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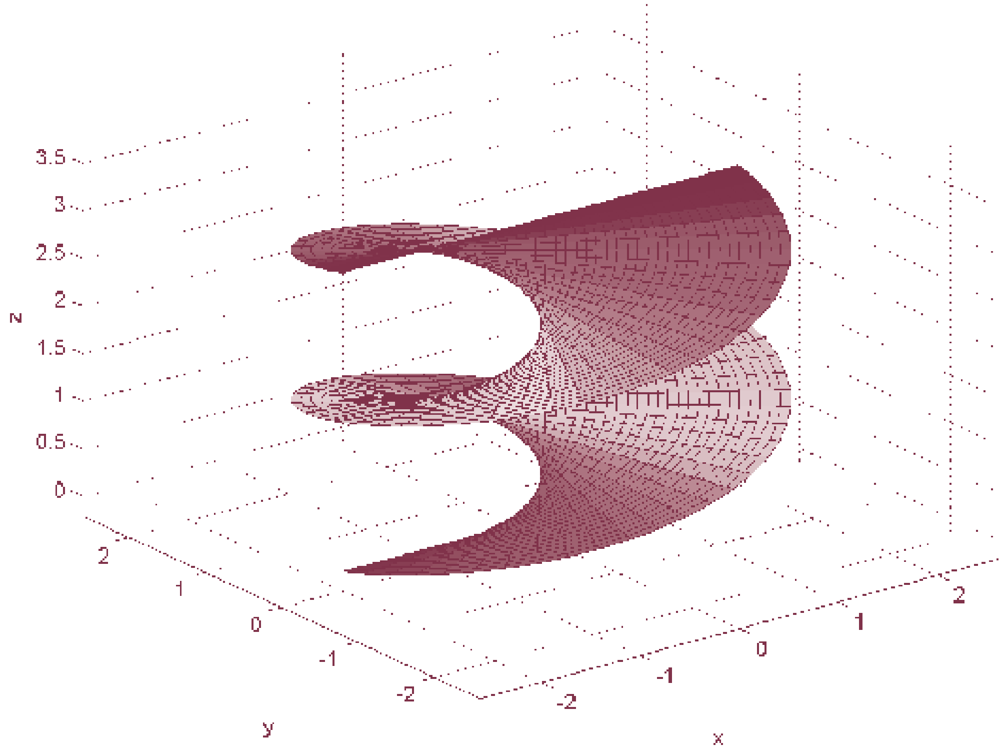

MATHEMATICA

4/2021

STUDIA
UNIVERSITATIS BABEŞ-BOLYAI
MATHEMATICA

4/2021

YEAR (LXVI) 2021
MONTH DECEMBER
ISSUE 4

S T U D I A
UNIVERSITATIS BABEŞ-BOLYAI

MATHEMATICA
4

Redacţia: 400084 Cluj-Napoca, str. M. Kogălniceanu nr. 1
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Stud. Univ. Babeş-Bolyai Math. 66(2021), No. 4, 605–611
DOI: 10.24193/subbmath.2021.4.01

On a property of the generalized Brauer pairs

Tiberiu Coconeţ

Abstract. In this paper we give a generalization of a result of Puig and Zhou to the context of group graded algebras. We use this generalization for an alternative approach of the proof of a result involving group graded basic Morita equivalences.

Mathematics Subject Classification (2010): 20C20, 20C05.

Keywords: Blocks, Brauer pairs, defect groups, normal subgroups, graded algebras, permutation algebras.

1. Introduction

The extended Brauer quotient for G-interior algebras was first introduced in [5]. The construction is useful for extending to the blocks of the normalizers of the local pointed groups the local equivalences induced by a basic Morita equivalence. The construction was then generalized in [4] to the case of H-interior G-algebras, where H is normal in G. Further remarks and properties are given in [2] and [3] for the case of G/H-graded algebras, serving for the generalization of the main rezult of [5], which was achived in [3].

In the current paper we generalize [6, Lemma 4.18] to the case of graded algebras. We obtain this generalization by employing a generalized Brauer pair and we first state and prove, without referring to graded algebras, Theorem 4.2. Secondly we give Corollary 4.4. This approach is necessary due to some technical difficulties involving the blocks of the generalized Brauer pairs. On the other hand the mentioned generalization can be used for giving an alternative proof for the main result of [2]. We discuss these situations in section 4.

In Sections 2 and 3 we introduce the standard notations and the basic results needed for the main result.

This work was supported by a grant of the Romanian Ministry of Education and Research, CNCS - UEFISCDI, project number PN-III-P1-1.1-TE-2019-0136, within PNCDI III.

2. Assumptions and general settings

Throughout this paper $\mathcal{O}$ denotes a discrete complete valuation ring with $k := \mathcal{O}/J(\mathcal{O})$ its residue class field, of characteristic p. G is always a finite group such that p divides its order. All modules and algebras are finitely generated or of finite dimension. A G-algebra over $\mathcal{O}$ is an algebra A such that there exists a group homomorphism $G \mapsto \mathrm{Aut}_{\mathcal{O}}(A)$ compatible with the algebra structure of A. For any $a \in A$ and any $g \in G$ we denoted by a^g the result of g acting on a.

A G-interior algebra over $\mathcal{O}$ is an algebra A endowed with a group homomorphims $G \to A^*$ such that $g \mapsto g \cdot 1_A = 1_A \cdot g$, where A^* denotes the units of A. Any G-interior algebra is becomes a G-algebra via conjugation with the elements of G. Explicitly for any $a \in A$ and $g \in G$ we have $a^g = g^{-1} \cdot a \cdot g$.

For all subgroups K in G, on a G-algebra A, we denote by A^K the set of all K-fixed elements of A. If L is also a subgroup of G with $L \leq K$ then the $\mathcal{O}$-module homomorphism

$$\mathrm{Tr}_L^K : A^L \to A^K \text{ with } \mathrm{Tr}_L^K(a) = \sum_{g \in K/L} a^g,$$

for all $a \in A^L$, is called the relative trace map. In this sum g runs through a set of representatives of K/L. For a p-subgroup P in G we denote by $A(P)$ the quotient

$$A^P / \sum_{R<P} A_R^P$$

and by Br_P the $N_G(P)$-algebra epimorphism

$$A^P \to A(P).$$

Any G-interior algebra is naturally endowed with a $G \times G$-module structure given by

$$(g_1, g_2) \cdot a = g_1 \cdot a \cdot g_2^{-1}$$

for any $g_1, g_2 \in G$ and $a \in A$. If A' is another G-algebra a homomorphism of G-algebras is any homomorphism of $\mathcal{O}$-algebras $\psi : A \to A'$ that verifies $\psi(a^g) = \psi(a)^g$, for any $a \in A'$ and $g \in G$.

If H is a normal subgroup of G, an H-interior G-algebra A is an algebra simultaneously endowed with an H-interior and G-algebra structure with G acting on the natural algebra homomorphism

$$\mathcal{O}H \to A.$$

We refer the reader to [7] for the language of pointed groups and defect pointed groups on G-algebras. Throughout the proof of Theorem 4.2 we freely use the fact that, for any finite group G, the epimorphism of G-interior algebras

$$\mathcal{O}G \to kG \tag{2.1}$$

induces a defect group preserving bijection between the blocks of the two group algebras. Furthermore, if G is normal in some finite group $\tilde{G}$ the epimorphism of $\tilde{G}$-algebras 2.1 also induces a defect group preserving bijection between the $\tilde{G}$-invariant blocks of $\mathcal{O}G$ and those of kG. We also frequently use the results of [1].

3. The extended Brauer quotient

Let A be a G-interior algebra and consider P, a p subgroup of G. For any $\varphi \in \mathrm{Aut}(P)$ we set $\Delta_\varphi(P) : P \to P \times P$ the φ-twisted diagonal subgroup of $P \times P$ defined as

$$\Delta_\varphi(P) = \{(u, \varphi(u)) \mid u \in P\}.$$

Since A is a $G \times G$-module we use the definition of $\Delta_\varphi(P)$ to introduce the $\mathcal{O}$-module

$$A^{\Delta_\varphi(P)} = \{a \in A \mid (u, \varphi(u)) \cdot a = a \text{ for all } u \in P\}. \tag{3.1}$$

We briefly present here the construction of the extended Brauer quotient. Let R be a subgroup of P, we also introduce the $\mathcal{O}$-module

$$A^{\Delta_\varphi(P)}_{\Delta_\varphi(R)} = \{\mathrm{Tr}^{\Delta_\varphi(P)}_{\Delta_\varphi(R)}(a) \mid a \in A^{\Delta_\varphi(R)}\}. \tag{3.2}$$

It is easily checked that $A^{\Delta_\varphi(P)} \cdot A^{\Delta_\psi(P)} \subseteq A^{\Delta_{\varphi\psi}(P)}$ for all $\varphi, \psi \in \mathrm{Aut}(P)$. Denote by $\varphi_g \in \mathrm{Aut}(P)$ the group homomorphism $\varphi_g(u) = g^{-1}ug$ for al $g \in N_G(P)$ and $u \in P$. Then the external direct sum

$$N_A^{\mathrm{Aut}(P)}(P) = \bigoplus_{\varphi \in \mathrm{Aut}(P)} A^{\Delta_\varphi(P)} \tag{3.3}$$

is an $N_G(P)$-interior algebra via $N_G(P) \ni g \mapsto g \cdot 1_A \in A^{\Delta_{\varphi_g}(P)}$, while the sum

$$\mathrm{I} := \bigoplus_{\varphi \in \mathrm{Aut}(P)} \left(\sum_{R<P} A^{\Delta_\varphi(P)}_{\Delta_\varphi(R)} \right) \tag{3.4}$$

is a two-sided ideal of $N_A^{\mathrm{Aut}(P)}(P)$.

Definition 3.1. The extended Brauer quotient associated with the interior G-algebra A and the p-group P is the $N_G(P)$-interior algebra

$$\begin{aligned}\bar{N}_A^{\mathrm{Aut}(P)}(P) :&= N_A^{\mathrm{Aut}(P)}(P)/\mathrm{I} \\ &= \bigoplus_{\varphi \in \mathrm{Aut}(P)} A^{\Delta_\varphi(P)} / \left(\sum_{R<P} A^{\Delta_\varphi(P)}_{\Delta_\varphi(R)} \right) \\ &=: \bigoplus_{\varphi \in \mathrm{Aut}(P)} A(\Delta_\varphi(P))\end{aligned}$$

Remark 3.2. Definition 3.1 can be given for any subgroup of $\mathrm{Aut}(P)$. It was generalized to the case of H-interior G-algebras, where H is normal in G and $P \leq G$, in [4]. In this situation the authors restrict the constructions presented in 3.1 through 3.4 for all automorphisms $\varphi \in \mathrm{Aut}(P)$ that satisfy $\varphi(u) \in uH$ for all $u \in P$. Further generalizations of the extended Brauer quotient are given in [3]. We present them here. Assume H is normal in G and B is an H-interior G-algebra. Then

$$A := B \otimes_{\mathcal{O}H} \mathcal{O}G$$

is a G-interior algebra via $g \mapsto 1 \otimes g$ with the multiplication ginev by

$$(a \otimes g)(b \otimes h) = ab^{g^{-1}} \otimes gh \text{ for } a, b \in A \text{ and } gh \in G.$$

Obviously A is a $\bar{G} := G/H$-graded G-algebra with $A_{\bar{g}} := B \otimes g$ for any $g \in \bar{g}$ and any $\bar{g} \in \bar{G}$. Furthermore if $x \in G$ and $\bar{g} \in \bar{G}$ then $(A_{\bar{g}})^x = A_{\overline{g^x}}$. Set $\bar{P} = PH/H$. Any element $\varphi \in \mathrm{Aut}(P)$ determines an element $\bar{\varphi} \in \mathrm{Aut}(\bar{P})$ with $\bar{\varphi}(u) = \overline{\varphi(u)}$ for all $u \in P$. According to [3, Paragraph 4] quotient of the groups

$$\mathrm{Aut}_{\bar{G}}(P) = \{\varphi \in \mathrm{Aut}(P) \mid \overline{\varphi(u)} = \overline{u^g} \text{ for some } \bar{g} \in \bar{G} \text{ and for all } u \in P\}$$

and

$$\mathrm{Aut}_{\bar{1}}(P) = \{\varphi \in \mathrm{Aut}(P) \mid \overline{\varphi(u)} = \overline{u} \text{ for all } u \in P\}$$

determine the graded structure of $\bar{N}_A^{\mathrm{Aut}(P)}(P)$. In the case $B = \mathcal{O}H$,

$$\bar{N}_A^{\mathrm{Aut}(P)}(P) \simeq kN_G(P)$$

as $N_G(P)/C_H(P)$-graded $N_G(P)$-interior algebras.

4. A property of the generalized Brauer pairs

We use the notations above, and chose H a normal subgroup of G and b a block of $\mathcal{O}H$ that is G-invariant. Let Q denote a p-subgroup of G and consider the Brauer homomorphism with respect to Q

$$\mathrm{Br}_Q : \mathcal{O}H^Q \to kC_H(Q).$$

Let (Q, e_Q) be a (G, H, b)-Brauer pair. That is e_Q is a block of $kC_H(Q)$ that verifies $\mathrm{Br}_Q(b)e_Q = e_Q$. Since $C_H(Q)$ is a normal subgroup of $N_G(Q)$, the latter group acts on $Z(kC_H(Q))$ hence it makes sense to consider the stabilizer of e_Q in $N_G(Q)$, denoted $N_G(Q, e_Q)$. Consider the group homomorphism

$$N_G(Q) \to \mathrm{Aut}(Q) \tag{4.1}$$

Let T denote a subgroup of the image of $N_G(Q, e_Q)$ in $\mathrm{Aut}(Q)$. We denote by $N_G^T(Q)$ the inverse image of T via 4.1.
Finally we denote $N_H^T(Q) = N_G^T(Q) \cap H$ and $Q_1 = Q \cap N_H(Q)$ and we assume that Q_1 is not trivial.

Remark 4.1. As [6, 4.1] states for the case $H = G$ any block of $kC_H(Q)$ remains a block of kH where $C_G(Q) \leq H \leq N_G^T(Q)$. In our situation it might happen that a block of $kC_H(Q)$ is not primitive in $Z(kN_H^T(Q))$. It could happen that is also not primitive in $kN_H^T(Q)^{N_G^T(Q)}$, unless $Q \leq N_G^T(Q)$. In any case the injective homomorphism 4.3 can be constructed and used for its purpose in the proof.

Theorem 4.2. *There exists a point ν of $Q \cdot N_G^T(Q)$ on $\mathcal{O}Hb$ with $\mathrm{Br}_Q(\nu) = e_Q$, and for any local pointed group R_ϵ on $\mathcal{O}Hb$ satisfying $Q \leq R$ and $R_\epsilon \leq Q \cdot N_G^T(Q)_\nu$, there is a local pointed group $N_R^T(Q)_{\bar{\epsilon}}$ on $\mathcal{O}N_H^T(Q)e_Q$ such that there exists a $N_R^T(Q)$-interior algebra embedding*

$$(kN_G^T(Q))_{\bar{\epsilon}} \to \bar{N}^T_{(\mathcal{O}Gb)_\epsilon}(Q) \tag{4.2}$$

determined by the structural homomorphism

$$N_G^T(Q) \to \bar{N}_{\mathcal{O}Gb}^T(Q).$$

Moreover, R_ϵ *is a defect pointed group of* $Q \cdot N_G^T(Q)_\nu$ *on* $\mathcal{O}Hb$ *if and only if* $N_R^T(Q)_{\mathrm{Br}_Q(\epsilon)} \le N_G^T(Q)_{\{e_Q\}}$ *is a maximal local pointed group with this property over* $kC_H(Q)e_Q$*. In particular,* $N_R^T(Q)$ *is a defect group of* e_Q *as a primitive idempotent of* $(kC_H(Q))^{N_G^T(Q)}$ *if and only if* R *is a defect group of* e_Q *as a primitive idempotent of* $(k(Q_1 \cdot N_H^T(Q))^{Q \cdot N_G^T(Q)}$*.*

Proof. The block e_Q lies in $kC_H(Q)^{Q \cdot N_G^T(Q)}$, where it remains primitive. It is clear that it lifts to a point $\nu \subseteq (\mathcal{O}H)^{Q \cdot N_G^T(Q)}$ via Br_Q . Since $\mathrm{Br}_Q(b)e_Q = e_Q$ we obtain $b\nu = \nu$ giving $\nu \subseteq (\mathcal{O}Hb)^{Q \cdot N_G^T(Q)}$. For any T chosen as above e_Q is a primitive idempotent in $\mathcal{O}N_H^T(Q)^{Q \cdot N_G^T(Q)}$ and in $\mathcal{O}(Q_1 \cdot N_H^T(Q))^{Q \cdot N_G^T(Q)}$ according to [8, Lemma 2.1].

Let R_ϵ be as mentioned. Since Q is normal in R and since we deal with permutation algebras we get that $\mathrm{Br}_R^{\mathcal{O}H}(\epsilon) \neq 0$ implies $\mathrm{Br}_Q^{\mathcal{O}H}(\epsilon) \neq 0$. Furthermore $\mathrm{Br}_Q(\epsilon)e_Q = \mathrm{Br}_Q(\epsilon)$, we set $\tilde{\epsilon} = \mathrm{Br}_Q(\epsilon)$ and since $\mathrm{Br}_R^{kC_H(Q)}(\tilde{\epsilon}) \neq 0$ it is clear that $N_R^T(Q)_{\tilde{\epsilon}}$ is a local pointed group on $kC_H(Q)e_Q$. The injective algebra homomorphism

$$(kC_H(Q)e_Q)(N_R^T(Q)) \to (kN_H^T(Q)e_Q)(N_R^T(Q)) \tag{4.3}$$

determines a point $\bar{\epsilon} \subseteq (kN_H^T(Q)e_Q)^{N_R^T(Q)}$ satisfying

$$\tilde{\epsilon} \cdot \bar{\epsilon} = \bar{\epsilon} \text{ and } \mathrm{Br}_{N_R^T(Q)}^{kN_H^T(Q)}(\bar{\epsilon}) \neq 0. \tag{4.4}$$

The construction of the extended Brauer quotient and [4, Theorem 3.1] give the $N_R^T(Q)$-interior algebra isomorphism

$$\bar{N}_{(\mathcal{O}Gb)_\epsilon}^T(Q) \simeq (\bar{N}_{\mathcal{O}Gb}^T(Q))_{\tilde{\epsilon}}.$$

We apply [3, Remark 2.4] and 4.4 to obtain that the group homomorphism

$$N_G^T(Q) \to (\bar{N}_{\mathcal{O}Gb}^T(Q))^*,$$

given by

$$g \mapsto \overline{g \cdot b} \in (\overline{(\mathcal{O}Gb)^{\Delta_{\varphi_g}(Q)}})^*$$

induces the mentioned $N_R^T(Q)$-interior algebra embedding.
First assume that R_ϵ is a defect pointed group of $Q \cdot N_G^T(Q)_\nu$ on $\mathcal{O}Hb$.
Then $\nu \subseteq (\mathcal{O}Hb)_R^{Q \cdot N_G^T(Q)}$ and $\nu \cdot \epsilon = \epsilon$. Applying Br_Q on the last equality we get $e_Q \cdot \mathrm{Br}_Q(\epsilon) = \mathrm{Br}_Q(\epsilon)$ and $e_Q \in kC_H(Q)_{N_R^T(Q)}^{N_G^T(Q)}$. According to 4.4 we obtain $\mathrm{Br}_{N_R^T(Q)}^{kC_H(Q)}(e_Q) \neq 0$.
Conversely, by our assumptions we have $e_Q \cdot \mathrm{Br}_Q(\epsilon) = \mathrm{Br}_Q(\epsilon)$, $e_Q \in kC_H(Q)_{N_R^T(Q)}^{N_G^T(Q)}$, $\mathrm{Br}_{N_R^T(Q)}^{kC_H(Q)}(e_Q) \neq 0$, for some local point $\epsilon \subseteq (\mathcal{O}Hb)^{Q \cdot N_R^T(Q)}$. Using the $\mathcal{O}$-algebra epimorphism

$$(\mathcal{O}Hb)_{Q \cdot N_R^T(Q)}^{Q \cdot N_G^T(Q)} \to (kC_H(Q)e_Q)_{Q \cdot N_R^T(Q)}^{Q \cdot N_G^T(Q)}$$

we see that ν in relatively $R = Q \cdot N_R^T(Q)$ -projective and $\nu \cdot \epsilon = \epsilon$. This concludes the proof of the first equivalence.

We use [8, Lemma 2.1] again to obtain that $e_Q \in k(Q_1 \cdot N_H^T(Q))^{Q \cdot N_G^T(Q)}$ is still a primitive idempotent. The last assertion follows immediately from the above statements by using $Q_1 \cdot N_H^T(Q)$ in place of H and e_Q in place of ν. □

Remark 4.3. Although the block e_Q is not necessarily primitive in $Z(kN_H^T(Q))$ or in $kN_H^T(Q)^{N_G^T(Q)}$, the theorem above covers [6, Lemma 4.18] when applied in the case $H = G$, which gives $Q_1 = Q$, and by substituting $C_G(Q)$ for $N_G^T(Q)$ in the second statemen of the theorem. We can obtain another generalization as it follows from the next corollary.

Corollary 4.4. *There exists a point ν of $Q \cdot N_G^T(Q)$ on $\mathcal{O}Hb$ with $\mathrm{Br}_Q(\nu) = e_Q$ and for any local pointed group R_ϵ on $\mathcal{O}Hb$ satisfying $Q \leq R$ and $R_\epsilon \leq Q \cdot N_G^T(Q)_\nu$, there is a local pointed group $N_R^T(Q)_{\bar{\epsilon}}$ on $kC_H(Q)e_Q$ such that there exists a $N_R^T(Q)$-interior $N_G^T(Q)/C_H(Q)$-graded algebra isomorphism*

$$(kN_G^T(Q))_{\bar{\epsilon}} \to \bar{N}^T_{(\mathcal{O}Gb)_\epsilon}(Q) \tag{4.5}$$

determined by the structural homomorphism

$$N_G^T(Q) \to \bar{N}^T_{\mathcal{O}Gb}(Q).$$

Moreover, R_ϵ is a defect pointed group of $Q \cdot N_G^T(Q)_\nu$ on $\mathcal{O}Hb$ if and only if $N_R^T(Q)_{\mathrm{Br}_Q(\epsilon)} \leq N_G^T(Q)_{\{e_Q\}}$ is a maximal local pointed group with this property over $kC_H(Q)e_Q$. In particular, $N_R^T(Q)$ is a defect group of e_Q as a primitive idempotent of $(kC_H(Q))^{N_G^T(Q)}$ if and only if R is a defect group of e_Q as a primitive idempotent of $(kQ_1 \cdot C_H(Q))^{Q \cdot N_G^T(Q)}$.

Remark 4.5.
1. Notice that we can also consider T such that $N_G^T(Q) = N_G(Q, e_Q)$, but in this situation the statement of Corollary 4.4 only points out the $N_G(Q, e_Q)/C_H(Q)$-graded isomorphism 4.5 and the fact that the primitive idempontents of $kC_H(Q)^{N_G(Q,e_Q)}$ are the primitive idempotents of $(kQ_1C_H(Q))^{N_G(Q,e_Q)}$ admitting the same defect group R that includes Q.
2. [2, Lemma 6.6] is an important ingredient of the proof of the main result [2, Theorem 6.9], and it can be deduced from the above corollary when applied in the general situation. Indeed, if T is a defect group of e_Q as a primitive idempotent of $kC_H(Q)^{N_G^K(Q)}$ then $Q \cdot T$ is a defect group of ν as a primitive idempotent of $\mathcal{O}H^{Q \cdot N_G^K(Q)}$. At this point we denote by $\tilde{e}_Q$ the primitive idempotent of $\mathcal{O}C_H(Q)^{N_G^K(Q)}$ corresponding to e_Q via 2.1. Note that $\tilde{e}_Q$ is primitive in $\mathcal{O}C_H(Q)^{Q \cdot N_G^K(Q)}$ and $\mathrm{Br}_Q(\tilde{e}_Q) = e_Q$. Consider the decomposition

$$\mathcal{O}H^{Q \cdot N_G^K(Q)} = \mathcal{O}C_H(Q)^{Q \cdot N_G^K(Q)} \bigoplus \left(\sum_{Q' < Q} (\mathcal{O}H)^Q_{Q'} \right)^{Q \cdot N_G^K(Q)}.$$

Since the sum on the right lies in the radical of $\mathcal{O}H$ it follows that $\nu \subseteq \mathcal{O}C_H(Q)^{Q \cdot N_G^K(Q)}$ and then $\nu = \tilde{e}_Q$. This means that e_Q has defect group $Q \cdot T$

as a primitive idempotent of $kC_H(Q)^{Q\cdot N_G^K(Q)}$. At last, if $Q \subseteq N_G^K(Q)$ then up to a $N_G(Q)$-conjugate we have $Q \cdot T = T$ forcing $Q \subseteq T$.

References

[1] Broué, M., Puig, L., *Characters and local structure in G-algebras*, J. Algebra, **63** (1980), 306-317.

[2] Coconeţ, T., Marcus, A., *Group graded basic Morita equivalences*, J. Algebra, **489**(2017), 1-24.

[3] Coconeţ, T., Marcus, A., *Remarks on the extended Brauer quotient*, J. Algebra, **491**(2017), 78-89.

[4] Coconeţ, T., Todea, C.-C., *The extended Brauer quotient of N-interior G-algebras*, J. Algebra, **396**(2013), 10-17.

