing an element of $\mathcal{S}(\mathbb{D})$ with a member of $\operatorname{Aut}\mathbb{D}$ and renormalizing, an operation now known as a Koebe transform. Pommerenke [20] coined the term "linear-invariant" for families in $\mathcal{LS}(\mathbb{D})$ that are invariant under all Koebe transforms and defined the order of a linear-invariant family to be the supremum of the moduli of the second coefficients in the Taylor series expansions of its elements. Pfaltzgraff [16] generalized this notion to several complex variables as follows. For $\varphi \in \operatorname{Aut} \mathbb{B}_n$, the Koebe transform with respect to $\varphi$ is $\Lambda_\varphi \colon \mathcal{LS}(\mathbb{B}_n) \to \mathcal{LS}(\mathbb{B}_n)$ given by

$$\Lambda_\varphi[f](z) = D\varphi(0)^{-1} Df(\varphi(0))^{-1}[f(\varphi(z)) - f(\varphi(0))], \qquad f \in \mathcal{LS}(\mathbb{B}_n),\ z \in \mathbb{B}_n.$$

For $\varphi, \psi \in \operatorname{Aut} \mathbb{B}_n$, we have $\Lambda_{\varphi\circ\psi} = \Lambda_\psi \circ \Lambda_\varphi$. It follows that $\Lambda_{\varphi^{-1}} = \Lambda_\varphi^{-1}$. A linear-invariant family is a set $\mathcal{F} \subseteq \mathcal{LS}(\mathbb{B}_n)$ such that $\Lambda_\varphi[f] \in \mathcal{F}$ for all $f \in \mathcal{F}$ and $\varphi \in \operatorname{Aut} \mathbb{B}_n$. If $\mathcal{G} \subseteq \mathcal{LS}(\mathbb{B}_n)$, then the linear-invariant family generated by $\mathcal{G}$ is

$$\Lambda[\mathcal{G}] = \{\Lambda_\varphi[g] : \varphi \in \operatorname{Aut} \mathbb{B}_n,\ g \in \mathcal{G}\}.$$

The complexity inherent in the generalization to higher dimensions manifests when defining the order of a linear-invariant family $\mathcal{F} \subseteq \mathcal{LS}(\mathbb{B}_n)$. In [16], Pfaltzgraff defined the order of $\mathcal{F}$ to be

$$\operatorname{ord} \mathcal{F} = \frac{1}{2} \sup_{u\in\mathbb{S}_n} \sup_{f\in\mathcal{F}} |\operatorname{tr} D^2 f(0)(u,\cdot)| \in \left[\frac{n+1}{2}, \infty\right], \tag{1.2}$$

and proved the sharp lower bound is as given. (Any $f \in \mathcal{LS}(\mathbb{B}_n)$ has a Taylor series expansion of the form

$$f(z) = z + \frac{1}{2} D^2 f(0)(z,z) + o(\|z\|^2), \qquad z \in \mathbb{B}_n,$$

and hence the expression (1.2) reduces to the definition of order given by Pommerenke when $n = 1$.) He then proved the following volume-distortion theorem.

**Theorem 1.1.** *Let $\mathcal{F} \subseteq \mathcal{LS}(\mathbb{B}_n)$ be a linear-invariant family such that $\alpha = \operatorname{ord} \mathcal{F} < \infty$. For all $f \in \mathcal{F}$,*

$$\frac{(1-\|z\|)^{\alpha-(n+1)/2}}{(1+\|z\|)^{\alpha+(n+1)/2}} \le |Jf(z)| \le \frac{(1+\|z\|)^{\alpha-(n+1)/2}}{(1-\|z\|)^{\alpha+(n+1)/2}}, \qquad z \in \mathbb{B}_n.$$

When $n = 1$, this becomes Pommerenke's distortion theorem for linear-invariant families on $\mathbb{D}$. In particular, $\mathcal{S}(\mathbb{D})$ is a linear-invariant family such that $\operatorname{ord} \mathcal{S}(\mathbb{D}) = 2$ (due to the classical Bieberbach estimate [1]), and Theorem 1.1 reduces to the aforementioned Koebe distortion theorem. When $n \ge 2$, the linear-invariant family $\mathcal{S}(\mathbb{B}_n)$ has infinite order, but other families of interest have finite order. One such linear-invariant family is the family of convex mappings

$$\mathcal{K}(\mathbb{B}_n) = \{f \in \mathcal{S}(\mathbb{B}_n) : f(\mathbb{B}_n) \text{ is convex}\}$$

(it is compact; see [8]). When $n = 1$, $\operatorname{ord} \mathcal{K}(\mathbb{D}) = 1$ (the minimum possible order) and any linear-invariant family of order 1 on $\mathbb{D}$ must be a subset of $\mathcal{K}(\mathbb{D})$. For $n \ge 2$, things are not so nice. Pfaltzgraff and Suffridge [18] showed that $\operatorname{ord} \mathcal{K}(\mathbb{B}_n) > (n+1)/2$ (that is, $\mathcal{K}(\mathbb{B}_n)$ does not have minimum order), and the problem of determining the exact value of $\operatorname{ord} \mathcal{K}(\mathbb{B}_n)$ remains open. Partly motivated by this, Pfaltzgraff and Suffridge introduced [19] a second notion of order based on the operator norm of $D^2 f(0)/2$ for $f$ in a linear-invariant family. This notion of order has some advantages. For instance, the minimum norm order of a linear-invariant family on $\mathbb{B}_n$ is 1 regardless of $n$ and $\mathcal{K}(\mathbb{B}_n)$ has this norm order. Furthermore, having finite norm order implies a linear-invariant family is a normal family and growth estimates can be obtained. Nevertheless, the norm order is, in general, much more difficult to use and does not seem to be as compatible with the extension operators we will study.

The following two identities from [16] used to prove Theorem 1.1 will be useful in our work.

**Lemma 1.2.** *Let $f \in \mathcal{LS}(\mathbb{B}_n)$ and $\varphi \in \operatorname{Aut}\mathbb{B}_n$. If $g = \Lambda_\varphi[f]$, then*

$$\operatorname{tr} D^2 g(0)(w,\cdot) = \operatorname{tr} Df(\varphi(0))^{-1} D^2 f(\varphi(0))(D\varphi(0)w,\cdot) + \operatorname{tr} D\varphi(0)^{-1} D^2\varphi(0)(w,\cdot)$$

*for all $w \in \mathbb{C}^n$.*

**Lemma 1.3.** *For $a \in \mathbb{B}_n$, we have*

$$D\varphi_a(0)^{-1} D^2\varphi_a(0)(w,\cdot) = \langle w, a\rangle I_n + wa^*, \qquad w \in \mathbb{C}^n.$$

In addition, Godula, Liczberski, and Starkov [5] obtained the following converse to Theorem 1.1.

**Theorem 1.4.** *If $\mathcal{F} \subseteq \mathcal{LS}(\mathbb{B}_n)$ is a linear-invariant family with* $\operatorname{ord}\mathcal{F} < \infty$, *then*

$$\operatorname{ord}\mathcal{F} = \inf\left\{\alpha \geq \frac{n+1}{2} : |Jf(z)| \leq \frac{(1+\|z\|)^{\alpha-(n+1)/2}}{(1-\|z\|)^{\alpha+(n+1)/2}} \text{ for } z \in \mathbb{B}_n \text{ and } f \in \mathcal{F}\right\}.$$

Generally speaking, an extension operator is a function $\Phi\colon \mathcal{F} \to \mathcal{LS}(\mathbb{B}_{n+m})$, where $\mathcal{F} \subseteq \mathcal{LS}(\mathbb{B}_n)$, such that $\Phi[f](z,0) = (f(z),0)$ for all $f \in \mathcal{F}$ and $z \in \mathbb{B}_n$. (We will consistently write points in $\mathbb{C}^{n+m}$ as ordered pairs in $\mathbb{C}^n \times \mathbb{C}^m$.) The study of extension operators focuses on those for which $\Phi[f]$ inherits a useful characteristic of $f$ such as a geometric property of the mapping's range or the ability to embed the mapping in a Loewner chain. In this work, we will focus primarily on the operator

$$\Phi_{n,m,\beta}[f](z,w) = \big(f(z), [Jf(z)]^\beta w\big), \qquad f \in \mathcal{LS}(\mathbb{B}_n),\ (z,w) \in \mathbb{B}_{n+m},$$

for $\beta \in \mathbb{C}$. The branch of the power function taking 0 to 1 is used. The operator $\Phi_{1,n-1,1/2}$, $n \geq 2$, is the Roper–Suffridge extension operator, the first such operator studied. It was introduced in [21] where the authors showed $\Phi_{1,n-1,1/2}[\mathcal{K}(\mathbb{D})] \subseteq \mathcal{K}(\mathbb{B}_n)$. Graham and Kohr [9] showed $\Phi_{1,n-1,1/2}[\mathcal{S}^*(\mathbb{D})] \subseteq \mathcal{S}^*(\mathbb{B}_n)$, where

$$\mathcal{S}^*(\mathbb{B}_n) = \{f \in \mathcal{S}(\mathbb{B}_n) : f(\mathbb{B}_n) \text{ is starlike with respect to } 0\}.$$

Pfaltzgraff and Suffridge introduced the operator $\Phi_{n,1,1/(n+1)}$ in [17] in their study of linear-invariant families. Of note, Chirilă [3] first considered the perturbation $\Phi_{n,1,\beta}$ for $\beta \in [0, 1/(n+1)]$ in connection with Loewner theory and showed $\Phi_{n,1,\beta}[\mathcal{S}^*(\mathbb{B}_n)] \subseteq \mathcal{S}^*(\mathbb{B}_{n+1})$ for such $\beta$, generalizing the same inclusion for the original Pfaltzgraff–Suffridge extension operator given in [11] by Graham, Kohr, and Pfaltzgraff.

The image of a linear-invariant family under an extension operator will generally not be linear-invariant, but we may consider the linear-invariant family generated by the image. That approach was taken by Pfaltzgraff and Suffridge in [17] where they showed that if $\mathcal{F} \subseteq \mathcal{LS}(\mathbb{B}_n)$ is a linear-invariant family of finite order, then

$$\operatorname{ord}\Lambda[\Phi_{n,1,1/(n+1)}[\mathcal{F}]] = \frac{n+2}{n+1}\operatorname{ord}\mathcal{F}.$$

In particular, a family of minimum order on $\mathbb{B}_n$ is extended in this manner to a family of minimum order on $\mathbb{B}_{n+1}$. Graham, Hamada, Kohr, and Suffridge [7] used a different approach for a perturbation of the Roper–Suffridge extension operator, showing

$$\operatorname{ord}\Lambda[\Phi_{1,n-1,\beta}[\mathcal{F}]] = (1+(n-1)\beta)\operatorname{ord}\mathcal{F} + \frac{(n-1)(1-2\beta)}{2}$$

for a linear-invariant family $\mathcal{F} \subseteq \mathcal{LS}(\mathbb{D})$ of finite order, $\beta \in [0,1/2]$, and $n \geq 2$. Again, a family of minimum order is extended to a family of minimum order for such $\beta$.

In what follows, we generalize the work of Graham, Hamada, Kohr, and Suffridge to produce an order result of this type for the general operator $\Phi_{n,m,\beta}$, $\beta \in \mathbb{C}$, that will imply both of the above results and maintain the characteristic that families of minimum order on $\mathbb{B}_n$ extend to families of minimum order on $\mathbb{B}_{n+m}$ for $\beta \in [-1/m, 1/(n+1)]$. We will also observe that any linear-invariant family on $\mathbb{B}_n$ is extended to a family of minimum order on $\mathbb{B}_{n+m}$ when $\beta = -1/m$. In no other cases will a family of minimum order be produced. We will follow with some results showing how the generation of a linear-invariant family from the composition of a linear-invariant family on $\mathbb{B}_n$ with a member of a particular subgroup of $\operatorname{Aut}\mathbb{C}^n$ preserves order, allowing us to generate large linear-invariant families of minimum order and to observe results as above for other commonly studied extension operators.

## 2. The order of $\Lambda[\Phi_{n,m,\beta}[\mathcal{F}]]$

The following lemma is a generalization of [7, Lemma 4.1], and its proof uses a technique from [19].

**Lemma 2.1.** *Let $\mathcal{A} \subseteq \operatorname{Aut}\mathbb{B}_n$ be such that $\{\varphi(0) : \varphi \in \mathcal{A}\} = \mathbb{B}_n$. If $\mathcal{G} \subseteq \mathcal{LS}(\mathbb{B}_n)$, then*

$$\operatorname{ord}\Lambda[\mathcal{G}] = \frac{1}{2}\sup_{f\in\mathcal{G}}\sup_{\varphi\in\mathcal{A}}\sup_{u\in\mathbb{S}_n}|\operatorname{tr} D^2\Lambda_\varphi[f](0)(u,\cdot)|.$$

*Proof.* Let $g \in \Lambda[\mathcal{G}]$. There are $\psi \in \operatorname{Aut}\mathbb{B}_n$ and $f \in \mathcal{G}$ such that $g = \Lambda_\psi[f]$. Choose $\varphi \in \mathcal{A}$ such that $\varphi(0) = \psi(0)$. Then there is a $U \in \mathcal{U}(n)$ such that $\psi = \varphi \circ U|_{\mathbb{B}_n}$. Let $h = \Lambda_\varphi[f] \in \Lambda[\mathcal{G}]$. It follows that $g = \Lambda_U[h]$. That is, $g(z) = U^*h(Uz)$ for all $z \in \mathbb{B}_n$. Then $Dg(z)u = U^*Dh(Uz)Uu$ for $z \in \mathbb{B}_n$ and $u \in \mathbb{S}_n$. Differentiation of both sides with respect to $z$ yields

$$D^2g(z)(u,\cdot) = U^*D^2h(Uz)(Uu,\cdot)U, \qquad z \in \mathbb{B}_n,\ u \in \mathbb{S}_n.$$

It follows that

$$\sup_{u\in\mathbb{S}_n}|\operatorname{tr} D^2g(0)(u,\cdot)| = \sup_{u\in\mathbb{S}_n}|\operatorname{tr} D^2h(0)(u,\cdot)|$$

because the trace is a similarity invariant. This implies the result. □

The following lemma is a generalization of [7, Lemma 4.2], although our proof uses a somewhat different technique (see also [8]).

**Lemma 2.2.** *Let $f \in \mathcal{LS}(\mathbb{B}_n)$ and $g \in H(\mathbb{B}_n)$ such that $g(0) = 1$ and $g(z) \neq 0$ for all $z \in \mathbb{B}_n$. If $F \in H(\mathbb{B}_{n+m}, \mathbb{C}^{n+m})$ is defined by $F(z,w) = (f(z), g(z)w)$, then $F \in \mathcal{LS}(\mathbb{B}_{n+m})$ and*

$$\sup_{b\in\mathbb{B}_m} \sup_{(u,v)\in\mathbb{S}_{n+m}} |\operatorname{tr} D^2\Lambda_{\varphi_{(0,b)}}[F](0,0)((u,v),\cdot)| = \max\left\{ n+m+1, \sup_{(u,v)\in\mathbb{S}_{n+m}} |\operatorname{tr} D^2F(0,0)((u,v),\cdot)| \right\}.$$

*Proof.* Let $b \in \mathbb{B}_m$, and set $G = \Lambda_{\varphi_{(0,b)}}[F]$. In block matrix form, we have

$$DF(z,w) = \begin{bmatrix} Df(z) & 0 \\ wDg(z) & g(z)I_m \end{bmatrix}, \qquad (z,w) \in \mathbb{B}_{n+m}. \tag{2.1}$$

Clearly, $F \in \mathcal{LS}(\mathbb{B}_{n+m})$. For all $(z,w) \in \mathbb{B}_{n+m}$ and $(u,v) \in \mathbb{C}^{n+m}$, differentiation of $DF(z,w)(u,v) = (Df(z)u, wDg(z)u + g(z)v)$ with respect to $(z,w)$ gives

$$D^2F(z,w)((u,v),\cdot) = \begin{bmatrix} D^2f(z)(u,\cdot) & 0 \\ wD^2g(z)(u,\cdot) + vDg(z) & (Dg(z)u)I_m \end{bmatrix}. \tag{2.2}$$

Taking advantage of the fact that the operators in (2.1) and (2.2) are both lower block-triangular, we obtain

$$\operatorname{tr} DF(0,b)^{-1}D^2F(0,b)((u,v),\cdot) = \operatorname{tr} D^2f(0)(u,\cdot) + mDg(0)u, \qquad (u,v) \in \mathbb{C}^{n+m}.$$

A direct calculation using (1.1) reveals $D\varphi_{(0,b)}(0,0) = (0,b)(0,b)^* - T_{(0,b)}$.

If $P \in L(\mathbb{C}^{n+m}, \mathbb{C}^n)$ is given by $P(z,w) = z$ for $(z,w) \in \mathbb{C}^{n+m}$, then $PD\varphi_{(0,b)}(0,0)(u,v) = -s_b u$ for $(u,v) \in \mathbb{C}^{n+m}$. For such $(u,v)$, we obtain

$$\operatorname{tr} DF(0,b)^{-1}D^2F(0,b)(D\varphi_{(0,b)}(0,0)(u,v),\cdot) = -s_b(\operatorname{tr} D^2f(0)(u,\cdot) + mDg(0)u).$$

We note that $\operatorname{tr}(u,v)(0,b)^* = \langle v,b\rangle$ for $(u,v) \in \mathbb{C}^{n+m}$. For such $(u,v)$, Lemma 1.3 gives

$$\operatorname{tr} D\varphi_{(0,b)}(0,0)^{-1}D^2\varphi_{(0,b)}(0,0)((u,v),\cdot) = (n+m+1)\langle v,b\rangle,$$

and by Lemma 1.2, we find

$$\operatorname{tr} D^2G(0,0)((u,v),\cdot) = -s_b(\operatorname{tr} D^2f(0)(u,\cdot) + mDg(0)u) + (n+m+1)\langle v,b\rangle$$

for the given $b$.

If we write $s_x = \sqrt{1-x^2}$ for $x \in [0,1]$, we now have

$$\begin{aligned}
&\sup_{b\in\mathbb{B}_m} \sup_{(u,v)\in\mathbb{S}_{n+m}} |\operatorname{tr} D^2\Lambda_{\varphi_{(0,b)}}[F](0,0)((u,v),\cdot)| \\
&\quad = \sup_{b\in\mathbb{B}_m} \sup_{(u,v)\in\mathbb{S}_{n+m}} |-s_b(\operatorname{tr} D^2f(0)(u,\cdot) + mDg(0)u) + (n+m+1)\langle v,b\rangle| \\
&\quad = \sup_{b\in\mathbb{B}_m} \sup_{x\in[0,1]} \sup_{u\in\mathbb{S}_n} \sup_{v\in\mathbb{S}_m} |-s_b x(\operatorname{tr} D^2f(0)(u,\cdot) + mDg(0)u) \\
&\qquad\qquad + (n+m+1)s_x\langle v,b\rangle| \\
&\quad = \sup_{b\in\mathbb{B}_m} \sup_{x\in[0,1]} \left( xs_b \sup_{u\in\mathbb{S}_n} |\operatorname{tr} D^2f(0)(u,\cdot) + mDg(0)u| + (n+m+1)\|b\|s_x \right).
\end{aligned}$$

It is a simple exercise in calculus to see that the function $h\colon [0,1]\times[0,1]\to\mathbb{R}$ given by $h(x,y)=\alpha x\sqrt{1-y^2}+\beta y\sqrt{1-x^2}$, where $\alpha,\beta\geq 0$, attains its maximum value at (at least) one of the points $(1,0)$ or $(0,1)$. From (2.2), we see that

$$\sup_{(u,v)\in\mathbb{S}_{n+m}} |\operatorname{tr} D^2F(0,0)((u,v),\cdot)| = \sup_{u\in\mathbb{S}_n} |\operatorname{tr} D^2 f(0)(u,\cdot)+mDg(0)u|,$$

giving the result. □

We now present our main result.

**Theorem 2.3.** *Let* $\mathcal{F}\subseteq\mathcal{LS}(\mathbb{B}_n)$ *be a linear-invariant family and* $\beta\in\mathbb{C}$*. If* $\operatorname{ord}\mathcal{F}<\infty$*, then*

$$\operatorname{ord}\Lambda[\Phi_{n,m,\beta}[\mathcal{F}]] = |1+m\beta|\operatorname{ord}\mathcal{F}+\frac{m|1-\beta(n+1)|}{2}.$$

*If* $\operatorname{ord}\mathcal{F}=\infty$*, then*

$$\operatorname{ord}\Lambda[\Phi_{n,m,\beta}[\mathcal{F}]] = \begin{cases}\infty & \text{if } \beta\neq -1/m,\\ (n+m+1)/2 & \text{if } \beta=-1/m.\end{cases}$$

*Proof.* Let $f\in\mathcal{F}$ and $F=\Phi_{n,m,\beta}[f]$. Now choose $a\in\mathbb{B}_n$ and put $G=\Lambda_{\varphi_{(a,0)}}[F]$. For clarity, we write $\psi_a\in\operatorname{Aut}\mathbb{B}_n$ for the involution described in (1.1) to distinguish it from members of $\operatorname{Aut}\mathbb{B}_{n+m}$. Using (1.1), we obtain

$$\varphi_{(a,0)}(z,w)=\left(\psi_a(z),\frac{-s_a w}{1-\langle z,a\rangle}\right),\qquad (z,w)\in\mathbb{B}_{n+m}.$$

We calculate

$$D\varphi_{(a,0)}(0,0)=\begin{bmatrix} D\psi_a(0) & 0\\ 0 & -s_aI_m\end{bmatrix},\qquad DF(a,0)=\begin{bmatrix} Df(a) & 0\\ 0 & [Jf(a)]^\beta I_m\end{bmatrix}.$$

It follows that $G$ can be written in terms of $\Lambda_{\psi_a}[f]$ and a function $g\in H(\mathbb{B}_n)$ as follows:

$$G(z,w)=(\Lambda_{\psi_a}[f](z),g(z)w),\qquad g(z)=\frac{[Jf(\psi_a(z))]^\beta}{[Jf(a)]^\beta(1-\langle z,a\rangle)},\qquad (z,w)\in\mathbb{B}_{n+m}.$$

Now $G$ has the general form considered in Lemma 2.2, allowing us to take advantage of the second-derivative expression (2.2) to write

$$\operatorname{tr} D^2G(0,0)((u,v),\cdot)=\operatorname{tr} D^2\Lambda_{\psi_a}[f](0)(u,\cdot)+mDg(0)u,\qquad (u,v)\in\mathbb{S}_{n+m}.$$

If $\Omega\subseteq\mathbb{C}$ is open and $A\colon\Omega\to L(\mathbb{C}^n)$ is analytic and such that $A(\zeta)$ is invertible for all $\zeta\in\Omega$, then Jacobi's formula for differentiation of the determinant of $A$ (see [6]) is

$$\frac{d}{d\zeta}\det A(\zeta)=\det A(\zeta)\operatorname{tr}[A(\zeta)^{-1}A'(\zeta)],\qquad \zeta\in\Omega.$$

We apply this for $z \in \mathbb{B}_n$ and $u \in \mathbb{C}^n$ to obtain

$$\begin{aligned} D[Jf](z)u &= \sum_{k=1}^{n} u_k \frac{\partial}{\partial z_k} Jf(z) \\ &= \sum_{k=1}^{n} u_k Jf(z) \operatorname{tr}\left( Df(z)^{-1} \frac{\partial}{\partial z_k} Df(z) \right) \\ &= \sum_{k=1}^{n} u_k Jf(z) \operatorname{tr} Df(z)^{-1} D^2 f(z)(e_k, \cdot) \\ &= Jf(z) \operatorname{tr} Df(z)^{-1} D^2 f(z)(u, \cdot). \end{aligned}$$

Observe that Lemmas 1.2 and 1.3 show

$$\begin{aligned} \operatorname{tr} Df(a)^{-1} D^2 f(a)(D\psi_a(0)u, \cdot) &= \operatorname{tr} D^2 \Lambda_{\psi_a}[f](0)(u, \cdot) - \operatorname{tr}(\langle u, a\rangle I_n + ua^*) \\ &= \operatorname{tr} D^2 \Lambda_{\psi_a}[f](0)(u, \cdot) - (n+1)\langle u, a\rangle \end{aligned}$$

for $u \in \mathbb{C}^n$. For such $u$, we now can compute

$$\begin{aligned} Dg(0)u &= \frac{\beta D[Jf](a) D\psi_a(0)u}{Jf(a)} + \langle u, a\rangle \\ &= \beta \operatorname{tr} D^2 \Lambda_{\psi_a}[f](0)(u, \cdot) + (1 - \beta(n+1))\langle u, a\rangle. \end{aligned}$$

It follows that

$$\operatorname{tr} D^2 G(0,0)((u,v), \cdot) = (1 + m\beta) \operatorname{tr} D^2 \Lambda_{\psi_a}[f](0)(u, \cdot) + m(1 - \beta(n+1))\langle u, a\rangle$$

for all $(u, v) \in \mathbb{S}_{n+m}$.

From the above, we now have

$$\begin{aligned} &\sup_{f \in \mathcal{F}} \sup_{a \in \mathbb{B}_n} \sup_{(u,v) \in \mathbb{S}_{n+m}} |\operatorname{tr} D^2 \Lambda_{\varphi_{(a,0)}}[\Phi_{n,m,\beta}[f]](0,0)((u,v), \cdot)| \\ &\quad = \sup_{f \in \mathcal{F}} \sup_{a \in \mathbb{B}_n} \sup_{u \in \mathbb{S}_n} |(1 + m\beta) \operatorname{tr} D^2 \Lambda_{\psi_a}[f](0)(u, \cdot) + m(1 - \beta(n+1))\langle u, a\rangle|. \end{aligned} \tag{2.3}$$

In the case that $\alpha = \operatorname{ord} \mathcal{F} < \infty$, we use $\Lambda_{\psi_a}[f] \in \mathcal{F}$ for all $f \in \mathcal{F}$, (1.2), and (2.3) to see that

$$\begin{aligned} &\sup_{f \in \mathcal{F}} \sup_{a \in \mathbb{B}_n} \sup_{(u,v) \in \mathbb{S}_{n+m}} |\operatorname{tr} D^2 \Lambda_{\varphi_{(a,0)}}[\Phi_{n,m,\beta}[f]](0,0)((u,v), \cdot)| \\ &\quad \le 2\alpha|1 + m\beta| + m|1 - \beta(n+1)|. \end{aligned} \tag{2.4}$$

Let $\varepsilon \in (0, 2\alpha)$, and set $\theta = \arg(1 + m\beta)$ and $\eta = \arg(1 - \beta(n+1))$. There exist $g_0 \in \mathcal{F}$ and $u_0 \in \mathbb{S}_n$ such that

$$e^{i\theta} \operatorname{tr} D^2 g_0(0)(u_0, \cdot) \ge 2\alpha - \varepsilon.$$

Let $a_0 = te^{i\eta} u_0 \in \mathbb{B}_n$ for $t \in (0,1)$ and $f_0 = \Lambda_{\psi_{a_0}}[g_0]$. Since $\psi_{a_0}$ is an involution, we have $g_0 = \Lambda_{\psi_{a_0}}[f_0]$, and hence

$$\begin{aligned} &|(1 + m\beta) \operatorname{tr} D^2 \Lambda_{\psi_{a_0}}[f_0](0)(u_0, \cdot) + m(1 - \beta(n+1))\langle u_0, a_0\rangle| \\ &\quad \ge (2\alpha - \varepsilon)|1 + m\beta| + mt|1 - \beta(n+1)|. \end{aligned}$$

The arbitrary choices of $\varepsilon$ and $t$ and (2.3) show that equality is attained in (2.4).

In the case that $\operatorname{ord}\mathcal{F} = \infty$ and $\beta = -1/m$, we use (2.3) to see that

$$\begin{aligned} &\sup_{f\in\mathcal{F}} \sup_{a\in\mathbb{B}_n} \sup_{(u,v)\in\mathbb{S}_{n+m}} |\operatorname{tr} D^2\Lambda_{\varphi_{(a,0)}}[\Phi_{n,m,\beta}[f]](0,0)((u,v),\cdot)| \\ &\quad = m|1-\beta(n+1)| \\ &\quad = n+m+1. \end{aligned} \tag{2.5}$$

If $\operatorname{ord}\mathcal{F} = \infty$ and $\beta \neq -1/m$, letting $a = 0$ in the supremum in (2.3) shows

$$\begin{aligned} &\sup_{f\in\mathcal{F}} \sup_{a\in\mathbb{B}_n} \sup_{(u,v)\in\mathbb{S}_{n+m}} |\operatorname{tr} D^2\Lambda_{\varphi_{(a,0)}}[\Phi_{n,m,\beta}[f]](0,0)((u,v),\cdot)| \\ &\quad \geq \sup_{f\in\mathcal{F}} \sup_{u\in\mathbb{S}_n} |1+m\beta||\operatorname{tr} D^2 f(0)(u,\cdot)| \\ &\quad = \infty \end{aligned} \tag{2.6}$$

Now let $\mathcal{A} = \{\varphi_{(a,0)} \circ \varphi_{(0,b)} : a \in \mathbb{B}_n,\ b \in \mathbb{B}_m\} \subseteq \operatorname{Aut}\mathbb{B}_{n+m}$. For any $a \in \mathbb{B}_n$ and $b \in \mathbb{B}_m$, we have

$$(\varphi_{(a,0)} \circ \varphi_{(0,b)})(0,0) = \varphi_{(a,0)}(0,b) = (a, -s_a b).$$

As noted above, $\Lambda_{\varphi_{(a,0)}}[\Phi_{n,m,\beta}[f]]$ has the form in Lemma 2.2 for all $a \in \mathbb{B}_n$. Since $\{(a,-s_a b) : a \in \mathbb{B}_n,\ b \in \mathbb{B}_m\} = \mathbb{B}_{n+m}$, we use Lemma 2.1 along with Lemma 2.2 to find that

$$\begin{aligned} &\operatorname{ord}\Lambda[\Phi_{n,m,\beta}[\mathcal{F}]] \\ &\quad = \frac{1}{2} \sup_{f\in\mathcal{F}} \sup_{a\in\mathbb{B}_n} \sup_{b\in\mathbb{B}_m} \sup_{(u,v)\in\mathbb{S}_{n+m}} |\operatorname{tr} D^2(\Lambda_{\varphi_{(0,b)}} \circ \Lambda_{\varphi_{(a,0)}})[\Phi_{n,m,\beta}[f]](0,0)((u,v),\cdot)| \\ &\quad = \frac{1}{2} \sup_{f\in\mathcal{F}} \sup_{a\in\mathbb{B}_n} \max\Bigg\{ n+m+1, \\ &\qquad\qquad \sup_{(u,v)\in\mathbb{S}_{n+m}} |\operatorname{tr} D^2\Lambda_{\varphi_{(0,a)}}[\Phi_{n,m,\beta}[f]](0,0)((u,v),\cdot)| \Bigg\}. \end{aligned}$$

The result for the case $\operatorname{ord}\mathcal{F} = \infty$ immediately follows from (2.5) and (2.6).

If $\alpha = \operatorname{ord}\mathcal{F} < \infty$, the equality in (2.4) observed earlier implies

$$\operatorname{ord}\Lambda[\Phi_{n,m,\beta}[\mathcal{F}]] = \frac{1}{2}\max\{n+m+1, 2\alpha|1+m\beta| + m|1-\beta(n+1)|\}.$$

Using $\alpha \geq (n+1)/2$, we have

$$\begin{aligned} 2\alpha|1+m\beta| + m|1-\beta(n+1)| &\geq |n+1+m\beta(n+1)| + |m - m\beta(n+1)| \\ &\geq |n+1+m|. \end{aligned} \tag{2.7}$$

This gives the result in this case. □

**Example 2.4.** To utilize Theorem 2.3 in a concrete manner, one must possess knowledge of the order of a linear-invariant family $\mathcal{F} \subseteq \mathcal{LS}(\mathbb{B}_n)$. One may consult [5, 8, 17] among other references for examples of linear-invariant families of various orders. We will simply take note of the well-known results that when $n = 1$ and $\alpha \geq 1$, we have $\operatorname{ord} \Lambda[\{k_\alpha\}] = \alpha$, where $k_\alpha \in \mathcal{LS}(\mathbb{D})$ is the generalized Koebe function

$$k_\alpha(z) = \frac{1}{2\alpha}\left[\left(\frac{1+z}{1-z}\right)^\alpha - 1\right], \qquad z \in \mathbb{D},$$

where the principal branch of the power is used. (See [20].) In addition (see [2], for instance), if $g \in \mathcal{LS}(\mathbb{D})$ is given by

$$g(z) = \frac{1}{2}\left(1 - \exp\frac{-2z}{1-z}\right), \qquad z \in \mathbb{D},$$

then $\operatorname{ord} \Lambda[\{g\}] = \infty$.

We now observe an immediate consequence.

**Corollary 2.5.** *Let $\mathcal{F}$ be a linear-invariant family of minimum order on $\mathbb{B}_n$ and $\beta \in \mathbb{C}$. Then $\Lambda[\Phi_{n,m,\beta}[\mathcal{F}]]$ is a linear-invariant family of minimum order on $\mathbb{B}_{n+m}$ if and only if $\beta \in [-1/m, 1/(n+1)]$.*

*Proof.* Clearly, $\Lambda[\Phi_{n,m,\beta}[\mathcal{F}]]$ has minimum order if and only if both inequalities in (2.7) are equalities. This holds for the first inequality because $\alpha = (n+1)/2$.

Equality in the second inequality occurs if and only if either the complex numbers $n+1+m\beta(n+1)$ and $m - m\beta(n+1)$ have the same argument or one of them is equal to 0. This clearly occurs if and only if both numbers are nonnegative real numbers, which coincides with $\beta \in [-1/m, 1/(n+1)]$. □

**Remark 2.6.** It is worth considering the special case where $\beta = -1/m$ in Theorem 2.3. Indeed, for any linear-invariant family $\mathcal{F} \subseteq \mathcal{LS}(\mathbb{B}_n)$, the family $\Lambda[\Phi_{n,m,-1/m}[\mathcal{F}]]$ has minimum order on $\mathbb{B}_{n+m}$. This includes the family $\Lambda[\Phi_{n,m,-1/m}[\mathcal{LS}(\mathbb{B}_n)]]$. This is not as surprising as it may seem, for if $F \in \Phi_{n,m,-1/m}[\mathcal{LS}(\mathbb{B}_n)]$, we can calculate $JF(z,w) = 1$ for all $(z,w) \in \mathbb{B}_{n+m}$. It is known [5, 17] that the linear-invariant family generated by

$$\mathcal{G} = \{f \in \mathcal{LS}(\mathbb{B}_n) : Jf(z) = 1 \text{ for all } z \in \mathbb{B}_n\}$$

has minimum order.

## 3. Compositions with automorphisms of $\mathbb{C}^n$

Consider the following subgroup of $\operatorname{Aut} \mathbb{C}^n$ and subspace of $H(\mathbb{C}^n)$:

$$\operatorname{Aut}_1 \mathbb{C}^n = \{\Psi \in \operatorname{Aut} \mathbb{C}^n : \Psi(0) = 0,\ D\Psi(0) = I_n, \text{ and } J\Psi(z) = 1 \text{ for } z \in \mathbb{C}^n\},$$
$$H_0(\mathbb{C}^n) = \{G \in H(\mathbb{C}^n) : G(0) = 0 \text{ and } DG(0) = 0\}.$$

Notable members of $\operatorname{Aut}_1 \mathbb{C}^n$ for $n \geq 2$ are the normalized shears (see [22]) given by

$$\Psi_G(z) = (z_1 + G(\hat{z}), \hat{z}), \qquad z \in \mathbb{C}^n, \tag{3.1}$$

where $G \in H_0(\mathbb{C}^{n-1})$.

As a consequence of the following simple result, we see that the linear-invariant family generated by the composition a linear-invariant family with a member of

$\operatorname{Aut}_1 \mathbb{C}^n$ has the same order as the original family. For notational convenience, if $\mathcal{F} \subseteq H(\mathbb{B}_n, \mathbb{C}^n)$ and $\Psi \in H(\mathbb{C}^n, \mathbb{C}^n)$, then we write

$$\Psi \circ \mathcal{F} = \{\Psi \circ f : f \in \mathcal{F}\}.$$

The proof relies on the following observation.

**Remark 3.1.** From the work done in [5] leading to the conclusion of Theorem 1.4 above, the following deduction can be drawn: *If $\mathcal{F} \subseteq \mathcal{LS}(\mathbb{B}_n)$ is a linear-invariant family and there exist $\alpha \geq (n+1)/2$ and $r \in (0,1)$ such that*

$$|Jf(z)| \leq \frac{(1+\|z\|)^{\alpha-(n+1)/2}}{(1-\|z\|)^{\alpha+(n+1)/2}}, \qquad f \in \mathcal{F},\ z \in B_n(0;r),$$

*then* $\operatorname{ord} \mathcal{F} \leq \alpha$*.* Clearly, the analogous result also holds using the lower estimate of the Jacobian from Theorem 1.1.

**Proposition 3.2.** *If $\mathcal{F} \subseteq \mathcal{LS}(\mathbb{B}_n)$ and $\Psi \in \operatorname{Aut}_1 \mathbb{C}^n$, then* $\operatorname{ord} \Lambda[\Psi \circ \mathcal{F}] = \operatorname{ord} \Lambda[\mathcal{F}]$.

*Proof.* Let $g = \Lambda_\varphi[\Psi \circ f]$ for some $\varphi \in \operatorname{Aut} \mathbb{B}_n$ and $f \in \mathcal{F}$. For all $z \in \mathbb{B}_n$,

$$\begin{aligned} Jg(z) &= \frac{J\Psi(f(\varphi(z)))Jf(\varphi(z))J\varphi(z)}{J\Psi(f(\varphi(0)))Jf(\varphi(0))J\varphi(0)} \\ &= \frac{Jf(\varphi(z))J\varphi(z)}{Jf(\varphi(0))J\varphi(0)} \\ &= J\Lambda_\varphi[f](z). \end{aligned} \tag{3.2}$$

Suppose $\alpha = \operatorname{ord} \Lambda[\mathcal{F}] < \infty$. Any $g \in \Lambda[\Psi \circ \mathcal{F}]$ has the form above, and, by Theorem 1.1, we have

$$|Jg(z)| = |J\Lambda_\varphi[f](z)| \leq \frac{(1+\|z\|)^{\alpha-(n+1)/2}}{(1-\|z\|)^{\alpha+(n+1)/2}}, \qquad z \in \mathbb{B}_n,$$

using (3.2). By Remark 3.1, we conclude that $\operatorname{ord} \Lambda[\Psi \circ \mathcal{F}] \leq \alpha$. Since $\Psi^{-1} \in \operatorname{Aut}_1 \mathbb{C}^n$, we apply the same argument as above to obtain

$$\alpha = \operatorname{ord} \Lambda[\mathcal{F}] = \operatorname{ord} \Lambda[\Psi^{-1} \circ (\Psi \circ \mathcal{F})] \leq \operatorname{ord} \Lambda[\Psi \circ \mathcal{F}],$$

as needed in this case.

If $\operatorname{ord} \Lambda[\mathcal{F}] = \infty$, then let $\alpha \geq (n+1)/2$. By Remark 3.1, there exist $f \in \mathcal{F}$, $\varphi \in \operatorname{Aut} \mathbb{B}_n$, and $z \in \mathbb{B}_n$ such that

$$|Jg(z)| = |J\Lambda_\varphi[f](z)| > \frac{(1+\|z\|)^{\alpha-(n+1)/2}}{(1-\|z\|)^{\alpha+(n+1)/2}}$$

for $g = \Lambda_\varphi[\Psi \circ f] \in \Lambda[\Psi \circ \mathcal{F}]$ by (3.2). It follows from Theorem 1.1 that $\operatorname{ord} \Lambda[\Psi \circ \mathcal{F}] > \alpha$. The arbitrary choice of $\alpha$ implies the result in this case. □

**Remark 3.3.** Godula, Liczberski, and Starkov [5] used Theorem 1.4 in a similar manner to argue that if $f, g \in \mathcal{LS}(\mathbb{B}_n)$ are such that $Jf(z) = Jg(z)$ for all $z \in \mathbb{B}_n$, then $\operatorname{ord} \Lambda[\{f\}] = \operatorname{ord} \Lambda[\{g\}]$. While their proof depends on the families being assumed to have finite order, the case of infinite order can be addressed as we have in the proof of Proposition 3.2. Were this done, one could also prove Proposition 3.2 using their result.

A notable point of interest in the following corollary is that the family $\mathcal{G}$ is linear-invariant without needing to be generated by a smaller family.

**Corollary 3.4.** *Let $\mathcal{F} \subseteq \mathcal{LS}(\mathbb{B}_n)$ be a linear-invariant family. Then*

$$\mathcal{G} = \bigcup_{\Psi \in \operatorname{Aut}_1 \mathbb{C}^n} \Psi \circ \mathcal{F}$$

*is a linear-invariant family and* $\operatorname{ord} \mathcal{G} = \operatorname{ord} \mathcal{F}$.

*Proof.* Once we verify that $\mathcal{G}$ is a linear-invariant family, the order claim follows from Proposition 3.2. Let $f \in \mathcal{F}$, $\Psi \in \operatorname{Aut}_1 \mathbb{C}^n$, and $\varphi \in \operatorname{Aut} \mathbb{B}_n$. For any $a \in \mathbb{C}^n$, let $S_a \in \operatorname{Aut} \mathbb{C}^n$ be the translation given by $S_a(z) = z + a$. Let

$$\begin{aligned}\Psi_0 = D\varphi(0)^{-1} Df(\varphi(0))^{-1} D\Psi(f(\varphi(0)))^{-1} \circ S_{-\Psi(f(\varphi(0)))} \\ \circ \Psi \circ S_{f(\varphi(0))} \circ Df(\varphi(0)) D\varphi(0).\end{aligned}$$

Then it is elementary to see that $\Psi_0 \in \operatorname{Aut}_1 \mathbb{C}^n$. A direct calculation reveals $\Lambda_\varphi[\Psi \circ f] = \Psi_0 \circ \Lambda_\varphi[f] \in \mathcal{G}$, as needed. □

**Remark 3.5.** Proposition 3.2 and Corollary 3.4 are only interesting if $n \geq 2$. Indeed, if $n = 1$, then $\operatorname{Aut}_1 \mathbb{C} = \{I_1\}$ and thus $\mathcal{G} = \mathcal{F}$ in Corollary 3.4.

When $n \geq 2$, Corollary 3.4 makes clear that linear-invariant families of finite order on $\mathbb{B}_n$ are not, in general, normal families no matter the order. (This has previously been established; see [18], for instance.) Indeed, for any function $G \in H_0(\mathbb{C}^{n-1})$, consider the shear $\Psi_G$ as in (3.1). Let $\hat{P} \in L(\mathbb{C}^n, \mathbb{C}^{n-1})$ be given by $\hat{P}z = \hat{z}$ for $z \in \mathbb{C}^n$. For any element $f$ of the linear-invariant family $\mathcal{G}$ in Corollary 3.4 and $G$ as described here such that $G(\hat{P}f(z)) \neq 0$ for some $z \in \mathbb{B}_n$, we have that $\Psi_{tG} \circ f \in \mathcal{G}$ for all $t > 0$. Clearly, $\{\Psi_{tG} \circ f : t > 0\}$ is not locally uniformly bounded.

Note that this is in contrast to linear-invariant families with finite norm order, which must be normal families as noted in Section 1. Furthermore, since the norm order agrees with the order when $n = 1$, the above observation only holds for $n \geq 2$.

Many extension operators studied in conjunction with the theories of geometric mappings and Loewner chains are of the form $\Phi[f] = \Psi_G \circ \Phi_{1,n-1,\beta}[f]$ for $f \in \mathcal{LS}(\mathbb{D})$ where $G \in H_0(\mathbb{C}^{n-1})$ and $\beta \in [0, 1/2]$. (See [13] for instance.) We now see from Theorem 2.3 and Proposition 3.2 that the extension operator $\Phi_{n,m,\beta,G} \colon \mathcal{LS}(\mathbb{B}_n) \to \mathcal{LS}(\mathbb{B}_{n+m})$ given by

$$\Phi_{n,m,\beta,G}[f](z,w) = (f(z) + G([Jf(z)]^\beta w), [Jf(z)]^\beta w), \qquad (z,w) \in \mathbb{B}_{n+m},$$

where $\beta \in \mathbb{C}$ and $G \in H_0(\mathbb{C}^m)$, is such that for any linear-invariant family $\mathcal{F}$ on $\mathbb{B}_n$, we have

$$\operatorname{ord} \Lambda[\Phi_{n,m,\beta,G}[\mathcal{F}]] = \operatorname{ord} \Lambda[\Psi_G \circ \Phi_{n,m,\beta}[\mathcal{F}]] = \alpha|1 + m\beta| + \frac{m|1 - \beta(n+1)|}{2}$$

if $\alpha = \operatorname{ord} \mathcal{F} < \infty$ and, likewise,

$$\operatorname{ord} \Lambda[\Phi_{n,m,\beta,G}[\mathcal{F}]] = \begin{cases} \infty & \text{if } \beta \neq -1/m, \\ (n+m+1)/2 & \text{if } \beta = -1/m, \end{cases}$$

if $\operatorname{ord} \mathcal{F} = \infty$. In particular, this process produces families of minimum order on $\mathbb{B}_{n+m}$ when beginning with a linear-invariant family of minimum order on $\mathbb{B}_n$ for all

$G \in H_0(\mathbb{C}^m)$ and $\beta \in [-1/m, 1/(n+1)]$ (see Corollary 2.5) or when beginning with any linear-invariant family for all $G \in H_0(\mathbb{C}^m)$ and $\beta = -1/m$.

This is of note even when extending linear-invariant families on $\mathbb{D}$ (in which case Theorem 2.3 reduces to a result in [7] if $\beta \in [0, 1/2]$, as noted previously). Recall from Section 1 that $\mathcal{K}(\mathbb{B}_n)$ is a linear-invariant family that has minimum order if and only if $n = 1$. We see that $\Lambda[\Phi_{1,n-1,\beta,G}[\mathcal{K}(\mathbb{D})]]$ is a linear-invariant family of minimum order on $\mathbb{B}_n$ for any $\beta \in [-1/(n-1), 1/2]$ and $G \in H_0(\mathbb{C}^{n-1})$ if $n \geq 2$. For $G = 0$ and $\beta \geq 0$, this family is a subset of $\mathcal{K}(\mathbb{B}_n)$ if and only if $\beta = 1/2$, as shown in [7] (see also [10]). For $\beta = 1/2$, it was shown in [15] (see also [14]) that this family is a subset of $\mathcal{K}(\mathbb{B}_n)$ if and only if $G = Q$, where $Q\colon \mathbb{C}^{n-1} \to \mathbb{C}$ is a homogeneous polynomial of degree 2 such that

$$\|Q\| = \sup_{u \in \mathbb{S}_{n-1}} |Q(u)| \leq \frac{1}{2}.$$

Thus, for many choices of $\beta$ and $G$, $\Lambda[\Phi_{1,n-1,\beta,G}[\mathcal{K}(\mathbb{D})]]$ is a linear-invariant family of minimum order not lying in $\mathcal{K}(\mathbb{B}_n)$, while $\Lambda[\Phi_{1,n-1,1/2,Q}[\mathcal{K}(\mathbb{D})]]$ is a linear-invariant family of minimum order lying within $\mathcal{K}(\mathbb{B}_n)$ when $\|Q\| \leq 1/2$. We note that $\Lambda[\Phi_{1,n-1,1/2,Q}[\mathcal{K}(\mathbb{D})]]$ was noted by Kohr to have minimum order (without proof) for $\|Q\|$ of any size in [12].

In view of Corollary 3.4, we also see that

$$\bigcup_{\Psi \in \mathrm{Aut}_1 \mathbb{C}^n} \bigcup_{\beta \in [-1/(n-1), 1/2]} \Psi \circ \Lambda[\Phi_{1,n-1,\beta}[\mathcal{K}(\mathbb{D})]]$$

is a linear-invariant family of rather substantial size of minimum order on $\mathbb{B}_n$.

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# Polynomial convexity properties of closure of domains biholomorphic to balls

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*Dedicated to the memory of Professor Gabriela Kohr*

**Abstract.** We discuss the connections between the polynomial convexity properties of a domain biholomorphic to ball and its closure.

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**Keywords:** Biholomorphic image of ball, Runge pair.

## 1. Introduction

A classical theorem of Runge states that for every simply connected open subset $U$ of $\mathbb{C}$, the restriction morphism $\mathcal{O}(\mathbb{C}) \to \mathcal{O}(U)$ has dense image. As usual, the topology on the space of holomorphic functions is the topology of uniform convergence on compacts. We say then that $U$ is Runge in $\mathbb{C}$. This is not longer true in $\mathbb{C}^n$ for $n \geq 2$. It was shown in [13], [14], [15] that there are open subsets of $\mathbb{C}^n$ that are biholomorphic to a polydisc and are not Runge in $\mathbb{C}^n$. E. F. Wold proved in [16] that there are Fatou-Bieberbach domains that are not Runge and hence any open subset of $\mathbb{C}^n$, $n \geq 2$, is biholomorphic to a non-Runge open subset of $\mathbb{C}^n$. In [5] it was given an example of a bounded open subset of $\mathbb{C}^3$ which is biholomorphic to a ball and it is not Runge in any strictly larger open subset of $\mathbb{C}^3$.

In this short paper, motivated by [9], which in turn is based on [7], we want to discuss the possible connections between the polynomial convexity properties of $f(B^n)$ and $\overline{f(B^n)}$ where $f : B^n \to \mathbb{C}^n$ is biholomorphic map onto its image. More precisely we will show that, in general, there is no such connection.

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## 2. Results

We start be recalling a few basic notions.

**Definition 2.1.** Let $M$ be a complex manifold. By $\mathcal{O}(M)$ we will denote the set of holomorphic functions defined on $M$. If $K \subset M$ is a compact subset we denote by $\widehat{K}^M$ the holomorphically convex hull of $K$,

$$\widehat{K}^M = \{z \in M : |f(z)| \leq \sup_{x \in K} |f(x)|,\ \forall f \in \mathcal{O}(M)\}.$$

$K$ is called holomorphically convex in $M$ if $\widehat{K}^M = K$.

If $M = \mathbb{C}^n$, then $\widehat{K}^{\mathbb{C}^n}$ is the same as the polynomially covex hull of $K$,

$$\{z \in M : |f(z)| \leq \sup_{x \in K} |f(x)|,\ \forall \text{ polynomial function } f\}.$$

**Definition 2.2.** If $M$ is a Stein manifold and $U$ is a Stein open subset then $U$ is called Runge in $M$ if the restriction morphism $\mathcal{O}(M) \to \mathcal{O}(U)$ has dense image

It is well-known, see e.g. [8], that, in the above setting, the following statements are equivalent:

1. $U$ is Runge in $M$.
2. For every compact set $K \subset U$ we have $\widehat{K}^U = \widehat{K}^M$.
3. For every compact set $K \subset U$ we have $\widehat{K}^M \subset U$.

We recall that a Fatou-Bieberbach domain is a proper open subset of $\mathbb{C}^n$ which is biholomorphic to $\mathbb{C}^n$. We will need the precise statement of the main theorem of [16] mentioned in the introduction. This is the following.

**Theorem 2.3.** *There exits a Fatou-Bieberbach domain $\Omega \subset \mathbb{C} \times \mathbb{C}^*$ which is Runge in $\mathbb{C} \times \mathbb{C}^*$ but not in $\mathbb{C}^2$.*

We will move now to our discussion of the closure of domains in $\mathbb{C}^n$ that are biholomorphic to a ball. We denote by $B^n$ the unit ball in $\mathbb{C}^n$ centered at the origin. We will begin with some remarks.

**Remark 2.4.**

- If $U$ is a bounded Runge open subset of $\mathbb{C}$ then it is simply connected and hence biholomorphic to a disc. In general $\overline{U}$ might not be holomorphically convex. It is easy to give such an example. However, if $U$ has smooth boundary, then $\overline{U}$ is holomorphically convex.
- If $n \geq 2$ on can construct a bounded Runge open subset of $\mathbb{C}^n$ biholomorphic to a ball and with smooth boundary such that $\overline{U}$ is not holomorphically convex. One possible construction is the following: start with $F : B^2 \to \mathbb{C}^2$ biholomorphic onto its image such that $F(B^2)$ is not Runge in $\mathbb{C}^2$. Let $B(0,r) \subset \mathbb{C}^2$ be the ball centered at the origin and of radius $r$. It is easy to see that if $r$ is small enough then $F(B(0,r))$ is Runge. Let $r_0 = \sup\{r : F(B(0,r)) \text{ is Runge}\}$. Because an increasing union of Runge domains is Runge as well we have that $r_0 < 1$ and $F(B(0,r_0))$ is Runge. It was noticed in [10] that $\overline{F(B(0,r_0))}$ is not polynomially convex.

- The interior of a polynomially convex compact set is Runge. Hence if one is trying to find $F : B^2 \to \mathbb{C}^2$ which is a biholomorpism onto its image such that $F(B^2)$ is not Runge and $\overline{F(B^2)}$ is polynomially convex then one must have that the interior of $\overline{F(B^2)}$ is strictly larger then $F(B^2)$.

**Proposition 2.5.** *Suppose that $M$ is a connected complex manifold, $\overline{\Gamma}$ and $\overline{\Delta}$ two closed sets, $U$ and $V$ two open sets such that $\overline{\Gamma} \subset U \subset \overline{\Delta} \subset V$. Moreover, we assume that there exist an open set $\tilde{U} \subset \mathbb{C}^n$ containing a closed ball $\overline{B}$, a biholomorphism $F : \tilde{U} \to U$ such that $F(\overline{B}) = \overline{\Gamma}$, an open set $\tilde{V} \subset \mathbb{C}^n$ containing a closed polydisc $\overline{P}$, and a biholomorphism $G : \tilde{V} \to V$ such that $G(\overline{P}) = \overline{\Gamma}$. Then there exists an open and dense subset of $M$ which is biholomorphic to a ball and contains $\overline{\Gamma}$.*

*Proof.* This proposition is simply a consequence of some of the results and the proofs given in [3], [4] and [2]. For the reader's convenience, we we will recall the main steps needed to prove the proposition. Actually in [3] and [2] the authors prove more than density results: they obtain full-measure embeddings.

We recall that a complex manifold $M$ is called taut if for every complex manifold $N$ (in fact it suffices to work with the unit disc in $\mathbb{C}$, see [1]) the space of holomorphic maps from $N$ to $M$ is a normal family.

• It was noticed in [3] that in any complex manifold $M$ there exists $M_1 \subset M$ a Stein, dense, open subset.

• Another remark from [3] is that for any Stein manifold, $M_1$, there exists $M_2 \subset M_1$ a taut dense open subset.

• It was proved in [3] that in a taut manifold an increasing union of open sets each one biholomorphic to a polydisc is biholomorphic to a polydisc. A similar statement holds for an increasing union of balls instead of polydiscs.

• A consequence of Theorem II.4 in [4] is the following: if $\tilde{U} \subset \mathbb{C}^n$ is an open neighborhood of a closed polydisc $\overline{P}$, $F : \tilde{U} \to U$ is a biholomorphism onto an open subset $U$ of a complex manifold $M$, $\overline{\Delta} = F(\overline{P})$ and $x$ is any point in $M$ then there exists an open subset $\Delta_1$ of $M$, biholomorphic to a polydisc, such that $\overline{\Delta} \cup \{x\} \subset \Delta_1$.

• This last statement implies easily that if $\tilde{U} \subset \mathbb{C}^n$ is an open neighborhood of a closed polydisc $\overline{P}$, $F : \tilde{U} \to U$ is a biholomorphism onto an open subset $U$ of a complex manifold $M$ and $\overline{\Delta} = F(\overline{P})$ then there exists an increasing sequence of open subsets biholomorphic to polydiscs in $M$, $\Delta_1 = \Delta \Subset \Delta_2 \Subset \cdots$ such that $\bigcup \Delta_j$ is dense in $M$. Indeed, it suffices to consider a dense sequence $\{x_k\}_{k \geq 1} \subset M$ and to construct inductively the polydiscs such that $\{x_1, \ldots, x_k\} \subset \overline{\Delta}_k$.

It follows then from the previous statements that:

• If $M$ is any complex manifold, $\tilde{U} \subset \mathbb{C}^n$ is an open neighborhood of a closed polydisc $\overline{P}$, $F : \tilde{U} \to U$ is a biholomorphism onto an open subset $U$ of $M$ and $\overline{\Delta} = F(\overline{P})$ then there exists a dense open subset of $M$ biholomorphic to polydisc that contains $\overline{\Delta}$.

• Lemma 2.1 in [2] implies the following statement: suppose that $P$ is a polydisc in $\mathbb{C}^n$, $U$ is an open subset of $P$ such that there exists $\tilde{U} \subset \mathbb{C}^n$ an open neighborhood of a closed ball $\overline{B}$ and a biholomorphism $F : \tilde{U} \to U$. If $\overline{\Gamma} = F(\overline{B})$ and $x$ is any point in $P$ then there exists an open subset $\Gamma_1$ of $P$, biholomorphic to a ball, such that

$\overline{\Delta} \cup \{x\} \subset \Gamma_1$. As before we deduce that there exists an open and dense subset of $P$ that contains $\overline{\Gamma}$.

The conclusion of the proposition is now straightforward. $\square$

**Corollary 2.6.** *There exists $F : B^2 \to \mathbb{C}^2$ wich is biholomorphic onto its image and such that $F(B^2)$ is not Runge in $\mathbb{C}^2$, and that $\overline{F(B^2)}$ is a holomorphically convex compact subset of $\mathbb{C}^2$.*

*Proof.* Let $\Omega \subset \mathbb{C}^2$ be a Fatou-Bieberbach domain which is not Runge in $\mathbb{C}^2$. Such a domain exists by Theorem 2.3. Let also $F : \mathbb{C}^2 \to \Omega$ be a biholomorphism.

As $\Omega$ is not Runge in $\mathbb{C}^2$, there exists a compact $K \subset \Omega$ such that $\widehat{K}^{\mathbb{C}^2} \not\subset \Omega$. Choose a point $a \in \widehat{K}^{\mathbb{C}^2} \setminus \Omega$. Choose also a ball $B$ and a polydisc $P$ in $\mathbb{C}^2$ such that

$$F^{-1}(K) \subset B \subset \overline{B} \subset P,$$

and an open ball $U \subset \mathbb{C}^2$ such that $\{a\} \cup F(\overline{P}) \subset U$.

We apply now Proposition 2.5 for $M = U \setminus \{a\}$ and we deduce that there exists a dense open subset $\Gamma$ of $U \setminus \{a\}$ which is biholomorphic to a ball and contains $F(\overline{B})$. In particular it contains $K$ while it does not contain $a$. This implies that $\Gamma$ is not Runge in $\mathbb{C}^2$. The closure of $\Gamma$ is, of course, $\overline{U}$ which is polynomially convex. $\square$

Proposition 2.5 and Corollary 2.6 are geometric in nature in the sense that they are not concerned with the behaviour of the map $F : B^2 \to \mathbb{C}^2$ (except that it is biholomorphic onto its image). Our next theorem exhibits a somehow stranger behaviour of the map.

**Theorem 2.7.** *There exists $F : B^2 \to \mathbb{C}^2$ biholomorphic onto its image such that $F(B^2)$ is not Runge in $\mathbb{C}^2$ and for every open set $V \in \mathbb{C}^2$ with $V \cap \partial B^2 \neq \emptyset$ we have $\overline{F(B^2 \cap V)} \supset (\mathbb{C}^2 \setminus F(B))$.*

Before we prove the theorem, we need some preliminaries.

For the following definition, see [11].

**Definition 2.8.** A complex manifold $M$ has the density property if every holomorphic vector field on $M$ can be approximated locally uniformly by Lie combinations of complete vector fields.

Manifolds with the density property have been studied in [11] and [12]. In particular one has:

**Proposition 2.9.** $\mathbb{C} \times \mathbb{C}^*$ *has the density property.*

The following theorem is a particular case of Theorem 0.2 in [12]. If $M = \mathbb{C}^n$, it is Corollary 2.2 in [6].

**Theorem 2.10.** *Suppose that $M$ is a conected Stein manifold that satisfies the density property. Let $K$ be a holomorphically convex compact subset of $M$ and $g$ a metric on $M$. Suppose also given: $\varepsilon$ a positive number, $A$ a finite subset of $K$, and $\{x_1, \dots, x_s\}$, $\{y_1, \dots, y_s\}$ two finite subsets of $M \setminus K$ of same cardinality. Then there exists an automorphism $F : M \to M$ such that:*

1. $\sup_{x\in K} d_g(F(x),x) < \varepsilon$ *where* $d_g$ *is the distance induced by* $g$,
2. $F(a) = a$ *and* $\mathrm{d}F(a) = Id$ *for every* $a \in A$,
3. $F(x_j) = y_j$ *for every* $j = 1,\dots,s$.

We need also the following elementary lemma.

**Lemma 2.11.** *Suppose that* $U, V, \Omega$ *are connected open subsets of* $\mathbb{C}^n$ *with* $V \Subset U \Subset \Omega$. *Let* $r > 0$ *be such that there exists a ball* $B(x_0, r)$ *of radius* $r$ *with* $B(x_0, r) \subset V$ *and let* $\delta$ *be the distance between* $\overline{V}$ *and* $\partial U$. *If* $F : \Omega \to F(\Omega) \subset \mathbb{C}^n$ *is a biholomorphism onto its image and* $\sup_{x\in\overline{U}} \|F(x) - x\| < \min\{\delta, r\}$ *then* $\overline{V} \subset F(U)$.

*Proof.* Because $\sup_{x\in\overline{U}} \|F(x) - x\| < \delta$, we get that $F(\partial U) \cap \overline{V} = \emptyset$. In particular $V \subset F(U) \cup (\mathbb{C}^n \setminus \overline{U})$. At the same time $\sup_{x\in\overline{U}} \|F(x) - x\| < r$ implies that $F(x_0) \in B(x_0, r)$ and hence $F(U) \cap V \neq \emptyset$. As $V$ is connected, we deduce that $V \subset F(U)$. Finally, $F(\partial U) \cap \overline{V} = \emptyset$ implies that $\overline{V} \subset F(U)$. □

*Proof of Theorem* 2.7. We consider the Fatou-Beiberbach domain $\Omega \subset \mathbb{C} \times \mathbb{C}^*$ given by Theorem 2.3 which is Runge in $\mathbb{C} \times \mathbb{C}^*$ but not in $\mathbb{C}^2$. Let $K$ be a compact subset of $\Omega$ such that $\widehat{K}^{\mathbb{C}^2} \not\subset \mathbb{C} \times \mathbb{C}^*$. Let $F_0 : \mathbb{C}^2 \to \Omega$ be a Fatou-Beiberbach map. Of course we may assume that $F_0(B^2) \supset K$. We fix also a point $a \in K$.

We choose a strictly increasing sequence of open balls, $\{B_s\}_{s\geq -1}$, centered at the origin, such that $\bigcup_s B_s = B^2$ and such that $B_{-1} \supset F_0^{-1}(K)$.

We will construct inductively a sequence of automorphisms $\{H_s\}_{s\geq 0}$ of $\mathbb{C} \times \mathbb{C}^*$ such that, if we set $F_s = H_s \circ \cdots \circ H_0 \circ F_0 \in \mathcal{S}(B^2)$, then the map we are looking for will be $F = \lim_s F_s$. Note that $F(B^2)$ will be also a subset of $\mathbb{C} \times \mathbb{C}^*$ because $\mathbb{C} \times \mathbb{C}^*$ is Stein.

We have to make sure that the sequence converges to a nondegenerate map on $B^2$. At the same time we would like to have $F_0(B_{-1}) \subset F(B^2)$. If this is the case, we will have $K \subset F(B^2)$ and this will imply that $F(B^2)$ is not Runge in $\mathbb{C}^2$. In fact we will need more that that, namely we would like to have $F_s(\overline{B}_{s-1}) \subset F(B^2)$ for every $s$. To force this inclusion we will apply Lemma 2.11. Hence we will introduce a sequence of positive real numbers $\{\varepsilon_s\}_{s\geq 0}$ that will act as the bounds needed in that lemma.

For the remaining property, we will need to introduce an increasing sequence of of finite subsets of $B^2$, $\{A_s\}_s \in \mathbb{N}$, $A_s \subset A_{s+1}$ that will help "spreading" the image of $F$.

• We consider $\{x_n\}_{n\geq 1} \subset \partial B^2$ a dense sequence. For each $n \in \mathbb{N}$ we consider $\{x_n^p\}_{p\in\mathbb{N}} \subset B^2$ a sequence that converges to $x_n$. Moreover we assume that $x_n \neq x_m$ for $n \neq m$ and $x_n^p \neq x_m^q$ for $(n,p) \neq (m,q)$.

• We set $H_0$ to be the identity and $A_0 = \{a\}$, $\varepsilon_0 = 1$.

• We assume that we have constructed $H_0,\dots,H_s$, $A_0,\dots,A_s$, $\varepsilon_0,\dots,\varepsilon_s$ and that $H_j(a) = a$ for $j \leq s$ and we will construct $H_{s+1}$, $A_{s+1}$, and $\varepsilon_{s+1}$.

We choose $T_1^{s+1},\dots,T_{s+1}^{s+1}$ pairwise disjoint, finite, subsets of $\mathbb{C} \times \mathbb{C}^*$, , such that for every $j = 1,\dots,s+1$ we have

◇ $T_j^{s+1} \cap (F_s(\overline{B}_s) \cup F_s(A_s)) = \emptyset$ and

$\diamond \bigcup_{z\in T_j^{s+1}} B(z,\frac{1}{s}) \supset \{z\in\mathbb{C}^2\setminus F_s(B_s) : d(z,F_s(\overline{B}_s))\le s\}$.
Here $d(z,F_s(\overline{B}_s))$ stands for the distance between $z$ and the compact set $F_s(\overline{B}_s)$.

After we chose these finite sets $T_j^{s+1}$, we choose, for each $j=1,\dots,s+1$, a finite subset, $A_j^{s+1}$, of $\{x_j^p : p\in\mathbb{N}\}$ such that:

$\diamond$ $\#A_j^{s+1} = \#T_j^{s+1}$,

$\diamond$ $A_j^{s+1}\cap(\overline{B}_s\cup A_s)=\emptyset$,

$\diamond$ $\|x_j - x\| < \frac{1}{s}$ for every $x\in A_j^{s+1}$.

We set

$$A_{s+1} = A_s \cup \left(\bigcup_{j=1}^{s+1} A_j^{s+1}\right).$$

Let $\delta_s$ denote the distance between $F_s(\overline{B}_{s-1})$ and $\partial F_s(\overline{B}_s)$. $F_s(B_{s-1})$ is an open subset of $\mathbb{C}\times\mathbb{C}^*$. Let $r_s>0$ be such that there exists a ball of radius $r_s$ included in $F_s(B_{s-1})$.

We define

$$\varepsilon_{s+1} := \frac{1}{2^{s+1}}\min\{\delta_s, r_s, \varepsilon_0\dots,\varepsilon_s\}.$$

Because $H_j$, $j\le s$, are automorphisms of $\mathbb{C}\times\mathbb{C}^*$ we have that $F_s(B^2)$ is Runge in $\mathbb{C}\times\mathbb{C}^*$ and hence $F_s(\overline{B}_s)$ is holomorphically convex in $\mathbb{C}\times\mathbb{C}^*$. As $A_s$ is a finite set, $F_s(\overline{B}_s\cup A_s)$ is holomorphically convex in $\mathbb{C}\times\mathbb{C}^*$.

We apply Theorem 2.10 and we deduce that there exists an automorphism $H_{s+1}$ of $\mathbb{C}\times\mathbb{C}^*$ such that

1. $\|H_{s+1}(z)-z\| < \varepsilon_{s+1}$ for every $z\in F_s(\overline{B}_s)$,
2. $H_{s+1}(z)=z$ for every $z\in F_s(A_s)$ (in particular $H_{s+1}(a)=a$),
3. $\mathrm{d}H_{s+1}(a)=I_2$,
4. $H_{s+1}(F_s(A_j^{s+1})) = T_j^{s+1}$ for every $j=1,\dots s+1$.

Note now that property 1 implies that $F=\lim_s F_s$ (where $F_s = H_s\circ\cdots\circ H_0\circ F_0$) is holomorphic and property 3 that it is nondegenerate. Hence $F$ is biholomorphic on $B^2$. Also property 2, together with Lemma 2.11, imply that $F_s(\overline{B}_{s-1})\subset F(B^2)$ (in fact it implies that $F_s(\overline{B}_{s-1})\subset F(B_s)$) for every $s$. In particular $K\subset F(B^2)$ and therefore $F(B^2)$ is not Runge in $\mathbb{C}^2$.