[5] Puig, L., Zhou, Y., *A local property of basic Morita equivalences*, Math. Z., **256**(2007), 551-562.

[6] Puig, L., Zhou, Y., *A local property of basic Rickard equivalences*, J. Algebra, **322**(2009), 1946-1973.

[7] Thévenaz, J., *G-Algebras and Modular Representation Theory*, Clarendon Press, Oxford, 1995.

[8] Todea, C.-C., *On some saturated triples and hyperfocal subgroups*, J. Algebra, **499** (2018), 358-374.

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Stud. Univ. Babeş-Bolyai Math. 66(2021), No. 4, 613–627
DOI: 10.24193/subbmath.2021.4.02

Positivity of sums and integrals for n-convex functions via the Fink identity and new Green functions

Asif R. Khan and Josip Pečarić

Abstract. We consider positivity of sum $\sum_{i=1}^{n} p_i f(x_i)$ involving convex functions of higher order. Analogous for integral $\int_a^b p(x)f(g(x))dx$ is also given. Representation of a function f via the Fink identity and the Green function leads us to identities for which we obtain conditions for positivity of the mentioned sum and integral. We obtain bounds for integral remainders which occur in those identities as well as corresponding mean value theorems.

Mathematics Subject Classification (2010): 26A51, 26D15, 26D20.

Keywords: n-convex functions, Fink identity, Green function, Čebyšev functional.

1. Introduction

In [9] we proved various results related to general linear inequalities via Fink identity with and without Green function (see also [8]). Recently, in [2] the authors have introduced new Green type functions. Our main objective of present article is to further extend results of [9] using new definitions stated in [2].

To recall the definitions of generalized convex function and related concepts and results we refer to interested readers the following references [10], [5] and [15].

In the sequel we use the notation $AC[a, b]$ for class of absolutely continuous functions defined on a real interval $[a, b]$ and by $(\xi - s)_+^k$, $k \in \mathbb{N}_0$, we will mean the following

$$(\xi - s)_+^k = \begin{cases} (\xi - s)^k, & \text{if } \xi \geq s \\ 0, & \text{if } \xi < s. \end{cases}$$

Now we recall the Fink identity to prove many interesting results. The following theorem is proved by A. M. Fink in [4].

Proposition 1.1. *Let $a, b \in \mathbb{R}$, $f : [a,b] \to \mathbb{R}$, $n \geq 1$ and $f^{(n-1)}$ is absolutely continuous on $[a,b]$. Then*

$$f(x) = \frac{n}{b-a}\int_a^b f(t)\,dt - \sum_{k=1}^{n-1}\frac{n-k}{k!}\left(\frac{f^{(k-1)}(a)(x-a)^k - f^{(k-1)}(b)(x-b)^k}{b-a}\right)$$
$$+ \frac{1}{(n-3)!(b-a)}\int_a^b (x-t)^{n-1} P^{[a,b]}(t,x) f^{(n)}(t)\,dt, \tag{1.1}$$

where

$$P^{[a,b]}(t,x) = \begin{cases} t-a, & a \leq t \leq x \leq b, \\ t-b, & a \leq x < t \leq b. \end{cases} \tag{1.2}$$

Pečarić in [12] proved the following result (see also [15, p.262]):

Proposition 1.2. *The inequality*

$$\sum_{i=1}^{m} p_i f(x_i) \geq 0 \tag{1.3}$$

holds for all convex functions f if and only if the $m-$tuples

$$\mathbf{x} = (x_1, \ldots, x_m), \quad \mathbf{p} = (p_1, \ldots, p_m) \in \mathbb{R}^m$$

satisfy

$$\sum_{i=1}^{m} p_i = 0 \quad \text{and} \quad \sum_{i=1}^{m} p_i|x_i - x_k| \geq 0 \text{ for } k \in \{1, \ldots, m\}. \tag{1.4}$$

Since

$$\sum_{i=1}^{m} p_i|x_i - x_k| = 2\sum_{i=1}^{m} p_i(x_i - x_k)_+ - \sum_{i=1}^{m} p_i(x_i - x_k),$$

where $y_+ = \max(y, 0)$, it is easy to see that condition (1.4) is equivalent to

$$\sum_{i=1}^{m} p_i = 0, \quad \sum_{i=1}^{m} p_i x_i = 0 \quad \text{and} \quad \sum_{i=1}^{m} p_i(x_i - x_k)_+ \geq 0 \text{ for } k \in \{1, \ldots, m-1\}. \tag{1.5}$$

The following result is due to Popoviciu [16, 17] (see [15, 18] also).

Proposition 1.3. *Let $n \geq 2$. Inequality (1.3) holds for all n-convex functions $f : [a,b] \to \mathbb{R}$ if and only if the $m-$tuples $\mathbf{x} \in [a,b]^m$, $\mathbf{p} \in \mathbb{R}^m$ satisfy*

$$\sum_{i=1}^{m} p_i x_i^k = 0, \quad \text{for all } k \in \{0, 1, \ldots, n-1\} \tag{1.6}$$

$$\sum_{i=1}^{m} p_i(x_i - t)_+^{n-1} \geq 0, \quad \text{for every } t \in [a,b]. \tag{1.7}$$

Proposition 1.4. *Let $n \geq 2$, $p : [\alpha, \beta] \to \mathbb{R}$ and $g : [\alpha, \beta] \to [a,b]$. Then, the inequality*

$$\int_\alpha^\beta p(x) f(g(x))\,dx \geq 0 \tag{1.8}$$

holds for all n-convex functions $f : [a,b] \to \mathbb{R}$ if and only if

$$\begin{aligned} &\int_{\alpha}^{\beta} p(x)g(x)^k\,dx = 0, \qquad \textit{for all } k \in \{0,1,\dots,n-1\} \\ &\int_{\alpha}^{\beta} p(x)\,(g(x)-t)_+^{n-1}\,dx \geq 0, \qquad \textit{for every } t \in [a,b]. \end{aligned} \tag{1.9}$$

After this introductory section, we continue with section 2 where identities for

$$\sum_{i=1}^{n} p_i f(x_i) \text{ and } \int_a^b p(x)f(g(x))dx$$

are given using the Fink identity and new Green functions. Also we consider inequalities for n-convex functions which are based on these identities. Section 3 is devoted to estimations of functions A_k by using Čebyšev, Gr̈uss and Ostrowski type inequalities and the Hölder inequality. In the last section we give mean value theorems for functionals A_k, $k \in \{1,2\}$.

2. Popoviciu type identities and inequalities via the Fink identity and new Green functions

In this section we obtain some discrete and integral identities and the corresponding linear inequalities using new Green functions and applying the Fink identity. As a special choice of Abel-Gontscharoff polynomial for 'two-point right focal' interpolating polynomial for $n = 2$ could be stated as (see [13]):

$$f(\xi) = f(a) + (\xi - a)f'(b) + \int_a^b G_1(\xi,t)f''(t)dt, \tag{2.1}$$

where $G_1(s,t)$ is Green's function for 'two-point right focal problem' defined as

$$G_1(s,t) = \begin{cases} a-t, & a \leq t \leq s, \\ a-s, & s \leq t \leq b. \end{cases} \tag{2.2}$$

Motivated by Abel-Gontscharoff identity (2.1) and related Green's function (2.2), we recall some new types of Green functions $G_l : [a,b] \times [a,b] \to \mathbb{R}$, $(l = 2,3,4,)$ defined as in [2]:

$$G_2(s,t) = \begin{cases} s-b, & a \leq t \leq s, \\ t-b, & s \leq t \leq b. \end{cases} \tag{2.3}$$

$$G_3(s,t) = \begin{cases} s-a, & a \leq t \leq s, \\ t-a, & s \leq t \leq b. \end{cases} \tag{2.4}$$

$$G_4(s,t) = \begin{cases} b-t, & a \leq t \leq s, \\ b-s, & s \leq t \leq b. \end{cases} \tag{2.5}$$

In [2], it is also shown that all four Green functions are symmetric and continuous. Moreover, all functions are convex with respect to both variables s and t. From these functions we can obtain new identities, given in following lemma:

Lemma 2.1. *Let $f : [a, b] \to \mathbb{R}$ be twice differentiable function and G_l, $(l = 2, 3, 4)$ are defined in (2.3), (2.4) and (2.5). Then the following identities holds:*

$$f(\xi) = f(b) + (b - \xi)f'(a) + \int_a^b G_2(\xi, t)f''(t)dt, \tag{2.6}$$

$$f(\xi) = f(b) - (b - a)f'(b) + (\xi - a)f'(a) + \int_a^b G_3(\xi, t)f''(t)dt, \tag{2.7}$$

$$f(\xi) = f(a) + (b - a)f'(a) - (b - \xi)f'(b) + \int_a^b G_4(\xi, t)f''(t)dt. \tag{2.8}$$

We can easily obtain these identities by using integration by parts by using respective Green function. Now we state here main results related to the Fink identity and the Green function.

Theorem 2.2. *Fix $l \in \{1, 2, 3, 4\}$. Let $f : [a, b] \to \mathbb{R}$ be such that for $n \geq 3$, $f^{(n-1)}$ is absolutely continuous. Let $x_i, y_i \in [a, b]$, $p_i \in \mathbb{R}$ for $i \in \{1, \ldots, m\}$ be such that $\sum_{i=1}^m p_i = 0$ and $\sum_{i=1}^m p_i x_i = 0$ and let $P^{[a,b]}(t, x)$ be the same as defined in (1.2). If G_l are the Green functions as defined in (2.2) – (2.5), then we have*

$$\begin{aligned}\sum_{i=1}^m p_i f(x_i) &= \sum_{k=0}^{n-3} \left(\frac{n-k-2}{k!\,(b-a)}\right) \int_a^b \left(\sum_{i=1}^m p_i G_l(x_i, s)\right) \\ &\times \left(f^{(k+1)}(b)(s-b)^k - f^{(k+1)}(a)(s-a)^k\right) ds + \frac{1}{(n-3)!\,(b-a)} \\ &\times \int_a^b f^{(n)}(t) \left(\int_a^b \sum_{i=1}^m p_i G_l(x_i, s)(s-t)^{n-3} P^{[a,b]}(t, s)\, ds\right) dt. \end{aligned} \tag{2.9}$$

Proof. First consider four identities (2.1), (2.6), (2.7) and (2.8), and putting $x = \xi_i$ in all these identities, multiplying each with p_i, and then summing over each identity for $i \in \{1, \ldots, m\}$ and using conditions that $\sum_{i=1}^m p_i = 0$, $\sum_{i=1}^m p_i \xi_i = 0$ we get by fixing $l \in \{1, 2, 3, 4\}$

$$\sum_{i=1}^m p_i f(\xi_i) = \int_a^b \left(\sum_{i=1}^m p_i G_l(\xi_i, t)\right) f''(t)dt. \tag{2.10}$$

Differentiating Fink identity twice we easily get

$$\begin{aligned} f''(x) &= \sum_{k=0}^{n-3} \frac{n-k-2}{k!} \frac{f^{(k+1)}(b)(x-b)^k - f^{(k+1)}(a)(x-a)^k}{b-a} \\ &+ \frac{1}{(n-3)!\,(b-a)} \int_a^b (x-t)^{n-3} P^{[a,b]}(t, x) f^{(n)}(t)\, dt, \end{aligned} \tag{2.11}$$

and by using (2.11) in (2.10), we have

$$\sum_{i=1}^{m} p_i f(x_i) = \int_a^b \left(\sum_{i=1}^{m} p_i G_l(x_i, s) \right)$$
$$\times \sum_{k=0}^{n-3} \frac{n-k-2}{k!} \frac{f^{(k+1)}(b)(s-b)^k - f^{(k+1)}(a)(s-a)^k}{b-a} ds$$
$$+ \frac{1}{(n-3)!(b-a)} \int_a^b \sum_{i=1}^{m} p_i G_l(x_i, s) \left(\int_a^b (s-t)^{n-3} P^{[a,b]}(t, s) f^{(n)}(t) \, dt \right) ds.$$

Now by interchanging the integral and summation in the second term and by applying Fubini's theorem in the last term, we have (2.9). □

The following theorem is the integral version of Theorem 2.2.

Theorem 2.3. *Fix $l \in \{1, 2, 3, 4\}$. Let $f : [a, b] \to \mathbb{R}$ be such that for $n \geq 3$, $f^{(n-1)}$ is absolutely continuous on $[a, b]$ and let $p : [\alpha, \beta] \to \mathbb{R}$ and $g : [\alpha, \beta] \to [a, b]$ be integrable functions such that $\int_\alpha^\beta p(x)dx = 0$ and $\int_\alpha^\beta p(x)g(x)dx = 0$. Let $P^{[a,b]}(t, x)$ be the same as defined in (1.2). If G_l are the Green functions as defined in (2.2) – (2.5), then we have*

$$\int_\alpha^\beta p(x) f(g(x)) \, dx = \sum_{k=0}^{n-3} \frac{n-k-2}{k!(b-a)} \int_a^b \left(\int_\alpha^\beta p(x) G_l(g(x), s) \, dx \right)$$
$$\left(f^{(k+1)}(b)(s-b)^k - f^{(k+1)}(a)(s-a)^k \right) ds + \frac{1}{(n-3)!(b-a)}$$
$$\times \int_a^b f^{(n)}(t) \left(\int_a^b \left(\int_\alpha^\beta p(x) G_l(g(x), s) \, dx \right) (s-t)^{n-3} P^{[a,b]}(t, s) \, ds \right) dt. \quad (2.12)$$

Proof. Since the proof is similar to that of the previous theorem, we omit the details. □

Here we introduce some notations which will be used in rest of the paper:

$$\Omega_1^{[a,b]}(m, \mathbf{x}, \mathbf{p}, t) = \int_a^b \sum_{i=1}^{m} p_i G_l(x_i, s)(s-t)^{n-3} P^{[a,b]}(t, s) \, ds, \quad (2.13)$$

$$\Omega_2^{[a,b]}([\alpha, \beta], g, p, t) = \int_a^b \int_\alpha^\beta p(x) G_l(g(x), s) \, dx \, (s-t)^{n-3} P^{[a,b]}(t, s) \, ds. \quad (2.14)$$

$$A_1^{[a,b]}(m,\mathbf{x},\mathbf{p},f) = \sum_{i=1}^{m} p_i f(x_i) - \sum_{k=0}^{n-3}\left(\frac{n-k-2}{k!\,(b-a)}\right)\int_a^b \sum_{i=1}^{m} p_i G_l(x_i,s) \times \left(f^{(k+1)}(b)(s-b)^k - f^{(k+1)}(a)(s-a)^k\right) ds \tag{2.15}$$

$$A_2^{[a,b]}([\alpha,\beta],g,p,f) = \int_\alpha^\beta p(x) f(g(x))\,dx - \sum_{k=0}^{n-3}\left(\frac{n-k-2}{k!\,(b-a)}\right)\int_a^b \left(\int_\alpha^\beta p(x) G_l(g(x),s)\,dx\right) \times \left(f^{(k+1)}(b)(s-b)^k - f^{(k+1)}(a)(s-a)^k\right) ds. \tag{2.16}$$

The following theorem is our second main result of this section:

Theorem 2.4. *Let all the assumptions of Theorem* 2.2 *be satisfied and let for* $n \geq 3$, *the inequality*

$$\Omega_1^{[a,b]}(m,\mathbf{x},\mathbf{p},t) \geq 0 \tag{2.17}$$

holds. If f *is* n*-convex, then we have*

$$A_1^{[a,b]}(m,\mathbf{x},\mathbf{p},f) \geq 0. \tag{2.18}$$

If opposite inequality holds in (2.17)*, then* (2.18) *holds in the reverse direction.*

Proof. Since $f^{(n-1)}$ is absolutely continuous on $[a,b]$, $f^{(n)}$ exists almost everywhere. As f is n-convex, applying definition, we have, $f^{(n)}(x) \geq 0$ for all $x \in [a,b]$. Now by using $f^{(n)} \geq 0$ and (2.17) in (2.9), we have (2.18). □

Corollary 2.5. *Let all the assumptions of Theorem* 2.2 *be satisfied. In addition we let*

$$\sum_{i=1}^{m} p_i (x_i - x_k)_+ \geq 0 \quad \textit{for} \quad k \in \{1,\ldots,m\}.$$

Let n *be even and* $n > 3$*. If the function* $f:[a,b]\to\mathbb{R}$ *is* n*-convex, then inequality* (2.18) *is satisfied, i. e.*

$$\sum_{i=1}^{m} p_i f(x_i) \geq \sum_{k=0}^{n-3} \frac{n-k-2}{k!\,(b-a)}\int_a^b \sum_{i=1}^{m} p_i G_l(x_i,s) \times \left(f^{(k+1)}(b)(s-b)^k - f^{(k+1)}(a)(s-a)^k\right) ds. \tag{2.19}$$

Further if $f^{(k+1)}(a) \leq 0$ *and* $(-1)^k f^{(k+1)}(b) \geq 0$ *for* $k \in \{0,1,\ldots,n-3\}$ *then*

$$\sum_{i=1}^{m} p_i f(x_i) \geq 0.$$

Proof. We fix $l \in \{1,2,3,4\}$ and $n > 3$. As $\mathbf{x}$ and $\mathbf{p}$ are real m-tuples such that they satisfy the assumption (1.5), by using the convex function $x \mapsto G_l(x,s)$ in (1.3), we obtain

$$\sum_{i=1}^{m} p_i G_l(x_i,s) \geq 0. \tag{2.20}$$

For $a \leq s \leq t$, it is easy to see that

$$\int_a^t \sum_{i=1}^{m} p_i G_l(x_i, s)(s-t)^{n-3} P^{[a,b]}(t,s)\, ds \geq 0 \tag{2.21}$$

holds for even n. Now as f is n-convex for even n, by applying Theorem 2.4, we get (2.19).
If $a \leq s \leq b$ and $k \in \{0, \ldots, n-3\}$, then from assumptions $f^{(k+1)}(a) \leq 0$ and $(-1)^k f^{(k+1)}(b) \geq 0$ we have that

$$f^{(k+1)}(b)(s-b)^k - f^{(k+1)}(a)(s-a)^k \geq 0, \tag{2.22}$$

So, from inequalities (2.19), (2.20) and (2.22) the non-negativity of the right hand side of (2.19) is immediate. □

An integral version of our second main result states that:

Theorem 2.6. *Let all the assumptions of Theorem* 2.3 *be satisfied and let for* $n \geq 3$, *the inequality*

$$\Omega_2^{[a,b]}([\alpha, \beta], g, p, t) \geq 0 \tag{2.23}$$

holds. If f is n-convex, then we have

$$A_2^{[a,b]}([\alpha, \beta], g, p, f) \geq 0. \tag{2.24}$$

If opposite inequality holds in (2.14), *then* (2.24) *holds in the reverse direction.*

Proof. The idea of the proof is the same as that of the proof of Theorem 2.4. By using $f^{(n)} \geq 0$ and (2.14) in (2.12), we have (2.24). □

Corollary 2.7. *Let all the assumptions of Theorem* 2.3 *be satisfied. In addition we let*

$$\int_\alpha^\beta p(x)(g(x)-t)_+^{n-1}\, dx \geq 0, \qquad \textit{for every } t \in [a,b].$$

Let n be even and $n > 3$. If the function $f : [a,b] \to \mathbb{R}$ is n-convex, then we have

$$\int_\alpha^\beta p(x) f(g(x))\, dx \geq \sum_{k=0}^{n-3} \frac{n-k-2}{k!\,(b-a)} \int_a^b \left(\int_\alpha^\beta p(x) G_l(g(x), s)\, dx \right) \times \left(f^{(k+1)}(b)(s-b)^k - f^{(k+1)}(a)(s-a)^k \right) ds. \tag{2.25}$$

Further if $f^{(k+1)}(a) \leq 0$ and $(-1)^k f^{(k+1)}(b) \geq 0$ for $k \in \{0, 1 \ldots, n-3\}$, then the right hand side of (2.25) *is non-negative.*

Proof. The proof is analogous to the proof of Corollary 2.5 but instead of Theorem 2.4, we apply Theorem 2.6. □

3. Related inequalities for n-convex functions at a point

In this section we will give related results for the class of n-convex functions at a point introduced in [14].

Definition 3.1. Let I be an interval in $\mathbb{R}$, c a point in the interior of I and $n \in \mathbb{N}$. A function $f : I \to \mathbb{R}$ is said to be n-convex at point c if there exists a constant K such that the function

$$F(x) = f(x) - \frac{K}{(n-3)!}x^{n-1}$$

is $(n-1)$-concave on $I \cap (-\infty, c]$ and $(n-1)$-convex on $I \cap [c, \infty)$. A function f is said to be n-concave at point c if the function $-f$ is n-convex at point c.

Let e_i denote the monomials $e_i(x) = x^i$, $i \in \mathbb{N}_0$. First we state main results for discrete case.

Theorem 3.2. *Let $c \in (a,b)$, $\mathbf{x} \in [a,c]^m$, $\mathbf{y} \in [c,b]^l$, $\mathbf{p} \in \mathbb{R}^m$, $\mathbf{q} \in \mathbb{R}^l$ and $f : [a,b] \to \mathbb{R}$ be a function such that $f^{(n-1)}$ is absolutely continuous. Let $\Omega_1^{[\cdot,\cdot]}(\cdot,\cdot,\cdot,t)$ and $A_1^{[\cdot,\cdot]}(\cdot,\cdot,\cdot,f)$ be defined as in* (2.13) *and* (2.15) *and satisfy the following conditions:*

$$\Omega_1^{[a,c]}(m,\mathbf{x},\mathbf{p},t) \geq 0 \quad \textit{for every} \quad t \in [a,c], \tag{3.1}$$

$$\Omega_1^{[c,b]}(l,\mathbf{y},\mathbf{q},t) \geq 0 \quad \textit{for every} \quad t \in [c,b], \tag{3.2}$$

and

$$A_1^{[a,c]}(m,\mathbf{x},\mathbf{p},e_n) = A_1^{[c,b]}(l,\mathbf{y},\mathbf{q},e_n). \tag{3.3}$$

If f is $(n+1)$-convex at point c, then

$$A_1^{[a,c]}(m,\mathbf{x},\mathbf{p},f) \leq A_1^{[c,b]}(l,\mathbf{y},\mathbf{q},f). \tag{3.4}$$

If inequalities in (3.1) *and* (3.2) *are reversed, then* (3.4) *holds with the reverse sign of inequality.*

Proof. Let $F = f - \frac{K}{n!}e_n$ be as in Definition 3.1, *i. e.*, the function F is n-concave on $[a,c]$ and n-convex on $[c,b]$. Applying Theorem 2.4 to F on the interval $[a,c]$ and on the interval $[c,b]$ we have

$$A_1^{[a,c]}(m,\mathbf{x},\mathbf{p},F) \leq 0 \leq A_1^{[c,b]}(l,\mathbf{y},\mathbf{q},F).$$

Using definition of F we obtain

$$A_1^{[a,c]}(m,\mathbf{x},\mathbf{p},f) - \frac{K}{n!}A_1^{[a,c]}(m,\mathbf{x},\mathbf{p},e_n) \leq A_1^{[c,b]}(l,\mathbf{y},\mathbf{q},f) - \frac{K}{n!}A_1^{[c,b]}(l,\mathbf{y},\mathbf{q},e_n).$$

Since equality (3.3) is valid we get

$$A_1^{[a,c]}(m,\mathbf{x},\mathbf{p},f) \leq A_1^{[c,b]}(l,\mathbf{y},\mathbf{q},f). \qquad \square$$

Remark 3.3. A closer look at the proof of Theorem 3.2 gives us that a similar result hold if instead equality (3.3) we consider a positivity of the difference

$$K\left(A_k^{[c,b]}(l,\mathbf{y},\mathbf{q},e_n) - A_k^{[a,c]}(m,\mathbf{x},\mathbf{p},e_n)\right) \geq 0.$$

Corollary 3.4. *Let $j_1, j_2, n \in \mathbb{N}$, $2 \leq j_1, j_2 \leq n$ and let $f : [a,b] \to \mathbb{R}$ be $(n+1)$-convex at point c. Let m-tuples $\mathbf{x} \in [a,c]^m$ and $\mathbf{p} \in \mathbb{R}^m$ satisfy* (1.6) *and* (1.7) *with n replaced by j_1, let l-tuples $\mathbf{y} \in [c,b]^l$ and $\mathbf{q} \in \mathbb{R}^l$ satisfy*

$$\sum_{i=1}^{l} q_i y_i^k = 0, \qquad \text{for all } k = 0, 1, \ldots, j_2 - 1$$

$$\sum_{i=1}^{l} q_i (y_i - t)_+^{j_2 - 1} \geq 0, \qquad \text{for every } t \in [y_{(1)}, y_{(l-n+1)}]$$

and let (3.3) *holds. If $n - j_1$ and $n - j_2$ are even, then* (3.4) *holds.*

Remark 3.5. For idea of the proof see [8, pp. 171-172].

Integral analogous of previous theorem may be stated as:

Theorem 3.6. *Let $c \in (a,b)$ and let $g : [\alpha, \beta] \to [a,c]$, $p : [\alpha, \beta] \to \mathbb{R}$, $h : [\gamma, \delta] \to [c,b]$, $q : [\gamma, \delta] \to \mathbb{R}$ be integrable functions. Let $f : I \to \mathbb{R}$, $[a,b] \subset I$ be a function such that $f^{(n-1)}$ is absolutely continuous. Let $\Omega_2^{[\cdot,\cdot]}(\cdot,\cdot,\cdot,t)$ and $A_2^{[\cdot,\cdot]}(\cdot,\cdot,\cdot,f)$ be defined as in* (2.14) *and* (2.16) *satisfy the following conditions:*

$$\Omega_2^{[a,c]}([\alpha,\beta], g, p, t) \geq 0 \quad \text{for every} \quad t \in [a,c], \tag{3.5}$$

$$\Omega_2^{[c,b]}([\gamma,\delta], h, q, t) \geq 0 \quad \text{for every} \quad t \in [c,b], \tag{3.6}$$

and

$$A_2^{[a,c]}([\alpha,\beta], g, p, e_n) = A_2^{[c,b]}([\gamma,\delta], h, q, e_n). \tag{3.7}$$

If f is $(n+1)$-convex at point c (for $k = 3$, $n \geq 3$), then

$$A_2^{[a,c]}([\alpha,\beta], g, p, f) \leq A_2^{[c,b]}([\gamma,\delta], h, q, f). \tag{3.8}$$

If inequalities in (3.5) *and* (3.6) *are reversed, then* (3.8) *holds with the reverse sign of inequality.*

Corollary 3.7. *Let $j_1, j_2, n \in \mathbb{N}$, $2 \leq j_1, j_2 \leq n$ and let $f : [a,b] \to \mathbb{R}$ be $(n+1)$-convex at point c. Let integrable functions $g : [\alpha,\beta] \to [a,c]$, $p : [\alpha,\beta] \to \mathbb{R}$ satisfy* (1.9) *with n replaced by j_1, let $h : [\gamma,\delta] \to [c,b]$, $q : [\gamma,\delta] \to \mathbb{R}$ satisfy*

$$\int_\gamma^\delta q(x) h(x)^k \, dx = 0, \qquad \text{for all } k \in \{0, 1, \ldots, j_2 - 1\}$$

$$\int_\gamma^\delta q(x) \left(h(x) - t\right)_+^{j_2 - 1} dx \geq 0, \qquad \text{for every } t \in [c,b].$$

and let (3.7) *holds. If $n - j_1$ and $n - j_2$ are even, then* (3.4) *holds.*

4. Bounds for $A_k^{[\cdot,\cdot]}(\cdot,\cdot,\cdot,f)$ and R_n^k

Let $f, h : [a,b] \to \mathbb{R}$ be two Lebesgue integrable functions. We consider the Čebyšev functional

$$T(f,h) = \frac{1}{b-a}\int_a^b f(x)h(x)dx - \left(\frac{1}{b-a}\int_a^b f(x)dx\right)\left(\frac{1}{b-a}\int_a^b h(x)dx\right). \quad (4.1)$$

The following results can be found in [3]:

Proposition 4.1. *Let $f : [a,b] \to \mathbb{R}$ be a Lebesgue integrable function and let $h : [a,b] \to \mathbb{R}$ be an absolutely continuous function with $(\cdot - a)(b - \cdot)[h']^2 \in L[a,b]$. Then we have the inequality*

$$|T(f,h)| \le \frac{1}{\sqrt{2}}\left(\frac{1}{b-a}|T(f,f)|\int_a^b (x-a)(b-x)[h'(x)]^2 dx\right)^{1/2}. \quad (4.2)$$

The constant $\frac{1}{\sqrt{2}}$ in (4.2) is the best possible.

Proposition 4.2. *Let $h : [a,b] \to \mathbb{R}$ be a monotonic nondecreasing function and let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous function such that $f' \in L_\infty[a,b]$. Then we have the inequality*

$$|T(f,h)| \le \frac{1}{2(b-a)}\|f'\|_\infty \int_a^b (x-a)(b-x)dh(x). \quad (4.3)$$

The constant $\frac{1}{2}$ in (4.3) is the best possible.

We use the well-known Hölders inequality and bound for the Čebyšev functional $T(f,h)$. This bound is given in the following proposition in which the pre-Grüss inequality is given [11].

Proposition 4.3. *Let $f, h : [a,b] \to \mathbb{R}$ be Lebesgue integrable functions such that $fh : [a,b] \in L(a,b)$. If*

$$\gamma \le h(x) \le \Gamma \quad \textit{for} \quad x \in [a,b],$$

then

$$|T(f,h)| \le \frac{1}{2}(\Gamma - \gamma)\sqrt{T(f,f)}, \quad (4.4)$$

Now by using aforementioned results, we are going to obtain generalizations of the result proved in the previous section.

Remark 4.4. For the sake of brevity, in present and next sections at some places we will use the notations $A_k(f) = A_k^{[\cdot,\cdot]}(\cdot,\cdot,\cdot,f)$ and $\Omega_k(t) = \Omega_k^{[\cdot,\cdot]}(\cdot,\cdot,\cdot,t)$ for $k \in \{1,2\}$ as defined in Theorems 2.4 and 2.6.

Now, we are ready to state main results of this section:

Theorem 4.5. *Let $f : [a,b] \to \mathbb{R}$ be such that $f^{(n)}$ is an absolutely continuous function for $n \in \mathbb{N}$ with $(. - a)(b - .)[f^{(n+1)}]^2 \in L[a,b]$. Then it holds for $k \in \{1,2\}$*

$$A_k(f) = \frac{[f^{(n-1)}(b) - f^{(n-1)}(a)]}{(n-3)!(b-a)}\int_a^b \Omega_k(s)ds + R_n^k(f;a,b),$$

where the remainder $R_n^k(f;a,b)$ satisfies the estimation

$$|R_n^k(f;a,b)| \le \frac{1}{(n-3)!}\left(\frac{(b-a)}{2}\left|T(\Omega_k,\Omega_k)\int_a^b (s-a)(b-s)[f^{(n+1)}(s)]^2 ds\right|\right)^{1/2}. \tag{4.5}$$

Proof. Fix $k \in \{1,2\}$. If we apply Proposition 4.3 for $f \to \Omega_k$ and $h \to f^{(n)}$, then we obtain

$$\left|\frac{1}{b-a}\int_a^b \Omega_k(t)f^{(n)}(t)dt - \left(\frac{1}{b-a}\int_a^b \Omega_k(t)dt\right)\left(\frac{1}{b-a}\int_a^b f^{(n)}(t)dt\right)\right|$$
$$\le \frac{1}{\sqrt{2}}\left(\frac{1}{b-a}|T(\Omega_k,\Omega_k)|\int_a^b (t-a)(b-t)[f^{(n+1)}(t)]^2 dt\right)^{1/2}.$$

Therefore we have

$$\frac{1}{(n-3)!}\int_a^b \Omega_k(t)f^{(n)}(t)dt = \frac{\left[f^{(n-1)}(b)-f^{(n-1)}(a)\right]}{(n-3)!(b-a)}\int_a^b \Omega_k(t)dt + R_n^k(f;a,b).$$

where $R_n^k(f;a,b)$ satisfies inequality (4.5). Now from identities (2.9) and (2.12) for $k \in \{1,2\}$ respectively, we obtain (4.5). □

By using Proposition 4.2 we obtain the following Grüss type inequality.

Theorem 4.6. *Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n)}$ is an absolutely continuous function for $n\in\mathbb{N}$ with $(.-a)(b-.)[f^{(n+1)}]^2 \in L[a,b]$ with $f^{(n+1)} \ge 0$ on $[a,b]$. Then we have the representation* (4.5) *and the remainder $R_n^k(f;a,b)$ satisfies the following condition for $k \in \{1,2\}$,*

$$|R_n^k(f;a,b)| \le \frac{1}{(n-3)!}\|\Omega_k'\|_\infty\left\{\frac{b-a}{2}\left[f^{(n-1)}(b)+f^{(n-1)}(a)\right]\right.$$
$$\left. -\left[f^{(n-2)}(b)-f^{(n-2)}(a)\right]\right\}. \tag{4.6}$$

Proof. Fix $k \in \{1,2\}$. If we apply Proposition 4.2 for $f \to \Omega_k$ and $h \to f^{(n)}$, then we obtain

$$\left|\frac{1}{b-a}\int_a^b \Omega_k(t)f^{(n)}(t)dt - \left(\frac{1}{b-a}\int_a^b \Omega_k(t)dt\right)\left(\frac{1}{b-a}\int_a^b f^{(n)}(t)dt\right)\right|$$
$$\le \frac{1}{2(b-a)}\|\Omega_k'\|_\infty\int_a^b (t-a)(b-t)f^{(n+1)}(t)dt.$$

Since

$$\int_a^b (t-a)(b-t)f^{(n+1)}(t)dt = \int_a^b (2t-a-b)f^{(n)}(t)dt$$
$$= (b-a)\left[f^{(n-1)}(b)+f^{(n-1)}(a)\right] - 2\left[f^{(n-2)}(b)-f^{(n-2)}(a)\right]. \tag{4.7}$$

Therefore, by using the identities (2.9) and (2.12) for $k \in \{1,2\}$ respectively and (4.7) we deduce (4.6). □

Theorem 4.7. *For $k = 1$ we assume that $\mathbf{x}$ and $\mathbf{p}$ satisfy the assumptions of Theorem 2.2 and for $k = 2$ we assume that x and p satisfy the assumptions of Theorem 2.3. Let $k \in \{1, 2\}$. Let $f : I \to \mathbb{R}$, $[a, b] \subseteq I$, be such that $f^{(n)}$ is an absolutely continuous function and*

$$\gamma \leq f^{(n)}(x) \leq \Gamma \quad \text{for} \quad x \in [a, b].$$

Then

$$A_k(f) = \frac{\left[f^{(n-1)}(b) - f^{(n-1)}(a)\right]}{(n-3)!(b-a)} \int_a^b \Omega_k(t)dt + R_n^k(f; a, b), \tag{4.8}$$

where the remainder $R_n^k(f; a, b)$ satisfies the estimation

$$|R_n^k(f; a, b)| \leq \frac{b-a}{2(n-3)!}(\Gamma - \gamma)\sqrt{T(\Omega_k, \Omega_k)}. \tag{4.9}$$

Proof. Fix $k \in \{1, 2\}$. Using definition of A_k and result from the second section we have

$$\begin{aligned} A_k(f) &= \frac{1}{(n-3)!} \int_a^b f^{(n)}(t)\Omega_k(t)dt \\ &= \frac{1}{(n-3)!(b-a)} \int_a^b f^{(n)}(t)dt \int_a^b \Omega_k(t)dt + R_n^k(f; a, b) \\ &= \frac{\left[f^{(n-1)}(b) - f^{(n-1)}(a)\right]}{(n-3)!(b-a)} \int_a^b \Omega_k(t)dt + R_n^k(f; a, b), \end{aligned}$$

where

$$R_n^k(f; a, b) = \frac{1}{(n-3)!} \left(\int_a^b f^{(n)}(t)\Omega_k(t)dt - \frac{1}{b-a} \int_a^b f^{(n)}(s)ds \int_a^b \Omega_k(t)dt \right).$$

If we apply Proposition 4.3 for $f \to \Omega_k$ and $h \to f^{(n)}$, then we obtain

$$|R_n^k(f; a, b)| = |T(\Omega_k, f^{(n)})| \leq \frac{b-a}{2(n-3)!}(\Gamma - \gamma)\sqrt{T(\Omega_k, \Omega_k)}. \qquad \square$$

Theorem 4.8. *Let $k \in \{1, 2\}$. Let (q, r) be a pair of conjugate exponents, that is, $1 \leq q, r \leq \infty$, $\frac{1}{q} + \frac{1}{r} = 1$. Let $f^{(n)} \in L_q[a, b]$ for some $n \in \mathbb{N}$, $n > 1$. Further, for $k = 1$ we assume that $\mathbf{x}$ and $\mathbf{p}$ satisfy the assumptions of Theorem 2.2 and for $k = 2$ we assume that x and p satisfy the assumptions of Theorem 2.3. Then we have*

$$|A_k(f)| \leq \frac{1}{(n-3)!} \|f^{(n)}\|_q \|\Omega_k\|_r. \tag{4.10}$$

The constant on the right hand side of (4.10) is sharp for $1 < q \leq \infty$ and the best possible for $q = 1$.

Proof. Fix $k \in \{1, 2\}$. From definition of A_k and results from the second section, applying the Hölder inequality we get

$$|A_k(f)| = \left| \frac{1}{(n-3)!} \int_a^b f^{(n)}(t)\Omega_k(t)dt \right| \leq \|f^{(n)}\|_q \|\lambda_k\|_r$$

where we denoted $\frac{1}{(n-3)!}\Omega_k$ by λ_k.

The sharpness of the constant $\left(\int_a^b |\lambda_k(t)|^r\, ds\right)^{1/r}$ can be proved by considering the following function f for which the equality in (4.10) is obtained.

For $1 < q < \infty$ we take f to be such that $f^{(n)}(s) = sgn\lambda_k(t) \cdot |\lambda_k(t)|^{1/(q-1)}$, while for $q = \infty$, we define f such that $f^{(n)}(t) = sgn\lambda_k(t)$. The fact that (4.10) is the best possible for $q = 1$, can be proved as in [7, Thm 12]. □

5. Mean value results

In this section we consider mean value theorems involving A_k. Throughout the section we use this agreement that if $k \in \{1, 2\}$, then $n \geq 3$. Further $k = 1$ we assume that $\mathbf{x}$ and $\mathbf{p}$ satisfy the assumptions of Theorem 2.2 and for $k = 2$ we assume that $\mathbf{x}$ and $\mathbf{p}$ satisfy the assumptions of Theorem 2.3.

Theorem 5.1. *Let $k \in \{1, 2\}$ and let us consider A_k as a functional on $C^n[a, b]$. If corresponding conditions from set $\{(2.13), (2.14)\}$ related to the fixed k, hold, then there exists $\xi_k \in [a, b]$ such that*

$$A_k(f) = f^{(n)}(\xi_k)A_k(f_0), \tag{5.1}$$

where $f_0(x) = \frac{x^n}{n!}$.

Proof. Let us define functions

$$F_1(x) = Mf_0(x) - f(x)$$

and

$$F_2(x) = f(x) - Lf_0(x)$$

where L and M are minimum and maximum of the image of $[a, b]$, i.e.,

$$F^{(n)}([a, b]) = [L, M]$$

Then F_1 and F_2 are $n-$convex. Hence $A_k(F_1) \geq 0$ and $A_k(F_2) \geq 0$ and

$$LA_k(f_0) \leq A_k(f) \leq MA_k(f_0).$$

If $A_k(f_0) = 0$, then the statement obviously holds.
If $A_k(f_0) \neq 0$, then $\frac{A_k(f)}{A_k(f_0)} \in [L, M] = f^{(n)}([a, b])$, so there exist $\xi_k \in [a, b]$ such that

$$\frac{A_k(f)}{A_k(f_0)} = f^{(n)}(\xi_k).$$ □

Applying Theorem 5.1 on function $\omega = A_k(h)f - A_k(f)h$, we get the following result.

Theorem 5.2. *Let $k \in \{1, 2\}$ and let us consider A_k as a functional on $C^n[a, b]$. If corresponding conditions from set $\{(2.13), (2.14)\}$ related to the fixed k, hold, then there exists $\xi_k \in [a, b]$ such that*

$$\frac{A_k(f)}{A_k(h)} = \frac{f^{(n)}(\xi_k)}{h^{(n)}(\xi_k)}$$

assuming that both the denominators are non-zero.

Remark 5.3. If the inverse of $\frac{f^{(n)}}{h^{(n)}}$ exists, then from the above mean value theorems we can give generalized means

$$\xi_k = \left(\frac{f^{(n)}}{h^{(n)}}\right)^{-1}\left(\frac{A_k(f)}{A_k(h)}\right). \tag{5.2}$$

Remark 5.4. Using the same method as in [7], we can construct new families of exponentially convex functions and Cauchy type means.

References

[1] Baloch, I.A., Pečarić, J., Praljak, M., *Generalization of Levinson's inequality*, J. Math. Inequal., **9**(2015), 571-586.

[2] Butt, S.I., Mehmood, N., Pečarić, J., *New generalization of Popoviciu type inequalities via new Green functions and Taylor's formula*, submitted.

[3] Cerone, P., Dragomir, S.S., *Some new Owstrowski-type bounds for the Čebyšev functional and applications*, J. Math. Inequal., **8**(2014), no. 1, 159-170.

[4] Fink, A.M., *Bounds of the deviation of a function from its avereges*, Czechoslovak Math. J., **42**(117)(1992), 289-310.

[5] Khan, A.R., Latif, N., Pečarić, J.E., *Exponential convexity for majorization*, J. Inequal. Appl., **2012**(2012): 105, 1-13.

[6] Khan, A.R., Pečarić, J., Lipanovic, M.R., *n-Exponential convexity for Jensen-type inequalities*, J. Math. Inequal., **7**(2013), no. 3, 313-335.

[7] Khan, A.R., Pečarić, J.E., Praljak, M., *Popoviciu type inequalities for n-convex functions via extension of Montgomery identity*, submitted.

[8] Khan, A.R., Pečarić, J., Praljak, M., Varošanec, S., *General Linear Inequalities and Positivity/ Higher order convexity*, Monographs in inequalities 12, Element, Zagreb, 2017, pp. 269.

[9] Khan, A.R., Pečarić, J., Praljak, M., Varošanec, S., *Positivity of sums and integrals of $n-$convex functions via the Fink identity*, submitted.

[10] Khan, A.R., Pečarić, J.E., Varošanec, S., *Popoviciu type characterization of positivity of sums and integrals for convex functions of higher order*, J. Math. Inequal., **7**(2013), no. 2, 195-212.

[11] Matić, M., Pečarić, J., Ujević, N., *Some new Owstrowski-type bounds for the Čebyšev functional and applications*, Computer. Math. Appl., **39**(2000), no. 3-4, 161-175.

[12] Pečarić, J., *On Jessenss inequality for convex functions, III*, J. Math. Anal. Appl., **156**(1991), 231-239.

[13] Pečarić, J., Perušić, A., Smoljak, K., *Generalizations of Steffensen's inequality by Abel-Gontscharoff polynomial*, Khayyam J. Math., **1**(2015), no. 1, 45-61.

[14] Pečarić, J., Praljak, M., Witkowski, A., *Linear operator inequality for n-convex functions at a point*, Math. Ineq. Appl., **18**(2015), 1201-1217.

[15] Pečarić, J., Proschan, F., Tong, Y.L., *Convex Functions, Partial Orderings and Statistical Applications*, Academic Press, New York, 1992.

[16] Popoviciu, T., *Notes sur les fonctions convexes d'orde superieur III,* Mathematica (Cluj), **16**(1940), 74-86.

[17] Popoviciu, T., *Notes sur les fonctions convexes d'orde superieur IV,* Disqusitiones Math., **1**(1940), 163-171.

[18] Popoviciu, T., *Les Fonctions Convexes*, Herman and Cie, Editeurs, Paris 1944.

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Stud. Univ. Babeş-Bolyai Math. 66(2021), No. 4, 629–640
DOI: 10.24193/subbmath.2021.4.03

Fractional Hadamard and Fejér-Hadamard inequalities for exponentially m-convex function

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Abstract. Fractional integral operators play a vital role in the advancement of mathematical inequalities. The aim of this paper is to present the Hadamard and the Fejér-Hadamard inequalities for generalized fractional integral operators containing Mittag-Leffler function. Exponentially m-convexity is utilized to establish these inequalities. By fixing parameters involved in the Mittag-Leffler function Hadamard and the Fejér-Hadamard inequalities for various well known fractional integral operators can be obtained.

Mathematics Subject Classification (2010): 26B25, 26A33, 26A51, 33E12.

Keywords: Convex functions, exponentially m-convex functions, Hadamard inequality, Fejér-Hadamard inequality, fractional integral operators, Mittag-Leffler function.

1. Introduction

A function $f : [a, b] \to \mathbb{R}$ is said to be convex, if for all $x, y \in [a, b]$ and $z \in [0, 1]$, the following inequality holds:

$$f(zx + (1 - z)y) \leq zf(x) + (1 - z)f(y). \tag{1.1}$$

If inequality (1.1) is reversed, then f is said to be concave.
Convex functions are equivalently defined by well known Hadamard inequality stated as follows:

Theorem 1.1. *Let $f : [a, b] \to \mathbb{R}$ be a convex function such that $a < b$. Then the following inequality holds:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.2}$$

Fejér-Hadamard inequality is a weighted version of the Hadamard inequality established by Fejér [13].

Theorem 1.2. *Let $f : [a,b] \to \mathbb{R}$ be a convex function and $g : [a,b] \to \mathbb{R}$ be a non-negative, integrable and symmetric to $\frac{a+b}{2}$. Then the following inequality holds:*

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)dx \le \int_a^b f(x)g(x)dx \le \frac{f(a)+f(b)}{2}\int_a^b g(x)dx. \tag{1.3}$$

Many researchers are continuously working on inequalities (1.2) and (1.3), and have produced very interesting results for convex and related functions (for example see, [1, 5, 6, 7, 8, 9, 12, 11, 14, 21, 22]).
Next we define exponentially convex function.

Definition 1.3. [4, 7] A function $f : [a,b] \to \mathbb{R}$ is said to be exponentially convex, if for all $x, y \in [a,b]$ and $z \in [0,1]$, the following inequality holds:

$$e^{f(zx+(1-z)y)} \le ze^{f(x)} + (1-z)e^{f(y)}. \tag{1.4}$$

The concept of exponentially m-convex functions was introduced by Rashid et al. in [18]. It is defined as follows:

Definition 1.4. A function $f : [a,b] \to \mathbb{R}$ is said to be exponentially m-convex, where $m \in (0,1]$, if for all $x, y \in [a,b]$ and $z \in [0,1]$, the following inequality holds:

$$e^{f(zx+m(1-z)y)} \le ze^{f(x)} + m(1-z)e^{f(y)}. \tag{1.5}$$

Remark 1.5. If we take $m = 1$ in (1.5), then (1.4) is achieved.

Mittag-Leffler function was introduced by the Swedish mathematician [15]. It is defined as follows:

$$E_\sigma(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(\sigma n+1)},$$

where $\Gamma(.)$ is the gamma function and $t, \sigma \in \mathbb{C}$, $\Re(\sigma) > 0$.
In the solution of kinetic equations and fractional differential equations the Mittag-Leffler function arises naturally. It is generalized by many researchers due to it's importance. Recently in [3], Andrić et al. introduced generalized Mittag-Leffler function defined as follows:

Definition 1.6. [3] Let $\mu, \sigma, l, \rho, c \in \mathbb{C}$, $\Re(\mu), \Re(\sigma), \Re(l) > 0$, $\Re(c) > \Re(\rho) > 0$ with $p \ge 0$, $r > 0$ and $0 < q \le r + \Re(\mu)$. Then the extended generalized Mittag-Leffler function $E^{\rho,r,q,c}_{\mu,\sigma,l}(t;p)$ is defined by:

$$E^{\rho,r,q,c}_{\mu,\sigma,l}(t;p) = \sum_{n=0}^{\infty} \frac{\beta_p(\rho+nq, c-\rho)}{\beta(\rho, c-\rho)} \frac{(c)_{nq}}{\Gamma(\mu n+\sigma)} \frac{t^n}{(l)_{nr}}, \tag{1.6}$$

where β_p is the generalized beta function defined by:

$$\beta_p(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}e^{-\frac{p}{t(1-t)}}dt$$

and $(c)_{nq}$ is the Pochhammer symbol defined as $(c)_{nq} = \frac{\Gamma(c+nq)}{\Gamma(c)}$.

Remark 1.7. (1.6) is a generalization of the following Mittag-Leffler functions defined by many authors:
(i) taking $p = 0$, it reduces to the Salim-Faraj function $E^{\rho,r,q,c}_{\mu,\sigma,l}(t)$ defined in [20],
(ii) taking $l = r = 1$, it reduces the function $E^{\rho,q,c}_{\mu,\sigma}(t;p)$ defined by Rahman et al. in [17],
(iii) taking $p = 0$ and $l = r = 1$, it reduces to the Shukla-Prajapati function $E^{\rho,q}_{\mu,\sigma}(t)$ defined in [23] see also [24],
(iv) taking $p = 0$ and $l = r = q = 1$, it reduces to the Prabhakar function $E^{\rho}_{\mu,\sigma}(t)$ defined in [16].

The left-sided and right-sided generalized fractional integral operators containing Mittag-Leffler function (1.6) are defined as follows:

Definition 1.8. [3] Let $\omega, \mu, \sigma, l, \rho, c \in \mathbb{C}$, $\Re(\mu), \Re(\sigma), \Re(l) > 0$, $\Re(c) > \Re(\rho) > 0$ with $p \geq 0$, $r > 0$ and $0 < q \leq r + \Re(\mu)$. Let $f \in L_1[a,b]$ and $x \in [a,b]$. Then the generalized fractional integral operators $\epsilon^{\rho,r,q,c}_{\mu,\sigma,l,\omega,a^+} f$ and $\epsilon^{\rho,r,q,c}_{\mu,\sigma,l,\omega,b^-} f$ are defined by:

$$\left(\epsilon^{\rho,r,q,c}_{\mu,\sigma,l,\omega,a^+} f\right)(x;p) = \int_a^x (x-t)^{\sigma-1} E^{\rho,r,q,c}_{\mu,\sigma,l}(\omega(x-t)^{\mu};p) f(t)dt, \tag{1.7}$$

and

$$\left(\epsilon^{\rho,r,q,c}_{\mu,\sigma,l,\omega,b^-} f\right)(x;p) = \int_x^b (t-x)^{\sigma-1} E^{\rho,r,q,c}_{\mu,\sigma,l}(\omega(t-x)^{\mu};p) f(t)dt. \tag{1.8}$$

Remark 1.9. (1.7) and (1.8) are the generalization of the following fractional integral operators defined by many authors:
(i) taking $p = 0$, it reduces to the fractional integral operators defined by Salim-Faraj in [20],
(ii) taking $l = r = 1$, it reduces to the fractional integral operators defined by Rahman et al. in [17],
(iii) taking $p = 0$ and $l = r = 1$, it reduces to the fractional integral operators defined by Srivastava-Tomovski in [24],
(iv) taking $p = 0$ and $l = r = q = 1$, it reduces to the fractional integral operators defined by Prabhakar in [16],
(v) taking $p = \omega = 0$, it reduces to the right-sided and left-sided Riemann-Liouville fractional integrals.