It remains to check that for every $V\in\mathbb{C}^2$ with $V\cap\partial B^2\neq\emptyset$ we have $\overline{F(B^2\cap V)}\supset(\mathbb{C}^2\setminus F(B))$. Fix then such an open set $V$ and a point $p\in\mathbb{C}^2\setminus F(B^2)$. We recall that the sequence $\{x_n\}$ was chose to be dense in $\partial B^2$. Let $x_j\in V\cap\partial B^2$. Let $m\in\mathbb{N}$ be large enough such that $m>j$, $\|p-a\|<m$, and $B(x_j,\frac{1}{m})\subset V$.

We distinguish now two cases:

a) $p\notin F_m(\overline{B}_m)$. Note that $\|p-a\|<m$ implies, in particular that $d(p,F_m(\overline{B}_m))<m$. According to our choice of $T_j^{m+1}$, there exists a point $z\in T_j^{m+1}$ such that $\|p-z\|<\frac{1}{m}$. By property 4 in the construction of $\{H_s\}$, there exists $x\in A_j^{s+1}$ such that $H_{m+1}(F_m(x))=z$. According to the choice of $A_j^{s+1}$, we have that

$\|x_j - x\| < \frac{1}{m}$ and hence $x \in V$. Note also that property 2 in the construction of $\{H_s\}$ implies that $F(x) = z$.

b) $p \in F_m(\overline{B}_m)$. Since $F_{m+1}(\overline{B}_m) \subset F(B^2)$ and $p \notin F(B^2)$, we have that $p \notin F_{m+1}(\overline{B}_m)$. Let $q = H_{m+1}(p)$. It follows that $q \in F_{m+1}(\overline{B}_m)$. At the same time, property 1 in the construction of $\{H_s\}$ implies that $\|q - p\| < \frac{1}{2^{m+1}}$. It follows that $d(p, F_{m+1}(\overline{B}_m)) < \frac{1}{2^{m+1}}$ and therefore $d(p, \partial F_{m+1}(B_m)) < \frac{1}{2^{m+1}}$. Let $v \in \partial F_{m+1}(B_m))$ be such that $\|p - v\| < \frac{1}{2^{m+1}}$. However $\partial F_{m+1}(B_m)) = H_{m+1}(\partial F_m(B_m)$ and we let $u \in \partial F_m(B_m)$ such that $H_{m+1}(u) = v$. We have then $\|u - v\| < \frac{1}{2^{m+1}}$. We use again our choice of $T_j^{m+1}$ and we find a point $z \in T_j^{m+1}$ such that $\|u - z\| < \frac{1}{m}$. Hence $\|p - z\| < \frac{1}{m} + \frac{1}{2^m}$. As above we obtain a point $x \in V$ such that $F(x) = z$.

In both cases we found $x \in V$ such that $\|p - F(x)\| < \frac{1}{m} + \frac{1}{2^m}$. As $m$ can be chosen arbitrarily large, this finishes the proof. □

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# Operators of the $\alpha$-Bloch space on the open unit ball of a JB*-triple

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*Dedicated to the memory of Professor Gabriela Kohr*

**Abstract.** Let $\mathbb{B}_X$ be a bounded symmetric domain realized as the open unit ball of a JB*-triple $X$ which may be infinite dimensional. In this paper, we characterize the bounded weighted composition operators from the Hardy space $H^\infty(\mathbb{B}_X)$ into the $\alpha$-Bloch space $\mathcal{B}^\alpha(\mathbb{B}_X)$ on $\mathbb{B}_X$. Later, we show the multiplication operator from $H^\infty(\mathbb{B}_X)$ into $\mathcal{B}^\alpha(\mathbb{B}_X)$ is bounded. Also, we give the operator norm of the bounded multiplication operator.

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## 1. Introduction

Let $\mathbb{U} = \{z \in \mathbb{C}; |z| < 1\}$ be the unit disc in $\mathbb{C}$, and let $H(\mathbb{U}, \mathbb{C})$ be the set of all holomorphic function from $\mathbb{U}$ to $\mathbb{C}$. In 1925, Bloch [4] showed that a holomorphic function $f \in H(\mathbb{U}, \mathbb{C})$ with $f'(0) = 1$ biholomorphically maps onto a disc, called a schlicht disc, with the radius $r(f)$ greater than some positive absolute constant. Then, the 'best possible' constant $\mathbf{B}$ for all such functions, that is,

$$\mathbf{B} = \inf\{r(f) : f \in H(\mathbb{U}, \mathbb{C}),\ f'(0) = 1\},$$

is called the Bloch constant.

Let $r_f(z)$ be the radius of the largest schlicht disc around $f(z)$. W. Seidel and J. L. Walsh [33] gave that $r_f(z) < (1 - |z|^2)lf'(z)|$. Then, we considered the space $\mathcal{B}$ of all holomorphic functions $f : \mathbb{U} \to \mathbb{C}$ satisfying

$$\|f\|_{\mathcal{B},s} = \sup_{z\in\mathbb{U}} (1 - |z|^2)|f'(z)| < \infty$$

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endowed with the norm $\|f\|_{\mathcal{B}} = |f(0)| + \|f\|_{\mathcal{B}.s}$. The Banach space $\mathcal{B} = (\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ is called the Bloch space $\mathcal{B}(\mathbb{U})$ on $\mathbb{U}$.

In 1975, Hahn [12] introduced the concept of a Bloch mapping on a finite dimensional bounded homogeneous domain, under the name of normal mapping of finite order. Later, Timoney [35] gave several equivalent definitions for Bloch functions on a finite dimensional bounded homogeneous domain Bergman metric plays an essential role in the definition and equivalent conditions for Bloch functions using the Bergman metric. Chu, Hamada, Honda and Kohr [10] generalized to Bloch functions on a bounded symmetric domain realized as the open unit ball of a JB*-triple $X$, which may be infinite dimensional, using the Kobayashi metric. We remark that the Bergman metric is not available in infinite dimensional bounded domains.

Also, operators of the Bloch space have been studied. Ohno [30] studied the weighted composition operators from the Hardy space $H^\infty(\mathbb{U})$ to the Bloch space $\mathcal{B}(\mathbb{U})$ on $\mathbb{U}$ in $\mathbb{C}$. Li and Stević [25, 26], Zhang and Chen [37] investigated weighted composition operators from $H^\infty(\mathbb{U})$ to the $\alpha$-Bloch space. Allen and Colonna [2] gave a characterization of the bounded weighted composition operators from $H^\infty(\Omega)$ to the Bloch space on a bounded homogeneous domain $\Omega$, and gave some estimates for the operator norm. Xiong [36] proved that the composition operator is bounded on the Bloch space and gave estimates for its operator norm. Furthermore, Xiong [36] obtained several necessary conditions for the composition operator to be an isometry. Colonna, Easley and Singman [11] obtained sharper estimates for the operator norm of the multiplication operators from $H^\infty(\mathbb{U})$ to the Bloch space on a bounded symmetric domain.

One of the main purposes of this paper is to generalize the above results for $\alpha$-Bloch mappings to any bounded symmetric domain realized as the unit ball $\mathbb{B}_X$ of a JB*-triple $X$ which may be infinite dimensional. Kaup [23] showed that the bounded symmetric domains in complex Banach spaces are exactly the open unit balls of JB*-triples which are complex Banach spaces equipped with a Jordan triple structure. Hamada and Kohr [19] gave a definition of $\alpha$-Bloch mappings on $\mathbb{B}_X$ which is a generalization of $\alpha$-Bloch functions on the unit disc in $\mathbb{C}$ by using the Bergman operator of the underlying finite dimensional JB*-triple $X$. When $\alpha = 1$, it is equivalent to the definition of Bloch mappings on $\mathbb{B}^n$ by Liu [27]. Honda [21, 22] gave a characterization of the bounded weighted composition operators from $H^\infty(\mathbb{B}_X)$ into the $\alpha$-Bloch space $\mathcal{B}^\alpha(\mathbb{B}_X)$ when $\mathbb{B}_X$ is the open unit ball of a finite dimensional JB*-triple $X$.

In this paper, when $X$ is a JB*-triple which may be infinite dimensional, we will characterize the bounded weighted composition operators from $H^\infty(\mathbb{B}_X)$ into the $\alpha$-Bloch space $\mathcal{B}^\alpha(\mathbb{B}_X)$. We also give estimates for the operator norm. Later, we show that the multiplication operators from $H^\infty(\mathbb{B}_X)$ into $\mathcal{B}^\alpha(\mathbb{B}_X)$ is bounded. Finally, we show that there exist no isometric multiplication operators.

## 2. Preliminaries

Let $\mathbb{B}_X$ be the unit ball of a complex Banach space $X$. Let $Y$ be a complex Banach space and let $H(\mathbb{B}_X, Y)$ denote the set of all holomorphic mappings from $\mathbb{B}_X$ into $Y$.

Let $L(X, Y)$ denote the set of all continuous linear operators from $X$ into $Y$. Let $I_X$ be the identity in $L(X) = L(X, X)$. For a linear operator $A \in L(X, Y)$, we denote the operator norm $\|A\|_{X,Y}$ of $A$ by

$$\|A\|_{X,Y} = \sup\left\{\|Az\|_Y : \|z\|_X = 1\right\},$$

where $\|\cdot\|_X$ and $\|\cdot\|_Y$ are the norms on $X$ and $Y$, respectively. Let $\|\cdot\|_e$ denote the Euclidean norm on $\mathbb{C}^n$. For $A \in L(X)$, we write $\|A\| = \|A\|_{X,X}$. In the case $Y = \mathbb{C}^n$ is the Euclidean space, we write $\|A\|_{X,e}$ for $A \in L(X, \mathbb{C}^n)$.

A complex Banach space $X$ is called a JB*-triple if it is a complex Banach space equipped with a continuous Jordan triple product

$$X \times X \times X \ni (x, y, z) \mapsto \{x, y, z\} \in X$$

satisfying

1. $\{x, y, z\}$ is symmetric bilinear in the outer variables, but conjugate linear in the middle variable,
2. $\{a, b, \{x, y, z\}\} = \{\{a, b, x\}, y, z\} - \{x, \{b, a, y\}, z\} + \{x, y, \{a, b, z\}\}$,
3. $x\Box x \in L(X, X)$ is a hermitian operator with spectrum $\geqq 0$,
4. $\|\{x, x, x\}\|_X = \|x\|_X^3$

for $a, b, x, y, z \in X$, where the *box operator* $x\Box y : X \to X$ is defined by

$$x\Box y(\cdot) = \{x, y, \cdot\}.$$

**Example 2.1.** A complex Hilbert space $H$ with inner product $\langle\cdot,\cdot\rangle$ is a JB*-triple with the triple product

$$\{x, y, z\} = \frac{1}{2}(\langle x, y\rangle z + \langle z, y\rangle x)$$

for $x, y, z \in H$.

For every $x, y \in X$, let $B(x, y)$ be the Bergman operator defined by

$$B(x, y) = I_X - 2x\Box y + Q_x Q_y \in L(X),$$

where $Q_a : X \to X$ is the conjugate linear operator defined by $Q_a(x) = \{a, x, a\}$. When $\|x\Box y\| < 1$, the fractional power $B(x, y)^r \in L(X)$ exists for every $r \in \mathbb{R}$, since the spectrum of $B(x, y)$ lies in $\{\zeta \in \mathbb{C} : |\zeta - 1| < 1\}$ (cf. [23, p.517]).
Let $\mathbb{B}_X$ be the unit ball of a JB*-triple $X$. we have

$$\|B(a, a)\| \leq \|B(a, a)^{1/2}\|^2 \leq 1 \tag{2.1}$$

for $a \in \mathbb{B}_X$ (cf. [6, p194]). For each $a \in \mathbb{B}_X$, let $g_a$ be the Möbius transformation on $\mathbb{B}_X$ defined by

$$g_a(x) = a + B(a, a)^{1/2}(I_X + x\Box a)^{-1}x.$$

Then $g_a$ is a biholomorphic mapping of $\mathbb{B}_X$ onto itself with $g_a(0) = a$, $g_a(-a) = 0$, $g_{-a} = g_a^{-1}$, $Dg_a(0) = B(a, a)^{1/2}$ and $Dg_{-a}(a) = B(a, a)^{-1/2}$. Moreover, we have

$$\|B(a, a)^{-1/2}\| = \frac{1}{1 - \|a\|_X^2} \tag{2.2}$$

by [24, Corollary 3.6] (see also [6, Proposition 3.2.13]).

Let $z \in \mathbb{B}_X, x \in X$. We define the infinitesimal Kobayashi metric $\kappa(z,x)$ on $\mathbb{B}_X$ by

$$\kappa(z,x) = \inf\left\{|\eta| > 0 : \exists\phi \in H(U, \mathbb{B}_X), \phi(0) = z, D\phi(0)\eta = x\right\},$$

where $U$ is the unit disc in $\mathbb{C}$. Then $\kappa(0,x) = \|x\|$, $\kappa(z,\alpha x) = |\alpha|\kappa(z,x)$ for all $\alpha \in \mathbb{C}$ and

$$\kappa(z,x) = \|Dg_{-z}(z)x\|_X = \|B(z,z)^{-1/2}x\|_X \le \frac{\|x\|_X}{1-\|z\|_X^2} \quad (z \in \mathbb{B}_X, x \in X). \quad (2.3)$$

Let $\mathrm{Aut}(\mathbb{B}_X)$ denote the family of holomorphic automorphisms of $\mathbb{B}_X$.
Bloch functions on bounded homogeneous domains in $\mathbb{C}^n$ were first defined by Hahn [12]. In [34, Theorem 3.4], Timoney gave some equivalent definitions. Chu, Hamada, Honda and Kohr [10] generalized them to Bloch functions on a bounded symmetric domain realized as the open unit ball of a JB*-triple $X$. Recently, Hamada and Kohr [19] gave the definition of $\alpha$-Bloch mappings on the open unit ball $\mathbb{B}_X$ in a finite dimensional JB*-triple $X$ using Bergman operator $B(a,a)$. We will define $\alpha$-Bloch mappings on the open unit ball $\mathbb{B}_X$ in a JB*-triple $X$ which may be infinite dimensional, using the Kobayashi metric $\kappa(z,u)$ as the following.

**Definition 2.2.** Let $\mathbb{B}_X$ be a bounded symmetric domain realized as the open unit ball of a JB*-triple $X$, and let $\alpha > 0$. A holomorphic mapping $f \in H(\mathbb{B}_X, \mathbb{C}^n)$ is called an $\alpha$-Bloch mapping if

$$\|f\|_\alpha + \|f(0)\|_e < +\infty,$$

where $\|f\|_\alpha$ denotes the $\alpha$-Bloch semi-norm of $f$ defined by

$$\|f\|_\alpha = \sup\{Q_f^\alpha(z) : z \in \mathbb{B}_X\} < +\infty,$$

$$Q_f^\alpha(z) = \sup\left\{\frac{\|Df(z)u\|_e}{\kappa(z,u)^\alpha} : u \in X\setminus\{0\}, \|u\|_X = 1\right\}.$$

Let $\mathcal{B}^\alpha_{X,n}(\mathbb{B}_X)$ be the space of $\alpha$-Bloch mappings $f : \mathbb{B}_X \to \mathbb{C}^n$. For $f \in \mathcal{B}^\alpha_{X,n}(\mathbb{B}_X)$ the norm $\|\cdot\|_{\mathcal{B}^\alpha_n}$ is given by $\|f\|_{\mathcal{B}^\alpha_n} = \|f\|_\alpha + \|f(0)\|_e$. Then, the space $(\mathcal{B}^\alpha_{X,n}(\mathbb{B}_X), \|\cdot\|_{\mathcal{B}^\alpha_n})$ is a complex Banach space. We write $\mathcal{B}^\alpha(\mathbb{B}_X) = \mathcal{B}^\alpha_{X,1}(\mathbb{B}_X)$ and $\|f\|_{\mathcal{B}^\alpha} = \|f\|_{\mathcal{B}^\alpha_X}$ for $f \in \mathcal{B}^\alpha(\mathbb{B}_X)$.
Let $f : \mathbb{B}_X \to \mathbb{C}$ be a holomorphic function and let $\alpha > 0$. We say that $f$ is an $\alpha$-Bloch function on $\mathbb{B}_X$ if $f \in \mathcal{B}^\alpha(\mathbb{B}_X)$. Then, by the definitions of $Q_f^\alpha(z)$ and $\|f\|_{\mathcal{B}^\alpha}$,

$$Q_f^\alpha(z) = \sup\left\{\frac{|Df(z)u|}{\kappa(z,u)^\alpha} : u \in X\setminus\{0\}, \|u\|_X = 1\right\}, \|f\|_{\mathcal{B}^\alpha} = |f(0)| + \|f\|_\alpha.$$

Moreover, we have

$$\begin{aligned} & \sup\left\{\frac{|Df(z)x|}{\kappa(z,x)} : x \in X\setminus\{0\}\right\} \\ = \; & \sup\left\{\frac{\|x\|_X|Df(z)\frac{x}{\|x\|_X}|}{\|x\|_X\kappa(z,\frac{x}{\|x\|_X})} : x \in X\setminus\{0\}\right\} \\ = \; & \sup\left\{\frac{|Df(z)u|}{\kappa(z,u)} : u \in X\setminus\{0\}, \|u\|_X = 1\right\} \\ = \; & Q_f^1(z). \end{aligned}$$

In the case that $\alpha = 1$, 1-Bloch mappings are equivalent to Bloch mappings in the sense of Chu, Hamada, Honda and Kohr [10].

**Remark 2.3.** Let $\alpha > 0$, $f = (f_1, \dots, f_n) \in H(\mathbb{B}_X, \mathbb{C}^n)$. Then $f$ is an $\alpha$-Bloch mapping if and only if each $f_j$ is an $\alpha$-Bloch function. If $\alpha = 1$, then $f_j$ is a Bloch function on $\mathbb{B}_X$ if and only if $\|D(f_j \circ g)(0)\|_{X,e}$ is uniformly bounded for $g \in \mathrm{Aut}(\mathbb{B}_X)$.

**Lemma 2.4.** *Let $\mathbb{B}_X$ be a bounded symmetric domain realized as the unit ball of a JB*-triple $X$. If $f \in H(\mathbb{B}_X, \mathbb{C}^n)$ is an $\alpha$-Bloch mapping, then we have*

$$\|Df(z)\|_{X,e} \le \frac{Q_f^\alpha(z)}{(1-\|z\|_X^2)^\alpha} \le \frac{\|f\|_\alpha}{(1-\|z\|_X^2)^\alpha}, \quad z \in \mathbb{B}_X.$$

*Proof.* Since $(1-\|z\|_X^2)\kappa(z,u) \le 1$ from (2.3), we have

$$\begin{aligned}\|Df(z)\|_{X,e} &= \sup_{\|u\|_X=1} \|Df(z)u\|_e \\ &= \sup_{\|u\|_X=1} \left(\frac{1-\|z\|_X^2}{\kappa(z,u)} \frac{\kappa(z,u)}{1-\|z\|_X^2}\right)^\alpha \|Df(z)u\|_e \\ &\le \frac{1}{(1-\|z\|_X^2)^\alpha} \sup_{\|u\|_X=1} \frac{\|Df(z)u\|_e}{\kappa(z,u))^\alpha} \\ &= \frac{Q_f^\alpha(z)}{(1-\|z\|_X^2)^\alpha} \le \frac{\|f\|_\alpha}{(1-\|z\|_X^2)^\alpha}.\end{aligned}$$

This completes the proof. □

**Remark 2.5.** By Lemma 2.4, $\alpha$-Bloch mappings are bounded on $B_X$ for $\alpha \in (0,1)$.

**Lemma 2.6.** *Let $\mathbb{B}_X$ and $\mathbb{B}_Y$ be bounded symmetric domains realized as the unit balls of JB*-triples $X$ and $Y$, respectively. Let $\alpha \ge 1$, $f \in H(\mathbb{B}_Y, \mathbb{C}^n)$ and $\varphi \in H(\mathbb{B}_X, \mathbb{B}_Y)$. Then*

$$\frac{\|D(f\circ\varphi)(z)u\|_e}{\kappa_X(z,u)^\alpha} \le Q_f^\alpha(\varphi(z)) \quad (u \in X, \|u\|_X = 1).$$

*Proof.* Let $z \in \mathbb{B}_X$ and $u \in X$ with $\|u\|_X = 1$ be fixed.
If $D\varphi(z)u = 0$, then $D(f\circ\varphi)(z)u = Df(\varphi(z))D\varphi(z)u = 0$. So the above estimate holds.
If $D\varphi(z)u \ne 0$, then we have

$$\frac{\|D(f\circ\varphi)(z)u\|_e}{\kappa_X(z,u)^\alpha} \le \frac{\|Df(\varphi(z))D\varphi(z)u\|_e}{\kappa_Y(\varphi(z), D\varphi(z)u)^\alpha} \le Q_f^\alpha(\varphi(z)).$$

This completes the proof. □

By Lemma 2.6 and the definition of $Q_f^\alpha(z)$ that we have the following.

**Proposition 2.7.** *Let $\mathbb{B}_X$ and $\mathbb{B}_Y$ be bounded symmetric domains realized as the unit balls of JB*-triples $X$ and $Y$, respectively. Let $\alpha \ge 1$, $f \in H(\mathbb{B}_Y, \mathbb{C}^n)$.*

(i) *If $\varphi \in H(\mathbb{B}_X, \mathbb{B}_Y)$, then $Q_{f\circ\varphi}^\alpha(z) \le Q_f^\alpha(\varphi(z))$ for each $z \in \mathbb{B}_X$.*
(ii) *If $\varphi$ is a biholomorphic mapping from $\mathbb{B}_X$ onto $\mathbb{B}_Y$, then $Q_{f\circ\varphi}^\alpha(z) = Q_f^\alpha(\varphi(z))$ for each $z \in \mathbb{B}_X$.*

Using the above lemmas, we have the following useful lemma.

**Lemma 2.8.** *Let $\mathbb{B}_X$ be the unit ball of JB*-triples $X$. Let $f \in H(\mathbb{B}_X, \mathbb{C}^n)$.*
(i) $Q_f^{\beta}(z) \leq Q_f^{\alpha}(z)$ *holds for* $z \in \mathbb{B}_X$, $\alpha \leq \beta$.
(ii) $Q_f^{\alpha}(0) = Q_f^1(0)$ *holds for* $\alpha \geq 1$.
(iii) *Let $g_{-z}$ be the Möbius transformation on $\mathbb{B}_X$ for $z \in \mathbb{B}_X$. Then, for $\alpha \geq 1$,*

$$Q_{f\circ g_{-z}}^{\alpha}(z) = Q_f^{\alpha}(0).$$

*Proof.* (i) Since $\|u\|_X = \|B(z,z)^{1/2}B(z,z)^{-1/2}u\|_X \leq \|B(z,z)^{1/2}\| \, \|B(z,z)^{-1/2}u\|_X$ for $u \in X$ with $\|u\|_X = 1$, we have, from (2.1),

$$\frac{1}{\|B(z,z)^{-1/2}u\|_X} \leq \|B(z,z)^{1/2}\| \leq 1.$$

Hence, it follows from (2.3) that

$$\begin{aligned}\frac{\|Df(z)u\|_e}{\kappa(z,u)^{\beta}} &= \frac{\|Df(z)u\|_e}{\kappa(z,u)^{\beta-\alpha}\kappa(z,u)^{\alpha}} \\ &= \frac{\|Df(z)u\|_e}{\kappa(z,u)^{\alpha}}\left(\frac{1}{\|B(z,z)^{-1/2}u\|_X}\right)^{\beta-\alpha} \\ &\leq \frac{\|Df(z)u\|_e}{\kappa(z,u)^{\alpha}} \leq Q_f^{\alpha}(z).\end{aligned}$$

(ii) Since $\kappa(0,u) = \|u\| = 1$, we have

$$\frac{\|Df(0)u\|_e}{\kappa(0,u)^{\alpha}} = \|Df(0)u\|_e = \frac{\|Df(0)u\|_e}{\kappa(0,u)^{1}}.$$

(iii) By Proposition 2.7 (ii), we have $Q_{f\circ g_{-z}}^{\alpha}(z) = Q_f^{\alpha}(g_{-z}(z)) = Q_f^{\alpha}(0)$. □

**Remark 2.9.**

(1) Any $\alpha$-Bloch mapping on $\mathbb{B}_X$ is also a $\beta$-Bloch mapping on $\mathbb{B}_X$ for $\alpha \leq \beta$.
(2) Since 1-Bloch functions are equivalent to Bloch functions in the sense of Chu, Hamada, Honda and Kohr [10], it follows that any Bloch function is also an $\alpha$-Bloch function, for $\alpha \geq 1$.

For $f \in H(\mathbb{B}_X, \mathbb{C})$, we denote the operator norm by

$$\|f\|_{\infty} = \sup\{|f(z)| : z \in \mathbb{B}_X\}.$$

Let

$$H^{\infty}(\mathbb{B}_X) = \{f \in H(\mathbb{B}_X, \mathbb{C}) : \|f\|_{\infty} < +\infty\}$$

be the space, called the Hardy space, of bounded holomorphic functions on $\mathbb{B}_X$. When the target is the unit disc in $\mathbb{C}$, Chu, Hamada, Honda and Kohr [10, Lemma 3.12] obtained the following Schwarz Pick lemma.

**Lemma 2.10.** *Let $f \in H^{\infty}(\mathbb{B}_X)$ be such that $\|f\|_{\infty} \leq 1$. Then we have*

$$\|Df(z)\|_{X,e} \leq \frac{1-|f(z)|^2}{1-\|z\|_X^2}, \quad z \in \mathbb{B}_X.$$

Using the above lemma, we obtain the following.

**Lemma 2.11.** *For $\alpha \geq 1$, $H^\infty(\mathbb{B}_X) \subset \mathcal{B}^\alpha(\mathbb{B}_X)$ and the inclusion mapping*

$$i : H^\infty(\mathbb{B}_X) \to \mathcal{B}^\alpha(\mathbb{B}_X)$$

*is a linear operator satisfying*

$$\|f\|_\alpha \leq \|f\|_\infty.$$

*Proof.* Let $f \in H^\infty(\mathbb{B}_X)$. We may assume $\|f\|_\infty = 1$. Since $Q_f^\alpha(z) \leq Q_f^1(z)$ for $z \in \mathbb{B}_X$ by Lemma 2.8, we have

$$\|f\|_\alpha \leq \|f\|_1 = \sup\{Q_f^1(z) : z \in \mathbb{B}_X\}.$$

Let $g_a$ be the Möbius transformation on $\mathbb{B}_X$ for $a \in \mathbb{B}_X$. Then, we have $f \circ g_a \in H^\infty(\mathbb{B}_X)$ and $\|f \circ g_a\|_\infty \leq 1$. By Lemmma 2.10, we have for $\|u\|_X = 1$,

$$\begin{aligned}
\frac{\|Df(a)u\|_e}{\kappa(a,u)^1} &= \frac{\|Df(g_a(g_{-a}(a))Dg_a(g_{-a}(a))Dg_{-a}(a)u\|_e}{\|Dg_{-a}(a)u\|_X} \\
&\leq \|Df(a) \circ Dg_a(0)\|_{X,e} \\
&= \|D(f \circ g_a)(0)\|_{X,e} \\
&\leq 1 - |f \circ g_a(0)|^2 \\
&\leq 1 = \|f\|_\infty.
\end{aligned}$$

□

## 3. Weighted composition operators

Let $\psi \in H(\mathbb{B}_X, \mathbb{C})$ and $\varphi \in H(\mathbb{B}_X, \mathbb{B}_X)$. The weighted composition operator $W_{\psi,\varphi} : H(\mathbb{B}_X, \mathbb{C}) \to H(\mathbb{B}_X, \mathbb{C})$ is defined by

$$W_{\psi,\varphi}(f)(z) = \psi(z)(f \circ \varphi)(z), \quad f \in H(\mathbb{B}_X, \mathbb{C}), z \in \mathbb{B}_X.$$

We set

$$\begin{aligned}
\theta_\varphi^\alpha(z) &= \sup\{Q_{f\circ\varphi}^\alpha(z) : f \in H^\infty(\mathbb{B}_X), \|f\|_\infty \leq 1\}, \\
\theta_{\psi,\varphi}^\alpha &= \sup_{z \in \mathbb{B}_X} |\psi(z)|\theta_\varphi^\alpha(z) \qquad .
\end{aligned}$$

**Theorem 3.1.** *Let $W_{\psi,\varphi} : H^\infty(\mathbb{B}_X) \to \mathcal{B}^\alpha(\mathbb{B}_X)$ be the weighted composition operator for $\alpha \geq 1$, $\psi \in H(\mathbb{B}_X, \mathbb{C})$, $\varphi \in H(\mathbb{B}_X, \mathbb{B}_X)$. Then,*

1. *(1) $W_{\psi,\varphi}$ is bounded if and only if $\theta_{\psi,\varphi}^\alpha$ is finite and $\psi \in \mathcal{B}^\alpha(\mathbb{B}_X)$,*
2. *(2) if $W_{\psi,\varphi}$ is bounded, then the following inequalities hold.*

$$\max\{\|\psi\|_{\mathcal{B}^\alpha}, \theta_{\psi,\varphi}^\alpha\} \leq \|W_{\psi,\varphi}\| \leq \|\psi\|_{\mathcal{B}^\alpha} + \theta_{\psi,\varphi}^\alpha.$$

*Proof.* We will prove the following inequality :

$$\theta_{\psi,\varphi}^\alpha \leq \|W_{\psi,\varphi}\|. \tag{3.1}$$

We may assume that $\theta^{\alpha}_{\psi,\varphi} > 0$. Let $\varepsilon \in (0, \theta^{\alpha}_{\psi,\varphi})$ and fixed. Then, there exist $a \in \mathbb{B}_X$ and $f \in H^{\infty}(\mathbb{B}_X)$ with $\|f\|_{\infty} \leq 1$ such that

$$|\psi(a)| Q^{\alpha}_{f\circ\varphi}(a) > \theta^{\alpha}_{\psi,\varphi} - \varepsilon.$$

Now, by the maximum principle for holomorphic functions, $|f(z)| < 1$ for all $z \in \mathbb{B}_X$. Let $g : U \to U$ be a biholomorphic function defined by

$$g(\zeta) = \frac{\zeta - f(\varphi(a))}{1 - \overline{f(\varphi(a))}\zeta}, \quad \zeta \in U.$$

Then, we obtain $\|g \circ f\|_{\infty} \leq 1$, $g(f(\varphi(a))) = 0$ and $g'(f(\varphi(a))) = 1/(1 - |f(\varphi(a))|^2)$. Hence

$$\begin{aligned} \|W_{\psi,\varphi}\| &\geq \|W_{\psi,\varphi}(g \circ f)\|_{\mathcal{B}^{\alpha}} \\ &\geq Q^{\alpha}_{\psi(g\circ f\circ\varphi)}(a) \\ &= \frac{|\psi(a)|}{1 - |f(\varphi(a))|^2} Q^{\alpha}_{f\circ\varphi}(a) \\ &> \theta^{\alpha}_{\psi,\varphi} - \varepsilon. \end{aligned}$$

Since $\varepsilon \in (0, \theta^{\alpha}_{\psi,\varphi})$ is arbitrary, we obtain (3.1).

(1) Now, we assume $W_{\psi,\varphi}$ is bounded. Then, from (3.1), we have $\theta^{\alpha}_{\psi,\varphi}$ is finite. Also, using the constant function $\mathbf{1} \in H^{\infty}(\mathbb{B}_X)$,

$$\|\psi\|_{\mathcal{B}^{\alpha}} = \|\psi(\mathbf{1} \circ \varphi)\|_{\mathcal{B}^{\alpha}} = \|W_{\psi,\varphi}(\mathbf{1})\|_{\mathcal{B}^{\alpha}} \leq \|W_{\psi,\varphi}\|. \tag{3.2}$$

Hence $\psi \in \mathcal{B}^{\alpha}(\mathbb{B}_X)$.
Conversely, we assume that $\theta^{\alpha}_{\psi,\varphi}$ is finite and $\psi \in \mathcal{B}^{\alpha}(\mathbb{B}_X)$. For $f \in H^{\infty}(\mathbb{B}_X)$ with $\|f\|_{\infty} \leq 1$,

$$Q^{\alpha}_{\psi(f\circ\varphi)}(z) \leq |\psi(z)| Q^{\alpha}_{f\circ\varphi}(z) + |f \circ \varphi(z)| Q^{\alpha}_{\psi}(z) \leq \theta^{\alpha}_{\psi,\varphi} + \|\psi\|_{\alpha}.$$

So, we have

$$\|\psi(f \circ \varphi)\|_{\alpha} = \sup_{z\in\mathbb{B}_X} Q^{\alpha}_{\psi(f\circ\varphi)}(z) \leq \theta^{\alpha}_{\psi,\varphi} + \|\psi\|_{\alpha}.$$

It follows from this that

$$\|W_{\psi,\varphi}(f)\|_{\mathcal{B}^{\alpha}} = \|\psi(f \circ \varphi)\|_{\alpha} + |\psi(0) f(\varphi(0))| \leq \theta^{\alpha}_{\psi,\varphi} + \|\psi\|_{\mathcal{B}^{\alpha}}. \tag{3.3}$$

(2) This follows from (3.1), (3.2), and (3.3). □

**Remark 3.2.** Honda [22] obtained when $X$ is a finite dimensional JB*-triple. The case $\alpha = 1$ was obtained by Hamada in [13].

## 4. Multiplication operators

For each $x \in X \setminus \{0\}$, we define

$$T(x) = \{l_x \in X^* : \; l_x(x) = \|x\|, \; \|l_x\| = 1\}.$$

By the Hahn-Banach theorem, $T(x)$ is nonempty.

Let $\psi \in H(\mathbb{B}_X, \mathbb{C})$. The multiplication operator $M_\psi : H(\mathbb{B}_X, \mathbb{C}) \to H(\mathbb{B}_X, \mathbb{C})$ is defined by

$$M_\psi(f)(z) = \psi(z)f(z)$$

for $f \in H(\mathbb{B}_X, \mathbb{C}), z \in \mathbb{B}_X$. We can give the operator norm of the bounded multiplication operator $M_\psi$ from the Hardy space $H^\infty(\mathbb{B}_X)$ to the $\alpha$-Bloch space $\mathcal{B}^\alpha(\mathbb{B}_X)$. The following theorem gives an answer to [11, Conjecture, p.628]. The case $\alpha = 1$ was obtained by Hamada [13].

**Theorem 4.1.** *Let $\psi \in H(\mathbb{B}_X, \mathbb{C})$, $\alpha \geq 1$. Then,*

(1) *$M_\psi : H^\infty(\mathbb{B}_X) \to \mathcal{B}^\alpha(\mathbb{B}_X)$ is bounded if and only if $\psi \in H^\infty(\mathbb{B}_X)$,*
(2) *if $M_\psi$ is bounded, then the following equality holds.*

$$\|M_\psi\| = |\psi(0)| + \|\psi\|_\infty,$$

*where* $\|M_\psi\| = \sup\left\{\|M_\psi(f)\|_{\mathcal{B}^\alpha} : f \in H^\infty(\mathbb{B}_X), \|f\|_\infty = 1\right\}.$

*Proof.* (1) If $\psi \in H^\infty(\mathbb{B}_X)$, then, by Lemma 2.11, we have, for $f \in H^\infty(\mathbb{B}_X)$ with $\|f\|_\infty = 1$,

$$\begin{aligned}\|M_\psi(f)\|_{\mathcal{B}^\alpha} &= |M_\psi(f)(0)| + \|M_\psi(f)\|_\alpha \\ &= |\psi(0)f(0)| + \|\psi f\|_\alpha \\ &\leq |\psi(0)| + \|\psi\|_\infty.\end{aligned}$$

This implies that

$$\|M_\psi\| \leq |\psi(0)| + \|\psi\|_\infty. \tag{4.1}$$

Hence, $M_\psi$ is bounded.

Conversely, assume that $M_\psi$ is bounded. Fix $a \in \mathbb{B}_X \setminus \{0\}$. We set $f(z) = l_a(z)$, where $l_a \in T(a)$. Let $g_a$ be the Möbius transformation on $\mathbb{B}_X$. Then $\|f \circ g_a\|_\infty = 1$, $(f \circ g_a)(-a) = 0$. By Lemma 2.8, we have

$$\begin{aligned}\|M_\psi(f \circ g_a)\|_\alpha &= \|\psi(f \circ g_a)\|_\alpha \\ &\geq Q^\alpha_{\psi(f\circ g_a)}(-a) \\ &= |\psi(-a)|Q^\alpha_{f\circ g_a}(-a) \\ &= |\psi(-a)|Q^\alpha_f(0) \\ &= |\psi(-a)|Q^1_f(0) \\ &= |\psi(-a)|.\end{aligned}$$

Therefore, we have

$$\|M_\psi(f \circ g_a)\|_{\mathcal{B}^\alpha} = \|\psi(f \circ g_a)\|_{\mathcal{B}^\alpha} \geq |\psi(0)|\|a\| + |\psi(-a)|.$$

It follows from this that

$$\|M_\psi\| \geq |\psi(0)| + \|\psi\|_\infty. \tag{4.2}$$

Thus, $\psi \in H^\infty(\mathbb{B}_X)$.

(2) This follows from (4.1) and (4.2). □

Allen and Colonna [2, Theorem 6.2] proved the following theorem when $\mathbb{B}_X$ is the unit disc in $\mathbb{C}$. Colonna, Easley and Singman [11, Theorem 5.1] generalized it when $\mathbb{B}_X$ is a finite dimensional bounded symmetric domain which satisfies some assumption. By using the Bloch norm introduced in section 2, we can generalize to any bounded symmetric domain. This result gives a positive answer to [11, Conjecture, p.629]. The case $\alpha = 1$ was obtained by Hamada [13].

**Theorem 4.2.** *Let $\mathbb{B}_X$ be the unit ball of a JB*-triple $X$. Then, there exist no isometric multiplication operators from $H^\infty(\mathbb{B}_X)$ to $\mathcal{B}^\alpha(\mathbb{B}_X)$ for $\alpha \geq 1$.*

*Proof.* Assume that $\psi \in H(\mathbb{B}_X, \mathbb{C})$, and $M_\psi$ is an isometric multiplication operator from $H^\infty(\mathbb{B}_X)$ to $\mathcal{B}^\alpha(\mathbb{B}_X)$. It follows from this that

$$\|M_\psi(f)\|_{\mathcal{B}^\alpha} = \|f\|_\infty \quad \text{for } f \in H^\infty(\mathbb{B}_X). \tag{4.3}$$

We set $f_a(z) = l_a(z) \in T(a)$ for some $a \in X \setminus \{0\}$. Since $f_a \in H^\infty(\mathbb{B}_X)$, by Lemma 2.11 and (4.3), we obtain

$$1 = \|f_a\|_\infty = \|M_\psi(f_a)\|_{\mathcal{B}^\alpha} = \|\psi f_a\|_\alpha \leq \|\psi f_a\|_\infty \leq \|\psi\|_\infty.$$

Moreover, by Theorem 4.1, $|\psi(0)| + \|\psi\|_\infty = \|M_\psi\| = 1$. So, we have $\psi(0) = 0$ and $\|\psi\|_\infty = 1$. It follows this and (4.3) that

$$\|\psi^2\|_{\mathcal{B}^\alpha} = \|\psi\psi\|_{\mathcal{B}^\alpha} = \|M_\psi(\psi)\|_{\mathcal{B}^\alpha} = \|\psi\|_\infty = 1.$$

On the other hand, we set $F = \psi^2$. Let $g_z$, $g_{-z}$ be the Möbius transformations on $\mathbb{B}_X$ for $z \in \mathbb{B}_X$. Then, by Lemma 2.8, 2.10,

$$\begin{aligned}
Q_F^\alpha(z) &= Q^\alpha_{F\circ g_z \circ g_{-z}}(z) = Q^\alpha_{F\circ g_z}(0) \\
&= \sup\left\{ \frac{\|D(F\circ g_z)(0)u\|_e}{\kappa(0,u)^\alpha} : u \in X\setminus\{0\}, \|u\|_X = 1 \right\}. \\
&\leq \sup\left\{ \frac{2|\psi(g_z(0))|\|D(\psi\circ g_z)(0)u\|_e}{\|u\|_X^\alpha} : u \in X\setminus\{0\}, \|u\|_X = 1 \right\}. \\
&\leq 2|\psi(g_z(0))|(1 - |(\psi(g_z(0))|^2) \\
&\leq 2 \sup_{x\in[0,1]} \max(x - x^3) = \frac{4}{3\sqrt{3}}.
\end{aligned}$$

It follows from this that $\|\psi^2\|_{\mathcal{B}^\alpha} = \|F\|_{\mathcal{B}^\alpha} \leq \frac{4}{3\sqrt{3}} < 1$. This is a contradiction. □

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# The Fekete–Szegö problem for spirallike mappings and non-linear resolvents in Banach spaces

Mark Elin and Fiana Jacobzon

*Dedicated to the memory of Professor Gabriela Kohr*

**Abstract.** We study the Fekete–Szegö problem on the open unit ball of a complex Banach space. Namely, the Fekete–Szegö inequalities are proved for the class of spirallike mappings relative to an arbitrary strongly accretive operator, and some of its subclasses. Next, we consider families of non-linear resolvents for holomorphically accretive mappings vanishing at the origin. We solve the Fekete–Szegö problem over these families.

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**Keywords:** Fekete–Szegö inequality, holomorphically accretive mapping, spirallike mapping, non-linear resolvent.

## 1. Introduction

Let $X$ be a complex Banach space equipped with the norm $\|\cdot\|$ and let $X^*$ be the dual space of $X$. We denote by $\mathbb{B}$ the open unit ball in $X$. For each $x \in X \setminus \{0\}$, denote

$$T(x) = \{\ell_x \in X^* : \|\ell_x\| = 1 \text{ and } \ell_x(x) = \|x\|\}. \tag{1.1}$$

According to the Hahn–Banach theorem (see, for example, [25, Theorem 3.2]), $T(x)$ is nonempty and may consists of a singleton (for instance, in the case of Hilbert space), or, otherwise, of infinitely many elements. Its elements $\ell_x \in T(x)$ are called support functionals at the point $x$.

Let $Y$ be a Banach space (possibly, different from $X$). The set of all holomorphic mappings from $\mathbb{B}$ into $Y$ will be denoted by $\mathrm{Hol}(\mathbb{B}, Y)$. It is well known (see, for

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example, [20, 9, 15, 24]) that if $f \in \mathrm{Hol}(\mathbb{B}, Y)$, then for every $x_0 \in \mathbb{B}$ and all $x$ in some neighborhood of $x_0 \in \mathbb{B}$, the mapping $f$ admits the Taylor series representation:

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} D^n f(x_0) \left[ (x - x_0)^n \right], \tag{1.2}$$

where $D^n f(x_0) : \prod_{k=1}^{n} X \to Y$ is a bounded symmetric $n$-linear operator that is called the $n$-th Fréchet derivative of $f$ at $x_0$. Also we write $D^n f(x_0) \left[ (x - x_0)^n \right]$ for $D^n f(x_0)[x - x_0, \ldots, x - x_0]$. One says that $f$ is normalized if $f(0) = 0$ and $Df(0) = \mathrm{Id}$, the identity operator on $X$.

Recall that a holomorphic mapping $f : \mathbb{B} \to X$ is called biholomorphic if the inverse $f^{-1}$ exists and is holomorphic on the image $f(\mathbb{B})$. A mapping $f \in \mathrm{Hol}(\mathbb{B}, X)$ is said to be locally biholomorphic if for each $x \in \mathbb{B}$ there exists a bounded inverse for the Fréchet derivative $Df(x)$, see [9, 15].

In the one-dimensional case, where $X = \mathbb{C}$ and $\mathbb{B} = \mathbb{D}$ is the open unit disk in $\mathbb{C}$, one usually writes $a_n(x - x_0)^n$ instead of $\frac{1}{n!} D^n f(x) \left[ (x - x_0)^n \right]$ in (1.2). The classical Fekete–Szegö problem [12] for a given subclass $\mathcal{F} \subset \mathrm{Hol}(\mathbb{D}, \mathbb{C})$ is to find

$$\sup_{f \in \mathcal{F}} \left| a_3 - \nu a_2^2 \right|, \quad \text{where} \ \ f(z) = z + a_2 z^2 + a_3 z^3 + \ldots .$$

In multi-dimensional settings various analogs of the classical Fekete–Szegö problem for different classes of holomorphic mappings have been studied by many mathematicians. Nice survey of the current state of the art and references can be found in [19] and [22].

H. Hamada, G. Kohr and M. Kohr in [19] introduced a new quadratic functional that generalizes the Fekete–Szegö functional to infinite-dimensional settings. Moreover, they estimated this functional over several classes of holomorphic mappings, including starlike mappings and non-linear resolvents of normalized holomorphically accretive mappings.

The aim of this paper is to extend the method used in [19] and solve the Fekete–Szegö problem over the classes of spirallike mappings and resolvents of non-normalized holomorphically accretive mappings. Along the way we generalize some results in [19] and [6].

Spirallike mappings in Banach spaces were first introduced and studied in the mid 1970's by K. Gurganus and T. J. Suffridge. This study has evolved into a coherent theory thanks to the influential contributions of Gabriela Kohr and her co-authors (I. Graham, H. Hamada, M. Kohr and others) over the past decades (some details can be found below). As for non-linear resolvents, they seem to have been among the last issues that caught her attention. Progress on this topic is reflected in [13, 19].

## 2. Preliminaries

Recall that for a densely defined linear operator $A$ with the domain $D_A \subset X$, the set $V(A) = \{\ell_x(Ax) : x \in D_A, \|x\| = 1, \, \ell_x \in T(x)\}$ is called the numerical range of $A$.

**Definition 2.1.** Let $A \in L(X)$ be a bounded linear operator on $X$. Then $A$ is called accretive if

$$\Re \ell_x(Ax) \geq 0$$

for all $x \in X \setminus \{0\}$, or, what is the same, if $m(A) \geq 0$, where

$$m(A) := \inf \{\Re \lambda : \lambda \in V(A)\}.$$

If for some $k > 0$,

$$\Re \ell_x(Ax) \geq k\|x\|$$

for all $x \in X \setminus \{0\}$, the operator $A$ is called strongly accretive.

The notion of accretivity was extended by Harris [20] to involve holomorphic mappings (see also [24, 9]).

**Definition 2.2.** Let $h \in \mathrm{Hol}(\mathbb{B}, X)$. This mapping $h$ is said to be holomorphically accretive if

$$m(h) := \liminf_{s \to 1^-} \big(\inf \{\Re \ell_x(h(sx)) : \ \|x\| = 1, \ \ell_x \in T(x)\}\big) \geq 0.$$

In the case where the last lower limit $m(h)$ is positive, $h$ is called strongly holomorphically accretive.

**Remark 2.3.** According to [9, Proposition 2.3.2] if $h(0) = 0$ then $V(A) \subset \overline{\mathrm{conv}}\, V(h)$, where $A = Dh(0)$, in particular, $m(A) \geq m(h)$. Consequently, if $h$ is holomorphically accretive, its linear part at zero $A$ is accretive too. Furthermore, for such mappings Proposition 2.5.4 in [9] implies that $h$ is holomorphically accretive if and only if $\Re \ell_x(h(x)) \geq 0$ for all $x \in \mathbb{B} \setminus \{0\}$.

The main feature of the class of holomorphically accretive mappings is that they generate semigroups of holomorphic self-mappings on $\mathbb{B}$, so they are of most importance in dynamical systems [24, 9]. A very fruitful characterization of holomorphically accretive mappings is:

**Proposition 2.4 (Theorem 7.3 in** [24]**, see also** [9]**).** *A mapping $h \in \mathrm{Hol}(\mathbb{B}, X)$ is holomorphically accretive if and only if it satisfies the so-called range condition (RC), that is, $(\mathrm{Id} + rh)(\mathbb{B}) \supseteq \mathbb{B}$ for each $r > 0$, and the inverse mapping $J_r := (\mathrm{Id} + rh)^{-1}$ is a well-defined holomorphic self-mapping of $\mathbb{B}$.*

The mapping $J_r$ that occurs in this proposition is called the *non-linear resolvent of $h$*. In other words, the non-linear resolvent is the unique solution $w = J_r(x) \in \mathbb{B}$ of the functional equation

$$w + rh(w) = x \in \mathbb{B}, \quad r > 0.$$

Assuming $h(0) = 0$, one sees that $J_r(0) = 0$ for all $r > 0$.
If, in addition, $A = Dh(0)$, then $DJ_r(0) = (\mathrm{Id} + rA)^{-1}$. Furthermore, the accretivity of $A$ mentioned in Remark 2.3, implies $DJ_r(0)$ is strongly contractive because $\|(\mathrm{Id} + rA)^{-1}\| < 1$.

We use the following classes (see [15] and references therein):

$$\begin{aligned}
\mathcal{N} &= \{h \in \mathrm{Hol}(\mathbb{B}, X) \colon h(0) = 0, \Re \ell_x(h(x)) > 0, \ x \in \mathbb{B} \setminus \{0\}, \ell_x \in T(x)\}, \\
\mathcal{M} &= \{h \in \mathcal{N}, Dh(0) = \mathrm{Id}\}
\end{aligned}$$

and (see [14])

$$\mathcal{N}_A \quad := \quad \{h \in \mathcal{N} : \; Dh(0) = A\} \,. \tag{2.1}$$

To proceed, we note that the inclusion $h \in \mathcal{N}$ can be expressed as $\ell_x(h(x)) \in g_0(\mathbb{D})$, $x \in \mathbb{B}\backslash\{0\}$, where $g_0(z) = \frac{1+z}{1-z}$ . At the same time, $\overline{V(A)}$ is a compact subset of the open right half-plane, hence the inclusion $\ell_x(h(x)) \in g_0(\mathbb{D})$ is imprecise. It can be improved by using other functions $g \prec g_0$, bearing in mind that $g(\mathbb{D})$ should contain $V(A)$ by Remark 2.3.

Throughout this paper we suppose that the following conditions hold

**Assumption 1.** A linear operator $A$ is bounded and strongly accretive. A function $g = g_A \in \mathrm{Hol}(\mathbb{D}, \mathbb{C})$ satisfies $g \prec g_0$ and $\overline{V(A)} \subset g(\mathbb{D})$. Therefore $\Delta := g^{-1}(V(A))$ is compactly embedded in $\mathbb{D}$.

**Definition 2.5 (cf.** [2, 27]**).** Let $A$ and $g$ satisfy Assumption 1. Denote

$$\mathcal{N}_A(g) := \left\{ h \in \mathcal{N}_A : \frac{\ell_x(h(x))}{\|x\|} \in g(\mathbb{D}), x \in \mathbb{B}\backslash\{0\}, \ell_x \in T(x) \right\}. \tag{2.2}$$

We now consider specific choices of $g$ providing some properties of semigroups generated by $h \in \mathcal{N}_A(g)$:

(a) $g_1^\alpha(z) := \left(\frac{1+z}{1-z}\right)^\alpha, \alpha \in (0,1)$: It can be shown that the semigroup generated by every $h \in \mathcal{N}_A(g_1^\alpha)$ can be analytically extended with respect to parameter $t$ to the sector $|\arg t| < \frac{\pi(1-\alpha)}{2}$; for the one-dimensional case see [11];

(b) $g_2^\alpha(z) := \alpha + (1-\alpha)\frac{1+z}{1-z}, \alpha \in (0, m(A))$: it follows from Lemma 3.3.2 in [8] that the semigroup $\{u(t,x)\}_{t\geq 0}$ generated by any element of $\mathcal{N}_A(g_2^\alpha)$ satisfies the estimate $\|u(t,x)\| \leq e^{-t\alpha}\|x\|$ uniformly on the whole $\mathbb{B}$;

(c) $g_3^\alpha(z) := \frac{1-z}{1-(2\alpha-1)z}, \alpha \in (0,1)$, maps $\mathbb{D}$ onto a disk $\Delta$ tangent the imaginary axis. In a sense this choice is dual to the previous one (in the one-dimensional case such duality was investigated in [1]);

In what follows we will refer to these functions as $g_0, g_1^\alpha, g_2^\alpha, g_3^\alpha$.

Another area where holomorphically accretive mappings are widely used is geometric function theory. The study of spirallike mappings is a good example of this fruitful connection.

**Definition 2.6 (see** [26, 15, 8, 24]**).** Let $A$ be a strongly accretive operator. A biholomorphic mapping $f \in \mathrm{Hol}(\mathbb{B}, X)$ is said to be spirallike relative to $A$ if its image is invariant under the action of the semigroup $\{e^{-tA}\}_{t\geq 0}$, that is, $e^{-tA}f(x) \in f(\mathbb{B})$ for all $t \geq 0$ and $x \in \mathbb{B}$. The set of all spirallike mappings relative to $A$ is denoted by $\widehat{S}_A(\mathbb{B})$.

If $f$ is spirallike relative to $A = e^{-i\beta}\,\mathrm{Id}$ for some $|\beta| < \frac{\pi}{2}$, then $f$ is said to be spirallike of type $\beta$. In the particular case where $\beta = 0$, spirallike mappings relative $A = \mathrm{Id}$ are called starlike.

The following result is well known (see, for example, Proposition 2.5.3 in [8] and references therein).

**Proposition 2.7.** *Let $A \in L(X)$ be strongly accretive, and let $f \in \mathrm{Hol}(\mathbb{B}, X)$ be a normalized and locally biholomorphic mapping. Then $f \in \widehat{S}_A(\mathbb{B})$ if and only if the mapping $h := (Df)^{-1}Af$ belongs to $\mathcal{N}_A$.*

This proposition *inter alia* implies that a spirallike mapping $f$ relative to $A$ linearizes the semigroup $u(t,x)$ generated by $h = (Df)^{-1}Af$ in the sense that $f \circ u(t, f^{-1}(x)) = e^{-tA}x$ on $f(\mathbb{B})$. In the one-dimensional case, any linear operator is scalar, hence can be chosen to be $A = e^{i\beta}\,\mathrm{Id}$. In this case the inclusion $h = (Df)^{-1}Af \in \mathcal{N}_A$ is equivalent to $\Re\left(e^{-i\beta}\frac{zf'(z)}{f(z)}\right) > 0$. This is the standard definition of spirallike functions of type $\beta$ on $\mathbb{D}$ (see, for example, [5, 15]).

Moreover, according to Proposition 2.7, it is relevant to consider biholomorphic functions $g \in \mathrm{Hol}(\mathbb{D}, \mathbb{C})$ satisfying Assumption 1 and to distinguish subclasses of $\widehat{S}_A(\mathbb{B})$ letting

$$\widehat{S}_g(\mathbb{B}) := \left\{ f \in \widehat{S}_A(\mathbb{B}) : (Df)^{-1}Af \in \mathcal{N}_A(g) \right\}. \tag{2.3}$$

In particular, $\widehat{S}_{g_0}(\mathbb{B}) = \widehat{S}_A(\mathbb{B})$. Further, $\widehat{S}_{g_1^\alpha}(\mathbb{B})$ consists of mappings that are spirallike relative to operator $e^{i\beta}A$ with any $|\beta| < 1-\alpha$. The classes $\widehat{S}_{g_2^\alpha}(\mathbb{B})$ and $\widehat{S}_{g_3^\alpha}(\mathbb{B})$ are also of specific interest. For instance, if $A = e^{i\beta}\,\mathrm{Id}$ and $\alpha = \lambda\cos\beta$, the class $\widehat{S}_{g_3^\alpha}(\mathbb{B})$ of spirallike mappings of type $\beta$ of order $\lambda$ is a widely studied object. The intersection $\widehat{S}_{g_2^\alpha}(\mathbb{B}) \bigcap \widehat{S}_{g_3^\alpha}(\mathbb{B})$ consists of strongly spirallike mappings (for an equivalent definition and properties of these mappings see [17, 18, 3]).

## 3. Auxiliary lemmata

Our first auxiliary result essentially coincides with Theorem 2.12 in [19]. We present it in a somewhat more general form.

**Lemma 3.1.** *Let $p(z) = a + p_1 z + p_2 z^2 + o(z^2)$ and $\phi(z) = a + b_1 z + b_2 z^2 + o(z^2)$ be holomorphic functions on $\mathbb{D}$ such that $\phi \prec p$. Then for every $\mu \in \mathbb{C}$ the following sharp inequality holds:*

$$|b_2 - \mu b_1^2| \le \max\left(|p_1|, |p_2 - \mu p_1^2|\right).$$

*Proof.* Since $\phi \prec p$, there is a function $\omega \in \Omega$ such that $\phi = p \circ \omega$.
Let $\omega(z) = c_1 z + c_2 z^2 + o(z^2)$. Then

$$b_1 = p_1 c_1 \quad \text{and} \quad b_2 = p_2 c_1^2 + p_1 c_2.$$

Therefore

$$b_2 - \mu b_1^2 = (p_2 - \mu p_1^2)c_1^2 + p_1 c_2.$$

Because the inequality $|c_2| \le 1 - |c_1|^2$ holds and is sharp (see, for example, [5]), one concludes that $|b_2 - \mu b_1^2|$ is bounded by a convex hull of $|p_1|$ and $|p_2 - \mu p_1^2|$. The result follows. □

**Lemma 3.2.** *Let $h \in \mathrm{Hol}(\mathbb{B}, X)$ with $h(0) = 0$ and $B \in L(X)$ with $\rho := \|B\| \le 1$. For any $x \in \partial\mathbb{B}$ and $\ell \in X^*$ denote*

$$\varphi(t) := \frac{\ell\left(h(tBx)\right)}{t}, \qquad t \in \mathbb{D} \setminus \{0\}.$$

(i) *The function $\varphi$ can be analytically extended to the disk $\frac{1}{\rho}\mathbb{D}$ with the Taylor expansion $\varphi(t) = b_0 + b_1 t + b_2 t^2 + o(t^2)$, where $b_0 = \ell(Dh(0)Bx)$,*

$$b_1 = \frac{1}{2!}\,\ell\left(D^2h(0)[(Bx)^2]\right) \quad \text{and} \quad b_2 = \frac{1}{3!}\,\ell\left(D^3h(0)[(Bx)^3]\right). \tag{3.1}$$

(ii) *If, in addition, $\ell \in T(Bx)$ and $h \in \mathcal{N}_A(g)$, then $\varphi(\mathbb{D}) \subset \rho\widehat{g}(\rho\mathbb{D})$, where*

$$\widehat{g}(t) = g\left(\frac{\tau - t}{1 - t\overline{\tau}}\right) \ \text{and}\ \tau = g^{-1}\left(\frac{\ell(Dh(0)Bx)}{\|Bx\|}\right).$$

*Proof.* The function $\varphi$ is holomorphic whenever $\|tBx\| < 1$, that is, for $|t| < \frac{1}{\rho} \le \frac{1}{\|Bx\|}$. Represent $h$ by the Taylor series (1.2). A straightforward calculation proves (i). Recall that $h \in \mathcal{N}_A(g)$, hence Definition 2.5 implies $\frac{\varphi(t)}{\|Bx\|} \in g(\mathbb{D}) = \widehat{g}(\mathbb{D})$ as $|t| < \frac{1}{\rho}$. Therefore the function $\widehat{g}^{-1}(\frac{\varphi(\cdot)}{\|Bx\|})$ maps the disk of radius $\frac{1}{\rho}$ into $\mathbb{D}$ and preserves zero. By the Schwarz Lemma $\widehat{g}^{-1}(\frac{\varphi(t)}{\|Bx\|}) \le \rho|t|$. Thus $\varphi \prec \|Bx\|\widehat{g}(\rho\,\cdot)$. The proof is complete. □

A mapping $f \in \mathrm{Hol}(\mathbb{B}, X)$ is said to be of one-dimensional type if it takes the form $f(x) = s(x)x$ for some $s \in \mathrm{Hol}(\mathbb{B}, \mathbb{C})$. Such mappings were studied by many authors (see, for example, [23, 10, 4] and references therein).

**Lemma 3.3.** *Let $f \in \mathrm{Hol}(\mathbb{B}, X)$ be a mapping of one-dimensional type. Then for every $n \in \mathbb{N}$ the entire mapping $x \mapsto D^n f(0)[x^n]$ is also of one-dimensional type. Therefore for any $x \in \partial\mathbb{B}$, $\ell_x \in T(x)$ and constants $\mu_j \in \mathbb{C}$, $j = 1, 2, \ldots,$ we have*

$$\left|\ell_x\left(\sum_{j=1}^{n} \mu_j D^j f(0)[x^j]\right)\right| = \left\|\sum_{j=1}^{n} \mu_j D^n f(0)[x^j]\right\|.$$

*Proof.* The first assertion is evident (for detailed calculation see [7]). To prove the second one we note that there is a function $F \in \mathrm{Hol}(X, \mathbb{C})$ such that

$$\sum_{j=1}^{n} \mu_j D^j f0)[x^j] = F(x)x.$$

Thus for any $x \in \partial\mathbb{B}$ we have

$$\left\|\sum_{j=1}^{n} \mu_j D^j f(0)[x^j]\right\| = |F(x)|\|x\| \qquad \text{and}$$

$$\ell_x\left(\sum_{j=1}^{n} \mu_j D^j f(0)[x^j]\right) = F(x)\ell_x(x) = F(x),$$

which completes the proof. □

## 4. Fekete–Szegö inequalities for spirallike mappings

In what follows $A$ and $g$ satisfy Assumption 1, and the class $\widehat{S}_g(\mathbb{B})$ is defined by formula (2.3).

**Theorem 4.1.** *Let $x \in \partial\mathbb{B}$, $\ell_x \in T(x)$ and $\tau = g^{-1}(\ell_x(Ax))$. Assume that*

$$g\left(\frac{\tau - z}{1 - z\overline{\tau}}\right) = q_0 + q_1 z + q_2 z^2 + o(z^2).$$

*Given $f \in \mathrm{Hol}(\mathbb{B}, X)$ denote*

$$\begin{aligned}
\widetilde{a}_2^2 &= \frac{1}{2}\ell_x\left(D^2 f(0)\left[x, D^2 f(0)[x, Ax]\right] - \frac{1}{2}D^2 f(0)\left[x, AD^2 f(0)[x^2]\right]\right), \\
a_2 &= \frac{1}{2!}\ell_x\left(2D^2 f(0)[x, Ax] - AD^2 f(0)[x^2]\right), \qquad (4.1)\\
a_3 &= \frac{1}{2\cdot 3!}\ell_x\left(3D^3 f(0)[x^2, Ax] - AD^3 f(0)[x^3]\right).
\end{aligned}$$

*If $f \in \widehat{S}_g(\mathbb{B})$, then for any $\nu \in \mathbb{C}$ we have*

$$\left|a_3 - (\nu - 1)a_2^2 - \widetilde{a}_2^2\right| \le \frac{|q_1|}{2}\max\left\{1, \left|\frac{q_2}{q_1} + 2(\nu - 1)q_1\right|\right\}. \qquad (4.2)$$

**Remark 4.2.** It can be directly calculated that $q_1 = -g'(\tau)(1 - |\tau|^2)$ and

$$\frac{q_2}{q_1} = \overline{\tau} - \frac{g''(\tau)}{2g'(\tau)}(1 - |\tau|^2).$$

Thus the right-hand side in (4.2) can be expressed by the hyperbolic and pre-Schwarzian derivatives of $g$.

*Proof.* Let $h(x) = [Df(x)]^{-1} Af(x)$. Recall that $f$ is a normalized biholomorphic mapping. Let the Taylor expansion of $f$ be

$$f(x) = x + \frac{1}{2!}D^2 f(0)[x^2] + \frac{1}{3!}D^3 f(0)[x^3] + o(\|x\|^3), \qquad (4.3)$$

so that

$$Df(x)[w] = w + D^2 f(0)[x, w] + \frac{1}{2}D^3 f(0)[x^2, w] + o(\|x\|^2). \qquad (4.4)$$

Take the Taylor expansion $h(z) = Ax + \frac{1}{2}D^2 h(0)[x^2] + \frac{1}{6}D^3 h(0)[x^3] + o(\|x\|^3)$ and substitute it together with (4.3)–(4.4) into the equality

$$Df(x)[h(x)] = Af(x).$$

This gives us

$$\begin{aligned}
& Ax + \frac{1}{2}D^2 h(0)[x^2] + \frac{1}{6}D^3 h(0)[x^3] + D^2 f(0)[x, Ax] \\
+ & \frac{1}{2}D^2 f(0)[x, D^2 h(0)x^2] + \frac{1}{2}D^3 f(0)[x^2, Ax] + o(\|x\|^3) \\
= & Ax + \frac{1}{2}AD^2 f(0)[x^2] + \frac{1}{6}AD^3 f(0)[x^3] + o(\|x\|^3).
\end{aligned}$$

Equating terms of the same order leads to

$$\frac{1}{2}D^2h(0)[x^2] + D^2f(0)[x, Ax] = \frac{1}{2}AD^2f(0)[x^2]$$

and

$$\frac{1}{6}D^3h(0)[x^3] + \frac{1}{2}D^2f(0)[x, D^2h(0)x^2] + \frac{1}{2}D^3f(0)[x^2, Ax] = \frac{1}{6}AD^3f(0)[x^3].$$

In turn, these equalities imply

$$D^2h(0)[x^2] = AD^2f(0)[x^2] - 2D^2f(0)[x, Ax]$$

and

$$\begin{aligned} &D^3h(0)[x^3] = AD^3f(0)[x^3] - 3D^2f(0)[x, D^2h(0)x^2] - 3D^3f(0)[x^2, Ax] \\ &= AD^3f(0)[x^3] - 3D^3f(0)[x^2, Ax] \\ &-3D^2f(0)\left[x, AD^2f(0)[x^2]\right] + 6D^2f(0)\left[x, D^2f(0)[x, Ax]\right]. \end{aligned}$$

Recall that $\ell_x(Ax) \in V(A) \subset g(\mathbb{D})$, so $\tau \in \Delta$ is well-defined. Similarly to the proof of the Theorem 3.1 in [19], denote

$$\varphi(t) = \begin{cases} \frac{\ell_x(h(tx))}{t}, & t \in \mathbb{D} \setminus \{0\}, \\ \ell_x(Ax), & t = 0. \end{cases}$$

Then $\varphi \in \mathrm{Hol}(\mathbb{D}, \mathbb{C})$ by assertion (i) of Lemma 3.2 with $B = \mathrm{Id}$,

$$b_1 = \frac{1}{2!}\ell_x\left(D^2h(0)[x^2]\right) \quad \text{and} \quad b_2 = \frac{1}{3!}\ell_x\left(D^3h(0)[x^3]\right).$$

Using $a_2$, $\widetilde{a}_2^2$ and $a_3$ defined in (4.1) we get

$$b_1 = -a_2 \quad \text{and} \quad b_2 = 2\widetilde{a}_2^2 - 2a_3.$$

Therefore,

$$\left|a_3 - \widetilde{a}_2^2 - (\nu - 1)a_2^2\right| = \frac{1}{2}\left|b_2 - 2(1 - \nu)b_1^2\right|.$$

Also, by assertion (ii) of the same Lemma 3.2, $\varphi \prec \widehat{g}$, $\widehat{g}(t) = g(\frac{\tau - t}{1 - \overline{\tau}t})$.
To this end we apply Lemma 3.1 with $p = \widehat{g}$ and $\mu = 2(1-\nu)$ and obtain estimate (4.2). □

There are two ways to make the above result more explicit: to fix some concrete forms of the function $g$, or to put additional restrictions on the mapping $f$. We start with some concrete choices of $g$.

Recall that for every strongly accretive operator $A$ and every spirallike mapping $f$ relative to $A$, the mapping $h := (Df)^{-1} Af$ is holomorphically accretive. Hence one can always choose $g = g_0$, where $g_0(z) = \frac{1+z}{1-z}$ is defined above. Denoting $\ell := \ell_x(Ax)$ and using Remark 4.2, we conclude that every spirallike mapping relative to $A$ satisfies

$$\left|a_3 - (\nu - 1)a_2^2 - \widetilde{a}_2^2\right| \le \Re\ell \cdot \max\left(1, |1 + 4(\nu - 1)\Re\ell|\right). \tag{4.5}$$

In the one-dimensional case, this inequality coincides with the result of Theorem 1 in [21] for $\lambda = 0$. By choosing other $g \prec g_0$ functions and denoting $\ell := \ell_x(Ax)$ as above, more precise estimates can be obtained.