As shown in [2] also [10], for the constant function we have:

$$\left(\epsilon^{\rho,r,q,c}_{\mu,\sigma,l,\omega,a^+} 1\right)(x;p) = (x-a)^{\sigma} E^{\rho,r,q,c}_{\mu,\sigma+1,l}(\omega(x-a)^{\mu};p) := G_{\sigma,\omega,a^+}(x;p), \tag{1.9}$$

and

$$\left(\epsilon^{\rho,r,q,c}_{\mu,\sigma,l,\omega,b^-} 1\right)(x;p) = (b-x)^{\sigma} E^{\rho,r,q,c}_{\mu,\sigma+1,l}(\omega(b-x)^{\mu};p) := G_{\sigma,\omega,b^-}(x;p), \tag{1.10}$$

which we use in our results.

In the upcoming section, first we prove the Hadamard inequality for exponentially m-convex functions via fractional integral operators defined in (1.7) and (1.8). Also, the Fejér-Hadamard inequality for these operators is obtained. We mention results for particular fractional integral operators associated with (1.7) and (1.8).

2. Fractional Hadamard and Fejér-Hadamard inequalities for generalized fractional integral operators

First we give the fractional Hadamard inequality for exponentially m-convex functions via generalized fractional integral operators.

Theorem 2.1. *Let $\omega, \mu, \sigma, l, \rho, c \in \mathbb{C}$, $\Re(\mu), \Re(\sigma), \Re(l) > 0$, $\Re(c) > \Re(\rho) > 0$ with $p \geq 0$, $r > 0$ and $0 < q \leq r + \Re(\mu)$. Let $f : [a, mb] \subset \mathbb{R} \to \mathbb{R}$ be a function such that $f \in L_1[a, mb]$ with $a < mb$. If f is exponentially m-convex function, then the following inequalities hold:*

$$\begin{aligned} & e^{f\left(\frac{a+mb}{2}\right)} G_{\sigma,\bar{\omega},a^+}(mb;p) \\ & \leq \frac{\left(\epsilon^{\rho,r,q,c}_{\mu,\sigma,l,\bar{\omega},a^+} e^f\right)(mb;p) + m^{\sigma+1}\left(\epsilon^{\rho,r,q,c}_{\mu,\sigma,l,\bar{\omega}m^\mu,b^-} e^f\right)\left(\frac{a}{m};p\right)}{2} \\ & \leq \frac{m^{\sigma+1}}{2(mb-a)}\left[\left(e^{f(a)} - m^2 e^{f\left(\frac{a}{m^2}\right)}\right) G_{\sigma+1,\bar{\omega}m^\mu,b^-}\left(\frac{a}{m};p\right)\right. \\ & \left. +(mb-a)\left(e^{f(b)} + m e^{f\left(\frac{a}{m^2}\right)}\right) G_{\sigma,\bar{\omega}m^\mu,b^-}\left(\frac{a}{m};p\right)\right] \end{aligned} \tag{2.1}$$

where $m \in (0, 1]$ and

$$\bar{\omega} = \frac{\omega}{(mb-a)^\mu}. \tag{2.2}$$

Proof. Since f is exponentially m-convex, we have

$$e^{f\left(\frac{x+my}{2}\right)} \leq \frac{e^{f(x)} + m e^{f(y)}}{2} \quad \forall\ x, y \in [a, mb] \ \ and \ \ m \in (0, 1]. \tag{2.3}$$

Putting $x = za + m(1-z)b$ and $y = (1-z)\frac{a}{m} + zb$ in (2.3), we get

$$2e^{f\left(\frac{a+mb}{2}\right)} \leq e^{f(za+m(1-z)b)} + m e^{f((1-z)\frac{a}{m}+zb)}. \tag{2.4}$$

Also from exponentially m-convexity of f, we have

$$\begin{aligned} & e^{f(za+m(1-z)b)} + m e^{f((1-z)\frac{a}{m}+zb)} \\ & \leq z e^{f(a)} + m(1-z)e^{f(b)} + m\left(m(1-z)e^{f\left(\frac{a}{m^2}\right)} + z e^{f(b)}\right) \\ & = z\left(e^{f(a)} - m^2 e^{f\left(\frac{a}{m^2}\right)}\right) + m\left(e^{f(b)} + m e^{f\left(\frac{a}{m^2}\right)}\right). \end{aligned} \tag{2.5}$$

Multiplying both sides of (2.4) with $z^{\sigma-1}E^{\rho,r,q,c}_{\mu,\sigma,l}(\omega z^\mu;p)$ and integrating over $[0, 1]$, we have

$$\begin{aligned} & 2e^{f\left(\frac{a+mb}{2}\right)} \int_0^1 z^{\sigma-1}E^{\rho,r,q,c}_{\mu,\sigma,l}(\omega z^\mu;p)dz \\ & \leq \int_0^1 z^{\sigma-1}E^{\rho,r,q,c}_{\mu,\sigma,l}(\omega z^\mu;p)e^{f(za+m(1-z)b)}dz \\ & + m\int_0^1 z^{\sigma-1}E^{\rho,r,q,c}_{\mu,\sigma,l}(\omega z^\mu;p)e^{f((1-z)\frac{a}{m}+zb)}dz. \end{aligned} \tag{2.6}$$

Putting $u = za + m(1-z)b$ and $v = (1-z)\frac{a}{m} + zb$ in (2.6), we get

$$2e^{f\left(\frac{a+mb}{2}\right)}\int_a^{mb}(mb-u)^{\sigma-1}E^{\rho,r,q,c}_{\mu,\sigma,l}(\bar{\omega}(mb-u)^{\mu};p)du$$
$$\leq \int_a^{mb}(mb-u)^{\sigma-1}E^{\rho,r,q,c}_{\mu,\sigma,l}(\bar{\omega}(mb-u)^{\mu};p)e^{f(u)}du$$
$$+ m^{\sigma+1}\int_{\frac{a}{m}}^{b}\left(v-\frac{a}{m}\right)^{\sigma-1}E^{\rho,r,q,c}_{\mu,\sigma,l}\left(m^{\mu}\bar{\omega}(v-\frac{a}{m})^{\mu};p\right)e^{f(v)}dv.$$

By using (1.7), (1.8) and (1.9), first inequality of (2.1) is achieved.
Now multiplying both sides of (2.5) with $z^{\sigma-1}E^{\rho,r,q,c}_{\mu,\sigma,l}(\omega z^{\mu};p)$ and integrating over $[0,1]$, we have

$$\int_0^1 z^{\sigma-1}E^{\rho,r,q,c}_{\mu,\sigma,l}(\omega z^{\mu};p)e^{f(za+m(1-z)b)}dz \tag{2.7}$$
$$+ m\int_0^1 z^{\sigma-1}E^{\rho,r,q,c}_{\mu,\sigma,l}(\omega z^{\mu};p)e^{f((1-z)\frac{a}{m}+zb)}dz$$
$$\leq \left(e^{f(a)} - m^2e^{f(\frac{a}{m^2})}\right)\int_0^1 z^{\sigma}E^{\rho,r,q,c}_{\mu,\sigma,l}(\omega z^{\mu};p)dz$$
$$+ m\left(e^{f(b)} + me^{f(\frac{a}{m^2})}\right)\int_0^1 z^{\sigma-1}E^{\rho,r,q,c}_{\mu,\sigma,l}(\omega z^{\mu};p)dz.$$

Putting $u = za + m(1-z)b$ and $v = (1-z)\frac{a}{m} + zb$ in (2.7), we get

$$\int_a^{mb}(mb-u)^{\sigma-1}E^{\rho,r,q,c}_{\mu,\sigma,l}(\bar{\omega}(mb-u)^{\mu};p)e^{f(u)}du$$
$$+ m^{\sigma+1}\int_{\frac{a}{m}}^{b}\left(v-\frac{a}{m}\right)^{\sigma-1}E^{\rho,r,q,c}_{\mu,\sigma,l}\left(m^{\mu}\bar{\omega}(v-\frac{a}{m})^{\mu};p\right)e^{f(v)}dv$$
$$\leq \frac{m^{\sigma+1}}{(mb-a)}\left[\left(e^{f(a)} - m^2e^{f(\frac{a}{m^2})}\right)\int_{\frac{a}{m}}^{b}\left(v-\frac{a}{m}\right)^{\sigma}E^{\rho,r,q,c}_{\mu,\sigma,l}\left(m^{\mu}\bar{\omega}(v-\frac{a}{m})^{\mu};p\right)dv\right.$$
$$\left.+(mb-a)\left(e^{f(b)} + me^{f(\frac{a}{m^2})}\right)\int_{\frac{a}{m}}^{b}\left(v-\frac{a}{m}\right)^{\sigma-1}E^{\rho,r,q,c}_{\mu,\sigma,l}\left(m^{\mu}\bar{\omega}(v-\frac{a}{m})^{\mu};p\right)dv\right].$$

By using (1.7), (1.8) and (1.10), second inequality of (2.1) is achieved. □

Corollary 2.2. *Suppose that assumptions of Theorem* 2.1 *hold and let* $m = 1$. *Then following inequalities for exponentially convex function hold:*

$$e^{f\left(\frac{a+b}{2}\right)}G_{\sigma,\omega^*,a^+}(b;p) \leq \frac{\left(\epsilon^{\rho,r,q,c}_{\mu,\sigma,l,\omega^*,a^+}e^f\right)(b;p) + \left(\epsilon^{\rho,r,q,c}_{\mu,\sigma,l,\omega^*,b^-}e^f\right)(a;p)}{2}$$
$$\leq \frac{e^{f(a)} + e^{f(b)}}{2}G_{\sigma,\omega^*,b^-}(a;p)$$

where

$$\omega^* = \frac{\omega}{(b-a)^{\mu}}. \tag{2.8}$$

In [19], S. Rashid et. al. prove the following Hadamard inequality for exponentially m-convex function which has several misprints.

Theorem 2.3. *Let $f : [a, mb] \subset \mathbb{R} \to \mathbb{R}$ be a function such that $f \in L_1[a, mb]$ with $a < mb$. If f is exponentially m-convex function, then the following inequalities hold:*

$$2e^{f\left(\frac{a+mb}{2}\right)} G_{\sigma,\left(\frac{a+mb}{2}\right)^+}(mb;p) \tag{2.9}$$
$$\leq \left(\epsilon^{\rho,r,q,c}_{\mu,\sigma,l,\bar{\omega}2^\mu,\left(\frac{a+mb}{2}\right)^+} e^f\right)(mb;p) + m^{\sigma+1}\left(\epsilon^{\rho,r,q,c}_{\mu,\sigma,l,\bar{\omega}2^\mu,\left(\frac{a+mb}{2m}\right)^-} e^f\right)\left(\frac{a}{m};p\right)$$
$$\leq \frac{a}{(mb-a)}\left(e^{f(a)} - m^2 e^{f\left(\frac{a}{m^2}\right)}\right) G_{\sigma+1,\left(\frac{a+mb}{2}\right)^+}(mb;p)$$
$$+ m^{\sigma+1}\left(e^{f(b)} + m e^{f\left(\frac{a}{m^2}\right)}\right) G_{\sigma,\left(\frac{a+mb}{2m}\right)^-}.$$

The correct form of the above theorem is stated and proved in the following theorem.

Theorem 2.4. *Let $\omega, \mu, \sigma, l, \rho, c \in \mathbb{C}$, $\Re(\mu), \Re(\sigma), \Re(l) > 0$, $\Re(c) > \Re(\rho) > 0$ with $p \geq 0$, $r > 0$ and $0 < q \leq r + \Re(\mu)$. Let $f : [a, mb] \subset \mathbb{R} \to \mathbb{R}$ be a function such that $f \in L_1[a, mb]$ with $a < mb$. If f is exponentially m-convex function, then the following inequalities hold:*

$$e^{f\left(\frac{a+mb}{2}\right)} G_{\sigma,\bar{\omega}2^\mu,\left(\frac{a+mb}{2}\right)^+}(mb;p) \tag{2.10}$$
$$\leq \frac{\left(\epsilon^{\rho,r,q,c}_{\mu,\sigma,l,\bar{\omega}2^\mu,\left(\frac{a+mb}{2}\right)^+} e^f\right)(mb;p) + m^{\sigma+1}\left(\epsilon^{\rho,r,q,c}_{\mu,\sigma,l,\bar{\omega}(2m)^\mu,\left(\frac{a+mb}{2m}\right)^-} e^f\right)\left(\frac{a}{m};p\right)}{2}$$
$$\leq \frac{m^{\sigma+1}}{2(mb-a)}\left[\left(e^{f(a)} - m^2 e^{f\left(\frac{a}{m^2}\right)}\right) G_{\sigma+1,\bar{\omega}(2m)^\mu,\left(\frac{a+mb}{2m}\right)^-}\left(\frac{a}{m};p\right)\right.$$
$$\left.+(mb-a)\left(e^{f(b)} + m e^{f\left(\frac{a}{m^2}\right)}\right) G_{\sigma,\bar{\omega}(2m)^\mu,\left(\frac{a+mb}{2m}\right)^-}\left(\frac{a}{m};p\right)\right]$$

where $m \in (0, 1]$ and $\bar{\omega}$ is defined in (2.2).

Proof. Putting $x = \frac{z}{2}a + m\frac{(2-z)}{2}b$ and $y = \frac{z}{2}b + \frac{(2-z)}{2}\frac{a}{m}$ in (2.3), we get

$$2e^{f\left(\frac{a+mb}{2}\right)} \leq e^{f\left(\frac{z}{2}a + m\frac{(2-z)}{2}b\right)} + m e^{f\left(\frac{z}{2}b + \frac{(2-z)}{2}\frac{a}{m}\right)}. \tag{2.11}$$

Multiplying both sides of (2.11) with $z^{\sigma-1}E^{\rho,r,q,c}_{\mu,\sigma,l}(\omega z^\mu;p)$ and integrating over $[0, 1]$, we have

$$2e^{f\left(\frac{a+mb}{2}\right)} \int_0^1 z^{\sigma-1}E^{\rho,r,q,c}_{\mu,\sigma,l}(\omega z^\mu;p)dz \tag{2.12}$$
$$\leq \int_0^1 z^{\sigma-1}E^{\rho,r,q,c}_{\mu,\sigma,l}(\omega z^\mu;p)e^{f\left(\frac{z}{2}a + \frac{(2-z)}{2}b\right)}dz$$
$$+ m\int_0^1 z^{\sigma-1}E^{\rho,r,q,c}_{\mu,\sigma,l}(\omega z^\mu;p)e^{f\left(\frac{z}{2}b + \frac{(2-z)}{2}\frac{a}{m}\right)}dz.$$

Putting $u = \frac{z}{2}a + m\frac{(2-z)}{2}b$ and $v = \frac{z}{2}b + \frac{(2-z)}{2}\frac{a}{m}$ in (2.12), we get

$$\begin{aligned}
&2e^{f\left(\frac{a+mb}{2}\right)} \int_{\frac{a+mb}{2}}^{mb} (mb-u)^{\sigma-1} E_{\mu,\sigma,l}^{\rho,r,q,c}(2^{\mu}\bar{\omega}(mb-u)^{\mu};p)du \\
&\leq \int_{\frac{a+mb}{2}}^{mb} (mb-u)^{\sigma-1} E_{\mu,\sigma,l}^{\rho,r,q,c}(2^{\mu}\bar{\omega}(mb-u)^{\mu};p)e^{f(u)}du \\
&+ m^{\sigma+1} \int_{\frac{a}{m}}^{\frac{a+mb}{2m}} \left(v-\frac{a}{m}\right)^{\sigma-1} E_{\mu,\sigma,l}^{\rho,r,q,c}\left((2m)^{\mu}\bar{\omega}(v-\frac{a}{m})^{\mu};p\right) e^{f(v)}dv.
\end{aligned}$$

By using (1.7), (1.8) and (1.9), first inequality of (2.10) is achieved.
From exponentially m-convexity of f, we have

$$\begin{aligned}
&e^{f(\frac{z}{2}a+m\frac{(2-z)}{2}b)} + me^{f(\frac{z}{2}b+\frac{(2-z)}{2}\frac{a}{m})} \\
&\leq \frac{z}{2}e^{f(a)} + m\frac{(2-z)}{2}e^{f(b)} + m\left(\frac{z}{2}e^{f(b)} + m\frac{(2-z)}{2}e^{f(\frac{a}{m^2})}\right) \\
&= \frac{z}{2}\left(e^{f(a)} - m^2e^{f(\frac{a}{m^2})}\right) + m\left(e^{f(b)} + me^{f(\frac{a}{m^2})}\right).
\end{aligned} \tag{2.13}$$

Multiplying both sides of (2.13) with $z^{\sigma-1}E_{\mu,\sigma,l}^{\rho,r,q,c}(\omega z^{\mu};p)$ and integrating over $[0,1]$, we have

$$\begin{aligned}
&\int_0^1 z^{\sigma-1}E_{\mu,\sigma,l}^{\rho,r,q,c}(\omega z^{\mu};p)e^{f(\frac{z}{2}a+m\frac{(2-z)}{2}b)}dz \\
&+ m\int_0^1 z^{\sigma-1}E_{\mu,\sigma,l}^{\rho,r,q,c}(\omega z^{\mu};p)e^{f(\frac{z}{2}b+\frac{(2-z)}{2}\frac{a}{m})}dz \\
&\leq \frac{1}{2}\left(e^{f(a)} - m^2e^{f(\frac{a}{m^2})}\right)\int_0^1 z^{\sigma}E_{\mu,\sigma,l}^{\rho,r,q,c}(\omega z^{\mu};p)dz \\
&+ m\left(e^{f(b)} + me^{f(\frac{a}{m^2})}\right)\int_0^1 z^{\sigma-1}E_{\mu,\sigma,l}^{\rho,r,q,c}(\omega z^{\mu};p)dz.
\end{aligned} \tag{2.14}$$

Putting $u = \frac{z}{2}a + m\frac{(2-z)}{2}b$ and $v = \frac{z}{2}b + \frac{(2-z)}{2}\frac{a}{m}$ in (2.14), we get

$$\begin{aligned}
&\int_{\frac{a+mb}{2}}^{mb} (mb-u)^{\sigma-1}E_{\mu,\sigma,l}^{\rho,r,q,c}(2^{\mu}\bar{\omega}(mb-u)^{\mu};p)e^{f(u)}du \\
&+ m^{\sigma+1}\int_{\frac{a}{m}}^{\frac{a+mb}{2m}} \left(v-\frac{a}{m}\right)^{\sigma-1} E_{\mu,\sigma,l}^{\rho,r,q,c}\left((2m)^{\mu}\bar{\omega}(v-\frac{a}{m})^{\mu};p\right)e^{f(v)}dv \\
&\leq \frac{m^{\sigma+1}}{mb-a}\left[\left(e^{f(a)} - m^2e^{f(\frac{a}{m^2})}\right)\int_{\frac{a}{m}}^{\frac{a+mb}{2m}} \left(v-\frac{a}{m}\right)^{\sigma} E_{\mu,\sigma,l}^{\rho,r,q,c}\left((2m)^{\mu}\bar{\omega}(v-\frac{a}{m})^{\mu};p\right)dv\right. \\
&\left.+(mb-a)\left(e^{f(b)} + me^{f(\frac{a}{m^2})}\right)\int_{\frac{a}{m}}^{\frac{a+mb}{2m}} \left(v-\frac{a}{m}\right)^{\sigma-1} E_{\mu,\sigma,l}^{\rho,r,q,c}\left((2m)^{\mu}\bar{\omega}(v-\frac{a}{m})^{\mu};p\right)dv\right].
\end{aligned}$$

By using (1.7), (1.8) and (1.10), second inequality of (2.10) is achieved. □

Corollary 2.5. *Suppose that assumptions of Theorem 2.4 hold and let $m = 1$. Then following inequalities for exponentially convex function hold:*

$$\begin{aligned} & e^{f\left(\frac{a+b}{2}\right)} G_{\sigma,\omega^* 2^\mu,\left(\frac{a+b}{2}\right)^+}(b;p) \\ & \leq \frac{\left(\epsilon^{\rho,r,q,c}_{\mu,\sigma,l,\omega^* 2^\mu,\left(\frac{a+b}{2}\right)^+} e^f\right)(b;p) + \left(\epsilon^{\rho,r,q,c}_{\mu,\sigma,l,\omega^* 2^\mu,\left(\frac{a+b}{2}\right)^-} e^f\right)(a;p)}{2} \\ & \leq \frac{e^{f(a)} + e^{f(b)}}{2} G_{\sigma,\omega^* 2^\mu,\left(\frac{a+b}{2}\right)^-}(a;p) \end{aligned}$$

where ω^* is defined in (2.8).

Remark 2.6. If we take $\omega = p = 0$ in (2.10), then [18, Theorem 3.3] is obtained.

In the following we give Fejér-Hadamard inequality for exponentially m-convex functions via generalized fractional integral operators.

Theorem 2.7. *Let $\omega, \mu, \sigma, l, \rho, c \in \mathbb{C}$, $\Re(\mu), \Re(\sigma), \Re(l) > 0$, $\Re(c) > \Re(\rho) > 0$ with $p \geq 0$, $r > 0$ and $0 < q \leq r + \Re(\mu)$. Let $f : [a, mb] \subset \mathbb{R} \to \mathbb{R}$ be a function such that $f \in L_1[a, mb]$ with $a < mb$. Also, let $g : [a, mb] \to \mathbb{R}$ be a function which is non-negative and integrable. If f is exponentially m-convex function and $f(v) = f(a + mb - mv)$, then the following inequalities hold:*

$$\begin{aligned} & e^{f\left(\frac{a+mb}{2}\right)} \left(\epsilon^{\rho,r,q,c}_{\mu,\sigma,l,\bar{\omega}m^\mu,b^-} e^g\right)\left(\frac{a}{m};p\right) \\ & \leq \frac{(1+m)\left(\epsilon^{\rho,r,q,c}_{\mu,\sigma,l,\bar{\omega}m^\mu,b^-} e^f e^g\right)\left(\frac{a}{m};p\right)}{2} \\ & \leq \frac{m}{2(mb-a)} \left[\left(e^{f(a)} - m^2 e^{f\left(\frac{a}{m^2}\right)}\right)\left(\epsilon^{\rho,r,q,c}_{\mu,\sigma+1,l,\bar{\omega}m^\mu,b^-} e^g\right)\left(\frac{a}{m};p\right)\right. \\ & \left. +(mb-a)\left(e^{f(b)} + m e^{f\left(\frac{a}{m^2}\right)}\right)\left(\epsilon^{\rho,r,q,c}_{\mu,\sigma,l,\bar{\omega}m^\mu,b^-} e^g\right)\left(\frac{a}{m};p\right)\right] \end{aligned} \tag{2.15}$$

where $m \in (0, 1]$ and $\bar{\omega}$ is defined in (2.2).

Proof. Multiplying both sides of (2.4) with $z^{\sigma-1}E^{\rho,r,q,c}_{\mu,\sigma,l}(\omega z^\mu;p)e^{g((1-z)\frac{a}{m}+zb)}$ and integrating over $[0, 1]$, we have

$$\begin{aligned} & 2e^{f\left(\frac{a+mb}{2}\right)} \int_0^1 z^{\sigma-1}E^{\rho,r,q,c}_{\mu,\sigma,l}(\omega z^\mu;p)e^{g((1-z)\frac{a}{m}+zb)}dz \\ & \leq \int_0^1 z^{\sigma-1}E^{\rho,r,q,c}_{\mu,\sigma,l}(\omega z^\mu;p)e^{f(za+m(1-z)b)}e^{g((1-z)\frac{a}{m}+zb)}dz \\ & + m\int_0^1 z^{\sigma-1}E^{\rho,r,q,c}_{\mu,\sigma,l}(\omega z^\mu;p)e^{f((1-z)\frac{a}{m}+zb)}e^{g((1-z)\frac{a}{m}+zb)}dz. \end{aligned} \tag{2.16}$$

Putting $v=(1-z)\frac{a}{m}+zb$ in (2.16), we get

$$2e^{f\left(\frac{a+mb}{2}\right)}\int_{\frac{a}{m}}^{b}\left(v-\frac{a}{m}\right)^{\sigma-1}E_{\mu,\sigma,l}^{\rho,r,q,c}\left(m^{\mu}\bar{\omega}(v-\frac{a}{m})^{\mu};p\right)e^{g(v)}dv$$
$$\leq\int_{\frac{a}{m}}^{b}\left(v-\frac{a}{m}\right)^{\sigma-1}E_{\mu,\sigma,l}^{\rho,r,q,c}\left(m^{\mu}\bar{\omega}(v-\frac{a}{m})^{\mu};p\right)e^{f(a+mb-mv)}e^{g(v)}dv$$
$$+m\int_{\frac{a}{m}}^{b}\left(v-\frac{a}{m}\right)^{\sigma-1}E_{\mu,\sigma,l}^{\rho,r,q,c}\left(m^{\mu}\bar{\omega}(v-\frac{a}{m})^{\mu};p\right)e^{f(v)}e^{g(v)}dv.$$

By using (1.7), (1.8), (1.9) and given condition $f(v)=f(a+mb-mv)$, first inequality of (2.15) is achieved.
Now multiplying both sides of (2.5) with $z^{\sigma-1}E_{\mu,\sigma,l}^{\rho,r,q,c}(\omega z^{\mu};p)e^{g((1-z)\frac{a}{m}+zb)}$ and integrating over $[0,1]$, we have

$$\int_{0}^{1}z^{\sigma-1}E_{\mu,\sigma,l}^{\rho,r,q,c}(\omega z^{\mu};p)e^{f(za+m(1-z)b)}e^{g((1-z)\frac{a}{m}+zb)}dz$$
$$+m\int_{0}^{1}z^{\sigma-1}E_{\mu,\sigma,l}^{\rho,r,q,c}(\omega z^{\mu};p)e^{f((1-z)\frac{a}{m}+zb)}e^{g((1-z)\frac{a}{m}+zb)}dz.$$
$$\leq\left(e^{f(a)}-m^{2}e^{f(\frac{a}{m^{2}})}\right)\int_{0}^{1}z^{\sigma}E_{\mu,\sigma,l}^{\rho,r,q,c}(\omega z^{\mu};p)e^{g((1-z)\frac{a}{m}+zb)}dz$$
$$+m\left(e^{f(b)}+me^{f(\frac{a}{m^{2}})}\right)\int_{0}^{1}z^{\sigma-1}E_{\mu,\sigma,l}^{\rho,r,q,c}(\omega z^{\mu};p)e^{g((1-z)\frac{a}{m}+zb)}dz.$$

From above second inequality of (2.15) is achieved. □

Corollary 2.8. *Suppose that assumptions of Theorem 2.7 hold and let $m=1$. Then following inequalities for exponentially convex function hold:*

$$e^{f\left(\frac{a+b}{2}\right)}\left(\epsilon_{\mu,\sigma,l,\omega^{*},b^{-}}^{\rho,r,q,c}e^{g}\right)(a;p)\leq\left(\epsilon_{\mu,\sigma,l,\omega^{*},b^{-}}^{\rho,r,q,c}e^{f}e^{g}\right)(a;p)$$
$$\leq\frac{e^{f(a)}+e^{f(b)}}{2}\left(\epsilon_{\mu,\sigma,l,\omega^{*},b^{-}}^{\rho,r,q,c}e^{g}\right)(a;p)$$

where ω^{*} is defined in (2.8).