Assume, for example, that $\ell_x(h(x))$ belongs to some sector of the form $\left\{w : |\arg w| < \frac{\pi\alpha}{2}\right\}$, $\alpha \in (0,1)$, for all $x \in \mathbb{B}$, where $h = (Df)^{-1} Af$. Then one can set $g = g_1^\alpha$ and to get

**Corollary 4.3.** *Every* $f \in \widehat{S}_{g_1^\alpha}(\mathbb{B})$ *satisfies*

$$\left|a_3 - (\nu-1)a_2^2 - \widetilde{a}_2^2\right| \le \alpha|\ell| \cos\arg \ell^{\frac{1}{\alpha}} \cdot \max\left\{1, Q_{1,\alpha}\right\},$$

*where* $Q_{1,\alpha} = \Re \ell^{\frac{1}{\alpha}} \left| 4\alpha(\nu-1)\ell^{\frac{\alpha-1}{\alpha}} + \frac{1}{\ell^{\frac{1}{\alpha}}} \left(\alpha + i \tan\arg \ell^{\frac{1}{\alpha}}\right)\right|.$

Also assuming that $\frac{\ell_x(h(x))}{\|x\|}$ is bounded away from the imaginary axis, namely,

$$\Re \frac{\ell_x(h(x))}{\|x\|} > \alpha, \ \alpha \in (0,1),$$

we choose $g = g_2^\alpha$. In this situation, we have

**Corollary 4.4.** *Every* $f \in \widehat{S}_{g_2^\alpha}(\mathbb{B})$ *satisfies*

$$\left|a_3 - (\nu-1)a_2^2 - \widetilde{a}_2^2\right| \le \Re\ell \cdot \max\left\{1, Q_{2,\alpha}\right\},$$

*where* $Q_{2,\alpha} = \left|1 + 4(\nu-1)(1-\alpha)\Re\ell\right|.$

In particular, taking $\alpha = 0$, we return to inequality (4.5) for all spirallike mappings relative to the linear operator $A$.

Another interesting (and, as we mentioned, dual) case occurs when $\frac{\ell_x(h(x))}{\|x\|}$ lies in some circle tangent to the imaginary axis. We can then set $g = g_3^\alpha$.

**Corollary 4.5.** *Every* $f \in \widehat{S}_{g_3^\alpha}(\mathbb{B})$ *satisfies*

$$\left|a_3 - (\nu-1)a_2^2 - \widetilde{a}_2^2\right| \le (\Re\ell - |\ell|^2\alpha) \cdot \max\left\{1, Q_{3,\alpha}\right\},$$

*where* $Q_{3,\alpha} = \left|1 - 2\overline{\ell}\alpha + 4(\nu-1)(\Re\ell - |\ell|^2\alpha)\right|.$

Recall that for $A = e^{i\beta}\,\mathrm{Id}$, the class $\widehat{S}_{g_3^\alpha}(\mathbb{B})$ consists of so-called spirallike mappings of type $\beta$ of order $\alpha$.

**Remark 4.6.** It is worth mentioning that even for the the case in which $A$ is a scalar operator, the estimates above (starting from (4.5)) are new. Since the class of spirallike mappings contains the class of starlike mappings, these estimates generalize Corollary 3.4 (i)–(iv) in [19] for starlike mappings.

In the rest of this section we deal with mappings $f$ that satisfy:

**Assumption 2.** There exists a function $\kappa : \partial\mathbb{B} \to \mathbb{C}$ such that

$$D^2 f(0)[x^2] = \kappa(x)x, \quad x \in \partial\mathbb{B}. \tag{4.6}$$

The Fréchet derivatives of $f$ of second and third order $D^2 f(0)$ and $D^3 f(0)$ commute with the linear operator $A$ in the sense that

$$D^k f(0)[x^{k-1}, Ax] = AD^k f(0)[x^k], \quad k = 2, 3. \tag{4.7}$$

Condition (4.6) holds automatically for one-dimensional type mappings (spirallike mappings of one-dimensional type were studied, for instance, in [10, 22, 7]), while condition (4.7) holds automatically whenever $A$ is a scalar operator.

In turn, relations (4.7) in Assumption 2 imply that formulae (4.1) become

$$\begin{aligned} a_2 &= \frac{1}{2!}\ell_x\left(AD^2f(0)[x^2]\right), \\ \widetilde{a}_2^2 &= \frac{1}{4}\ell_x\left(AD^2f(0)[x, D^2f(0)[x^2]]\right), \\ a_3 &= \frac{1}{3!}\ell_x\left(AD^3f(0)[x^3]\right). \end{aligned} \tag{4.8}$$

**Corollary 4.7.** *If $f \in \widehat{S}_A(\mathbb{B})$ satisfies Assumption 2, then for any $\nu \in \mathbb{C}$,*

$$\left|a_3 - \left(\nu - 1 + \frac{1}{\ell_x(Ax)}\right)a_2^2\right| \le \frac{|q_1|}{2}\max\left\{1, \left|\frac{q_2}{q_1} + 2(\nu-1)q_1\right|\right\}. \tag{4.9}$$

*Proof.* Indeed, denote $\alpha = \ell_x(Ax)$.Then $a_2 = \frac{1}{2}\kappa(x)\alpha$ and

$$\begin{aligned} \widetilde{a}_2^2 &= \frac{1}{4}\ell_x\left(AD^2f(0)[x, \kappa(x)x]\right) = \frac{1}{4}\cdot\kappa(x)\ell_x\left(AD^2f(0)[x^2]\right) \\ &= \frac{1}{4}\cdot\kappa(x)\ell_x\left(A\kappa(x)x\right) = \frac{\alpha}{4}\cdot(\kappa(x))^2. \end{aligned}$$

Thus $\widetilde{a}_2^2 = \frac{1}{\alpha}a_2^2$ and hence

$$|a_3 - (\nu-1)a_2^2 - \widetilde{a}_2^2| = \left|a_3 - \left(\nu - 1 + \frac{1}{\alpha}\right)a_2^2\right|.$$

So, estimate (4.9) follows from Theorem 4.1. □

Let $A$ be a scalar operator. Without loss of generality, we assume $A = e^{i\beta}\,\mathrm{Id}$, $|\beta| < \frac{\pi}{2}$. Then it follows from Assumption 2 that formulae (4.1) (or (4.8)) become

$$a_2 = \frac{1}{2!}\kappa(x)e^{i\beta} \quad \widetilde{a}_2^2 = \left(\frac{1}{2!}\kappa(x)\right)^2 e^{i\beta}, \quad a_3 = \frac{1}{3!}\ell_x\left(D^3f(0)[x^3]\right)e^{i\beta}.$$

These relations and Lemma 3.3 imply immediately

**Corollary 4.8.** *If $f \in \mathrm{Hol}(\mathbb{B}, X)$ is a spirallike mapping of type $\beta$, that satisfies Assumption 2. Then for any $\mu \in \mathbb{C}$ we have*

$$\left|a_3 - \mu a_2^2\right| \le \frac{|q_1|}{2}\max\left\{1, \left|\frac{q_2}{q_1} + 2(\mu - e^{-i\beta})q_1\right|\right\}.$$

*If, in addition, $f$ is of one-dimensional type, then for any $x \in \partial\mathbb{B}$ we have*

$$\begin{aligned} &\left\|\frac{1}{3!}D^3f(0)[x^3] - \mu\cdot\frac{1}{2!}D^2f(0)\left[x, \frac{1}{2!}D^2f(0)[x^2]\right]\right\| \\ &\qquad \le \frac{|q_1|}{2}\max\left\{1, \left|\frac{q_2}{q_1} + 2(\mu - e^{-i\beta})q_1\right|\right\}. \end{aligned}$$

The last estimate coincides with Theorem 2 in [7].

## 5. Fekete–Szegö inequalities for normalized non-linear resolvents

As above, we suppose that $A \in L(X)$ and $g \in \mathrm{Hol}(\mathbb{D}, \mathbb{C})$ satisfy Assumption 1 and $h \in \mathcal{N}_A(g)$. In this section we concentrate on the non-linear resolvent

$$J_r := (\mathrm{Id} + rh)^{-1}, \ r > 0,$$

that is well-defined self-mappings of the open unit ball $\mathbb{B}$ that solves the functional equation

$$J_r(x) + rh(J_r(x)) = x \in \mathbb{B}, \quad r > 0. \tag{5.1}$$

**Lemma 5.1.**

(a) *For any $r > 0$, the operator $B_r := DJ_r(0) = (\mathrm{Id} + rA)^{-1}$ is strongly contractive, that is, $\rho_r := \|B_r\| < 1$.*

(b) *If $h$ is of one-dimensional type, then $A$ is a scalar operator and $J_r$, $r > 0$, is of one-dimensional type too.*

*Proof.* Assertion (a) follows from the strong accretivity of $A$.
Since $h$ is of one-dimensional type, it has the form $h(x) = s(x)x$, where $s \in \mathrm{Hol}(\mathbb{B}, \mathbb{C})$. Therefore $A = Dh(0) = s(0)\,\mathrm{Id}$. In addition, (5.1) implies

$$x = J_r(x) + rs(J_r(x))J_r(x) = (1 + rs(J_r(x)))J_r(x),$$

that is, $J_r(x)$ is collinear to $x$. □

Further, it is natural to consider the family of normalized resolvents $(\mathrm{Id} + rA)J_r$ and to study the Fekete–Szegö problem for these mappings.

We now present the main result of this section.

**Theorem 5.2.** *Let $h \in \mathcal{N}_A(g)$ and $J_r$ be the nonlinear resolvent of $h$ for some $r > 0$. For $x \in \partial\mathbb{B}$ and $\ell_r := \ell_{B_r x} \in T(B_r x)$, let*

$$\begin{aligned}
\widetilde{a}_2^2 &:= \ell_r \left( (\mathrm{Id} + rA) \frac{1}{2!} D^2 J_r(0) \left[ x, (\mathrm{Id} + rA) \frac{1}{2!} D^2 J_r(0)[x^2] \right] \right), \\
a_2 &:= \ell_r \left( (\mathrm{Id} + rA) \frac{1}{2!} D^2 J_r(0)[x^2] \right), \\
a_3 &:= \ell_r \left( (\mathrm{Id} + rA) \frac{1}{3!} D^3 J_r(0)[x^3] \right).
\end{aligned} \tag{5.2}$$

*Then for any $\nu \in \mathbb{C}$ we have*

$$\left| a_3 - 2\widetilde{a}_2^2 - (\nu - 2)a_2^2 \right| \le r|q_1| \|B_r x\| \rho_r^2 \max\left(1, Q_r(x)\right), \tag{5.3}$$

*where*

$$Q_r(x) := \left| \frac{q_2}{q_1} - (2 - \nu) r q_1 \|B_r x\| \right| \tag{5.4}$$

*and $q_1, q_2$ are the Taylor coefficients of $\widehat{g}(t) = g\left(\frac{\tau - t}{1 - t\overline{\tau}}\right)$ with $\tau = g^{-1}\left(\frac{\ell_r(AB_r x)}{\|B_r x\|}\right)$.*

*Proof.* Denote $x_r := B_r x$. Using the functional equation (5.1), one finds

$$(I + rA)\, D^2 J_r(0)[x, y] = -r D^2 h(0)\, [x_r, B_r y]$$

and

$$\begin{aligned} (\mathrm{Id} + rA) \frac{1}{2!} D^2 J_r(0)[x^2] &= -r \frac{1}{2!} D^2 h(0) \left[(x_r)^2\right], \\ (\mathrm{Id} + rA) \frac{1}{3!} D^3 J_r(0)[x^3] &= -r \frac{1}{3!} B_r D^3 h(0) \left[(x_r)^3\right] \\ &+ 2r^2 \cdot \frac{1}{2!} B_r D^2 h(0) \left[x_r, B_r \frac{1}{2!} D^2 h(0) \left[(x_r)^2\right]\right]. \end{aligned}$$

Thus the quantities $a_2, \widetilde{a}_2^2$ and $a_3$ defined by (5.2) can be expressed by the Fréchet derivatives of $h$:

$$\begin{aligned} \widetilde{a}_2^2 &= r^2 \frac{1}{2!} \ell_r \left( D^2 h(0) \left[x_r, \frac{1}{2!} B_r D^2 h(0) \left[(x_r)^2\right]\right]\right) \\ a_2 &= -r \frac{1}{2!} \ell_r \left( D^2 h(0) \left[(x_r)^2\right]\right) \\ a_3 &= -r \frac{1}{3!} \ell_r \left( D^3 h(0) \left[(x_r)^3\right]\right) \\ &+ 2r^2 \ell_r \left( \frac{1}{2!} D^2 h(0) \left[x_r, \frac{1}{2!} B_r D^2 h(0) \left[(x_r)^2\right]\right]\right). \end{aligned} \tag{5.5}$$

Denote

$$\varphi(t) = \begin{cases} \frac{\ell_r(h(t x_r))}{t}, & t \in \mathbb{D} \setminus \{0\}, \\ \ell_r(A x_r), & t = 0. \end{cases}$$

By assertion (i) of Lemma 3.2 with $B = B_r$, the function $\varphi$ is analytic in the disk of radius $\frac{1}{\rho_r}$ and

$$b_1 = \frac{1}{2!} \cdot \ell_r \left( D^2 h(0)[(x_r)^2]\right) \quad \text{and} \quad b_2 = \frac{1}{3!} \cdot \ell_r \left( D^3 h(0)[(x_r)^3]\right). \tag{5.6}$$

Comparing formulae (5.6) and (5.5) we see that

$$b_1 = -\frac{1}{r} a_2 \qquad \text{and} \qquad b_2 = -\frac{1}{r}(a_3 - 2\widetilde{a}_2^2).$$

Therefore,

$$\left|a_3 - \widetilde{a}_2^2 - (\nu - 2) a_2^2\right| = r \left|b_2 - r(2 - \nu) b_1^2\right|.$$

Also, by assertion (ii) of Lemma 3.2, $\varphi \prec \|x_r\| \widehat{g}(\rho_r \cdot)$.
To complete the proof we apply Lemma 3.1 with $p = \|x_r\| \widehat{g}(\rho_r \cdot)$ and $\mu = r(2 - \nu)$. □

From now on, for any $x \in \partial\mathbb{B}$ we will adopt the notations $x_r = B_r x$ and $\ell_r := \ell_{x_r} \in T(x_r)$. To compare our results with the previous ones we consider some special cases. If, for example, $A = \lambda\,\mathrm{Id}$, $\Re\lambda > 0$, is a scalar operator, then $B_r = \frac{1}{1+\lambda r}\,\mathrm{Id}$, $x_r = \frac{1}{1+\lambda r} x$ and $\rho_r = \|x_r\| = \frac{1}{|1+\lambda r|}$. Thus

$$\tau = g^{-1} \left( \frac{\ell_r(\lambda x_r)}{\|x_r\|} \right) = g^{-1}(\lambda). \tag{5.7}$$

Thus inequality (5.3) takes the form

$$\left|a_3 - 2\widetilde{a}_2^2 - (\nu-2)a_2^2\right| \le \frac{|q_1|r}{|1+\lambda r|^3} \max\left(1, \left|\frac{q_2}{q_1} - \frac{q_1 r}{|1+\lambda r|}(2-\nu)\right|\right), \tag{5.8}$$

where $q_1, q_2$ are the Taylor coefficients of $\widehat{g}(t) = g\left(\frac{\tau - t}{1-t\overline{\tau}}\right)$ with $\tau = g^{-1}(\lambda)$.

**Corollary 5.3.** *Assume that* $A = \lambda\,\mathrm{Id}$, $\Re\lambda > 0$ *and* $g = g_0$. *Then for any* $\nu \in \mathbb{C}$ *we have*

$$\left|a_3 - 2\widetilde{a}_2^2 - (\nu-2)a_2^2\right| \le \frac{|1+\lambda^2|r}{|1+\lambda r|^3} \max\left(1, \left|\lambda - (2-\nu)r\frac{1+\lambda^2}{|1+\lambda r|}\right|\right). \tag{5.9}$$

*Proof.* Since $g = g_0$, formula (5.7) is $\tau = g^{-1}(\lambda) = \frac{\lambda-1}{\lambda+1}$. Thus $q_1 = -(1+\lambda^2)$ and $q_2 = \lambda(1+\lambda^2)$. Then (5.9) follows from (5.8). □

For $A = \mathrm{Id}$, Corollary 5.3 coincides with [19, Theorem 5.6].

Another interesting case occurs when $h$ satisfies Assumption 2.

**Corollary 5.4.** *If* $h \in \mathcal{N}_A(g)$ *satisfies Assumption* 2, *then*

$$\left|a_3 - (\nu - 2 + 2\delta)a_2^2\right| \le r|q_1|\|x_r\|\rho_r^2 \max\left(1, Q_r(x)\right), \tag{5.10}$$

*where* $Q_r(x)$ *is defined by* (5.4) *and* $\delta = \frac{\ell_r(B_r x_r)}{\|x_r\|^2}$.

*Proof.* Since $h$ satisfies condition (4.6), there exists a function $\kappa : \partial\mathbb{B} \to \mathbb{C}$ such that $D^2h(0)[x^2] = \kappa(x)x$, $x \in \partial\mathbb{B}$. Thus,

$$\begin{aligned} a_2^2 &= \frac{r^2}{4}\left(\ell_r\left(D^2h(0)\left[(x_r)^2\right]\right)\right)^2 = \frac{r^2}{4}\left(\ell_r\left(\kappa(x_r)x_r\right)\right)^2 \\ &= \left(\frac{r}{2}\kappa(x_r)\right)^2 \|x_r\|^2. \end{aligned}$$

At the same time,

$$\begin{aligned} \widetilde{a}_2^2 &= r^2\frac{1}{2!}\ell_r\left(D^2h(0)\left[x_r, \frac{1}{2!}B_r D^2h(0)\left[(x_r)^2\right]\right]\right) \\ &= r^2\frac{1}{4}\ell_r\left(D^2h(0)\left[x_r, B_r\kappa(x_r)x_r\right]\right) \\ &= r^2\frac{1}{4}\kappa(x_r)\ell_r\left(D^2h(0)\left[x_r, B_r x_r\right]\right). \end{aligned}$$

The mapping $h$ also satisfies condition (4.7), then

$$\begin{aligned} \widetilde{a}_2^2 &= r^2\frac{1}{4}\kappa(x_r)\ell_r\left(B_r D^2h(0)\left[(x_r)^2\right]\right) \\ &= r^2\frac{1}{4}\kappa(x_r)\ell_r\left(B_r\kappa(x_r)x_r\right) = \left(\frac{r}{2}\kappa(x_r)\right)^2 \ell_r\left(B_r x_r\right). \end{aligned}$$

Now estimate (5.10) follows from the relation $\widetilde{a}_2^2 = \delta a_2^2$ with $\delta = \frac{\ell_r(B_r x_r)}{\|x_r\|^2}$. □

If $h$ is of a one-dimensional type, then $A = \lambda\,\mathrm{Id}$ for some $\lambda \in \mathbb{C}$ by Lemma 5.1. In this case formula (5.10) gets a simpler form.

**Corollary 5.5.** *If $h \in \mathcal{N}_A(g)$ is one-dimensional type with $A = \lambda \operatorname{Id}$, then for any $\nu \in \mathbb{C}$ we have*

$$\left\| (\operatorname{Id} + rA)\frac{1}{3!}D^3J_r(0)[x^3] - \mu(\operatorname{Id} + rA)\frac{1}{2!}D^2J_r(0)\left[x, (\operatorname{Id} + rA)\frac{1}{2!}D^2J_r(0)[x^2]\right]\right\|$$
$$= \left|a_3 - \mu a_2^2\right| \le \frac{r|q_1|}{|1+\lambda r|^3} \cdot \max\left(1, \left|\frac{q_2}{q_1} - (2\delta - \mu)\frac{rq_1}{|1+\lambda r|}\right|\right)$$

*with* $\delta = \frac{|1+\lambda r|}{1+\lambda r}$.

In particular, if $A = \operatorname{Id}$ and $g = g_0$, this coincides with [19, Corollary 5.7].

*Proof.* By Lemma 3.3, there is a function $\kappa$ such that $\frac{1+\lambda r}{2!}D^2J_r(0)[x^2] = \kappa(x)x$. Then the left-hand term equals to

$$\left\| \frac{1+r\lambda}{3!}D^3J_r(0)[x^3] - \mu\frac{1+r\lambda}{2!}\kappa(x)D^2J_r(0)[x^2]\right\|.$$

Lemma 3.3 states that this is equal to

$$\left|\ell_x\left(\frac{1+r\lambda}{3!}D^3J_r(0)[x^3] - \mu\frac{1+r\lambda}{2!}\kappa(x)D^2J_r(0)[x^2]\right)\right|$$
$$= \quad |a_3 - \mu a_2\kappa(x)| = \left|a_3 - \mu a_2^2\right|.$$

Set $\mu = \nu - 2 + 2\delta$. Then we proceed by Corollary 5.4:

$$\le \quad \frac{r|q_1|}{|1+\lambda r|^3} \cdot \max\left(1, \left|\frac{q_2}{q_1} - (2 - \nu)\frac{rq_1}{|1+\lambda r|}\right|\right)$$
$$= \quad \frac{r|q_1|}{|1+\lambda r|^3} \cdot \max\left(1, \left|\frac{q_2}{q_1} - (2\delta - \mu)\frac{rq_1}{|1+\lambda r|}\right|\right).$$

□

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# A note on Bloch functions

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*Dedicated to the memory of Professor Gabriela Kohr*

**Abstract.** We construct a natural linear isomorphism between the little Bloch space $\mathcal{B}_0$ and the Banach space $c_0$ of complex null sequences. This paper is written for the special issue of Studia Universitatis Babeş-Bolyai Mathematica in memory of Professor Gabriela Kohr.

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## 1. Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the complex open unit disc. A holomorphic function $f : \mathbb{D} \longrightarrow \mathbb{C}$ satisfying

$$|f|_{\mathcal{B}} := \sup_{z\in\mathbb{D}}(1 - |z|^2)|f'(z)| < \infty$$

is known as a *Bloch function*, where $|\cdot|_{\mathcal{B}}$ is called the *Bloch semi-norm*. Obviously, bounded holomorphic functions on $\mathbb{D}$, complex polynomials in particular, are Bloch functions, but also, unbounded Bloch functions abound. With the usual addition and scalar multiplication, the Bloch functions on $\mathbb{D}$ form a Banach space $\mathcal{B}$, called the *Bloch space*, in the *Bloch norm* $\|\cdot\|_{\mathcal{B}}$ defined by

$$\|f\|_{\mathcal{B}} = |f(0)| + |f|_{\mathcal{B}} \qquad (f \in \mathcal{B}).$$

The following subspace

$$\mathcal{B}_0 := \{f \in \mathcal{B} : \lim_{|z|\to 1}(1 - |z|^2)|f'(z)| = 0\}$$

of $\mathcal{B}$ is often called the *little Bloch space*.

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It is well-known that $\mathcal{B}_0$ is linearly isomorphic to the Banach space $c_0$ of complex null sequences, and a common recourse of its proof is a result in [7, Theorem 7] asserting that $c_0$ is linearly isomorphic to the Banach space

$$h_0=\{h : h \text{ is complex harmonic on } \mathbb{D}, \sup_{|z|<1}(1-|z|^2)|h(z)|<\infty, \lim_{|z|\to 1}(1-|z|^2)|h(z)|=0\}$$

which is equipped with the norm

$$\|h\| = \sup_{|z|<1}(1-|z|^2)|h(z)| \qquad (h \in h_0).$$

This result implies that $\mathcal{B}_0$ is linearly isomorphic to $c_0$ since $\mathcal{B}_0$ is linearly isomorphic to a complemented subspace of $h_0$ and by [6], every infinite dimensional complemented subspace of $c_0$ is linearly isomorphic to $c_0$.

However, besides invoking [6] in this proof, the isomorphism between $h_0$ and $c_0$ in [7] is obtained from a composition of mappings on various Banach spaces, involving a series of non-trivial lemmas. In this note, we show directly that $\mathcal{B}_0$ is linearly isomorphic to $c_0$ by exhibiting an explicit linear isomorphism between them.

## 2. Isomorphism of Bloch space

We first explain why $\mathcal{B}_0$ is linearly isomorphic to a complemented subspace of $h_0$. A function $h : \mathbb{D} \longrightarrow \mathbb{C}$ is called *complex harmonic* if its real and imaginary parts are both real harmonic functions. Such a function can be written as $h = f + \overline{g}$, where $f$ and $g$ are holomorphic functions, and the symbol '$-$' denotes the complex conjugation. Plainly, holomorphic functions are complex harmonic. Let

$$\mathcal{A}_0 = \{h : h \text{ is holomorphic on } \mathbb{D}, \sup_{|z|<1}(1-|z|^2)|h(z)| < \infty, \lim_{|z|\to 1}(1-|z|^2)|h(z)| = 0\}$$

which forms a complex Banach space with the norm

$$\|h\|_{\mathcal{A}_0} = \sup_{|z|<1}(1-|z|^2)|h(z)| \qquad (h \in \mathcal{A}_0)$$

and the preceding remark implies that $\mathcal{A}_0$ is a complemented subspace of $h_0$. On the other hand, the map

$$f \in \mathcal{B}_0 \mapsto f' \in \mathcal{A}_0 \tag{2.1}$$

is a linearly isometry and therefore $\mathcal{B}_0$ is isomorphic to a complemented subspace of $h_0$.

In view of (2.1), to construct a linear isomorphism between $\mathcal{B}_0$ and $c_0$, it suffices to build one between $\mathcal{A}_0$ and $c_0$.

We will denote the elements in $c_0$ by bold letters such as

$$\boldsymbol{a} = (a_0, a_1, a_2, \ldots) \in c_0$$

and make use of the fact that $\mathcal{B}_0$ is the $\|\cdot\|_{\mathcal{B}}$-closure of polynomials in $\mathcal{B}$. Further, if $f(x) = \sum_{k=0}^{\infty} b_k z^k$ belongs to $\mathcal{B}_0$, then $\lim_{k\to\infty} b_k = 0$, by a remark following [1, Lemma 3.1].

Given a sequence $(f_n)$ in $\mathcal{B}_0$ converging to $f \in \mathcal{B}$ (in the Bloch norm), we have

$$\text{(i) } (f_n) \text{ converges to } f \text{ locally uniformly on } \mathbb{D}, \tag{2.2}$$

$$\text{(ii) } \lim_{|z|\to 1}(1-|z|^2)|f_n'(z)| = 0, \text{ uniformly in } n \tag{2.3}$$

(cf. [1, p.14]).

The norm of each $\boldsymbol{a} = (a_k) \in c_0$ is given by $\|\boldsymbol{a}\|_{c_0} = \sup_k |a_k|$. Let $c_{00}$ be the subspace of $c_0$, consisting of elements $\boldsymbol{a} = (a_k)$ with $a_k = 0$ except a finite number of indices $k$.

**Lemma 2.1.** *The linear map* $\varphi : c_{00} \longrightarrow \mathcal{A}_0$ *defined by*

$$\varphi(\boldsymbol{a})(z) = \sum_k a_k z^k \qquad (z \in \mathbb{D}, \boldsymbol{a} = (a_k) \in c_{00})$$

*is continuous.*

*Proof.* We have

$$\|\varphi(\boldsymbol{a})\|_{\mathcal{A}_0} = \sup\left\{(1-|z|^2)\left|\sum_k a_k z^k\right| : |z| < 1\right\}$$

where

$$\left|\sum_k a_k z^k\right| \leq (\sup_k |a_k|)(1+|z|+|z|^2+\cdots) = \frac{\|\boldsymbol{a}\|_{c_0}}{1-|z|}$$

and hence

$$\|\varphi(\boldsymbol{a})\|_{\mathcal{A}_0} \leq \sup\{(1+|z|)\|\boldsymbol{a}\|_{c_0} : |z| < 1\} \leq 2\|\boldsymbol{a}\|_{c_0}. \qquad \square$$

Since $c_{00}$ is dense in $c_0$, the map $\varphi$ in Lemma 2.1 extends to a continuous linear map, *still denoted by* $\varphi$, from $c_0$ to $\mathcal{A}_0$. We show that this map is actually a linear isomorphism.

**Theorem 2.2.** *The extension* $\varphi : c_0 \longrightarrow \mathcal{A}_0$ *of the map in Lemma* 2.1 *is a linear homeomorphism.*

*Proof.* We begin by showing that $\varphi$ is injective. Let $\boldsymbol{a} \in c_0$ and $\varphi(\boldsymbol{a}) = 0$. We show $\boldsymbol{a} = 0$. By definition of the map $\varphi$, there is a sequence $(\boldsymbol{a}_n)$ in $c_{00}$ norm converging to $\boldsymbol{a}$ such that $\lim_n \varphi(\boldsymbol{a}_n) = 0$ in $\mathcal{A}_0$, where

$$\boldsymbol{a}_n = (a_{nk}) = (a_{n0}, a_{n1}, \ldots, a_{nk}, \ldots).$$

By virtue of (2.1) and (2.2), the sequence $\varphi(\boldsymbol{a}_n)$ of functions converges to 0 locally uniformly on $\mathbb{D}$, where

$$\varphi(\boldsymbol{a}_n)(z) = \sum_k a_{nk} z^k \qquad (z \in \mathbb{D}).$$

For $k = 0, 1, 2, \ldots$, the $k$-th derivative $\varphi(\boldsymbol{a}_n)^{(k)}$ converges to 0 locally uniformly, as $n \to \infty$. It follows that

$$k!|a_{nk}| = |\varphi(\boldsymbol{a}_n)^{(k)}(0)| \leq \sup\{|\varphi(\boldsymbol{a}_n)^{(k)}(z)| : |z| \leq 1/2\} \to 0$$

as $n \to \infty$. Given $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $n \geq n_0$ implies

$$\sup\{|\varphi(\boldsymbol{a}_n)^{(k)}(z)| : |z| \leq 1/2\} < k!\varepsilon$$

and hence $|a_{nk}| < \varepsilon$ for $n \geq n_0$. Therefore we have

(i) $\lim_n a_{nk} = 0$ for each $k$,
(ii) $\lim_n \lim_k a_{nk} = 0$ since $\boldsymbol{a}_n \in c_{00}$.

By [5, IV.13.10], the sequence $(\boldsymbol{a}_n)$ converges weakly to 0 in $c_0$ and hence $\boldsymbol{a} = 0$.

Finally, we show that $\varphi$ is surjective. Let $f \in \mathcal{A}_0$. By (2.1), there is a sequence $(p_n)$ of polynomials such that $\|p_n - f\|_{\mathcal{A}_0} \to 0$ as $n \to \infty$. Write

$$p_n(z) = \sum_k a_{nk} z^k.$$

Then $p_n = \varphi(\boldsymbol{a}_n)$ where $\boldsymbol{a}_n = (a_{nk}) \in c_{00}$.

As before, $(p_n - f)$ converges locally uniformly to 0 on $\mathbb{D}$, and we have

$$\sup\{|(p_m - p_n)'(z)| : |z| \leq 1/2\} \leq 2 \sup\{|(p_m - p_n)(z)| : |z| \leq 1/2\}$$

from the Cauchy formula. Iterating this inequality yields

$$\begin{aligned} k!|a_{mk} - a_{nk}| &= |(p_m - p_n)^{(k)}(0)| \leq \sup\{|(p_m - p_n)^{(k)}(z)| : |z| \leq 1/2\} \\ &\leq 2^k \sup\{|(p_m - p_n)(z)| : |z| \leq 1/2\} \to 0 \quad (k = 0, 1, 2, \dots) \end{aligned}$$

as $m, n \to \infty$. It follows that the sequence $(a_{nk})_{n=1}^{\infty}$ converges to some $a_k \in \mathbb{C}$ for each $k$, and for some $m_0 \in \mathbb{N}$ and for all $k$, we have

$$|a_{m_0 k} - a_{nk}| \leq \frac{2^k}{k!} \quad \text{whenever} \quad n \geq m_0.$$

Since $(a_{m_0 k}) \in c_{00}$, there is some $k_0$ such that $a_{m_0 k} = 0$ for $k \geq k_0$, which gives

$$|a_{nk}| \leq 2^k / k!$$

for $n \geq m_0$ and $k \geq k_0$, Hence $|a_k| \leq 2^k/k!$ for $k \geq k_0$ and $\lim_k a_k = 0$.

By [5, IV.13.10] again, the following properties

(i) $\lim_n a_{nk} = a_k$ for each $k$,
(ii) $\lim_n \lim_k a_{nk} = 0 = \lim_k a_k$

imply that $(\boldsymbol{a}_n)$ converges weakly to $\boldsymbol{a} = (a_k)$ in $c_0$. Since $\varphi$ is weakly continuous, the sequence $\varphi(\boldsymbol{a}_n)$ converges weakly to $\varphi(\boldsymbol{a})$ in $\mathcal{A}_0$. On the other hand, $\varphi(\boldsymbol{a}_n)$ norm converges $f$ in $\mathcal{A}_0$ and hence $\varphi(\boldsymbol{a}) = f$. This proves surjectivity of $\varphi$.

By the open mapping theorem, the map $\varphi : c_0 \longrightarrow \mathcal{A}_0$ is a linear homeomorphism which completes the proof. □

It has been shown in [1] that the second dual space $\mathcal{B}_0^{**}$ is linearly isomorphic to $\mathcal{B}$. It follows that $\mathcal{B}$ is linearly isomorphic to the Banach space $\ell_\infty$ of bounded complex sequences.

## 3. Bloch functions of several complex variables

The concept of a Bloch function has been extended to higher and infinite dimensions by several authors. We refer to [2] for references of these extensions. The various definitions of Bloch functions on bounded symmetric domains in these references are all equivalent to the one given in [3] and below.

We recall that a *bounded symmetric domain* is a bounded domain $D$ in a complex Banach space $V$ such that each point $p \in D$ admits a (unique) symmetry $s_p : D \longrightarrow D$ which, by definition, is a biholomorphic map such that $p$ is an isolated fixed-point of $s_p$ and $s_p \circ s_p$ is the identity map on $D$. Further details of infinite dimensional bounded symmetric domains including their realisation as the open unit ball of a complex Banach space with a Jordan structure, alias JB*-triple, can be found in [2].

**Definition 3.1.** Let $D$ be a bounded symmetric domain realised as the open unit ball of a JB*-triple $V$ and let $\operatorname{Aut} D$ be the automorphism group of $D$, consisting of biholomorphisms of $D$. The *Bloch semi-norm* of a holomorphic map $f : D \longrightarrow \mathbb{C}^d$ is defined by

$$|f|_{\mathcal{B}} = \sup\{\|(f \circ g)'(0)\| : g \in \operatorname{Aut} D\}$$

where $d \in \mathbb{N}$ and $\mathbb{C}^d$ is equipped with the Euclidean norm. We call $f$ a *Bloch map* if $|f|_{\mathcal{B}} < \infty$. A Bloch map $f : D \longrightarrow \mathbb{C}$ is often called a *Bloch function.*

We note that on the unit disc $\mathbb{D}$, the two definitions of the Bloch semi-norm $|\cdot|_{\mathcal{B}}$ given previously coincide, that is,

$$\sup_{z\in\mathbb{D}}(1 - |z|^2)|f'(z)| = \sup\{|(f \circ g)'(0)| : g \in \operatorname{Aut} \mathbb{D}\}.$$

On higher dimensional domains $D$, however, they are not equal, even on the bidisc, although we always have

$$\sup_{z\in D}(1 - \|z\|^2)\|f'(z)\| \le \sup\{\|(f \circ g)'(0)\| : g \in \operatorname{Aut} D\}.$$

The following example has been given in [4].

**Example 3.2.** Let $f : \mathbb{D} \times \mathbb{D} \longrightarrow \mathbb{C}$ be defined by

$$f(z_1, z_2) = (1 - z_2) \log \frac{1}{1 - z_1}, \qquad (z_1, z_2) \in \mathbb{D} \times \mathbb{D}.$$

Then we have

$$\sup_{(z_1,z_2)\in\mathbb{D}\times\mathbb{D}} (1 - \|(z_1, z_2)\|^2)\|f'(z_1, z_2)\| < \infty$$

where $\|(z_1, z_2)\| = \max\{|z_1|, |z_2|\}$, but in contrast

$$\sup\{\|(f \circ g)'(0)\| : g \in \operatorname{Aut}(\mathbb{D} \times \mathbb{D})\} = \infty.$$

As in the one dimensional case, the Bloch functions on $D$ form a Banach space $\mathcal{B}(D)$ in the following Bloch norm:

$$\|f\|_{\mathcal{B}} = \|f(0)\| + |f|_{\mathcal{B}} \qquad (f \in \mathcal{B}(D)).$$

One can also define the *little Bloch space* $\mathcal{B}_0(D)$ as the closure of the polynomials in $\mathcal{B}(D)$ and likewise, we have

$$\mathcal{B}_0(D) = \{f \in \mathcal{B}(D) : \lim_{\|z\|\to 1}(1 - \|z\|^2)\|f'(z)\| = 0\}$$

if $D$ is the open unit ball of a Hilbert space $V$ (cf. [2, Theorem 4.3.11]). While it is known that the little Bloch space $\mathcal{B}_0(B_d)$ of a $d$-dimensional Euclidean ball $B_d \subset \mathbb{C}^d$ is linearly isomorphic to $c_0$, as in the case of $\mathbb{D}$ by similar arguments, the little Bloch space $\mathcal{B}_0(B)$ of the open unit ball $B$ of a non-separable Hilbert space is not linearly isomorphic to $c_0$ since $\mathcal{B}_0(B)$ is not separable.

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# Janowski subclasses of starlike mappings

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*Dedicated to the memory of Professor Gabriela Kohr*

**Abstract.** In this paper, two subclasses of biholomorphic starlike mappings named Janowski starlike and Janowski almost starlike with complex parameters are introduced and studied. We determine $M$ such that holomorphic mappings $f$ which satisfy the condition $\|Df(z) - I\| \leq M$, $z \in B^n$, are Janowski starlike, respectively Janowski almost starlike. We also derive sufficient conditions for normalized holomorphic mappings (expressed in terms of their coefficient bounds) to belong to one of the subclasses of mappings mentioned above.

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## 1. Introduction and preliminaries

The main reason for studying the properties of subclasses of biholomorphic mappings in several variables is the fact that many of the classical results regarding the class of univalent functions in one complex variable cannot be extended (without imposing supplementary restrictions) to higher dimensions.

In this paper we generalize the Janowski starlike and almost starlike classes of biholomorphic mappings studied in [4].

In [7], W. Janowski introduced the following class of univalent normalized functions defined on the unit disk $U$ of the complex plane.

If $A, B \in \mathbb{R}$, $-1 \leq B < A < 1 \leq 1$, then

$$S^*[A, B] = \left\{ f \in \mathcal{H}(U),\ f(0) = 0,\ f'(0) = 1,\ \frac{zf'(z)}{f(z)} \prec \frac{1 + Az}{1 + Bz} \right\}.$$

Closely related to Janowski starlike class of functions is the following class of univalent functions [17, 18].

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If $a, b \in \mathbb{R}$, $a \geq b$ then

$$S^*(a,b) = \left\{ f \in \mathcal{H}(U),\ f(0) = 0,\ f'(0) = 1,\ \left| \frac{zf'(z)}{f(z)} - a \right| < b \right\}.$$

Various results concerning the classes $S^*[A,B]$ and $S^*(a,b)$ can be found in [7, 17, 18] (and the references therein).

Later, in [10] the authors introduced and studied the class $S^*[A,B]$ for some complex parameters $A$ and $B$, $A \neq B$ which satisfy one of the following two conditions:

$$|A| \leq 1, |B| < 1 \text{ and } \Re(1 - A\overline{B}) \geq |A - B| \tag{1.1}$$

$$|A| \leq 1, |B| = 1 \text{ and } 1 - A\overline{B} > 0. \tag{1.2}$$

Many results related to the class $S^*[A,B]$ for $A, B \in \mathbb{C}$ may be found in [1, 10].

The main goal of the present paper is to generalize the Janowski mappings studied in [4] by introducing the $n$-dimensional version of the class $S^*[A,B]$ with complex parameters $A$ and $B$ that satisfy equivalent conditions to (1.1) and (1.2).

To this end, we recall some notions of function theory in several complex variables that will be used throughout the paper.

We denote by $\mathbb{C}^n$ the space of $n$ complex variables $z = (z_1, z_2, \ldots, z_n)$ with the Euclidean inner product $\langle z, w \rangle = \sum_{j=1}^n z_j \overline{w_j}$ and the Euclidean norm $\|z\| = \langle z, z \rangle^{1/2}$. The open unit ball $\{z \in \mathbb{C}^n :\ \|z\| < 1\}$ is denoted by $B^n$. By $\mathcal{L}(\mathbb{C}^n)$ we denote the the space of linear continuous operators from $\mathbb{C}^n$ into $\mathbb{C}^n$ with the standard operator norm, $\|A\| = \sup\{\|A(z)\| :\ \|z\| = 1\}$. $I_n = I$ is the identity in $\mathcal{L}(\mathbb{C}^n)$.

Let $\mathcal{H}(B^n)$ be the set of holomorphic mappings from $B^n$ into $\mathbb{C}^n$. If $f \in \mathcal{H}(B^n)$ we say that $f$ is normalized if $f(0) = 0$ and the complex Jacobian matrix of $f$ at $z = 0$, $Df(0)$, is the identity operator $I$. If $f \in \mathcal{H}(B^n)$ is a normalized mapping, then

$$f(z) = z + \sum_{k=2}^{\infty} A_k(z^k),\ z \in B^n,$$

where $A_k(w^k) = \frac{1}{k!} D^k f(0)(w^k)$, and $D^k f(0)(w^k)$ is the $k^{th}$ order Fréchet derivative of $f$ at $z = 0$.

A holomorphic mapping $f : B^n \longrightarrow \mathbb{C}^n$ is said to be biholomorphic if the inverse $f^{-1}$ exists and is biholomorphic on the open set $f(B^n)$. Any holomorphic and injective mapping on $B^n$ is biholomorphic on $B^n$. Let $S(B^n)$ be the set of normalized biholomorphic mappings on $B^n$. If $f \in \mathcal{H}(B^n)$, we say that $f$ is locally biholomorphic on $B^n$ if $\det Df(z) \neq 0$, $z \in B^n$. Let $\mathcal{L}S(B^n)$ be the set of normalized locally biholomorphic mappings on $B^n$.

A locally biholomorphic mapping $f$ with $f(0) = 0$ is starlike [19] if and only if

$$\text{Re}\ \langle (Df(z))^{-1} f(z), z \rangle > 0,\ z \in B^n \setminus \{0\}.$$

We denote by $S^*(B^n)$ the set of biholomorphic normalized starlike mappings.

We present next the notions of starlikeness of order $\alpha \in [0,1)$ (see [2, 9]) and almost starlikeness of order $\alpha \in [0,1)$ (see [8, 6]). We denote by $S^*_\alpha(B^n)$ the class of mappings that are starlike of order $\alpha$ on $B^n$ and by $\mathcal{A}S^*_\alpha(B^n)$ the class of almost starlike mappings of order $\alpha$ on $B^n$.

**Definition 1.1.** Let $\alpha \in [0,1)$.

$$S^*_\alpha(B^n) = \left\{ f \in \mathcal{L}S(B^n) : \operatorname{Re} \frac{\|z\|^2}{\langle (Df(z))^{-1} f(z), z\rangle} > \alpha, z \in B^n \setminus \{0\} \right\}$$

$$\mathcal{A}S^*_\alpha(B^n) = \left\{ f \in \mathcal{L}S(B^n) : \operatorname{Re} \frac{\langle (Df(z))^{-1} f(z), z\rangle}{\|z\|^2} > \alpha, z \in B^n \setminus \{0\} \right\}.$$

It is obvious that $S^*_0(B^n) = \mathcal{A}S^*_0(B^n) = S^*(B^n)$.

In [4] we introduced the following classes of starlike mappings on $B^n$.

**Definition 1.2.** Let $a, b \in \mathbb{R}$ such that $|a-1| < b \le a$.

$$S^*(a,b,B^n) = \left\{ f \in \mathcal{L}S(B^n) : \left| \frac{\|z\|^2}{\langle (Df(z))^{-1} f(z), z\rangle} - a \right| < b,\ z \in B^n \setminus \{0\} \right\}$$

$$\mathcal{A}S^*(a,b,B^n) = \left\{ f \in \mathcal{L}S(B^n) : \left| \frac{\langle (Df(z))^{-1} f(z), z\rangle}{\|z\|^2} - a \right| < b,\ z \in B^n \setminus \{0\} \right\}.$$

If in the previous definition we take $a = b$, then $S^*(a,a,B^n) = \mathcal{A}S^*_{1/2a}(B^n)$ and $\mathcal{A}S^*(a,a,B^n) = S^*_{1/2a}(B^n)$.

Various results concerning the classes $S^*(a,b,B^n)$ and $\mathcal{A}S^*(a,b,B^n)$ can be found in [4, 11, 14, 15].

The following set of normalized mappings is the generalization to $n$-complex variables of the well-known Carathéodory class of one variable holomorphic functions with positive real part on the unit disk of complex plane.

$$\mathcal{M} = \{h \in \mathcal{H}(B^n) :\ h(0) = 0,\ Dh(0) = I,\ \operatorname{Re} \langle h(z), z\rangle > 0,\ z \in B^n \setminus \{0\}\}.$$

The class $\mathcal{M}$, introduced in [16] plays a fundamental role in the study of the Loewner differential equation (see for example [3, 5, 6] and the references therein). Also, it is closely related to certain subclasses of biholomorphic mappings on $B$, such as starlike mappings, mappings with parametric representation [5], etc.
Some subclasses of the class $\mathcal{M}$ are presented next.
Let $g \in \mathcal{H}(U)$ be an univalent function, such that $g(0) = 1$ and $\operatorname{Re} g(\zeta) > 0$ on $U$.
Let $\mathcal{M}_g$ be the class of holomorphic mappings given by

$$\mathcal{M}_g = \left\{ h \in \mathcal{H}(B^n) : h(0) = 0,\ Dh(0) = I,\ \frac{1}{\|z\|^2}\langle h(z), z\rangle \in g(U),\ z \in B \setminus \{0\} \right\}.$$

It is clear that $\mathcal{M}_g \subseteq \mathcal{M}$ and for $g(\zeta) = \frac{1-\zeta}{1+\zeta}$ it follows that $\mathcal{M}_g = \mathcal{M}$. Particular choices of the function $g$ provide various subclasses of class $\mathcal{M}$.

If $g$ is a univalent function with $g(0) = 1$ and positive real part on $U$ we denote by $S^*_g(B^n)$ the subset of $S^*(B^n)$ consisting of the normalized locally biholomorphic mappings $f$ such that $(Df(z))^{-1} f(z) \in \mathcal{M}_g$.

In this paper, our main concern is the case when the function $g \in \mathcal{H}(U)$ has positive real part on $U$ and is of the following particular form: $g(\zeta) = \frac{1+A\zeta}{1+B\zeta}$, with $A, B \in \mathbb{C}$, $A \neq B$.

## 2. Janowski subclasses of starlike mappings

In this section we introduce and study two subclasses of biholomorphic mappings named Janowski starlike and Janowski almost starlike with complex parameters.

Let $a \in \mathbb{C}, b \in \mathbb{R}$ such that $|a-1| < b \leq \Re a$. We denote by $S^*(a,b,B^n)$ the class of Janowski starlike mappings on $B^n$ and by $\mathcal{A}S^*(a,b,B^n)$ the class of Janowski almost starlike mappings on $B^n$.

**Definition 2.1.** Let $a \in \mathbb{C}, b \in \mathbb{R}$ such that $|a-1| < b \leq \Re a$.

$$S^*(a,b,B^n) = \left\{ f \in \mathcal{L}S(B^n) : \left| \frac{\|z\|^2}{\langle (Df(z))^{-1} f(z), z \rangle} - a \right| < b,\ z \in B^n \setminus \{0\} \right\}$$

$$\mathcal{A}S^*(a,b,B^n) = \left\{ f \in \mathcal{L}S(B^n) : \left| \frac{\langle (Df(z))^{-1} f(z), z \rangle}{\|z\|^2} - a \right| < b,\ z \in B^n \setminus \{0\} \right\}.$$

In the next remark we present the relationships between the two subclasses of starlike mappings introduced above.

**Remark 2.2.** Let $a \in \mathbb{C}, b \in \mathbb{R}$ such that $|a-1| < b \leq \Re a$.

Then the following assertions are true:

(i) $S^*(a,b,B^n) = \mathcal{A}S^*\left(\frac{\overline{a}}{|a|^2-b^2}, \frac{b}{|a|^2-b^2}, B^n\right)$, if $b < |a|$;

(ii) $S^*(a,a,B^n) = \mathcal{A}S^*_{\frac{1}{2a}}(B^n)$, if $b = |a|$;

(iii) $\mathcal{A}S^*(a,b,B^n) = S^*\left(\frac{\overline{a}}{|a|^2-b^2}, \frac{b}{|a|^2-b^2}, B^n\right)$, if $b < |a|$;

(iv) $\mathcal{A}S^*(a,a,B^n) = S^*_{\frac{1}{2a}}(B^n)$, if $b = |a|$.

*Proof.* The assertions (i) and (iii) are immediate consequences of the fact that the disk of center $a$ and radius $b$ is mapped by the function $1/\zeta$ onto the disk of center $\frac{\overline{a}}{|a|^2-b^2}$ and radius $\frac{b}{|a|^2-b^2}$.

The assertions (ii) and (iv) are immediate consequences of the fact that the disk of center $a$ and radius $a$ is mapped by the function $1/\zeta$ onto the half-plane $\left\{\zeta \mid \Re\zeta > \frac{1}{2a}\right\}$. □

The following remark (see also [10]) presents the conditions satisfied by the complex parameters $A, B,\ A \neq B$ such that $g(\zeta) = \frac{1+A\zeta}{1+B\zeta},\ \zeta \in U$, is a holomorphic function with positive real part on $U$.

**Remark 2.3.** Let $A, B \in \mathbb{C},\ A \neq B$ and let $g \in \mathcal{H}(U)$ be the function defined by $g(\zeta) = \frac{1+A\zeta}{1+B\zeta}$. If $g$ has positive real part on $U$, then the complex parameters $A$ and $B$ satisfy one of the following conditions:

$$|B| < 1 \text{ and } \Re(1 - A\overline{B}) \geq |A - B| \tag{2.1}$$

$$|B| = 1 \text{ and } -1 \leq A\overline{B} < 1 \tag{2.2}$$

*Proof.* The fact that $g \in \mathcal{H}(U)$ immediately implies that $|B| \leq 1$.

The function $g$ maps the unit disk $U$ either onto an open disk (when $|B| < 1$) or onto a half-plane (when $|B| = 1$). It remains for us to determine the conditions satisfied by $A$ and $B$ such that the image $g(U)$ to be situated in the right half-plane.

When $|B| < 1$, $g$ maps the unit disk $U$ onto the open unit disk given by

$$\left| g(\zeta) - \frac{1 - A\overline{B}}{1 - |B|^2} \right| < \frac{|A - B|}{1 - |B|^2},\ \zeta \in U.$$

The above disk is in the right half-plane if

$$\Re g(\zeta) > \frac{\Re(1 - A\overline{B}) - |A - B|}{1 - |B|^2} \geq 0,\ \zeta \in U,$$

hence (2.1) is fulfilled.

When $|B| = 1$, $g$ maps the unit disk $U$ onto a half-plane, which has to be situated in the right half-plane, hence the image of the unit circle is a vertical line. Therefore, $\Re g(\overline{B}) = \Re g(i\overline{B})$, wherefrom we obtain that $A\overline{B} \in \mathbb{R}$ and $g(\overline{B}) = \frac{1+A\overline{B}}{2}$. In this case $\Re g(\zeta) > 0, \zeta \in U$, if and only if $1 = \Re g(0) > \frac{1+A\overline{B}}{2} \geq 0$, hence (2.2) is fulfilled. $\square$

We next determine the function $g$ such that $\mathcal{A}S^*(a, b, B^n) = S_g^*(B^n)$.

**Remark 2.4.** (i) Let $a \in \mathbb{C}$ and $b \in \mathbb{R}$ such that $|a - 1| < b \leq \Re a$. Then

$$\mathcal{A}S^*(a, b, B^n) = S_g^*(B^n), \text{where } g(\zeta) = \frac{1 + \frac{a-|a|^2+b^2}{b}\zeta}{1 + \frac{1-\overline{a}}{b}\zeta},\ \zeta \in U \tag{2.3}$$

(ii) Let $A, B \in \mathbb{C}$, $A \neq B$, and let $g : U \longrightarrow \mathbb{C}$, $g(\zeta) = \frac{1+A\zeta}{1+B\zeta}$ be a holomorphic function with positive real part on $U$. Then:

$$S_g^*(B^n) = \mathcal{A}S^*\left(\frac{1 - A\overline{B}}{1 - |B|^2}, \frac{|A - B|}{1 - |B|^2}, B^n\right) \text{ if } |B| < 1 \text{ and } \Re(1 - A\overline{B}) \geq |A - B| \tag{2.4}$$

$$S_g^*(B^n) = \mathcal{A}S^*_{\frac{1+A\overline{B}}{2}}(B^n) \text{ if } |B| = 1 \text{ and } -1 < A\overline{B} < 1.$$

*Proof.* To prove (2.3) we have to determine $A, B \in \mathbb{C}$, $A \neq B$, $|B| < 1$, $\Re(1 - A\overline{B}) \geq |A - B|$ such that $a = \frac{1-A\overline{B}}{1-|B|^2}$ and $b = \frac{|A-B|}{1-|B|^2}$.
Straightforward computations lead to the following values:

$$B = \frac{1 - \overline{a}}{b} \frac{|A - B|}{\overline{A} - \overline{B}} = \frac{1 - \overline{a}}{b} e^{i\phi},\ A = \frac{b^2 + a - |a|^2}{b} e^{i\phi},\ \phi = \arg(A - B).$$

Since the image of $U$ under the function $g$ is invariant to the rotations of the unit disk, without loss of generality we can assume that $B = \frac{1-\overline{a}}{b}$ and hence $A = \frac{b^2+a-|a|^2}{b}$. By direct computations we obtain that

$$1 - A\overline{B} = \frac{\overline{a}(b^2 - |a - 1|^2)}{b^2} \text{ and } B = \frac{b^2 - |a - 1|^2}{b}.$$

Therefore the desired inequality $\Re(1 - A\overline{B}) \geq |A - B|$ is equivalent to $\Re\overline{a} \geq b$, which is true. The inequality $|B| < 1$ results immediately from $|a - 1| < b$.
The equalities in (2.4) easily follow from the fact that the unit disk is mapped by the function $g(\zeta) = \frac{1+A\zeta}{1+B\zeta}$ onto the disk centered at $a = \frac{1-A\overline{B}}{1-|B|^2}$ with radius $b = \frac{|A-B|}{1-|B|^2}$ when $|B| < 1$, respectively onto the half-plane $\left\{\zeta \mid \mathrm{Re}\ \zeta > \frac{1+A\overline{B}}{2}\right\}$ when $|B| = 1$. $\square$

We next determine the function $g$ such that $S^*(a,b,B^n) = S_g^*(B^n)$.

**Remark 2.5.** (i) Let $a \in \mathbb{C}$ and $b \in \mathbb{R}$ such that $|a-1| < b \leq \Re a$. Then

$$S^*(a,b,B^n) = S_g^*(B^n), \text{where } g(\zeta) = \frac{1 + \frac{\overline{a}-1}{b}\zeta}{1 + \frac{|a|^2 - a - b^2}{b}\zeta}, \zeta \in U. \tag{2.5}$$

(ii) Let $A, B \in \mathbb{C}$, $A \neq B$, and let $g : U \longrightarrow \mathbb{C}$, $g(\zeta) = \frac{1+A\zeta}{1+B\zeta}$ be a holomorphic function with positive real part on $U$. Then:

$$S_g^*(B^n) = S^*\left(\frac{1-A\overline{B}}{1-|A|^2}, \frac{|A-B|}{1-|A|^2}, B^n\right) \text{ for } |A| < 1 \text{ and } \Re(1-A\overline{B}) \geq |A-B| \tag{2.6}$$

$$S_g^*(B^n) = S^*_{\frac{1+A\overline{B}}{2}}(B^n) \text{ if } |A| = 1 \text{ and } -1 < A\overline{B} < 1.$$

*Proof.* The equality in (2.5) is an immediate consequence of (2.3) and assertion (iii) from Remark 2.2. The equalities in (2.6) can be justified by using similar arguments to those presented in the proof of (ii), Remark 2.4. □

## 3. Sufficient conditions for Janowski starlikeness

In this section we obtain sufficient conditions for normalized holomorphic mappings to belong to $S^*(a,b,B^n)$, respectively $\mathcal{A}S^*(a,b,B^n)$, where $a \in \mathbb{C}, b \in \mathbb{R}$, $|a-1| < b \leq \Re a$.

**Theorem 3.1.** *Let $f(z) = z + \sum_{k=2}^{\infty} A_k(z^k)$ be a holomorphic mapping on $B^n$ and let $g : U \to \mathbb{C}$ be the function defined by $g(\zeta) = \frac{1+A\zeta}{1+B\zeta}$, $\zeta \in U$, where $A, B \in \mathbb{C}$, $A \neq B, |B| < 1$ and $\Re(1 - A\overline{B}) \geq |A-B|$. If*

$$\|Df(z) - I\| \leq M, \quad z \in B^n \tag{3.1}$$

*where $M$ is defined by*

$$M = \frac{2|A-B|(1-|B|)}{2|1-A\overline{B}| + 2|A-B| + (1-|B|^2)} \tag{3.2}$$

*then $f \in S_g^*(B^n)$.*

*Proof.* In order that $f$ to be in $S_g^*(B^n)$ it is sufficient to show that

$$\left\| \frac{1-|B|^2}{|A-B|}(Df(z))^{-1}f(z) - \frac{1-A\overline{B}}{|A-B|}z \right\| < \|z\|, \forall z \in B^n \setminus \{0\}.$$

If $h$ is the holomorphic function defined by

$$h(z) = \frac{1-|B|^2}{|A-B|}(Df(z))^{-1}f(z) - \frac{1-A\overline{B}}{|A-B|}z, z \in B^n$$

then $h(0) = 0$, $\|Dh(0)\| = |B| < 1$ hence $\lim_{z \to 0} \frac{\|h(z)\|}{\|z\|} < 1$.
Therefore, it suffices to prove that $\|h(z)\| < \|z\|$ for all $z \in B^n \setminus \{0\}$.

If the previous inequality is not true, then there exists a point $z_0 \in B^n \setminus \{0\}$ such that $\|h(z_0)\| = \|z_0\|$. By denoting $w_0 = (Df(z_0))^{-1}f(z_0)$, after direct computations we obtain

$$\|w_0\| \leq \frac{|1-A\overline{B}|+|A-B|}{1-|B|^2}\|z_0\|. \tag{3.3}$$

Since $\|Df(z_0) - I\| \leq M$, we have first $\|Df(z_0)w_0 - w_0\| \leq M\|w_0\|$ and hence

$$\|f(z_0) - w_0\| \leq M\|w_0\| \tag{3.4}$$

Now we prove that

$$\|f(z) - z\| < \frac{M}{2}, \forall z \in B^n. \tag{3.5}$$

If not, then there exists a point $z_1 \in B^n \setminus \{0\}$ such that

$$\max_{\|z\|\leq\|z_1\|} \|f(z) - z\| = \|f(z_1) - z_1\| = \frac{M}{2} \tag{3.6}$$

According to Lemma1 [12], there exists a real number $t \geq 2$ such that

$$\langle Df(z_1)(z_1) - z_1, f(z_1) - z_1\rangle = t\|f(z_1) - z_1\|^2.$$

In view of the relations (3.1) and (3.6), the previous equality implies

$$t\frac{M^2}{4} \leq \|Df(z_1)(z_1) - z_1\| \cdot \|f(z_1) - z_1\| \leq \frac{M^2}{2}\|z_1\| < \frac{M^2}{2}, \tag{3.7}$$

hence $t < 2$, which is a contradiction.
Therefore, the relation (3.5) is true and by applying the Schwarz's Lemma we obtain

$$\|f(z) - z\| \leq \frac{M}{2}\|z\|^2, \ \forall z \in B^n. \tag{3.8}$$

By using the relations (3.4) and (3.8) we obtain first that

$$M\|w_0\| \geq \|z_0 - w_0\| - \|f(z_0) - z_0\| > \|z_0 - w_0\| - \frac{M}{2}\|z_0\|,$$

and then

$$M < \frac{\|z_0 - w_0\|}{\frac{\|z_0\|}{2} + \|w_0\|}. \tag{3.9}$$

On the other hand, by using the fact that $\|h(z_0)\| = \|z_0\|$, (3.3) and (3.2) we get

$$\frac{\|z_0 - w_0\|}{\frac{\|z_0\|}{2} + \|w_0\|} = \frac{\left\|w_0 - \frac{|1-A\overline{B}|}{1-|B|^2}z_0 + \frac{|1-A\overline{B}|}{1-|B|^2}z_0 - z_0\right\|}{\frac{\|z_0\|}{2} + \|w_0\|} \geq M$$

Because the previous inequality contradicts (3.9), the assumption that there exists a point $z_0 \in B^n \setminus \{0\}$ such that $\|h(z_0)\| = \|z_0\|$ is false, hence $\|h(z)\| < \|z\|$ for all $z \in B^n \setminus \{0\}$. This completes the proof. □

Let $a \in \mathbb{C}$ and $b \in \mathbb{R}$ such that $|a-1| < b \leq \Re a$.

By taking $A = \frac{a-|a|^2+b^2}{b}$ and $B = \frac{1-\overline{a}}{b}$ in Theorem 3.1 and by using Remark 2.4 we obtain the following sufficient condition for a holomorphic mapping to belong to $\mathcal{A}S^*(a, b, B^n)$

**Theorem 3.2.** *Let* $f(z) = z + \sum_{k=2}^{\infty} A_k(z^k)$ *be a holomorphic mapping on* $B^n$ *and let* $a \in \mathbb{C}, b \in \mathbb{R}$ *such that* $|a-1| < b \leq \Re a$. *If*

$$\|Df(z) - I\| \leq \frac{2(b - |1-a|)}{2(|a|+b)+1}, \quad z \in B^n$$

*then* $f \in \mathcal{A}S^*(a, b, B)$.

If we take $A = \frac{\overline{a}-1}{b}$ and $B = \frac{|a|^2-a-b^2}{b}$ in Theorem 3.1 and use Remark 2.5 we get the following sufficient condition for a holomorphic mapping to be in $S^*(a, b, B^n)$.

**Theorem 3.3.** *Let* $f(z) = z + \sum_{k=2}^{\infty} A_k(z^k)$ *be a holomorphic mapping on* $B^n$ *and let* $a \in \mathbb{C}, b \in \mathbb{R}$ *such that* $|a-1| < b \leq \Re a$ *and* $b < |a|$. *If*

$$\|Df(z) - I\| \leq \frac{2(b - ||a|^2 - a + b^2|)}{(|a|+b)(|a|-b+2)}, \quad z \in B^n$$

*then* $f \in S^*(a, b, B^n)$.

Next theorems present sufficient conditions that are expressed in terms of coefficients bounds of the considered mappings.

**Theorem 3.4.** *Let* $f(z) = z + \sum_{k=2}^{\infty} A_k(z^k)$ *be a holomorphic mapping on* $B^n$ *and let* $g : U \to \mathbb{C}$ *be the univalent function defined by* $g(\zeta) = \dfrac{1+A\zeta}{1+B\zeta}$, $\zeta \in U$, *where* $A, B \in \mathbb{C}$, $A \neq B, |B| < 1$ *and* $\Re(1 - A\overline{B}) \geq |A - B|$. *If*

$$\sum_{k=2}^{\infty} \left( \left| k\frac{1-A\overline{B}}{|A-B|} - \frac{1-|B|^2}{|A-B|} \right| + k \right) \|A_k\| \leq 1 - |B| \tag{3.10}$$

*then* $f \in S_g^*(B^n)$.

*Proof.* From the inequality (3.10) it follows that

$$\sum_{k=2}^{\infty} k\|A_k\| \leq \frac{2}{2\frac{|1-A\overline{B}|}{|A-B|} - \frac{1-|B|^2}{|A-B|} + 2} \sum_{k=2}^{\infty} \left( \left| k\frac{1-A\overline{B}}{|A-B|} - \frac{1-|B|^2}{|A-B|} \right| + k \right) \|A_k\| < 1.$$

By direct computation of Fréchet derivatives of $f$ we obtain

$$\|Df(z) - I\| = \left\| \sum_{k=2}^{\infty} kA_k(z^{k-1}, \cdot) \right\| \leq \sum_{k=2}^{\infty} k\|A_k\| < 1, \ z \in B^n.$$

Hence we obtain that $Df(z) = I - (I - Df(z))$ is an invertible linear operator and

$$\|(Df(z))^{-1}\| \leq \frac{1}{1 - \|I - Df(z)\|} \leq \frac{1}{1 - \sum_{k=2}^{\infty} k\|A_k\|}, \ z \in B^n. \tag{3.11}$$

For every $z \in B^n \setminus \{0\}$, we have

$$\left\| \frac{1-|B|^2}{|A-B|} f(z) - \frac{1-A\overline{B}}{|A-B|} Df(z)(z) \right\|$$
$$< \|z\| \left( |B| + \sum_{k=2}^{\infty} \left| k\frac{|1-A\overline{B}|}{|A-B|} - \frac{1-|B|^2}{|A-B|} \right| \|A_k\| \right).$$

By using the inequality (3.11) and the previous inequality we obtain

$$\frac{1}{\|z\|^2}\left|\left\langle \frac{1-|B|^2}{|A-B|}(Df(z))^{-1}f(z), z\right\rangle - \frac{1-A\overline{B}}{|A-B|}\|z\|^2\right|$$
$$< \frac{|B| + \sum_{k=2}^{\infty}\left|k\frac{|1-A\overline{B}|}{|A-B|} - \frac{1-|B|^2}{|A-B|}\right|\|A_k\|}{1-\sum_{k=2}^{\infty}\|A_k\|k} \leq 1,$$

where for the last inequality we have used the relation (3.10).
In conclusion

$$\left|\left\langle \frac{(Df(z))^{-1}f(z)}{\|z\|^2}, z\right\rangle - \frac{1-A\overline{B}}{1-|B|^2}\right| < \frac{|A-B|}{1-|B|^2},\ z \in B^n \setminus \{0\},$$

which implies that $f \in S_g^*(B^n)$ as desired.

□

Let $a \in \mathbb{C}$ and $b \in \mathbb{R}$ such that $|a-1| < b \leq \Re a$.

By taking $A = \frac{a-|a|^2+b^2}{b}$ and $B = \frac{1-\overline{a}}{b}$ in Theorem 3.4 we obtain the following sufficient condition for a holomorphic mapping to be in $\mathcal{A}S^*(a, b, B^n)$.

**Theorem 3.5.** *Let $f(z) = z + \sum_{k=2}^{\infty} A_k(z^k)$ be a holomorphic mapping on $B^n$ and let $a \in \mathbb{C}, b \in \mathbb{R}$ such that $|a-1| < b \leq \Re a$. If*

$$\sum_{k=2}^{\infty}(|ka-1| + kb)\|A_k\| \leq b - |1-a|$$

*then $f \in \mathcal{A}S^*(a, b, B)$.*

If in the previous theorem we take $a = b = \frac{1}{2\alpha}$, $0 < \alpha < 1$, we get Theorem 2.2 [13].

If we take $A = \frac{\overline{a}-1}{b}$ and $B = \frac{|a|^2-a-b^2}{b}$ in Theorem 3.4 we get the following sufficient condition for a holomorphic mapping to be in $S^*(a, b, B^n)$.

**Theorem 3.6.** *Let $f(z) = z + \sum_{k=2}^{\infty} A_k(z^k)$ be a holomorphic mapping on $B^n$ and let $a \in \mathbb{C}, b \in \mathbb{R}$ such that $|a-1| < b \leq \Re a$ and $b < |a|$. If*

$$\sum_{k=2}^{\infty}\left(\left|k\overline{a} - (|a|^2 - b^2)\right| + kb\right)\|A_k\| \leq b - \left||a|^2 - a + b^2\right|$$

*then $f \in S^*(a, b, B^n)$.*

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# On some cluster sets problems

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*Dedicated to the memory of Professor Gabriela Kohr*

**Abstract.** We generalize some cluster sets theorems of Tsuji and Iversen from plane holomorphic mappings to the class of ring mappings.

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## 1. Introduction

A classical problem in complex analysis is the study of boundary behaviour of analytic mappings and a special case is the study of cluster sets (see the book of Noshiro [26]).

We shall extend some theorems of Iversen and Tsuji concerning cluster sets of holomorphic plane mappings in the class of so called ring mappings or mappings satisfying modular inequalities. Such mappings were intensively studied in the last 20 years (see the book of Martio, Ryazanov, Srebro, Yakubov [21], or some papers of Cristea [2-9], Golberg, Salimov, Sevost'yanov, Ryazanov [11-14], [16], [19-21], [27-33]).

The class of ring mappings preserves many geometric properties of the well-known quasiregular mappings, the last class being itself the best known extension of the plane analytic mappings.