Theorem 2.9. *Let $\omega,\mu,\sigma,l,\rho,c\in\mathbb{C}$, $\Re(\mu),\Re(\sigma),\Re(l)>0$, $\Re(c)>\Re(\rho)>0$ with $p\geq0$, $r>0$ and $0<q\leq r+\Re(\mu)$. Let $f,g:[a,mb]\subset\mathbb{R}\to\mathbb{R}$ be the functions such that $f,g\in L_{1}[a,mb]$ with $a<mb$. If f and g are exponentially m-convex functions, then the following inequality holds:*

$$\left(\epsilon_{\mu,\sigma,l,\bar{\omega},mb^{-}}^{\rho,r,q,c}e^{f}\right)(a;p)+\left(\epsilon_{\mu,\sigma,l,\bar{\omega},a^{+}}^{\rho,r,q,c}e^{g}\right)(mb;p) \tag{2.17}$$
$$\leq\frac{1}{(mb-a)}\Big[\left(e^{g(a)}+me^{f(b)}\right)G_{\sigma+1,\bar{\omega},a^{+}}(mb;p)$$
$$+\left(e^{f(a)}+me^{g(b)}\right)\left\{(mb-a)G_{\sigma,\bar{\omega},a^{+}}(mb;p)-G_{\sigma+1,\bar{\omega},a^{+}}(mb;p)\right\}\Big]$$

where $m\in(0,1]$ and $\bar{\omega}$ is defined in (2.2).

Proof. Since f and g are exponentially m-convex, we have

$$e^{f((1-z)a+mzb)}+e^{g(za+m(1-z)b)}\leq(1-z)\left(e^{f(a)}+me^{g(b)}\right)+z\left(e^{g(a)}+me^{f(b)}\right). \tag{2.18}$$

Multiplying both sides of (2.18) with $z^{\sigma-1}E^{\rho,r,q,c}_{\mu,\sigma,l}(\omega z^{\mu};p)$ and integrating over $[0,1]$, we have

$$\begin{aligned}&\int_0^1 z^{\sigma-1}E^{\rho,r,q,c}_{\mu,\sigma,l}(\omega z^{\mu};p)e^{f((1-z)a+mzb)}dz \\ &+\int_0^1 z^{\sigma-1}E^{\rho,r,q,c}_{\mu,\sigma,l}(\omega z^{\mu};p)e^{g(za+m(1-z)b)}dz \\ &\leq\left(e^{f(a)}+me^{g(b)}\right)\int_0^1(1-z)z^{\sigma-1}E^{\rho,r,q,c}_{\mu,\sigma,l}(\omega z^{\mu};p)dz \\ &+\left(e^{g(a)}+me^{f(b)}\right)\int_0^1 z^{\sigma}E^{\rho,r,q,c}_{\mu,\sigma,l}(\omega z^{\mu};p)dz.\end{aligned} \tag{2.19}$$

Putting $u=(1-z)a+mzb$ and $v=za+m(1-z)b$ in (2.19), then by using (1.7), (1.8) and (1.9), inequality (2.17) is achieved. □

Corollary 2.10. *Suppose that assumptions of Theorem* 2.9 *hold and let* $m=1$*. Then following inequality for exponentially convex function holds:*

$$\begin{aligned}&\left(\epsilon^{\rho,r,q,c}_{\mu,\sigma,l,\omega^*,b^-}e^{f}\right)(a;p)+\left(\epsilon^{\rho,r,q,c}_{\mu,\sigma,l,\omega^*,a^+}e^{g}\right)(b;p) \\ &\leq\frac{1}{(b-a)}\Big[\left(e^{g(a)}+e^{f(b)}\right)G_{\sigma+1,\omega^*,a^+}(b;p) \\ &+\left(e^{f(a)}+e^{g(b)}\right)\left\{(b-a)G_{\sigma,\omega^*,a^+}(b;p)-G_{\sigma+1,\omega^*,a^+}(b;p)\right\}\Big].\end{aligned}$$

where ω^* *is defined in (*2.8*).*

Remark 2.11. If we take $\omega=p=0$ in (2.17), then [18, Theorem 3.2] is obtained.

Concluding remarks. The aim of this paper is to establish two versions of the fractional Hadamard inequalities for exponentially m-convex functions via generalized fractional integral operators. Further, a generalized version of the Hadamard inequality so called Fejér-Hadamard inequality is proved. The results of this paper are hold for various associated fractional integral operators.

Acknowledgment. The research work of the Ghulam Farid is supported by Higher Education Commission of Pakistan under NRPU 2016, Project No. 5421.

References

[1] Abbas, G., Farid, G., *Hadamard and Fejér-Hadamard type inequalities for harmonically convex functions via generalized fractional integrals*, J. Anal., **25**(2017), no. 1, 107-119.

[2] Andrić, M., Farid, G., Mehmood, S., Pečarić, J., *Pólya-Szegö and Chebyshev types inequalities via an extended generalized Mittag-Leffler function*, Math. Ineq. Appl., **22**(2019), no. 4, 1365-1377.

[3] Andrić, M., Farid, G., Pečarić, J., *A further extension of Mittag-Leffler function*, Fract. Calc. Appl. Anal., **21**(2018), no. 5, 1377-1395.

[4] Antczak, T., (p, r)*-invex sets and functions*, J. Math. Anal. Appl., **263**(2001), 355-379.

[5] Chen, F., *On Hermite-Hadamard type inequalities for Riemann-Liouville fractional integrals via two kinds of convexity*, Chin. J. Math., **2014**(2014), pp 7.

[6] Chen, H., Katugampola, U.N., *Hermite-Hadamard-Fejér type inequalities for generalized fractional integrals*, J. Math. Anal. Appl., **446**(2017), 1274-1291.

[7] Dragomir, S.S., Gomm, I., *Some Hermite-Hadamard type inequalities for functions whose exponentials are convex*, Stud. Univ. Babeş-Bolyai Math., **60**(2015), no. 4, 527-534.

[8] Farid, G., *Hadamard and Fejér-Hadamard inequalities for generalized fractional integral involving special functions*, Konuralp J. Math., **4**(2016), no. 1, 108-113.

[9] Farid, G., *A Treatment of the Hadamard inequality due to* m*-convexity via generalized fractional integral*, J. Fract. Calc. Appl., **9**(2018), no. 1, 8-14.

[10] Farid, G., Khan, K.A., Latif, N., Rehman, A.U., Mehmood, S., *General fractional integral inequalities for convex and* m*-convex functions via an extended generalized Mittag-Leffler function*, J. Inequal. Appl., **2018**(2018), 243 pp.

[11] Farid, G., Rehman, A.U., Mehmood, S., *Hadamard and Fejér-Hadamard type integral inequalities for harmonically convex functions via an extended generalized Mittag-Leffler function*, J. Math. Comput. Sci., **8**(2018), no. 5, 630-643.

[12] Farid, G., Rehman, A.U., Tariq, B., *On Hadamard-type inequalities for* m*-convex functions via Riemann-Liouville fractional integrals*, Stud. Univ. Babeş-Bolyai, Math., **62**(2017), no. 2, 141-150.

[13] Fejér, L., *Überdie Fourierreihen II*, Math. Naturwiss Anz Ungar. Akad. Wiss., **24**(1906), 369-390.

[14] Kang, S.M., Farid, G., Nazeer, W., Mehmood, S., (h, m)*-convex functions and associated fractional Hadamard and Fejér-Hadamard inequalities via an extended generalized Mittag-Leffler function*, J. Inequal. Appl., **2019**(2019), 78 pp.

[15] Mittag-Leffler, G.M., *Sur la nouvelle fonction* $E_\sigma(x)$, Comptes Rendus Hebdomadaires des Séances de l' Académie des Sciences Paris, **137**(1903), 554-558.

[16] Prabhakar, T.R., *A singular integral equation with a generalized Mittag-Leffler function in the kernel*, Yokohama Math. J., **19**(1971), 7-15.

[17] Rahman, G., Baleanu, D., Qurashi, M.A., Purohit, S.D., Mubeen, S., Arshad, M., *The extended Mittag-Leffler function via fractional calculus*, J. Nonlinear Sci. Appl., **10**(2017), 4244-4253.

[18] Rashid, S., Noor, M.A., Noor, K.I., *Fractional exponentially* m*-convex functions and inequalities*, Int. J. Anal. Appl., **17**(2019), no. 3, 464-478.

[19] Rashid, S., Safdar, F., Akdemir, A.O., Noor, M.A., Noor, K.I., *Some new fractional integral inequalities for exponentially* m*-convex functions via extended generalized Mittag-Leffler function*, J. Inequal. Appl., **2019**(2019), 299 pp.

[20] Salim, T.O., Faraj, A.W., *A generalization of Mittag-Leffler function and integral operator associated with integral calculus*, J. Frac. Calc. Appl., **3**(2012), no. 5, 1-13.

[21] Sarikaya, M.Z., Set, E., Yaldiz, H., Basak, N., *Hermite- Hadamard inequalities for fractional integrals and related fractional inequalities*, J. Math. Comput. Model, **57**(2013), no. 9, 2403-2407.

[22] Sarikaya, M.Z., Yildirim, H., *On Hermite-Hadamard type inequalities for Riemann-Liouville fractional integrals*, Miskolc Math. Notes, **17**(2016), no. 2, 1049-1059.

[23] Shukla, A.K., Prajapati, J.C., *On a generalization of Mittag-Leffler function and its properties*, J. Math. Anal. Appl., **336**(2007), 797-811.

[24] Srivastava, H.M., Tomovski, Z., *Fractional calculus with an integral operator containing generalized Mittag-Leffler function in the kernal*, Appl. Math. Comput., **211**(2009), no. 1, 198-210.

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Stud. Univ. Babeş-Bolyai Math. 66(2021), No. 4, 641–657
DOI: 10.24193/subbmath.2021.4.04

On (p, q)-Opial type inequalities for (p, q)-calculus

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Abstract. In this paper, we establish some (p, q)-Opial type inequalities and generalization of (p, q)-Opial type inequalities.

Mathematics Subject Classification (2010): 26D10, 26D15, 81S25.

Keywords: Opial inequality, Hölder's inequality.

1. Introduction

(p, q)-Calculus is more general from q-calculus. There have been many studies on (p, q)-calculus. Recently, Tunç and Göv [27, 28, 29] studied the concept of (p, q)-derivatives and (p, q)-integrals over the intervals of $[a, b] \subset \mathbb{R}$ and settled a number of (p, q) analogues of some well-known results like Hölder inequality, Minkowski inequality, Hermite-Hadamard inequality and Ostrowski inequality, Cauchy-Bunyakovsky-Schwarz, Gruss, Gruss- Cebysev and other integral inequalities using classical convexity. The most recently, Alp et al. in [3], proved q-Hermite-Hadamard inequality, some new q-Hermite-Hadamard inequalities, and generalized q-Hermite-Hadamard inequality, also they studied some integral inequalities which provide quantum estimates for the left part of the quantum analogue of Hermite-Hadamard inequality through q-differentiable convex and quasi-convex functions. See [10], [12], [13], [14], [15] for q and (p, q)-analysis.

Inequalities which involve integrals of functions and their derivatives, whose study has a history of about one century, are of great importance in mathematics, with far-reaching applications in the theory of differential equations, approximations and probability, among others. This class of inequalities includes the Wirtinger, Lyapunov, Landau-Kolmogorov, and Hardy types to which an abundance of literature, including several monographs, have been devoted. Of these inequalities, the earliest one which appeared in print is believed to be a Wirtinger type inequality by L. Sheeffer in 1885 (actually before the result by Wirtinger), which found its motivation in the calculus of variations. Improvements, generalizations, extensions, discretizations, and new applications of these inequalities are constantly being found, making their study an extremely prolific field. These inequalities and their manifold manifestations occupy a central position in mathematical analysis and its applications [1].

In the year 1960, Opial [17], [18] established the following interesting integral inequalities:

Theorem 1.1. *Let $x(t) \in C^{(1)}[0,h]$ be such that $x(t) > 0$ in $(0,h)$. Then, the following inequalities holds:*

i) If $x(0) = x(h) = 0$, then

$$\int_0^h |x(t)x'(t)|\, dt \leq \frac{h}{4} \int_0^h |x'(t)|^2\, dt. \tag{1.1}$$

ii) If $x(0) = 0$, then

$$\int_0^h |x(t)x'(t)|\, dt \leq \frac{h}{2} \int_0^h |x'(t)|^2\, dt. \tag{1.2}$$

In (1.1), the constant $h/4$ is the best possible.

Opial's inequality and its generalizations, extensions and discretizations, play a fundamental role in establishing the existence and uniqueness of initial and boundary value problems for ordinary and partial differential equations as well as difference equations. Over the last twenty years a large number of papers have been appeared in the literature which deals with the simple proofs, various generalizations and discrete analogues of Opial inequality and its generalizations, see [5], [7], [8], [11], [20], [22], [23], [24], [30], [4], [9], [16].

In this paper we obtain (p,q)-Opial type inequalities on (p,q)-quantum integral. If $p,q \to 1^-$ are taken, all the results we have obtained provide valid results for classical analysis.

2. Preliminaries and definitions of (p,q)-calculus

Throughout this paper, let $[a,b] \subset \mathbb{R}$ is an interval, $0 < q < p \leq 1$ are constants. The following definitions and theorems for (p,q)- derivative and (p,q)- integral are given in [27, 28].

Definition 2.1. [27, 28]For a continuous function $f : [a,b] \to \mathbb{R}$ then (p,q)- derivative of f at $t \in [a,b]$ is characterized by the expression

$$_aD_{p,q}f\,(t) \;=\; \frac{f\,(pt + (1-p)\,a) - f\,(qt + (1-q)\,a)}{(p-q)\,(t-a)},\ t \neq a. \tag{2.1}$$

Since $f : [a,b] \to \mathbb{R}$ is a continuous function, thus we have

$$_aD_{p,q}f\,(a) \;= \lim_{t\to a}\ {_aD_{p,q}f\,(t)}\ .$$

The function f is said to be (p,q)- differentiable on $[a,b]$ if $_aD_{p,q}f\,(t)$ exists for all $t \in [a,b]$. If $a = 0$ in (2.1), then $_0D_{p,q}f\,(t) = D_{p,q}f\,(t)$, where $D_{p,q}f\,(t)$ is familiar

(p,q)- derivative of f at $t \in [a,b]$ defined by the expression (see [6, 13, 21])

$$D_{p,q}f(t) = \frac{f(pt) - f(qt)}{(p-q)t}, \ t \neq 0. \tag{2.2}$$

Note also that if $p = 1$ in (2.2), then $D_q f(x)$ is familiar q- derivative of f at $x \in [a,b]$ defined by the expression (see [14])

$$D_q f(t) = \frac{f(t) - f(qt)}{(1-q)t}, \ t \neq 0. \tag{2.3}$$

Corollary 2.2. [21]*For f, g are two functions the rule of multiplicative derivative $D_{p,q}f(t)$ is*

$$D_{p,q}f(t)g(t) = f(pt)D_{p,q}g(t) + g(qt)D_{p,q}f(t).$$

We will use the following proposition throughout our work:

Proposition 2.3.

$$D_{p,q}x^n(t) = \sum_{i=0}^{n-1} x^{n-1-i}(pt)x^i(qt)D_{p,q}x(t) \tag{2.4}$$

Proof. By using rule of multiplicative derivative $D_{p,q}f(t)$ we have

$$\begin{aligned}
D_{p,q}x^n(t) &= D_{p,q}\left[x^{n-1}(t)x(t)\right] \\
&= x^{n-1}(pt)D_{p,q}x(t) + x(qt)D_{p,q}x^{n-1}(t) \\
&= x^{n-1}(pt)D_{p,q}x(t) + x(qt)\left[x^{n-2}(pt)D_{p,q}x(t) + x(qt)D_{p,q}x^{n-2}(t)\right] \\
&= \left[x^{n-1}(pt) + x^{n-2}(pt)\right]D_{p,q}x(t) + x^2(qt)D_{p,q}x^{n-2}(t) \\
&= \left[x^{n-1}(pt) + x^{n-2}(pt) + x^{n-3}(pt)\right]D_{p,q}x(t) + x^3(qt)D_{p,q}x^{n-3}(t) \\
&\ldots \\
&= \sum_{i=0}^{n-1} x^{n-1-i}(pt)x^i(qt)D_{p,q}x(t)
\end{aligned}$$

□

Definition 2.4. [27, 28]. Let $f : [a,b] \to \mathbb{R}$ be a continuous function. The definite (p,q) integral on $[a,b]$ is delineated as

$$\int_a^t f(x) \ {}_a d_{p,q}x = (p-q)(t-a)\sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} f\left(\frac{q^n}{p^{n+1}}t + \left(1 - \frac{q^n}{p^{n+1}}\right)a\right) \tag{2.5}$$

for $t \in [a, pb + (1-p)a]$. If $c \in (a,t)$, then the (p,q)- definite integral on $[c,t]$ is expressed as

$$\int_c^t f(x) \ {}_a d_{p,q}x = \int_a^t f(x) \ {}_a d_{p,q}x - \int_a^c f(x) \ {}_a d_{p,q}x \ . \tag{2.6}$$

If $p=1$ in (2.5), then one can get the classical q- definite integral on $[a,b]$ defined by (see [25, Definition 2.2])

$$\int_a^t f(x) \quad {}_a d_q x \quad = (1-q)(t-a)\sum_{n=0}^{\infty} q^n f(q^n t + (1-q^n)a). \tag{2.7}$$

If $a=0$ in (2.5), then one can get the classical (p,q)- definite integral defined by (see [21, Definition 4.])

$$\int_0^t f(x) \quad {}_0 d_{p,q} x \quad = \int_0^t f(x) \quad d_{p,q} x \quad = (p-q)\, t \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} f\left(\frac{q^n}{p^{n+1}} t\right). \tag{2.8}$$

Note also that if $p=1$ in (2.8), then one can get the classical q- definite integral defined by (see [25, Definition 2.2])

$$\int_0^t f(x) \quad {}_0 d_q x \quad = \int_0^t f(x) \quad d_q x \quad = (1-q)\, t \sum_{n=0}^{\infty} q^n f(q^n t). \tag{2.9}$$

3. Main results

First we will prove the (p,q)-Opial inequalities below and some results

Theorem 3.1 ((p,q)-Opial Inequality). *Let $x(t) \in C^{(1)}[0,h]$ be such that*

$$x(0) = x(h) = 0,$$

and $x(t) > 0$ in $(0,h)$. Then, the following inequality holds:

$$\int_0^h |x(pt) + x(qt)|\, |D_{p,q} x(t)|\, d_{p,q} t \le \frac{h}{p+q} \int_0^h |D_{p,q} x(t)|^2\, d_{p,q} t. \tag{3.1}$$

Proof. Let choosing $y(t)$ and $z(t)$ functions as

$$y(t) = \int_0^t |D_{p,q} x(s)|\, d_{p,q} s \tag{3.2}$$

$$z(t) = \int_t^h |D_{p,q} x(s)|\, d_{p,q} s,$$

such that

$$|D_{p,q} x(t)| = D_{p,q} y(t) = -D_{p,q} z(t) \tag{3.3}$$

and for $t \in [0,h]$, it follows that

$$\begin{aligned} |x(t)| &= \left| \int_0^t D_{p,q}x(s)\, d_{p,q}s \right| \leq \int_0^t |D_{p,q}x(s)|\, d_{p,q}s = y(t) \\ |x(t)| &= \left| \int_t^h D_{p,q}x(s)\, d_{p,q}s \right| \leq \int_t^h |D_{p,q}x(s)|\, d_{p,q}s = z(t). \end{aligned} \tag{3.4}$$

$$\begin{aligned} |x(qt)| &= \left| \int_0^{qt} D_{p,q}x(s)\, d_{p,q}s \right| \leq \int_0^{qt} |D_{p,q}x(s)|\, d_{p,q}s = y(qt) \\ |x(qt)| &= \left| \int_{qt}^h D_{p,q}x(s)\, d_{p,q}s \right| \leq \int_{qt}^h |D_{p,q}x(s)|\, d_{p,q}s = z(qt). \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} |x(pt)| &= \left| \int_0^{pt} D_{p,q}x(s)\, d_{p,q}s \right| \leq \int_0^{pt} |D_{p,q}x(s)|\, d_{p,q}s = y(pt) \\ |x(pt)| &= \left| \int_{pt}^h D_{p,q}x(s)\, d_{p,q}s \right| \leq \int_{pt}^h |D_{p,q}x(s)|\, d_{p,q}s = z(pt). \end{aligned} \tag{3.6}$$

Now let calculating the following (p,q)-integral by using partial (p,q)-integration method

$$\int_0^{\frac{h}{p+q}} y(pt) D_{p,q}y(t)\, d_{p,q}t = y^2\left(\frac{h}{p+q}\right) - \int_0^{\frac{h}{p+q}} y(qt) D_{p,q}y(t)\, d_{p,q}t$$

and then

$$\int_0^{\frac{h}{p+q}} \{y(pt) + y(qt)\} D_{p,q}y(t)\, d_{p,q}t = y^2\left(\frac{h}{p+q}\right). \tag{3.7}$$

By using (3.3), (3.4), (3.5), (3.6) and (3.7) we have the following inequality

$$\begin{aligned} \int_0^{\frac{h}{p+q}} |x(pt) + x(qt)|\, |D_{p,q}x(t)|\, d_{p,q}t &\leq \int_0^{\frac{h}{p+q}} \{|x(pt)| + |x(qt)|\}\, |D_{p,q}x(t)|\, d_{p,q}t \\ &\leq \int_0^{\frac{h}{p+q}} \{y(pt) + y(qt)\} D_{p,q}y(t)\, d_{p,q}t \\ &= y^2\left(\frac{h}{p+q}\right). \end{aligned} \tag{3.8}$$

Similarly we can write that

$$\begin{aligned} \int_{\frac{h}{p+q}}^{h} |x(pt)+x(qt)|\,|D_{p,q}x\,(t)|\,d_{p,q}t &\le \int_{\frac{h}{p+q}}^{h} \{|x(pt)|+|x(qt)|\}\,|D_{p,q}x\,(t)|\,d_{p,q}t \\ &\le -\int_{\frac{h}{p+q}}^{h} \{z(pt)+z(qt)\}\,D_{p,q}z\,(t)\,d_{p,q}t \\ &= z^2\left(\frac{h}{p+q}\right). \end{aligned} \tag{3.9}$$

Adding (3.8) and (3.9), we find that

$$\int_0^h |x(pt)+x(qt)|\,|D_{p,q}x\,(t)|\,d_{p,q}t \le y^2\left(\frac{h}{p+q}\right)+z^2\left(\frac{h}{p+q}\right).$$

Finally using the Cauchy-Schwarz inequality, we get

$$\begin{aligned} y^2\left(\frac{h}{p+q}\right) &= \left[\int_0^{\frac{h}{p+q}} |D_{p,q}x\,(t)|\,d_{p,q}t\right]^2 \\ &= \left[\left(\int_0^{\frac{h}{p+q}} 1^2 d_{p,q}t\right)^{1/2}\left(\int_0^{\frac{h}{p+q}} |D_{p,q}x\,(t)|^2\,d_{p,q}t\right)^{1/2}\right]^2 \\ &= \frac{h}{p+q}\int_0^{\frac{h}{p+q}} |D_{p,q}x\,(t)|^2\,d_{p,q}t. \end{aligned} \tag{3.10}$$

Similarly we have

$$z^2\left(\frac{h}{p+q}\right) = \left[\int_{\frac{h}{p+q}}^{h} |D_{p,q}x\,(t)|\,d_{p,q}t\right]^2 = \frac{h}{p+q}\int_{\frac{h}{p+q}}^{h} |D_{p,q}x\,(t)|^2\,d_{p,q}t. \tag{3.11}$$

Therefore, from (3.10) and (3.11) we obtain that

$$\int_0^h |x(pt)+x(qt)|\,|D_{p,q}x\,(t)|\,d_{p,q}t \le \frac{h}{p+q}\int_0^h |D_{p,q}x\,(t)|^2\,d_{p,q}t$$

and the proof is completed. □

Remark 3.2. In Theorem 3.1 if we take $p \to 1^-$, we recapture the following q-Opial inequality in [2]:

$$\int_0^h |x(t)+x(qt)|\,|D_q x(t)|\,d_q t \le \frac{h}{1+q}\int_0^h |D_q x(t)|^2\,d_q t.$$

Remark 3.3. In Theorem 3.1 if we take $p \to 1^-$ and $q \to 1^-$, we recapture the (1.1) inequality.

Theorem 3.4. *Let $x(t) \in C^{(1)}[0,h]$ be such that $x(0)=0$ and $x(t)>0$ in $(0,h)$. Then, the following inequality holds:*

$$\int_0^h |x(pt)+x(qt)|\,|D_{p,q} x(t)|\,d_{p,q} t \le h\int_0^h |D_{p,q} x(t)|^2\,d_{p,q} t. \tag{3.12}$$

Proof. Let choosing $y(t)$ functions as (3.2) such that

$$\begin{aligned} |x(t)| &\le y(t) \\ |D_{p,q}x(t)| &= D_{p,q}y(t) \end{aligned} \tag{3.13}$$

and then

$$\int_0^h y(pt)D_{p,q}y(t)\,d_{p,q}t = y^2(h) - \int_0^h y(qt)D_{p,q}y(t)\,d_{p,q}t,$$

i.e

$$\int_0^h \{y(pt)+y(qt)\}\,D_{p,q}y(t)\,d_{p,q}t = y^2(h). \tag{3.14}$$

Now by using Cauchy-Schwarz inequality for $y^2(h)$, we have

$$y^2(h) = \left[\int_0^h |D_{p,q}x(s)|\,d_{p,q}s\right]^2 \le h\int_0^h |D_{p,q}x(s)|^2\,d_{p,q}s.$$

Finally by using (3.13), then we have

$$\begin{aligned} \int_0^h |x(pt)+x(qt)|\,|D_{p,q}x(t)|\,d_{p,q}t &\le \int_0^h \{y(pt)+y(qt)\}\,D_{p,q}y(t)\,d_{p,q}t \\ &\le h\int_0^h |D_{p,q}x(t)|^2\,d_{p,q}t \end{aligned}$$

and the proof is completed. □

Remark 3.5. In Theorem 3.4 if we take $p \to 1^-$, we recapture the following q-Opial inequality in [2]:

$$\int_0^h |x(t) + x(qt)| \, |D_q x(t)| \, d_q t \leq h \int_0^h |D_q x(t)|^2 \, d_q t.$$

Remark 3.6. In Theorem 3.4 if we take $q \to 1^-$, we recapture the (1.2) inequality.

Theorem 3.7. *Let $k(t)$ be a nonnegative and continuous function on $[0,h]$ and $x(t) \in C^{(1)}[0,h]$ be such that $x(0) = x(h) = 0$, and $x(t) > 0$ in $(0,h)$. Then, the following inequality holds:*

$$\int_0^h k(t) \, |x(pt) + x(qt)| \, |D_{p,q} x(t)| \, d_{p,q} t \leq \left(h \int_0^h k^2(t) \, d_{p,q} t \right)^{\frac{1}{2}} \int_0^h |D_{p,q} x(t)|^2 \, d_{p,q} t$$

Proof. In proof of Theorem 3.1, we obtained that

$$|x(t)| \leq y(t) \quad \text{and} \quad |x(t)| \leq z(t)$$

Thus we get

$$\begin{aligned} |x(pt)| &\leq \frac{y(pt) + z(pt)}{2} \\ &= \frac{\int_0^{pt} D_{p,q} x(s) \, d_{p,q} s + \int_{pt}^h D_{p,q} x(s) \, d_{p,q} s}{2} \\ &= \frac{1}{2} \int_0^h |D_{p,q} x(s)| \, d_{p,q} s. \end{aligned} \tag{3.15}$$

$$\begin{aligned} |x(qt)| &\leq \frac{y(qt) + z(qt)}{2} \\ &= \frac{\int_0^{qt} D_{p,q} x(s) \, d_{p,q} s + \int_{qt}^h D_{p,q} x(s) \, d_{p,q} s}{2} \\ &= \frac{1}{2} \int_0^h |D_{p,q} x(s)| \, d_{p,q} s. \end{aligned} \tag{3.16}$$

By using the (3.15) and from Cauchy-Schwarz inequality for (p,q)-integral,

$$\begin{aligned} &\int_0^h k(t)\left|x(pt)\right|^2 d_{p,q}t \qquad (3.17)\\ \leq\ & \frac{1}{4}\int_0^h k(t)\left[\int_0^h \left|D_{p,q}x(s)\right| d_{p,q}s\right]^2 d_{p,q}t \\ \leq\ & \frac{1}{4}\left(\int_0^h k(t)\,d_{p,q}t\right)\left(\int_0^h d_{p,q}s\right)\left(\int_0^h \left|D_{p,q}x(s)\right|^2 d_{p,q}s\right) \\ \leq\ & \frac{h}{4}\left(\int_0^h k(t)\,d_{p,q}t\right)\left(\int_0^h \left|D_{p,q}x(t)\right|^2 d_{p,q}t\right). \end{aligned}$$

Similarly from (3.16) we have

$$\int_0^h k(t)\left|x(qt)\right|^2 d_{p,q}t \leq \frac{h}{4}\left(\int_0^h k(t)\,d_{p,q}t\right)\left(\int_0^h \left|D_{p,q}x(t)\right|^2 d_{p,q}t\right). \qquad (3.18)$$

From Cauchy-Schwarz inequality and (3.17), we have

$$\begin{aligned} &\int_0^h k(t)\left|x(pt)\,D_{p,q}x(t)\right| d_{p,q}t \qquad (3.19)\\ \leq\ & \left(\int_0^h k^2(t)\left|x(pt)\right|^2 d_{p,q}t\right)^{\frac{1}{2}}\left(\int_0^h \left|D_{p,q}x(t)\right|^2 d_{p,q}t\right)^{\frac{1}{2}} \\ \leq\ & \left(\frac{h}{4}\left(\int_0^h k^2(t)\,d_{p,q}t\right)\left(\int_0^h \left|D_{p,q}x(t)\right|^2 d_{p,q}t\right)\right)^{\frac{1}{2}}\left(\int_0^h \left|D_{p,q}x(t)\right|^2 d_{p,q}t\right)^{\frac{1}{2}} \\ \leq\ & \frac{1}{2}\left(h\int_0^h k^2(t)\,d_{p,q}t\right)^{\frac{1}{2}}\left(\int_0^h \left|D_{p,q}x(t)\right|^2 d_{p,q}t\right). \end{aligned}$$

Similarly, by using (3.18) we can write

$$\begin{aligned} &\int_0^h k(t)\left|x(qt)\,D_{p,q}x(t)\right| d_{p,q}t \qquad (3.20)\\ \leq\ & \frac{1}{2}\left(h\int_0^h k^2(t)\,d_{p,q}t\right)^{\frac{1}{2}}\left(\int_0^h \left|D_{p,q}x(t)\right|^2 d_{p,q}t\right) \end{aligned}$$

Finally by adding (3.19) and (3.20) we have

$$\begin{aligned}
&\int_0^h k(t)\,|x(pt)+x(qt)|\,|D_{p,q}x(t)|\,d_{p,q}t \\
\leq\ &\int_0^h k(t)\,\{|x(pt)|+|x(qt)|\}\,|D_{p,q}x(t)|\,d_{p,q}t \\
\leq\ &\left(h\int_0^h k^2(t)\,d_{p,q}t\right)^{\frac{1}{2}}\left(\int_0^h |D_{p,q}x(t)|^2\,d_{p,q}t\right)
\end{aligned}$$

which is complete the proof. □

Remark 3.8. In Theorem 3.7 if we take $p \to 1^-$, we obtain the following inequality in [2]:

$$\int_0^h k(t)\,|x(t)+x(qt)|\,|D_q x(t)|\,d_q t \leq \left(h\int_0^h k^2(t)\,d_q t\right)^{\frac{1}{2}}\int_0^h |D_q x(t)|^2\,d_q t$$

Remark 3.9. In Theorem 3.7 if we take $p \to 1^-$ and $q \to 1^-$, we recapture the following inequality

$$\int_0^h k(t)\,|x(t)x'(t)|\,dt \leq \left(\frac{h}{4}\int_0^h k^2(t)\,dt\right)^{\frac{1}{2}}\left(\int_0^h |x'(t)|^2\,dt\right)$$

which is proved by Trable in [26].

Theorem 3.10. *Let $x(t) \in C^{(1)}[0,h]$ be such that $x(0)=x(h)=0$, and $x(t)>0$ in $(0,h)$. Then, the following inequality holds:*

$$\int_0^h |x(s)|^{m(R+r)}\,d_{p,q}s \leq \frac{[K(m)]^{(R+r)}}{p^{R-1}}\int_0^h |D_{p,q}x(s)|^{m(R+r)}\left|\sum_{i=0}^{R+r-1}\left(\frac{x(qs)}{x(ps)}\right)^i\right|^{m(R+r)} d_{p,q}s \tag{3.21}$$

where

$$K(m)=\int_0^h \left[t^{1-m}+(h-t)^{1-m}\right]^{-1} d_{p,q}t.$$

Proof. Firstly we can write (p,q)-derivative of $x^n(t)$ from (2.4)

$$D_{p,q}x^n(t)=\sum_{i=0}^{n-1} x^{n-1-i}(pt)x^i(qt)D_{p,q}x(t) \tag{3.22}$$

using (3.22) we have

$$\int_0^t D_{p,q}x^{R+r}(s)d_{p,q}s = x^{R+r}(t) \tag{3.23}$$

on the other hand we can write

$$\int_0^t D_{p,q}x^{R+r}(s)d_{p,q}s = \int_0^t \sum_{i=0}^{R+r-1} x^{R+r-1-i}(ps)x^i(qs)D_{p,q}x\,(s)\,d_{p,q}s. \tag{3.24}$$

From (3.23)-(3.24) we get

$$x^{R+r}(t) = \int_0^t \sum_{i=0}^{R+r-1} x^{R+r-1-i}(ps)x^i(qs)D_{p,q}x\,(s)\,d_{p,q}s. \tag{3.25}$$

Similarly, we can write

$$x^{R+r}(t) = -\int_t^h \sum_{i=0}^{R+r-1} x^{R+r-1-i}(ps)x^i(qs)D_{p,q}x\,(s)\,d_{p,q}s. \tag{3.26}$$

Using the Hölder's inequality for (p,q)-integral with indices m, $\frac{m}{m-1}$ in (3.25) and (3.26), we have

$$\begin{aligned} & |x(t)|^{m(R+r)} \qquad (3.27)\\ \leq\ & \left(\int_0^t \left|\sum_{i=0}^{R+r-1} x^{R+r-1-i}(ps)x^i(qs)D_{p,q}x\,(s)\right| d_{p,q}s\right)^m \\ \leq\ & \left(\int_0^t \left|\sum_{i=0}^{R+r-1} x^{R+r-1-i}(ps)x^i(qs)\right|^m |D_{p,q}x\,(s)|^m\,d_{p,q}s\right)\left(\int_0^t d_{p,q}s\right)^{m-1} \\ \leq\ & t^{m-1}\left(\int_0^t \left|\sum_{i=0}^{R+r-1} x^{R+r-1-i}(ps)x^i(qs)\right|^m |D_{p,q}x\,(s)|^m\,d_{p,q}s\right). \end{aligned}$$

Similarly, we get

$$\begin{aligned} & |x(t)|^{m(R+r)} \qquad (3.28)\\ \leq\ & (h-t)^{m-1}\left(\int_0^t \left|\sum_{i=0}^{R+r-1} x^{R+r-1-i}(ps)x^i(qs)\right|^m |D_{p,q}x\,(s)|^m\,d_{p,q}s\right). \end{aligned}$$

Multiplying the (3.27) and (3.28) respectively by t^{1-m} and $(h-t)^{1-m}$ and summing these inequalities, we have

$$\left[t^{1-m}+(h-t)^{1-m}\right]|x(t)|^{m(R+r)} \tag{3.29}$$
$$\leq \left(\int_0^h \left|\sum_{i=0}^{R+r-1} x^{R+r-1-i}(ps)x^i(qs)\right|^m |D_{p,q}x(s)|^m d_{p,q}s\right)$$

and for $t\in[0,h]$ we get

$$|x(t)|^{m(R+r)} \leq \left[t^{1-m}+(h-t)^{1-m}\right]^{-1} \tag{3.30}$$
$$\times\left(\int_0^h \left|\sum_{i=0}^{R+r-1} x^{R+r-1-i}(ps)x^i(qs)\right|^m |D_{p,q}x(s)|^m d_{p,q}s\right)$$
$$=\left[t^{1-m}+(h-t)^{1-m}\right]^{-1}$$
$$\times\left(\int_0^h |x(ps)|^{m(R+r-1)}\left|\sum_{i=0}^{R+r-1}\left(\frac{x(qs)}{x(ps)}\right)^i\right|^m |D_{p,q}x(s)|^m d_{p,q}s\right)$$
$$=\left[t^{1-m}+(h-t)^{1-m}\right]^{-1}$$
$$\times\left(\int_0^h |x(ps)|^{mR/r}|D_{p,q}x(s)|^m |x(ps)|^{m(R+r-1)-mR/r}\left|\sum_{i=0}^{R+r-1}\left(\frac{x(qs)}{x(ps)}\right)^i\right|^m d_{p,q}s\right).$$

Integrating (3.30) on $[0,h]$ and using the Hölder's inequality for (p,q)-integral with indices r, $\frac{r}{r-1}$ we have

$$\int_0^h |x(t)|^{m(R+r)} d_{p,q}t \leq \int_0^h \left[t^{1-m}+(h-t)^{1-m}\right]^{-1} d_{p,q}t \tag{3.31}$$
$$\times\left(\int_0^h |x(ps)|^{mR/r}|D_{p,q}x(s)|^m |x(ps)|^{m(R+r-1)-mR/r}\left|\sum_{i=0}^{R+r-1}\left(\frac{x(qs)}{x(ps)}\right)^i\right|^m d_{p,q}s\right)$$
$$\leq K(m)\left(\int_0^h |x(ps)|^{mR}|D_{p,q}x(s)|^{mr}\left|\sum_{i=0}^{R+r-1}\left(\frac{x(qs)}{x(ps)}\right)^i\right|^{mr} d_{p,q}s\right)^{\frac{1}{r}}$$
$$\times\left(\int_0^h |x(ps)|^{m(R+r)} d_{p,q}s\right)^{\frac{r-1}{r}}$$

which by dividing the both sides of (3.31) with $\left(\int_0^h |x(ps)|^{m(R+r)} d_{p,q}s\right)^{\frac{r-1}{r}}$ and taking the rth power on both sides of resulting inequaliy. Finally by using the Hölder's inequality for (p,q)-integral with indices $\frac{R+r}{R}$, $\frac{R+r}{r}$ then, we get

$$\begin{aligned}
&\int_0^h |x(t)|^{m(R+r)}\, d_{p,q}t \qquad (3.32)\\
\leq\ & [K(m)]^r \left(\int_0^h |x(ps)|^{mR} |D_{p,q}x(s)|^{mr} \left|\sum_{i=0}^{R+r-1}\left(\frac{x(qs)}{x(ps)}\right)^i\right|^{mr} d_{p,q}s\right)\\
\leq\ & [K(m)]^r \left(\int_0^h |x(ps)|^{m(R+r)}\, d_{p,q}s\right)^{\frac{R}{R+r}}\\
&\times \left(\int_0^h |D_{p,q}x(s)|^{m(R+r)} \left|\sum_{i=0}^{R+r-1}\left(\frac{x(qs)}{x(ps)}\right)^i\right|^{m(R+r)} d_{p,q}s\right)^{\frac{r}{R+r}}
\end{aligned}$$

which by dividing the both sides of (3.32) with $\left(\int_0^h |x(ps)|^{m(R+r)}\, d_{p,q}s\right)^{\frac{R}{R+r}}$

$$\begin{aligned}
&\int_0^h |x(t)|^{m(R+r)}\, d_{p,q}t \left(\int_0^h |x(ps)|^{m(R+r)}\, d_{p,q}s\right)^{\frac{-R}{R+r}} \qquad (3.33)\\
\leq\ & [K(m)]^r \left(\int_0^h |D_{p,q}x(s)|^{m(R+r)} \left|\sum_{i=0}^{R+r-1}\left(\frac{x(qs)}{x(ps)}\right)^i\right|^{m(R+r)} d_{p,q}s\right)^{\frac{r}{R+r}}
\end{aligned}$$

Here since

$$\int_0^h |x(ps)|^{m(R+r)}\, d_{p,q}s = \frac{1}{p}\int_0^{ph} |x(s)|^{m(R+r)}\, d_{p,q}s$$

from $|x(s)|^{m(R+r)} \geq 0$ and $ph \leq h$ we can say

$$\int_0^h |x(ps)|^{m(R+r)}\, d_{p,q}s = \frac{1}{p}\int_0^{ph} |x(s)|^{m(R+r)}\, d_{p,q}s \leq \frac{1}{p}\int_0^{h} |x(s)|^{m(R+r)}\, d_{p,q}s$$

so

$$\begin{aligned}
\left(\int_0^h |x(ps)|^{m(R+r)}\, d_{p,q}s\right)^{\frac{-R}{R+r}} &= \left(\frac{1}{p}\int_0^{ph} |x(s)|^{m(R+r)}\, d_{p,q}s\right)^{\frac{-R}{R+r}} \qquad (3.34)\\
&\geq \left(\frac{1}{p}\int_0^{h} |x(s)|^{m(R+r)}\, d_{p,q}s\right)^{\frac{-R}{R+r}}.
\end{aligned}$$

From (3.34)

$$\begin{aligned} & \int_0^h |x(t)|^{m(R+r)} d_{p,q}t \left(\int_0^h |x(ps)|^{m(R+r)} d_{p,q}s \right)^{\frac{-R}{R+r}} \\ \geq & \int_0^h |x(t)|^{m(R+r)} d_{p,q}t \left(\frac{1}{p} \int_0^h |x(s)|^{m(R+r)} d_{p,q}s \right)^{\frac{-R}{R+r}} \\ = & \ p^{\frac{R}{R+r}} \left(\int_0^h |x(s)|^{m(R+r)} d_{p,q}s \right)^{\frac{r}{R+r}}. \end{aligned} \tag{3.35}$$

From (3.33) and (3.35) we get

$$\begin{aligned} & p^{\frac{R}{R+r}} \left(\int_0^h |x(s)|^{m(R+r)} d_{p,q}s \right)^{\frac{r}{R+r}} \\ \leq & [K(m)]^r \left(\int_0^h |D_{p,q}x(s)|^{m(R+r)} \left| \sum_{i=0}^{R+r-1} \left(\frac{x(qs)}{x(ps)} \right)^i \right|^{m(R+r)} d_{p,q}s \right)^{\frac{r}{R+r}}. \end{aligned} \tag{3.36}$$

Finally by taking the $\frac{R+r}{r}$th power on both sides of (3.36) we have

$$\begin{aligned} & \int_0^h |x(s)|^{m(R+r)} d_{p,q}s \\ \leq & \frac{[K(m)]^{(R+r)}}{p^{R-1}} \int_0^h |D_{p,q}x(s)|^{m(R+r)} \left| \sum_{i=0}^{R+r-1} \left(\frac{x(qs)}{x(ps)} \right)^i \right|^{m(R+r)} d_{p,q}s \end{aligned}$$

and the proof is completed. □

Remark 3.11. In Theorem 3.10 if we take $p \to 1^-$, we obtain the following inequality in [2]:

$$\int_0^h |x(s)|^{m(R+r)} d_qs \leq [K(m)]^{(R+r)} \int_0^h |D_qx(s)|^{m(R+r)} \left| \sum_{i=0}^{R+r-1} \left(\frac{x(qs)}{x(s)} \right)^i \right|^{m(R+r)} d_qs$$

which is proved by Pachpatte in [19].

Remark 3.12. In Theorem 3.10 if we take $p \to 1^-$ and $q \to 1^-$, we recapture the following result

$$\int_0^h |x(t)|^{m(R+r)} dt \leq [(R+r)^m K(m)]^{(R+r)} \int_0^h |x'(s)|^{m(R+r)} ds$$

which is proved by Pachpatte in [19].

Theorem 3.13. *Let $x(t)$ be absulately continuous on $[0,h]$, and $x(0)=0$. Further let $\alpha \geq 0$. Then, the following inequality holds:*

$$\int_0^h \left| \sum_{i=0}^{\alpha} x^{\alpha-i}(pt)x^i(qt)D_{p,q}x(t) \right| d_{p,q}t \leq h^\alpha \int_0^h |D_{p,q}x(s)|^{\alpha+1} d_{p,q}s.$$

Proof. By (p,q)-derivative of $x^n(t)$ from (2.4) we have

$$D_{p,q}y^{\alpha+1}(t) = \sum_{i=0}^{\alpha} y^{\alpha-i}(pt)y^i(qt)D_{p,q}y(t). \tag{3.37}$$

and choosing $y(t)$ as

$$y(t) = \int_0^t |D_{p,q}x(s)| \, d_{p,q}s \tag{3.38}$$

such that

$$|x(t)| \leq y(t).$$

From (3.37) we get

$$\begin{aligned} \int_0^h \left| \sum_{i=0}^{\alpha} x^{\alpha-i}(pt)x^i(qt)D_{p,q}x(t) \right| d_{p,q}t &\leq \int_0^h \sum_{i=0}^{\alpha} y^{\alpha-i}(pt)y^i(qt)D_{p,q}y(t) \, d_{p,q}t \qquad (3.39) \\ &= \int_0^h D_{p,q}y^{\alpha+1}(t) \, d_{p,q}t \\ &= y^{\alpha+1}(h). \end{aligned}$$

By using the Hölder's inequality and (3.39) with (3.38) for (p,q)-integral with indices $\alpha+1$, $\frac{\alpha+1}{\alpha}$, we get

$$\begin{aligned} y^{\alpha+1}(h) &= \left[\int_0^h |D_{p,q}x(s)| \, d_{p,q}s \right]^{\alpha+1} \\ &\leq \left[\left(\int_0^h d_{p,q}s \right)^{\frac{\alpha}{\alpha+1}} \left(\int_0^h |D_{p,q}x(s)|^{\alpha+1} d_{p,q}s \right)^{\frac{1}{\alpha+1}} \right]^{\alpha+1} \\ &= h^\alpha \int_0^h |D_{p,q}x(s)|^{\alpha+1} d_{p,q}s \end{aligned}$$

and

$$\int_0^h \left| \sum_{i=0}^{\alpha} x^{\alpha-i}(pt)x^i(qt)D_{p,q}x(t) \right| d_{p,q}t \leq h^\alpha \int_0^h |D_{p,q}x(s)|^{\alpha+1} d_{p,q}s$$

which is completes the proof. ☐

Remark 3.14. In Theorem 3.13 if we take $p \to 1^-$, we obtain the following inequality in [2]:

$$\int_0^h \left| \sum_{i=0}^{\alpha} x^{\alpha-i}(t)x^i(qt)D_q x(t) \right| d_q t \leq h^\alpha \int_0^h |D_q x(s)|^{\alpha+1} d_q s.$$

Remark 3.15. In Theorem 3.13 if we take $p \to 1^-$ and $q \to 1^-$, we recapture the following result

$$\int_0^h |x^\alpha(t)x'(t)|\, dt \leq \frac{h^\alpha}{\alpha+1} \int_0^h |x'(s)|^{\alpha+1} ds$$

which is proved by Hua in [11].

References

[1] Agarwal, R.P., *Difference Equations and Inequalities*, Marcel Dekker Inc., New York, 1992.

[2] Alp, N., Bilişik, C.C., Sarikaya, M.Z., *q-Opial type inequality for quantum integral*, Filomat, **33**(13)(2019), 4175-4184.

[3] Alp, N., Sarikaya, M. Z., Kunt, M., Iscan, I., *q-Hermite Hadamard inequalities and quantum estimates for midpoint type inequalities via convex and quasi-convex functions*, Journal of King Saud University-Science, **30**(2018), 193-203.

[4] Beesack, P.R., Das, K.M., *Extensions of Opial's inequality*, Pacific J. Math., **26**(1968), 215-232.

[5] Budak, H., Sarikaya, M.Z., *New inequalities of Opial type for conformable fractional integrals*, Turkish J. Math., **41**(2017), 1164-1173.

[6] Bukweli-Kyemba, J.D., Hounkonnou, M.N., *Quantum deformed algebras: coherent states and special functions*, arXiv:1301.0116v1, 2013.

[7] Cheung, W.S., *Some new Opial-type inequalities*, Mathematika, **37**(1990), 136-142.

[8] Cheung, W.S., *Some generalized Opial-type inequalities*, J. Math. Anal. Appl., **162**(1991), 317-321.

[9] Das, K.M., *An inequality similar to Opial's inequality*, Proc. Amer. Math. Soc., **22**(1969), 378-387.