Given a domain $D \subset \mathbb{R}^n$, we denote by $A(D)$ the set of all path families from $D$ and if $\Gamma \in A(D)$ we set $F(\Gamma) = \{\rho : \mathbb{R}^n \to [0,\infty]$ Borel maps $\mid \int_\gamma \rho ds \geq 1$ for every $\gamma \in \Gamma$ locally rectifiable$\}$. If $E, F \subset \overline{D}$, we set $\Delta(E,F,D) = \{\gamma : [0,1] \to \overline{D}$ path such that $\gamma(0) \in E$, $\gamma(1) \in F$ and $\gamma((0,1)) \subset D\}$ and if $x \in \mathbb{R}^n$ and $0 < a < b$ we set $\Gamma_{x,a,b} = \Delta(\overline{B}(x,a) \cap D, S(x,b) \cap D, C_{x,a,b} \cap D)$. Here $C_{x,a,b} = B(x,b) \setminus \overline{B}(x,a)$.

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We define for $p>1$ and $\omega : D \to [0,\infty]$ measurable and finite a.e. the $p$-modulus of weight $\omega$

$$M_\omega^p(\Gamma) = \inf_{\rho \in F(\Gamma)} \int_{\mathbb{R}^n} \omega(x)\rho^p(x)dx \ \ for \ \ \Gamma \in A(D).$$

For $\omega = 1$ we have the classical $p$-modulus

$$M_p(\Gamma) = \inf_{\rho \in F(\Gamma)} \int_{\mathbb{R}^n} \rho^p(x)dx \ \ for \ \ \Gamma \in A(D).$$

We see that $M_\omega^p(\Gamma_1) \le M_\omega^p(\Gamma_2)$ if $\Gamma_1 > \Gamma_2$ and

$$M_\omega^p\left(\bigcup_{i=1}^{\infty} \Gamma_i\right) \le \sum_{i=1}^{\infty} M_\omega^p(\Gamma_i)$$

for $\Gamma_1, \Gamma_2, ..., \Gamma_i, ... \in A(D)$. Here, if $\Gamma_1, \Gamma_2 \in A(D)$, we say that $\Gamma_1 > \Gamma_2$ if every path $\gamma_1 \in \Gamma_1$ has a subpath $\gamma_2 \in \Gamma_2$.

We say that a mapping $f : D \to \mathbb{R}^n$ is open if carries open sets to open sets and we say that $f$ is discrete if either $f^{-1}(y) = \phi$ or $f^{-1}(y)$ is a discrete subset of $D$ for every $y \in \mathbb{R}^n$. If $f : D \to \mathbb{R}^n$ is continuous, open and discrete, then for every path $p : [0,1] \to \mathbb{R}^n$ and every $x \in D$ such that $f(x) = p(0)$ there exists $0 < a \le 1$, a path $q : [0,a) \to D$ such that $q(0) = x$ and $f \circ q = p|[0,a)$ and such a path is maximal with this property (we say that $f$ has the property of path lifting and $q : [0,a) \to D$ is a maximal lifting of the path $p : [0,1] \to \mathbb{R}^n$ from the point $x \in D$ such that $f(x) = p(0)$). We say that $D$ is locally connected at $x \in \partial D$ if there exists $(U_m)_{m\in\mathbb{N}}$ a fundamental system of neighbourhoods of $x$ such that $U_m \cap D$ is connected for every $m \in \mathbb{N}$. Let $p > 1$. We say that $E = (A, C)$ is a condenser if $C \subset A \subset \mathbb{R}^n$, $A$ is open and $C$ is compact and we define

$$cap_\omega^p(E) = \inf \int_{\mathbb{R}^n} \omega(x)\rho^p(x)dx,$$

where $u \in C_0^\infty(A)$ and $u \ge 1$ on $C$. For $\omega = 1$ we have the classical $p$-capacity. If $E = (A, C)$ is a condenser and $\Gamma_E = \Delta(C, \partial A, A)$, then $M_p(\Gamma_E) = cap_p(E)$. If $C \subset \mathbb{R}^n$ is compact, we say that $cap_p(C) = 0$ if $cap_p((A, C)) = 0$ for some bounded open set $A \subset \mathbb{R}^n$ and if $C \subset \mathbb{R}^n$ is arbitrary, we say that $cap_p(C) = 0$ if $cap_p(K) = 0$ for every compact $K \subset C$.

If $K \subset \mathbb{R}^n$, we say that $M_\omega^p(K) = 0$ if $M_\omega^p(\Gamma) = 0$, where $\Gamma = \{\gamma : [0,1) \to \mathbb{R}^n$ path $|\gamma$ has at least a limit point in $K\}$. Here, for an open path $\gamma : [0,1) \to \mathbb{R}^n$ we say that a point $x \in \overline{\mathbb{R}^n}$ is a limit point of $\gamma$ if there exist $t_m \to 1$ such that $\gamma(t_m) \to x$.

The following capacity inequality is proved in [17]:

$$cap_p((A,C)) \ge C_1\left(\frac{d(C)^p}{\mu_n(A)^{1-n+p}}\right)^{\frac{1}{n-1}} \text{ if } n-1 < p \le n. \tag{1.1}$$

Here $d(C)$ is the diameter of $C$, $\mu_n$ is the Lebesgue measure in $\mathbb{R}^n$ and $C_1$ is a constant which does not depends on $n$. We set $V_n$ the volume of the unit ball from

$\mathbb{R}^n$. We also use a modular inequality from [1]:

$$M_p(\Delta(E_1, E_2, C_{x,a,b})) \geq \frac{C(n,p)}{n-p}(b^{n-p} - a^{n-p}) \text{ if } n-1 < p < n \tag{1.2}$$

$$M_p(\Delta(E_1, E_2, C_{x,a,b}) \geq C(n) \ln\left(\frac{b}{a}\right) \text{ if } p = n$$

where $E_1 \cap S(x,r) \neq \phi$, $E_2 \cap S(x,r) \neq \phi$ for every $a < r < b$ and $x \in \mathbb{R}^n$.

Here $C(n,p)$ is a constant depending only on $n$ and $p$. Throughout this paper $C(n,p)$ means a constant depending only on $n$ and $p$.

It is easy to see that the condition $M_\omega^p(E) = 0$ holds for instance if

$$\int_{\overline{D}} \omega(z)^{\frac{m}{m-p}} dz < \infty$$

and $cap_m(E) = 0$ and $p < m \leq n$ (see [8] page 4 or [12]).

Let us speak about the objects of cluster sets theory that will be used in this paper. Let $D \subset \mathbb{R}^n$ be open, $f : D \to \mathbb{R}^n$ and $x \in E \subset \partial D$. We set $C(f,x) = \{z \in \overline{\mathbb{R}^n} |$ there exists $x_m \in D$, $x_m \to x$ such that $f(x_m) \to z\}$ and we set $C(f,E) = \bigcup_{x \in E} C(f,x)$. Let $F : \overline{D} \to \mathcal{P}(\mathbb{R}^n)$ be defined by $F(x) = f(x)$ if $x \in D$, $F(x) = C(f,x)$ if $x \in \partial D$ and if $(U_m)_{m \in \mathbb{N}}$ is a fundamental system of neighbourhoods of $x$ such that $U_{m+1} \subset U_m$ for every $m \in \mathbb{N}$, we set $C(f,x,E) = \bigcap_{m=1}^{\infty} F(U_m \cap (E \setminus \{x\}))$. We set for $x \in \overline{D}$ the range of values $R_D(f,x) = \bigcap_{r>0} f(B(x,r) \cap D)$. If $x \in \partial D$, we set $A(f,x) = \{z \in \mathbb{R}^n |$ there exists $\gamma : [0,1) \to D$ path such that $\gamma(0) \in D$, $\lim_{t \to 1} \gamma(t) = x$ and $\lim_{t \to 1} f(\gamma(t)) = z\}$. If $\gamma : [0,1] \to D$ is a path, we set $|\gamma| = \{z \in \mathbb{R}^n |$ there exists $t \in [0,1]$ such that $z = \gamma(t)\}$.

The theory of cluster sets was also studied for quasiregular mappings by many mathematicians like Näkki, Vuorinen, Martio, Zoric [22-25], [34-36], [18], [37].

In some recent papers [8-9] we studied classes of continuous, open, discrete mappings $f : D \subset \mathbb{R}^n \to \mathbb{R}^n$ for which the following modular inequality holds:

$$M_p(f(\Gamma)) \leq \gamma(M_\omega^q(\Gamma)) \text{ for every } \Gamma \in A(D), \tag{1.3}$$

where $n \geq 2$, $q > 1$, $n-1 < p \leq n$, $\omega : D \to [0,\infty]$ is measurable and finite a.e. and $\gamma : (0,\infty) \to (0,\infty)$ is increasing with $\lim_{t \to 0} \gamma(t) = 0$. We briefly say that such mappings satisfy condition (1.3). It must be mentioned that if $f : D \to \mathbb{R}^n$ is quasiregular, then the Poletski modular inequality holds, namely $M_n(f(\Gamma)) \leq K M_n(\Gamma)$ for every $\Gamma \in A(D)$ and a fixed constant $K \geq 1$.

It is surprisingly that for the mappings satisfying relation (1.3) with $p \neq n$, $n-1 < p < n$ we can give analogous of Liouville, Montel, Picard type theorems, boundary extension results and estimates of the modulus of continuity. Interesting examples of such mappings satisfying relation (1.3) which are not quasiregular may be found in [12]. Also, the case $q = n$, $p = n$ was extensively studied in [20].

We continue our researches and related researches on these mappings from [3], [8-9] and [12].

In this paper we study the cluster sets and the boundary cluster sets for such mappings. Tsuji proved the following theorem for analytic mappings see Theorem 3.5 in [26]).

**Theorem A.** *Let $D \subset \mathbb{C}$ be a domain, $E \subset \partial D$ compact with $cap_2(E) = 0$, $x \in E \cap \overline{(\partial D \setminus E)}$ and let $f \in H(D)$. Then if the open set $Q = C(f,x) \setminus C(f,x,\partial D \setminus E)$ is nonempty, it results that $cap_2(Q \setminus R_D(f,x)) = 0$.*

We prove the following extension of Tsuji's theorem for mappings satisfying relation (1.3):

**Theorem 1.1.** *Let $E \subset \partial D$, $x \in E \cap \overline{\partial D \setminus E}$ such that $D$ is locally connected at $x$, $\omega \in L^1_{loc}(D)$ such that $M^q_\omega(E) = 0$ and let $f : D \to \mathbb{R}^n$ satisfying condition (1.3). Then, if the open set $Q = C(f,x) \setminus C(f,x,\partial D \setminus E)$ is nonempty, it results that $cap_p(Q \setminus R_D(f,x)) = 0$.*

**Theorem 1.2.** *Let $E \subset \partial D$, $x \in E \cap \overline{(\partial D \setminus E)}$ such that $D$ is locally connected at $x$, $\omega \in L^1_{loc}(D)$ such that $M^q_\omega(E) = 0$ and let $f : D \to \mathbb{R}^n$ be bounded satisfying condition (1.3) such that there exists $K \subset \mathbb{R}^n$ compact such that $C(f,x,\partial D \setminus E) \subset K$ and $\mathbb{R}^n \setminus K$ is connected. Then $C(f,x) \subset K$.*

The following theorem is due to Iversen and Tsuji (see Theorem 3.2 in [26]).

**Theorem B.** *Let $D \subset \mathbb{C}$ be a domain, $E \subset \partial D$ compact, with $cap_2(E) = 0$, $x \in E \cap \overline{(\partial D \setminus E)}$ and let $f \in H(D)$ be bounded. Then $\limsup\limits_{y \to x} |f(y)| = \limsup\limits_{\substack{y \to x \\ y \in \partial D \setminus E}} (\limsup\limits_{z \to y} |f(z)|)$.*

We prove:

**Theorem 1.3.** *Let $E \subset \partial D$, $x \in E \cap \overline{(\partial D \setminus E)}$ such that $D$ is locally connected at $x$, $\omega \in L^1_{loc}(D)$ such that $M^q_\omega(E) = 0$ and let $f : D \to \mathbb{R}^n$ be bounded satisfying condition (1.3). Then $\limsup\limits_{y \to x} |f(y)| = \limsup\limits_{\substack{y \to x \\ y \in \partial D \setminus E}} (\limsup\limits_{z \to y} |f(z)|$.*

Using the method from [3] and [10] we prove the following generalization of some theorems of Noshiro [26] and Martio and Rickman [20]:

**Theorem 1.4.** *Let $E \subset \partial D$ such that $\dim \partial D \geq 1$, $\dim E = 0$ and $M^q_\omega(E) = 0$, let $x \in (\partial D \setminus E)'$ and $z \in C(f,x) \setminus (C(f,x,\partial D \setminus E) \cup R_D(f,x))$ and $f : D \to \mathbb{R}^n$ satisfying condition (1.3). Then either $x \in E$ and $z \in A(f,x)$ or there exists $x_k \in E$ such that $x_k \in E$, $x_k \to x$ and $z \in A(f,x_k)$ for every $k \in \mathbb{N}$.*

We also see that our extensions given to the theorems of Tsuji and Iversen-Tsuji are effective even for plane analytic mappings, since we don't impose the exceptional set $E \subset \partial D$ to be compact.

## 2. Proofs of the results

*Proof of Theorem 1.1.* Let $y \in Q$, $\delta_y = d(y, \partial Q)$ and $0 < \delta < \frac{\delta_y}{3}$ and let

$$F_r = C(f, \overline{B}(x,r) \cap ((\partial D \setminus E) \setminus \{x\})) \text{ for } r > 0.$$

Since $C(f,x,\partial D \setminus E) = \bigcap\limits_{r>0} \overline{F}_r$, there exists $r_0 > 0$ such that $\overline{F}_r \cap B(y, \frac{5\delta}{2}) = \phi$ for every $0 < r < r_0$. Let $0 < r < r_0$ be fixed and let $\epsilon > 0$. We can find $0 < b < r$ such that $B(x,b) \cap D$ is connected and $M^q_\omega(\Gamma_{x,b,r}) < \epsilon$. Let $x_m$, $z_m \to x$ be such that $x_m, z_m \in B(x,b) \cap D$, $d(f(z_m), y) > \frac{5\delta}{2}$ and $d(f(x_m), y) < \frac{\delta}{2}$ for every $m \in \mathbb{N}$.

Since $B(x,b)\cap D$ is connected, there exists a path $q_m : [0,1] \to B(x,b)\cap D$ such that $q_m(0) = x_m$, $q_m(1) = z_m$ and $|q_m| \subset B(x,b)\cap D$ for every $m \in \mathbb{N}$.

Let us fix such $m \in \mathbb{N}$ and let $p_m = f \circ q_m$. Since $p_m(0) = f(q_m(o)) = f(x_m)$ and $p_m(1) = f(q_m(1)) = f(z_m)$, we see that $d(|p_m|) > 2\delta$ and that there exists a subpath $\gamma_m$ of $p_m$ such that $f(x_m) \in |\gamma_m|$, $d(|\gamma_m|) > \frac{\delta}{2}$ and $|\gamma_m| \subset B(y,\delta)$.
Suppose that there exists a compact set $F \subset B(y,\delta) \setminus f(B(x,r))$ such that

$$cap_p(B(y,2\delta),F) = \delta_0 > 0.$$

Let $\Gamma'_m = \Delta(|\gamma_m|, F, B(y,2\delta))$ and let $\rho \in F(\Gamma'_m)$.
Let $\Delta_0 = \Delta(F, S(y,2\delta), B(y,2\delta))$ and $\Delta_m = \Delta(|\gamma_m|, S(y,2\delta), B(y,2\delta))$.
If $3\rho \in F(\Delta_0) \cup F(\Delta_m)$, then

$$\int\limits_{\mathbb{R}^n} \rho^p(x)dx \geq \frac{M_p(\Delta_0)}{3^p} = \frac{\delta_0}{3^p}$$

and using relation (1.1) we have

$$\int\limits_{\mathbb{R}^n} \rho^p(x)dx \geq \frac{M_p(\Delta_m)}{3^p} \geq \frac{1}{3^p}\left(\frac{C_1 d(|\gamma_m|)^p}{2^n V_n (2\delta)^{1-n+p}}\right)^{\frac{1}{n-1}} \geq C(n,p)\delta.$$

If $3\rho \notin F(\Delta_0) \cup F(\Delta_m)$, we can find paths $\lambda_1 \in \Delta_0$, $\lambda_2 \in \Delta_m$ such that

$$\int\limits_{\lambda_k} \rho ds < 1 \text{ for } k = 1,2.$$

Let $\Gamma = \Delta(|\lambda_1|, |\lambda_2|, C_{y,\delta,2\delta})$. Using relation (1.2) we see that

$$M_p(\Gamma) \geq C(n,p)((2\delta)^{n-p} - \delta^{n-p}) = C(n,p)\delta^{n-p}.$$

Let $\gamma_0 \in \Gamma$. We can find subpaths $\alpha_1$ of $\lambda_1$ and $\beta_1$ of $\lambda_2$ such that the path

$$\gamma = \alpha_1 \vee \gamma_o \vee \beta_1 \in \Gamma'_m$$

and since $\rho \in F(\Gamma'_m)$, we see that

$$1 \leq \int\limits_{\gamma} \rho ds \leq \int\limits_{\alpha_1} \rho ds + \int\limits_{\gamma_0} \rho ds + \int\limits_{\beta_1} \rho ds \leq \frac{1}{3} + \int\limits_{\gamma_0} \rho ds + \frac{1}{3}$$

and hence

$$1 \leq \int\limits_{\gamma_0} 3\rho ds.$$

It results that if $3\rho \notin F(\Delta_0) \cup F(\Delta_m)$, then

$$\int\limits_{\mathbb{R}^n} \rho^n(x)dx \geq \frac{1}{3^p} M_p(\Gamma) \geq C(n,p)\delta^{n-p}.$$

We therefore proved that

$$M_p(\Gamma'_m) \geq \frac{1}{3^p} \min\{\delta_0, C(n,p)\delta, C(n,p)\delta^{n-p}\} = \rho > 0$$

and the constant $\rho$ does not depends on $m$.

Let $\Gamma_m$ be the family of all maximal liftings of some path from $\Gamma'_m$ starting from some point of $|q_m|$. Let $S_m = \Delta(|q_m|, E, B(x,r)\cap D)$ and $T_m = \Delta(|q_m|, \partial D\setminus E, B(x,r)\cap D)$. Let $\beta : [0,1] \to \mathbb{R}^n$, $\beta \in \Gamma'_m$ and let $\alpha : [0,a) \to B(x,r)\cap D$, $0 < a < 1$ be a maximal lifting of $\beta$ with $\alpha(0) \in |q_m|$. Such a path $\alpha$ exists due to the openness and discreteness of the mapping $f$ and since $\lim\limits_{t\to 1}\beta(t) \notin f(B(x,r)\cap D)$. Also, using the openness of the mapping $f$ we see that any limit point of the open path $\alpha : [0,a) \to B(x,r)\cap D$ either belongs to $\partial D$, or intersects $S(x,r)$.
We see that $\Gamma_m \subset \Gamma_{x,b,r} \cup S_m \cup T_m$ and let us show that $T_m = \phi$. Indeed, if $\alpha : [0,a) \to B(x,r)\cap D$ is a maximal lifting of some path $\beta \in \Gamma'_m$ which has a limit point $z \in \partial D\setminus E$, let $t_k \nearrow a$. Then $f(\alpha(t_k)) = \beta(t_k) \in |\beta|$ and $\beta \in \Gamma'_m$ and hence $f(\alpha(t_k)) \in B(y, 2\delta)$ for $k \in \mathbb{N}$. On the other side, we see that $\beta(t_k) \in B(C(f,x,\overline{B}(x,r)\cap((\partial D\setminus E)|\{x\})),\delta)$ for $k$ great enough and since $B(y,3\delta)\cap C(f,x,\overline{B}(x,r)\cap((\partial D\setminus E)\setminus\{x\})) = \phi$, we reached a contradiction.
We proved that $T_m = \phi$ and hence $\Gamma_m \subset \Gamma_{x,b,r} \cup S_m$ and we also see that $\Gamma'_m > f(\Gamma_m)$. We have

$$\begin{aligned} 0 < \rho < M_p(\Gamma'_m) &\le M_p(f(\Gamma_m)) \\ &\le M_p(f(\Gamma_{x,b,r} \cup S_m)) \\ &\le M_p f((\Gamma_{x,b,r})) + M_p(f(S_m)) \\ &\le \gamma(M^q_\omega(\Gamma_{x,b,r})) + \gamma(M^q_\omega(S_m)) \le \gamma(\epsilon). \end{aligned}$$

Letting now $\epsilon$ small enough such that $\gamma(\epsilon) < \rho$, we reached a contradiction.

We proved that $cap_p(B(y,2\delta), F) = 0$ for every $0 < \delta < \frac{\delta_y}{3}$ and every set $F \subset B(y,\delta)\setminus f(B(x,r)\cap D)$ and this implies that $cap_p(B(y,\frac{1}{3}\delta_y)\setminus f(B(x,r)\cap D)) = 0$. Since this holds for every $r > 0$, it results that $cap_p(B(y,\frac{1}{3}\delta_y)\setminus R_D(f,x)) = 0$. Let $W = \{y \in Q | cap_p(B(y,\frac{1}{3}\delta_y)\setminus R_D(f,x)) = 0\}$. We see that $W$ is an open subset of $Q$ and using the preceeding arguments we show that $W$ is also a closed subset of $Q$ and since $Q$ is connected, we see that $Q = W$ and the theorem is now proved.

*Proof of Theorem 1.2.* Suppose that $C(f,x)\cap(\mathbb{R}^n\setminus K) \ne \phi$ and let $Q \supset \mathbb{R}^n\setminus K$ be the unbounded component of $C(f,x)\setminus C(f,x,\partial D\setminus E)$. We see that $Q \ne \phi$ and using Theorem 1 we find a set $F \subset \mathbb{R}^n$ with $cap_p(F) = 0$ and $f(D) \supset Q\setminus F \supset \mathbb{R}^n\setminus(K\cup F)$. This implies that $f$ is unbounded and we reached a contradiction. We therefore proved that $C(f,x) \subset K$.

*Proof of Theorem 1.3.* Let $M = \limsup\limits_{\substack{y\to x\\ y\in\partial D\setminus E}}(\limsup\limits_{z\to y}|f(z)|)$.

Since $f$ is bounded, we see that $M < \infty$ and since $n \ge 2$ it results that $\mathbb{R}^n\setminus\overline{B}(0,M)$ is connected. Now $C(f,x,\partial D\setminus E) \subset \overline{B}(0,M)$ and from Theorem 2 we find that $C(f,x) \subset \overline{B}(0,M)$. Then $\limsup\limits_{y\to x}|f(y)| \le M$ and the theorem is now proved.

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# Polynomial estimates for solutions of parametric elliptic equations on complete manifolds

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*We dedicate this paper to the memory of Professor Gabriela Kohr*

**Abstract.** Let $P : \mathcal{C}^\infty(M;E) \to \mathcal{C}^\infty(M;F)$ be an order $\mu$ differential operator with coefficients $a$ and $P_k := P : H^{s_0+k+\mu}(M;E) \to H^{s_0+k}(M;F)$. We prove polynomial norm estimates for the solution $P_0^{-1}f$ of the form

$$\|P_0^{-1}f\|_{H^{s_0+k+\mu}(M;E)} \leq C \sum_{q=0}^{k} \|P_0^{-1}\|^{q+1} \, \|a\|^q_{W^{|s_0|+k}} \, \|f\|_{H^{s_0+k-q}},$$

(thus in higher order Sobolev spaces, which amounts also to a parametric regularity result). The assumptions are that $E, F \to M$ are Hermitian vector bundles and that $M$ is a complete manifold satisfying the Fréchet Finiteness Condition (FFC), which was introduced in (Kohr and Nistor, Annals of Global Analysis and Geometry, 2022). These estimates are useful for uncertainty quantification, since the coefficient $a$ can be regarded as a vector valued random variable. We use these results to prove integrability of the norm $\|P_k^{-1}f\|$ of the solution of $P_k u = f$ with respect to suitable Gaussian measures.

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## 1. Introduction

### 1.1. A short summary

Let $M$ be a Riemannian manifold and $E, F \to M$ be Hermitian vector bundles. We let $\nabla$ denote a generic connection on vector bundles. On $E$ and $F$, $\nabla$ is given, whereas on $TM$ we consider the Levi-Civita connection. Let

$$\nabla^j = \nabla \circ \nabla \circ \ldots \circ \nabla : \mathcal{C}^\infty(M;E) \to \mathcal{C}^\infty(M;T^{*\otimes j}M \otimes E)$$

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be the $j$-times composition of the covariant derivative on $E$, namely the composition of the maps $\nabla : \mathcal{C}^\infty(M; T^{*\otimes i}M \otimes E) \to \mathcal{C}^\infty(M; T^{*\otimes(i+1)}M \otimes E)$.

In this paper, we study an order $\mu \geq 1$ differential operator

$$P \;:=\; a \cdot \nabla^{tot} \;:=\; \sum_{j=0}^{\mu} a^{[j]} \nabla^j \tag{1.1}$$

acting on sections of $E$ and with values sections of $F$. This setting allows us to consider *systems* of partial differential operators (PDEs). Let $s_0 \in \mathbb{Z}$ (fixed throughout the paper) and $k \in \mathbb{Z}_+$ and let

$$P_k \;:=\; a \cdot \nabla^{tot} : H^{s_0+k}(M; E) \to H^{s_0+k+\mu}(M; F) \tag{1.2}$$

denote the operator induced by $P$ on the indicated Sobolev spaces (which is defined and continuous provided that $a$ is smooth enough, see Lemma 2.2). Our *main result* is to prove polynomial bounds for $\|P_k^{-1} f\|_{H^{s_0+k+\mu}}$ in terms of $\|P_0^{-1}\|$ and the norm of the coefficient $a$ under suitable hypotheses on $M$, $E$, and $F$ (Theorem 1.3).

We apply these estimates to the integrability of the norm of $P_k^{-1}$ (in $\mathcal{L}(H^{s_0+k}(M; F), H^{s_0+k+\mu}(M; E))$) with respect to suitable Gaussian measures. This type of estimate is useful for uncertainty quantification, see [3, 7, 11, 13, 15, 16, 20].

Our results apply to complete manifolds $M$ that satisfy the Fréchet Finiteness Condition (FFC), a condition that was introduced in [14] and will be recalled shortly. It was proved in Lemma 3.1 of [9] that manifolds with bounded geometry satisfy (FFC). Consequently, every open subset of a manifold with bounded geometry satisfies (FFC). In particular, all compact manifolds and all euclidean spaces satisfy (FFC).

### 1.2. Basic concepts

To formulate our result more precisely, we need to introduce some notation and terminology and to remind some basic definitions. If $E \to M$ is a vector bundle, then $\mathcal{M}(M; E)$ denotes the set of measurable sections of $E$. We shall use in the following $\nabla$-*differential operators* [14]. To define them, let the *truncated Fock space* $\mathcal{F}_\mu^M(E)$ be defined by

$$\mathcal{F}_\mu^M(E) \;:=\; \oplus_{j=0}^{\mu} T^{*\otimes j}M \otimes E\,. \tag{1.3}$$

**Definition 1.1.** Let $E, F \to M$ be vector bundles, with $E$ endowed with a connection and let $a = (a^{[0]}, a^{[1]}, \ldots, a^{[\mu]})$ be a measurable section of $\mathrm{Hom}(\mathcal{F}_\mu^M(E); F)$, $\nabla^0 := id$. A $\nabla$–*differential operator (on $E$ with values in $F$)* is a map (see Equation (1.1))

$$P \;=\; a \cdot \nabla^{tot} \;:=\; \sum_{j=0}^{\mu} a^{[j]} \nabla^j : \mathcal{C}^\infty(M; E) \to \mathcal{M}(M; F)\,.$$

The order of $P$, denoted $\mathrm{ord}(P)$, is the least $\mu$ for which such a writing exists.

If $E$ is Hermitian, we let

$$W_\nabla^{k,\infty}(M; E) \;:=\; \{u \in \mathcal{M}(M; E) \mid \nabla^j u \in L^p(M; E), 0 \leq j \leq k\}$$

be the space of sections of $E$ whose first $k$ covariant derivatives are bounded, as in [2, 12, 14]. For $k \leq 0$ and $k \notin \mathbb{Z}$, we proceed by duality and interpolation (see [14], for instance). We let $H^s(M;E) := W_\nabla^{s,2}(M;E)$ and $W_\nabla^{\infty,\infty}(M;E) := \cap_{k\geq 0} W_\nabla^{k,\infty}(M;E)$. Recall the Fréchet Finiteness Conditions (FFC), see [14, Definition 5.8].

**Definition 1.2.** Let $M$ be a Riemannian manifold with a metric $g$. We say that $(M,g)$ satisfies the *Fréchet finiteness condition* (FFC) if there exist $N \in \mathbb{N}$ and an isometric (vector bundle) embedding $\Phi : TM \to M \times \mathbb{R}^N$ ,, $\Phi \in W_\nabla^{\infty,\infty}(M; \mathrm{Hom}(TM;\mathbb{R}^N))$ , where, in order to define the Sobolev space $W_\nabla^{\infty,\infty}(M; \mathrm{Hom}(TM;\mathbb{R}^N))$, we consider the trivial connection on the vector bundle $M \times \mathbb{R}^N \to M$.

### 1.3. Statement of the main result

It is known that the operator $P_k := a\cdot\nabla^{tot} : H^{s_0+k+\mu}(M;E) \to H^{s_0+k}(M;F)$ of Equation (1.2) is well-defined and continuous if $a \in W_\nabla^{|s_0+k|,\infty}(M; \mathrm{Hom}(\mathcal{F}_\mu^M(E);F))$ (see Lemma 2.2). A vector bundle $E$ is said to have *totally bounded curvature* if its curvature tensor is in $W^{\infty,\infty}(M;\Lambda^2T^*M \otimes \mathrm{End}(E))$. We are ready to state our main result. Recall that, throughout this paper, we have fixed $s_0 \in \mathbb{Z}$.

**Theorem 1.3.** *Let us assume that $M$ is a complete manifold satisfying (FFC) and that $E$ and $F$ have totally bounded curvature. Let $a \in W_\nabla^{|s_0|+k,\infty}(M; \mathrm{Hom}(\mathcal{F}_\mu^M(E);F))$ and $P_k := a\cdot\nabla^{tot} : H^{s_0+k+\mu}(M;E) \to H^{s_0+k}(M;F)$, $k \in \mathbb{Z}_+$. Let us assume that $P_0$ is invertible. Then $\mathfrak{C} := \|P_0^{-1}\|\,\|a\|_{W^{|s_0|+k}} \geq 1$. Let $f \in H^{s_0+k}(M;F)$, so $P_0^{-1}f \in H^{s_0+\mu}(M;E)$ is defined, then, in fact, $P_0^{-1}f \in H^{s_0+k+\mu}(M;E)$, and*

$$\|P_0^{-1}f\|_{H^{s_0+k+\mu}} \lesssim \|P_0^{-1}\| \sum_{q=0}^{k} \mathfrak{C}^q \,\|f\|_{H^{s_0+k-q}} \,. \tag{$I_k$}$$

*Consequently, $P_k$ is an isomorphism with $\|P_k^{-1}\| \leq \|P_0^{-1}\|^{k+1}\,\|a\|_{W^{|s_0|+k}}^k$.*

For operators in divergence form, we obtain a slightly better result in that we may allow lower regularity for $a$, as in [18]. A consequence of our results is the integrability of $\|P_k^{-1}f\|_{H^{k+\mu}}$ for operators of divergence form of *order* $2m$ with respect to certain measures of Gaussian type on the set of coefficients $a$, see Theorem 5.3. We stress that a particular, but important, special case of our results is when $M$ is compact without boundary. The case of bounded domains is discussed in [19].

### 1.4. Contents of the paper

The main result is stated in the Introduction states, where we also recall some needed concepts. Section 2 contains some preliminary material, including a version of Nirenberg's trick following [6, 18]. The third section is devoted to proving that a totally bounded vector field (i.e. one in $W_\nabla^{\infty,\infty}(M;TM)$) integrates to a global one-parameter groups of diffeomorphisms of $M$ and of automorphisms of our Sobolev spaces. The forth section is devoted to the proof of the main result (Theorem 1.3) following the method from [18]. The integrability of $\|P_k^{-1}f\|_{H^{s_0+k+\mu}}$ with respect to suitable Gaussian measures on the space of coefficients is proved in the last section.

## 2. Operators and Nirenberg's trick

Here we describe the ingredients needed to formulate our main result in more detail. We also recall some needed results, including an extension of Nirenberg's trick. See [4, 5, 8, 17, 21] for concepts and results that are not discussed in this article.

### 2.1. Operators and their norms

The following notation will be used throughout the paper. It was already used in the statement of the main result. We fix $\mu \in \mathbb{N} = \{1, 2, \ldots\}$, which will be the order of the operator $P := a \cdot \nabla^{tot}$ that we study. We also fix throughout this paper $s_0 \in \mathbb{Z}$, which will be the order of the minimal regularity Sobolev spaces where we assume the invertibility of $P$. We let $\sigma := s_0 + k$, to simplify the notation. We shall usually write $\|u\|_{W^k}$ (or even $\|a\|_k$) instead of $\|u\|_{W^{k,\infty}(M;E)}$ and $\|u\|_{H^s}$ instead of $\|u\|_{H^s(M;E)}$. Recall that if $X$ and $Y$ are two normed spaces, then $\mathcal{L}(X, Y)$ denotes the space of linear, continuous operators $X \to Y$. Let $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$.

**Notation 2.1.** Let $k \in \mathbb{Z}_+$ and $s_0$, $\mu$, and $\sigma := s_0 + k$ as above. We shall write $\|T_1\|_k := \|T_1\|_{\mathcal{L}(H^{\sigma+\mu}(M;E);H^{\sigma}(M;F))}$ and $|||T_2|||_k := \|T_2\|_{\mathcal{L}(H^{\sigma}(M;F);H^{\sigma+\mu}(M;E))}$. We shall write $D_1 \lesssim D_2$ if there is $C_{\Phi,k,M,E,F} > 0$ such that $D_1 \leq C_{\Phi,k,M,E,F} D_2$, where $C_{\Phi,k,M,E,F}$ depends only on $\Phi$, $k$, $M$, $E$, and $F$, where $\Phi$ is as in the Definition 1.2, $k$ some other parameter, usually related to the order of the Sobolev spaces involved, $M$ is our manifold and $E, F \to M$ are the vector bundle involved. The next result follows from [14, Proposition 3.7].

**Lemma 2.2.** *Let $E, F, E_j$ be Hermitian vector bundles, $j = 1, 2, 3$.*
*Let $b_j \in W^{k,\infty}_{\nabla}(M; \mathrm{Hom}(E_j, E_{j+1}))$, $j = 1, 2$, and $f \in H^s(M; E_1)$.*

*(i)* $\|b_2 b_1\|_{W^k} \lesssim \|b_2\|_{W^k} \|b_1\|_{W^k}$.

*(ii)* $\|b_1 f\|_{H^s} \lesssim \|b_1\|_{W^k} \|f\|_{H^s}$, *if* $|s| \leq k$.

*(iii) Let $P_k := a \cdot \nabla^{tot} : H^{\sigma+\mu}(M; E) \to H^{\sigma}(M; F)$. Then $P_k$ is continuous with norm $\|P_k\|_k \lesssim \|a\|_{W^{|s_0|+k}}$, if $a \in W^{|s_0|+k,\infty}_{\nabla}(M; \mathrm{Hom}(\mathcal{F}^M_m(E); F))$.*

*Here $k, \mu \in \mathbb{Z}_+$, $s, s_0 \in \mathbb{Z}$, and $\sigma := s_0 + k$.*

As in [18], we obtain the following simple lemma.

**Lemma 2.3.** *We use the notation of Lemma 2.2 and assume $P_k$ is invertible. Then* $1 \lesssim |||P_k^{-1}|||_k \, \|a\|_{W^{|s_0|+k}}$.

### 2.2. Nirenberg's trick

In this section we recall a version of Nirenberg's trick, as it is formalized in [6, 18]. We write $t \searrow 0$ if $t \to 0$ and $t > 0$. Let $X$ and $Y$ be two Banach spaces, recall that a family $(T_t)_{t \geq 0}$ in $\mathcal{L}(X, Y)$ converges strongly to $T$ for $t \searrow 0$ if $\lim_{t \searrow 0} \|T_t u - Tu\|_Y = 0$, for all $u \in X$. We shall need also the following basic concept.

**Definition 2.4.** A family of operators $(S(t))_{t \geq 0}$ of $\mathcal{L}(X) := \mathcal{L}(X, X)$ is *a strongly continuous semigroup* on $X$ if the following conditions are satisfied: $S(0) = id_X$, for all $t \geq 0$ and $r \geq 0$, $S(t + r) = S(t)S(r)$, and, for all $x \in X$, $\lim_{t \to 0} \|S(t)x - x\|_X = 0$. Then *the infinitesimal generator of* $(S(t))_{t \geq 0}$ is the operator $(L_S, \mathcal{D}(L_S))$ defined by

$$\mathcal{D}(L_S) \ := \ \{x \in X \mid \ L_S x \ := \ \lim_{t \to 0} t^{-1}\big(S(t)x - x\big) \text{ exists in } X\}\,.$$

The following lemma [6, 18] will play an essential role in what follows. The version here is a simplified one compared to the ones in the aforementioned articles.

**Proposition 2.5.** *Let $T : X \to Y$ be an invertible bounded operator between two Banach spaces and let $S_X(t) \in \mathcal{L}(X)$ and $S_Y(t) \in \mathcal{L}(Y)$ be two strongly continuous semigroups of operators. We assume that, for each $t \in \mathbb{R}$, there exists $T_t \in \mathcal{L}(X,Y)$ such that $T_t S_X(t) = S_Y(t)T$. Suppose that $t^{-1}(T_t - T)$ converges strongly to $Q \in \mathcal{L}(X,Y)$ for $t \searrow 0$. Then, for all $v$ in $\mathcal{D}(L_Y)$, we have*

$$T^{-1}v \in \mathcal{D}(L_{S_X}) \quad \textit{and} \quad L_{S_X} T^{-1} v = T^{-1} L_{S_Y} v - T^{-1} Q T^{-1} v.$$

In our case, at least one of the assumptions of this result will be easy to check.

**Remark 2.6.** In our applications, both $S_X$ and $S_Y$ will extend to *groups* of operators. Thus, $S_X(t)$ and $S_Y(t)$ are defined for $t \in \mathbb{R}$ with the usual group laws:

$$S_X(t)S_X(t') = S_X(t+t') \text{ and } S_Y(t)S_Y(t') = S_Y(t+t').$$

Therefore, the existence of $T_t$ satisfying $T_t S_X(t) = S_Y(t)T$ is guaranteed simply by taking $T_t := S_Y(t) T S_X(-t)$.

## 3. Groups of diffeomorphisms and Sobolev spaces

In this section, we systematically use vector fields to define our Sobolev spaces.

### 3.1. Vector fields and Sobolev spaces

Assume $M$ satisfies (FFC). Let $\Phi : TM \to M \times \mathbb{R}^N$ be as in Definition 1.2 and $e_j$, $j = 1, \ldots, N$, be the canonical basis of $\mathbb{R}^N$. Then $Z_1, Z_2, \ldots, Z_N \in \mathcal{W}_b(M) := W_\nabla^{\infty,\infty}(M;TM)$ will continue to denote a Fréchet system of generators of $\mathcal{W}_b(M)$ as $\mathcal{C}_\mathrm{b}^\infty$-module, as in [14], that is,

$$Z_j \ := \ \Phi^T(e_j)\,, \tag{3.1}$$

We shall need the following proposition from [14].

**Proposition 3.1.** *Let us assume that $M$ satisfies (FFC) and let $\{Z_j\}$ be a Fréchet system of generators of $\mathcal{W}_b(M)$, as in Equation (3.1). Let $\ell \in \mathbb{N}$ and $1 \le p \le \infty$. Then*

$$W_\nabla^{\ell,p}(M;E) \ = \ \{\, u \mid \nabla^E_{Z_{k_1}} \nabla^E_{Z_{k_2}} \ldots \nabla^E_{Z_{k_j}} u \in L^p(M;E),\ j \le \ell,\ 1 \le k_i \le N \,\}.$$

We shall need the following standard consequence. (See also [18].)

**Lemma 3.2.** *Let $Z_0 u := u$ and $Z_k u := \nabla_{Z_k}(u)$, for simplicity, with $Z_j$ as in Equation (3.1). Let $s \in \mathbb{Z}_+$ and*

$$\|u\|' \ := \ \sum_{i=0}^{N} \|Z_i u\|_{H^s}\,.$$

*Then $\|u\|'$ defines an equivalent norm on $H^{s+1}(M;E)$.*

*Proof.* This follows right away from Proposition 3.1. □

### 3.2. Diffeomorphism groups

We shall need the fact that vector fields $X \in W^{\infty,\infty}_{\nabla}(M;TM)$ integrate to global diffeomorphisms groups and then that these diffeomorphism groups lift to automorphism groups of vector bundles.

**Proposition 3.3.** *Let $X \in W^{\infty,\infty}_{\nabla}(M;TM)$ (that is, $X$ is a totally bounded smooth vector field on $M$). Assume that $M$ is complete, then $X$ generates a one-parameter group of diffeomorphisms $\phi_t : M \to M$, $t \in \mathbb{R}$.*

A one-parameter group of diffeomorphisms (of $M$) will also be called a "flow (on $M$)."

*Proof.* The proof is almost the same as the one of the existence of global geodesics on complete Riemannian manifolds. First, the existence of a local family $\phi_t$ is a classical result in ordinary differential equations and differential geometry. Then, for each $x \in M$, the curve $\phi_t(x)$ is an integral curve of the vector field $X$ and is defined at least on some interval $(-\epsilon, \epsilon)$, where $\epsilon > 0$ may depend on $x$. We need to show that this curve extends indefinitely for each $x$ under the assumption that our manifold $M$ is complete. We shall proceed by contradiction. For any given $x \in M$, let $I \subset \mathbb{R}$ be a maximal interval on which the integral curve $\phi_t(x)$, $t \in I$, is defined. Let us assume $I \neq \mathbb{R}$ and let $a \in \overline{I} \smallsetminus I$. Let also $t_n \in I$, $t_n \to a$. As $X$ is bounded, we have $dist(\phi_{t_n}(x), \phi_{t_m}(x)) \le \|X\|_{L^\infty}|t_n - t_m|$, for all $n, m \in \mathbb{N}$. Hence, the sequence $\phi_{t_n}(x)$ is a Cauchy sequence. Since we have assumed that $M$ is complete, this sequence has a limit $y \in M$. Therefore $\lim_{t\to a, t\in I} \phi_t(a) = y$ exists. Then the local existence of the flow generated by $X$ in a neighborhood of $y$ will allow us to extend the flow $\phi_t(x)$ for $t$ past $a$ by setting $\phi_{a+t}(x) = \phi_t(y)$ for $|t|$ small. This is a contradiction and hence our result is proved. □

**Lemma 3.4.** *We use the notation and the assumptions of Proposition* 3.3*, in particular, $X \in W^{\infty,\infty}_{\nabla}(M;TM)$ generates the flow $\phi_t : M \to M$, $t \in \mathbb{R}$. For any vector bundle $E$ endowed with a connection, the parallel transport $\tau_t$ along the integral curves of $X$ generates a one-parameter group of automorphisms of $\mathcal{C}^\infty(M;E)$.*

*Proof.* This is a classical result. Indeed, the definition of the parallel transport $\tau_t$ amounts to solving a linear system of ordinary differential equations (ODEs) along each of the integral curves $\phi_t(x)$ of $\phi$. We know by Proposition 3.3 that the integral curves of $X$ extend indefinitely, so $\phi_t(x)$ is defined for all $t \in \mathbb{R}$ and $x \in M$. Hence we have global solutions for the system of ODEs defining the parallel transport (since the integral curves of $\phi$ extend indefinitely and the ODE system is linear). □

In what follows, one should distinguish between the parallel transport $\tau_t$ and the map $\phi_{t*} : \mathcal{C}^\infty(M;T) \to \mathcal{C}^\infty(M;T)$, where $T$ is a tensor bundle on $M$ (tensor product of $TM$ and $T^*M$ or canonical subbundles) and $\phi_{t*}$ is the map induced by the diffeomorphism $\phi_t : M \to M$. We shall use this construction for $T = TM$ and $T = \mathbb{R}$ (plain functions). Of course, if $\alpha$ is a function, then $\tau_t(\alpha) = \phi_{t*}(\alpha)$.

**Theorem 3.5.** *We use the notation and the assumptions of Proposition* 3.3 *and Lemma* 3.4*. Let $k \in \mathbb{Z}_+$. Let us assume also that $E \to M$ is a Hermitian vector bundle endowed with a metric-preserving connection with totally bounded curvature. Then the parallel* *transport* $\tau = (\tau_t^*)_{t\in\mathbb{R}}$ *defines a one-parameter group of continuous operators on all spaces* $W^{k,p}_{\nabla}(M;E)$, $1 \le p \le \infty$, *such that, if* $k \ge 1$ *and* $Z \in W^{k-1,\infty}_{\nabla}(M;TM)$, *then*

$$\begin{aligned} A(t;X,Z) \ &:= \ \tau_t(Z) - \phi_{t*}(Z) \in W^{k-1,\infty}_{\nabla}(M;TM)\,, \\ &\phi_{t*} \text{ is bounded on } W^{k-1,\infty}(M;TM) \quad \text{and} \\ B(t;X,Z) \ &:= \ \tau_t \nabla_Z \tau_{-t} - \nabla_{\phi_t^*(Z)} \in W^{k-1,\infty}_{\nabla}(M;\mathrm{End}(E))\,, \end{aligned} \tag{3.2}$$

*with* $\mathcal{C}^\infty$*-dependence for* $A$ *and* $B$ *on* $t \in \mathbb{R}$ *if* $Z \in W^{\infty,\infty}_{\nabla}(M;TM)$. *If* $p < \infty$, *the resulting group* $(\tau_t)_{t\in\mathbb{R}}$ *is strongly continuous. The infinitesimal generator* $L_\tau$ *of* $\tau$ *acting on* $W^{k+1,p}(M;E)$ *satisfies* $L_\tau\xi = \nabla_X\xi$ *for* $\xi \in W^{k+1,p}(M;E)$.

*Proof.* We shall prove our result by induction on $k$. Let us assume $k = 0$. Since the connection on $E$ is metric-preserving, the parallel transport will be isometric between the fibers of $E$. To obtain the desired result on the boundedness of the induced operator, it is enough to notice that the volume form is increased by at most a bounded factor since $\mathrm{div}(X)$ is bounded (i.e. in $L^\infty$), by the results of [14]. (For $k = 0$ there is nothing to check about $\phi_{t*}$ or the functions $A$ and $B$ of Equation (3.2).)

Let us assume now that the result is true for $k-1 \ge 0$ and let us prove it for $k$. We will first prove the result for $A$, then the boundedness of $\phi_{t*}$ on the $W^{k-1,\infty}_{\nabla}(M;TM)$ spaces, then the result for $B$ and, finally, we will check the boundedness of $\tau_t$ on the $W^{k,p}_{\nabla}$ spaces.

Let us prove the result for $A(t;X,Z)$. If, furthermore, $Z \in W^{k,p}(M;TM)$ (slightly better regularity than in the statement), then

$$\begin{aligned} \partial_t A(t;X,Z) \ &= \ \partial_t\big(\tau_t(Z) - \phi_{t*}(Z)\big) \\ &= \ \tau_t(\nabla_X(Z)) - \phi_{t*}([X,Z]) \\ &= \ \tau_t([X,Z]) - \phi_{t*}([X,Z]) + \tau_t(\nabla_Z(X)) \\ &= \ A(t;X,[X,Z]) + \tau_t(\nabla_Z(X))\,. \end{aligned}$$

We shall use this relation for all $Z = Z_j$, $j = 1,\dots,N$, where $\{Z_j\}$ is a Fréchet system of generators of $W^{\infty,\infty}_{\nabla}(M;TM)$, as in Equation (3.1). We have $X, Z_j \in W^{\infty,\infty}_{\nabla}(M;TM)$, and hence $\nabla_{Z_j}(X) \in W^{\infty,\infty}_{\nabla}(M;TM)$ [14]. The induction hypothesis tells us that $\tau_t$ is bounded on the space $W^{k-1,\infty}_{\nabla}(M;TM)$, which gives then that $\tau_t(\nabla_{Z_j}(X)) \in W^{k-1,\infty}_{\nabla}(M;TM)$. We then express each $[X,Z_j] = \sum_{ji} C_{ji}Z_i$, with $C_{ji} \in W^{\infty,\infty}_{\nabla}(M)$, as in [14]. This yields an inhomogeneouse linear system of ODEs in $W^{k-1,p}(M;TM)$ for $A(t;X,Z_j)$ with free term $\tau_t(\nabla_{Z_j}(X)) \in W^{k-1,\infty}_{\nabla}(M;TM)$. Since $A(0;X,Z) = 0$ and since $\tau_t$ preserves $W^{k-1,\infty}_{\nabla}(M;TM)$, we obtain the desired result that $A(t;X,Z_j) \in W^{k-1,\infty}_{\nabla}(M;TM)$ for $Z = Z_j$. Next, we use use the linearity of $A(t;X,Z)$ in $Z \in W^{k-1,p}(M;TM)$ (same regularity now as in the statement) and express $Z = \sum_{j=1}^N \alpha_j Z_j$ as a linear combination of the Fréchet system of generators $\{Z_j\}$, $j = 1,\dots,N$ with coefficients $\alpha_j \in W^{k-1,\infty}_{\nabla}(M)$. We also notice that

$A(t;X,\alpha Z)=\tau_t(\alpha)A(t;X,\alpha Z)$ for $\alpha$ a function, by the definition of $A$. This gives

$$A(t;X,Z) \;=\; \sum_{j=1}^{N} A(t;X,\alpha_j Z_j) \;=\; \tau_t(\alpha_j)\sum_{j=1}^{N} A(t;X,Z_j) \in W_{\nabla}^{k-1,\infty}(M;TM)\,,$$

where we have used again the induction hypothesis for $\tau_t$ acting on $W_{\nabla}^{k-1,\infty}(M)$.

Let us check next that $\phi_{t*}$ is bounded on the spaces $W_{\nabla}^{k-1,\infty}(M;TM)$. Let $Z\in W_{\nabla}^{k-1,\infty}(M;TM)$. The result we have just proved for $A$ gives

$$\phi_{t*}(Z) \;:=\; \tau_t(Z)-A(t;X,Z)\in W_{\nabla}^{k-1,\infty}(M;TM)$$

Hence $\phi_{t*}$ maps $W_{\nabla}^{k-1,\infty}(M;TM)$ to itself. Since $\phi_{t*}$ is continuous in the sense of distributions (by Lemma 3.4) it has closed graph, and hence it is continuous.

Let us now turn to the study of the term $B(t;X,Z)$, which will be similar. Let $\Omega$ be the curvature of $E$. Our assumption that $E$ has totally bounded curvature amounts to $\Omega\in W_{\nabla}^{\infty,\infty}(M;\Lambda^2T^*M\otimes \mathrm{End}(E))$. Again for $Z$ with a little bit more regularity, we have

$$\begin{aligned}\partial_t B(t;X,Z) \;&=\; \partial_t\big[\tau_t\nabla_Z\tau_{-t}-\nabla_{\phi_{t*}(Z)}\big]\\ &=\; \tau_t\big(\nabla_X\nabla_Z-\nabla_Z\nabla_X\big)\tau_{-t}-\nabla_{\phi_{t*}([X,Z])}\\ &=\; \tau_t\big(\nabla_{[X,Z]}+\Omega(X,Z)\big)\tau_{-t}-\nabla_{\phi_{t*}([X,Z])}\\ &=\; B(t;X,[X,Z])+\tau_t\Omega(X,Z)\tau_{-t}\,.\end{aligned}$$

We know by the induction hypothesis (boundedness of $\tau_t$ on $W_{\nabla}^{k-1,\infty}$ and $E$ with totally bounded curvature) that $\tau_t\Omega(X,Z)\tau_{-t}\in W_{\nabla}^{k-1,\infty}(M;\mathrm{End}(E))$. We complete the discussion for $B$ as we did for $A$. That is, we notice first that $B(t;X,Z)$ is linear in $Z$ and that $B(t;X,\alpha Z)=\tau_t(\alpha)B(t;X,Z)$ for $\alpha$ a smooth enough function. We then use the last displayed equation for $Z$ ranging through the vectors of a Fréchet system of generators $\{Z_j\}$. We next express $[X,Z_j]=\sum_j \alpha_j Z_j$, and we use linearity to obtain an inhomogeneous linear system of ODEs for $B(t;X,Z_j)$. Since $B(0,X,Z)=0$, since $\tau_t$ preserves $W_{\nabla}^{k-1,\infty}(M;TM)$, and since $\tau_t\Omega(X,Z)\tau_{-t}\in W_{\nabla}^{k-1,\infty}(M;\mathrm{End}(E))$, we obtain $B(t;X,Z_j)\in W_{\nabla}^{k-1,\infty}(M;\mathrm{End}(E))$. By using this relation, the linearity of $B(t;X,Z)$ in $Z$, and by expressing $Z$ as a linear combination with $W_{\nabla}^{k-1,\infty}(M)$ coefficients of the Fréchet basis $\{Z_j\}$, $j=1,\dots,N$, we obtain the desired result that $B(t;X,Z) \;:=\; \tau_t\nabla_Z\tau_{-t}-\nabla_{\phi_{t*}(Z)}\in W_{\nabla}^{k-1,\infty}(M;\mathrm{End}(E))$.

It remains to prove that $\tau_t$ is maps continuously $W_{\nabla}^{k,p}(M;E)$ to itself. Let us prove this without checking the continuity in $t$. Let $\xi\in W_{\nabla}^{k,p}(M;E)$. By Lemma 3.2, it is enough to check that $\nabla_{Z_j}\tau_{-t}(\xi)\in W_{\nabla}^{k-1,p}(M;E)$ for all $j=1,\dots,N$. We have just proved that $\phi_{t*}(Z_j)\in W_{\nabla}^{k-1,\infty}(M;TM)$. Hence $\phi_{t*}(Z_j)$ can be expressed as a linear combination of the vectors $Z_i$ with coefficients in $W_{\nabla}^{k-1,\infty}(M)$ and therefore $\nabla_{\phi_{t*}(Z_j)}\xi\in W_{\nabla}^{k-1,p}(M;E)$. The relation we proved for $B$, namely $B(t;X,Z)\in W_{\nabla}^{k-1,\infty}(M;\mathrm{End}(E))$ gives also $B(t;X,Z)\xi\in W_{\nabla}^{k-1,p}(M;E)$. Putting all this together we obtain

$$\tau_t \nabla_{Z_j} \tau_{-t}\xi \;=\; \nabla_{\phi_{t*}(Z_j)}\xi + B(t;X,Z)\xi \in W^{k-1,p}_{\nabla}(M;E)\,.$$

The desired result follows by multiplying the last equation to the left with $\tau_{-t}$ (which is bounded on $W^{k-1,p}_{\nabla}(M;E)$ by induction). Hence $\tau_t$ maps $W^{k,\infty}_{\nabla}(M;TM)$ to itself. Since $\tau_t$ is continuous in the sense of distributions (Lemma 3.4) it has closed graph, and hence $\tau_t : W^{k,\infty}_{\nabla}(M;TM) \to W^{k,\infty}_{\nabla}(M;TM)$ is continuous.

The smoothness of $A$ and $B$ as functions of $t$ follows from the fact that the free term of the linear equations defining them are smooth functions of $t$ and from the smoothness of $\tau_t(\alpha)$ if $\alpha \in W^{\infty,\infty}_{\nabla}(M)$.

Finally, the continuity of the group $\tau_t$ follows from the continuity on smooth sections with compact support and the density of those in the spaces $W^{k,p}_{\nabla}$ for $p<\infty$. The statement on the infinitesimal generator is proved by the same argument. □

The following consequence will allow us to check the hypotheses of Proposition 2.5.

**Proposition 3.6.** *Let $a \in W^{k+1,\infty}_{\nabla}(M;\mathrm{Hom}(\mathcal{F}^M_\mu(E);F))$ and $X \in W^{\infty,\infty}_{\nabla}(M;TM)$. We assume $E$ to have totally bounded curvature and we use the notation in Theorem 3.5. Let $\tau = (\tau_t)_{t\in\mathbb{R}}$ be the one-parameter group defined by parallel transport on any given Sobolev space $H^k$ or $W^{k,\infty}_{\nabla}$.*

- *(i) $[\nabla_X, a\cdot\nabla^{tot}] = \nabla_X(a)\cdot\nabla^{tot} + a\cdot[\nabla_X,\nabla^{tot}]$ is a differential operator of order $\le \mu$ with coefficients in $W^{k,\infty}_{\nabla}(M;\mathrm{Hom}(\mathcal{F}^M_\mu(E);F))$.*
- *(ii) $\|[\nabla_X, a\cdot\nabla^{tot}]\|_{\mathcal{L}(H^{k+\mu}(M;E);\mathcal{L}(H^k(M;F))} \lesssim \|a\|_{W^{k+1}_{\nabla}}$.*
- *(iii) $t^{-1}\big[\tau_t(a\cdot\nabla^{tot})\tau_{-t} - a\cdot\nabla^{tot}\big]$ converges strongly to $[\nabla_X, a\cdot\nabla^{tot}]$ in $\mathcal{L}(H^{k+\mu}(M;E);\mathcal{L}(H^k(M;F))$.*

*Proof.* (i) is a straightforward calculation. To prove (ii), we notice that $\nabla_X(a) \in W^{k,\infty}_{\nabla}(M;\mathrm{Hom}(\mathcal{F}^M_\mu(E);F))$ and that $[\nabla_X,\nabla^{tot}]$ is a $\nabla$-differential operator with $W^{\infty,\infty}_{\nabla}$ coefficients. Therefore (ii) follows right away from (i) and Lemma 2.2. To prove (iii), let us first notice that the operators $\tau_t$ are bounded in view of Theorem 3.5. We then compute for $\xi \in H^{k+\mu}(M;E)$:

$$\frac{1}{t}\big[\tau_t(a\cdot\nabla^{tot})\tau_{-t} - a\cdot\nabla^{tot}\big]\xi \;=\; \frac{1}{t}\big[\tau_t(a\cdot\nabla^{tot})\tau_{-t} - \tau_t a\tau_{-t}\cdot\nabla^{tot}$$

$$+\,\tau_t a\tau_{-t}\cdot\nabla^{tot} - a\cdot\nabla^{tot}\big]\xi \;=\; \tau_t a\tau_{-t}\cdot\frac{1}{t}\big[\tau_t\nabla^{tot}\tau_{-t} - \nabla^{tot}\big]\xi + \frac{1}{t}\big[\tau_t a\tau_{-t} - a\big]\cdot\nabla^{tot}\xi$$

$$\to a\cdot[\nabla_X,\nabla^{tot}]\xi + \nabla_X(a)\cdot\nabla^{tot}\xi\,.$$

The proof that $\lim_{t\to 0}\frac{1}{t}\big[\tau_t a\tau_{-t} - a\big]\xi = \nabla_X(a)\xi$, $\xi \in H^{k+\mu}(M;E)$ is as in [18]. □

These results give the action of certain diffeomorphism groups on Sobolev spaces on manifolds with bounded geometry [1, 2, 9, 10].

**Corollary 3.7.** *We use the notation of Proposition 3.6. Suppose that $a \in W^{|s_0|+k+1,\infty}_{\nabla}(M;\mathrm{Hom}(\mathcal{F}^M_\mu(E);F))$ and that $P = a\cdot\nabla^{tot} : H^{\sigma+\mu}(M;E) \to H^\sigma(M;F)$ is an isomorphism. Then, for all $f \in H^{\sigma+1}(M;E)$ we have*

$$\nabla_X(P^{-1}f) \;=\; P^{-1}(\nabla_X f) - P^{-1}[\nabla_X,P]P^{-1}f\,.$$

*Proof.* As in [18], the proof of this result is a direct and immediate consequence of Propositions 2.5 and 3.6 and of Theorem 3.5. Indeed, let us consider the spaces $H^{\sigma+\mu}(M;E)$ and $H^{\sigma}(M;F)$ and the operator $T := P = a \cdot \nabla^{tot}$ and the groups of automorphisms $\tau = (\tau_t)_{t\in\mathbb{R}}$ generated by the parallel transport along the flow defined by $X$. It was assumed that $T := P := a \cdot \nabla$ is bijective. Proposition 3.6, parts (i) and (iii) shows that the other two hypotheses of Proposition 2.5 are satisfied with $Q := [\nabla_X, a \cdot \nabla^{tot}]$. Proposition 2.5 then gives the result. □

## 4. Proof of the main theorem

Let us now give the proof of the main result (Theorem 1.3). Our proof follows the method of [18]. See also [19] for other similar results.

*Proof of Theorem* 1.3. In this proof, we shall write $H^j$ instead of $H^j(M;E)$ or $H^j(M;F)$, and $W_\nabla^{j,\infty}$ instead of $W_\nabla^{j,\infty}(M;\mathrm{Hom}(E;F))$, to simplify the notation. Thus we shall write $\|f\|_{H^j} = \|f\|_{H^j(M;E)}$ and so on. Moreover, we shall write $\|a\|_j = \|a\|_{W^{j,\infty}(M;E)}$. The relation $\mathfrak{C} \geq 1$ follows from Lemma 2.3. To prove the relation $(I_k)$ we shall proceed by induction on $k \geq 0$ using Corollary 3.7 (which is Proposition 2.5 applied to the setting that we need).

The case $k = 0$ (that is, the relation $(I_0)$) follows right away from the definition of the norm of $P_0^{-1}$. Let us now prove the result for $k+1$ assuming that it is true for $k$. As before, we shall write $\sigma := s_0 + k$ for the sake of brevity, and we shall use the result of Lemma 3.2 on norm equivalences. Let then $f \in H^{\sigma+1} = H^{s_0+k+1}(M;F)$. Since $f \in H^\sigma$ as well, the induction hypothesis gives that $P_0^{-1} f \in H^{\sigma+\mu}$. The induction step is then to prove that $P_0^{-1} f \in H^{\sigma+1+\mu}$ and that $(I_{k+1})$ is satisfied.

Let $Z_1, Z_2, \ldots, Z_N \in \mathcal{W}_b(M) := W_\nabla^{\infty,\infty}(M;TM)$ be a Fréchet system of generators of $\mathcal{W}_b(M)$ as $\mathcal{C}_{\mathrm{b}}^\infty$-module, for instance the system given in Equation (3.1). Let also $Z_0 := id$, to simplify the notation, as before. Also, we shall write $Z_\ell$ instead of $\nabla_{Z_\ell}$. Corollary 3.7 for $P_0 : H^{\sigma+\mu} \to H^\sigma$ and the parallel transport automorphisms groups generated by $Z_\ell$, $\ell \geq 1$, (which exist due to Theorem 3.5) give

$$\|Z_\ell(P_0^{-1} f)\|_{H^{\sigma+\mu}} \leq \|P_0^{-1}(Z_\ell f)\|_{H^{\sigma+\mu}} + \|Qf\|_{H^{\sigma+\mu}}\,, \tag{4.1}$$

where $Q := P_0^{-1}[Z_\ell, P_0]P_0^{-1} : H^\sigma(M) \to H^{\sigma+\mu}(M)$. For every $\ell = 0, \ldots, N$, we have $Z_\ell f \in H^\sigma$, and hence we can use by the induction hypothesis the relation $(I_k)$ with $f$ replaced with $Z_\ell f$ to obtain for $\ell \geq 1$:

$$\begin{aligned} \|P_0^{-1}(Z_\ell f)\|_{H^{\sigma+\mu}} &\lesssim \sum_{q=0}^{k} |||P_0^{-1}|||_0^{q+1}\, \|a\|_{|s_0|+k}^q \|Z_\ell f\|_{H^{\sigma-q}} \\ &\lesssim \sum_{q=0}^{k} |||P_0^{-1}|||_0^{q+1}\, \|a\|_{|s_0|+k}^q \|f\|_{H^{\sigma+1-q}}\,. \end{aligned} \tag{4.2}$$

To estimate the term $\|Qf\|_{H^{\sigma+\mu}}$, we shall use the relation $(I_k)$ twice. First, for $g := [Z_\ell, P]P^{-1}f \in H^\sigma(M)$, the induction hypothesis $(I_k)$ gives the relation

$$\|Qf\|_{H^{\sigma+\mu}} := \|P_0^{-1}(g)\|_{H^{\sigma+\mu}} \lesssim \sum_{q=0}^{k} |||P_0^{-1}|||_0^{q+1}\, \|a\|_{|s_0|+k}^q \|g\|_{H^{\sigma-q}}\,. \tag{4.3}$$

Moreover, for $q$ fixed in $\{0, 1, \ldots, k\}$, using Proposition 3.6(ii), we get:

$$\begin{aligned} \|g\|_{H^{\sigma-q}} &= \|[Z_\ell, P_0]P_0^{-1}f\|_{H^{\sigma-q}} \lesssim \|a\|_{|s_0|+k+1-q} \|P_0^{-1}(f)\|_{H^{\sigma-q+\mu}} \\ &\lesssim \|a\|_{|s_0|+k+1-q} \sum_{s=0}^{k-q} |||P_0^{-1}|||_0^{s+1}\, \|a\|_{|s_0|+k-q}^s \|f\|_{H^{\sigma-q-s}} \\ &\lesssim \sum_{s=0}^{k-q} |||P_0^{-1}|||_0^{s+1}\, \|a\|_{|s_0|+k+1}^{s+1} \|f\|_{H^{\sigma-q-s}}\,. \end{aligned} \tag{4.4}$$

Consequently, the equations (4.3) and (4.4) give,

$$\|Qf\|_{H^{\sigma+\mu}} \lesssim \sum_{q=0}^{k} \sum_{s=0}^{k-q} |||P_0^{-1}|||_0^{q+s+2}\, \|a\|_{|s_0|+k+1}^{q+s+1} \|f\|_{H^{\sigma-q-s}}\,. \tag{4.5}$$

Then, by substituting $p = q + s + 1$, we get that

$$\|Qf\|_{H^{\sigma+\mu}} \lesssim \sum_{p=1}^{k+1} |||P_0^{-1}|||_0^{p+1}\, \|a\|_{|s_0|+k+1}^{p} \|f\|_{H^{\sigma+1-p}}\,. \tag{4.6}$$

Then, by using the equations (4.2) and (4.6) to estimate the two right-hand side terms in (4.1), we obtain that:

$$\|Z_\ell(P^{-1}f)\|_{H^{\sigma+\mu}} \lesssim \sum_{q=0}^{k+1} |||P_0^{-1}|||_0^{q+1}\, \|a\|_{|s_0|+k+1}^{q} \|f\|_{H^{\sigma+1-q}}\,. \tag{4.7}$$

To use the estimate of Lemma 3.2, need to estimate $\|Z_0 P_0^{-1} f\|_{H^{\sigma+\mu}} = \|P_0^{-1} f\|_{H^{\sigma+\mu}}$, which we will do by using $(I_k)$ to obtain

$$\begin{aligned} \|Z_0(P_0^{-1}f)\|_{H^{\sigma+\mu}} &\lesssim \sum_{q=0}^{k} |||P_0^{-1}|||_0^{q+1}\, \|a\|_{|s_0|+k}^{q} \|f\|_{H^{\sigma-q}} \\ &\lesssim \sum_{q=0}^{k} |||P_0^{-1}|||_0^{q+1}\, \|a\|_{|s_0|+k+1}^{q} \|f\|_{H^{\sigma+1-q}}\,. \end{aligned}$$

We then take the sum of this last equation with all the equations 4.7, for $\ell = 1, \ldots, N$. As desired, Lemma 3.2 gives

$$\|P_0^{-1}f\|_{H^{\sigma+1+\mu}} \lesssim \sum_{q=0}^{k+1} |||P_0^{-1}|||_0^{q+1}\, \|a\|_{|s_0|+k+1}^{q} \|f\|_{H^{\sigma+1-q}}\,, \tag{4.8}$$

which is exactly the relation $(I_{k+1})$ we were also looking for. This reasoning also gives $P_0^{-1} f \in H^{\sigma+1+\mu}$. Using $\mathfrak{C} \geq 1$ and bounding $\|f\|_{H^{\sigma-q}}$ with $\|f\|_{H^\sigma}$, we also obtain the desired inequality for $|||P_{k+1}^{-1}|||$. This completes the proof of Theorem 1.3. □

## 5. Remarks and applications

The method of our theorem gives a stronger result than the more straightforward estimates expounded below.