[10] Gauchman, H., *Integral inequalities in q-calculus*, Comput. Math. Appl., **47**(2004), 281-300.

[11] Hua, L.K., *On an inequality of Opial*, Scientia Sinica, **14**(1965), 789-790.

[12] Jackson, F.H., *On a q-definite integrals*, Quarterly J. Pure Appl. Math., **41**(1910), 193-203.

[13] Jagannathan, R., Srinivasa Rao, K., *Tow-parameter quantum algebras, twin-basic numbers, and associated generalized hypergeometric series*, arXiv:math/0602613v, 2006.

[14] Kac, V., Cheung, P., *Quantum Calculus*, Springer, 2001.

[15] Kunt, M., Iscan, I., Alp, N., Sarikaya, M.Z., *(p,q)-Hermite-Hadamard inequalities and (p,q)-estimates for midpoint type inequalities via convex and quasi-convex functions,* Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat. RACSAM, DOI 10.1007/s13398-017-0402-y(2017).

[16] Mitrinovic, D.S., *Analytic Inequalities*, Springer, Berlin, 1970.

[17] Olech, C., *A simple proof of a certain result of Z. Opial*, Ann. Polon. Math., **8**(1960), 61-63.

[18] Opial, Z., *Sur une inegalité*, Ann. Polon. Math., **8**(1960), 29-32.

[19] Pachpatte, B.G., *A note on some new Opial type integral inequalities*, Octogan Math. Mag., **7**(1999), 80-84.

[20] Pachpatte, B.G., *On Opial-type integral inequalities*, J. Math. Anal. Appl., **120**(1986), 547-556.

[21] Sadjang, P.N., *On the fundamental theorem of (p,q)-calculus and some (p,q)-Taylor formulas*, arXiv:1309.3934v1, 2013.

[22] Saker, S.H., Abdou, M.D., Kubiaczyk, I., *Opial and Polya type inequalities via convexity*, Fasc. Math., **60**(2018), no. 1, 145-159.

[23] Sarıkaya, M.Z., Bilişik, C.C., *Some Opial type inequalities for conformable fractional integrals*, AIP Conference Proceedings, 1991, 020013 (2018); doi: 10.1063/1.5047886.

[24] Sarıkaya, M.Z., Bilişik, C.C., *On a new Hardy type inequalities involving fractional integrals via Opial type inequalities*, 7-9 October, 2016, Giresun, Turkey.

[25] Sudsutad, W., Ntouyas, S.K., Tariboon, J., *Quantum integral inequalities for convex functions*, J. Math. Inequal., **9**(3)(2015), 781-793.

[26] Trable, J., *On a boundary value problem for system of ordinary differential equatins of second order*, Zeszyty Nauk. Univ. Jagiello. Prace Mat., **15**(1971), 159-168.

[27] Tunç, M., Göv, E., *(p,q)-Integral inequalities*, RGMIA Res. Rep. Coll., **19**(2016), Art. 97, 1-13.

[28] Tunç, M., Göv, E., *Some integral inequalities via (p,q)-calculus on finite intervals*, RGMIA Res. Rep. Coll., **19**(2016), Art. 95, 1-12.

[29] Tunç, M., Göv, E., *(p,q)-integral inequalities for convex functions*, RGMIA Res. Rep. Coll., **19**(2016), Art. 98, 1-12.

[30] Wong, J.S.W., *A discrete analogue of Opial's inequality*, Canad. Math. Bull., **10**(1967), 115-118.

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Stud. Univ. Babeş-Bolyai Math. 66(2021), No. 4, 659–666
DOI: 10.24193/subbmath.2021.4.05

Coefficient bounds for new subclasses of analytic and m-fold symmetric bi-univalent functions

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Abstract. In the present paper, we introduce and study two new subclasses of analytic and m-fold symmetric bi-univalent functions defined in the open unit disk U. Furthermore, for functions in each of the subclasses introduced here, we obtain upper bounds for the initial coefficients $|a_{m+1}|$ and $|a_{2m+1}|$. Also, we indicate certain special cases for our results.

Mathematics Subject Classification (2010): 30C45, 30C50.

Keywords: Analytic function, univalent function, m-fold symmetric bi-univalent function, coefficient bound.

1. Introduction

Denote by $\mathcal{A}$ the class of functions f of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. Let S be the subclass of $\mathcal{A}$ consisting in functions of the form (1.1) which are also univalent in U. The Koebe one-quarter theorem (see [4]) states that the image of U under every function $f \in S$ contains a disk of radius $\frac{1}{4}$. Therefore, every function $f \in S$ has an inverse f^{-1} which satisfies $f^{-1}(f(z)) = z$, $(z \in U)$ and $f(f^{-1}(w)) = w$, $(|w| < r_0(f), r_0(f) \geq \frac{1}{4})$, where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots . \tag{1.2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in U if both f and f^{-1} are univalent in U. We denote by Σ the class of bi-univalent functions in U given by (1.1). For a brief history and interesting examples in the class Σ see [14], (see also [6, 7, 10, 11, 12]).

For each function $f \in S$, the function $h(z) = (f(z^m))^{\frac{1}{m}}$, $(z \in U, m \in \mathbb{N})$ is univalent and maps the unit disk U into a region with m-fold symmetry. A function is said to be m-fold symmetric (see [8]) if it has the following normalized form:

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1}, \ (z \in U, m \in \mathbb{N}). \tag{1.3}$$

Let S_m stands for the class of m-fold symmetric univalent functions in U, which are normalized by the series expansion (1.3). In fact, the functions in the class S are one-fold symmetric.

In [15] Srivastava et al. defined m-fold symmetric bi-univalent functions analogues to the concept of m-fold symmetric univalent functions. They gave some important results, such as each function $f \in \Sigma$ generates an m-fold symmetric bi-univalent function for each $m \in \mathbb{N}$. Furthermore, for the normalized form of f given by (1.3), they obtained the series expansion for f^{-1} as follows:

$$\begin{aligned} g(w) &= w - a_{m+1} w^{m+1} + \left[(m+1)a_{m+1}^2 - a_{2m+1}\right] w^{2m+1} \\ &- \left[\frac{1}{2}(m+1)(3m+2)a_{m+1}^3 - (3m+2)a_{m+1}a_{2m+1} + a_{3m+1}\right] w^{3m+1} + \cdots , \end{aligned} \tag{1.4}$$

where $f^{-1} = g$. We denote by Σ_m the class of m-fold symmetric bi-univalent functions in U. It is easily seen that for $m = 1$, the formula (1.4) coincides with the formula (1.2) of the class Σ. Some examples of m-fold symmetric bi-univalent functions are given as follows:

$$\left(\frac{z^m}{1-z^m}\right)^{\frac{1}{m}}, \ \left[\frac{1}{2}\log\left(\frac{1+z^m}{1-z^m}\right)\right]^{\frac{1}{m}} \ \textit{and} \ \left[-\log\left(1-z^m\right)\right]^{\frac{1}{m}}$$

with the corresponding inverse functions

$$\left(\frac{w^m}{1+w^m}\right)^{\frac{1}{m}}, \ \left(\frac{e^{2w^m}-1}{e^{2w^m}+1}\right)^{\frac{1}{m}} \ \textit{and} \ \left(\frac{e^{w^m}-1}{e^{w^m}}\right)^{\frac{1}{m}},$$

respectively.

Recently, many authors investigated bounds for various subclasses of m-fold bi-univalent functions (see [1, 2, 5, 13, 15, 16, 17]).

The purpose of the present investigation is to introduce the new subclasses $\mathcal{AS}_{\Sigma_m}(\gamma, \lambda; \alpha)$ and $\mathcal{AS}^*_{\Sigma_m}(\gamma, \lambda; \beta)$ of Σ_m and find estimates on the coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ for functions in each of these new subclasses.

We will require the following lemma in proving our main results.

Lemma 1.1. [3] *If $h \in \mathcal{P}$, then $|c_k| \leq 2$ for each $k \in \mathbb{N}$, where $\mathcal{P}$ is the family of all functions h analytic in U for which*

$$Re\,(h(z)) > 0, \quad (z \in U),$$

where

$$h(z) = 1 + c_1 z + c_2 z^2 + \cdots , \quad (z \in U).$$

2. Coefficient bounds for the function class $\mathcal{AS}_{\Sigma_m}(\gamma, \lambda; \alpha)$

Definition 2.1. A function $f \in \Sigma_m$ given by (1.3) is said to be in the class $\mathcal{AS}_{\Sigma_m}(\gamma, \lambda; \alpha)$ if it satisfies the following conditions:

$$\left| arg\left(\left[(1-\lambda)\frac{zf'(z)}{f(z)} + \lambda\left(1+\frac{zf''(z)}{f'(z)}\right)\right]^{\gamma}\right)\right| < \frac{\alpha\pi}{2} \tag{2.1}$$

and

$$\left| arg\left(\left[(1-\lambda)\frac{wg'(w)}{g(w)} + \lambda\left(1+\frac{wg''(w)}{g'(w)}\right)\right]^{\gamma}\right)\right| < \frac{\alpha\pi}{2}, \tag{2.2}$$

$$\left(z, w \in U, 0 < \alpha \leq 1,\ 0 \leq \gamma \leq 1,\ 0 \leq \lambda \leq 1,\ m \in \mathbb{N}\right),$$

where the function $g = f^{-1}$ is given by (1.4).

In particular, for one-fold symmetric bi-univalent functions, we denote the class $\mathcal{AS}_{\Sigma_1}(\gamma, \lambda; \alpha) = \mathcal{AS}_{\Sigma}(\gamma, \lambda; \alpha)$.

Remark 2.2. It should be remarked that the classes $\mathcal{AS}_{\Sigma_m}(\gamma, \lambda; \alpha)$ and $\mathcal{AS}_{\Sigma}(\gamma, \lambda; \alpha)$ are a generalization of well-known classes consider earlier. These classes are:
(1) For $\lambda = 0$ and $\gamma = 1$, the class $\mathcal{AS}_{\Sigma_m}(\gamma, \lambda; \alpha)$ reduce to the class $S^{\alpha}_{\Sigma_m}$ which was considered by Altinkaya and Yalçin [1];
(2) For $\gamma = 1$, the class $\mathcal{AS}_{\Sigma}(\gamma, \lambda; \alpha)$ reduce to the class $M_{\Sigma}(\alpha, \lambda)$ which was introduced by Liu and Wang [9];
(3) For $\lambda = 0$ and $\gamma = 1$, the class $\mathcal{AS}_{\Sigma}(\gamma, \lambda; \alpha)$ reduce to the class $S^{*}_{\Sigma}(\alpha)$ which was given by Brannan and Taha [3].

Theorem 2.3. *Let $f \in \mathcal{AS}_{\Sigma_m}(\gamma, \lambda; \alpha)$ $(0 < \alpha \leq 1,\ 0 \leq \gamma \leq 1,\ 0 \leq \lambda \leq 1,\ m \in \mathbb{N})$ be given by (1.3). Then*

$$|a_{m+1}| \leq \frac{2\alpha}{m\sqrt{2\alpha\gamma(1+\lambda m) + \gamma(\gamma-\alpha)\left(1+\lambda m\right)^2}} \tag{2.3}$$

and

$$|a_{2m+1}| \leq \frac{2\alpha^2(m+1)}{m^2\gamma^2\left(1+\lambda m\right)^2} + \frac{\alpha}{m\gamma(1+2\lambda m)}. \tag{2.4}$$

Proof. It follows from conditions (2.1) and (2.2) that

$$\left[(1-\lambda)\frac{zf'(z)}{f(z)} + \lambda\left(1+\frac{zf''(z)}{f'(z)}\right)\right]^{\gamma} = [p(z)]^{\alpha} \tag{2.5}$$

and

$$\left[(1-\lambda)\frac{wg'(w)}{g(w)} + \lambda\left(1+\frac{wg''(w)}{g'(w)}\right)\right]^{\gamma} = [q(w)]^{\alpha}, \tag{2.6}$$

where $g = f^{-1}$ and p, q in $\mathcal{P}$ have the following series representations:

$$p(z) = 1 + p_m z^m + p_{2m} z^{2m} + p_{3m} z^{3m} + \cdots \tag{2.7}$$

and

$$q(w) = 1 + q_m w^m + q_{2m} w^{2m} + q_{3m} w^{3m} + \cdots. \tag{2.8}$$

Comparing the corresponding coefficients of (2.5) and (2.6) yields

$$m\gamma(1+\lambda m)a_{m+1} = \alpha p_m, \tag{2.9}$$

$$\begin{aligned} &m\left[2\gamma(1+2\lambda m)a_{2m+1} - \gamma\left(\lambda m^2+2\lambda m+1\right)a_{m+1}^2\right] \\ &+\frac{m^2}{2}\gamma(1+\lambda m)(\gamma-1)(1+\lambda m)a_{m+1}^2 \\ &= \alpha p_{2m} + \frac{\alpha(\alpha-1)}{2}p_m^2, \end{aligned} \tag{2.10}$$

$$-m\gamma(1+\lambda m)a_{m+1} = \alpha q_m \tag{2.11}$$

and

$$\begin{aligned} &m\left[\gamma\left(3\lambda m^2+2(\lambda+1)m+1\right)a_{m+1}^2 - 2\gamma(1+2\lambda m)a_{2m+1}\right] \\ &+\frac{m^2}{2}\gamma(1+\lambda m)(\gamma-1)(1+\lambda m)a_{m+1}^2 = \alpha q_{2m} + \frac{\alpha(\alpha-1)}{2}q_m^2. \end{aligned} \tag{2.12}$$

Making use of (2.9) and (2.11), we obtain

$$p_m = -q_m \tag{2.13}$$

and

$$2m^2\gamma^2\left(1+\lambda m\right)^2 a_{m+1}^2 = \alpha^2(p_m^2+q_m^2). \tag{2.14}$$

Also, from (2.10), (2.12) and (2.14), we find that

$$\begin{aligned} &m^2\left[2\gamma\left(1+\lambda m\right)+\gamma(\gamma-1)\left(1+\lambda m\right)^2\right]a_{m+1}^2 \\ &= \alpha(p_{2m}+q_{2m}) + \frac{\alpha(\alpha-1)}{2}\left(p_m^2+q_m^2\right) \\ &= \alpha(p_{2m}+q_{2m}) + \frac{m^2\gamma^2(\alpha-1)\left(1+\lambda m\right)^2}{\alpha}a_{m+1}^2. \end{aligned}$$

Therefore, we have

$$a_{m+1}^2 = \frac{\alpha^2(p_{2m}+q_{2m})}{m^2\left[2\alpha\gamma(1+\lambda m)+\gamma(\gamma-\alpha)\left(1+\lambda m\right)^2\right]}. \tag{2.15}$$

Now, taking the absolute value of (2.15) and applying Lemma 1.1 for the coefficients p_{2m} and q_{2m}, we deduce that

$$|a_{m+1}| \le \frac{2\alpha}{m\sqrt{2\alpha\gamma(1+\lambda m)+\gamma(\gamma-\alpha)\left(1+\lambda m\right)^2}}.$$

This gives the desired estimate for $|a_{m+1}|$ as asserted in (2.3).
In order to find the bound on $|a_{2m+1}|$, by subtracting (2.12) from (2.10), we get

$$2m\gamma(1+2\lambda m)\left[2a_{2m+1}-(m+1)a_{m+1}^2\right] = \alpha\left(p_{2m}-q_{2m}\right) + \frac{\alpha(\alpha-1)}{2}\left(p_m^2-q_m^2\right). \tag{2.16}$$

It follows from (2.13), (2.14) and (2.16) that

$$a_{2m+1}=\frac{\alpha^2(m+1)\left(p_m^2+q_m^2\right)}{4m^2\gamma^2\left(1+\lambda m\right)^2}+\frac{\alpha\left(p_{2m}-q_{2m}\right)}{4m\gamma(1+2\lambda m)}. \tag{2.17}$$

Taking the absolute value of (2.17) and applying Lemma 1.1 once again for the coefficients p_m, p_{2m}, q_m and q_{2m}, we obtain

$$|a_{2m+1}|\leq\frac{2\alpha^2(m+1)}{m^2\gamma^2\left(1+\lambda m\right)^2}+\frac{\alpha}{m\gamma(1+2\lambda m)},$$

which completes the proof of Theorem 2.3. □

For one-fold symmetric bi-univalent functions, Theorem 2.3 reduce to the following corollary:

Corollary 2.4. *Let* $f\in\mathcal{AS}_\Sigma(\gamma,\lambda;\alpha)$ $(0<\alpha\leq 1,\ 0\leq\gamma\leq 1,\ 0\leq\lambda\leq 1)$ *be given by (*1.1*). Then*

$$|a_2|\leq\frac{2\alpha}{\sqrt{2\alpha\gamma(1+\lambda)+\gamma(\gamma-\alpha)\left(1+\lambda\right)^2}}$$

and

$$|a_3|\leq\frac{4\alpha^2}{\gamma^2\left(1+\lambda\right)^2}+\frac{\alpha}{\gamma(1+2\lambda)}.$$

3. Coefficient bounds for the function class $\mathcal{AS}^*_{\Sigma_m}(\gamma,\lambda;\beta)$

Definition 3.1. A function $f\in\Sigma_m$ given by (1.3) is said to be in the class $\mathcal{AS}^*_{\Sigma_m}(\gamma,\lambda;\beta)$ if it satisfies the following conditions:

$$Re\left\{\left[(1-\lambda)\frac{zf'(z)}{f(z)}+\lambda\left(1+\frac{zf''(z)}{f'(z)}\right)\right]^\gamma\right\}>\beta \tag{3.1}$$

and

$$Re\left\{\left[(1-\lambda)\frac{wg'(w)}{g(w)}+\lambda\left(1+\frac{wg''(w)}{g'(w)}\right)\right]^\gamma\right\}>\beta, \tag{3.2}$$

$$\left(z,w\in U,0\leq\beta<1,\ 0\leq\gamma\leq 1,\ 0\leq\lambda\leq 1,\ m\in\mathbb{N}\right),$$

where the function $g=f^{-1}$ is given by (1.4).

In particular, for one-fold symmetric bi-univalent functions, we denote the class

$$\mathcal{AS}^*_{\Sigma_1}(\gamma,\lambda;\beta)=\mathcal{AS}^*_{\Sigma}(\gamma,\lambda;\beta).$$

Remark 3.2. It should be remarked that the classes $\mathcal{AS}^*_{\Sigma_m}(\gamma,\lambda;\beta)$ and $\mathcal{AS}^*_{\Sigma}(\gamma,\lambda;\beta)$ are a generalization of well-known classes consider earlier. These classes are:
(1) For $\lambda=0$ and $\gamma=1$, the class $\mathcal{AS}^*_{\Sigma_m}(\gamma,\lambda;\beta)$ reduce to the class $S^\beta_{\Sigma_m}$ which was considered by Altinkaya and Yalçin [1];
(2) For $\gamma=1$, the class $\mathcal{AS}^*_{\Sigma}(\gamma,\lambda;\beta)$ reduce to the class $B_\Sigma(\beta,\tau)$ which was introduced by Liu and Wang [9];
(3) For $\lambda=0$ and $\gamma=1$, the class $\mathcal{AS}^*_{\Sigma}(\gamma,\lambda;\beta)$ reduce to the class $S^*_\Sigma(\beta)$ which was given by Brannan and Taha [3].

Theorem 3.3. *Let* $f \in \mathcal{AS}^{*}_{\Sigma_m}(\gamma, \lambda; \beta)$ $(0 \leq \beta < 1,\ 0 \leq \gamma \leq 1,\ 0 \leq \lambda \leq 1,\ m \in \mathbb{N})$ *be given by (1.3). Then*

$$|a_{m+1}| \leq \frac{2}{m}\sqrt{\frac{1-\beta}{2\gamma(1+\lambda m)+\gamma(\gamma-1)\left(1+\lambda m\right)^2}} \tag{3.3}$$

and

$$|a_{2m+1}| \leq \frac{2(m+1)\left(1-\beta\right)^2}{m^2\gamma^2\left(1+\lambda m\right)^2} + \frac{1-\beta}{m\gamma(1+2\lambda m)}. \tag{3.4}$$

Proof. It follows from conditions (3.1) and (3.2) that there exist $p, q \in \mathcal{P}$ such that

$$\left[(1-\lambda)\frac{zf'(z)}{f(z)} + \lambda\left(1+\frac{zf''(z)}{f'(z)}\right)\right]^{\gamma} = \beta + (1-\beta)p(z) \tag{3.5}$$

and

$$\left[(1-\lambda)\frac{wg'(w)}{g(w)} + \lambda\left(1+\frac{wg''(w)}{g'(w)}\right)\right]^{\gamma} = \beta + (1-\beta)q(w), \tag{3.6}$$

where $p(z)$ and $q(w)$ have the forms (2.7) and (2.8), respectively. Equating coefficients (3.5) and (3.6) yields

$$m\gamma(1+\lambda m)a_{m+1} = (1-\beta)p_m, \tag{3.7}$$

$$m\left[2\gamma(1+2\lambda m)a_{2m+1} - \gamma\left(\lambda m^2+2\lambda m+1\right)a_{m+1}^2\right] + \frac{m^2\gamma}{2}(1+\lambda m)(\gamma-1)(1+\lambda m)a_{m+1}^2 = (1-\beta)p_{2m}, \tag{3.8}$$

$$-m\gamma(1+\lambda m)a_{m+1} = (1-\beta)q_m \tag{3.9}$$

and

$$m\left[\gamma\left(3\lambda m^2+2(\lambda+1)m+1\right)a_{m+1}^2 - 2\gamma(1+2\lambda m)a_{2m+1}\right] + \frac{m^2\gamma}{2}(1+\lambda m)(\gamma-1)(1+\lambda m)a_{m+1}^2 = (1-\beta)q_{2m}. \tag{3.10}$$

From (3.7) and (3.9), we get

$$p_m = -q_m \tag{3.11}$$

and

$$2m^2\gamma^2\left(1+\lambda m\right)^2 a_{m+1}^2 = \left(1-\beta\right)^2\left(p_m^2+q_m^2\right). \tag{3.12}$$

Adding (3.8) and (3.10), we obtain

$$m^2\left[2\gamma\left(1+\lambda m\right)+\gamma(\gamma-1)\left(1+\lambda m\right)^2\right]a_{m+1}^2 = (1-\beta)(p_{2m}+q_{2m}). \tag{3.13}$$

Therefore, we have

$$a_{m+1}^2 = \frac{(1-\beta)(p_{2m}+q_{2m})}{m^2\left[2\gamma\left(1+\lambda m\right)+\gamma(\gamma-1)\left(1+\lambda m\right)^2\right]}.$$

Applying Lemma 1.1 for the coefficients p_{2m} and q_{2m}, we obtain

$$|a_{m+1}| \leq \frac{2}{m}\sqrt{\frac{1-\beta}{2\gamma(1+\lambda m)+\gamma(\gamma-1)\left(1+\lambda m\right)^2}}.$$

This gives the desired estimate for $|a_{m+1}|$ as asserted in (3.3).
In order to find the bound on $|a_{2m+1}|$, by subtracting (3.10) from (3.8), we get

$$2m\gamma(1+2\lambda m)\left[2a_{2m+1}-(m+1)a_{m+1}^2\right]=(1-\beta)\left(p_{2m}-q_{2m}\right).$$

or equivalently

$$a_{2m+1}=\frac{m+1}{2}a_{m+1}^2+\frac{(1-\beta)\left(p_{2m}-q_{2m}\right)}{4m\gamma(1+2\lambda m)}.$$

Upon substituting the value of a_{m+1}^2 from (3.12), it follows that

$$a_{2m+1}=\frac{(m+1)\left(1-\beta\right)^2\left(p_m^2+q_m^2\right)}{4m^2\gamma^2\left(1+\lambda m\right)^2}+\frac{(1-\beta)\left(p_{2m}-q_{2m}\right)}{4m\gamma(1+2\lambda m)}.$$

Applying Lemma 1.1 once again for the coefficients p_m, p_{2m}, q_m and q_{2m}, we obtain

$$|a_{2m+1}|\leq\frac{2(m+1)\left(1-\beta\right)^2}{m^2\gamma^2\left(1+\lambda m\right)^2}+\frac{1-\beta}{m\gamma(1+2\lambda m)}.$$

which completes the proof of Theorem 3.3. □

For one-fold symmetric bi-univalent functions, Theorem 3.3 reduce to the following corollary:

Corollary 3.4. *Let* $f\in\mathcal{AS}_\Sigma^*(\gamma,\lambda;\beta)$ $(0\leq\beta<1,\ 0\leq\gamma\leq1,\ 0\leq\lambda\leq1)$ *be given by* (1.1)*. Then*

$$|a_2|\leq2\sqrt{\frac{1-\beta}{2\gamma(1+\lambda)+\gamma(\gamma-1)\left(1+\lambda\right)^2}}$$

and

$$|a_3|\leq\frac{4\left(1-\beta\right)^2}{\gamma^2\left(1+\lambda\right)^2}+\frac{1-\beta}{\gamma(1+2\lambda)}.$$

Acknowledgement. The present work has received financial support through the project: Entrepreneurship for innovation through doctoral and postdoctoral research, POCU/360/6/13/123886 co-financed by the European Social Fund, through the Operational Program for Human Capital 2014- 2020 (for the author Páll-Szabó Ágnes Orsolya).

References

[1] Altinkaya, Ş., Yalçin, S., *Coefficient bounds for certain subclasses of m-fold symmetric bi-univalent functions*, Journal of Mathematics, Art. ID **241683**(2015), 1-5.

[2] Altinkaya, Ş., Yalçin, S., *On some subclasses of m-fold symmetric bi-univalent functions*, Commun. Fac. Sci. Univ. Ank. Series A1, **67**(2018), no. 1, 29-36.