**Remark 5.1.** As in [18], Corollary 3.7 and the Lemmas 2.2 and 3.2 give

$$|||P_0^{-1}|||_{k+1} \leq |||P_0^{-1}|||_k^2 \|a\|_{|s_0|+k+1} \,. \tag{5.1}$$

An induction argument in $k$ then gives

$$|||P_0^{-1}|||_k \ := \ \|P_k^{-1}\| \leq \|P_0^{-1}\|^{2^k} \|a\|_{|s_0|+k}^{2^k-1} \,, \tag{5.2}$$

which is, obviously, much weaker than the result of Theorem 1.3. Nevertheless, this type of result (which follows the method of [6]) is also sufficient for many applications.

Let $A \in W_\nabla^{k+2m,\infty}(M; \mathrm{End}(\mathcal{F}_m^M(E)))$, $k, m \geq 0$. It is known then from [14] that $(\nabla^{tot})^* A \nabla^{tot}$ is a $\nabla$-differential operator of order $2m$ with coefficients in $W_\nabla^{k+2m,\infty}(M; \mathrm{End}(\mathcal{F}_m^M(E)))$. We will let $\mathrm{Re}\, A := \frac{1}{2}\big(A + A^*\big)$, with $A^*$ the adjoint of $A$. We write $A \geq \gamma I$ if, for any complex vector $\xi$ on which $A$ acts, we have $(A\xi, \xi) \geq \gamma \|\xi\|^2$, pointwise (that is, as functions on $M$). From now on, we shall assume that $s_0 = -m$. Theorem 1.3 applied to the operator $(\nabla^{tot})^* A \nabla^{tot}$ (and $s_0 = -m$) then yields the following result.

**Theorem 5.2.** *Let $A \in W_\nabla^{k+2m,\infty}(M; \mathrm{End}(\mathcal{F}_\mu^M(E)))$ be such that $\mathrm{Re}\, A \geq \gamma I$, $\gamma > 0$. Let $\mathcal{P}_k = (\nabla^{tot})^* A \nabla^{tot} : H^{k+m}(M; E) \to H^{k-m}(M; E)$. Then $\mathcal{P}_0$ is invertible with norm $|||\mathcal{P}_0^{-1}|||_0 \leq \gamma^{-1}$. Moreover, given $f \in H^{k-m}(M; E)$, we have that*

$$\|\mathcal{P}_0^{-1} f\|_{H^{k+m}} \lesssim \sum_{q=0}^{k} \gamma^{-(q+1)} \|A\|_{W^{k+2m}}^q \|f\|_{H^{k-m}} \,.$$

*In particular, $\mathcal{P}_k$ is invertible and $|||\mathcal{P}_k^{-1}|||_k \leq \gamma^{-k-1} \|A\|_{W^{k+2m}}^k$.*

*Proof.* Let $u \in H^m(M; E)$. Then

$$\mathrm{Re}(\mathcal{P}_0 u, u) \ := \ \mathrm{Re}(A\nabla^{tot} u, \nabla^{tot} u) \geq \gamma(\nabla^{tot} u, \nabla^{tot} u) \geq \gamma \|u\|_{H^m}^2 \,.$$

So $\mathcal{P}_0$ is invertible and $|||\mathcal{P}^{-1}|||_0 \leq \gamma^{-1}$, by the Lax-Milgram lemma. We also notice that $1 \leq \mathfrak{C} \lesssim \gamma^{-1} \|A\|_{W^{k+2m}}$. The proof is then completed by using Theorem 1.3. □

We obtain the following consequence.

**Theorem 5.3.** *We use the setting of Theorem 1.3. Let $X = (X_1, X_2, \ldots, X_K)$ be a vector Gaussian random variable with covariance $\sigma = (\sigma_{ij}) > 0$. Let*

$$A_1, A_2, \ldots, A_K \in W_\nabla^{k+2m,\infty}(M; \mathrm{End}(\mathcal{F}_\mu^M(E))) \,,$$

*with $A_j \geq \gamma I$, $\gamma > 0$. We note $A := \sum_{j=1}^K e^{X_j} A_j$. Then $|||\mathcal{P}_k^{-1}|||_k$ is integrable.*

*Proof.* We proceed as in [18]. We have

$$\mathrm{Re}(A) := \mathrm{Re}\left(\sum_{j=1}^{K} e^{X_j} A_j\right) \geq \gamma \sum_{j=1}^{K} e^{X_j} .$$

Likewise $\|A\|_{W^{k+2m}} \leq \sum_{j=1}^{K} e^{X_j} \|A_j\|_{W^{k+2m}}$ . According to Theorem 5.2, we get

$$
\begin{aligned}
|||\mathcal{P}_k^{-1}|||_k &\leq \left(\gamma \sum_{j=1}^{K} e^{X_j}\right)^{-k-1} \|A\|_{W^{k+2m}}^{K} \\
&\leq \left(\gamma \sum_{j=1}^{K} e^{X_j}\right)^{-k-1} \left(\sum_{j=1}^{K} e^{X_j} \|A_j\|_{W^{k+2m}}\right)^{k} \\
&\leq C\Big(\sum_{j=1}^{K} e^{X_j}\Big)^{-1},
\end{aligned}
$$

which is integrable, because $\left(\sum_{j=1}^{K} e^{x_j}\right)^{-1} \leq e^{|x_1|+\ldots+|x_K|}$ is integrable with respect to the measure of density $e^{-(\sigma x,x)} \leq e^{-\epsilon\|x\|^2}$, where $C$ depends on $\|A_j\|_{W^{k+2m}}$, $k$ and $K$. □

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# Microscopic behavior of the solutions of a transmission problem for the Helmholtz equation. A functional analytic approach

Tuğba Akyel and Massimo Lanza de Cristoforis

*Dedicated to the memory of Professor Gabriela Kohr*

**Abstract.** Let $\Omega^i$, $\Omega^o$ be bounded open connected subsets of $\mathbb{R}^n$ that contain the origin. Let $\Omega(\epsilon) \equiv \Omega^o \setminus \epsilon\overline{\Omega^i}$ for small $\epsilon > 0$. Then we consider a linear transmission problem for the Helmholtz equation in the pair of domains $\epsilon\Omega^i$ and $\Omega(\epsilon)$ with Neumann boundary conditions on $\partial\Omega^o$. Under appropriate conditions on the wave numbers in $\epsilon\Omega^i$ and $\Omega(\epsilon)$ and on the parameters involved in the transmission conditions on $\epsilon\partial\Omega^i$, the transmission problem has a unique solution $(u^i(\epsilon,\cdot), u^o(\epsilon,\cdot))$ for small values of $\epsilon > 0$. Here $u^i(\epsilon,\cdot)$ and $u^o(\epsilon,\cdot)$ solve the Helmholtz equation in $\epsilon\Omega^i$ and $\Omega(\epsilon)$, respectively. Then we prove that if $\xi \in \overline{\Omega^i}$ and $\xi \in \mathbb{R}^n \setminus \Omega^i$ then the rescaled solutions $u^i(\epsilon, \epsilon\xi)$ and $u^o(\epsilon, \epsilon\xi)$ can be expanded into a convergent power expansion of $\epsilon$, $\kappa_n \epsilon \log \epsilon$, $\delta_{2,n} \log^{-1} \epsilon$, $\kappa_n \epsilon \log^2 \epsilon$ for $\epsilon$ small enough. Here $\kappa_n = 1$ if $n$ is even and $\kappa_n = 0$ if $n$ is odd and $\delta_{2,2} \equiv 1$ and $\delta_{2,n} \equiv 0$ if $n \geq 3$.

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## 1. Introduction

In this paper we consider a linear transmission problem for the Helmholtz equation in a domain with a small inclusion. Problems of this type are motivated by the analysis of time-harmonic Maxwells Equations and we continue an analysis of [1] by analyzing the microscopic behavior of the solutions. For related problems for the Helmholtz equation, we refer to the papers [3] of Ammari, Vogelius and Volkov, [2] of

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Ammari, Iakovleva and Moskow, [4] of Ammari and Volkov, [12] of Hansen, Poignard and Vogelius, and [25] of Vogelius and Volkov.

First we introduce a problem with no hole (and no transmission), and then we consider the case with the hole. We consider $m \in \mathbb{N} \setminus \{0\}$, $n \in \mathbb{N} \setminus \{0,1\}$, $\alpha \in ]0,1[$ and the following assumption.

$$\begin{aligned}&\textit{Let } \Omega \textit{ be a bounded open connected subset of } \mathbb{R}^n \textit{ of class } C^{m,\alpha}. \\ &\textit{Let } \mathbb{R}^n \setminus \overline{\Omega} \textit{ be connected. Let } 0 \in \Omega\,.\end{aligned} \tag{1.1}$$

Now let $\Omega^o$ be as in (1.1). Let

$$k_o \in \mathbb{C} \setminus ]-\infty, 0]\,, \qquad \Im k_o \geq 0\,. \tag{1.2}$$

We also assume that $k_o^2$ is not a Neumann eigenvalue for $-\Delta$ in $\Omega^o$. Then if

$$g^o \in C^{m-1,\alpha}(\partial\Omega^o)\,, \tag{1.3}$$

the Neumann problem

$$\begin{cases} \Delta u^o + k_o^2 u^o = 0 & \text{in } \Omega^o\,, \\ \frac{\partial}{\partial \nu_{\Omega^o}} u^o = g^o & \text{on } \partial\Omega^o \end{cases}$$

has a unique solution $\tilde{u}^o \in C^{m,\alpha}(\overline{\Omega}^o)$ (see for example Colton and Kress [9, Thm. 3.20] and classical Schauder regularity theory).

We now perturb singularly our problem. To do so, we consider another subset $\Omega^i$ of $\mathbb{R}^n$ as in (1.1). Then there exists

$$\epsilon_0 \in ]0,1[ \text{ such that } \epsilon\overline{\Omega^i} \subseteq \Omega^o \qquad \forall \epsilon \in [-\epsilon_0, \epsilon_0]\,.$$

A known topological argument shows that $\Omega(\epsilon) \equiv \Omega^o \setminus \epsilon\overline{\Omega^i}$ is connected, and that $\mathbb{R}^n \setminus \overline{\Omega(\epsilon)}$ has exactly the two connected components $\epsilon\Omega^i$ and $\mathbb{R}^n \setminus \overline{\Omega^o}$, and that

$$\partial\Omega(\epsilon) = (\epsilon\partial\Omega^i) \cup \partial\Omega^o \qquad \forall \epsilon \in ]-\epsilon_0, \epsilon_0[\setminus\{0\}\,.$$

Obviously, the outward unit normal $\nu_\epsilon$ to $\partial\Omega(\epsilon)$ satisfies the equality

$$\begin{aligned}&\nu_\epsilon(x) = -\nu_{\Omega^i}(x/\epsilon)\,\mathrm{sgn}(\epsilon) \qquad \forall x \in \epsilon\partial\Omega^i\,, \\ &\nu_\epsilon(x) = \nu_{\Omega^o}(x) \qquad \forall x \in \partial\Omega^o\,,\end{aligned}$$

for all $\epsilon \in ]-\epsilon_0, \epsilon_0[\setminus\{0\}$, where $\mathrm{sgn}(\epsilon) = 1$ if $\epsilon > 0$, $\mathrm{sgn}(\epsilon) = -1$ if $\epsilon < 0$. Then we introduce the constants

$$m^i, m^o \in ]0, +\infty[\,, \qquad a \in ]0, +\infty[\,,\ b \in \mathbb{R}\,,$$

and

$$k_i \in \mathbb{C} \setminus ]-\infty, 0]\,, \qquad \Im k_i \geq 0\,, \tag{1.4}$$

and the datum

$$g^i \in C^{m-1,\alpha}(\partial\Omega^i)\,. \tag{1.5}$$

Then we consider the transmission problem

$$\begin{cases} \Delta u^i + k_i^2 u^i = 0 & \text{in } \epsilon\Omega^i\,, \\ \Delta u^o + k_o^2 u^o = 0 & \text{in } \Omega(\epsilon)\,, \\ u^o(x) - a u^i(x) = b & \forall x \in \epsilon\partial\Omega^i\,, \\ -\frac{1}{m^i}\frac{\partial}{\partial \nu_{\epsilon\Omega^i}} u^i(x) + \frac{1}{m^o}\frac{\partial}{\partial \nu_{\epsilon\Omega^i}} u^o(x) = g^i(x/\epsilon) & \forall x \in \epsilon\partial\Omega^i\,, \\ \frac{\partial}{\partial \nu_{\Omega^o}} u^o = g^o & \text{on } \partial\Omega^o\,, \end{cases} \tag{1.6}$$

in the unknown $(u^i, u^o) \in C^{m,\alpha}(\epsilon\overline{\Omega^i}) \times C^{m,\alpha}(\overline{\Omega(\epsilon)})$ for $\epsilon \in ]0, \epsilon_0[$. By [1, Thm. 4.61], there exists $\epsilon' \in ]0, \epsilon_0[$ such that problem (1.6) has a unique solution $(u^i(\epsilon, \cdot), u^o(\epsilon, \cdot)) \in C^{m,\alpha}(\epsilon\overline{\Omega^i}) \times C^{m,\alpha}(\overline{\Omega(\epsilon)})$. In [1, Thm. 5.1], we have analyzed the behavior of $u^o(\epsilon, \cdot)$ as $\epsilon$ approaches 0 and we have shown that if $x \in \Omega^o \setminus \{0\}$, then $u^o(\epsilon, x)$ can be expanded into a convergent power expansion of $\epsilon$, $\kappa_n \epsilon \log \epsilon$, $\delta_{2,n} \log^{-1} \epsilon$ for $\epsilon$ small enough. Here $\kappa_n = 1$ if $n$ is even and $\kappa_n = 0$ if $n$ is odd and $\delta_{2,2} \equiv 1$ and $\delta_{2,n} \equiv 0$ if $n \geq 3$. In this paper we plan to consider the 'microscopic' behavior of our family of solutions, i.e., the behavior of the rescaled family

$$\{(u^i(\epsilon, \epsilon\cdot), u^o(\epsilon, \epsilon\cdot))\}_{\epsilon \in ]0, \epsilon'[}$$

when $\epsilon$ is small enough. More precisely, we plan to answer the following two questions

(i) Let $\xi$ be fixed in $\overline{\Omega^i}$. What can be said on the map $\epsilon \mapsto u^i(\epsilon, \epsilon\xi)$ when $\epsilon > 0$ is close to 0?

(ii) Let $\xi$ be fixed in $\mathbb{R}^n \setminus \Omega^i$. What can be said on the map $\epsilon \mapsto u^o(\epsilon, \epsilon\xi)$ when $\epsilon > 0$ is close to 0?

Questions of this type have long been investigated for linear problems on domains with small holes with the methods of asymptotic analysis, which aim at proving complete asymptotic expansions in terms of the parameter $\epsilon$. Although we cannot provide here a complete list of contributions, we mention the early works of of Cherepanov [6], [7] and the books of Nayfeh [22], Van Dyke [24], and Cole [8]. Then the description of the method of matching outer and inner asymptotic expansions of Il'in [13] and the Compound Expansion Method of Mazya, Nazarov and Plamenewskii [21] where the authors introduce a systematic approach for analyzing general Douglis and Nirenberg elliptic boundary value problems in domains with perforations and corners.

To analyze the problem and answer the above questions we resort to the Functional Analytic Approach (see reference [11] with Dalla Riva and Musolino) and we exploit the corresponding results of [1] and we prove that if $\xi \in \overline{\Omega^i}$ and $\xi \in \mathbb{R}^n \setminus \Omega^i$ then $u^i(\epsilon, \epsilon\xi)$ and $u^o(\epsilon, \epsilon\xi)$ can be expanded into a convergent power expansion of $\epsilon$, $\kappa_n \epsilon \log \epsilon$, $\delta_{2,n} \log^{-1} \epsilon$, $\kappa_n \epsilon \log^2 \epsilon$ for $\epsilon$ small enough, respectively (see Theorem 5.1).

## 2. Preliminaries and notation

For standard definitions of Calculus in normed spaces, we refer to Cartan [5] and to Prodi and Ambrosetti [23]. The symbol $\mathbb{N}$ denotes the set of natural numbers including 0. Throughout the paper,

$$n \in \mathbb{N} \setminus \{0, 1\}.$$

Let $\mathbb{D} \subseteq \mathbb{R}^n$. Then $\overline{\mathbb{D}}$ denotes the closure of $\mathbb{D}$ and $\partial\mathbb{D}$ denotes the boundary of $\mathbb{D}$. For all $R > 0$, $x \in \mathbb{R}^n$, $x_j$ denotes the $j$-th coordinate of $x$, $|x|$ denotes the Euclidean modulus of $x$ in $\mathbb{R}^n$, and $\mathbb{B}_n(x, R)$ denotes the ball $\{y \in \mathbb{R}^n : |x - y| < R\}$. Let $\Omega$ be an open subset of $\mathbb{R}^n$. Then we find convenient to set

$$\Omega^+ \equiv \Omega, \qquad \Omega^- \equiv \mathbb{R}^n \setminus \overline{\Omega}.$$

The space of $m$ times continuously differentiable complex-valued functions on $\Omega$ is denoted by $C^m(\Omega, \mathbb{C})$, or more simply by $C^m(\Omega)$. Let $r \in \mathbb{N} \setminus \{0\}$, $f \in (C^m(\Omega))^r$. The

$s$-th component of $f$ is denoted $f_s$ and the Jacobian matrix of $f$ is denoted $Df$. Let $\eta \equiv (\eta_1, \dots, \eta_n) \in \mathbb{N}^n$, $|\eta| \equiv \eta_1 + \dots + \eta_n$. Then $D^\eta f$ denotes $\frac{\partial^{|\eta|} f}{\partial x_1^{\eta_1} \dots \partial x_n^{\eta_n}}$. The subspace of $C^m(\Omega)$ of those functions $f$ such that $f$ and its derivatives $D^\eta f$ of order $|\eta| \leq m$ can be extended with continuity to $\overline{\Omega}$ is denoted $C^m(\overline{\Omega})$. The subspace of $C^m(\overline{\Omega})$ whose functions have $m$-th order derivatives that are Hölder continuous with exponent $\alpha \in ]0,1]$ is denoted $C^{m,\alpha}(\overline{\Omega})$, (cf. *e.g.* [11, §2.11]). Let $\mathbb{D} \subseteq \mathbb{R}^n$. Then $C^{m,\alpha}(\overline{\Omega}, \mathbb{D})$ denotes the set $\left\{ f \in \left(C^{m,\alpha}(\overline{\Omega})\right)^n : \ f(\overline{\Omega}) \subseteq \mathbb{D} \right\}$. We say that a bounded open subset of $\mathbb{R}^n$ is of class $C^m$ or of class $C^{m,\alpha}$, if it is a manifold with boundary imbedded in $\mathbb{R}^n$ of class $C^m$ or $C^{m,\alpha}$, respectively (cf. *e.g.*, [11, §2.13]). For standard properties of the functions of class $C^{m,\alpha}$ both on a domain of $\mathbb{R}^n$ or on a manifold imbedded in $\mathbb{R}^n$ we refer to [11, §2.11, 2.12, 2.14, 2.20] (see also [14, §2, Lem. 3.1, 4.26, Thm. 4.28], [18, §2].) We retain the standard notation of $L^p$ spaces and of corresponding norms. We note that throughout the paper 'analytic' means 'real analytic'.

## 3. Some basic facts in potential theory

In the sequel, arg and log denote the principal branch of the argument and of the logarithm in $\mathbb{C} \setminus ]-\infty, 0]$, respectively. Then we have

$$\arg(z) = \Im \log(z) \in ]-\pi, \pi[ \qquad \forall z \in \mathbb{C} \setminus ]-\infty, 0] .$$

Then we set

$$J_\nu^\sharp(z) \equiv \sum_{j=0}^{\infty} \frac{(-1)^j z^j (1/2)^{2j} (1/2)^\nu}{\Gamma(j+1)\Gamma(j+\nu+1)} \qquad \forall z \in \mathbb{C}, \tag{3.1}$$

for all $\nu \in \mathbb{C} \setminus \{-j : j \in \mathbb{N} \setminus \{0\}\}$. Here $(1/2)^\nu = e^{\nu \log(1/2)}$. As is well known, if $\nu \in \mathbb{C} \setminus \{-j : j \in \mathbb{N} \setminus \{0\}\}$ then the function $J_\nu^\sharp(\cdot)$ is entire and

$$J_\nu^\sharp(z^2) = e^{-\nu \log z} J_\nu(z) \qquad \forall z \in \mathbb{C} \setminus ]-\infty, 0] ,$$

where $J_\nu(\cdot)$ is the Bessel function of the first kind of index $\nu$ (cf. *e.g.*, Lebedev [20, Ch. 1, §5.3].) If $\nu \in \mathbb{N}$, we set

$$\begin{aligned} N_\nu^\sharp(z) \equiv & -\frac{2^\nu}{\pi} \sum_{0 \leq j \leq \nu - 1} \frac{(\nu - j - 1)!}{j!} z^j (1/2)^{2j} \\ & - \frac{z^\nu}{\pi} \sum_{j=0}^{\infty} \frac{(-1)^j z^j (1/2)^{2j} (1/2)^\nu}{j!(\nu + j)!} \left( 2 \sum_{0 < l \leq j} \frac{1}{l} + \sum_{j < l \leq j + \nu} \frac{1}{l} \right) \qquad \forall z \in \mathbb{C} . \end{aligned}$$

As one can easily see, the $N_\nu^\sharp(\cdot)$ is an entire holomorphic function of the variable $z \in \mathbb{C}$ and

$$N_\nu(z) = \frac{2}{\pi} (\log(z) - \log 2 + \gamma) J_\nu(z) + z^{-\nu} N_\nu^\sharp(z^2) \qquad \forall z \in \mathbb{C} \setminus ]-\infty, 0] ,$$

where $\gamma$ is the Euler-Mascheroni constant, and where $N_\nu(\cdot)$ is the Neumann function of index $\nu$, also known as Bessel function of second kind and index $\nu$ (cf. *e.g.*, Lebedev [20, Ch. 1, §5.5].) Let $k \in \mathbb{C}\setminus]-\infty,0]$, $n \in \mathbb{N}\setminus\{0,1\}$, $a_n \in \mathbb{C}$. Then we set

$$b_n \equiv \begin{cases} \pi^{1-(n/2)}2^{-1-(n/2)} & \text{if } n \text{ is even}, \\ (-1)^{\frac{n-1}{2}}\pi^{1-(n/2)}2^{-1-(n/2)} & \text{if } n \text{ is odd}, \end{cases}$$

and

$$\tilde{S}_{k,a_n}(x) = \begin{cases} k^{n-2}\left\{a_n + \frac{2b_n}{\pi}(\log k - \log 2 + \gamma) + \frac{2b_n}{\pi}\log|x|\right\} \\ \quad \times J^\sharp_{\frac{n-2}{2}}(k^2|x|^2) + b_n|x|^{2-n}N^\sharp_{\frac{n-2}{2}}(k^2|x|^2) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } n \text{ is even}, \\ a_n k^{n-2}J^\sharp_{\frac{n-2}{2}}(k^2|x|^2) + b_n|x|^{2-n}J^\sharp_{-\frac{n-2}{2}}(k^2|x|^2) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } n \text{ is odd}, \end{cases} \tag{3.2}$$

for all $x \in \mathbb{R}^n \setminus \{0\}$. As it is known and can be easily verified, the family $\{\tilde{S}_{k,a_n}\}_{a_n\in\mathbb{C}}$ coincides with the family of all radial fundamental solutions of $\Delta + k^2$.

Now we need to consider two specific fundamental solutions. For the first, which we denote by $S_{h,n}$, we need to choose $a_n$ so that the resulting fundamental solution can be extended to an entire holomorphic function of the variable $k \in \mathbb{C}$. Then we introduce the following theorem. For a proof we refer to the paper [19, Prop. 3.3] with Rossi.

**Theorem 3.1.** *Let $n \in \mathbb{N}\setminus\{0,1\}$. Let $S_{h,n}(\cdot,\cdot)$ be the map from $(\mathbb{R}^n\setminus\{0\})\times\mathbb{C}$ to $\mathbb{C}$ defined by*

$$S_{h,n}(x,k) \equiv \begin{cases} b_n\left\{\frac{2}{\pi}k^{n-2}J^\sharp_{\frac{n-2}{2}}(k^2|x|^2)\log|x| \right. \\ \qquad\qquad \left. +|x|^{-(n-2)}N^\sharp_{\frac{n-2}{2}}(k^2|x|^2)\right\} & \text{if } n \text{ is even}, \\ b_n|x|^{-(n-2)}J^\sharp_{-\frac{n-2}{2}}(k^2|x|^2) & \text{if } n \text{ is odd}, \end{cases}$$

*for all $(x,k) \in (\mathbb{R}^n\setminus\{0\})\times\mathbb{C}$. Then the following statements hold.*

(i) *$S_{h,n}(\cdot,k)$ is a fundamental solution of $\Delta+k^2$ for all $k\in\mathbb{C}$ and $S_{h,n}(\cdot,0)$ coincides with the classical fundamental solution $S_n$ of $\Delta$, i.e.,*

$$S_{h,n}(x,0) = S_n(x) \equiv \begin{cases} \frac{1}{s_n}\log|x| & \forall x \in \mathbb{R}^n\setminus\{0\}, \quad \text{if } n=2, \\ \frac{1}{(2-n)s_n}|x|^{2-n} & \forall x \in \mathbb{R}^n\setminus\{0\}, \quad \text{if } n>2, \end{cases}$$

*where $s_n$ denotes the $(n-1)$ dimensional measure of $\partial\mathbb{B}_n(0,1)$.*

(ii) *$S_{h,n}(\cdot,k)$ is real analytic in $\mathbb{R}^n\setminus\{0\}$. Moreover, if $x\in\mathbb{R}^n\setminus\{0\}$, then the map $S_{h,n}(x,\cdot)$ is holomorphic in $\mathbb{C}$.*

Next we introduce the second fundamental solution that we need. Let $k \in \mathbb{C}\setminus]-\infty,0]$, $\Im k \geq 0$. As well known in scattering theory, a function $u \in C^1(\mathbb{R}^n\setminus\{0\})$ satisfies the outgoing $k$-radiation condition provided that

$$\lim_{x\to\infty}|x|^{\frac{n-1}{2}}\left(Du(x)\frac{x}{|x|} - iku(x)\right) = 0\,.$$

Classically, one can prove that the fundamental solution of (3.2) satisfies the outgoing $k$-radiation condition if and only if

$$a_n \equiv \left\{ \begin{array}{ll} -ib_n & \text{if } n \text{ is even} \\ -e^{-i\frac{n-2}{2}\pi}b_n & \text{if } n \text{ is odd} \end{array} \right\} = -i\pi^{1-(n/2)}2^{-1-(n/2)} \tag{3.3}$$

Then we introduce the following definition.

**Definition 3.2.** Let $n \in \mathbb{N} \setminus \{0,1\}$. Let $k \in \mathbb{C} \setminus ]-\infty,0]$. We denote by $S_{r,n}(\cdot,k)$ the function from $\mathbb{R}^n \setminus \{0\}$ to $\mathbb{C}$ defined by

$$S_{r,n}(x,k) \equiv \tilde{S}_{k,a_n}(x) \qquad \forall x \in \mathbb{R}^n \setminus \{0\}\,,$$

with $a_n$ as in (3.3) (cf. (3.2).)

As we have said above, if $k \in \mathbb{C} \setminus ]-\infty,0]$ and $\Im k \geq 0$, then $S_{r,n}(\cdot,k)$ satisfies the outgoing $k$-radiation condition. The subscript $r$ stands for 'radiation'. Now we introduce the function $\gamma_n$ from $\mathbb{C}$ to $\mathbb{C}$ defined by setting

$$\gamma_n(z) \equiv \left\{ \begin{array}{ll} [-i + \frac{2}{\pi}(z - \log 2 + \gamma)]b_n & \text{if } n \text{ is even}\,, \\ -e^{-i\frac{n-2}{2}\pi}b_n & \text{if } n \text{ is odd}\,, \end{array} \right. \tag{3.4}$$

for all $z \in \mathbb{C}$. Then we have

$$S_{r,n}(x,k) = S_{h,n}(x,k) + \gamma_n(\log k)k^{n-2}J^{\sharp}_{\frac{n-2}{2}}(k^2|x|^2) \qquad \forall x \in \mathbb{R}^n \setminus \{0\}\,,$$

for all $k \in \mathbb{C} \setminus ]-\infty,0]$. Next we introduce the layer potential operators corresponding to a fundamental solution or to a smooth kernel.

**Definition 3.3.** Let $n \in \mathbb{N} \setminus \{0,1\}$, $k \in \mathbb{C}$. Let $S$ be either a fundamental solution of $\Delta + k^2$ or a real analytic function from $\mathbb{R}^n$ to $\mathbb{C}$. Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$ of class $C^{1,\alpha}$. Let $\mu \in C^0(\partial\Omega)$. Then we introduce the following notation.

(i) We denote by $v_\Omega[\mu,S]$ the function from $\mathbb{R}^n$ to $\mathbb{C}$ defined by

$$v_\Omega[\mu,S](x) \equiv \int_{\partial\Omega} S(x-y)\mu(y)\,d\sigma_y \qquad \forall x \in \mathbb{R}^n\,.$$

Then we denote by $v^+_\Omega[\mu,S]$, by $v^-_\Omega[\mu,S]$ and by $V_\Omega[\mu,S]$, the restriction of $v_\Omega[\mu,S]$ to $\overline{\Omega}$, to $\overline{\Omega^-}$ and to $\partial\Omega$, respectively.

(ii) We denote by $W^t_\Omega[\mu,S]$ the function from $\partial\Omega$ to $\mathbb{C}$ defined by

$$W^t_\Omega[\mu,S](x) \equiv \int_{\partial\Omega} \frac{\partial}{\partial\nu_{\Omega,x}} S(x-y)\mu(y)\,d\sigma_y \qquad \forall x \in \partial\Omega\,,$$

where

$$\frac{\partial}{\partial\nu_{\Omega,x}} S(x-y) \equiv DS(x-y)\nu_\Omega(x) \qquad \forall (x,y) \in \partial\Omega \times \partial\Omega, x \neq y\,.$$

If $k \in \mathbb{C} \setminus ]-\infty,0]$, we set

$$v_\Omega[\mu,k] \equiv v_\Omega[\mu, S_{r,n}(\cdot,k)]\,,$$

and we use corresponding abbreviations for $V_\Omega$, $v^\pm_\Omega$, $W^t_\Omega$. If $k \in \mathbb{C}$, we set

$$v_{\Omega,h}[\mu,k] = v_\Omega[\mu, S_{h,n}(\cdot,k)]\,,$$

and we use corresponding abbreviations for $V_{\Omega,h}$, $v^{\pm}_{\Omega,h}$, $W^t_{\Omega,h}$. If $\lambda \in \mathbb{C}$, we set

$$v_{\Omega,J}[\mu,\lambda] = v_{\Omega}[\mu, J^{\sharp}_{\frac{n-2}{2}}(\lambda|\cdot|^2)]\,,$$

and we use corresponding abbreviations for $V_{\Omega,J}$, $v^{\pm}_{\Omega,J}$, $W^t_{\Omega,J}$. For the regularity results on acoustic layer potentials that we need, we refer the reader to [10] (which is a generalization of [19]), to [16, Thm. A.3] and to [1, §3]. If $m \in \mathbb{N}\setminus\{0\}$, $\alpha \in ]0,1[$, we set

$$\tilde{W}^t_{\Omega,J}[\mu,\lambda](x) \equiv 2\int_{\partial\Omega} (J^{\sharp}_{\frac{n-2}{2}})'(\lambda(x-y)(x-y))(x-y)\nu_{\Omega}(x)\mu(y)\,d\sigma_y\,,$$

for all $x \in \partial\Omega$ and for all $(\mu,\lambda) \in C^{m-1,\alpha}(\partial\Omega)\times\mathbb{C}$. Then we have

$$W^t_{\Omega,J}[\mu,\lambda](x) = \lambda\tilde{W}^t_{\Omega,J}[\mu,\lambda](x) \qquad \forall x \in \partial\Omega\,,$$

for all $(\mu,\lambda) \in C^{m-1,\alpha}(\partial\Omega)\times\mathbb{C}$. By our abbreviations, we have

$$v^{\pm}_{\Omega}[\mu,k] = v^{\pm}_{\Omega,h}[\mu,k] + \gamma_n(\log k)k^{n-2}v^{\pm}_{\Omega,J}[\mu,k^2] \tag{3.5}$$

on $\overline{\Omega^{\pm}}$ for all $\mu$ in $C^{m-1,\alpha}(\partial\Omega)$ and $k \in \mathbb{C}\setminus]-\infty,0]$ (cf. [1, Cor. 3.25]).

Next we observe that the fundamental solution $S_{r,n}$ satisfies the following scaling property, which can be verified by exploiting the definition of $S_{r,n}$ and elementary computations.

**Lemma 3.4.** *Let $n \in \mathbb{N}\setminus\{0,1\}$, $k \in \mathbb{C}\setminus]-\infty,0]$. Then the following equalities hold*

$$\begin{aligned} \epsilon^{n-2}S_{r,n}(\epsilon x,k) &= S_{r,n}(x,\epsilon k)\,,\\ \epsilon^{n-1}DS_{r,n}(\epsilon x,k) &= DS_{r,n}(x,\epsilon k) \end{aligned}$$

*for all $x \in \mathbb{R}^n\setminus\{0\}$, $\epsilon \in ]0,+\infty[$.*

Then we note that the following elementary equality holds

$$\gamma_n(\log(\epsilon k)) = \frac{2b_n}{\pi}\kappa_n\log\epsilon + \gamma_n(\log k)\,, \tag{3.6}$$

for all $k \in \mathbb{C}\setminus]-\infty,0]$ and $\epsilon \in ]0,+\infty[$ (cf. (3.4).)

## 4. Existence of a family of solutions $\{(u^i(\epsilon,\cdot),u^o(\epsilon,\cdot))\}_{\epsilon\in]0,\epsilon'[}$

We first transform problem (1.6) into a problem for integral equations on the boundaries $\partial\Omega^i$ and $\partial\Omega^o$. To do so, we first set

$$Y_{m-1,\alpha} \equiv C^{m-1,\alpha}(\partial\Omega^i)_0\times\mathbb{C}\times C^{m-1,\alpha}(\partial\Omega^i)_0\times\mathbb{C}\times C^{m-1,\alpha}(\partial\Omega^o)\,,$$

where

$$C^{m-1,\alpha}(\partial\Omega^i)_0 \equiv \left\{\theta \in C^{m-1,\alpha}(\partial\Omega^i) : \int_{\partial\Omega^i}\theta\,d\sigma = 0\right\}$$

and we mention that we can choose $\theta^\sharp \in C^{m-1,\alpha}(\partial\Omega^i)$ such that

$$\theta^\sharp \text{ is real valued}\,, \int_{\partial\Omega^i} \theta^\sharp \, d\sigma = 1\,, \qquad -\frac{1}{2}\theta^\sharp + W^t_{\Omega^i,h}[\theta^\sharp, 0] = 0 \quad \text{on } \partial\Omega^i \tag{4.1}$$

and accordingly that

$$v^\sharp \equiv V_{\Omega^i,h}[\theta^\sharp, 0] \text{ is constant on } \partial\Omega^i \tag{4.2}$$

(cf. *e.g.*, [11, Prop. 6.18, Thms. 6.24, 6.25], [15, Thm. 5.1]). Then we also have

$$V_{\Omega^i,J}[\theta^\sharp, 0] = J^\sharp_{\frac{n-2}{2}}(0) \qquad \text{on } \partial\Omega^i \tag{4.3}$$

and

$$\int_{\partial\Omega^i} \frac{1}{2}\phi + W^t_{\Omega^i,h}[\phi, 0] d\sigma = \int_{\partial\Omega^i} \phi \; d\sigma \qquad \forall \phi \in C^{m-1,\alpha}(\partial\Omega^i)$$

(cf. *e.g.*, [11, Lem. 6.11]). To shorten our notation, we find convenient to introduce the polynomial function $\varrho_n$ from $\mathbb{R}^2$ to $\mathbb{R}$ defined by

$$\varrho_n(\epsilon, \epsilon_1) \equiv [(1-\delta_{2,n})\epsilon^{n-3} + \delta_{2,n}][(1-\delta_{2,n})\epsilon_1 + \kappa_n \delta_{2,n}] \qquad \forall (\epsilon, \epsilon_1) \in \mathbb{R}^2\,, \tag{4.4}$$

and we observe that

$$\varrho_n(\epsilon, \kappa_n \epsilon \log \epsilon) = \epsilon^{n-2} \kappa_n \frac{\log \epsilon}{\log^{\delta_{2,n}} \epsilon} \qquad \forall \epsilon \in ]0,1[\,. \tag{4.5}$$

Then we set

$$Z_{m-1,\alpha} \equiv C^{m,\alpha}(\partial\Omega^i) \times C^{m-1,\alpha}(\partial\Omega^i) \times C^{m-1,\alpha}(\partial\Omega^o)$$

and we introduce the map $\mathcal{M} \equiv (\mathcal{M}_l)_{l=1,2,3}$ from $]-\epsilon_0, \epsilon_0[\times \mathbb{R}^2 \times Y_{m-1,\alpha}$ to $Z_{m-1,\alpha}$ defined by

$$\begin{aligned}
\mathcal{M}_1[\epsilon, \epsilon_1, \epsilon_2, \zeta, c^i, \varsigma^i, c, \theta^o](\xi) &\equiv \int_{\partial\Omega^i} S_{h,n}(\xi - \eta, \epsilon k_o)\varsigma^i(\eta)\, d\sigma_\eta \\
&+\epsilon^{n-1} k_o^n \left[\frac{2b_n}{\pi}\epsilon_1 + \epsilon\gamma_n(\log k_o)\right] \int_0^1 \frac{\partial}{\partial\lambda} V_{\Omega^i,J}[\varsigma^i, t\epsilon^2 k_o^2](\xi)\, dt \\
&+ \int_{\partial\Omega^i} S_{h,n}(\xi-\eta, \epsilon k_o) c^i \theta^\sharp(\eta)\, d\sigma_\eta \\
&+\epsilon^{n-1} k_o^n \left[\frac{2b_n}{\pi}\epsilon_1 + \epsilon\gamma_n(\log k_o)\right] c^i \int_0^1 \frac{\partial}{\partial\lambda} V_{\Omega^i,J}[\theta^\sharp, t\epsilon^2 k_o^2](\xi)\, dt \\
&+\epsilon^{n-2} k_o^{n-2} \gamma_n(\log k_o) c^i V_{\Omega^i,J}[\theta^\sharp, 0](\xi) + \int_{\partial\Omega^o} S_{r,n}(\epsilon\xi - y, k_o)\theta^o(y)\, d\sigma_y \\
&-a \int_{\partial\Omega^i} S_{h,n}(\xi-\eta, \epsilon k_i)\zeta(\eta)\, d\sigma_\eta \\
&-a\epsilon^{n-1} k_i^n \left[\frac{2b_n}{\pi}\epsilon_1 + \epsilon\gamma_n(\log k_i)\right] \int_0^1 \frac{\partial}{\partial\lambda} V_{\Omega^i,J}[\zeta, t\epsilon^2 k_i^2](\xi)\, dt \\
&-(k_o^{n-2}/k_i^{n-2}) c^i \int_{\partial\Omega^i} S_{h,n}(\xi-\eta, \epsilon k_i)\theta^\sharp(\eta)\, d\sigma_\eta \\
&-a \int_{\partial\Omega^i} S_{h,n}(\xi-\eta, \epsilon k_i) c[(1-\delta_{2,n}) + \epsilon_2]\theta^\sharp(\eta)\, d\sigma_\eta
\end{aligned} \tag{4.6}$$

$$-\epsilon^{n-1}k_o^{n-2}c^ik_i^2\left[\frac{2b_n}{\pi}\epsilon_1+\epsilon\gamma_n(\log k_i)\right]\int_0^1\frac{\partial}{\partial\lambda}V_{\Omega^i,J}[\theta^\sharp,t\epsilon^2k_i^2](\xi)\,dt$$

$$-a\epsilon^{n-1}k_i^n\left[\frac{2b_n}{\pi}\epsilon_1+\epsilon\gamma_n(\log k_i)\right]c[(1-\delta_{2,n})+\epsilon_2]$$

$$\times\int_0^1\frac{\partial}{\partial\lambda}V_{\Omega^i,J}[\theta^\sharp,t\epsilon^2k_i^2](\xi)\,dt-\epsilon^{n-2}k_o^{n-2}c^i\gamma_n(\log k_i)V_{\Omega^i,J}[\theta^\sharp,0](\xi)$$

$$-ak_i^{n-2}\left[\frac{2b_n}{\pi}\varrho_n(\epsilon,\epsilon_1)+\epsilon^{n-2}[(1-\delta_{2,n})+\epsilon_2]\gamma_n(\log k_i)\right]cV_{\Omega^i,J}[\theta^\sharp,0](\xi)-b,\ \forall\xi\in\partial\Omega^i,$$

$$\mathcal{M}_2[\epsilon,\epsilon_1,\epsilon_2,\zeta,c^i,\varsigma^i,c,\theta^o](\xi) \tag{4.7}$$

$$\equiv-\frac{1}{m^i}\Big\{-\frac{1}{2}\left(\zeta(\xi)+a^{-1}(k_o^{n-2}/k_i^{n-2})c^i\theta^\sharp(\xi)+c[(1-\delta_{2,n})+\epsilon_2]\theta^\sharp(\xi)\right)$$

$$+\int_{\partial\Omega^i}DS_{h,n}(\xi-\eta,\epsilon k_i)\nu_{\Omega^i}(\xi)$$

$$\times\left(\zeta(\eta)+a^{-1}(k_o^{n-2}/k_i^{n-2})c^i\theta^\sharp(\eta)+c[(1-\delta_{2,n})+\epsilon_2]\theta^\sharp(\eta)\right)\,d\sigma_\eta$$

$$+\epsilon^{n-1}k_i^n\left[\frac{2b_n}{\pi}\epsilon_1+\epsilon\gamma_n(\log k_i)\right]$$

$$\times\tilde{W}^t_{\Omega^i,J}[\zeta+a^{-1}(k_o^{n-2}/k_i^{n-2})c^i\theta^\sharp+c[(1-\delta_{2,n})+\epsilon_2]\theta^\sharp,\epsilon^2k_i^2](\xi)\Big\}$$

$$-\frac{1}{m^o}\Big\{-\frac{1}{2}\left(\varsigma^i(\xi)+c^i\theta^\sharp(\xi)\right)$$

$$-\int_{\partial\Omega^i}DS_{h,n}(\xi-\eta,\epsilon k_o)\nu_{\Omega^i}(\xi)\left(\varsigma^i(\eta)+c^i\theta^\sharp(\eta)\right)\,d\sigma_\eta$$

$$-\epsilon^{n-1}k_o^n\left[\frac{2b_n}{\pi}\epsilon_1+\epsilon\gamma_n(\log k_o)\right]\tilde{W}^t_{\Omega^i,J}[\varsigma^i+c^i\theta^\sharp,\epsilon^2k_o^2](\xi)$$

$$-\epsilon\int_{\partial\Omega^o}DS_{r,n}(\epsilon\xi-y,k_o)\nu_{\Omega^i}(\xi)\theta^o(y)\,d\sigma_y\Big\}-\epsilon g^i(\xi),\ \forall\xi\in\partial\Omega^i,$$

$$\mathcal{M}_3[\epsilon,\epsilon_1,\epsilon_2,\zeta,c^i,\varsigma^i,c,\theta^o](x) \tag{4.8}$$

$$\equiv-\frac{1}{2}\theta^o(x)+\int_{\partial\Omega^i}DS_{r,n}(x-\epsilon\eta,k_o)\nu_{\Omega^o}(x)\left(\varsigma^i(\eta)+c^i\theta^\sharp(\eta)\right)\,d\sigma_\eta\epsilon^{n-2}$$

$$+\int_{\partial\Omega^o}DS_{r,n}(x-y,k_o)\nu_{\Omega^o}(x)\theta^o(y)\,d\sigma_y-g^o(x),\ \forall x\in\partial\Omega^o$$

for all $(\epsilon,\epsilon_1,\epsilon_2,\zeta,c^i,\varsigma^i,c,\theta^o)\in]-\epsilon_0,\epsilon_0[\times\mathbb{R}^2\times Y_{m-1,\alpha}$. Here $\frac{\partial}{\partial\lambda}V_{\Omega^i,J}$ denotes the partial differential of the analytic map $V_{\Omega^i,J}[\cdot,\cdot]$ with respect to its second argument (cf. [1, Thm. 3.22]). Then we have the following statement of [1, Thms. 4.18, 4.47] that shows that for $\epsilon\in]0,\epsilon_0[$ small problem (1.6) is equivalent to equation $\mathcal{M}[\epsilon,\epsilon_1,\epsilon_2,\zeta,c^i,\varsigma^i,c,\theta^o]=0$ provided that we choose $\epsilon_1=\kappa_n\epsilon\log\epsilon$, $\epsilon_2=\frac{\delta_{2,n}}{\log\epsilon}$.

**Theorem 4.1.** *Let* $m \in \mathbb{N} \setminus \{0\}$, $n \in \mathbb{N} \setminus \{0,1\}$, $\alpha \in ]0,1[$. *Let* $\Omega^i$, $\Omega^o$ *be as in* (1.1). *Let* $m^i$, $m^o$, $a \in ]0,+\infty[$, $b \in \mathbb{R}$. *Let* $g^i$, $g^o$ *be as in* (1.3), (1.5). *Let* $k_i$, $k_o$ *be as in* (1.2), (1.4). *Assume that* $k_o^2$ *is not a Neumann eigenvalue for* $-\Delta$ *in* $\Omega^o$. *Let* $\theta^\sharp \in C^{m-1,\alpha}(\partial\Omega^i)$ *be as in* (4.1). *Let* $\mathcal{M} \equiv (\mathcal{M}_l)_{l=1,2,3}$ *be the map from* $]-\epsilon_0, \epsilon_0[\times \mathbb{R}^2 \times Y_{m-1,\alpha}$ *to* $Z_{m-1,\alpha}$ *defined by* (4.6)–(4.8). *Then the following statements hold.*

(i) *If* $\epsilon = \epsilon_1 = \epsilon_2 = 0$, *then equation*

$$\mathcal{M}[0,0,0,\zeta,c^i,\varsigma^i,c,\theta^o] = 0 \tag{4.9}$$

*has one and only one solution* $(\tilde{\zeta},\tilde{c}^i,\tilde{\varsigma}^i,\tilde{c},\tilde{\theta}^o)$ *in* $Y_{m-1,\alpha}$. *Moreover,* $\tilde{c}^i = 0$.

(ii) *There exists* $\epsilon^* \in ]0,\epsilon_0[$ *such that the map from the subset of* $Y_{m-1,\alpha}$ *consisting of the* 5*-tuples* $(\zeta,c^i,\varsigma^i,c,\theta^o)$ *that solve the equation*

$$\mathcal{M}[\epsilon,\kappa_n\epsilon\log\epsilon,\frac{\delta_{2,n}}{\log\epsilon},\zeta,c^i,\varsigma^i,c,\theta^o] = 0$$

*onto the set of solutions* $(u^i,u^o)$ *in* $C^{m,\alpha}(\epsilon\overline{\Omega^i}) \times C^{m,\alpha}(\overline{\Omega(\epsilon)})$, *which satisfy problem* (1.6), *which takes* $(\zeta,c^i,\varsigma^i,c,\theta^o)$ *to the pair of functions*

$$(u^i[\epsilon,\zeta,c^i,\varsigma^i,c,\theta^o], u^o[\epsilon,\zeta,c^i,\varsigma^i,c,\theta^o])$$

*defined by*

$$\begin{aligned} u^i[\epsilon,\zeta,c^i,\varsigma^i,c,\theta^o](x) &= \frac{1}{\epsilon}v^+_{\epsilon\Omega^i}[\zeta(\cdot/\epsilon),k_i](x) \\ &+\frac{1}{\epsilon}\Big(a^{-1}\big(k_o^{n-2}/k_i^{n-2}\big)c^i + \frac{c}{(\log\epsilon)^{\delta_{2,n}}}\Big)v^+_{\epsilon\Omega^i}[\theta^\sharp(\cdot/\epsilon),k_i](x) \qquad \forall x \in \epsilon\overline{\Omega^i}\,, \\ u^o[\epsilon,\zeta,c^i,\varsigma^i,c,\theta^o](x) &= v^+_{\Omega^o}[\theta^o,k_o](x) + \frac{1}{\epsilon}v^-_{\epsilon\Omega^i}[\varsigma^i(\cdot/\epsilon),k_o](x) \\ &+\frac{c^i}{\epsilon}v^-_{\epsilon\Omega^i}[\theta^\sharp(\cdot/\epsilon),k_o](x) \qquad \forall x \in \overline{\Omega(\epsilon)}\,, \end{aligned} \tag{4.10}$$

*is a bijection.*

The equation (4.9) can be shown to be equivalent to a boundary value problem in the sense of the following statement of [1, Thm. 4.32].

**Theorem 4.2.** *Let* $m \in \mathbb{N} \setminus \{0\}$, $n \in \mathbb{N} \setminus \{0,1\}$, $\alpha \in ]0,1[$. *Let* $\Omega^i$, $\Omega^o$ *be as in* (1.1). *Let* $m^i$, $m^o$, $a \in ]0,+\infty[$, $b \in \mathbb{R}$. *Let* $g^i$, $g^o$ *be as in* (1.3), (1.5). *Let* $k_i$, $k_o$ *be as in* (1.2), (1.4). *Assume that* $k_o^2$ *is not a Neumann eigenvalue for* $-\Delta$ *in* $\Omega^o$. *Then the limiting boundary value problem*

$$\begin{cases} \Delta u_1^{i,r} = 0 & \text{in } \Omega^i\,, \\ \Delta u_1^{o,r} = 0 & \text{in } \Omega^{i-}\,, \\ \Delta u^o + k_o^2 u^o = 0 & \text{in } \Omega^o\,, \\ u_1^{o,r}(x) + u^o(0) - a u_1^{i,r}(x) = b & \forall x \in \partial\Omega^i\,, \\ -\frac{1}{m^i}\frac{\partial}{\partial\nu_{\Omega^i}}u_1^{i,r}(x) + \frac{1}{m^o}\frac{\partial}{\partial\nu_{\Omega^i}}u_1^{o,r}(x) = 0 & \forall x \in \partial\Omega^i\,, \\ \frac{\partial}{\partial\nu_{\Omega^o}}u^o = g^o & \text{on } \partial\Omega^o\,, \\ \lim_{\xi\to\infty} u_1^{o,r}(\xi) = 0\,, \end{cases}$$

*has one and only one solution* $(\tilde{u}_1^{i,r}, \tilde{u}_1^{o,r}, \tilde{u}^o)$ *in*

$$C^{m,\alpha}(\overline{\Omega^i}) \times C^{m,\alpha}_{\mathrm{loc}}(\overline{\Omega^{i-}}) \times C^{m,\alpha}(\overline{\Omega^o}),$$

*which is delivered by the following formulas*

$$\begin{aligned} &\tilde{u}_1^{i,r} = v^+_{\Omega^i,h}[\tilde{\zeta}, 0] + \tilde{C} \quad \text{in } \overline{\Omega^i}, \qquad \tilde{u}_1^{o,r} = v^-_{\Omega^i,h}[\tilde{\varsigma}^i, 0] \quad \text{in } \overline{\Omega^{i-}}, \\ &\tilde{u}^o = v^+_{\Omega^o}[\tilde{\theta}^o, k_o] \qquad \text{in } \overline{\Omega^o} \end{aligned} \tag{4.11}$$

*where* $(\tilde{\zeta}, \tilde{c}^i, \tilde{\varsigma}^i, \tilde{c}, \tilde{\theta}^o)$ *is the only solution in* $Y_{m-1,\alpha}$ *of equation* (4.9) *and*

$$\tilde{C} = \left(\frac{\delta_{2,n}}{2\pi} + (1 - \delta_{2,n})\, v^\sharp\right) \tilde{c}$$

*(see* (4.2) *for the constant* $v^\sharp \equiv V_{\Omega^i,h}[\theta^\sharp, 0]$*).*

Next we turn to equation $\mathcal{M} = 0$. One can show that one can solve equation $\mathcal{M}[\epsilon, \epsilon_1, \epsilon_2, \zeta, c^i, \varsigma^i, c, \theta^o] = 0$ in the unknown $(\zeta, c^i, \varsigma^i, c, \theta^o)$ in terms of $(\epsilon, \epsilon_1, \epsilon_2)$ by mean of the following statement of [1, Thm. 4.53, Rmk. 4.58].

**Theorem 4.3.** *Let* $m \in \mathbb{N} \setminus \{0\}$, $n \in \mathbb{N} \setminus \{0, 1\}$, $\alpha \in ]0, 1[$. *Let* $\Omega^i$, $\Omega^o$ *be as in* (1.1). *Let* $m^i$, $m^o$, $a \in ]0, +\infty[$, $b \in \mathbb{R}$. *Let* $g^i$, $g^o$ *be as in* (1.3), (1.5). *Let* $k_i$, $k_o$ *be as in* (1.2), (1.4). *Assume that* $k_o^2$ *is not a Neumann eigenvalue for* $-\Delta$ *in* $\Omega^o$. *Let* $\epsilon_* \in ]0, \epsilon_0[$ *be as in Theorem* 4.1. *Let* $\mathcal{M} \equiv (\mathcal{M}_l)_{l=1,2,3}$ *be the map from* $]-\epsilon_0, \epsilon_0[ \times \mathbb{R}^2 \times Y_{m-1,\alpha}$ *to* $Z_{m-1,\alpha}$ *defined by* (4.6)–(4.8). *Then there exists* $\epsilon' \in ]0, \epsilon_*[$, *an open neighbourhood* $\tilde{U}$ *of* $(0, 0)$ *in* $\mathbb{R}^2$ *and an open neighbourhood* $\tilde{V}$ *of* $(\tilde{\zeta}, \tilde{c}^i, \tilde{\varsigma}^i, \tilde{c}, \tilde{\theta}^o)$ *in* $Y_{m-1,\alpha}$ *and a real analytic map*

$$(Z, C^i, S^i, C, \Theta^o)$$

*from* $]-\epsilon', \epsilon'[ \times \tilde{U}$ *to* $\tilde{V}$ *such that*

$$\left(\kappa_n \epsilon \log \epsilon, \frac{\delta_{2,n}}{\log \epsilon}\right) \in \tilde{U}, \ \forall \epsilon \in ]0, \epsilon'[,$$

*and such that the set of zeros of* $\mathcal{M}$ *in* $]-\epsilon', \epsilon'[ \times \tilde{U} \times \tilde{V}$ *coincides with the graph of the map* $(Z, C^i, S^i, C, \Theta^o)$. *In particular,*

$$\big(Z[0,0,0], C^i[0,0,0], S^i[0,0,0], C[0,0,0], \Theta^o[0,0,0]\big) = (\tilde{\zeta}, \tilde{c}^i, \tilde{\varsigma}^i, \tilde{c}, \tilde{\theta}^o),$$

*where* $(\tilde{\zeta}, \tilde{c}^i, \tilde{\varsigma}^i, \tilde{c}, \tilde{\theta}^o)$ *is the only solution in* $Y_{m-1,\alpha}$ *of equation* (4.9). *Moreover,*

$$\frac{\partial C^i}{\partial \epsilon_1}[0,0,0] = 0, \ \frac{\partial C^i}{\partial \epsilon_2}[0,0,0] = 0. \tag{4.12}$$

For the sake of brevity, we set

$$\Xi_n[\epsilon] \equiv \left(\kappa_n \epsilon \log \epsilon, \frac{\delta_{2,n}}{\log \epsilon}\right), \qquad \forall \epsilon \in ]0, 1[. \tag{4.13}$$

Then we have the following existence and uniqueness theorem for problem (1.6) for $\epsilon \in ]0, \epsilon'[$ (cf. [1, Thm. 4.61].)

**Theorem 4.4.** *Let* $m \in \mathbb{N} \setminus \{0\}$, $n \in \mathbb{N} \setminus \{0,1\}$, $\alpha \in ]0,1[$. *Let* $\Omega^i$, $\Omega^o$ *be as in* (1.1). *Let* $m^i$, $m^o$, $a \in ]0,+\infty[$, $b \in \mathbb{R}$. *Let* $g^i$, $g^o$ *be as in* (1.3), (1.5). *Let* $k_i$, $k_o$ *be as in* (1.2), (1.4). *Assume that* $k_o^2$ *is not a Neumann eigenvalue for* $-\Delta$ *in* $\Omega^o$.

*Let* $\epsilon' \in ]0,\epsilon_0[$ *be as in Theorem* 4.3. *If* $\epsilon \in ]0,\epsilon'[$, *then the transmission problem* (1.6) *has one and only one solution* $(u^i(\epsilon,\cdot), u^o(\epsilon,\cdot)) \in C^{m,\alpha}(\epsilon\overline{\Omega^i}) \times C^{m,\alpha}(\overline{\Omega(\epsilon)})$ *and the following formula holds*

$$
\begin{aligned}
&u^i(\epsilon,\cdot) \\
&\quad = u^i[\epsilon, Z[\epsilon,\Xi_n[\epsilon]], C^i[\epsilon,\Xi_n[\epsilon]], S^i[\epsilon,\Xi_n[\epsilon]], C[\epsilon,\Xi_n[\epsilon]], \Theta^o[\epsilon,\Xi_n[\epsilon]]](\cdot) \\
&u^o(\epsilon,\cdot) \\
&\quad = u^o[\epsilon, Z[\epsilon,\Xi_n[\epsilon]], C^i[\epsilon,\Xi_n[\epsilon]], S^i[\epsilon,\Xi_n[\epsilon]], C[\epsilon,\Xi_n[\epsilon]], \Theta^o[\epsilon,\Xi_n[\epsilon]]](\cdot)
\end{aligned}
\tag{4.14}
$$

*for all* $\epsilon \in ]0,\epsilon'[$ *(cf.* (4.10)*).*

## 5. Microscopic representation for $\{(u^i(\epsilon,\cdot), u^o(\epsilon,\cdot))\}_{\epsilon\in]0,\epsilon'[}$

We now analyze the microscopic behavior of our family of solutions, i.e., the behavior of the rescaled family $\{(u^i(\epsilon,\epsilon\cdot), u^o(\epsilon,\epsilon\cdot))\}_{\epsilon\in]0,\epsilon'[}$.

**Theorem 5.1.** *With the assumptions of Theorem* 4.3, *the following statements hold.*

(i) *There exist real analytic maps* $\mathcal{U}_1^i$, $\mathcal{U}_2^i$ *from* $]-\epsilon',\epsilon'[\times\tilde{U}$ *to* $C^{m,\alpha}(\overline{\Omega^i})$ *such that*

$$
u^i(\epsilon,\epsilon\xi) = \mathcal{U}_1^i[\epsilon,\Xi_n[\epsilon]] + (\kappa_n \epsilon \log^2 \epsilon)\mathcal{U}_2^i[\epsilon,\Xi_n[\epsilon]] \quad \forall \xi \in \overline{\Omega^i}
$$

*for all* $\epsilon \in ]0,\epsilon'[$ *(cf.* (4.13) *for the definition of* $\Xi_n$*). Moreover,*

$$
\mathcal{U}_1^i[0,0,0] = \tilde{u}_1^{i,r}, \qquad \mathcal{U}_2^i[0,0,0] = 0,
$$

*where* $\tilde{u}_1^{i,r}$ *has been defined in Theorem* 4.2.

(ii) *Let* $\Omega_m$ *be a bounded open subset of* $\mathbb{R}^n \setminus \overline{\Omega^i}$. *Then there exist* $\epsilon_m \in ]0,\epsilon'[$, *and two real analytic maps* $\mathcal{U}_{m,1}^o$, $\mathcal{U}_{m,2}^o$ *from* $]-\epsilon_m,\epsilon_m[\times\tilde{U}$ *to* $C^{m,\alpha}(\overline{\Omega_m})$ *such that*

$$
\epsilon\overline{\Omega_m} \subseteq \Omega^o \ \forall \epsilon \in ]-\epsilon_m,\epsilon_m[,
$$

$$
\begin{aligned}
u^o(\epsilon,\epsilon\xi) = {} & \mathcal{U}_{m,1}^o[\epsilon,\Xi[\epsilon]](\xi) \\
& + (\kappa_n \epsilon \log^2 \epsilon)\mathcal{U}_{m,2}^o[\epsilon,\Xi[\epsilon]](\xi) \qquad \forall \xi \in \overline{\Omega_m},\ \epsilon \in ]0,\epsilon_m[.
\end{aligned}
$$

*Moreover,*

$$
\mathcal{U}_{m,1}^o[0,0,0](\xi) = \tilde{u}^o(0) + \tilde{u}_1^{o,r}(\xi), \quad \mathcal{U}_{m,2}^o[0,0,0](\xi) = 0 \quad \forall \xi \in \overline{\Omega_m},
$$

*where* $\tilde{u}^o$ *and* $\tilde{u}_1^{o,r}$ *are as in Theorem* 4.2.

*Proof.* By the first formulas of (4.10) and (4.14), we have

$$
\begin{aligned}
u^i(\epsilon,\epsilon\xi) = {} & \epsilon^{n-2}\int_{\partial\Omega^i} S_{r,n}(\epsilon\xi-\epsilon\eta,k_i) Z[\epsilon,\Xi_n[\epsilon]](\eta)\,d\sigma_\eta \\
& + \epsilon^{n-2} a^{-1}\left(k_o^{n-2}/k_i^{n-2}\right) C^i[\epsilon,\Xi_n[\epsilon]] \int_{\partial\Omega^i} S_{r,n}(\epsilon\xi-\epsilon\eta,k_i)\theta^\sharp(\eta)\,d\sigma_\eta \\
& + \epsilon^{n-2}\log^{-\delta_{2,n}}\epsilon\ C[\epsilon,\Xi_n[\epsilon]] \int_{\partial\Omega^i} S_{r,n}(\epsilon\xi-\epsilon\eta,k_i)\theta^\sharp(\eta)\,d\sigma_\eta \quad \forall \xi \in \overline{\Omega^i}
\end{aligned}
$$

for all $\epsilon \in ]0, \epsilon'[$. Then Lemma 3.4 implies that

$$u^i(\epsilon, \epsilon\xi) = v^+_{\Omega^i}[Z[\epsilon, \Xi_n[\epsilon]], \epsilon k_i](\xi) \\ +a^{-1}\left(k_o^{n-2}/k_i^{n-2}\right) C^i[\epsilon, \Xi_n[\epsilon]] v^+_{\Omega^i}[\theta^\sharp, \epsilon k_i](\xi) \\ + \log^{-\delta_{2,n}} \epsilon \; C[\epsilon, \Xi_n[\epsilon]] v^+_{\Omega^i}[\theta^\sharp, \epsilon k_i](\xi) \quad \forall \xi \in \overline{\Omega^i}$$

for all $\epsilon \in ]0, \epsilon'[$. Then equality (3.5) implies that

$$u^i(\epsilon, \epsilon\xi) = v^+_{\Omega^i,h}[Z[\epsilon, \Xi_n[\epsilon]], \epsilon k_i](\xi) \\ +\gamma_n(\log(\epsilon k_i))\epsilon^{n-2} k_i^{n-2} v^+_{\Omega^i,J}[Z[\epsilon, \Xi_n[\epsilon]], \epsilon^2 k_i^2](\xi) \\ +a^{-1}\left(k_o^{n-2}/k_i^{n-2}\right) C^i[\epsilon, \Xi_n[\epsilon]] \\ \times \left(v^+_{\Omega^i,h}[\theta^\sharp, \epsilon k_i](\xi) + \gamma_n(\log(\epsilon k_i))\epsilon^{n-2} k_i^{n-2} v^+_{\Omega^i,J}[\theta^\sharp, \epsilon^2 k_i^2](\xi)\right) \\ + \log^{-\delta_{2,n}} \epsilon \; C[\epsilon, \Xi_n[\epsilon]] \\ \times \left(v^+_{\Omega^i,h}[\theta^\sharp, \epsilon k_i](\xi) + \gamma_n(\log(\epsilon k_i))\epsilon^{n-2} k_i^{n-2} v^+_{\Omega^i,J}[\theta^\sharp, \epsilon^2 k_i^2](\xi)\right)$$

for all $\xi \in \overline{\Omega^i}$ and $\epsilon \in ]0, \epsilon'[$. By equality (3.6), we have

$$u^i(\epsilon, \epsilon\xi) = v^+_{\Omega^i,h}[Z[\epsilon, \Xi_n[\epsilon]], \epsilon k_i](\xi) \\ + \left[\frac{2b_n}{\pi}\kappa_n \log\epsilon + \gamma_n(\log k_i)\right] \epsilon^{n-2} k_i^{n-2} v^+_{\Omega^i,J}[Z[\epsilon, \Xi_n[\epsilon]], \epsilon^2 k_i^2](\xi) \\ +a^{-1}\left(k_o^{n-2}/k_i^{n-2}\right) C^i[\epsilon, \Xi_n[\epsilon]] \Bigg( v^+_{\Omega^i,h}[\theta^\sharp, \epsilon k_i](\xi) \\ + \left[\frac{2b_n}{\pi}\kappa_n \log\epsilon + \gamma_n(\log k_i)\right] \epsilon^{n-2} k_i^{n-2} v^+_{\Omega^i,J}[\theta^\sharp, \epsilon^2 k_i^2](\xi)\Bigg) \\ + \log^{-\delta_{2,n}} \epsilon \; C[\epsilon, \Xi_n[\epsilon]] \Bigg( v^+_{\Omega^i,h}[\theta^\sharp, \epsilon k_i](\xi) \\ + \left[\frac{2b_n}{\pi}\kappa_n \log\epsilon + \gamma_n(\log k_i)\right] \epsilon^{n-2} k_i^{n-2} v^+_{\Omega^i,J}[\theta^\sharp, \epsilon^2 k_i^2](\xi)\Bigg) \qquad \forall \xi \in \overline{\Omega^i}$$

for all $\epsilon \in ]0, \epsilon'[$. Since there exists an entire function $J^{\sharp,1}_{\frac{n-2}{2}}$ such that

$$J^\sharp_{\frac{n-2}{2}}(z) = J^\sharp_{\frac{n-2}{2}}(0) + z J^{\sharp,1}_{\frac{n-2}{2}}(z) \qquad \forall z \in \mathbb{C} \tag{5.1}$$

(see (3.1)), then we have

$$v^+_{\Omega^i,J}[Z[\epsilon, \Xi_n[\epsilon]], \epsilon^2 k_i^2](\xi) = \int_{\partial\Omega^i} J^\sharp_{\frac{n-2}{2}}(\epsilon^2 k_i^2 |\xi-\eta|^2) Z[\epsilon, \Xi_n[\epsilon]](\eta) d\sigma_\eta \tag{5.2}$$

$$= J^\sharp_{\frac{n-2}{2}}(0) \int_{\partial\Omega^i} Z[\epsilon, \Xi_n[\epsilon]](\eta) d\sigma_\eta \\ +\epsilon^2 k_i^2 \int_{\partial\Omega^i} |\xi-\eta|^2 J^{\sharp,1}_{\frac{n-2}{2}}(\epsilon^2 k_i^2 |\xi-\eta|^2) Z[\epsilon, \Xi_n[\epsilon]](\eta) d\sigma_\eta \\ = \epsilon^2 k_i^2 \int_{\partial\Omega^i} |\xi-\eta|^2 J^{\sharp,1}_{\frac{n-2}{2}}(\epsilon^2 k_i^2 |\xi-\eta|^2) Z[\epsilon, \Xi_n[\epsilon]](\eta) d\sigma_\eta \qquad \forall \epsilon \in ]0, \epsilon'[.$$

Indeed, $\int_{\partial\Omega^i} Z[\epsilon,\Xi_n[\epsilon]](\eta)d\sigma_\eta = 0$ for all $\epsilon \in ]0,\epsilon'[$. Also, the Fundamental Theorem of Calculus and equality $C^i[0,0,0] = \tilde{c}^i = 0$ imply that

$$
\begin{aligned}
&(\kappa_n \log\epsilon)\ C^i[\epsilon,\kappa_n\epsilon\log\epsilon,\delta_{2,n}/\log\epsilon] \\
&\quad = (\kappa_n\epsilon\log\epsilon)\int_0^1 \frac{\partial C^i}{\partial t_1}[s\epsilon,s\kappa_n\epsilon\log\epsilon,s\delta_{2,n}/\log\epsilon]\,ds \\
&\qquad +\kappa_n\epsilon\log^2\epsilon\int_0^1 \frac{\partial C^i}{\partial t_2}[s\epsilon,s\kappa_n\epsilon\log\epsilon,s\delta_{2,n}/\log\epsilon]\,ds \\
&\qquad +\delta_{2,n}\int_0^1 \frac{\partial C^i}{\partial t_3}[s\epsilon,s\kappa_n\epsilon\log\epsilon,s\delta_{2,n}/\log\epsilon]\,ds \qquad \forall\epsilon\in]0,\epsilon'[.
\end{aligned} \tag{5.3}
$$

Next introduce the following analytic functions

$$
C_1[\epsilon,\epsilon_1,\epsilon_2] \equiv \epsilon_1\int_0^1 \frac{\partial C^i}{\partial t_1}[s\epsilon,s\epsilon_1,s\epsilon_2]\,ds + \delta_{2,n}\int_0^1 \frac{\partial C^i}{\partial t_3}[s\epsilon,s\epsilon_1,s\epsilon_2]\,ds
$$

and

$$
C_2[\epsilon,\epsilon_1,\epsilon_2] \equiv \int_0^1 \frac{\partial C^i}{\partial t_2}[s\epsilon,s\epsilon_1,s\epsilon_2]\,ds \qquad \forall(\epsilon,\epsilon_1,\epsilon_2)\in]-\epsilon',\epsilon'[\times\tilde{U}.
$$

By equality (5.3), we have

$$
\begin{aligned}
&(\kappa_n\log\epsilon)\ C^i[\epsilon,\kappa_n\epsilon\log\epsilon,\delta_{2,n}/\log\epsilon] \\
&\quad = C_1[\epsilon,\kappa_n\epsilon\log\epsilon,\delta_{2,n}/\log\epsilon] + \kappa_n\epsilon\log^2\epsilon\, C_2[\epsilon,\kappa_n\epsilon\log\epsilon,\delta_{2,n}/\log\epsilon].
\end{aligned} \tag{5.4}
$$

By (5.2), (5.4), and by the elementary equality

$$
\log^{-\delta_{2,n}}\epsilon = (1-\delta_{2,n}) + \frac{\delta_{2,n}}{\log\epsilon} \qquad \forall\epsilon\in]0,1[\,,
$$

we have

$$
\begin{aligned}
u^i(\epsilon,\epsilon\xi) &= v^+_{\Omega^i,h}[Z[\epsilon,\Xi_n[\epsilon]],\epsilon k_i](\xi) \\
&+\frac{2b_n}{\pi}k_i^n\kappa_n\epsilon^n\log\epsilon\int_{\partial\Omega^i}|\xi-\eta|^2 J^{\sharp,1}_{\frac{n-2}{2}}(\epsilon^2k_i^2|\xi-\eta|^2)Z[\epsilon,\Xi_n[\epsilon]](\eta)d\sigma_\eta \\
&+\gamma_n(\log k_i)k_i^{n-2}\epsilon^{n-2}v^+_{\Omega^i,J}[Z[\epsilon,\Xi_n[\epsilon]],\epsilon^2k_i^2](\xi) \\
&+a^{-1}\left(k_o^{n-2}/k_i^{n-2}\right)C^i[\epsilon,\Xi_n[\epsilon]]v^+_{\Omega^i,h}[\theta^\sharp,\epsilon k_i](\xi) \\
&+a^{-1}k_o^{n-2}\frac{2b_n}{\pi}C_1[\epsilon,\Xi_n[\epsilon]]\,\epsilon^{n-2}v^+_{\Omega^i,J}[\theta^\sharp,\epsilon^2k_i^2](\xi) \\
&+a^{-1}k_o^{n-2}\frac{2b_n}{\pi}\kappa_n\epsilon\log^2\epsilon\, C_2[\epsilon,\Xi_n[\epsilon]]\epsilon^{n-2}v^+_{\Omega^i,J}[\theta^\sharp,\epsilon^2k_i^2](\xi) \\
&+a^{-1}k_o^{n-2}C^i[\epsilon,\Xi_n[\epsilon]]\gamma_n(\log k_i)\epsilon^{n-2}v^+_{\Omega^i,J}[\theta^\sharp,\epsilon^2k_i^2](\xi) \\
&+\left[(1-\delta_{2,n})+\frac{\delta_{2,n}}{\log\epsilon}\right]C[\epsilon,\Xi_n[\epsilon]]v^+_{\Omega^i,h}[\theta^\sharp,\epsilon k_i](\xi) \\
&+\frac{2b_n}{\pi}\varrho_n(\epsilon,\kappa_n\epsilon\log\epsilon)k_i^{n-2}C[\epsilon,\Xi_n[\epsilon]]v^+_{\Omega^i,J}[\theta^\sharp,\epsilon^2k_i^2](\xi) \\
&+\left[(1-\delta_{2,n})+\frac{\delta_{2,n}}{\log\epsilon}\right]\gamma_n(\log k_i)C[\epsilon,\Xi_n[\epsilon]]\epsilon^{n-2}k_i^{n-2}v^+_{\Omega^i,J}[\theta^\sharp,\epsilon^2k_i^2](\xi),\ \forall\xi\in\overline{\Omega^i}
\end{aligned}
$$

for all $\epsilon \in ]0, \epsilon'[$ (see (4.5)). Thus, we find natural to set

$$\begin{aligned}
\mathcal{U}_1^i[\epsilon, \epsilon_1, \epsilon_2](\xi) &\equiv v^+_{\Omega^i,h}[Z[\epsilon, \epsilon_1, \epsilon_2], \epsilon k_i](\xi) \\
&+ \frac{2b_n}{\pi} k_i^n \epsilon_1 \epsilon^{n-1} \int_{\partial\Omega^i} |\xi - \eta|^2 J^{\sharp,1}_{\frac{n-2}{2}}(\epsilon^2 k_i^2 |\xi - \eta|^2) Z[\epsilon, \epsilon_1, \epsilon_2](\eta) d\sigma_\eta \\
&+ \gamma_n (\log k_i) \epsilon^{n-2} k_i^{n-2} v^+_{\Omega^i,J}[Z[\epsilon, \epsilon_1, \epsilon_2], \epsilon^2 k_i^2](\xi) \\
&+ a^{-1} \left(k_o^{n-2}/k_i^{n-2}\right) C^i[\epsilon, \epsilon_1, \epsilon_2] v^+_{\Omega^i,h}[\theta^\sharp, \epsilon k_i](\xi) \\
&+ a^{-1} k_o^{n-2} \frac{2b_n}{\pi} C_1[\epsilon, \epsilon_1, \epsilon_2]\, \epsilon^{n-2} v^+_{\Omega^i,J}[\theta^\sharp, \epsilon^2 k_i^2](\xi) \\
&+ a^{-1} k_o^{n-2} C^i[\epsilon, \epsilon_1, \epsilon_2] \gamma_n (\log k_i) \epsilon^{n-2} v^+_{\Omega^i,J}[\theta^\sharp, \epsilon^2 k_i^2](\xi) \\
&+ \left[(1 - \delta_{2,n}) + \epsilon_2\right] C[\epsilon, \epsilon_1, \epsilon_2] v^+_{\Omega^i,h}[\theta^\sharp, \epsilon k_i](\xi) \\
&+ \frac{2b_n}{\pi} \varrho_n(\epsilon, \epsilon_1) k_i^{n-2} C[\epsilon, \epsilon_1, \epsilon_2] v^+_{\Omega^i,J}[\theta^\sharp, \epsilon^2 k_i^2](\xi) + \left[(1 - \delta_{2,n}) + \epsilon_2\right] \\
&\times \gamma_n (\log k_i) C[\epsilon, \epsilon_1, \epsilon_2] \epsilon^{n-2} k_i^{n-2} v^+_{\Omega^i,J}[\theta^\sharp, \epsilon^2 k_i^2](\xi) \quad \forall \xi \in \overline{\Omega^i}, \\
\mathcal{U}_2^i[\epsilon, \epsilon_1, \epsilon_2](\xi) &\equiv a^{-1} k_o^{n-2} \frac{2b_n}{\pi} C_2[\epsilon, \epsilon_1, \epsilon_2] \epsilon^{n-2} v^+_{\Omega^i,J}[\theta^\sharp, \epsilon^2 k_i^2](\xi) \quad \forall \xi \in \overline{\Omega^i}
\end{aligned} \tag{5.5}$$

for all $(\epsilon, \epsilon_1, \epsilon_2) \in ]\epsilon', \epsilon'[ \times \tilde{U}$ (see Theorem 4.3). By [1, Thm. 3.21 (i)], $v^+_{\Omega^i,h}[\cdot,\cdot]$ defines a real analytic map from $C^{m-1,\alpha}(\partial\Omega^i) \times \mathbb{C}$ to $C^{m,\alpha}(\overline{\Omega^i})$. Then Theorem 4.3 implies that the map from $]-\epsilon', \epsilon'[ \times \tilde{U}$ to $C^{m,\alpha}(\overline{\Omega^i})$ which takes $(\epsilon, \epsilon_1, \epsilon_2)$ to $v^+_{\Omega^i,h}[Z[\epsilon, \epsilon_1, \epsilon_2], \epsilon k_i]$ is real analytic. Similarly, the map from $]-\epsilon', \epsilon'[ \times \tilde{U}$ to $C^{m,\alpha}(\overline{\Omega^i})$ which takes $(\epsilon, \epsilon_1, \epsilon_2)$ to $v^+_{\Omega^i,h}\left[\theta^\sharp, \epsilon k_i\right]$ is real analytic. Since $|\xi - \eta|^2 J^{\sharp,1}_{\frac{n-2}{2}}(\epsilon^2 k_i^2 |\xi - \eta|^2)$ is analytic in the variable $(\xi, \eta, \epsilon)$ in an open neighbourhood of $\overline{\Omega^i} \times \partial\Omega^i \times ]-\epsilon', \epsilon'[$ then Proposition 4.1 (i) of paper [17] with Musolino on integral operators with real analytic kernel implies that the map from $]-\epsilon', \epsilon'[ \times L^1(\partial\Omega^i)$ to $C^{m,\alpha}(\overline{\Omega^i})$ that takes $(\epsilon, f)$ to the function $\int_{\partial\Omega^i} |\xi - \eta|^2 J^{\sharp,1}_{\frac{n-2}{2}}(\epsilon^2 k_i^2 |\xi - \eta|^2) f(\eta) d\sigma_\eta$ of the variable $\xi \in \partial\Omega^i$ is real analytic. Since $Z$ is real analytic and $C^{m-1,\alpha}(\partial\Omega^i)$ is continuously imbedded into $L^1(\partial\Omega^i)$, we conclude that the map from $]-\epsilon', \epsilon'[ \times \tilde{U}$ to $C^{m,\alpha}(\overline{\Omega^i})$ which takes $(\epsilon, \epsilon_1, \epsilon_2)$ to the second summand of the right-hand side of (5.5) is analytic. By [1, Thm. 3.22 (ii)], $v^+_{\Omega^i,J}[\cdot,\cdot]$ defines a real analytic map from $C^{m-1,\alpha}(\partial\Omega^i) \times \mathbb{C}$ to $C^{m,\alpha}(\overline{\Omega^i})$. Then Theorem 4.3 implies that the map from $]-\epsilon', \epsilon'[ \times \tilde{U}$ to $C^{m,\alpha}(\overline{\Omega^i})$ which takes $(\epsilon, \epsilon_1, \epsilon_2)$ to $v^+_{\Omega^i,J}\left[Z[\epsilon, \epsilon_1, \epsilon_2], \epsilon^2 k_i^2\right]$ is real analytic. Similarly, the map from $]-\epsilon', \epsilon'[ \times \tilde{U}$ to $C^{m,\alpha}(\overline{\Omega^i})$ which takes $(\epsilon, \epsilon_1, \epsilon_2)$ to $v^+_{\Omega^i,J}[\theta^\sharp, \epsilon^2 k_i^2]$ is real analytic. Finally $C^i$ is real analytic by Theorem 4.3 and thus, $C_1$ and $C_2$ are real analytic as well. Hence $\mathcal{U}_1^i$ and $\mathcal{U}_2^i$ are real analytic. Moreover, (4.3), (4.4) and Theorems 4.2, 4.3 imply that

$$\mathcal{U}_1^i[0,0,0](\xi) = v^+_{\Omega^i,h}[Z[0,0,0],0](\xi) + \gamma_n (\log k_i) \delta_{2,n} k_i^{n-2} v^+_{\Omega_i,J}[Z[0,0,0],0](\xi)$$

$$+ a^{-1} \left(k_o^{n-2}/k_i^{n-2}\right) C^i[0,0,0] v^+_{\Omega^i,h}[\theta^\sharp, 0](\xi) + a^{-1} k_o^{n-2} \frac{2b_n}{\pi} C_1[0,0,0]\, \delta_{2,n} v^+_{\Omega^i,J}[\theta^\sharp, 0](\xi)$$

$$+ a^{-1} k_o^{n-2} C^i[0,0,0] \gamma_n (\log k_i) \delta_{2,n} v^+_{\Omega^i,J}[\theta^\sharp, 0](\xi) + (1 - \delta_{2,n}) C[0,0,0] v^+_{\Omega^i,h}[\theta^\sharp, 0](\xi)$$

$$
\begin{aligned}
&+\frac{2b_n}{\pi}\delta_{2,n}k_i^{n-2}C[0,0,0]v^+_{\Omega^i,J}[\theta^\sharp,0](\xi)\\
&+(1-\delta_{2,n})\gamma_n(\log k_i)C[0,0,0]\delta_{2,n}k_i^{n-2}v^+_{\Omega^i,J}[\theta^\sharp,0](\xi)=v^+_{\Omega^i,h}[Z[0,0,0],0](\xi)\\
&+\left((1-\delta_{2,n})v^+_{\Omega^i,h}[\theta^\sharp,0](\xi)+\delta_{2,n}k_i^{n-2}\frac{2b_n}{\pi}J^\sharp_{\frac{n-2}{2}}(0)\right)C[0,0,0]\\
&+a^{-1}k_o^{n-2}\frac{2b_n}{\pi}C_1[0,0,0]\,\delta_{2,n}J^\sharp_{\frac{n-2}{2}}(0)\\
&=v^+_{\Omega^i,h}[\tilde{\zeta},0](\xi)+\tilde{C}+a^{-1}k_o^{n-2}\frac{2b_n}{\pi}C_1[0,0,0]\,\delta_{2,n}J^\sharp_{\frac{n-2}{2}}(0)\\
&=\tilde{u}_1^{i,r}(\xi)+a^{-1}k_o^{n-2}\frac{2b_n}{\pi}C_1[0,0,0]\,\delta_{2,n}J^\sharp_{\frac{n-2}{2}}(0)\\
&\mathcal{U}_2^i[0,0,0](\xi)=a^{-1}k_o^{n-2}\frac{2b_n}{\pi}C_2[0,0,0]\delta_{2,n}J^\sharp_{\frac{n-2}{2}}(0),\ \forall\xi\in\overline{\Omega^i}.
\end{aligned}
$$

By the definition of $C_1$, $C_2$ and by equality (4.12), we have

$$
C_1[0,0,0]=C_2[0,0,0]=0\,, \tag{5.6}
$$

and thus the proof of (i) is complete. We now consider statement (ii). Let $\epsilon_m$ be such that

$$
\epsilon\overline{\Omega_m}\subseteq\Omega^o\qquad\forall\epsilon\in[-\epsilon_m,\epsilon_m].
$$

Then we have

$$
\overline{\Omega_m}\subseteq\frac{1}{\epsilon}\overline{\Omega(\epsilon)}\qquad\forall\epsilon\in[-\epsilon_m,\epsilon_m]\setminus\{0\}.
$$

By the second formulas of (4.10) and (4.14), we have

$$
\begin{aligned}
u^o(\epsilon,\epsilon\xi)=&\int_{\partial\Omega^o}S_{r,n}(\epsilon\xi-y,k_o)\Theta^o[\epsilon,\Xi_n[\epsilon]](y)d\sigma_y\\
&+\epsilon^{n-2}\int_{\partial\Omega^i}S_{r,n}(\epsilon\xi-\epsilon\eta,k_o)S^i[\epsilon,\Xi_n[\epsilon]](\eta)\,d\sigma_\eta\\
&+\epsilon^{n-2}\int_{\partial\Omega^i}S_{r,n}(\epsilon\xi-\epsilon\eta,k_o)C^i[\epsilon,\Xi_n[\epsilon]]\theta^\sharp(\eta)\,d\sigma_\eta\qquad\xi\in\overline{\Omega_m}
\end{aligned}
$$

for all $\epsilon\in]0,\epsilon_m[$. By Lemma 3.4 and equalities (3.5), (3.6), we have

$$
\begin{aligned}
u^o(\epsilon,\epsilon\xi)=&\int_{\partial\Omega^o}S_{r,n}(\epsilon\xi-y,k_o)\Theta^o[\epsilon,\Xi_n[\epsilon]](y)d\sigma_y\\
&+v^-_{\Omega^i,h}[S^i[\epsilon,\Xi_n[\epsilon]],\epsilon k_o](\xi)+\left[\frac{2b_n}{\pi}\kappa_n\log\epsilon+\gamma_n(\log k_o)\right]\epsilon^{n-2}k_o^{n-2}\\
&\times v^-_{\Omega^i,J}[S^i[\epsilon,\Xi_n[\epsilon]],\epsilon^2k_o^2](\xi)\\
&+C^i[\epsilon,\Xi_n[\epsilon]]\Big(v^-_{\Omega^i,h}[\theta^\sharp,\epsilon k_o](\xi)+\left[\frac{2b_n}{\pi}\kappa_n\log\epsilon+\gamma_n(\log k_o)\right]\epsilon^{n-2}k_o^{n-2}\\
&\times v^-_{\Omega^i,J}[\theta^\sharp,\epsilon^2k_o^2](\xi)\Big)\qquad\forall\xi\in\overline{\Omega_m}
\end{aligned}
$$

for all $\epsilon \in ]0, \epsilon_m[$. Thus by exploiting (5.1) and (5.4) we find natural to set

$$
\begin{aligned}
&\mathcal{U}^o_{m,1}[\epsilon, \epsilon_1, \epsilon_2](\xi) \qquad (5.7)\\
&\equiv \int_{\partial\Omega^o} S_{r,n}(\epsilon\xi - y, k_o)\Theta^o[\epsilon, \epsilon_1, \epsilon_2](y)d\sigma_y + v^-_{\Omega^i,h}[S^i[\epsilon, \epsilon_1, \epsilon_2], \epsilon k_o](\xi)\\
&+\frac{2b_n}{\pi}\epsilon_1\epsilon^{n-1}k_o^n \int_{\partial\Omega^i} |\xi - \eta|^2 J^{\sharp,1}_{\frac{n-2}{2}}(\epsilon^2 k_o^2|\xi - \eta|^2)S^i[\epsilon, \epsilon_1, \epsilon_2](\eta)d\sigma_\eta\\
&+\gamma_n(\log k_o)\epsilon^{n-2}k_o^{n-2}v^-_{\Omega^i,J}[S^i[\epsilon, \epsilon_1, \epsilon_2], \epsilon^2 k_o^2](\xi)\\
&+C^i[\epsilon, \epsilon_1, \epsilon_2]v^-_{\Omega^i,h}[\theta^\sharp, \epsilon k_o](\xi)\\
&+\gamma_n(\log k_o)\epsilon^{n-2}k_o^{n-2}C^i[\epsilon, \epsilon_1, \epsilon_2]v^-_{\Omega^i,J}[\theta^\sharp, \epsilon^2 k_o^2](\xi)\\
&+\frac{2b_n}{\pi}\epsilon^{n-2}k_o^{n-2}C_1[\epsilon, \epsilon_1, \epsilon_2]v^-_{\Omega^i,J}[\theta^\sharp, \epsilon^2 k_o^2](\xi) \qquad \forall \xi \in \overline{\Omega_m}
\end{aligned}
$$

$$
\mathcal{U}^o_{m,2}[\epsilon, \epsilon_1, \epsilon_2](\xi) \equiv \epsilon^{n-2}k_o^{n-2}\frac{2b_n}{\pi}C_2[\epsilon, \epsilon_1, \epsilon_2]v^-_{\Omega^i,J}[\theta^\sharp, \epsilon^2 k_o^2](\xi) \qquad \forall \xi \in \overline{\Omega_m}
$$

for all $(\epsilon, \epsilon_1, \epsilon_2) \in ]-\epsilon', \epsilon[\times \tilde{U}$. Since $S_{r,n}(\epsilon\xi - y, k_o)$ is real analytic in the variable $(\xi, y, \epsilon)$ in an open neighbourhood of $\overline{\Omega_m} \times \partial\Omega^o \times ]-\epsilon_m, \epsilon_m[$ then by Proposition 4.1 (i) of [17] on the integral operators with real analytic kernel, the map from $]-\epsilon_m, \epsilon_m[\times L^1(\partial\Omega^o)$ to $C^{m,\alpha}(\overline{\Omega_m})$ which takes a pair $(\epsilon, h)$ to $\int_{\partial\Omega^o} S_{r,n}(\epsilon\cdot - y, k_o)h(y)d\sigma_y$ is analytic. Since $\Theta^o$ is real analytic and $C^{m-1,\alpha}(\partial\Omega^o)$ is continuously imbedded into $L^1(\partial\Omega^o)$, we conclude that the map from $]-\epsilon_m, \epsilon_m[\times\tilde{U}$ to $C^{m,\alpha}(\overline{\Omega_m})$ which takes $(\epsilon, \epsilon_1, \epsilon_2)$ to the first summand of the right-hand side of (5.7) is real analytic. By [1, Thm. 3.22 (ii)], $v^-_{\Omega^i,h}[\cdot, \cdot]$ defines a real analytic map from $C^{m-1,\alpha}(\partial\Omega^i) \times \mathbb{C}$ to $C^{m,\alpha}(\overline{\Omega_m})$. Then Theorem 4.3 implies that the map from $]-\epsilon', \epsilon'[\times\tilde{U}$ to $C^{m,\alpha}(\overline{\Omega_m})$ which takes $(\epsilon, \epsilon_1, \epsilon_2)$ to $v^-_{\Omega^i,h}\left[S^i[\epsilon, \epsilon_1, \epsilon_2], \epsilon k_o\right]$ is real analytic. Similarly, the map $]-\epsilon_m, \epsilon_m[\times\tilde{U}$ to $C^{m,\alpha}(\overline{\Omega_m})$ which takes $(\epsilon, \epsilon_1, \epsilon_2)$ to $v^-_{\Omega^i,h}[\theta^\sharp, \epsilon k_o]$ is real analytic. Since $|\xi - \eta|^2 J^{\sharp,1}_{\frac{n-2}{2}}(\epsilon^2 k_o^2|\xi - \eta|)$ is analytic in the variable $(\xi, \eta, \epsilon)$ in an open neighbourhood of $\overline{\Omega_m} \times \partial\Omega^i \times ]-\epsilon_m, \epsilon_m[$ then by Proposition 4.1 (i) of [17] on integral operators with real analytic kernel the map from $]-\epsilon_m, \epsilon_m[\times L^1(\partial\Omega^i)$ to $C^{m,\alpha}(\overline{\Omega_m})$ that takes $(\epsilon, h)$ to the function

$$
\int_{\partial\Omega^i} |\xi - \eta|^2 J^{\sharp,1}_{\frac{n-2}{2}}(\epsilon^2 k_o^2|\xi - \eta|^2)h(\eta)d\sigma_\eta \qquad \forall \xi \in \partial\Omega^i
$$

is real analytic. Since $S^i$ is real analytic and $C^{m-1,\alpha}(\partial\Omega^i)$ is continuously imbedded into $L^1(\partial\Omega^i)$, we conclude that the map from $]-\epsilon_m, \epsilon_m[\times\tilde{U}$ to $C^{m,\alpha}(\overline{\Omega_m})$ which takes $(\epsilon, \epsilon_1, \epsilon_2)$ to the third summand of the right-hand side of (5.7) is real analytic. By [1, Thm. 3.22 (iii)], $v^-_{\Omega^i,J}[\cdot, \cdot]$ defines a real analytic map from $C^{m-1,\alpha}(\partial\Omega^i) \times \mathbb{C}$ to $C^{m,\alpha}(\overline{\Omega_m})$. Then Theorem 4.3 (ii) implies that the map from $]-\epsilon_m, \epsilon_m[\times\tilde{U}$ to $C^{m,\alpha}(\overline{\Omega_m})$ which takes $(\epsilon, \epsilon_1, \epsilon_2)$ to $v^-_{\Omega^i,J}\left[S^i[\epsilon, \epsilon_1, \epsilon_2], \epsilon^2 k_o^2\right]$ is real analytic. Similarly, the map from $]-\epsilon_m, \epsilon_m[\times\tilde{U}$ to $C^{m,\alpha}(\overline{\Omega_m})$ which takes $(\epsilon, \epsilon_1, \epsilon_2)$ to $v^-_{\Omega^i,J}[\theta^\sharp, \epsilon^2 k_o^2]$ is real analytic. Finally, $C^i$ is real analytic by Theorem 4.3 and thus $C_1$ and $C_2$ are analytic. Hence $\mathcal{U}^o_{m,1}$ and $\mathcal{U}^o_{m,2}$ are analytic. Then by (4.11), Theorem 4.3 and equality (5.6), we have

$$\begin{aligned}\mathcal{U}^o_{m,1}[0,0,0](\xi) &= \int_{\partial\Omega^o} S_{r,n}(-y,k_o)\Theta^o[0,0,0](y)d\sigma_y + v^-_{\Omega^i,h}[S^i[0,0,0],0](\xi)\\ &+\gamma_n(\log k_o)\delta_{2,n}k_o^{n-2}v^-_{\Omega^i,J}[S^i[0,0,0],0](\xi) + C^i[0,0,0]v^-_{\Omega^i,h}[\theta^\sharp,0](\xi)\\ &+\gamma_n(\log k_o)\delta_{2,n}k_o^{n-2}C^i[0,0,0]v^-_{\Omega^i,J}[\theta^\sharp,0](\xi)\\ &+\frac{2b_n}{\pi}\delta_{2,n}k_o^{n-2}C_1[0,0,0]v^-_{\Omega^i,J}[\theta^\sharp,0](\xi)\\ &= v^+_{\Omega^o}[\tilde\theta^o,k_o](0) + v^-_{\Omega^i,h}[\tilde\varsigma^i,0](\xi) + \frac{2b_n}{\pi}\delta_{2,n}k_o^{n-2}C_1[0,0,0]J^\sharp_{\frac{n-2}{2}}(0)\\ &= \tilde u^o(0) + \tilde u_1^{o,r}(\xi)\,,\\ \mathcal{U}^o_{m,2}[0,0,0](\xi) &= 0 \qquad \forall\xi\in\overline{\Omega_m}\,.\end{aligned}$$

□

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# Distortion theorems for homeomorphic Sobolev mappings of integrable $p$-dilatations

Elena Afanas'eva, Anatoly Golberg and Ruslan Salimov

*Dedicated to the memory of Professor Gabriela Kohr – outstanding mathematician and person*

**Abstract.** We study the distortion features of homeomorphisms of Sobolev class $W^{1,1}_{\rm loc}$ admitting integrability for $p$-outer dilatation. We show that such mappings belong to $W^{1,n-1}_{\rm loc}$, are differentiable almost everywhere and possess absolute continuity in measure. In addition, such mappings are both ring and lower $Q$-homeomorphisms with appropriate measurable functions $Q$. This allows us to derive various distortion results like Lipschitz, Hölder, logarithmic Hölder continuity, etc. We also establish a weak bounded variation property for such class of homeomorphisms.

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**Keywords:** Sobolev classes, Lusin's $(N)$-property, finitely Lipschitz mappings, ring $Q$-homeomorphisms, lower $Q$-homeomorphisms, Lipschitz continuity, Hölder continuity, bounded variation.

## 1. Introduction

Geometric Function Theory which lies at the core of two distinguished fields of Mathematics, namely, Geometry and Analysis, has various fundamental applications. One of appeals relates to the distortion theory of mappings.

The main claim of the present paper is developing the theory of mappings and solving some important problems in this field of geometric function theory of several real variables.

Various relations between absolute continuity, bounded variation, Sobolev spaces, etc. in higher dimensions attract an attention of many mathematicians during last decades. It is the well-known fact that homeomorphisms of Sobolev classes $W^{1,p}$ are differentiable almost everywhere (a.e.) under $p > n-1$. The border case $p = n-1$, in general case, fails to guarantee this crucial property, but assuming an appropriate

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additional restriction on mappings many far advanced regularity properties can be reached. We refer here to recent papers [2], [6], [7], [19], [24] and monograph [11].

The idea to study various properties of mappings involving only a geometric description is a key approach in Geometric Function Theory and goes back to the classical works of Grötzsch, Köbe and Ahlfors-Beurling. This method relies on the invariance/quasi-invariance of the conformal modulus under conformal/quasiconformal mappings. The classes of ring and lower $Q$-homeomorphisms provide a modern tool for studying various properties of mappings including regularity, removability, boundary correspondence and others; see, e.g. [1], [2], [4], [5], [9], [10], [13], [22], [23] and monograph [18].

We study the distortion features of homeomorphisms of Sobolev class $W^{1,1}_{\rm loc}$ in $\mathbb{R}^n$ admitting appropriate integrability for $p$-outer dilatation. It is shown that such mappings belong to $W^{1,n-1}_{\rm loc}$, are differentiable a.e., possess absolute continuities in measure with respect to the $n$-dimensional Lebesque measure $m$ and $(n-1)$-dimensional Hausdorff measure $\mathcal{H}^{n-1}$ in $\mathbb{R}^n$, and have a bounded variation. In addition, such mappings are both ring and lower $Q$-homeomorphisms with the corresponding measurable functions $Q$. This allows us to derive various distortion results like Lipschitz, Hölder, logarithmic Hölder continuity, etc. The range of real parameter $p$ is the interval $[n, n+1/(n-2))$ for $n \geq 2$. It means that for the planer case we deal with $[2,\infty)$.

## 2. Sobolev classes and absolute continuity

**2.1.** Obviously the notion of absolute continuity is strongly connected with Sobolev classes in $\mathbb{R}^n$. We recall the definition of Sobolev spaces $W^{1,p}$, $p \geq 1$, following [11].

Let $\Omega \subset \mathbb{R}^n$ be open and $u \in L^1_{\rm loc}(\Omega)$. A function $v \in L^1_{\rm loc}(\Omega)$ is called a *weak derivative of* $u$ if

$$\int\limits_{\Omega} \varphi(x) v(x)\, dm(x) \;=\; -\int\limits_{\Omega} u(x) \nabla\varphi(x)\, dm(x)$$

for every $\varphi \in {\rm C}^\infty_C(\Omega)$. The function $v$ is referred to $Du$. For $1 \leq p \leq \infty$, the Sobolev space is defined by

$$W^{1,p}(\Omega) \;=\; \{u \in L^p(\Omega) : Du \in L^p(\Omega)\}$$

with the norm

$$\|u\|_{W^{1,p}(\Omega)} \;=\; \left(\int\limits_{\Omega} |u|^p + \int\limits_{\Omega} |Du|^p\right)^{1/p}.$$

Here ${\rm C}_C$ denotes the collection of continuous functions with compact support. A mapping $f : \Omega \to \mathbb{R}^n$ belongs to $W^{1,p}(\Omega)$ if its each component $f_j$, $j = 1, \ldots, n$ is a $W^{1,p}$-function.

A mapping $f \in L^1(\Omega)$ is of bounded variation, $f \in {\rm BV}(\Omega)$, if the coordinate functions of $f$ belong to the space ${\rm BV}(\Omega)$. This means that the distributional derivatives of each coordinate function $f_j$ are measures with finite total variations in $\Omega$; see, e.g. [6].

**2.2.** It is well known that if $f \in W^{1,1}_{\rm loc}(D)$ of a domain $D \subset \mathbb{R}^n$, $n \geq 2$, then $f$ has partial derivatives a.e. For $n = 2$, in addition, $f$ is differentiable a.e. Thus, for any $f \in W^{1,1}_{\rm loc}(D)$ we denote by $f'(x)$ its Jacobi matrix. The quantities $\|f'(x)\| = \sup_{|h|=1} |f'(x)h|$ and $l(f'(x)) = \inf_{|h|=1} |f'(x)h|$ can be regarded as a maximal stretching and a minimal stretching of $f$ at $x$, respectively. At a point of nondegenerate differentiability, i.e. $J_f(x) = \det f'(x) \neq 0$, the outer and inner dilatations are defined by

$$K_O(x,f) = \frac{\|f'(x)\|^n}{J_f(x)}\,, \qquad K_I(x,f) = \frac{J_f(x)}{l^n(f'(x))}\,,$$

respectively, extended to points where $J_f(x) = 0$ by $K_O(x,f) = K_I(x,f) = 1$.

Pick real $p$, $p \geq 1$, we consider $p$-counterparts of the above quantities determined as

$$K_{O,p}(x,f) = \frac{\|f'(x)\|^p}{J_f(x)}\,, \qquad K_{I,p}(x,f) = \frac{J_f(x)}{l^p(f'(x))}\,,$$

whereas $J_f(x) \neq 0$. Define $K_{O,p}(x,f) = K_{I,p}(x,f) = 1$, if $f'(x) = 0$, and $K_{O,p}(x,f) = K_{I,p}(x,f) = \infty$ otherwise. We call these quantities the *$p$-outer and $p$-inner dilatations* of $f$ at $x$, respectively.

**2.3.** Let $D$ be a domain in $\mathbb{R}^n$ for some $n \geq 2$. A mapping $f : D \to \mathbb{R}^n$ is called *quasiregular* (or a *mapping of bounded distortion* by Reshetnyak) if $f \in W^{1,n}(D)$ and there exists a constant $K \geq 1$ such that $K_O(x,f) \leq K$ a.e. in $D$.

A crucial extension of quasiregularity relates to the class of mappings of finite distortion where the uniform boundedness of $K_O(x,f)$ is relaxed by its finiteness. We say that a mapping $f : D \to \mathbb{R}^n$ has *finite distortion* if $f \in W^{1,1}(D)$, $J_f(x) \in L^1_{\rm loc}(D)$ and there is a function $K : D \to [1,\infty]$ with $K(x) < \infty$ a.e. such that $K_O(x,f) \leq K(x)$ a.e. in $D$.

Note that for homeomorphisms of finite distortion the condition $J_f(x) \in L^1_{\rm loc}(D)$ can be removed. Many analytic and topological properties for quasiregular mappings and mappings of finite distortion can be derived from their definitions. For the latter such properties obviously depend on appropriate restrictions on $K(x)$; see, e.g. [11], [14] and references therein.

Here we recall some analytic features for mappings of Sobolev spaces $W^{1,p}$. Each mapping $f$ of $W^{1,p}$ has a representative $g$ (i.e. $g = f$ a.e.) which is differentiable a.e. for the case when $p > n$ and $n \geq 2$. On the other hand, there are mappings of $W^{1,p}$, $p \leq n$, which are not continuous at any point and, therefore, are differentiable nowhere. Homeomorphisms of Sobolev classes $W^{1,p}$, $p > n-1$ for $n > 2$ or $p \geq 1$ for $n = 2$ provide differentiability a.e. So, the case when $p = n-1$ is crucial for higher dimensions.

The following recent results given in [24] are of special interest since they relate to the border case $p = n-1$.

**Proposition 2.1.** Suppose that $D$ is a domain in $\mathbb{R}^n$, $n \geq 2$. Let $f \in W^{1,n-1}_{\rm loc}(D)$ be a continuous, discrete and open mapping satisfying $K_O(\cdot,f) \in L^1_{\rm loc}(D)$. Then $f$ is differentiable a.e. in $D$.

The second result in [24] ensuring differentiability a.e. relies on integrability of $p$-outer dilatation.

**Proposition 2.2.** Suppose that $D$ is a domain in $\mathbb{R}^n$, $n \geq 2$. Let $f \in W^{1,n-1}_{\rm loc}(D)$ be a continuous, discrete and open mapping satisfying $K_{O,q}(\cdot, f) \in L^1_{\rm loc}(D)$ for some $n-1 < q \leq n$. Then $f$ is differentiable a.e. in $D$.

One more result on mappings of Sobolev class $W^{1,n-1}_{\rm loc}$ provided rich regularity properties for their inverses can be found in [12]; cf. [15].

**Proposition 2.3.** Let $f : D \to D'$ be a $W^{1,n-1}_{\rm loc}$-homeomorphism of finite inner distortion. If $K_I(\cdot, f) \in L^1_{\rm loc}(D)$ then $f^{-1} \in W^{1,n}_{\rm loc}(D')$.

**2.4.** Absolute continuity, more precisely, absolute continuity in measure is called the Lusin $(N)$-property, or preservation of sets of zero measure.

Let $\Omega \subset \mathbb{R}^n$ be an open set and $f : \Omega \to \mathbb{R}^n$ be a mapping. We say that $f$ possesses the *Lusin* $(N)$*-property* on a set $\Omega' \subset \Omega$ if the implication

$$mE = 0 \qquad \Longrightarrow \qquad mf(E) = 0$$

holds for each subset $E$ of $\Omega'$.

The above definition can be extended to the $k$-dimensional Hausdorff measure $\mathcal{H}^k$, $k = 1, \ldots, n-1$, by replacing the $n$-dimensional Lebesgue measure $m$ to $\mathcal{H}^k$. In this case, we deal with the Lusin $(N)$-property with respect to the $k$-dimensional Hausdorff measure.

Let $\Omega \subset \mathbb{R}^n$ be an open set. We say that $f : \Omega \to \mathbb{R}^n$ satisfies the *Lusin* $(N^{-1})$*-property* if for each $E \subset f(\Omega)$ such that $mE = 0$ we have $mf^{-1}(E) = 0$.

The Lusin $(N)$-property is satisfied for general Sobolev mappings for the case $p > n$. The limiting case $p = n$ for homeomorphisms also guarantees the Lusin $(N)$-property.

The following statement provides a sufficient condition for mappings of finite distortion to satisfy the Lusin $(N^{-1})$-property; see [11].

**Proposition 2.4.** Let a continuous mapping $f \in W^{1,1}(\Omega)$ be a mapping of finite distortion with $K_f^{1/(n-1)} \in L^1(\Omega)$. If the multiplicity of $f$ is essentially bounded by a constant $N$ and $f$ is not constant, then $J_f(x) > 0$ a.e. in $\Omega$, and hence $f$ satisfies the Lusin $(N^{-1})$-property.

The exponent $1/(n-1)$ is crucial and cannot be reduced even for homeomorphisms.

**Remark 2.5.** Let $a < 1/(n-1)$. There exists a Lipschitz homeomorphism $f$ of finite distortion $f \in W^{1,1}((-1,1)^n)$ and $K_f^a \in L^1((-1,1)^n)$, for which the Lusin $(N^{-1})$-property fails; see again [11].

For $K_{O,q}(x)$-distortion function, $q \leq n$, the Lusin $(N^{-1})$-property can be derived assuming $K_{O,q} \in L^{1/(q-1)}$; see [11, Thm 5.14].

**Remark 2.6.** Note that the Lusin $(N^{-1})$-property is equivalent that the Jacobian does not vanish a.e.; cf. [20].

The following important statements proved in [6] provide the Lusin $(N)$-property w.r.t. the $(n-1)$-dimensional Hausdorff measure for homeomorphisms of Sobolev classes with the border exponent.

**Proposition 2.7.** Let $f \in W^{1,n-1}_{\rm loc}((-1,1)^n)$ be a homeomorphism. Then for almost every $y \in (-1,1)$ the restriction of $f$ on $(-1,1)^{n-1} \times \{y\}$ satisfies the $(n-1)$-dimensional Lusin $(N)$-property, i.e. for every $E \subset (-1,1)^{n-1} \times \{y\}$, $\mathcal{H}^{n-1}E = 0$ implies $\mathcal{H}^{n-1}f(E) = 0$.

Replacing the cube $(-1,1)^n$ to a ball $B(x_0,r)$ and the hyperplanes $(-1,1)^{n-1} \times \{y\}$ to spheres $S(x,r)$, one gets

**Proposition 2.8.** Let $f \in W^{1,n-1}(B(x_0,r))$ be a homeomorphism. Then for almost every $r \in (0,r_0)$ the mapping $f : S(x,r) \to \mathbb{R}^n$ satisfies the $(n-1)$-dimensional Lusin $(N)$-property, i.e. for every $E \subset S(x,r)$, $\mathcal{H}^{n-1}E = 0$ implies $\mathcal{H}^{n-1}f(E) = 0$.

## 3. Moduli of curve and surface families

**3.1.** For a Borel function $\rho : \mathbb{R}^n \to [0,\infty]$, its *integral over a $k$-dimensional surface* $\mathcal{S}$ (a continuous mapping $\mathcal{S} : D_{\mathcal{S}} \to \mathbb{R}^n$, $D_{\mathcal{S}}$ is a domain in $\mathbb{R}^k$, $k = 1,\ldots,n-1$) is determined by

$$\int\limits_{\mathcal{S}} \rho \, d\mathcal{A} := \int\limits_{\mathbb{R}^n} \rho(y)\, N(\mathcal{S},y)\, d\mathcal{H}^k y\,,$$

where $N(\mathcal{S},y)$ stands for the multiplicity function of $\mathcal{S}$, namely, the multiplicity of covering the point $y$ by the surface $\mathcal{S}$, $N(\mathcal{S},y) = \operatorname{card} \mathcal{S}^{-1}(y)$, which is measurable with respect to the Hausdorff measure $\mathcal{H}^k$; see, e.g. [21, Theorem II (7.6)].

A Borel function $\rho : \mathbb{R}^n \to [0,\infty]$ is called *admissible* for the family of $k$-dimensional surfaces $\Gamma$ in $\mathbb{R}^n$, $k = 1,2,...,n-1$, abbr. $\rho \in \operatorname{adm}\Gamma$, if

$$\int\limits_{\mathcal{S}} \rho^k \, d\mathcal{A} \geq 1 \qquad \forall\, \mathcal{S} \in \Gamma\,. \tag{3.1}$$

By the *$k$-dimensional Hausdorff area* of a Borel set $B$ in $\mathbb{R}^n$ (or simply *area* of $B$ in the case $k = n-1$) associated with the surface $\mathcal{S} : \omega \to \mathbb{R}^n$, we mean

$$\mathcal{A}_{\mathcal{S}}(B) \;=\; \mathcal{A}^k_{\mathcal{S}}(B) \;:=\; \int\limits_{B} N(\mathcal{S},y)\, d\mathcal{H}^k y\,,$$

(cf. [8, Ch. 3.2.1]). The surface $\mathcal{S}$ is called *rectifiable (quadrable)*, if $\mathcal{A}_{\mathcal{S}}(\mathbb{R}^n) < \infty$ (see, e.g [18, Ch. 9.2]).

The *modulus of family* $\Gamma$ (conformal modulus) is defined by

$$\mathcal{M}(\Gamma) \;:=\; \inf_{\rho \,\in\, \operatorname{adm}\Gamma} \int\limits_{D} \rho^n(x)\, dm(x)\,. \tag{3.2}$$

Replacing the exponent $n$ in (3.2) by real $p$, $p \geq 1$, we arrive at the quantity which is called $p$-modulus $\mathcal{M}_p(\Gamma)$ of the family $\Gamma$.

We say that a property $P$ holds for a.a. $\mathcal{S} \in \Gamma$, if the corresponding modulus of a subfamily $\Gamma_*$ of $\mathcal{S} \in \Gamma$, for which $P$ is not true, vanishes. Following [18], a Borel function $\rho : \mathbb{R}^n \to [0, \infty]$ is called *extensively admissible* for the family $\Gamma$ of $k$-dimensional surfaces $\mathcal{S}$ in $\mathbb{R}^n$, abbr. $\rho \in \mathrm{ext}_p\mathrm{adm}\,\Gamma$, if the admissibility condition (3.1) is fulfilled only for a.a. $\mathcal{S} \in \Gamma$.

**3.2.** A significance of moduli of curve/surface families follows mainly from the fact that the conformal modulus remains invariant under conformal mappings, whereas $p$-modulus is invariant under isometries. Various inequalities for moduli form the basis for the geometric part of quasiconformality/quasiregularity. We recall that the quasiinvariance of the conformal modulus completely characterizes quasiconformality. The same property for the $p$-modulus provides quasiisometry.

The following notions successfully extend the above classes of mappings including quasiconformality/quasiisometry.

Denote by $\Delta(E, F; G)$ a family of all curves $\gamma : [0, 1] \to \mathbb{R}^n$, which join arbitrary sets $E$ and $F$ located in $G \subset \mathbb{R}^n$, i.e. $\gamma(0) \in E$, $\gamma(1) \in F$ and $\gamma(t) \in G$ for all $t \in (0, 1)$.

Let $D$ be a domain in $\mathbb{R}^n$, $n \geq 2$, and $Q : D \to [0, \infty]$ be a measurable function. We say that a homeomorphism $f : D \to D'$ is a *ring $Q$-homeomorphism with respect to $p$-modulus at* $x_0 \in D$, $p > 1$, if the following inequality

$$\mathcal{M}_p\left(\Delta\left(f(S_1), f(S_2); f(D)\right)\right) \leq \int\limits_{\mathbb{A}} Q(x) \cdot \eta^p(|x - x_0|)\, dm(x) \tag{3.3}$$

holds for any ring $\mathbb{A} = \mathbb{A}(x_0, r_1, r_2)$, $0 < r_1 < r_2 < d_0 = \mathrm{dist}\,(x_0\,, \partial D)$ and any measurable function $\eta : (r_1, r_2) \to [0, \infty]$ such that

$$\int\limits_{r_1}^{r_2} \eta(r)\, dr = 1\,. \tag{3.4}$$

We also say that a homeomorphism $f : D \to \mathbb{R}^n$ is a *ring $Q$-homeomorphism with respect to $p$-modulus in* $D$, if inequality (3.3) is valid for any $x_0 \in D$.

Now instead of an upper bound for the modulus, we consider a lower integral estimate. Let $D$ be a bounded domain in $\mathbb{R}^n$, $n \geq 2$ and $x_0 \in D$. Given a Lebesgue measurable function $Q : D \to [0, \infty]$, a homeomorphism $f : D \to \mathbb{R}^n$ is called a *lower $Q$-homeomorphism with respect to $p$-modulus at* $x_0$ if

$$\mathcal{M}_p(f(\Sigma_\varepsilon)) \geq \inf_{\rho \in \mathrm{exp}_p\mathrm{adm}\ \Sigma_\varepsilon} \int\limits_{D_\varepsilon(x_0)} \frac{\rho^p(x)}{Q(x)}\, dm(x)\,,$$

where

$$D_\varepsilon(x_0) = D \cap \{x \in \mathbb{R}^n : \varepsilon < |x - x_0| < \varepsilon_0\}, \quad 0 < \varepsilon < \varepsilon_0, \quad 0 < \varepsilon_0 < \sup_{x \in D} |x - x_0|,$$

and $\Sigma_\varepsilon$ denotes the family of all pieces of spheres centered at $x_0$ of radii $r$, $\varepsilon < r < \varepsilon_0$, located in $D$.

Similarly to above, a homeomorphism $f : D \to \mathbb{R}^n$ is called a *lower $Q$-homeomorphism with respect to $p$-modulus in $D$* if it is a lower $Q$-homeomorphism at each point $x_0 \in D$.

The following relationship between the ring and lower $Q$-homeomorphisms with respect to $p$-modulus has been established in [9].

**Proposition 3.1.** Every lower $Q$-homeomorphism $f : D \to \mathbb{R}^n$ at $x_0 \in D$ with respect to $p$-modulus, $p > n-1$ and $Q \in L_{\mathrm{loc}}^{\frac{n-1}{p-n+1}}$, is a ring $\widetilde{Q}$-homeomorphism with respect to $\alpha$-modulus at $x_0$ with $\widetilde{Q} = Q^{\frac{n-1}{p-n+1}}$ and $\alpha = p/(p-n+1)$.

## 4. Auxiliary results

**4.1.** As was mentioned above, homeomorphisms of Sobolev space $W^{1,n-1}$ are of special interest, since, in general case, they need not be differentiable a.e., although for any $W^{1,p}$, $p > n-1$, this crucial regularity property holds. We show that the integrability of $p$-outer dilatation with an appropriate degree for Sobolev homeomorphisms guaranties differentiability a.e.

**Theorem 4.1.** Let $f : D \to \mathbb{R}^n$ satisfying $f \in W_{\mathrm{loc}}^{1,1}(D)$ and $K_{O,p} \in L_{\mathrm{loc}}^{\frac{n-1}{p-n+1}}(D)$, $p \in [n, n+1/(n-2))$. Then $f \in W_{\mathrm{loc}}^{1,n-1}(D)$ and $f$ is differentiable in $D$ a.e. Moreover, $f^{-1} \in W_{\mathrm{loc}}^{1,n}(f(D))$, and, therefore, $f$ possesses the Lusin $(N^{-1})$-property with respect to the $n$-dimensional Lebesgue measure.

*Proof.* Denote by $E$ any compact set in $D$. Then the Hölder inequality with exponents $\alpha = \frac{p}{p-n+1}$ and $\beta = \frac{p}{n-1}$ provides

$$\int\limits_E \|f'(x)\|^{n-1}\,dm(x) = \int\limits_E K_{O,p}^{\frac{n-1}{p}}(x,f)\cdot J_f^{\frac{n-1}{p}}(x)\,dm(x)$$
$$\le \left(\int\limits_E K_{O,p}^{\frac{n-1}{p-n+1}}(x,f)dm(x)\right)^{\frac{p-n+1}{p}} \left(\int\limits_E J_f(x)\,dm(x)\right)^{\frac{n-1}{p}} < \infty,$$

and, therefore, $f \in W_{\mathrm{loc}}^{1,n-1}(D)$.

Now pick $\alpha = p/(p-n+1)$ for arbitrary $p \in [n, n+1/(n-2))$. Then $n-1 < \alpha \le n$ and at a point of nondegenerate differentiability, we have

$$K_{I,\alpha}(x,f) = \frac{J_f(x)}{l^\alpha(f'(x))} = \frac{J_f^{\frac{p}{p-n+1}}(x)}{J_f^{\frac{n-1}{p-n+1}}(x)\,l^{\frac{p}{p-n+1}}(f'(x))} \le \frac{\|f'(x)\|^{\frac{n-1}{p-n+1}}}{J_f^{\frac{n-1}{p-n+1}}(x)} = K_{O,p}^{\frac{n-1}{p-n+1}}(x,f).$$

Thus, $K_{I,\alpha}(x,f)$ is locally integrable in $D$. Applying Proposition 2.2 (Tengvall's theorem from [24]), we reach that $f$ is differentiable a.e. in $D$.

To reach the last assertions of Theorem 4.1, note first that for $p = n$ we obtain $K_O \in L_{\mathrm{loc}}^{n-1}$. Further by the well-known relations between $K_I$ and $K_O$-distortion functions, namely $K_I \le K_O^{n-1}$, one can apply Proposition 2.3. This yields $f^{-1} \in W_{\mathrm{loc}}^{1,n}$, and, hence $f^{-1}$ possesses the Lusin $(N)$-property. □

**4.2.** Absolute continuity in measure for homeomorphisms of $W_{\mathrm{loc}}^{1,1}$ with integrable $p$-outer dilatation is established in the following statements.

**Theorem 4.2.** Let $D$ be a domain in $\mathbb{R}^n$ and $f : D \to \mathbb{R}^n$ be a homeomorphism of Sobolev class $W^{1,1}_{\rm loc}$ and $K_{O,p} \in L^{\frac{n-1}{p-n+1}}_{\rm loc}(D)$, $p \in [n, n+1/(n-2))$. Then $f$ satisfies the Lusin $(N)$-property w.r.t. the $(n-1)$-dimensional Hausdorff measure on pieces $S_r \cap D$ of almost all spheres $S_r$ centered at an arbitrary point $x_0 \in D$. In addition, on all such pieces $\mathcal{H}^{n-1} f(E) = 0$ holds whereas $f' = 0$ on a measurable set $E$.

*Proof.* By Theorem 4.2, the homeomorphism $f$ is differentiable a.e. in $D$ and belongs to $W^{1,n-1}_{\rm loc}(D)$. This allows us to apply Proposition 2.7 and obtain the Lusin $(N)$-property with respect to the $\mathcal{H}^{n-1}$-measure; cf. [2].

The last assertion of Theorem 4.2 follows from the equality

$$\mathcal{H}^{n-1} f(E) = \int\limits_E J_{n-1,f}(x)\, d\mathcal{A},$$

where $J_{n-1,f}(x)$ stands for the $(n-1)$-dimensional Jacobian of the mapping $f$ on $S_r \cap D$, and $E \subset S_r \cap D$ is a measurable set. Then from the evident estimate we have

$$\mathcal{H}^{n-1} f(E) \le \int\limits_E \|f'(x)\|^{n-1}\, d\mathcal{A},$$

which completes the proof. □

The above theorem with Proposition 2.8 yields

**Corollary 4.3.** Let $D$ be a domain in $\mathbb{R}^n$ and $f : D \to \mathbb{R}^n$ be a homeomorphism of Sobolev class $W^{1,1}_{\rm loc}$ and $K_{O,p} \in L^{\frac{n-1}{p-n+1}}_{\rm loc}(D)$, $p \in [n, n+1/(n-2))$. Then $f$ satisfies the Lusin $(N)$-property w.r.t. the $(n-1)$-dimensional Hausdorff measure on $\mathcal{P} \cap D$ of almost all hyperplanes $\mathcal{P}$ which are parallel to the coordinate hyperplanes. In addition, on all such intersections $\mathcal{H}^{n-1} f(E) = 0$ holds whereas $f' = 0$ on a measurable set $E$.

**4.3.** Here we obtain relationships between Sobolev homeomorphisms with integrable $p$-outer dilatation and classes of mappings admitting modular presentations (ring and lower $Q$-homeomorphisms).

**Theorem 4.4.** Let $f : D \to \mathbb{R}^n$ be a homeomorphism of Sobolev class $W^{1,1}_{\rm loc}$ and $K_{O,p} \in L^{\frac{n-1}{p-n+1}}_{\rm loc}(D)$, $p \in [n, n+1/(n-2))$. Then $f$ is a lower $Q$-homeomorphism with respect to $p$-modulus at arbitrary $x_0 \in D$ with $Q(x) = K_{O,p}(x,f)$ in $D$. Moreover, $f$ is a ring $\widetilde{Q}$-homeomorphism with respect to $\alpha$-modulus in D with $\widetilde{Q}(x) = K^{(n-1)/(p-n+1)}_{O,p}(x,f)$, where $\alpha = p/(p-n+1)$.

*Proof.* Due to Theorem 4.1, $f$ is differentiable a.e. in $D$. Moreover, the Lusin $(N^{-1})$-property yields that $J_f(x)$ does not vanish a.e. in $D$. Denote by $E$ a Borel set of all points $x$ in $D$, where $f$ has a total differential $f'(x)$ and Jacobian $J_f(x) \neq 0$, and by $\widetilde{E}$ a set of all points at which $f$ has a total differential $f'(x)$ but $J_f(x) = 0$. Then both sets $E_0 := D \setminus (E \cup \widetilde{E})$ and $\widetilde{E}$ have zero $n$-dimensional Lebesgue measure.

Now applying a Kirszbraun type theorem one concludes that the set $E$ can be presented as a countable union of piecewise distinct Borel sets $E_l$, $l = 1, 2, \ldots$, such that $f_l = f|_{E_l}$ are bi-Lipschitz homeomorphisms; see, e.g. Lemma 3.2.2 and Theorems 3.1.4 and 3.1.8 in [8]. Since the set $E_0$ has zero Lebesgue $n$-measure, applying [18, Theorem 9.1] yields that $\mathcal{H}^{n-1}(f(E_0)\cap S_r^{'}) = 0$ and $\mathcal{H}^{n-1}(f(\widetilde{E})\cap S_r^{'}) = 0$ for almost all images $S_r^{'} = f(S_r)$ of spheres $S_r$ in the sense of $p$-modulus of surface families. Fix arbitrarily $x_0 \in D$ and note that

$$\mathcal{H}^{n-1}(f(E_0)\cap S_r^{'}) \;=\; 0 \quad \text{and} \quad \mathcal{H}^{n-1}(f(\widetilde{E})\cap S_r^{'}) = 0 \tag{4.1}$$

for almost all $r \in (\varepsilon, \varepsilon_0)$ by [13, Theorem 4.1].

For an arbitrary admissible function $\varrho' \in \operatorname{adm} f(\Sigma_\varepsilon)$ extended by $\varrho' \equiv 0$ outside of $f(D)$, we define

$$\varrho(x) \;:=\; \varrho'(f(x))\|f'(x)\|$$

for $x \in E$ and set $\varrho \equiv 0$ otherwise.

Note that

$$f(\Sigma_\varepsilon) \;=\; f(D)\cap S_r^{'} \;=\; \bigcup_{l=0}^{\infty}(f(E_l)\cap S_r^{'})\cup(f(\widetilde{E})\cap S_r^{'}).$$

Since $\varrho' \in \operatorname{adm} f(\Sigma_\varepsilon)$ and due to (4.1),

$$\begin{aligned}
1 \;\le\; \int\limits_{f(\Sigma_\varepsilon)} (\varrho'(y))^{n-1}\, d\mathcal{A}' \;&=\; \sum_{l=0}^{\infty}\;\int\limits_{f(E_l)\cap S_r'} (\varrho'(y))^{n-1} N(y,f,E_l\cap S_r)\, d\mathcal{H}^{n-1}y \\
&\quad+ \int\limits_{f(\widetilde{E})\cap S_r'} (\varrho'(y))^{n-1} N(y,f,\widetilde{E}\cap S_r)\, d\mathcal{H}^{n-1}y \\
&= \sum_{l=1}^{\infty}\;\int\limits_{f(E_l)\cap S_r'} (\varrho'(y))^{n-1} N(y,f,E_l\cap S_r)\, d\mathcal{H}^{n-1}y
\end{aligned} \tag{4.2}$$

for almost all $r \in (\varepsilon, \varepsilon_0)$. Here $N(y,f,A)$ denotes the multiplicity function, i.e. $N(y,f,A) = \operatorname{card}\{x\in A | f(x)=y\}$. Recall that for homeomorphisms $N(y,f,A)=1$.

Arguing piecewise on $E_l$, $l = 1,2,\ldots$, and using [8, Theorem 3.2.5] with the Lusin's $(N)$-property w.r.t. the $(n-1)$-dimensional Hausdorff measure (Theorem 4.2), we get

$$\begin{aligned}
\int\limits_{E_l\cap S_r} \varrho^{n-1}(x)\, d\mathcal{A} \ge \int\limits_{E_l\cap S_r} (\varrho'(f(x)))^{n-1} J_{n-1,f}(x)\, d\mathcal{A} \\
= \int\limits_{f(E_l)\cap S_r'} (\varrho'(y))^{n-1} N(y,f,E_l\cap S_r)\, d\mathcal{H}^{n-1}y
\end{aligned}$$

for a.a. $r \in (\varepsilon, \varepsilon_0)$, which together with (4.2) implies $\varrho \in \operatorname{ext}_p\operatorname{adm}\Sigma_\varepsilon$.

Now applying on each $E_l$ the change of variables formula with the countable additivity of integrals, we have

$$\sum_l \int\limits_{E_l\cap S_r} \frac{\varrho^p(x)}{K_{O,p}(x,f)}\, dm(x) \;=\; \int\limits_{D\cap S_r} \frac{\varrho^p(x)}{K_{O,p}(x,f)}\, dm(x) \;\le\; \int\limits_{f(\Sigma_\varepsilon)} (\varrho^{'}(y))^p\, dm(y)\,.$$

Thus, $f$ is a lower $Q$-homeomorphism with $Q(x) = K_{O,p}(x, f)$.

By Proposition 3.1, $f$ is also a ring $\widetilde{Q}$-homeomorphism with respect to $\alpha$-modulus in $D$ with $\widetilde{Q}(x) = K_{O,p}^{(n-1)/(p-n+1)}(x, f)$, where $\alpha = p/(p-n+1)$, which completes the proof. □

**4.4.** Several properties related to absolute continuity and bounded variation can be derives as consequences from the above results.

Combining [6, Theorem 1.1] and Theorem 4.1, one derives

**Corollary 4.5.** Let $\Omega \in \mathbb{R}^n$ be an open set and $f : \Omega \to \mathbb{R}^n$ be a homeomorphism of $W^{1,1}_{\rm loc}(\Omega)$ with $K_{O,p} \in L_{\rm loc}^{\frac{n-1}{p-n+1}}(\Omega)$. Then $f^{-1} \in {\rm BV}_{\rm loc}(f(\Omega))$.

Applying [6, Theorem 1.2] with Theorem 4.1 provides the following sufficient condition for a homeomorphism to be a bi-Sobolev mapping.

**Corollary 4.6.** Let $\Omega \in \mathbb{R}^n$ be an open set and $f : \Omega \to \mathbb{R}^n$ be a homeomorphism of finite distortion with $K_{O,p} \in L_{\rm loc}^{\frac{n-1}{p-n+1}}(\Omega)$. Then $f^{-1} \in W^{1,1}_{\rm loc}(f(\Omega))$ and $f^{-1}$ is of finite distortion.

The following corollaries can be derived from [19] and [7], respectively, replacing the condition $f \in W^{1,n-1}$ by $f \in W^{1,1}$ with the appropriate integrability of $p$-outer dilatation.

**Corollary 4.7.** *Let $f : D \to D'$ be a homeomorphism of $W^{1,1}_{\rm loc}(D)$ with finite inner distortion such that $K_{O,p} \in L^{\frac{n-1}{p-n+1}}(D)$, $p \in [n, n+1/(n-2))$. Then $\|(f^{-1}(y))'\| \in L^n(D')$ and $\int_{D'} \|(f^{-1}(y))'\|^n \, dm(y) = \int_D K_I(x, f) \, dm(x)$.*

**Corollary 4.8.** Let $f : D \to D'$ be a homeomorphism of finite inner distortion and $f \in W^{1,1}(D)$ with $K_{O,p} \in L_{\rm loc}^{\frac{n-1}{p-n+1}}(D)$, $p \in [n, n+1/(n-2))$. Assume that $u \in W^{1,\infty}_{\rm loc}(D)$. Then $u \circ f^{-1} \in W^{1,1}_{\rm loc}(D')$.

## 5. Distortion theorems

In this section, we provide distortion type theorems whose proofs mainly rely on Theorem 4.4.

**5.1.** We start with Hölder's continuity. Theorem 4.4 yields that every homeomorphism of Sobolev class $W^{1,1}_{\rm loc}$ in $\mathbb{R}^n$, $n \geq 2$, with $K_{O,p}$ integrable in degree $(n-1)/(p-n+1)$ is a lower $Q$-homeomorphism with respect to $p$-modulus. Then by Theorem 4.2 in [22], one gets

**Theorem 5.1.** Let $D$ and $D'$ be two domains in $\mathbb{R}^n$, $n \geq 2$, and $f : D \to D'$ be a homeomorphism of Sobolev class $W^{1,1}_{\rm loc}$ with $K_{O,p} \in L_{\rm loc}^{\frac{n-1}{p-n+1}}(D)$, $p \in (n, n+1/(n-2))$. Assume that for some real $\lambda > 1$, $\sigma > 0$, and $C_{x_0} > 0$ the following condition holds

$$\varepsilon^{\sigma} \int\limits_{\varepsilon}^{\lambda\varepsilon} \frac{dr}{\| K_{O,p}\|_{\frac{n-1}{p-n+1}}(x_0, r)} \geq C_{x_0} \qquad \forall\, \varepsilon \in \left(0, \frac{{\rm dist}\,(x_0, \partial D)}{\lambda^2}\right),$$

where

$$\| K_{O,p}\|_{\frac{n-1}{p-n+1}}(x_0,r) = \left( \int\limits_{S(x_0,r)} [K_{O,p}(x,f)]^{\frac{n-1}{p-n+1}} \, d\mathcal{A} \right)^{\frac{p-n+1}{n-1}} . \tag{5.1}$$

Then the estimate

$$|f(x)-f(x_0)| \leq \nu_0 \, C_{x_0}^{-\frac{1}{p-n}} \, |x-x_0|^{\frac{\sigma}{p-n}}$$

is valid for all $x \in B(x_0,\delta_0)$, where $\nu_0$ is a positive constant depending only on $n$, $p$, $\lambda$ and $\sigma$.

**Corollary 5.2.** In particular, if for some $\lambda > 1$ and $C_{x_0} > 0$ the condition

$$\varepsilon^{p-n} \int\limits_{\varepsilon}^{\lambda\varepsilon} \frac{dr}{\| K_{O,p}(x,f)\|_{\frac{n-1}{p-n+1}}(x_0,r)} \geq C_{x_0}$$

holds for some $\varepsilon \in \left(0, \mathrm{dist}\,(x_0,\partial D)/\lambda^2\right)$, then $f$ is Lipschitz continuous, i.e.

$$|f(x)-f(x_0)| \leq \nu_0 \, C_{x_0}^{-\frac{1}{p-n}} \, |x-x_0|$$

for all $x \in B(x_0,\delta_0)$; here $\nu_0 = \nu_0(n,p,\lambda) > 0$.

Now Theorem 4.4 together with Theorem 4.4 in [22] yields

**Theorem 5.3.** Let $D$ and $D'$ be two domains in $\mathbb{R}^n$, $n \geq 2$. Suppose that $f : D \to D'$ is a homeomorphism of Sobolev class $W^{1,1}_{\mathrm{loc}}$ with $K_{O,p} \in L^{\frac{n-1}{p-n+1}}_{\mathrm{loc}}(D)$, $p \in (n, n+1/(n-2))$. If $K_{O,p}(x,f) \in L^{\alpha}(B(x_0,\delta_0))$, $\delta_0 \leq \mathrm{dist}\,(x_0,\partial D)/4$, $\alpha > n/(p-n)$, then

$$|f(x)-f(x_0)| \leq \nu_0 \|K_{O,p}(f)\|_{\alpha}^{\frac{1}{p-n}} |x-x_0|^{1-\frac{n}{\alpha(p-n)}}$$

for all $x \in B(x_0,\delta_0)$, where $\|K_{O,p}(f)\|_{\alpha} = \left(\int_{B(x_0,\delta_0)} K^{\alpha}_{O,p}(x,f)\, dm(x)\right)^{1/\alpha}$ denotes an $L^{\alpha}$-norm over $(B(x_0,\delta_0))$, and $\nu_0$ stands for a positive constant depending only on $n$, $p$ and $\alpha$.

**5.2.** A logarithmic Hölder continuity is much weaker than the Hölder one; see, e.g. [10]. Here we first apply Theorem 4.4 and then Theorem 5.2 from [22], in order to reach a logarithmic type of distance distortions for homeomorphic Sobolev mappings of $W^{1,1}_{\mathrm{loc}}$ in $\mathbb{R}^n$, $n \geq 2$.

**Theorem 5.4.** Let $D$ and $D'$ be two domains in $\mathbb{R}^n$, $n \geq 2$, and $f : D \to D'$ be a homeomorphism of Sobolev class $W^{1,1}_{\mathrm{loc}}$ with $K_{O,p} \in L^{\frac{n-1}{p-n+1}}_{\mathrm{loc}}(D)$. If $p \in (n, n+1/(n-2))$, $\| K_{O,p}\|_{\frac{n-1}{p-n+1}}(x_0,r) \neq \infty$ for a.a. $r \in (0,d_0)$, $d_0 = \mathrm{dist}(x_0,\partial D)$, and for some real $\kappa \in [0, p/(p-n+1))$, $C_{x_0} > 0$, the upper bound

$$\int\limits_{\mathbb{A}(x_0,\varepsilon_1,\varepsilon_2)} \frac{[K_{O,p}(x,f)]^{\frac{n-1}{p-n+1}} \, dm(x)}{|x-x_0|^{\frac{p}{p-n+1}}} \leq C_{x_0} \ln^{\kappa}\left(\frac{\varepsilon_2}{\varepsilon_1}\right)$$

holds for any $0 < \varepsilon_1 < \varepsilon_2 < d_0$, then

$$|f(x) - f(x_0)| \le \nu_0 \, C_{x_0}^{\gamma} \, \ln^{-\theta} \frac{1}{|x - x_0|}$$

for all $x \in B(x_0, \delta_0)$, where $\delta_0 \le \min\{1, \operatorname{dist}^4 (x_0, \partial D)\}$,

$$\gamma = \frac{p - n + 1}{(n-1)(p-n)}, \quad \theta = \frac{p - \kappa(p - n + 1)}{(n-1)(p-n)}$$

and $\nu_0$ is a positive constant depending only on $n$, $p$ and $\kappa$.

Theorem 5.4 with Corollary 5.1 and Theorem 5.3 in [22] imply two following statements.

**Corollary 5.5.** Let $D$ and $D'$ be two domains in $\mathbb{R}^n$, $n \ge 2$. Suppose that $f : D \to D'$ is a homeomorphism of Sobolev class $W^{1,1}_{\mathrm{loc}}$ with $K_{O,p} \in L^{\frac{n-1}{p-n+1}}_{\mathrm{loc}}(D)$. If $K_{O,p}(x, f) \in L^{n/(p-n)}(B(x_0, \delta_0))$, $\delta_0 \le \min\{1, \operatorname{dist}^4(x_0, \partial D)\}$ and $p \in (n, n + 1/(n-2)))$, then

$$|f(x) - f(x_0)| \le \nu_0 \, \|K_{O,p}(f)\|^{\frac{1}{p-n}}_{\frac{n}{p-n}} \, \ln^{-\frac{p}{n(p-n)}} \frac{1}{|x - x_0|}$$

for all $x \in B(x_0, \delta_0)$, where

$$\|K_{O,p}(f)\|_{\frac{n}{p-n}} = \left( \int\limits_{B(x_0,\delta_0)} K^{\frac{n}{p-n}}_{O,p}(x, f) \, dm(x) \right)^{\frac{p-n}{n}}$$

stands for a norm in $L^{n/(p-n)}(B(x_0, \delta_0))$ and $\nu_0$ is a positive constant depending only on $n$ and $p$.

**Corollary 5.6.** Let $D$ and $D'$ be two domains in $\mathbb{R}^n$, $n \ge 2$. Suppose that $f : D \to D'$ is a homeomorphism of Sobolev class $W^{1,1}_{\mathrm{loc}}$ with $K_{O,p} \in L^{\frac{n-1}{p-n+1}}_{\mathrm{loc}}(D)$. If $p \in (n, n + 1/(n-2))$ and for some $k_{x_0} > 0$, the growth estimate

$$\|K_{O,p}\|_{\frac{n-1}{p-n+1}}(x_0, r) \le k_{x_0} r$$

holds for a.a. $r \in (0, \delta_0)$, $\delta_0 \le \min\{1, \operatorname{dist}^4 (x_0, \partial D)\}$, then

$$|f(x) - f(x_0)| \le \nu_0 \, \kappa^{\frac{1}{p-n}}_{x_0} \, \ln^{-\frac{1}{p-n}} \frac{1}{|x - x_0|},$$

for all $x \in B(x_0, \delta_0)$, where $\nu_0 > 0$ depends only on $n$ and $p$.

**5.3.** The finitely Lipschitz homeomorphisms have some very important and interesting properties. They can fail to belong to $W^{1,1}_{\mathrm{loc}}$, however, they possess the Lusin $(N)$-property with respect to the Hausdorff measure $\mathcal{H}^k$, $k = 1, \ldots, n$; see [18] (and [1] in more general settings).

Recall that a mapping is called Lipschitz in a domain $D \subset \mathbb{R}^n$, if there exists a constant $L$ such that $|f(x) - f(y)| \le L|x - y|$ for ant $x, y \in D$. Consider a quantity

$$L(x_0, f) \;=\; \limsup_{x \to x_0} \frac{|f(x) - f(x_0)|}{|x - x_0|}\,,$$

and say that a mapping $f$ is finitely Lipschitz in $D$ if $L(x_0, f) < \infty$ at any $x_0 \in D$; see, e.g. [18]. The quantity $L(x_0, f)$ can be treated as a maximal stretching of $f$ at $x_0$, and the condition $L(x_0, f) < \infty$ at any $x_0 \in D$ provides (by the well-known Stepanoff's theorem) differentiability a.e.

**Theorem 5.7.** Let $D$ and $D'$ be two domains in $\mathbb{R}^n$, $n \ge 2$. Assume that $f : D \to D'$ is a homeomorphism of Sobolev class $W^{1,1}_{\rm loc}$ with $K_{O,p} \in L^{\frac{n-1}{p-n+1}}_{\rm loc}(D)$. If $p \in (n, n + 1/(n - 2))$ and

$$k_p(x_0) \;=\; \limsup_{\varepsilon \to 0} \left( \fint_{B(x_0,\varepsilon)} [K_{O,p}(x, f)]^{\frac{n-1}{p-n+1}}\, dm(x) \right)^{\frac{p-n+1}{n-1}} \;<\; \infty\,,$$

then

$$L(x_0, f) \;\le\; \nu_0\, k_p^{\frac{1}{p-n}}(x_0) < \infty\,, \tag{5.2}$$

where $\nu_0$ is a positive constant depending on $n$ and $p$.

The proof of this theorem follows from Theorem 4.4 and Theorem 6.1 with Lemma 5.3 in [23]. For the reader convenience, we provide here the main ideas of proof.

*Sketch of the proof.* By Theorem 4.4, $f$ is a lower $Q$-homeomorphism with respect to $p$-modulus with $Q(x) = K_{O,p}(x, f)$, and $f$ is a ring $\widetilde{Q}$-homeomorphism with respect to $\alpha$-modulus in $D$ with $\widetilde{Q}(x) = K^{(n-1)/(p-n+1)}_{O,p}(x, f)$, where $\alpha = p/(p - n + 1)$.

Pick arbitrary $x_0 \in D$. Then for arbitrary spherical ring $\mathbb{A} = \mathbb{A}(x_0, r_1, r_2)$, $0 < r_1 < r_2 < d_0 = {\rm dist}\,(x_0\,, \partial D)$ one obtains

$$\mathcal{M}_\alpha\left(\Delta\left(f(S_1), f(S_2); f(D)\right)\right) \;\le\; \left( \int_{r_1}^{r_2} \frac{dr}{\|K_{O,p}(x, f)\|_{(n-1)/(p-n+1)}(r)} \right)^{\frac{1-n}{p-n+1}}\,, \tag{5.3}$$

applying the relation between $p$-modulus of the family of $(n-1)$-dimensional separating surfaces and $\alpha$-modulus of the family of joining curves in $f(\mathbb{A})$ together with [9, Thm 6.1]. Here $\|K_{O,p}(x, f)\|_{(n-1)/(p-n+1)}(r)$ is defined by (5.1).

By Hölder's inequality, the right-hand side in (5.3) can be estimated from above by $(r_2 - r_1)^{p/(n-1-p)} \int_{\mathbb{A}} K^{(n-1)/(p-n+1)}_{O,p} dm(x)$. Choosing first $r_1 = 2\varepsilon$ and $r_2 = 4\varepsilon$ and applying the well-known connection between $\alpha$-capacity of condenser and $\alpha$-modulus together with the lower bound in terms of the $n$-dimensional Lebesgue measure, one gets

$$\frac{mf(B(x_0, 2\varepsilon))}{mB(x_0, 2\varepsilon)} \;\le\; c_1 \left( \fint_{B(x_0,4\varepsilon)} [K_{O,p}(x, f)]^{\frac{n-1}{p-n+1}}\, dm(x) \right)^{\frac{n(p-n+1)}{n(p-n+1)-p}}\,.$$

Now we pick $r_1 = \varepsilon$ and $r_2 = 2\varepsilon$ and apply again the connection between $\alpha$-capacity of condenser and $\alpha$-modulus together with the lower bound in terms of the diameter of $f(B(x_0, r))$, then

$$\frac{\operatorname{diam} f(B(x_0,\varepsilon)}{\varepsilon} \le c_2 \left(\frac{mf(B(x_0,2\varepsilon))}{mB(x_0,2\varepsilon)}\right)^{j_1} \left(\fint_{B(x_0,4\varepsilon)} [K_{O,p}(x,f)]^{\frac{n-1}{p-n+1}}\, dm(x)\right)^{j_2},$$

where $j_1 = ((1-n)(p-n+1)+p)/p$, $j_2 = (n-1)(p-n+1)/p$, and $c_1$ and $c_2$ are constants.

Finally, combining two last estimates and passing to the limsup as $\varepsilon \to 0$, we obtain the desirable conclusion (5.2). □

**Corollary 5.8.** *Let $D$ and $D'$ be two domains in $\mathbb{R}^n$, $n \ge 2$. Assume that $f : D \to D'$ is a homeomorphism of Sobolev class $W^{1,1}_{\mathrm{loc}}$ with $K_{O,p} \in L^{\frac{n-1}{p-n+1}}_{\mathrm{loc}}(D)$. If $p \in (n, n+1/(n-2))$ and*

$$\limsup_{\varepsilon\to 0} \fint_{B(x_0,\varepsilon)} [K_{O,p}(x,f)]^{\frac{n-1}{p-n+1}}\, dm(x) < \infty$$

*for all $x_0 \in D$, then $f$ is finitely Lipschitz in $D$.*

The finiteness of $k_p(x_0)$ is a necessary condition (in somewhat sense). One can illustrate it by the following example.

**Example.** Assume that $n \ge 3$ and $p \in (n, n+1/(n-2))$, and consider an automorphism $f : \mathbb{B}^n \to \mathbb{B}^n$ of the unit ball $\mathbb{B}^n$ in $\mathbb{R}^n$ of such a form

$$f(x) = \frac{x}{|x|} \left( \int\limits_{|x|}^{1} \frac{dt}{t^{p-n+1} \ln^{\frac{p-n+1}{n-1}}(e/t)} \right)^{-\frac{1}{p-n}}, \quad x \neq 0, \tag{5.4}$$

extended by $f(0) = 0$.

Passing to the spherical coordinates in the image $(\rho, \psi_i)$ and in the inverse image $(r, \varphi_i)$, $i = 1, \dots, n-1$, one can rewrite (5.4) by

$$f(x) = \left\{ \rho = \left( \int\limits_{r}^{1} \frac{dt}{t^{p-n+1} \ln^{\frac{p-n+1}{n-1}}(e/t)} \right)^{-\frac{1}{p-n}}, 0 < r < 1,\ \rho(0) = 0 \right\}.$$

Since $\rho$ depends only on $r$, $\psi_i = \varphi_i$, whereas $0 \le \varphi_i \le \pi, i = 1, \dots, n-2$, and $0 \le \varphi_{n-1} \le 2\pi$. In this case, the stretchings are equal

$$\frac{d\rho}{dr}, \frac{\rho}{r}\frac{d\psi_1}{d\varphi_1}, \dots, \frac{\rho}{r}\frac{\sin\psi_i}{\sin\varphi_i}\frac{d\psi_1}{d\varphi_1}, \quad i = 2, \dots, n-1;$$

see, e.g. [16]. By a direct calculation,

$$\frac{\rho}{r} = \frac{1}{r} \left( \int\limits_{r}^{1} \frac{dt}{t^{p-n+1} \ln^{\frac{p-n+1}{n-1}}(e/t)} \right)^{-\frac{1}{p-n}},$$

$$\frac{d\rho}{dr} = \frac{1}{(p-n)r^{p-n+1}\ln^{\frac{p-n+1}{n-1}}(e/r)}\left(\int\limits_r^1 \frac{dt}{t^{p-n+1}\ln^{\frac{p-n+1}{n-1}}(e/t)}\right)^{-\frac{p-n+1}{p-n}} .$$

Since $\rho/r \geq d\rho/dr$ and

$$\left(\frac{\rho}{r}\right)^{p-n+1} = \frac{d\rho}{dr}(p-n)\ln^{\frac{p-n+1}{n-1}}(e/r) ,$$

one gets,

$$K_{O,p}(x,f) = \frac{(\rho/r)^p}{(\rho/r)^{n-1}d\rho/dr} = (p-n)\ln^{\frac{p-n+1}{n-1}}(e/|x|) .$$

Clearly,

$$\limsup_{\varepsilon\to 0} ⨍_{B_\varepsilon} [K_{O,p}(x,f)]^{\frac{n-1}{p-n+1}}\, dm(x) = \infty ,$$

where $B_\varepsilon = \{x \in \mathbb{R}^n : |x| < \varepsilon\}$.

On the other hand, by L'Hôpital's rule,

$$\lim_{x\to 0} \frac{|f(x)|}{|x|} = \infty ,$$

thus, $f$ fails to be finitely Lipschitz at the origin.

## 6. Bounded variation and differentiability a.e.

In this section, we show the connection between Sobolev topological mappings with integrable $p$-outer dilations and homeomorphisms of a weaker bounded variation.

**6.1.** In 1999, Jan Malý [17] introduced a multidimensional bounded variation by the following way. Given a mapping $f : \Omega \to \mathbb{R}^m$ and an open set $G \subset \Omega$, the *n-variation* of $f$ on $G$ is defined by

$$V^n(f,G) = \sup\Big\{\sum_j \big(\mathrm{osc}_{B(x_j,r_j)} f\big)^n : \{B(x_j,r_j)\} \text{ is a disjoint family of balls in } G\Big\},$$

where $\mathrm{osc}_B f = \sup\{|f(x)-f(y)| : x,y \in B\}$ for a ball $B$. We say that $f$ has a bounded *n-variation* in $\Omega$ if $V^n(f,\Omega) < \infty$. By $\mathrm{BV}^n(\Omega)$ we denote the class of all mappings with bounded $n$-variation with the seminorm $||f||_{\mathrm{BV}^n(\Omega)} = (V_n(f,\Omega))^{1/n}$.

This class provides a proper subset of Sobolev class $W^{1,n}$, and, moreover, has rich regularity properties as differentiability a.e., the Lusin $(N)$-condition, etc.

Later the Malý's definition has been extended to a $p$-counterpart of $n$-variation in [3], $1 \leq p < n$.

We say that $f$ has bounded *p-variation* (abbr., $f \in \mathrm{BV}^p(\Omega)$) if there exist $M > 0$ and $\eta$ such that

$$\sum_j (\mathrm{osc}_{B(x_j,r_j)} f)^p r_j^{n-p} < M$$

for each disjoint system of balls $\{B(x_j,r_j)\}$ in $\Omega$ such that $r_j < \eta$.

Several important properties and relationships between $\mathrm{BV}^p$ and Sobolev classes are also established in [3]. One of them is differentiability a.e. Note that although $p \geq 1$ has been assumed, in fact, it is enough to require for $p$ to be only positive.

**6.2.** Here we show that homeomorphisms of $W^{1,1}_{\rm loc}$ with $K_{O,p} \in L^{\frac{n-1}{p-n+1}}_{\rm loc}$ belong to $\mathrm{BV}^{n-\alpha}$, $\alpha = p/(p-n+1)$.

**Theorem 6.1.** Let $f : D \to \mathbb{R}^n$ be a homeomorphism of Sobolev class $W^{1,1}_{\rm loc}$ and $K_{O,p} \in L^{\frac{n-1}{p-n+1}}(D)$, $p \in [n, n+1/(n-2))$. Then $f \in \mathrm{BV}^{n-\alpha}(D)$, $\alpha = p/(p-n+1)$.

*Proof.* Due to Theorem 4.4, $f$ is a ring $\widetilde{Q}$-homeomorphism in $D$ with $\widetilde{Q}(x) = K^{\alpha-1}_{O,p}(x,f)$, where $\alpha = p/(p-n+1)$. Since for $\eta(r) = 1/(r_2 - r_1)$, condition (3.4) holds, we have for $r_1 = 2r$ and $r_2 = 4r$,

$$\mathcal{M}_\alpha \left(\Delta \left(f(S_1), f(S_2); f(D)\right)\right) \,\leq\, (2r)^{-\alpha} \int\limits_{\mathbb{A}} K^{\alpha-1}_{O,p}(x,f)\, dm(x)\,. \tag{6.1}$$

On the other hand, by estimate (15) in [10], one gets the following lower bound for the above modulus (or, equivalently, for the $\alpha$-capacity of condenser)

$$\mathcal{M}_\alpha \left(\Delta \left(f(S_1), f(S_2); f(D)\right)\right) \,\geq\, C_1 \left[m f(B_{2r})\right]^{\frac{n-\alpha}{n}}\,, \tag{6.2}$$

where $C_1$ is a positive constant depending only on $n$ and $\alpha$. Here and throughout the proof, we denote (by simplicity) by $B_\varepsilon$ a ball in $\mathbb{R}^n$ of radius $\varepsilon$. Combining both (6.1)–(6.2), we reach the estimate for the image of $B_{2r}$,

$$m f(B_{2r}) \,\leq\, C_2 r^{\frac{\alpha n}{\alpha - n}} \left(\int\limits_{B_{4r}} K^{\alpha-1}_{O,p}(x,f)\, dm(x)\right)^{\frac{n}{n-\alpha}}\,; \tag{6.3}$$

here $C_2 = C_2(n,\alpha) > 0$.

Now we apply the following double inequality letting $r_1 = r$ and $r_2 = 2r$,

$$C_3 \left[\frac{(\operatorname{diam} f(B_r))^\alpha}{(m\, f(B_{2r}))^{1-n+\alpha}}\right]^{1/(n-1)} \leq \mathcal{M}_\alpha \left(\Delta \left(f(S_1), f(S_2); f(D)\right)\right) \leq r^{-\alpha} \int\limits_{\mathbb{A}} K^{\alpha-1}_{O,p}(x,f)\, dm(x)$$

with a positive constant $C_3$ depending only on $n$ and $\alpha$; cf. (18) in [10]. This derives the following upper bound

$$\operatorname{diam} f(B_r) \,\leq\, C_4 \left[m\, f(B_{2r})\right]^{\frac{\alpha-n+1}{\alpha}} r^{1-n} \left(\int\limits_{B_{2r}} K^{\alpha-1}_{O,p}(x,f)\, dm(x)\right)^{\frac{n-1}{\alpha}}\,,$$

which together with (6.3) gives

$$\operatorname{diam} f(B_r) \,\leq\, C_5 r^{\frac{\alpha}{\alpha-n}} \left(\int\limits_{B_{4r}} K^{\alpha-1}_{O,p}(x,f)\, dm(x)\right)^{\frac{1}{n-\alpha}}\,;$$

$C_5 = C_5(n, \alpha) > 0$. Rewriting the last inequality as

$$[\operatorname{diam} f(B_r)]^{n-\alpha}\, r^{\alpha} \;\leq\; C_6 \int\limits_{B_{4r}} K_{O,p}^{\alpha-1}(x,f)\, dm(x)\,,$$

and summarizing over each disjoint system of balls $\{B(x_j, 4r_j)\}$, we complete the proof. □

Taking into account our remark on differentiability a.e. of homeomorphisms of $\mathrm{BV}^p$, $0 < p \leq n$, we obtain an alternative proof of the corresponding part of Theorem 4.1.

**Corollary 6.2.** Let $f : D \to \mathbb{R}^n$ be a homeomorphism of Sobolev class $W^{1,1}_{\mathrm{loc}}$ and $K_{O,p} \in L^{\frac{n-1}{p-n+1}}(D)$, $p \in [n, n+1/(n-2))$. Then $f$ is differentiable a.e. in $D$.

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# A criterion of univalence in $\mathbb{C}^n$ in terms of the Schwarzian derivative

Rodrigo Hernández

*Dedicated to the memory of Professor Gabriela Kohr*

**Abstract.** Using the Loewner Chain Theory, we obtain a new criterion of univalence in $C^n$ in terms of the Schwarzian derivative for locally biholomorphic mappings. We as well derive explicitly the formula of this Schwarzian derivative using the numerical method of approximation of zeros due by Halley.

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**Keywords:** Univalence criterion, Schwarzian derivative, Loewner chain, Halley method.

## 1. Introduction

The *Schwarzian derivative* of a locally univalent analytic function $f$ in a simply connected domain $\Omega$ of the complex plane $\mathbb{C}$ is

$$Sf = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2 . \tag{1.1}$$

The quotient $f''/f'$, denoted by $Pf$, is the *pre-Schwarzian derivative* of the function $f$.

These two operators come up naturally as the values of the derivatives of the generating functions of two particular methods for approximating zeros, as we now explain.

It is well know that the Newton (or the Newton-Raphson method) is a technique to approximate the zero of a function $f$ via the sequence

$$z_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}, \quad n \geq 1 , \tag{1.2}$$

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starting with a guess $z_0$, say.

The function $g(z) = z - f(z)/f'(z)$ is called the *generating function* of the Newton iteration method. It is not difficult to prove that the following identities hold, assuming that $f(\alpha) = 0$:

$$g(\alpha) = \alpha, \quad g'(\alpha) = 0\,, \quad g''(\alpha) = Pf(\alpha)\,.$$

The Halley Method can be derived by applying the Newton Method to the function $\dfrac{f}{\sqrt{|f'|}} = f|f'|^{-1/2}$. In this case, (1.2) becomes

$$z_{n+1} = z_n - \frac{2f(z_n)f'(z_n)}{2f'(z_n)^2 - f(z_n)f''(z_n)}\,, \quad n \geq 1\,.$$

The generating function $h$ of the method is given by

$$h(z) = z - \frac{2f(z)f'(z)}{2f'(z)^2 - f(z)f''(z)},$$

which satisfies

$$h(\alpha) = \alpha,\, h'(\alpha) = h''(\alpha) = 0,\, h'''(\alpha) = -Sf(\alpha)\,,$$

where $Sf$ is the Schwarzian derivative (1.1).

In this paper, we analyze the extension of the two methods mentioned to several complex variables and, in particular, we show that the definition of the Schwarzian derivative for locally biholomorphic mappings introduced in [3] (which derive from Oda's definition given in [8]) is precisely the value of the third coefficient of the generating function of the corresponding Halley method in several variables. That is, this operator is a third order differential tensor $Sf(z) \in \mathcal{L}^3(\mathbb{C}^n)$.

In addition to this, and using the Loewner chain theory, we obtain a criterion of univalence for locally biholomorphic mappings in the unit ball of $\mathbb{C}^n$ in terms of the Schwarzian derivative.

## 2. Preliminaries

### 2.1. Several complex variables

As usual, $\mathbb{C}^n$ is set of points $z = (z_1, \ldots, z_n)$, where $z_i \in \mathbb{C}, i = 1, \ldots, n$. The inner (dot) product and the norm are defined, respectively, by $z \cdot w = \sum_{i=1}^n z_i w_i$ and $|z| = (z \cdot \overline{z})^{1/2}$.

We denote by $\mathcal{L}^k(\mathbb{C}^n)$ the space of continuous $k$-linear operators from $\mathbb{C}^n$ into $\mathbb{C}^n$. For $T \in \mathcal{L}^k(\mathbb{C}^n)$, we write $T\langle\cdot, \ldots, \cdot\rangle$ to denote its placeholders. When $k = 1$ we simply write $\mathcal{L}(\mathbb{C}^n)$ for the space of linear maps and also write $Tu$ instead of $T\langle u\rangle$ for any linear map $T$. The identity linear operator in $\mathbb{C}^n$ is denoted by $I_n$.

The standard operator norm in $\mathcal{L}^k(\mathbb{C}^n)$ is given by

$$\|T\| = \max_{u_1,\ldots,u_k \in \mathbb{C}^n} \left| T\left\langle \tfrac{u_1}{|u_1|}, \ldots, \tfrac{u_k}{|u_k|} \right\rangle \right|.$$

Let $\Omega$ be a domain in $\mathbb{C}^n$ and let $f$ be a mapping defined in $\Omega$ with values in $\mathcal{L}(\mathbb{C}^n)$. If $f$ is $k$-times (Fréchet) differentiable with respect to $z \in \Omega$ then its $k$-th derivative, denoted by $D^k f(z)$, is a symmetric mapping in $\mathcal{L}^{k+1}(\mathbb{C}^n)$, meaning that the value $D^k f(z)\langle u_1, \ldots, u_{k+1}\rangle$ remains unchanged after any permutation of the entries $u_1, \ldots, u_{k+1}$ (see Theorem 14.6 in [6]).

The *product rule* for the derivative of the product $fg$ of two differentiable mappings $f, g : \Omega \to \mathcal{L}(\mathbb{C}^n)$, equals

$$D(fg)(z)\langle \cdot, \cdot\rangle \;=\; Df(z)\langle \cdot, g(z)\,\cdot\rangle + f(z)\,Dg(z)\langle \cdot, \cdot\rangle, \quad z \in \Omega. \tag{2.1}$$

Using the product rule, it is easy to check that if $f(z)$ is invertible for every $z \in \Omega$ then the derivative of $g(z) = f(z)^{-1}$ is given by

$$Dg(z)\langle \cdot, \cdot\rangle \;=\; -g(z)Df(z)\langle \cdot, g(z)\,\cdot\rangle, \qquad z \in \Omega. \tag{2.2}$$

If $f : \Omega \to \mathcal{L}(\mathbb{C}^n)$ is holomorphic in $\Omega$, then $f$ can be writen in terms of Taylor's formula centered at some $\alpha \in \Omega$:

$$f(z) \;=\; \sum_{k=0}^{\infty} \frac{1}{k!} D^k f(\alpha)\langle z-\alpha\rangle^k, \qquad |z-\alpha| < \delta(\alpha),$$

where $\delta(\alpha)$ denotes the distance from $\alpha$ to the boundary of $\Omega$ (see Theorem 7.13 in [6]). Here, the notation $D^k f(\alpha) w^k$ should be understood as $D^k f(\alpha)\langle w, \ldots, w, \cdot\rangle$, with the point $w$ repeated $k$ times and one placeholder left without being evaluated.

**2.2. Schwarzian derivative in several complex variables**

Let $f$ be a holomorphic mapping in the simply connected domain $\Omega \subset \mathbb{C}^n$. It is well known that $f$ is locally univalent (biholomorphic) in $\Omega$ if and only if its Jacobian, $J_f = \det(Df)$, has no zeros (see Lewy [5]). For such functions, the *pre-Schwarzian derivative* and the *Schwarzian derivative* are linear operators defined, for any $u$ and $v$ in $\mathbb{C}^n$, as follows:

$$P_f(z)\langle u, v\rangle = Df(z)^{-1} D^2 f(z)\langle u, v\rangle \tag{2.3}$$

and

$$\begin{aligned} S_f(z)\langle u, v\rangle \;&=\; P_f(z)\langle u, v\rangle \\ &-\; \frac{1}{n+1}\left((-\nabla \log J_f(z)\cdot u)v + (\nabla \log J_f(z)\cdot v)u\right). \end{aligned} \tag{2.4}$$

The pre-Schwarzian derivative (2.3) was introduced by J. Pfaltzgraff in [10]. The Schwarzian derivative (2.4) was presented in [3]. This higher dimensional Schwarzian derivative splits into two operators $S_f$ and $S_f^0$ of order two and three, respectively. The fact that, unlike in one single variable, an operator of purely order two must appear is consistent with the fact that the dimension of the group to be anihilated by the Schwarzian, namely the special linear projective group in dimension one or higher, is not big enough to prescribe all jets up to order 2 of a given mapping.

More concretely, we would like to mention that T. Oda in [8] defined the Schwarzian derivative $S_{ij}^k$ of a locally biholomorphic mapping $f(z) = (f_1, \ldots, f_n)$

by

$$S_{ij}^k f = \sum_{l=1}^{n} \frac{\partial^2 f_l}{\partial z_i \partial z_j} \frac{\partial z_k}{\partial f_l} - \frac{1}{n+1} \left( \delta_i^k \frac{\partial}{\partial z_j} + \delta_j^k \frac{\partial}{\partial z_i} \right) \log J_f \,, \tag{2.5}$$

where $i, j, k = 1, 2, \ldots, n$, and $\delta_i^k$ are the Kronecker symbols. For $n > 1$ the Schwarzian derivatives have the following properties:

$$S_{ij}^k f = 0 \quad \text{for all} \quad i, j, k = 1, 2, \ldots, n \quad \text{iff} \quad f(z) = M(z)$$

for some Möbius transformation

$$M(z) = \left( \frac{l_1(z)}{l_0(z)}, \ldots, \frac{l_n(z)}{l_0(z)} \right) ,$$

where $l_i(z) = a_{i0} + a_{i1} z_1 + \cdots + a_{in} z_n$ with $det(a_{ij}) \neq 0$. Furthermore, for a composition

$$S_{ij}^k (f \circ g)(z) = S_{ij}^k g(z) + \sum_{l,m,r=1}^{n} S_{lm}^r f(w) \frac{\partial w_l}{\partial z_i} \frac{\partial w_m}{\partial z_j} \frac{\partial z_k}{\partial w_r} \,, \; w = g(z) \,.$$

Thus, if $f$ is a Mobius transformation then $S_{ij}^k(f \circ g) = S_{ij}^k g$. The $S_{ij}^0 f$ coefficients are given by

$$S_{ij}^0 f(z) = J_f^{1/(n+1)} \left( \frac{\partial^2}{\partial z_i \partial z_j} J_f^{-1/(n+1)} - \sum_{k=1}^{n} \frac{\partial}{\partial z_k} J_f^{-1/(n+1)} S_{ij}^k f(z) \right) . \tag{2.6}$$

By using these Schwarzian derivatives in Oda's paper [8], the following definition, which coincides with (2.4), was presented in [3].

**Definition 2.1.** *The Schwarzian derivative $S_f$ of a locally biholomorphic mapping $f : \mathbb{C}^n \to \mathbb{C}^n$ equals*

$$S_f(z)(v, v) = \left( v^t \mathbb{S}^1 f(z) v \,, \ldots, v^t \mathbb{S}^n f(z) v \right) ,$$

*where $\vec{v} \in \mathbb{C}^n$ and the $n \times n$ matrix operator $\mathbb{S}^k f$, $k = 1, \ldots, n$, are given by*

$$\mathbb{S}^k f = (S_{ij}^k f) \,, \quad i, j = 1, \ldots, n.$$

It was proved in [3] that it is possible to recover the mapping $f$ from its Schwarzian components. More explicitly, consider the following overdetermined system of partial differential equations,

$$\frac{\partial^2 u}{\partial z_i \partial z_j} = \sum_{k=1}^{n} P_{ij}^k(z) \frac{\partial u}{\partial z_k} + P_{ij}^0(z) u \,, \quad i, j = 1, 2, \ldots, n \,, \tag{2.7}$$

where $z = (z_1, z_2, ..., z_n) \in \Omega$ and $P_{ij}^k(z)$ are holomorphic functions in $\Omega$, for $i, j, k = 0, \ldots, n$. The system (2.7) is called *completely integrable* if there are $n+1$ (maximun) linearly independent solutions, and is said to be in *canonical form* (see [11]) if the coefficients satisfy

$$\sum_{j=1}^{n} P_{ij}^j(z) = 0 \,, \quad i = 1, 2, \ldots, n.$$

T. Oda proved that (2.7) is a completely integrable system in canonical form if and only if $P_{ij}^k = S_{ij}^k f$ for a locally biholomorphic mapping $f = (f_1, \ldots, f_n)$, where $f_i = u_i/u_0$ for $1 \leq i \leq n$ and $u_0, u_1, \ldots, u_n$ is a set of linearly independent solutions of the system. For a given mapping $f$, $u_0 = J_f^{-\frac{1}{n+1}}$ is always a solution of (2.7) with $P_{ij}^k = S_{ij}^k f$.

## 3. On the Schwarzian derivative and the Halley method of approximation of zeros

As was mentioned in the introduction, the Halley Method, to find zeros of a function $f : \mathbb{C} \to \mathbb{C}$, can be obtained by applying the Newton Method to $f \cdot |f'|^{-\frac{1}{2}}$.

By considering a function $f : \mathbb{C}^n \to \mathbb{C}^n$, now with $n \geq 1$, whis is locally biholomorphic in a simply connected domain $\Omega$ with $f(\alpha) = 0$ for some $\alpha \in \Omega$, the *Newton Iteration Method* is given by

$$z_{n+1} = z_n - Df(z_n)^{-1}\langle f(z_n)\rangle, \quad n \geq 0.$$

The generating function of this method, then, equals

$$F(z) = z - Df(z)^{-1}\langle f(z)\rangle \, . \tag{3.1}$$

By applying the Newton iteration method to $g = f \cdot J_f^{-\frac{1}{n+1}}$, the corresponding function in (3.1) becomes

$$G(z) = z - Dg(z)^{-1}\langle g(z)\rangle \, . \tag{3.2}$$

The next theorem shows how this function in (3.2) is related to the Schwarzian derivative in several complex variables.

**Theorem 3.1.** *Let $f : \Omega \subset \mathbb{C}^n \to \mathbb{C}^n$ be a locally biholomorphic mapping defined in the simply connected domian $\Omega$, such that $f(\alpha) = 0$ for some $\alpha \in \Omega$. Then*

$$G(\alpha) = \alpha, \quad DG(\alpha) = 0, \quad D^2G(\alpha) = S_f(\alpha)\, ,$$

*where $S_f$ is given by equation (2.4).*

*Proof.* By (3.2), we have that $G(\alpha) = \alpha$ (since $g(\alpha) = 0$). Moreover, a straightforward calculation shows that (suppressing the variable $z$),

$$DG = \mathrm{Id} + D(Dg^{-1})\langle g, \cdot\rangle - \mathrm{Id} = -Dg^{-1}D^2g\langle Dg^{-1}\langle g\rangle, \cdot\rangle,$$

which gives $DG(\alpha) = 0$.

Notice that

$$Dg = J_f^{-\frac{1}{n+1}} Df - \frac{J_f^{-\frac{1}{n+1}} f}{(n+1)J_f}(\nabla J_f)^t = J_f^{-\frac{1}{n+1}} Df - \frac{g}{n+1}(\nabla \log J_f)^t.$$

Now let $u$ and $v$ be two vectors in $\mathbb{C}^n$. Then

$$\begin{aligned} D^2g\langle u, v\rangle &= J_f^{-\frac{1}{n+1}} D^2 f\langle u, v\rangle - \frac{J_f^{-\frac{1}{n+1}} Df(u)}{n+1}\nabla \log J_f \cdot v \\ &- \frac{J_f^{-\frac{1}{n+1}} Df(v)}{n+1}\nabla \log J_f \cdot u - \frac{J_f^{-\frac{1}{n+1}} f}{n+1} u(\mathrm{Hess} \log J_f)v^t. \end{aligned}$$

On the other hand, differentiating $DF$ and using equations (2.1) and (2.2), we obtain

$$\begin{aligned} D^2F\langle\cdot,\cdot\rangle &= Dg^{-1}D^2g\langle\cdot, Dg^{-1}D^2g\langle Dg^{-1}\langle g\rangle,\cdot\rangle\rangle - Dg^{-1}D^3g\langle Dg^{-1}\langle g\rangle,\cdot,\cdot\rangle \\ &+ Dg^{-1}D^2g\langle Dg^{-1}D^2g\langle Dg^{-1}\langle g\rangle,\cdot\rangle,\cdot\rangle - Dg^{-1}D^2g\langle\cdot,\cdot\rangle. \end{aligned}$$

Therefore,

$$\begin{aligned} D^2F(\alpha)\langle u,v\rangle &= -Dg^{-1}(\alpha)D^2g(\alpha)\langle u,v\rangle \\ &= -Df(\alpha)^{-1}D^2f\langle u,v\rangle - \frac{\nabla\log J_f\cdot u}{n+1}v - \frac{\nabla\log J_f\cdot v}{n+1}u. \\ &= -S_f(\alpha)\langle u,v\rangle. \end{aligned}$$

This ends the proof of the theorem. □

## 4. Univalence Criterion on the unit ball of $\mathbb{C}^n$

In this section we obtain the main result in this paper. Namely, we get a new sufficient condition for the univalence of a locally biholomorphic mapping defined in the unit ball $\mathbb{B}^n$ of $\mathbb{C}^n$ in terms of the Schwarzian derivative defined by (2.4).

Recall that in the setting of functions in the complex plane, there are different applications of the Loewner chain theory to get univalence criteria. In particular, we can mention that for a given locally univalent function $f$ defined in the unit disk $\mathbb{D}$ in the complex plane, the conditions

$$|P_f(z)| \le \frac{1}{1-|z|^2} \quad \text{or} \quad |S_f(z)| \le \frac{2}{(1-|z|^2)^2}, \quad \forall z\in\mathbb{D},$$

are sufficient to guarantee the univalence of $f$ in $\mathbb{D}$.

These sufficient conditions for the global univalence of a function defined on the unit disk are due to Becker and Nehari, respectively (see [1] and [7]). Actually, these results can be proved by using the Loewner chain theory, as is shown in the great survey book "*Geometric Function Theory in one and higher dimension*" by Gabriela Kohr and Ian Graham [2].

The generalization of the classical Loewner chain theory to several complex variables was first introduced by J. Pfaltzgraff in [9]. We again refer the reader to [2, Ch. 8] for a beautiful review of this theory. Pfaltzgraff himself generalized the analogous of the Becker criterion of univalence to several variables in [10]. Specifically, it is proved in [10] that given a locally univalent function $f:\mathbb{B}^n\to\mathbb{C}^n$ normalized by the conditions $f(0)=0$ and $Df(0)=$Id, if the inequality

$$(1-\|z\|^2)\|Df(z)^{-1}D^2f\langle z,\cdot\rangle\| \le 1$$

holds for all $z\in\mathbb{B}^n$, then $f$ is univalent in the unit ball. The reader can be found in [4] another univalence criterion that involves the Schwarzian derivative in several complex variables.

Here is the new criterion of univalence, now in terms of the Schwarzian derivative of functions in several complex variables.

**Theorem 4.1.** *Let $f : \mathbb{B}^n \to \mathbb{C}^n$ be a locally biholomorphic mapping normalized by $f(0) = 0$ and $Df(0) = Id$. Assume that $f$ satisfies the following inequality (where $S_f$ and $S_f^0$ are evaluated in $z$) for all $z \in \mathbb{B}^n$, where $p = \nabla \log J_f(z)$:*

$$\begin{aligned} \frac{\|S_f\langle z,\cdot\rangle\|}{(1-\|z\|^2)} &+ \left\| \frac{(z\cdot p)S_f\langle z,\cdot\rangle - (S_f\langle z,z\rangle \cdot p)\, Id}{n+1} + S_f^0\langle z,z\rangle Id \right\| \\ &\leq \frac{1}{(1-\|z\|^2)^2}. \end{aligned} \tag{4.1}$$

*Then, $f$ is univalent in $\mathbb{B}^n$.*

*Proof.* We shall prove that

$$f(z,t) = \frac{u(e^{-t}z) + (e^{2t}-1)Du(e^{-t}z)\langle e^{-t}z\rangle}{u_0(e^{-t}z) + (e^{2t}-1)\nabla u_0(e^{-t}z)\cdot e^{-t}z}, \tag{4.2}$$

where $u = (u_1, \dots, u_n)$ and $u_i$, $i = 0, \dots, n$, are the independent solutions of (2.7) with $P_{ij}^k = S_{ij}^k f$ (where $S_{ij}^k f$ are defined by (2.5) and (2.6)) and $u_0 = J_f^{-\frac{1}{n+1}}$, is a Loewner chain in $\mathbb{B}^n$. To do so, we will show that $f(z,t)$ satisfies the hypothesis of Theorem 8.1.6. in [2, p. 308] by following the same arguments as in the proof of [2, Thm. 8.4.1].

By differentiating $f(z,t)$ with respect to the variable $z$, we have

$$\begin{aligned} Df(z,t) &= \frac{e^{-t}Du + (e^{2t}-1)(e^{-t}Du + e^{-t}D^2u\langle e^{-t}z,\cdot\rangle)}{u_0 + (e^{2t}-1)\nabla u_0\cdot e^{-t}z} \\ &- \frac{(u + (e^{2t}-1)Du\langle e^{-t}z\rangle)(e^t\nabla u_0 + e^{-t}(e^{2t}-1)\text{Hess } u_0(e^{-t}z,\cdot)}{(u_0 + (e^{2t}-1)\nabla u_0\cdot e^{-t}z)^2}, \end{aligned}$$

where all the functions $u_0$, $u$, $\nabla u_0$, $Du$, and $D^2u$ are evaluated at $e^{-t}z$.

By (2.7) we have that $D^2u\langle\cdot,\cdot\rangle = Du \cdot S_f\langle\cdot,\cdot\rangle + S_f^0\langle\cdot,\cdot\rangle u$ and Hess $u_0(\cdot,\cdot) = S_f\langle\cdot,\cdot\rangle \cdot \nabla u_0 + S_f^0\langle\cdot,\cdot\rangle u_0$. Therefore, we get

$$e^tDf(z,t) = \frac{u_0Du - u\nabla u_0 + (e^{2t}-1)A + (e^{2t}-1)^2B}{(u_0 + (e^{2t}-1)\nabla u_0\cdot e^{-t}z)^2}, \tag{4.3}$$

where $A$ is the differential operator (evaluated in $e^{-t}z$) given by

$$\begin{aligned} A &= (e^{-t}z\cdot\nabla u_0)\, Du + u_0[Du + DuS_f\langle e^{-t}z,\cdot\rangle] \\ &- u[\nabla u_0 + S_f\langle e^{-t}z,\cdot\rangle\cdot\nabla u_0\rangle] - Du\langle e^{-t}z\rangle\nabla u_0. \end{aligned}$$

Notice that

$$A\langle e^{-t}z\rangle = [u_0Du - u(\nabla u_0)^t]\langle e^{-t}z + S_f\langle e^{-t}z, e^{-t}z\rangle\rangle.$$

In the same way, the linear operator $B$ given by

$$\begin{aligned} B &= [Du + DuS_f\langle e^{-t}z,\cdot\rangle + uS_f^0\langle e^{-t}z,\cdot\rangle]\nabla u_0\cdot e^{-t}z \\ &- Du\langle e^{-t}z\rangle[\nabla u_0\cdot(\cdot) + \nabla u_0\cdot S_f\langle e^{-t}z,\cdot\rangle + S_f^0\langle e^{-t}z,\cdot\rangle u_0] \end{aligned},$$

satisfies

$$\begin{aligned}B\langle e^{-t}z\rangle &= (\nabla u_0\cdot e^{-t}z)Du\langle S_f\langle e^{-t}z,e^{-t}z\rangle\rangle\\ &- (S_f\langle e^{-t}z,e^{-t}z\rangle\cdot\nabla u_0)Du\langle e^{-t}z\rangle\\ &- S_f^0\langle e^{-t}z,e^{-t}z\rangle[u_0Du-u(\nabla u_0)^t]\langle e^{-t}z\rangle.\end{aligned}$$

On the other hand, $u_0 f = u$, and then, $u_0 Du - u\nabla u_0 = u_0^2 Df$. Therefore, the derivative $Df(z,t)$ in (4.3) of the function $f(z,t)$ in (4.2) satisfies

$$e^tDf(z,t)\langle e^{-t}z\rangle = \frac{u_0^2Df\langle e^{-t}z\rangle\rangle + (e^{2t-1})A\langle e^{-t}z\rangle + (e^{2t}-1)^2B\langle e^{-t}z\rangle}{(u_0+(e^{2t}-1)\nabla u_0\cdot e^{-t}z)^2}.$$

Using that

$$A\langle e^{-t}z\rangle = u_0^2Df\langle e^{-t}z+S_f\langle e^{-t}z,e^{-t}z\rangle\rangle,$$

and

$$\begin{aligned}B\langle e^{-t}z\rangle &= (\nabla u_0\cdot e^{-t}z)u_0Df\langle S_f\langle e^{-t}z,e^{-t}z\rangle\rangle\\ &- S_f^0\langle e^{-t}z,e^{-t}z\rangle u_0^2Df\langle e^{-t}z\rangle\\ &- \nabla u_0\cdot S_f\langle e^{-t}z,e^{-t}z\rangle u_0Df\langle e^{-t}z\rangle\,,\end{aligned}$$

it follows that

$$\begin{aligned}e^tDf(z,t)\langle e^{-t}z\rangle &= Df\,\langle\, e^{-t}z+(e^{2t}-1)(e^{-t}z+S_f\langle e^{-t}z,e^{-t}z\rangle)\\ &+ (e^{2t}-1)^2(\nabla\log u_0\cdot e^{-t}zS_f\langle e^{-t}z,e^{-t}z\rangle\\ &- \nabla\log u_0\cdot S_f\langle e^{-t}z,e^{-t}z\rangle e^{-t}z\\ &- S_f^0\langle e^{-t}z,e^{-t}z\rangle e^{-t}z)\,\rangle\,(1+(e^{2t}-1)\langle\nabla\log u_0,e^{-t}z\rangle)^{-2}\\ &= e^{2t}Df\,[\,\mathrm{Id}+(1-e^{-2t})S_f\langle e^{-t}z,\cdot\rangle\\ &+ e^{2t}(1-e^{-2t})^2(\nabla\log u_0\cdot e^{-t}zS_f\langle e^{-t}z,\cdot\rangle\\ &- \nabla\log u_0\cdot S_f\langle e^{-t}z,e^{-t}z\rangle\mathrm{Id})\\ &- S_f^0(e^{-t}z,e^{-t}z)\mathrm{Id}\,]\,\langle e^{-t}z\rangle/(1+(e^{2t}-1)\nabla\log u_0\cdot e^{-t}z)^2\\ &= \frac{e^{2t}Df[\mathrm{Id}-E(z,t)]\langle e^{-t}z\rangle}{(1+(e^{2t}-1)\nabla\log u_0\cdot e^{-t}z)^2},\end{aligned}$$

where

$$\begin{aligned}E(z,t) &= (e^{-2t}-1)S_f\langle e^{-t}z,\cdot\rangle - e^{2t}(1-e^{2t})^2(\nabla\log u_0\cdot e^{-t}zS_f\langle e^{-t}z,\cdot\rangle\\ &- \nabla\log u_0\cdot S_f\langle e^{-t}z,e^{-t}z\rangle\mathrm{Id} - S_f^0\langle e^{-t}z,e^{-t}z\rangle\mathrm{Id}).\end{aligned}$$

Since

$$\frac{\partial e^{-t}z}{dz} = e^{-t} \quad \text{and} \quad \frac{\partial e^{-t}z}{dt} = -e^{-t}z,$$

we have

$$\frac{\partial f}{\partial t}(z,t) = \frac{e^{2t}Df\,[\mathrm{Id}+E(z,t)]\,\langle e^{-t}z\rangle}{(1+(e^{2t}-1)\nabla\log u_0\cdot e^{-t}z)^2}.$$

Notice that $(n+1)\log u_0 = -\log J_f$ and that $(1-e^{-2t}) < 1 - \|e^{-t}z\|^2$. Hence, using (4.1), we conclude that $\|E(z,t)\| < 1$. As a consequence, we see that $\mathrm{Id} - E(z,t)$ is an invertible operator. Therefore, its follows that

$$\frac{\partial f}{\partial t}(z,t) = Df(z,t)\,(\mathrm{Id} - E(z,t))^{-1}\,(\mathrm{Id} + E(z,t))\,\langle z\rangle.$$

Thus, $f(z,t)$ satisfies the differential equation

$$\frac{\partial f}{\partial t}(z,t) = Df(z,t)h(z,t), \quad z \in \mathbb{B}^n, \quad t \geq 0,$$

where $h(z,t) = [\mathrm{Id} - E(z,t)]^{-1}[\mathrm{Id} + E(z,t)]\langle z\rangle$. This shows that $f(z,t)$ is a Loewner chain with the intial value $f(z,0) = f(z)$, which completes the proof. □

**Remark 4.2.** If $f$ is a locally univalent analytic function defined in the unit disk $\mathbb{D} \subset \mathbb{C}$, the correspondig $S^k_{ij}f$ and $S^0_{ij}f$ given, respectively, by (2.5) and (2.6), satisfy that $S^k_{ij}f = 0$ and

$$S^0_{11}f = -\frac{1}{2}Sf, \quad S^0_{12}f = 0, \quad S^0_{22}f = 0.$$

Therefore, univalence criterion given by inequality (4.1) becomes the classical criterion of univalence in the unit disk due to Nehari: if such function $f$ satisfies that

$$|Sf(z)| \leq \frac{2}{(1-|z|^2)^2}$$

for all $|z| < 1$, then $f$ is univalent in the unit disk.

**Corollary 4.3.** *Let $f$ be as in previous theorem have constant (non-zero) Jacobian $J_f$. If*

$$(1-\|z\|^2)\|S_f(z)\langle z,\cdot\rangle\| \leq 1,$$

*then $f$ is univalent in $\mathbb{B}^n$.*

*Proof.* Since $J_f$ is a constant, $\nabla \log J_f = 0$. Furthermore, in this case, the correspondig solution $u_0$ in the proof of Theorem 4.1 is a constant too. Then the system (2.7) asserts that $S^0_{ij}f$ are identically zero for all $i$, $j$, and $k$. Thus, the inequality (4.1) equals

$$(1-\|z\|^2)\|S_f(z)\langle z,\cdot\rangle\| \leq 1.$$

A direct application of Theorem 4.1 ends the proof. □

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# On convolution, convex, and starlike mappings

Martin Chuaqui and Brad Osgood

*Dedicated to the memory of Professor Gabriela Kohr*

**Abstract.** Let $C$ and $S^*$ stand for the classes of convex and starlike mapping in $\mathbb{D}$, and let $\overline{\mathrm{co}(C)}$, $\overline{\mathrm{co}(S^*)}$ denote the closures of the respective convex hulls. We derive characterizations for when the convolution of mappings in $\overline{\mathrm{co}(C)}$ is convex, as well as when the convolution of mappings in $\overline{\mathrm{co}(S^*)}$ is starlike. Several characterizations in terms of convolution are given for convexity within $\overline{\mathrm{co}(C)}$ and for starlikeness within $\overline{\mathrm{co}(S^*)}$. We also obtain a correspondence via convolution between $C$ and $S^*$, as well as correspondences between the subclasses of convex and starlike mappings that have $n$-fold symmetry.

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**Keywords:** Convolution, convex mapping, starlike mapping, convex polygon, slit mapping, Pólya-Schoenberg conjecture.

## 1. Introduction

The present paper is motivated by our interest in convex mappings of the unit disk, $\mathbb{D}$, in particular a representation formula for the pre-Schwarzian of such mappings that has been very useful in studying, for example, Schwarz-Christoffel mappings onto convex polygons. The famous Pólya-Schoenberg conjecture, resolved in 1973 by Ruscheweyh and Sheil-Small, [9], can be formulated in terms of this representation formula and leads to an open problem, stated in Section 4, regarding a certain product in the unit ball of $H^\infty(\mathbb{D})$.

We also revisit some classical themes related to convolution of holomorphic mappings in $\mathbb{D}$, with a particular focus in the classes $C$ and $S^*$ of convex and starlike mappings. The analysis will carry over naturally to the closures of the convex hulls $\overline{\mathrm{co}(C)}$ and $\overline{\mathrm{co}(S^*)}$. In Section 2, we will derive necessary and sufficient conditions for $f*g$ to be convex when $f, g \in \overline{\mathrm{co}(C)}$, with two corollaries characterizing the mappings

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in $\overline{\text{co}(C)}$ that are convex. In the same vein, we will characterize when $f * g$ is starlike for $f, g \in \overline{\text{co}(S^*)}$, as well as give necessary and sufficient conditions for $f \in \overline{\text{co}(S^*)}$ to be starlike. Many more characterizations are probably possible.

In the Section 3 we will derive via convolution the classical theorem of Alexander for the correspondence between convex and starlike mappings, with the interesting special case of the correspondence between mappings onto convex polygons and starlike slit mappings. We also establish correspondences via convolution for the subclasses of $C$ and $S^*$ having $n$-fold symmetry.

We recall the definition of the convolution of two holomorphic functions. In terms of power series, if

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad g(z) = \sum_{n=0}^{\infty} b_n z^n$$

then their convolution is

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n.$$

As well,

$$(f * g)(z) = \frac{1}{2\pi i} \int_{|\zeta|=\rho} f(\zeta) g(z\zeta^{-1}) \frac{d\zeta}{\zeta}, \quad |z| < \rho. \tag{1.1}$$

## 2. Convex Hulls

Let $\overline{\text{co}(C)}$ and $\overline{\text{co}(S^*)}$ stand, respectively, for the closures of the convex hulls of convex and starlike mappings of $\mathbb{D}$. It was shown in [1] that $f \in \overline{\text{co}(C)}$ if and only if there exists a probability measure $\mu$ on $\partial\mathbb{D}$ such that

$$f(z) = \int\limits_{|\zeta|=1} \frac{z}{1 - z\zeta} d\mu\,,$$

and that $f \in \overline{\text{co}(S^*)}$ if and only if there exists a probability measure $\nu$ on $\partial\mathbb{D}$ such that

$$f(z) = \int\limits_{|\zeta|=1} \frac{z}{(1 - z\zeta)^2} d\nu\,.$$

To make this actionable we will need a number of explicit convolutions.

**Lemma 2.1.** *The following identities hold for functions of the variable $z \in \mathbb{D}$ and fixed parameters $\zeta, \xi \in \partial\mathbb{D}$.*

*i)* $\dfrac{z}{1 - z\zeta} * \dfrac{z}{1 - z\xi} = \dfrac{z}{1 - z\zeta\xi}$

*ii)* $\dfrac{z}{(1 - z\zeta)^2} * \dfrac{z}{1 - z\xi} = \dfrac{z}{(1 - z\zeta\xi)^2}$

*iii)* $\dfrac{z}{(1 - z\zeta)^2} * \dfrac{z\xi}{(1 - z\xi)^2} = \dfrac{z(1 + z\zeta\xi)}{(1 - z\zeta\xi)^3}$

*iv)* $\dfrac{z}{1-z\zeta} * \dfrac{1}{(1-z)^2} = \dfrac{z(2-z\zeta)}{(1-z\zeta)^3}$

*v)* $\dfrac{z}{1-z\zeta} * \dfrac{z^2}{(1-z)^2} = \dfrac{z^2\zeta}{(1-z\zeta)^2}$

*vi)* $\dfrac{z}{1-z\zeta} * \dfrac{z}{(1-z)^3} = \dfrac{z^2}{(1-z\zeta)^3}$

*vii)* $\dfrac{z}{1-z\zeta} * \dfrac{z^2}{(1-z)^3} = \dfrac{z^2\zeta}{(1-z\zeta)^3}$

*Proof.* The first identity follows directly from the power series of the functions convolved. For the remaining parts, we will use that $z(h_1 * h_2)' = h_1 * (zh_2')$.

*ii)* We have

$$\frac{z}{(1-z\zeta)^2} * \frac{z}{1-z\xi} = z\left(\frac{z}{1-z\zeta}\right)' * \frac{z}{1-z\xi} = z\left(\frac{z}{1-z\zeta} * \frac{z}{1-z\xi}\right)'$$
$$= z\left(\frac{z}{1-z\zeta\xi}\right)' = \frac{z}{(1-z\zeta\xi)^2}\,.$$

*iii)* Here we use that

$$\frac{z\xi}{(1-z\xi)^2} = z\left(\frac{z}{1-z\xi}\right)'\,.$$

*iv)* Since

$$\frac{1}{(1-z)^2} = 1 + \frac{z}{1-z} + \frac{z}{(1-z)^2}$$

the convolution is equal to

$$\frac{z}{1-z\zeta} + \frac{z}{(1-z\zeta)^2}\,,$$

which gives the result.

*v)* The identity follows at once from

$$\frac{z^2}{(1-z)^2} = 1 + \frac{2z}{(1-z)^2} - \frac{1}{(1-z)^2}\,.$$

*vi)* Here we write

$$\frac{z}{(1-z)^3} = \frac{z}{2}\left(\frac{1}{(1-z)^2}\right)'\,.$$

*vii)* The identity follows from

$$\frac{z^2}{(1-z)^3} = \frac{z}{(1-z)^3} - \frac{z}{(1-z)^2}\,.$$

□

We have a series of observations.

**Theorem 2.2.** *Let $f, g \in \overline{\mathrm{co}(C)}$ be represented by measures $\mu, \tau$, respectively. Then $f * g$ is convex if and only if*

$$\left| \iint\limits_{|\zeta|,|\xi|=1} \frac{z\zeta\xi}{(1-z\zeta\xi)^3} d\mu d\tau \right| \le \left| \iint\limits_{|\zeta|,|\xi|=1} \frac{1}{(1-z\zeta\xi)^3} d\mu d\tau \right| .$$

*Proof.* Let $f, g \in \overline{\mathrm{co}(C)}$. Differentiating $z(f*g)' = f*(zg')$ gives

$$z(f*g)'' + (f*g)' = (f*(zg'))' = [(zf')*(zg')]/z .$$

Write

$$\psi := 1 + z\frac{(f*g)''}{(f*g)'} = \frac{(zf')*(zg')}{z(f*g)'} = \frac{(zf')*(zg')}{f*(zg')} .$$

Because $\mathrm{Re}\{\psi\} \ge 0$ if and only if $|\psi - 1| \le |\psi + 1|$, we have that $f*g$ is convex if and only if

$$|(zf' - f)*(zg')| \le |(zf' + f)*(zg')| .$$

The correspondences in terms of the kernels are given by

$$zf' - f \longleftrightarrow \frac{z^2\zeta}{(1-z\zeta)^2} , \qquad zf' + f \longleftrightarrow \frac{z(2-z\zeta)}{(1-z\zeta)^2} ,$$

hence

$$(zf' - f)*(zg') \longleftrightarrow \frac{2z\zeta\xi}{(1-z\zeta)^3} , \qquad (zf' + f)*(zg') \longleftrightarrow \frac{2}{(1-z\zeta\xi)^3} ,$$

and the theorem follows. □

**Corollary 2.3.** *Let $f \in \overline{\mathrm{co}(C)}$ be represented by the measure $\mu$. Then $f$ is convex if and only if*

$$\left| \int\limits_{|\zeta|=1} \frac{z\zeta}{(1-z\zeta)^3} d\mu \right| \le \left| \int\limits_{|\zeta|=1} \frac{1}{(1-z\zeta)^3} d\mu \right| .$$

*Proof.* The corollary follows by letting $g = z/(1-z)$. □

If we let $\mu = \sum_{k=1}^{n} \alpha_k \delta_{\zeta_k}$ be a finite sum of delta functions at points $\zeta_k \in \partial\mathbb{D}$, then $f$ is convex if and only if

$$\left| \sum_{k=1}^{n} \frac{\alpha_k \zeta_k z}{(1-z\zeta_k)^3} \right| \le \left| \sum_{k=1}^{n} \frac{\alpha_k}{(1-z\zeta_k)^3} \right| .$$

This inequality characterizes the finite convex combinations of rotations of a half-plane mapping that are convex.

**Theorem 2.4.** *The function* $f \in \overline{\mathrm{co}(C)}$ *is convex if and only if*

$$\left| f(z) * \frac{z^2}{(1-z)^3} \right| \leq \left| f(z) * \frac{z}{(1-z)^3} \right| .$$

*If* $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, *then this holds if and only if*

$$\left| \sum_{k=1}^{\infty} k(k+1) a_{k+1} z^k \right| \leq \left| 2 + \sum_{k=1}^{\infty} (k+1)(k+2) a_{k+1} z^k \right| .$$

*Proof.* The first part of the theorem follows from parts vi) and vii) of Lemma 2.1, and the second follows directly from convolution. □

**Theorem 2.5.** *Let* $f, g \in \overline{\mathrm{co}(S^*)}$ *be represented by measures* $\mu, \tau$, *respectively. Then* $f * g$ *is starlike if and only if*

$$\left| \iint\limits_{|\zeta|,|\xi|=1} \frac{z\zeta\xi}{(1-z\zeta\xi)^2} d\mu d\tau \right| \leq \left| \iint\limits_{|\zeta|,|\xi|=1} \frac{2 - z\zeta\xi}{(1-z\zeta\xi)^2} d\mu d\tau \right| .$$

*Proof.* For $f, g \in \overline{\mathrm{co}(S^*)}$ let

$$\phi = z \frac{(f*g)'}{f*g} = \frac{f * (zg')}{f*g} .$$

Then $\mathrm{Re}\{\phi\} \geq 0$ if and only if

$$|f * (zg' - g)| \leq |f * (zg' + g)| ,$$

which proves the theorem from the representing kernels and Lemma 2.1. □

**Corollary 2.6.** *Let* $f \in \overline{\mathrm{co}(S^*)}$ *be represented by the measure* $\mu$. *Then* $f$ *is starlike if and only if*

$$\left| \int\limits_{|\zeta|=1} \frac{z\zeta}{(1-z\zeta)^2} d\mu \right| \leq \left| \int\limits_{|\zeta|=1} \frac{2 - z\zeta}{(1-z\zeta)^2} d\mu \right| .$$

*Proof.* As before, we let $g(z) = z/(1-z)$. □

**Theorem 2.7.** *The function* $f \in \overline{\mathrm{co}(S^*)}$ *is starlike if and only if*

$$\left| f(z) * \frac{z^2}{(1-z)^2} \right| \leq \left| f(z) * \frac{1}{(1-z)^2} \right| .$$

*If* $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, *then this holds if and only if*

$$\left| \sum_{k=2}^{\infty} (k-1) a_k z^k \right| \leq \left| z + \sum_{k=2}^{\infty} k a_k z^k \right| .$$

*Proof.* Parts iv) and v) of Lemma 1.1 give the first statement, which corresponds to the second inequality by convolving directly. □

## 3. Correspondences

Let

$$l(z) = \frac{z}{1-z}, \quad L_1(z) = \log\frac{1}{1-z}, \quad k_1(z) = \frac{z}{(1-z)^2}.$$

Note that $k_1 = zl'$ and $l = zL_1'$. We will use these functions together with convolution to recover Alexander's theorem relating convex and starlike mappings.

**Theorem 3.1.** *If $f \in C$ then $f * k_1 \in S^*$. Conversely, if $g \in S^*$ then $g * L_1 \in C$.*

*Proof.* Let $f \in C$. Then $f * k_1 = f * (zl') = z(f * k_1)' = zf'$, hence

$$1 + z\frac{f''}{f'} = z\frac{(f * k_1)'}{f * k_1}, \tag{3.1}$$

which shows the first claim.

On the other hand, if $g \in S^*$ then $z(g * L_1)' = g * (zL_1') = g * l = g$, and thus

$$1 + z\frac{(g * L_1)''}{(g * L_1)'} = z\frac{g'}{g},$$

which establishes the second claim. □

The special case when $f$ is a conformal mapping onto a convex polygon $\mathcal{P}$ is interesting. It follows from the Schwarz-Christoffel formula that

$$z\frac{f''}{f'} = -2\sum_{k=1}^{n}\frac{\beta_k z}{z - z_k},$$

where $z_k \in \partial\mathbb{D}$ are the pre-vertices, and $2\pi\beta_k$ are the exterior angles, satisfying $0 < \beta_k < 1$ with $\sum_{k=1}^{n}\beta_k = 1$. In [3] it was shown that

$$z\frac{f''}{f'} = \frac{2zB(z)}{1 - zB(z)}, \tag{3.2}$$

where $B(z)$ is a finite Blaschke product of degree $n-1$. Furthermore, the pre-vertices are the roots of the equation

$$zB(z) = 1.$$

**Theorem 3.2.** *Under the above convolutions, convex polygons correspond to slit mappings, with the number of vertices being equal to the number slits. If $g$ is the slit mapping corresponding to $f$ as above, then the pre-images $\zeta_k$ under $g$ of the finite endpoints of the slits are given by the root of the equation*

$$zB(z) = -1,$$

*while the pre-vertices $z_k$ of the polygon are mapped under $g$ to the point at infinity.*

*Proof.* Let $f$ map $\mathbb{D}$ onto a convex polygon $\mathcal{P}$. The correspondence of $f$ with a starlike mapping $g$ given by (3.1) is equivalent to

$$g = zf'.$$

On an open arc $A_k$ between the pre-vertices $z_k$ and $z_{k+1}$ we have that $\arg\{zf'\}$ is constant, hence so is $\arg\{g\}$. We conclude that $g(A_k)$ lies on a slit. Since $f'(z) \to \infty$ as $z$ approaches any pre-vertex, we see that $g = \infty$ at every pre-vertex $z_k$.

Let $\zeta_k \in A_k$ be the pre-image under $g$ of the finite endpoint of the slit $g^{-1}(f(A_k))$. Then $g'(\zeta_k) = 0$, hence

$$1 + \zeta_k \frac{f''}{f'}(\zeta_k) = 0 \,.$$

Using (3.2) we see that

$$\zeta_k B(\zeta_k) = -1 \,,$$

as claimed. □

**Corollary 3.3.** *Let $f$ be a conformal mapping onto a convex polygon, and let $B(z)$ be the associated Blaschke product in the representation (3.2). Then the roots of the equation*

$$zB(z) = -1$$

*correspond to the points between consecutive pre-vertices where $|f'|$ attains the minimum value on that arc.*

*Proof.* The mapping $g$ in the previous theorem is starlike, therefore $|g(\zeta_k)|$ is the minimum value of $|g|$ on the arc $A_k$. The corollary follows because $g = zf'$. □

By appropriately modifying $L_1, k_1$, these correspondences carry through to subspaces of convex and starlike mappings with symmetries. For this, we begin with the functions $L_2, k_2$ defined by the conditions $L_2(0) = k_2(0) = 0$ and

$$L_2'(z) = \frac{1}{2}\left(L_1'(z) + L_1'(-z)\right) \quad , \quad k_2'(z) = \frac{1}{2}\left(k_1'(z) + k_1'(-z)\right) \,.$$

Then

$$z(f * L_2)' = \frac{1}{2} f * \left(L_1'(z) + L_1'(-z)\right) = \frac{1}{2}\left(f * \frac{z}{1-z} + \frac{z}{1+z}\right) \,.$$

If $f$ is odd then

$$f * \frac{z}{1-z} = f * \frac{z}{1+z} \,,$$

and we are back in the case when $z(f * L_2)' = f$. Therefore, if $f \in \overline{\mathrm{co}(S^*)}$ is odd then $f * L_2 \in \overline{\mathrm{co}(C)}$, and is also odd because $L_2$ is odd. A similar analysis shows that if $g \in \overline{\mathrm{co}(C)}$ is odd, then $g * k_2 \in \overline{\mathrm{co}(S^*)}$ and is also odd.

For the general construction, we introduce the averaging operator $A_n$ defined by

$$A_n(f)(z) = \frac{1}{n} \sum_{k=1}^{n} \omega^{-k} f(\omega^k z) \,,$$

where $\omega = e^{\frac{2\pi i}{n}}$. An equivalent definition is that $A_n(f)$ satisfies $A_n(f)(0) = 0$ and

$$A_n(f)'(z) = \frac{1}{n} \sum_{k=1}^{n} f'(\omega^k z) \,.$$

We thus see that $L_2 = A_2(L_1)$ and $k_2 = A_2(k_1)$.

A function $f$ defined in $\mathbb{D}$ is said to be $n$-symmetric if $A_n(f) = f$. It is not difficult to see that $f$ is $n$-symmetric if and only if $f(\omega z) = \omega f(z)$, which holds if and only if $f(z) = z\sum_{k=0}^{\infty} a_k z^{kn}$. Furthermore,

$$A_n(f) * g = f * A_n(g)\,,$$

a fact that follows immediately from the respective power series expansions. We can also see that $A_n(f) * g$ is always $n$-symmetric. We finally let $L_n = A_n(L_1)$ and $k_n = A_n(k_1)$ stand for the symmetrization of $L_1$ and $k_1$, and define $C_n, S_n^*$ to be the set of convex and starlike mappings with $n$-fold symmetry. It is interesting to note that all the mappings $L_n$ are univalent since $\mathrm{Re}\{L_n'\} > 0$ in $\mathbb{D}$. On the other hand, already the function $k_2$ is not even locally univalent in $\mathbb{D}$.

**Theorem 3.4.** *If $f \in C_n$ then $f * k_n \in S_n^*$. If $g \in S_n^*$ then $g * L_n \in C_n$.*

*Proof.* If $f \in C_n$ then $z(f * k_n') = (f * k_n)' = (A_n(f) * k_1)' = (f * k_1)'$, which allows us to conclude that $f * k_n \in S^*$ as argued in Theorem 2.2. Since it is also also symmetric, the first claim follows. The second claim is established in similar fashion. □

These results carry through to establish correspondences between the spaces $\overline{\mathrm{co}(C)}$ and $\overline{\mathrm{co}(S^*)}$, and the respective subspaces with $n$-fold symmetry $\overline{\mathrm{co}(C)}_n = \{A_n(f) : f \in \overline{\mathrm{co}(C)}\}$ and $\overline{\mathrm{co}(S^*)}_n = \{A_n(g) : g \in \overline{\mathrm{co}(S^*)}\}$. We have:

**Theorem 3.5.** *If $f \in \overline{\mathrm{co}(C)}$ then $f * k_1 \in \overline{\mathrm{co}(S^*)}$. Conversely, if $g \in \overline{\mathrm{co}(S^*)}$ then $g * L_1 \in \overline{\mathrm{co}(S^*)}$.*

*Proof.* The proof is very similar to that of Theorem 3.1, and we will give the details for the first claim. If $f \in \overline{\mathrm{co}(C)}$ then for some probability measure $\mu$ we have

$$f(z) = \int\limits_{|\zeta|=1} \frac{z}{1 - z\zeta}\, d\mu\,.$$

Hence

$$\begin{aligned}(f * k_1)(z) &= \int\limits_{|\zeta|=1} \frac{z}{1-z\zeta} * k_1(z) d\mu \;=\; \int\limits_{|\zeta|=1} \frac{z}{1-z\zeta} * (zl'(z))\, d\mu \\ &= \int\limits_{|\zeta|=1} z\left(\frac{z}{1-z\zeta} * l(z)\right)' d\mu \;=\; \int\limits_{|\zeta|=1} z\left(\frac{z}{1-z\zeta}\right)' d\mu \\ &= \int\limits_{|\zeta|=1} \frac{z}{(1-z\zeta)^2}\, d\mu\,,\end{aligned}$$

showing that $f * k_1 \in \overline{\mathrm{co}(S^*)}$. □

We state without proof the last result in this section.

**Theorem 3.6.** *If $f \in \overline{\mathrm{co}(C)}_n$ then $f * k_n \in \overline{\mathrm{co}(S^*)}_n$. Conversely, if $g \in \overline{\mathrm{co}(S^*)}_n$ then $g * L_n \in \overline{\mathrm{co}(S^*)}_n$.*

## 4. On the Pólya-Schoenberg Conjecture

The Pólya-Schoenberg Conjecture (PSC) states that the convolution of convex mappings is again convex. Our point of departure for the discussion here is the representation formula for convex mappings $f$, namely

$$\frac{f''}{f'} = \frac{2z\phi}{1 - z\phi}\,, \tag{4.1}$$

for some holomorphic $\phi$ with $|\phi| \leq 1$ in $\mathbb{D}$. For example, for $\phi$ a unimodular constant, the resulting $f$ is a rotation of the half-plane mapping, while for $\phi = z$ we obtain the mapping onto a parallel strip. The representation formula can be derived from the classical characterization of convexity via positive real part, and Schwarz's lemma. Any choice of such a function $\phi$ will determine a unique (normalized) convex mapping.

Let now $f, g$ be convex and represented as above by functions $\phi, \psi$ bounded by 1 in $\mathbb{D}$. By PSC $f * g$ is also convex, and thus must be represented by another such function $\chi$, which can be thought of as determined by the functions $\phi$ and $\psi$. This dependence yields therefore a certain "product" in the unit ball of $H^\infty(\mathbb{D})$ inherited from convolution being associative and commutative. An independent proof that this product does indeed preserve the unit ball in $H^\infty(\mathbb{D})$ would provide an alternative proof of the PSC.

To be more precise, for $f$ convex we define the operator

$$\Phi(f) = \frac{f''/f'}{2 + zf''/f'}\,,$$

which comes from expressing $\phi$ in terms of $f$ in (4.1). As an example, we compute $\Phi(f * g)$ when $f, g$ are Möbius transformations. If

$$f(z) = \frac{az + b}{cz + d}\,,\ ad - bc = 1\,,$$

then a simple calculation yields

$$\Phi(f) = -\frac{c}{d} = \frac{1}{f^{-1}(\infty)}\,.$$

If

$$g(z) = \frac{\alpha z + \beta}{\gamma z + \delta}\,,\ \alpha\delta - \beta\delta = 1\,,$$

then

$$\begin{aligned} f * g &= \frac{1}{\gamma\delta} f\left(-(\gamma/\delta)z\right) - \frac{\alpha}{\gamma} f(0) \\ &= \frac{1}{\gamma\delta}\frac{(a\gamma/\delta)z - b}{(c\gamma/\delta)z - d} - \frac{\alpha}{\gamma} f(0) \end{aligned}$$

is again a Möbius transformation with denominator $c\gamma^2 z - \gamma\delta d$. Therefore

$$\Phi(f * g) = \frac{1}{(f * g)^{-1}(\infty)} = \frac{c\gamma}{d\delta} = \Phi(f)\Phi(g)\,.$$

That the convolution corresponds to the actual product in $H^\infty(\mathbb{D})$ is exceptional and can be readily seen not to hold in general. But we find the problem of understanding and determining the properties of the $\Phi$-operator in relation to convolution appealing.

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# The level sets of functions with bounded critical sets and bounded $\text{Hess}^+$ complements

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*Dedicated to the memory of Professor Gabriela Kohr*

**Abstract.** We denote by $\text{Hess}^+(f)$ the set of all points $p \in \mathbb{R}^n$ such that the Hessian matrix $H_p(f)$ of the $C^2$-smooth function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is positive definite. In this paper we prove several properties of real-valued functions of several variables by showing the connectedness of their level sets for sufficiently high levels, under the boundedness assumption on the critical set. In the case of three variables we also prove the convexity of the levels surfaces for sufficiently high levels, under the additional boundedness assumption on the $\text{Hess}^+$ complement. The selection of the *a priori* convex levels, among the connected regular ones, is done through the positivity of the Gauss curvature function which ensure an ovaloidal shape of the levels to be selected. The ovaloidal shape of a level set makes a diffeomorphism out of the associated Gauss map. This outcome Gauss map diffeomorphism is then extended to a smooth homeomorphism which is used afterwards to construct one-parameter families of smooth homeomorphisms of Loewner chain flavor.

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**Keywords:** The Hessian matrix, the $\text{Hess}^+$ region, curvature, Gauss curvature, convex curves, ovaloids.

## 1. Introduction

After analyzing in [12] the $\text{Hess}^+(f_a)$ region of the polynomial function

$$f_a : \mathbb{R}^2 \longrightarrow \mathbb{R},\ f_a(x, y) = (x^2 + y^2)^2 - 2a^2(x^2 - y^2)$$

and noticing that its complement is bounded, we provided in [2] a class of norm-coercive polynomial functions with large $\text{Hess}^+$ regions, as their $\text{Hess}^+$ complements happen to be bounded as well. A detailed analysis of the $\text{Hess}^+$ regions for some particular polynomial functions which happen to have bounded $\text{Hess}^+(f)$ complements

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along with bounded critical sets is done there. Basic properties of their level curves, such as regularity, connectedness even convexity of their level sets, for sufficiently large levels, are also pointed out. These properties are then proved to hold true for the whole class of norm-coercive functions of two variables with bounded $\mathrm{Hess}^+(f)$ complements. Since the convex hypersurfaces will be repeatedly used, let us mention that one way to consider convexity for regular hypersurfaces of $\mathbb{R}^n$ consists in their quality to stay on the same side of each of its tangent hyperplane [10, p. 174], [3, p. 37]. On the other hand a regular hypersurface could sometimes bound a convex set and such a hypersurface is also said to be convex [13] (see also [10, p. 175]). Apart from the mentioned source of examples, some sufficient conditions on two functions $f, g : \mathbb{R}^2 \longrightarrow \mathbb{R}$ with bounded $\mathrm{Hess}^+$ complements are provided in [2] in order for the product $fg$ to keep having bounded $\mathrm{Hess}^+$ complement.

In this paper we partially extend [2, Theorem 3.6] to real-valued functions of several variables by showing several properties of their level sets for sufficiently high levels, under the boundedness assumption on the critical set. Although we loose convexity of level hypersurfaces in the general case of several variables, we still prove the connectedness in this general case and recapture their convexity in the case of three variables for sufficiently high levels, under the additional boundedness assumption on the $\mathrm{Hess}^+$ complement.

The paper is organized as follows: In the second section, we prove the connectedness of the level sets, for sufficiently large levels, of a real valued function with several variables having bounded critical set. Section 3 is still devoted to properties of the level sets but for functions with three variables under the additional requirements on the function to be norm-coercive and to have bounded $\mathrm{Hess}^+$ complement. For this particular number of variables we recapture the convexity of the level sets for sufficiently large levels, a property we used to have in the case of two variables as well (see [2, 12]). In this case of three variables we still use the positivity test of the Gauss curvature function along a level set to select the *a priori* ovaloidal level sets, among the connected regular ones. For two variables we used the nonvanishing test of the curvature function. In section 4 we first use the outcome Gauss map diffeomorphism of the ovaloidal level sets of the norm-coercive functions with bounded critical set and bounded $\mathrm{Hess}^+$ complement to construct a smooth homeomorphism from the disc $D^3$ to some sub-level set bounded by an ovaloidal level set. Such smooth homeomorphisms are than used to construct one-parameter families of Loewner chain flavor (compare Theorem 4.1(2)(4) with the definition of Loewner chains [4, 8, 6, 7, 9] and Theorem 4.1(4) with [4, Theorem 1.6 (iv)]) by using a homothetic perturbation of the gradient vector field of the function itself which permutes the, sufficiently high, regular levels.

## 2. Properties of the level sets of functions with bounded critical set

In order to state the first result of this paper, we shall quickly recall the critical/regular points and critical/regular sets of real-valued functions in a similar fashion with [2]. If $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a Fréchet differentiable map, then the *rank* of $f$ at $x \in \mathbb{R}^n$ is defined as $\mathrm{rank}_x f := \mathrm{rank}(df)_x = \mathrm{rank}(Jf)_x$. Observe that $\mathrm{rank}_x f \leq \min\{m, n\}$ for every $x \in \mathbb{R}^n$. A point $x \in \mathbb{R}^n$ is said to be a *critical point* of $f$ if $\mathrm{rank}_x f < \min\{m, n\}$.

Otherwise $x$ is said to be a *regular point* of $f$. If $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a $C^1$-smooth map, then each point $x \in \mathbb{R}^n$ has an open neighbourhood, say $V_x \subseteq \mathbb{R}^n$, such that $\mathrm{rank}_y f \geq \mathrm{rank}_x f$, for all $y \in V_x$. In particular, once a point $x$ is regular, it has a whole neighbourhood of regular points. Indeed the Jacobian matrix $(Jf)_x$ has a non-zero minor of order $\mathrm{rank}_x f$ and all minors of $(Jf)_x$ of superior order are zero. But the nonzero minor of $(Jf)_x$ is nonzero on a whole open neighbourhood of $x$ since it is a continuous function. This shows that $\mathrm{rank}_y f = \mathrm{rank}_y (Jf)_y \geq \mathrm{rank}(Jf)_x = \mathrm{rank}_x f$, which are satisfied for $y$ in a whole neighbourhood of $x$. Consequently the set $R(f)$, of regular points of $f$, is open in $\mathbb{R}^n$, while the set $C(f)$, of critical points of $f$, is closed in $\mathbb{R}^n$. We denote by $B(f)$ the set $f(C(f))$ of critical values of $f$. Note that for a real valued function $f : U \longrightarrow \mathbb{R}$, the critical set of $f$ is the vanishing set of its gradient $\nabla f$.

**Theorem 2.1.** *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a smooth norm-coercive function. If $C(f)$ is bounded, then the c-level set $f^{-1}(c)$ of $f$ is a regular compact connected and orientable hypersurface for $c$ sufficiently large.*

The proof of Theorem 2.1 works along the same lines with the proof of [2, Theorem 3.6].

**Remark 2.2.** ([14, Theorem 2.5.7]) If $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a $C^1$-smooth convex function, then its critical set $C(f)$ is convex. Indeed the critical points of $f$ coincide with the global minimum points of $f$ (see also [2]).

Apart from the examples of real-valued polynomial functions of two variables

1. $f_a : \mathbb{R}^2 \longrightarrow \mathbb{R},\ f_a(x,y) = (x^2+y^2)^2 - 2a^2(x^2-y^2),\ (a>0);$
2. $g_a : \mathbb{R}^2 \longrightarrow \mathbb{R},\ g_a(x,y) = (x^2+y^2)^2 + 2a^2(x^2-y^2),\ (a>0);$
3. $f_a g_a : \mathbb{R}^2 \longrightarrow \mathbb{R},\ (f_a g_a)(x,y) = (x^2+y^2)^4 - 4a^4(x^2-y^2)^2,\ (a>0);$

which are not convex as their critical sets are discrete with at least two critical points, mentioned in [2], we consider here a polynomial function of three variables.

**Example 2.3.** $f : \mathbb{R}^3 \longrightarrow \mathbb{R},\ f_a(x,y,z) = (x^2+y^2+z^2)^2 - 8(x^2-y^2-z^2).$

This function is not convex as its critical set $C(f) = \{(-4,0,0),(0,0,0),(4,0,0)\}$ is obviously not convex. In the sections to come this function will be analyzed from the $\mathrm{Hess}^+(f)$-region point of view.

## 3. Levels of functions whose $\mathrm{Hess}^+$ complements are additionally bounded

Let $D$ be a nonempty open convex subset of $\mathbb{R}^n$, and let $f : D \to \mathbb{R}$ be a $C^2$-smooth function. The Hessian matrix of $f$ at an arbitrary point $x \in D$ will be denoted by $H_x(f)$. Recall that $H_f(x)$ is a symmetric matrix and it defines a symmetric bilinear functional

$$\mathcal{H}_f(x) : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R},\ \mathcal{H}_f(x)(u,v) := u \cdot H_x(f) \cdot v^T.$$

We are interested about the region

$$\mathrm{Hess}^+(f) = \{x \in D : H_x(f) \text{ is positive definite}\}.$$

**Example 3.1.** ([12]) For the polynomial function $f_a : \mathbb{R}^2 \longrightarrow \mathbb{R}$, given by

$$f_a(x, y) = (x^2 + y^2)^2 - 2a^2(x^2 - y^2),\ (a > 0)$$

we have

$$\mathrm{Hess}^+(f_a) = \{(x, y) \in \mathbb{R}^2 | 3(x^2 + y^2)^2 + 2a^2(x^2 - y^2) > a^4\} = g^{-1}_{a/\sqrt{3}}\left(a^4/3, +\infty\right), \quad (3.1)$$

where $g_b : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is given by $g_b(x, y) = (x^2 + y^2)^2 + 2b^2(x^2 - y^2),\ (b > 0)$.

Recall that the nondiscrete level sets of $f_a$ are the Cassini's ovals and its zero level curve, which is critical, is the Bernoulli's lemniscate.

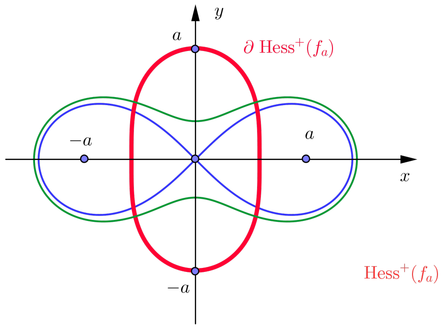


FIGURE 1. The boundary of $\mathrm{Hess}^+(f_a)$, the critical level $f_a^{-1}(0)$ and a positive regular value of $f_a$

**Example 3.2.** In this example we deal with a polynomial function of three variables whose restriction to $\mathbb{R}^2$ is $f_2$. More precisely, the $\mathrm{Hess}^+$ region of the polynomial function $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$, given by $f(x, y, z) = (x^2 + y^2 + z^2)^2 - 8(x^2 - y^2 - z^2)$, is $\mathrm{Hess}^+(f) = \{(x, y, z) \in \mathbb{R}^3 | 3(x^2 + y^2 + z^2)^2 + 8(x^2 - y^2 - z^2) > 16\}$.

The characteristic polynomial $CP_{H(f)}$ of the Hessian matrix

$$H(f) = \begin{bmatrix} 4(3x^2 + y^2 + z^2 - 4) & 8xy & 8xz \\ 8yx & 4(x^2 + 3y^2 + z^2 + 4) & 8yz \\ 8zx & 8zy & 4(x^2 + y^2 + 3z^2 + 4) \end{bmatrix} \quad (3.2)$$

is invariant under the action $\mathbb{R} \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3, (t, x) \mapsto t * x$, where $t * x := r(t) \cdot x^T$ and $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ is identified, on the right hand side, with the row matrix $[x_1\ x_2\ x_3]$ and $r(t)$ stands for the $3 \times 3$ matrix of the rotation around the $x$-axis of angle $t$.

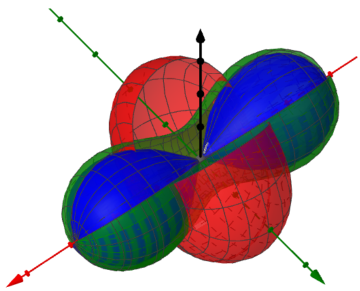

FIGURE 2. The 0-critical level of $f$, a nonconvex regular level of $f$ and the boundary of $\mathrm{Hess}^+(f)$

$$r(t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{bmatrix}$$

Thus, the action (3.2) is

$$t * (x_1, x_2, x_3) := \begin{bmatrix} x_1 \\ x_2 \cos t - x_3 \sin t \\ x_2 \sin t + x_3 \cos t \end{bmatrix}.$$

Indeed, the characteristic polynomial $CP_{H(f)}$ of the Hessian matrix of the polynomial function $\mathrm{Tr}H_{(x,y,z)}(f) = 4(5(x^2 + y^2 + z^2) + 4)$

$$\begin{aligned} A_{11} + A_{22} + A_{33} &= 16(7(x^2 + y^2 + z^2)^2 + 8(x^2 + y^2 + z^2) + 16x^2 - 16) \\ \frac{1}{4^3} \det H(f) &= 2x^2(x^2 + y^2 + z^2 + 4)^2 + (x^2 + y^2 + z^2 - 4)[(x^2 + y^2 + z^2)^2 \\ &\quad + 2(y^2 + z^2 + 4)(x^2 + y^2 + z^2) + 8(y^2 + z^2) + 16] \end{aligned}$$

are all invariant with respect to the action (3.2). Since $t * \left(x, \sqrt{y^2 + z^2}, 0\right) = (x, y, z)$ for every $t \in \mathbb{R}$ such that $\cos t = y/\sqrt{y^2 + z^2}$, $\sin t = z/\sqrt{y^2 + z^2}$ we get by performing elemetary computations

$$\begin{aligned} CP_{H_{(x,y,z)}(f)}(\lambda) &= CP_{H_{\left(x,\sqrt{y^2+z^2},0\right)}(f)}(\lambda) \\ &= (4(x^2 + y^2 + 3z^2 + 4) - \lambda) CP_{H_{\left(x,\sqrt{y^2+z^2}\right)}\left(f\big|_{\mathbb{R}^2}\right)} \\ &= (4(x^2 + y^2 + z^2 + 4) - \lambda) CP_{H_{\left(x,\sqrt{y^2+z^2}\right)}(f_2)}, \end{aligned}$$

where $f_2 : \mathbb{R}^2 \longrightarrow \mathbb{R}$, $f_2(u, v) = (u^2 + v^2)^2 - 8(u^2 - v^2)$. Consequently the Hessian matrix $H_f(x, y, z)$ is positive definite if and only if both eigenvalues of $H_{\left(x,\sqrt{y^2+z^2}\right)}(f_2)$, i.e. the roots of the characteristic polynomial,

$$CP_{H_{\left(x,\sqrt{y^2+z^2}\right)}\left(f\big|_{\mathbb{R}^2}\right)} = CP_{H_{\left(x,\sqrt{y^2+z^2}\right)}(f_2)}$$

are positive. Equivalently, the Hessian matrix $H_{f_2}\left(x, \sqrt{y^2 + z^2}\right)$ must be positive definite, as one eigenvalue $\lambda = 4(x^2 + y^2 + z^2 + 4)$ of $H_{(x,y,z)}(f)$ is positive. In other words

$$\begin{aligned} (x, y, z) \in \mathrm{Hess}^+(f) &\Longleftrightarrow \left(x, \sqrt{y^2 + z^2}\right) \in H_{f_2}\left(x, \sqrt{y^2 + z^2}\right) \\ &\overset{(3.1)}{\Longleftrightarrow} \left(x, \sqrt{y^2 + z^2}\right) \in \{(u, v) \in \mathbb{R}^2 | 3(u^2 + v^2)^2 + 8(u^2 - v^2) > 16\} \\ &\Longleftrightarrow 3(x^2 + y^2 + z^2)^2 + 8(x^2 - y^2 - z^2) > 16. \end{aligned}$$

**Definition 3.3.** ([10, p. 174], [3, p. 322]) A compact connected surface $S \subset \mathbb{R}^3$ is said to be an *ovaloid* if its Gauss curvature is everywhere positive.

According with the Hadamard-Stoker Theorem [10, Theorem 6.1, p. 175], the interior of every ovaloid along with the closure of its interior are convex sets.

**Theorem 3.4.** *Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ be a $C^2$-smooth norm-coercive function. If $C(f)$ and $\mathbb{R}^3 \setminus \mathrm{Hess}^+(f)$ are bounded, then the level surface $f^{-1}(c)$ is a compact connected regular ovaloid for $c$ sufficiently large.*

The compactness, connectedness and the regularity of the hypersurfaces $f^{-1}(c)$, for $c$ sufficiently large, follow via Theorem 2.1. In order to prove the ovaloidal shape of the level surfaces $f^{-1}(c)$ for $c$ sufficiently large, we need the following

**Lemma 3.5.** *The Gauss curvature function associated to the $C^2$-smooth function $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ is positive over the* $\mathrm{Hess}^+(f)$ *region.*

*Proof.* Indeed, according to [5], the Gauss curvature function associated to $f$ is

$$K_G = -\frac{\begin{vmatrix} H(f) & (\nabla f)^T \\ \nabla f & 0 \end{vmatrix}}{\|\nabla f\|^4} = \frac{\nabla f \cdot H^*(f) \cdot (\nabla f)^T}{\|\nabla f\|^4} = \det H(f) \frac{\nabla f \cdot H^{-1}(f) \cdot (\nabla f)^T}{\|\nabla f\|^4},$$

where $H^*(f)$ is the adjugate matrix of the Hessian matrix $H(f)$ of $f$. Its value at $x \in R(f) = \mathbb{R}^n \setminus C(f)$ is the Gauss curvature of the level surface $f^{-1}(f(x))$ at $x$. We next recall that $x \in \mathrm{Hess}^+(f)$ if and only if the eigenvalues of $H_x(f)$ are all positive, which implies that $\det H_x(f) > 0$ for $x \in \mathrm{Hess}^+(f)$. On the other hand the eigenvalues of $H_x^{-1}(f)$ are the inverse values of the eigenvalues of $H_x(f)$ and are therefore positive as well. Consequently the inverse matrix $H_x^{-1}(f)$ is also positive definite which implies that $\nabla_x f \cdot H_x^{-1}(f) \cdot (\nabla_x f)^T > 0$ and $K_G(x) > 0$ therefore. □

**Corollary 3.6.** *Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ be a $C^2$-smooth norm-coercive function. If $C(f)$ and $\mathbb{R}^3 \setminus \mathrm{Hess}^+(f)$ are bounded, then the level surface $f^{-1}(c)$ is diffeomorphic with the sphere $S^2$ and bounds an open convex set for $c$ sufficiently large.*

*Proof.* It follows immediately by combining Theorem 3.4 with the Hadamard-Stoker Theorem [10, Theorem 6.5, p. 178]. □

*Proof of Theorem* 3.4. The boundedness of $\mathbb{R}^3 \setminus \mathrm{Hess}^+(f)$ combined with its closedness imply its compactness. For $c > h_{\max}(f) := \max\left\{f(x) \;\; x \in \mathbb{R}^3 \setminus \mathrm{Hess}^+(f)\right\}$, the level surface $f^{-1}(c)$ is completely contained in $\mathrm{Hess}^+(f)$ and for $c > \mu_{\max}(f)$ the level surface $f^{-1}(c)$ is regular. Therefore the level surface $f^{-1}(c)$ is additionally an ovaloid for $c > \max\{h_{\max}(f), \mu_{\max}(f)\}$, besides its compactness, connectedness and regularity ensured by Theorem 2.1. Indeed, for such a value of $c$, the compact connected regular level surface $f^{-1}(c)$ is completely contained in $\mathrm{Hess}^+(f)$, where the Gauss curvature function associated to $f$ is, according to Lemma 3.5, positive. □

**Corollary 3.7.** *Every two level surfaces of a function $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ subject to the hypothesis of Theorem* 3.4*, above the level $\mu_{\max}(f)$, are connected regular hypersurfaces diffeomorphic to the unit sphere $S^2$ which bounds a convex open set for $c > \max\{h_{\max}(f), \mu_{\max}(f)\}$.*

*Proof.* The statement follows by combining Corollary 3.6 with the Non-Critical Neck Principle (see e.g. [11, p. 194]) and the proof of Theorem 3.4. □

**Corollary 3.8.** *The sublevel set $f^{-1}(-\infty, h_{\max}(f)]$ of a function $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ subject to the hypothesis of Theorem* 3.4 *is convex, whenever $h_{\max}(f) \geq \mu_{\max}(f)$.*

The two dimensional couterpart of Corollary 3.8 is [2, Corollary 3.8] whose proof works along the same lines.

**Example 3.9.** The first convex connected level curve of the function

$$f_a : \mathbb{R}^2 \longrightarrow \mathbb{R},\ f_a(x,y) = (x^2+y^2)^2 - 2a^2(x^2-y^2)\ (a>0),$$

is $f_a^{-1}(3a^4)$. It is the first positive regular level of $f_a$ which is completely contained in

$$\text{cl Hess}^+(f_a) = \{(x,y) \in \mathbb{R}^2 | 3(x^2+y^2)^2 + 2a^2(x^2-y^2) \geq a^4\} = g^{-1}_{a/\sqrt{3}}\left[a^4/3, +\infty\right),$$

and is illustrated in FIGURE 3 (see [12]).

The higher level curves of $f_a$ are also contained in cl $\text{Hess}^+(f_a)$ and they keep being convex. The convex level curves of $f_a$ were selected through the nonvanishing requirement of the determinant

$$\delta_a := \begin{vmatrix} H(f) & (\nabla f)^T \\ \nabla f & 0 \end{vmatrix} = \begin{vmatrix} (f_a)_{xx} & (f_a)_{xy} & (f_a)_x \\ (f_a)_{yx} & (f_a)_{yy} & (f_a)_y \\ (f_a)_x & (f_a)_y & 0 \end{vmatrix} = -4^3\frac{a^4+c}{2}\{3(x^2+y^2)^2 - c\}$$

as part of the curvature formula

$$\rho_C = -\frac{\begin{vmatrix} H(f) & (\nabla f)^T \\ \nabla f & 0 \end{vmatrix}}{\|\nabla f\|^3}$$

over the level curve $C = f_a^{-1}(c)$ of $f$. We can similarly select the level surfaces of a function $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ through the positiveness requirement on the determinant

$$\Delta = \begin{vmatrix} H(f) & (\nabla f)^T \\ \nabla f & 0 \end{vmatrix}$$

as part of the Gauss curvature formula

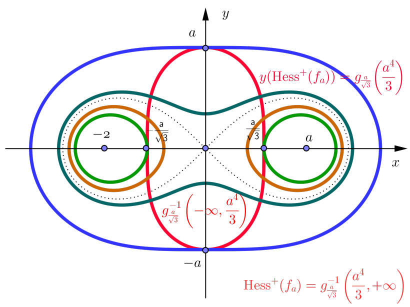


FIGURE 3. The boundary of $\text{Hess}^+(f_a)$, the last negative regular level and the first positive convex regular level of $f_a$ contained in cl $\text{Hess}^+(f_a)$, Positive nonconvex regular level and disconnected negative regular level of $f_a$ with convex components which are not contained in cl $\text{Hess}^+(f_a)$

$$K_G = -\frac{\begin{vmatrix} H(f) & (\nabla f)^T \\ \nabla f & 0 \end{vmatrix}}{\|\nabla f\|^3} = -\frac{\begin{vmatrix} f_{xx} & f_{xy} & f_{xz} & f_x \\ f_{yx} & f_{yy} & f_{yz} & f_y \\ f_{zx} & f_{zy} & f_{zz} & f_z \\ f_x & f_y & f_z & 0 \end{vmatrix}}{\|\nabla f\|^3}$$

of the level surface $S = f^{-1}(s)$. The theoretical basis for the convexity of the ovaloids is the Hadamard-Stoker [10, Theorem 6.5, p. 178].

**Example 3.10.** The first convex level of the function

$$f : \mathbb{R}^3 \longrightarrow \mathbb{R},\ f(x,y,z) = (x^2+y^2+z^2)^2 - 8(x^2-y^2-z^2),$$

is $f^{-1}(3 \cdot 16)$. It is also the first positive regular level of $f$ completely contained in

$$\text{cl Hess}^+(f) = \{(x,y,z) \in \mathbb{R}^3 \mid 3(x^2+y^2+z^2)^2 + 8(x^2-y^2-z^2) \geq 16\}.$$

Indeed $\Delta$ is invariant with respect to the one parameter group of rotations (3.2) as

$$\frac{1}{16^2}\Delta = [(x^2+y^2+z^2)^2-16]\{-x^2(x^2+y^2+z^2-4)[x^2+3(y^2+z^2)] - (y^2+z^2)[(y^2+z^2+4)^2-x^4]\}.$$

The detailed computations of $\Delta$ were done in [1]. Therefore, the obvious equality

$$\Delta(x,y,z) = \Delta(x, \sqrt{y^2+z^2}, 0)$$

can now be exploited to obtain

$$\frac{1}{16^2}\Delta = (x^2+3y^2+3z^2+4)\delta_2\left(x, \sqrt{y^2+z^2}\right)$$

Therefore, the Gauss curvature

$$K_G(x,y,z) = \frac{-\Delta}{\left(f_x^2+f_y^2+f_z^2\right)^{3/2}} = \frac{16^2(x^2+3y^2+3z^2+4)\left[-\delta_2\left(x,\sqrt{y^2+z^2}\right)\right]}{\left(f_x^2+f_y^2+f_z^2\right)^{3/2}}$$

is positive over $f^{-1}(s)$ if and only if over $f_2^{-1}(s) = f^{-1}(s)\cap(z=0)$
$\delta(x,\pm y) = \delta_2\left(x,\sqrt{y^2+z^2}\right) < 0.$

But, according to [12], the determinant $\delta_2$ is negative over $f_2^{-1}(s)$ if and only if one has $s^2 - 3\cdot 16 s > 0$, i.e. $s\in(-\infty,0]\cup[3\cdot 16,+\infty)$.

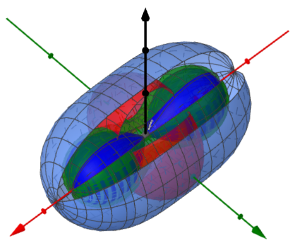

FIGURE 4. The 0-critical level of $f$, a nonconvex regular level of $f$, The boundary of $\mathrm{Hess}^+(f)$ and the first convex regular level of $f$

## 4. One parameter families of one-to-one smooth maps

In this section we shall exploit the Theorem 3.4 to produce some one parameter families of one-to-one smooth maps of Loewner chain flavor. In this respect we first consider a smooth norm-coercive function $f:\mathbb{R}^3\longrightarrow\mathbb{R}$ with bounded critical set and bounded $\mathrm{Hess}^+(f)$ complement region. Then the level surface $f^{-1}(c)$ is a compact connected regular ovaloid for every $c>\max\{h_{\max}(f),\mu_{\max}(f)\}$, one of which is fixed.

### 4.1. The Gauss map outcome smooth homeomorphism of an ovaloidal level set

Therefore, the inside of $f^{-1}(c)$, i.e. $\widehat{\overset{\circ}{M_c(f)}} = f^{-1}(-\infty,c) = f^{-1}[\min f, c)$, is convex open set and its closure $M_c(f) = f^{-1}(-\infty,c] = f^{-1}[\min f, c]$ is a compact convex set. Moreover, the restriction and co-restriction of the normalized gradient

$$G_c : f^{-1}(c)\longrightarrow S^2,\ G_c(x) = \frac{\nabla_x f}{\|\nabla_x f\|},$$

which is the Gauss map of the surface $f^{-1}(c)$ is, according with the Hadamard-Stoker theorem, a diffeomorphism (see [10, 178]). Therefore, the map

$$F^c_{x_0} : D^3 \longrightarrow M_c(f),\ F^c_{x_0}(x) = \begin{cases} x_0 + \exp \dfrac{\|x\|^4 - 1}{\|x\|^2} \left[ -x_0 + G_c^{-1}\left(\dfrac{x}{\|x\|}\right)\right] & \text{if } x \neq 0 \\ x_0 & \text{if } x = 0. \end{cases}$$

is a smooth homeomorphism for every $x_0 \in \widehat{\overset{\circ}{M_c(f)}}$. In order to justify this statement, we first observe that the map

$$\varphi : D^3 \longrightarrow [0,1],\ \varphi(x) = \begin{cases} \exp \dfrac{\|x\|^4 - 1}{\|x\|^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

is well-defined, smooth and for each of its level sphere

$$\varphi^{-1}(l) = S\left(0, \sqrt{\frac{(\ln l)^2 + \sqrt{(\ln l)^2 + 4}}{2}}\right),\ 0 \leq l \leq 1$$

the restriction and co-restriction

$$\varphi^{-1}(l) \longrightarrow x_0 + l(-x_0 + f^{-1}(c)),\ x \longmapsto x_0 + l \cdot \left(-x_0 + G_c^{-1}\left(\frac{x}{\|x\|}\right)\right)$$

is obviously a diffeomorphism, as $G_c^{-1}$ realizes a diffeomorphism between $S^2$ and $f^{-1}(c)$. In other words the restriction and co-restriction of $F^c_{x_0}$ to each of the leaves of the foliation

$$\{\varphi^{-1}(l)\}_{0<l\leq 1} \text{ of } D^3 \setminus \{0\} \tag{4.1}$$

to the corresponding leave of the foliation $\{x_0 + l(-x_0 + f^{-1}(c))\}_{0<l\leq 1}$ of $M_c(f) \setminus \{x_0\}$ is a diffeomorphism. On the other hand the restriction and co-restriction of $F^c_{x_0}$ to each of the leaves of the orthogonal foliation

$$\{]0x]\}_{x\in S^2} \tag{4.2}$$

is also a diffeomorphism onto the corresponding leave of the foliation

$$\left\{\left]x_0, G_c^{-1}\left(\frac{x}{\|x\|}\right)\right]\right\}_{x\in f^{-1}(c))},$$

of $M_c(f) \setminus \{x_0\}$, which is transversal to the foliation $\{x_0 + l(-x_0 + f^{-1}(c))\}_{0<l\leq 1}$, as for every $x \in S^2$ and $t \in ]0,1]$ we have

$$\frac{d}{dt}\varphi(tx) = \frac{d}{dt}\left(t^2\|x\|^2 - \frac{1}{t^2\|x\|^2}\right)\exp\frac{t^4\|x\|^4 - 1}{t^2\|x\|^2} = \left(2t + 2\frac{2}{t^3}\right)\exp\frac{t^4\|x\|^4 - 1}{t^2\|x\|^2} > 0.$$

Therefore $F^c_{x_0}$ is bijective and its Fréchet differential $(dF^c_{x_0})_x$ is an isomorphism at every point $x \in D^3 \setminus \{x_0\}$ as its restrictions to the orthogonal complement subpaces of $T_x(D^3)$, one of which is the tangent space to the leave through $x$ of the foliation (4.1) and its orthogonal complement, i.e. the tangent space to the leave through $x$ of the foliation (4.2), are one-to-one. Consequently $F^c_{x_0}$ is a global smooth homeomorphism on $D^3$ onto $M_c(f)$ and its restriction and co-restriction $D^3 \setminus \{0\} \longrightarrow M_c(f) \setminus \{x_0\},\ x \longmapsto F^c_{x_0}(x)$ is a diffeomorphism, but $(dF^c_{x_0})_0 = 0$.

### 4.2. Vector fields with bounded norm on $M_c(f)$ and their global flows

In this subsection we will point out quite a large family of diffeomorphisms from $D^3$ to $M_c(f)$ which is parametrized by $\mathbb{R} \times GL_3(\mathbb{R}) \times \widehat{\overset{\circ}{M_c(f)}}$ whenever $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ is a $C^2$-smooth function whose critical set $C(f)$ and $\mathrm{Hess}^+(f)$ complement are bounded. For such a function we rely on Theorem 3.4 to conclude that the level surface $f^{-1}(c)$ is a compact connected regular ovaloid for $c$ sufficiently large.

For every nonsingular linear operator $A : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$, i.e. its matrix representation $[A]$ is nonsingular, and every point $x_0 \in \widehat{\overset{\circ}{M_c(f)}}$ we consider the vector field $X \in \mathfrak{X}(\mathbb{R}^3)$ defiend by $X_x = A(x - x_0)$ as well as the smooth function $h := b \circ (F^c_{x_0})^{-1} : M_c(f) \longrightarrow \mathbb{R}$, where $x_0 \in M_c(f)$ and $b : D^3 \longrightarrow \mathbb{R}$ is a bump function such that $b\big|_{D^3_{1-\varepsilon}} \equiv 1$, $b\big|_{\mathbb{R}^3 \setminus D^3_{1-\varepsilon/2}} \equiv 0$ for some $\varepsilon > 0$ sufficiently small and $D^3_r$ stands for the disc centered at $0 \in \mathbb{R}^3$ of radius $r > 0$. Now the vector field $hX$ has obviously bounded norm, which make it completely integrable [11, Corollary 9.1.5, p. 183], i.e. there exists a global one-parameter group of diffeomorphisms $\Psi^{c,x_0} : \mathbb{R} \times M_c(f) \longrightarrow M_c(f)$ such that

$$\frac{d}{dt}\Psi^{c,x_0}_t(x) = X_{\Psi^{c,x_0}_t(x)},\ \forall x \in M_c(f),$$

where $\Psi^{c,x_0}_t : M_c(f) \longrightarrow M_c(f),\ \Psi^{c,x_0}_t(x) = \Psi^{c,x_0}(t,x)$ is a diffeomorphism for every $t \in \mathbb{R}$, which implies that $(d\Psi^{c,x_0}_t)_x : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is an isomorphism for every $x \in M_c(f)$. Indeed, the following properties

1. $\Psi^{c,x_0}_0 = id_{M_c(f)}$;
2. $\Psi^{c,x_0}_s \circ \Psi^{c,x_0}_t = \Psi^{c,x_0}_{s+t}$ for all $s, t \in \mathbb{R}$

hold. In fact $\Psi^{c,x_0}_t(x) = x_0 + Ae^{tA}(x - x_0)$ for every $x \in (F^c_{x_0})^{-1}(D_{1-\varepsilon})$, as can be easily seen, and $d_x\Psi^{c,x_0}_t(\cdot) = Ae^{tA}$ for every such $x$. On the other hand, $\Psi^{c,x_0}_t(x) = x$, for all $x \in f^{-1}(c - \varepsilon/3, c)$ and all $t \in \mathbb{R}$ as $(hX)_x = 0$ for all $x \in f^{-1}(c - \varepsilon/3, c)$.

### 4.3. A gradient homothetic vector field whose flow permutes the sublevel sets

In this subsection we additionally assume that $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ satisfies the Palais-Smale condition, i.e. every sequence $(x_n)$ such that $(df)_{x_n} \longrightarrow 0$ as $n \longrightarrow \infty$ has a convergent subsequence. For example every norm-coercive function such that $\|\nabla_x f\| \longrightarrow \infty$ as $x \longrightarrow \infty$ has this property. Since $(\nabla f)f = \|\nabla f\|^2$ , on the set $\mathbb{R}^3 \setminus C(f)$ of regular points, where $\nabla f \neq 0$, the smooth vector field

$$Y = \pm\frac{\nabla f}{\|\nabla f\|^2}$$

satisfies $Yf = \pm 1$. More generally if $F : \mathbb{R} \longrightarrow \mathbb{R}$ is any smooth bounded function vanishing in a neighborhood of $B(f) = f(C(f))$, then, following [11, Section 9.3], we consider the smooth vector field $X = (F \circ f)Y$ on $\mathbb{R}^3$ that vanishes in a neighborhood of $C(f)$, and $X(f) = \pm(F \circ f)$. We denote by $\Phi_t$ the flow on $\mathbb{R}^3$ generated by $X$. Let us choose $F : \mathbb{R} \longrightarrow \mathbb{R}$ to be a smooth, non-negative function that is identically one on a neighborhood of $[c,d]$ and zero outside $(c-\varepsilon/2, d+\varepsilon/2)$, where $d$ might be infinity. In the later case the intervals $[c,d]$ and $(c-\varepsilon/2, d+\varepsilon/2)$ are understood as $[c,+\infty)$ and $(c-\varepsilon/2,+\infty)$ respectively. Since $\|Y\| = 1/\|\nabla f\|$ and $\|X\| = \|\nabla f\| \cdot |F\circ f|$ along with the boundedness of $F$ and the vanishing of $|F\circ f|$ outside $f^{-1}([c-\varepsilon/2, d+\varepsilon/2])$, one can show, following the same lines with those in the proof of [11, Proposition 9.3.1] that the vector field $X$ has on $\mathbb{R}^3$ bounded length and hence its associated flow $\Phi_t$ generates a one-parameter group of diffeomorphisms of $\mathbb{R}^3$. We denote by $\gamma(t,c')$ the solution of the ordinary differential equation

$$\frac{d\gamma}{dt} = \pm F(\gamma) \tag{4.3}$$

with initial value $c'$. Since

$$\frac{d}{dt}(f\circ\Phi_t(x)) = X_{\Phi_t(x)}f = \pm(F\circ f)(\Phi_t(x)), \tag{4.4}$$

it follows that $f(\Phi_t(x)) = \gamma(t, f(x))$, and hence that $\Phi_t(f^{-1}(c')) = f^{-1}(\gamma(t,c'))$. On the other hand $\gamma(t,c')$ is bounded and its range is contained in the interval $[\min\{0, c-\varepsilon/2\}, \max\{0, d+\varepsilon/2\}]$, as $F$ vanishes on $\mathbb{R}\setminus(c-\varepsilon/2, d+\varepsilon/2)$, and the flow $\Phi_t$ permutes the level sets of $f$ on the other hand. From the definition of $\gamma(t,c')$ it follows that $\gamma(t,c') = c' \pm t$ for $c' \in [c,d]$ and $c' \pm t \in [c,d]$, while $\gamma(t,c') = c'$ if $c' > d+\varepsilon$ or $c' < c-\varepsilon$. Therefore, the range of $\gamma(t,c')$ is an interval with the endpoints $i_c, s_c$, where $i_c = \inf \gamma(\cdot, c)$ and $s_c = \sup\gamma(\cdot,c)$ for every $c' \in (c-\varepsilon/2, d+\varepsilon/2)$ and this range is the singleton $\{c'\}$ for every $c' \in \mathrm{Im}(f)\setminus(c-\varepsilon/2, d+\varepsilon/2)$, as $\gamma(t,c') = c'+t$ for $c' \in [c,d]$ and $c \le c'+t \le d$, while $\gamma(t,c') = c'$ if $c' \ge d+\varepsilon/2$ or $c' \le c-\varepsilon/2$.

### 4.4. One-parameter families of smooth homeomorphisms associated to the flow $\Phi_t$

This subsection is devoted to the one-parameter family of smooth homeomorphisms

$$G_s^{c,x_0} : D^3 \longrightarrow \mathbb{R}^3,\ G_s^{c,x_0} = \Phi_s \circ g,$$

where $c > \max\{h_{\max}(f), \mu_{\max}(f)\}$, $x_0 \in \widehat{\overset{\circ}{M_c(f)}}$, $\mu_{\max}(f) := \max f\big|_{C(f)}$ and $g : D^3 \longrightarrow \mathbb{R}^3$ is a smooth homeomorphism such that $M_c(f) = g(D^3)$. Examples of such smooth homeomorphisms are $F^c_{x_0}$ or $\Psi_t^{c,x_0}$ for some real parameter $t$. Note that for the later option $G_s^{c,x_0}$ is a diffeomorphism. We also consider $d > c$ and $\varepsilon > 0$ such that $[c-\varepsilon, c+\varepsilon]$ remains an interval of regular values of $f$

**Theorem 4.1.** *The one-parameter family* $\{G_s^{c,x_0}\}_{0\le s<+\infty}$ *has the following properties:*

1. *each function* $G_s^{c,x_0}$ *is a smooth homeomorphism (diffeomorphism for* $g = \Psi_t^{c,x_0}$*)*
2. $G_s^{c,x_0}(D^3) \subseteq G_t^{c,x_0}(D^3)$, $\forall 0 \le s < t < d \le +\infty$ *for the* $''+''$ *option in* (4.3).
3. $G_s^{c,x_0}(D^3) \supseteq G_t^{c,x_0}(D^3)$, $\forall 0 \le s < t < d \le +\infty$ *for the* $''-''$ *option in* (4.3).
4. $|G_s^{c,x_0}(x) - G_t^{c,x_0}(x)| \le \displaystyle\int_s^t \|\nabla_{\Phi_r(g(x))} f\| dr.$
5. $\displaystyle\bigcup_{t\ge 0} G_t^{c,x_0}(D^3) = \mathbb{R}^3$ *for the* $''+''$ *option in* (4.3) *and* $d = +\infty$.
6. $\widehat{\overset{\circ}{M_{s_c}(f)}} \subseteq \displaystyle\bigcup_{t\ge 0} G_t^{c,x_0}(D^3) \subseteq M_{s_c}(f)$ *for the* $''+''$ *option in* (4.3) *and* $d < +\infty$.

7. $\widehat{\overset{\circ}{M_{i_c}}(f)} \subseteq \bigcap_{t\geq 0} G_t^{c,x_0}(D^3) \subseteq M_{i_c}(f)$, *for the* $''-''$ *option in* (4.3) *and* $d < +\infty$.

*Proof.* (1) Obvious
(2) By using (4.4) for the $''+''$ option in (4.3) and taking into account that $F \equiv 1 > 0$ on $[c, +\infty)$ we deduce that the real-valued function of one real variable $f \circ \Phi_t$ is nondecreasing and for $t \geq 0$ we obtain that $(f \circ \Phi_{-t})(x) \leq (f \circ \Phi_0)(x) = f(x)$. This shows that $M_c(f)$ is invariant under the action of $\Phi_{-t}$, as $f(x) \leq c \Longrightarrow f(\Phi_{-t}(x)) \leq c$. But $\Phi_{-t}(M_c(f)) \subseteq M_c(f)$ is equivalent with $M_c(f) \subseteq \Phi_t(M_c(f))$. Now, for the required inclusion we have

$$G_s^{c,x_0}(D^3) \subseteq G_t^{c,x_0}(D^3) \Longleftrightarrow \Phi_s(g(D^3)) \subseteq \Phi_t(g(D^3)) \Longleftrightarrow g(D^3) \subseteq \Phi_{t-s}(g(D^3)) \Longleftrightarrow M_c(f) \subseteq \Phi_{t-s}(M_c(f)),$$

which holds true as $t - s \geq 0$.
(3) Similar with (2).
(4) Since $\dfrac{d}{dt}\Phi_t(x) = X_{\Phi_t(x)} \Longleftrightarrow \Phi_t(x) = x + \displaystyle\int_0^t X_{\Phi_r(x)} dr$ we have successively:

$$|G_s^{c,x_0}(x) - G_t^{c,x_0}(x)| = \left| \int_0^s X_{\Phi_r(x)} dr - \int_0^t X_{\Phi_r(x)} dr \right| = \left| \int_s^t X_{\Phi_r(x)} dr \right| \leq \int_s^t \| X_{\Phi_r(x)} dr \| = \int_s^t |(F \circ f)(\Phi_r(x))| \cdot \|\nabla_{\Phi_r(x)} f\| dr \leq \int_s^t \|\nabla_{\Phi_r(x)} f\| dr.$$

(5) Indeed, we have successively:

$$\bigcup_{t\geq 0} G_t^{c,x_0}(D^3) = \bigcup_{t\geq 0} \Phi_t \left( \widehat{\overset{\circ}{M_c}(f)} \cup f^{-1}(c) \right) = \bigcup_{t\geq 0} \Phi_t \left( \widehat{\overset{\circ}{M_c}(f)} \right) \cup \bigcup_{t\geq 0} \Phi_t(f^{-1}(c))$$
$$= \bigcup_{t\geq 0} \Phi_t \left( \widehat{\overset{\circ}{M_c}(f)} \right) \cup \bigcup_{t\geq 0} \Phi_t(f^{-1}(\gamma(t, c))$$
$$= \bigcup_{t\geq 0} \Phi_t \left( \widehat{\overset{\circ}{M_c}(f)} \right) \cup \bigcup_{t\geq 0} f^{-1}(c+t) = \bigcup_{t\geq 0} \widehat{\overset{\circ}{\Phi_t (M_c(f))}} \cup \bigcup_{t\geq 0} f^{-1}(c+t)$$

But $M_c(f) \subseteq \Phi_t(M_c(f))$ implies $\widehat{\overset{\circ}{M_c}(f)} \subseteq \widehat{\overset{\circ}{\Phi_t(M_c(f))}} = \Phi_t \left( \widehat{\overset{\circ}{M_c}(f)} \right)$. Therefore $\widehat{\overset{\circ}{\Phi_t (M_c(f))}} = \widehat{\overset{\circ}{M_c}(f)} \cup \left( \widehat{\overset{\circ}{\Phi_t (M_c(f))}} \setminus \widehat{\overset{\circ}{M_c}(f)} \right)$, namely

$$\bigcup_{t\geq 0} \widehat{\overset{\circ}{M_c}(f)} \cup f^{-1}(c+t) = \widehat{\overset{\circ}{M_c}(f)} \cup \bigcup_{t\geq 0} f^{-1}(c+t) = M_c(f) \cup \bigcup_{t>0} f^{-1}(c+t)$$
$$= \bigcup_{\min(f)\leq r\leq c} f^{-1}(r) \cup \bigcup_{t>0} f^{-1}(c+t)$$
$$= \bigcup_{t\geq 0} \left( \widehat{\overset{\circ}{\Phi_t (M_c(f))}} \setminus \widehat{\overset{\circ}{M_c}(f)} \right) \cup \mathbb{R}^3 = \mathbb{R}^3.$$

Thus

$$\begin{aligned}\bigcup_{t\geq 0} G_t^{c,x_0}(D^3) &= \bigcup_{t\geq 0} \overset{\circ}{\widehat{\Phi_t\left(M_c(f)\right)}} \cup \bigcup_{t\geq 0} f^{-1}(c+t)\\ &= \bigcup_{t\geq 0}\left(\overset{\circ}{\widehat{\Phi_t\left(M_c(f)\right)}} \setminus \overset{\circ}{\widehat{M_c(f)}}\right) \cup \bigcup_{t\geq 0} \overset{\circ}{\widehat{M_c(f)}} \cup f^{-1}(c+t)\\ &= \bigcup_{t\geq 0}\left(\overset{\circ}{\widehat{\Phi_t\left(M_c(f)\right)}} \setminus \overset{\circ}{\widehat{M_c(f)}}\right) \cup \mathbb{R}^3 = \mathbb{R}^3.\end{aligned}$$

(6) Since $\dfrac{d}{dt}(f\circ\Phi_t(x)) = (F\circ f)(\Phi_t(x)) \geq 0$, it follows that $\gamma(t, f(x)) = f(\Phi_t(x))$ is nondecreasing. Thus

$$\begin{aligned}\bigcup_{t\geq 0} G_t^{c,x_0}(D^3) = \bigcup_{t\geq 0}\Phi_t\left(M_c(f)\right) &= \bigcup_{t\geq 0}\Phi_t\left(\overset{\circ}{\widehat{M_c(f)}} \cup f^{-1}(c)\right)\\ = \bigcup_{t\geq 0}\Phi_t\left(\overset{\circ}{\widehat{M_c(f)}}\right) \cup \bigcup_{t\geq 0}\Phi_t(f^{-1}(c)) &= \bigcup_{t\geq 0}\Phi_t\left(\overset{\circ}{\widehat{M_c(f)}}\right) \cup \bigcup_{t\geq 0} f^{-1}(\gamma(t,c))\end{aligned}$$

Since $[c, s_c) \subseteq \{\gamma(t,c) \mid t \geq 0\} \subseteq [c, s_c]$ it follows that

$$\bigcup_{c\leq u<s_c} f^{-1}(u) \subseteq \bigcup_{t\geq 0} f^{-1}(\gamma(t,c)) \subseteq \bigcup_{c\leq v\leq s_c} f^{-1}(v) \tag{4.5}$$

The left hand side inclusion of (4.5) implies that

$$\bigcup_{t\geq 0}\Phi_t\left(\overset{\circ}{\widehat{M_c(f)}}\right) \cup \bigcup_{c\leq u<s_c} f^{-1}(u) \subseteq \bigcup_{t\geq 0}\Phi_t\left(\overset{\circ}{\widehat{M_c(f)}}\right) \cup \bigcup_{t\geq 0} f^{-1}(\gamma(t,c)) = \bigcup_{t\geq 0} G_t^{c,x_0}(D^3)$$

$$i.e. \bigcup_{t\geq 0}\left(\overset{\circ}{\widehat{\Phi_t\left(M_c(f)\right)}} \setminus \overset{\circ}{\widehat{M_c(f)}}\right) \cup \overset{\circ}{\widehat{M_c(f)}} \cup \bigcup_{c\leq u<s_c} f^{-1}(u) \subseteq \bigcup_{t\geq 0} G_t^{c,x_0}(D^3)$$

$$\Longleftrightarrow \bigcup_{t\geq 0}\left(\overset{\circ}{\widehat{\Phi_t\left(M_c(f)\right)}} \setminus \overset{\circ}{\widehat{M_c(f)}}\right) \cup \overset{\circ}{\widehat{M_{s_c}(f)}} \subseteq \bigcup_{t\geq 0} G_t^{c,x_0}(D^3) \Longrightarrow \overset{\circ}{\widehat{M_{s_c}(f)}} \subseteq \bigcup_{t\geq 0} G_t^{c,x_0}(D^3).$$

The right hand side inclusion of (4.5) implies that

$$\bigcup_{t\geq 0} G_t^{c,x_0}(D^3) = \bigcup_{t\geq 0}\Phi_t\left(\overset{\circ}{\widehat{M_c(f)}}\right) \cup \bigcup_{t\geq 0} f^{-1}(\gamma(t,c)) \subseteq \bigcup_{t\geq 0}\Phi_t\left(\overset{\circ}{\widehat{M_c(f)}}\right) \cup \bigcup_{c\leq v\leq s_c} f^{-1}(v)$$

$$i.e. \bigcup_{t\geq 0} G_t^{c,x_0}(D^3) \subseteq \bigcup_{t\geq 0}\Phi_t\left(\overset{\circ}{\widehat{M_c(f)}}\right) \cup \bigcup_{c\leq v\leq s_c} f^{-1}(v)$$

$$\Longleftrightarrow \bigcup_{t\geq 0} G_t^{c,x_0}(D^3) \subseteq \bigcup_{t\geq 0}\left(\overset{\circ}{\widehat{\Phi_t\left(M_c(f)\right)}} \setminus \overset{\circ}{\widehat{M_c(f)}}\right) \cup \overset{\circ}{\widehat{M_c(f)}} \cup \bigcup_{c\leq v\leq s_c} f^{-1}(v)$$

$$\Longleftrightarrow \bigcup_{t\geq 0} G_t^{c,x_0}(D^3) \subseteq \bigcup_{t\geq 0}\left(\overset{\circ}{\widehat{\Phi_t\left(M_c(f)\right)}} \setminus \overset{\circ}{\widehat{M_c(f)}}\right) \cup M_{s_c}(f) \Leftrightarrow \bigcup_{t\geq 0} G_t^{c,x_0}(D^3) \subseteq M_{s_c}(f),$$

as $\gamma(t,c) = f\left(\Phi_t\left(x\right)\right) \leq s_c$, for all $x \in M_c(f)$.

(7) Similar with (5). □

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# Geometric properties of normalized imaginary error function

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*Dedicated to the memory of Professor Gabriela Kohr*

**Abstract.** The error function takes place in a wide range in the fields of mathematics, mathematical physics and natural sciences. The aim of the current paper is to investigate certain properties such as univalence and close-to-convexity of normalized imaginary error function, which its region is symmetric with respect to the real axis. Some other outcomes are also obtained.

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**Keywords:** Univalent function, close-to-convex, error function, imaginary error function.

## 1. Introduction and preliminaries

The *error function* plays a significant role in different fields of science including statistics, probability, partial differential equations, and many engineering problems. Therefore, it is attracted much attentions in mathematics. Specially, several remarkable inequalities and related topics for the error function were reported, see for examples [10, 12, 14, 15]. The error function and its approximations are usually employed to forecast outcomes that hold with high or low probability.

The error function, which is denoted by the symbol erf and defined by [1, p. 297]

$$\operatorname{erf}(z)=\frac{2}{\sqrt{\pi}}\int_0^z \exp(-x^2)\mathrm{d}x=\frac{2}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{(-1)^k z^{2k+1}}{(2k+1)k!} \tag{1.1}$$

for every complex number $z\in\mathbb{C}$ and is a subject of intensive studies and recently applications. The integral of relation (1.1) cannot be assessed in closed form in terms of elementary functions, but by wringing the integrand $e^{-z^2}$ as Maclaurin series and

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integrating term by term, the error function's Maclaurin series is obtained as it is shown above. Also, the *imaginary error function*, denoted by the symbol erfi, has a very similar Maclaurin series, which is defined by

$$\operatorname{erfi}(z)=\frac{2}{\sqrt{\pi}}\int_0^z \exp(x^2)\mathrm{d}x=\frac{2}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{z^{2k+1}}{(2k+1)k!}, \tag{1.2}$$

for every $z\in\mathbb{C}$. Some inequalities and properties of error function can be seen in [3, 11].

Before presenting our results, some basic definitions are first stated. Let $\mathbb{U}=\{z\in\mathbb{C}:|z|<1\}$ and let $\mathcal{A}$ be the class of normalized analytic functions $f$ in $\mathbb{U}$ that has the following power series expansion

$$f(z)=z+\sum_{j=2}^{\infty}a_j z^j,\ z\in\mathbb{U}, \tag{1.3}$$

and denote by $\mathcal{S}$ the class of univalent functions that belong to $\mathcal{A}$.

In 2018, Ramachandran *et al.* [20] studied the normalized analytic error function, which is obtained from (1.1) and of the form

$$\operatorname{Erf}(z)=\frac{\sqrt{\pi z}}{2}\operatorname{erf}(\sqrt{z})=z+\sum_{j=2}^{\infty}\frac{(-1)^{j-1}}{(2j-1)(j-1)!}z^j,$$

and defined a class of analytic functions having the following representation

$$\operatorname{Erf}*\mathcal{A}=\left\{\mathcal{L}:\mathcal{L}(z)=(\operatorname{Erf}*f)(z)=z+\sum_{j=2}^{\infty}\frac{(-1)^{n-1}a_j}{(2j-1)(j-1)!}z^j, f\in\mathcal{A}\right\},$$

where the symbol "$*$" represents the Hadamard (or convolution) product, while Erf denotes the class that consists of the single function Erf.

Let Erfi be the normalized form of the error function which is obtained from (1.2) and defined by

$$\operatorname{Erfi}(z)=\frac{\sqrt{\pi z}}{2}\operatorname{erfi}(\sqrt{z})=z+\sum_{j=2}^{\infty}\frac{z^j}{(2j-1)(j-1)!}. \tag{1.4}$$

We denote by $\mathcal{S}^*(\gamma)$ and $\mathcal{C}(\gamma)$ the classes of $\mathcal{A}$ consisting of functions which are *starlike of order* $\gamma$ and *convex of order* $\gamma$, that is,

$$\mathcal{S}^*(\gamma):=\left\{f\in\mathcal{A}:\operatorname{Re}\frac{zf'(z)}{f(z)}>\gamma,\ z\in\mathbb{U}\right\}\ (0\le\gamma<1)$$

and

$$\mathcal{C}(\gamma):=\left\{f\in\mathcal{A}:\operatorname{Re}\frac{(zf'(z))'}{f'(z)}>\gamma,\ z\in\mathbb{U}\right\}\ (0\le\gamma<1)$$

respectively. Specifically, $\mathcal{S}^*:=\mathcal{S}^*(0)$ and $\mathcal{C}:=\mathcal{C}(0)$ are the classes of *starlike functions* and *convex functions* in $\mathbb{U}$, respectively.

Furthermore, we represent by $\mathcal{K}(\gamma)$ the subclass of $\mathcal{A}$ consisting of functions which are *close-to-convex of order* $\gamma$, that is,

$$\operatorname{Re}\frac{f'(z)}{g'(z)} > \gamma,\ z \in \mathbb{U},\ (0 \leq \gamma < 1)$$

for some function $g \in \mathcal{C}$. In particular, $\mathcal{K} := \mathcal{K}(0)$ is the class of *close-to-convex functions* in $\mathbb{U}$.

In the recent years a serious attention was attracted on the geometric and other related properties as univalence, convexity and starlikeness of various special functions like the normalized forms of Bessel, Struve and Lommel functions of the first kind. In this area, some authors obtained many applications in the Geometric Functions Theory for these special functions, and see for examples the articles [2, 4, 5, 6, 7, 8, 9, 16, 18, 19]. The aim of the present paper is to investigate some properties such as univalence and close-to-convexity of normalized imaginary error function, which maps the open unit disk in a domain that is symmetric with respect to the real axis. Some other outcomes are also presented.

## 2. Main results

In this section we study some geometric and other related properties of normalized imaginary error function in the open unit disk. To prove our outcomes we require the following lemmas.

**Lemma 2.1.** [13], (see also [21, p. 59]) *Let* $h(z) = \sum_{k=1}^{\infty} b_k z^{k-1}$ *be analytic in* $\mathbb{U}$ *such that* $\{b_k\}_{k\geq 1}$ *is a sequence with* $b_1 = 1$ *and* $b_k \geq 0$ *for all* $k \geq 2$*. If* $\{b_k\}_{k\geq 2}$ *is a convex decreasing sequence, that is,*

$$0 \geq b_{k+2} - b_{k+1} \geq b_{k+1} - b_k \quad \textit{for all} \quad k \geq 2.$$

*Then*

$$\operatorname{Re}\left(\sum_{k=1}^{\infty} b_k z^{k-1}\right) > \frac{1}{2} \quad (z \in \mathbb{U}).$$

**Lemma 2.2.** [17, Corollary 7] *Let* $h(z) = z + \sum_{k=2}^{\infty} b_k z^k$ *be analytic in* $\mathbb{U}$*. If*

$$1 \geq 2b_2 \geq \ldots \geq kb_k \geq (k+1)b_{k+1} \geq \ldots \geq 0,$$

*or*

$$1 \leq 2b_2 \geq \ldots \leq kb_k \leq (k+1)b_{k+1} \leq \ldots \leq 2,$$

*then the function* $h$ *is close-to-convex with respect to the convex function* $-\log(1-z)$ *in* $\mathbb{U}$*.*

Letting $q \to 1^-$ in Theorem 2.6 given in Sahoo and Sharma [22], we obtain the next lemma:

**Lemma 2.3.** *Let* $h(z)=z+\sum\limits_{k=1}^{\infty} b_{2k+1}z^{2k+1}$ *be an analytic and odd function* $\mathbb{U}$. *If*

$$1 \geq 3b_3 \geq \ldots \geq (2k+1)b_{2k+1} \geq \ldots \geq 0,$$

*or*

$$1 \leq 3b_3 \leq \ldots \leq (2k+1)b_{2k+1} \leq \ldots \leq 2,$$

*then the function* $h$ *is close-to-convex with respect to the function* $\frac{1}{2}\log\frac{1+z}{1-z}$ *, which is convex in* $\mathbb{U}$.

Now we obtain the following results applying the lemmas mentioned above.

**Theorem 2.4.** *The normalized error function* Erfi *is close-to-convex in* $\mathbb{U}$ *with respect to the convex function* $-\log(1-z)$.

*Proof.* From (1.4) it follows that

$$ja_j-(j+1)a_{j+1}=\frac{1}{j!}\left[\frac{j^2}{(2j-1)}-\frac{j+1}{(2j+1)}\right]=\frac{1}{(4k^2-1)j!}\nu(j),$$

where

$$\nu(j)=2j^3-j^2-j+1.$$

According to Lemma 2.2, it is enough to prove that $\nu(j)\geq 0$ for all $j\geq 1$. Setting

$$\varphi(x):=2x^3-x^2-x+1,\ x\geq 1,$$

the function $\varphi$ is strictly increasing on $[1,+\infty)$, hence $\min\{\varphi(x):x\geq 1\}=\varphi(1)=1$. Therefore, since

$$\nu(j)=2j^3-j^2-j+1\geq 0,$$

for all $j\geq 1$, we get our result. □

**Theorem 2.5.** *The normalized imaginary error odd function* $\operatorname{Eri}(z)=(\sqrt{\pi}/2)\operatorname{erfi}(z)$ *is close-to-convex in* $\mathbb{U}$ *with respect to the convex function* $(1/2)\log((1+z)/(1-z))$.

*Proof.* Since

$$\operatorname{Eri}(z)=\frac{\sqrt{\pi}}{2}\operatorname{erfi}(z)=z+\sum_{j=2}^{\infty} l_{2j-1}z^{2j-1}\quad (z\in\mathbb{U}),$$

where

$$l_{2j-1}=\frac{1}{(2j-1)(j-1)!},$$

then $l_{2k-1}>0$ for all $j\geq 2$. To prove our result, from Lemma 2.3, it is sufficient to show that $\{(2j-1)l_{2j-1}\}_{j\geq 2}$ is a non-increasing sequence. If $j\geq 2$, a simple computation shows that

$$(2j-1)l_{2j-1}-(2j+1)l_{2j+1}=\frac{2j-1}{(2j-1)(j-1)!}-\frac{2j+1}{(2j+1)j!}=\frac{1}{j!}(j-1)\geq 0.$$

Therefore, the sequence $\{(2j-1)l_{2j-1}\}_{j\geq 2}$ is a non-increasing and form Lemma 2.3 our result follows. □

**Remark 2.6.** According to Theorems 2.4 and 2.5, the both functions Erfi and $\mathrm{Eri}(z) = \frac{\sqrt{\pi}}{2}\,\mathrm{erfi}(z)$ are univalent in $\mathbb{U}$.

**Theorem 2.7.** *For the function* Erfi*, the following inequality holds:*

$$\mathrm{Re}\,\frac{\mathrm{Erfi}(z)}{z} > \frac{1}{2} \quad (z \in \mathbb{U}).$$

*Proof.* To prove of the result, according to Lemma 2.1 it is sufficient to show that

$$\{a_j\}_{j=2}^{\infty} = \left\{\frac{1}{(2j-1)(j-1)!}\right\}_{j=2}^{\infty}$$

is a convex decreasing sequence, i.e., we need to prove that

$$a_j - a_{j+1} \geq 0 \tag{2.1}$$

and

$$a_{j+2} - a_{j+1} \geq a_{j+1} - a_j \tag{2.2}$$

for all $j \geq 2$.

First, we will prove (2.2) which is equivalent to

$$a_j - 2a_{j+1} + a_{j+2} \geq 0,\ j \geq 2.$$

A simple computation shows that

$$a_j - 2a_{j+1} = \frac{1}{(2j-1)(j-1)!} - \frac{2}{(2j+1)j!} = \frac{1}{(4j^2-1)j!}\mu(j),$$

where

$$\mu(j) := 2j^2 - 3j + 2.$$

Setting

$$\psi(x) := 2x^2 - 3x + 2,\ x \geq 2,$$

the function $f$ is strictly increasing on $[2, +\infty)$, hence $\min\{\psi(x) : x \geq 2\} = \psi(2) = 4$. Therefore, $a_j - 2a_{j+1} \geq 0$ for all $j \geq 2$. Using the fact that $a_{j+1} > 0$ for all $j \geq 2$, it follows $a_j - a_{j+1} \geq a_{j+1} > 0$ which is (2.1). From $a_{j+2} > 0$ for $j \geq 2$, the above inequality implies that $a_j - 2a_{j+1} + a_{j+2} \geq 0$ for all $j \geq 2$. Hence (2.2) holds. Therefore, from Lemma 2.1 we deduce that

$$\mathrm{Re}\left(\sum_{j=1}^{\infty} a_j z^{j-1}\right) > \frac{1}{2} \quad (z \in \mathbb{U}),$$

which gives

$$\mathrm{Re}\,\frac{\mathrm{Erfi}(z)}{z} > \frac{1}{2} \quad (z \in \mathbb{U})$$

and hence the proof is completed. □

**Theorem 2.8.** *For the function* Erfi*, the following inequality holds:*

$$\mathrm{Re}\left(\mathrm{Erfi}(z)\right)' > \frac{1}{2},\ z \in \mathbb{U}.$$

*Proof.* From (1.4) we have

$$\big(\mathrm{Erfi}(z)\big)' = 1 + \sum_{j=2}^{\infty} \frac{j}{(2j-1)(j-1)!} z^{j-1} \quad (z \in \mathbb{U}).$$

Denoting

$$a_j = \frac{j}{(2j-1)(j-1)!}, \tag{2.3}$$

and according to Lemma 2.1 it is sufficient to show that the inequalities (2.1) and (2.2) hold for $a_j$ given by (2.3). Using the same method like in the proof of Theorem 2.5, it is enough to prove that the difference of two term, that is,

$$a_j - 2a_{j+1} = \frac{j}{(2j-1)(j-1)!} - \frac{2(j+1)}{(2j+1)j!} = \frac{1}{(4j^2-1)j!}\lambda(j),$$

where

$$\lambda(j) := 2j^3 - 3j^2 - 2j + 2,$$

is positive. Setting

$$\chi(x) := 2x^3 - 3x^2 - 2x + 2, \ x \geq 2,$$

we see that the function $\chi$ is strictly increasing on $[2, +\infty)$, and hence $\min\{\chi(x) : x \geq 2\} = \chi(2) = 2 > 0$. Since $a_j > 0$ for all $j \geq 2$, using the same methods like in the last part of the proof of Theorem 2.7, we obtain the desired result. □

## 3. Conclusion

In this paper we have considered the normalized imaginary error function in the open unit disk and we obtained some geometric properties including close-to-convexity for this function. Moreover, it was proved that the normalized imaginary error function and the normalized error function are univalent (close-to-convex) in the open unit disk.

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