[3] Brannan, D.A., Taha, T.S., *On Some classes of bi-univalent functions*, Stud. Univ. Babeş-Bolyai Math., **31**(1986), no. 2, 70-77.

[4] Duren, P.L., *Univalent Functions*, Grundlehren der Mathematischen Wissenschaften, Band 259, Springer Verlag, New York, Berlin, Heidelberg and Tokyo, 1983.

[5] Eker, S.S., *Coefficient bounds for subclasses of m-fold symmetric bi-univalent functions*, Turk. J. Math., **40**(2016), 641-646.

[6] Frasin, B.A., Aouf, M.K., *New subclasses of bi-univalent functions*, Appl. Math. Lett., **24**(2011), 1569-1573.

[7] Goyal, S.P., Goswami, P., *Estimate for initial Maclaurin coefficients of bi-univalent functions for a class defined by fractional derivatives*, J. Egyptian Math. Soc., **20**(2012), 179-182.

[8] Koepf, W., *Coefficients of symmetric functions of bounded boundary rotations*, Proc. Amer. Math. Soc., **105**(1989), 324-329.

[9] Li, X.F., Wang, A.P., *Two new subclasses of bi-univalent functions*, Int. Math. Forum, **7**(2012), 1495-1504.

[10] Srivastava, H.M., Bansal, D., *Coefficient estimates for a subclass of analytic and bi-univalent functions*, J. Egyptian Math. Soc., **23**(2015), 242-246.

[11] Srivastava, H.M., Bulut, S., Caglar, M., Yagmur, N., *Coefficient estimates for a general subclass of analytic and bi-univalent functions*, Filomat, **27**(2013), no. 5, 831-842.

[12] Srivastava, H.M., Eker, S.S., Ali, R.M., *Coefficient bounds for a certain class of analytic and bi-univalent functions*, Filomat, **29**(2015), 1839-1845.

[13] Srivastava, H.M., Gaboury, S., Ghanim, F., *Initial coefficient estimates for some subclasses of m-fold symmetric bi-univalent functions*, Acta Mathematica Scientia, **36B**(2016), no. 3, 863-871.

[14] Srivastava, H.M., Mishra, A.K., Gochhayat, P., *Certain subclasses of analytic and bi-univalent functions*, Appl. Math. Lett., **23**(2010), 1188-1192.

[15] Srivastava, H.M., Sivasubramanian, S., Sivakumar, R., *Initial coefficient bounds for a subclass of m-fold symmetric bi-univalent functions*, Tbilisi Math. J., **7**(2014), no. 2, 1-10.

[16] Srivastava, H.M., Wanas, A.K., *Initial Maclaurin coefficient bounds for new subclasses of analytic and m-fold symmetric bi-univalent functions defined by a linear combination*, Kyungpook Mathematical Journal, **59**(2019), no. 3, 493-503.

[17] Tang, H., Srivastava, H.M., Sivasubramanian, S., Gurusamy, P., *The Fekete-Szegö functional problems for some subclasses of m-fold symmetric bi-univalent functions*, J. Math. Inequal., **10**(2016), 1063-1092.

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Stud. Univ. Babeş-Bolyai Math. 66(2021), No. 4, 667–675
DOI: 10.24193/subbmath.2021.4.06

Strong subordination and superordination with sandwich-type theorems using integral operators

Parviz Arjomandinia and Rasoul Aghalary

Abstract. The notions of strong differential subordination and superordination have been studied recently by many authors. In the present paper, using these concepts, we obtain some preserving properties of certain nonlinear integral operator defined on the space of normalized analytic functions in $\mathbb{D}\times\overline{\mathbb{D}}$. The sandwich-type theorems and consequences of the main results are also considered.

Mathematics Subject Classification (2010): 30C45, 30C80.

Keywords: Univalent function, integral operator, strong differential subordination and superordination.

1. Introduction

Let $\mathcal{H} = \mathcal{H}(\mathbb{D})$ denote the class of analytic functions in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{H}^* = \mathcal{H}(\mathbb{D} \times \overline{\mathbb{D}})$ be the class of analytic functions in $\mathbb{D} \times \overline{\mathbb{D}}$. Suppose n is a positive integer and $\mathcal{A}^*_{n\xi}$ is the subclass of $\mathcal{H}^*$ consisting of functions $f(z, \xi)$ of the form

$$f(z, \xi) = z + a_{n+1}(\xi)z^{n+1} + a_{n+2}(\xi)z^{n+2} + \cdots , \ (z \in \mathbb{D},\ \xi \in \overline{\mathbb{D}}),$$

where the coefficients $a_k(\xi)$, $(k \geq n + 1)$ are analytic in $\overline{\mathbb{D}}$. For $n = 1$ we write $\mathcal{A}^*_\xi = \mathcal{A}^*_{1\xi}$. Also, if $n = 1$ and $a_k(\xi) = b_k$, then we obtain the usual class of normalized analytic functions $\mathcal{A}$ in $\mathbb{D}$.

For two functions $f, g \in \mathcal{H}$ we say that f is subordinate to g (or g is superordinate to f) and written as $f \prec g$ or $f(z) \prec g(z)$ if there exists an analytic function $w(z)$ in $\mathbb{D}$ such that

$$w(0) = 0, |w(z)| < 1 \text{ and } f(z) = g(w(z)).$$

If g is univalent in $\mathbb{D}$, then

$$f(z) \prec g(z) \Longleftrightarrow f(0) = g(0) \text{ and } f(\mathbb{D}) \subseteq g(\mathbb{D}).$$

Let $f(z,\xi)$ and $g(z,\xi)$ be analytic in $\mathbb{D}\times\overline{\mathbb{D}}$. The function $f(z,\xi)$ is said to be strongly subordinate to $g(z,\xi)$ (or $g(z,\xi)$ is strongly superordinate to $f(z,\xi)$) if there exists an analytic function $w(z)$ in $\mathbb{D}$ with $w(0)=0$ and $|w(z)|<1$ such that $f(z,\xi)=g(w(z),\xi)$ for all $\xi\in\overline{\mathbb{D}}$, (see [10]). In such a case we write

$$f(z,\xi)\prec\prec g(z,\xi),\ (z\in\mathbb{D},\xi\in\overline{\mathbb{D}}).$$

If $g(z,\xi)$, as a function of z, is univalent in $\mathbb{D}$ for all $\xi\in\overline{\mathbb{D}}$, then

$$f(z,\xi)\prec\prec g(z,\xi)\Longleftrightarrow f(0,\xi)=g(0,\xi),\xi\in\overline{\mathbb{D}} \text{ and } f(\mathbb{D}\times\overline{\mathbb{D}})\subseteq g(\mathbb{D}\times\overline{\mathbb{D}}).$$

When $f(z,\xi)\equiv f(z)$ and $g(z,\xi)\equiv g(z)$, the strong subordination becomes the usual notion of subordination.

The function $L:\mathbb{D}\times[0,+\infty)\times\overline{\mathbb{D}}\to\mathbb{C}$ is a subordination (or Loewner) chain if $L(z,t;\xi)$, as a function of z, is analytic and univalent in $\mathbb{D}$ for all $t\geq 0,\ \xi\in\overline{\mathbb{D}}$ and is continuously differentiable function of t on $[0,+\infty)$ for all $z\in\mathbb{D},\ \xi\in\overline{\mathbb{D}}$ and $L(z,t_1;\xi)\prec L(z,t_2;\xi)$ when $0\leq t_1\leq t_2$, (see [7]).

Suppose that $f(z,\xi),F(z,\xi)\in\mathcal{A}^*_{n\xi},\ f(z,\xi)\neq 0$ and $F(z,\xi)F'(z,\xi)\neq 0$ for all $z\in\mathbb{D}\setminus\{0\}$ and $\xi\in\overline{\mathbb{D}}$ with $F'(z,\xi)=\frac{\partial F(z,\xi)}{\partial z}$. We introduce the integral operator $I^*_{F,\beta}:\mathcal{A}^*_{n\xi}\to\mathcal{A}^*_{n\xi}$ as follows:

$$I^*_{F,\beta}(f)(z,\xi)=\left(\beta\int_0^z f^\beta(t,\xi)\frac{F'(t,\xi)}{F(t,\xi)}dt\right)^{1/\beta},(z\in\mathbb{D},\xi\in\overline{\mathbb{D}},Re\beta>0). \tag{1.1}$$

Note that all powers in (1.1) are principal ones.
When $f(z,\xi)\equiv f(z)$ and $F(z,\xi)\equiv F(z)$ the integral operator (1.1) becomes

$$I_{F,\beta}(f)(z)=\left(\beta\int_0^z f^\beta(t)\frac{F'(t)}{F(t)}dt\right)^{1/\beta}$$

which has been studied by Bulboaca [2].

The notions of strong subordination and superordination have been used by many authors (see, for example [1, 6, 8, 10]). Motivated by the recent works in the literature (see [2, 3, 4, 11]), in the present investigation we obtain some strong subordination and superordination preserving properties for the integral operator $I^*_{F,\beta}$ defined by (1.1) with the sandwich-type theorems. Applications of the main results are also mentioned.

To prove our main results we shall need the following lemmas.

Lemma 1.1. ([8]) *Let $p(z,\xi)$ be analytic in $\overline{\mathbb{D}}\times\overline{\mathbb{D}}$ and, as a function of z, univalent in $\overline{\mathbb{D}}$ for all $\xi\in\overline{\mathbb{D}}$ with $p(0,\xi)=a$, and let*

$$q(z,\xi)=a+a_n(\xi)z^n+a_{n+1}(\xi)z^{n+1}+\cdots\in\mathcal{H}^*$$

with $n\geq 1$ and $q(z,\xi)\not\equiv a$. If $q(z,\xi)$ is not strongly subordinate to $p(z,\xi)$ then there exist points $z_0=r_0e^{i\theta_0}\in\mathbb{D},\ \xi_0\in\partial\mathbb{D}$ and an $m\geq n\geq 1$ such that

$$q(z_0,\xi)=p(\xi_0,\xi),z_0q'(z_0,\xi)=m\xi_0p'(\xi_0,\xi),\ \xi\in\overline{\mathbb{D}}$$

and $q(\mathbb{D}_{r_0}\times\overline{\mathbb{D}_{r_0}})\subseteq p(\mathbb{D}\times\overline{\mathbb{D}})$ where $\mathbb{D}_{r_0}=\{z\in\mathbb{C}:|z|<r_0\}$.

Lemma 1.2. ([8]) *Let $h(z,\xi)$ be analytic in $\mathbb{D}\times\overline{\mathbb{D}}$,*

$$q(z,\xi)=a+a_n(\xi)z^n+a_{n+1}(\xi)z^{n+1}+\cdots\in\mathcal{H}^*,\ (z\in\mathbb{D},\xi\in\overline{\mathbb{D}},n\in\mathbb{N})$$

and $\psi:\mathbb{C}^2\times\overline{\mathbb{D}}\times\overline{\mathbb{D}}\longrightarrow\mathbb{C}$. Suppose that

$$\psi(q(z,\xi),tzq'(z,\xi);\zeta,\xi)\in h(\mathbb{D}\times\overline{\mathbb{D}})$$

for $z\in\mathbb{D}$, $\zeta\in\partial\mathbb{D}$, $\xi\in\overline{\mathbb{D}}$ and $0<t\le\frac{1}{n}\le 1$. If $p(z,\xi)$ is analytic in $\overline{\mathbb{D}}\times\overline{\mathbb{D}}$ and univalent in $\overline{\mathbb{D}}$ for all $\xi\in\overline{\mathbb{D}}$, $p(0,\xi)=a$ and $\psi(p(z,\xi),zp'(z,\xi);z,\xi)$ is analytic in $\mathbb{D}\times\overline{\mathbb{D}}$ and univalent in $\mathbb{D}$ for all $\xi\in\overline{\mathbb{D}}$, then

$$h(z,\xi)\prec\prec\psi(p(z,\xi),zp'(z,\xi);z,\xi)\Longrightarrow q(z,\xi)\prec\prec p(z,\xi).$$

Lemma 1.3. ([7], p. 4) *Let*

$$L(z,t;\xi)=a_1(t,\xi)z+a_2(t,\xi)z^2+\cdots,\ (z\in\mathbb{D},t\ge 0,\xi\in\overline{\mathbb{D}})$$

with $a_1(t,\xi)\neq 0$, $\lim\limits_{t\to+\infty}|a_1(t,\xi)|=+\infty$ for all $t\ge 0$, $\xi\in\overline{\mathbb{D}}$. Suppose that $L(z,t;\xi)$, as a function of z, is analytic in $\mathbb{D}$ and continuously differentiable function of t on $[0,+\infty)$ for all $z\in\mathbb{D}$ and $\xi\in\overline{\mathbb{D}}$. If $L(z,t;\xi)$ satisfies

$$Re\left(\frac{z\partial L/\partial z}{\partial L/\partial t}\right)>0,\ (z\in\mathbb{D},\ t\ge 0),$$

and

$$|L(z,t;\xi)|\le k_0|a_1(t,\xi)|,\ (|z|<r_0<1,\ t\ge 0),$$

for some positive constants k_0 and r_0, then $L(z,t;\xi)$ is a subordination chain.

Lemma 1.4. ([7], pp. 30-35, [9]) *Let $Rea>0$ and the function*

$$p(z,\xi)=a+a_n(\xi)z^n+a_{n+1}(\xi)z^{n+1}+\cdots,$$

is analytic in $\mathbb{D}\times\overline{\mathbb{D}}$. Suppose that the function $J:\mathbb{C}^2\times\mathbb{D}\times\overline{\mathbb{D}}\to\mathbb{C}$ satisfies the condition

$$Re\{J(is,t;z,\xi)\}\le 0,\ \left(s\in\mathbb{R},\ t\le\frac{-n(a^2+s^2)}{2\ Rea}\right).$$

If

$$Re\{J(p(z,\xi),zp'(z,\xi);z,\xi)\}>0,\quad (z\in\mathbb{D},\ \xi\in\overline{\mathbb{D}}),$$

then $Re\{p(z,\xi)\}>0$ in $\mathbb{D}\times\overline{\mathbb{D}}$.

From here and throughout the paper we will assume that $f,g,F,G\in\mathcal{A}^*_{n\xi}$, $f(z,\xi)\neq 0$, $g(z,\xi)\neq 0$, $F(z,\xi)F'(z,\xi)\neq 0, G(z,\xi)G'(z,\xi)\neq 0$ for all $z\in\mathbb{D}\backslash\{0\}$ and $\xi\in\overline{\mathbb{D}}$.

2. Main results

We begin with the following theorem which gives the sufficient conditions so that the integral operator $I^*_{F,\beta}$ are preserved under the strong subordination.

Theorem 2.1. *Let* $\left(z\left(\frac{I^*_{G,\beta}(g)(z,\xi)}{z}\right)^{\beta}\right)'(z,\xi) \neq 0$ *for all* $z \in \mathbb{D}$ *and* $\xi \in \overline{\mathbb{D}}$*. Suppose, also that*

$$Re\left\{1 + \frac{z\varphi''(z,\xi)}{\varphi'(z,\xi)}\right\} > -\delta \tag{2.1}$$

for all $z \in \mathbb{D}$ *and* $\xi \in \overline{\mathbb{D}}$*, with*

$$\varphi(z,\xi) = z\left(\frac{g(z,\xi)}{z}\right)^{\beta}\frac{zG'(z,\xi)}{G(z,\xi)},\ \delta = \frac{n(1+|\beta-1|^2 - |1-(\beta-1)^2|)}{4Re(\beta-1)},\ Re\beta > 1. \tag{2.2}$$

If $I^*_{F,\beta}, I^*_{G,\beta}$ *are the integral operators defined by* (1.1), then

$$z\left(\frac{f(z,\xi)}{z}\right)^{\beta}\frac{zF'(z,\xi)}{F(z,\xi)} \prec\prec z\left(\frac{g(z,\xi)}{z}\right)^{\beta}\frac{zG'(z,\xi)}{G(z,\xi)} \tag{2.3}$$

implies that

$$z\left(\frac{I^*_{F,\beta}(f)(z,\xi)}{z}\right)^{\beta} \prec\prec z\left(\frac{I^*_{G,\beta}(g)(z,\xi)}{z}\right)^{\beta}.$$

Proof. We define the functions H_1 and H_2 by

$$H_1(z,\xi) = z\left(\frac{I^*_{F,\beta}(f)(z,\xi)}{z}\right)^{\beta},\ H_2(z,\xi) = z\left(\frac{I^*_{G,\beta}(g)(z,\xi)}{z}\right)^{\beta}. \tag{2.4}$$

Note that H_1 and H_2 are analytic in $\mathbb{D}\times\overline{\mathbb{D}}$. First, we show that if the function $q(z,\xi)$ is defined by

$$q(z,\xi) = 1 + \frac{zH_2''(z,\xi)}{H_2'(z,\xi)},\ (z \in \mathbb{D}, \xi \in \overline{\mathbb{D}}) \tag{2.5}$$

then $Re\{q(z,\xi)\} > 0$ for all $z \in \mathbb{D}$ and $\xi \in \overline{\mathbb{D}}$. By a simple calculation, using (2.4) and (2.5), we obtain the following relation

$$1 + \frac{z\varphi''(z,\xi)}{\varphi'(z,\xi)} = q(z,\xi) + \frac{zq'(z,\xi)}{\beta - 1 + q(z,\xi)} \equiv Q(z,\xi). \tag{2.6}$$

From the definition of $q(z,\xi)$ and assumption of the theorem it is clear that $q(z,\xi)$ is analytic in $\mathbb{D}\times\overline{\mathbb{D}}$ and $q(0,\xi) = Q(0,\xi) = 1$. Now we define the function $J : \mathbb{C}^2 \to \mathbb{C}$ by

$$J(u,v) = u + \frac{v}{u+\beta-1} + \delta.$$

From the above relations we obtain $Re\{J(q(z,\xi), zq'(z,\xi))\} > 0$ for all $z \in \mathbb{D}$ and $\xi \in \overline{\mathbb{D}}$. Next, we show that

$$Re\{J(is,t)\} \leq 0,\ \left(s \in \mathbb{R},\ t \leq \frac{-n(1+s^2)}{2},\ n \in \mathbb{N}\right).$$

We have

$$Re\{J(is,t)\} = Re\left\{is + \frac{t}{is+\beta-1} + \delta\right\} = t\frac{Re(\beta-1)}{|\beta-1+is|^2} + \delta$$
$$\leq -\frac{I_\delta(s)}{2|\beta-1+is|^2}$$

where

$$I_\delta(s) = (nRe(\beta-1) - 2\delta)s^2 - 4\delta(Im(\beta-1))s$$
$$+ (nRe(\beta-1) - 2\delta|\beta-1|^2).$$

The definition of δ shows that $nRe(\beta-1) \geq 2\delta$ and

$$(nRe(\beta-1) - 2\delta|\beta-1|^2)(nRe(\beta-1) - 2\delta) - 4\delta^2(Im(\beta-1))^2 = 0.$$

Therefore

$$I_\delta(s) = (nRe(\beta-1) - 2\delta)\left(s - \frac{2\delta(Im(\beta-1))}{nRe(\beta-1)-2\delta}\right)^2 \geq 0,$$

and we obtain $Re\{J(is,t)\} \leq 0$. By using Lemma 1.4, with $a = 1$, we conclude that

$$Re\{q(z,\xi)\} = Re\left\{1 + \frac{zH_2''(z,\xi)}{H_2'(z,\xi)}\right\} > 0$$

and $H_2(z,\xi)$, as a function of z, is convex (univalent) function in $\mathbb{D}$ for all $\xi \in \overline{\mathbb{D}}$. Next, we prove that $H_1(z,\xi) \prec\prec H_2(z,\xi)$. Without loss of generality we can assume that $H_2(z,\xi)$ is analytic in $\overline{\mathbb{D}} \times \overline{\mathbb{D}}$ and univalent in $\overline{\mathbb{D}}$ for all $\xi \in \overline{\mathbb{D}}$. The function $L : \mathbb{D} \times [0,+\infty) \times \overline{\mathbb{D}} \longrightarrow \mathbb{C}$ given by

$$L(z,t;\xi) = \frac{\beta-1}{\beta}H_2(z,\xi) + \frac{1+t}{\beta}zH_2'(z,\xi)$$

is analytic in $\mathbb{D}$ for all $t \geq 0$ and $\xi \in \overline{\mathbb{D}}$, and is continuously differentiable function of t on $[0,+\infty)$ for all $z \in \mathbb{D}$ and $\xi \in \overline{\mathbb{D}}$. Since H_2 is convex in $\mathbb{D}$ for all $\xi \in \overline{\mathbb{D}}$ and $Re\beta > 1$, we have

$$a_1(t,\xi) = \left.\frac{\partial L}{\partial z}\right|_{z=0} = 1 + \frac{t}{\beta} \neq 0.$$

Also, $\lim_{t\to+\infty} |a_1(t,\xi)| = +\infty$ for all $\xi \in \overline{\mathbb{D}}$. A simple calculation shows that

$$Re\left\{\frac{z\partial L/\partial z}{\partial L/\partial t}\right\} = Re(\beta-1) + (1+t)Re\left\{1 + \frac{zH_2''(z,\xi)}{H_2'(z,\xi)}\right\} > 0$$

for all $\xi \in \overline{\mathbb{D}}$.

From the definition of $L(z,t;\xi)$, for all $t \geq 0$ and arbitrary (fixed) point $\xi_0 \in \overline{\mathbb{D}}$, we have

$$\frac{|L(z,t;\xi_0)|}{|a_1(t,\xi_0)|} = \frac{|(\beta-1)H_2(z,\xi_0) + (1+t)zH_2'(z,\xi_0)|}{|\beta+t|}$$
$$\leq |\beta-1||H_2(z,\xi_0)| + |H_2'(z,\xi_0)|. \tag{2.7}$$

We know that $|H_2(z,\xi_0)|$ and $|H_2'(z,\xi_0)|$ are both continuous real-valued functions in each subdisk $|z| \le r_0 < 1$. So, there exist positive numbers k_1 and k_2 such that

$$\frac{|L(z,t;\xi_0)|}{|a_1(t,\xi_0)|} \le |\beta-1|k_1 + k_2 = k_0, \quad (|z| \le r_0 < 1,\ t \ge 0).$$

Therefore, by Lemma 1.3, $L(z,t;\xi)$ is a subordination chain and we have

$$\varphi(z,\xi) = L(z,0;\xi) \prec L(z,t;\xi)$$

for $t \ge 0$ and $\xi \in \overline{\mathbb{D}}$. From the last relation we see that

$$L(\zeta,t;\xi) \notin L(\mathbb{D}\times\{0\}\times\{\xi\}) = \varphi(\mathbb{D}\times\{\xi\}) \tag{2.8}$$

where $\zeta \in \partial\mathbb{D}, t \ge 0$ and $\xi \in \overline{\mathbb{D}}$. Now, suppose that $H_1(z,\xi)$ is not strongly subordinate to $H_2(z,\xi)$. Then by Lemma 1.1 there exist points $z_0 \in \mathbb{D}, \xi_0 \in \partial\mathbb{D}$ and $t \ge 0$ such that

$$H_1(z_0,\xi) = H_2(\xi_0,\xi), z_0H_1'(z_0,\xi) = (1+t)\xi_0H_2'(\xi_0,\xi)$$

for all $\xi \in \overline{\mathbb{D}}$. So we obtain

$$\begin{aligned} L(\xi_0,t;\xi) &= \frac{\beta-1}{\beta}H_2(\xi_0,\xi) + \frac{1+t}{\beta}\xi_0H_2'(\xi_0,\xi) \\ &= \frac{\beta-1}{\beta}H_1(z_0,\xi) + \frac{1}{\beta}z_0H_1'(z_0,\xi) \\ &= \left(\frac{f(z_0,\xi)}{z_0}\right)^\beta \frac{{z_0}^2F'(z_0,\xi)}{F(z_0,\xi)}. \end{aligned}$$

Condition (2.3) then shows that $L(\xi_0,t;\xi) \in \varphi(\mathbb{D}\times\{\xi\})$ for all $\xi \in \overline{\mathbb{D}}$. But this contradicts (2.8) and we conclude that $H_1(z,\xi) \prec\prec H_2(z,\xi)$. □

Next, we investigate the dual problem of Theorem 2.1. In this case the subordinations are replaced by superordinations.

Theorem 2.2. *Let* $\left(z\left(\frac{I^*_{G,\beta}(g)(z,\xi)}{z}\right)^\beta\right)'(z,\xi) \neq 0$ *for all* $z \in \mathbb{D}$ *and* $\xi \in \overline{\mathbb{D}}$. *Suppose, also that*

$$Re\left\{1 + \frac{z\varphi''(z,\xi)}{\varphi'(z,\xi)}\right\} > -\delta, \quad (z \in \mathbb{D},\ \xi \in \overline{\mathbb{D}}),$$

where δ *and* $\varphi(z,\xi)$ *are given by* (2.2) and $Re\beta > 1$. In addition, assume that

$$\psi(z,\xi) = z\left(\frac{f(z,\xi)}{z}\right)^\beta \frac{zF'(z,\xi)}{F(z,\xi)}$$

as a function of z, is univalent in $\mathbb{D}$ for all $\xi \in \overline{\mathbb{D}}$ and that $z\left(\frac{I^*_{F,\beta}(f)(z,\xi)}{z}\right)^\beta$, as a function of z, is univalent in $\overline{\mathbb{D}}$ for all $\xi \in \overline{\mathbb{D}}$. If $I^*_{F,\beta}$ and $I^*_{G,\beta}$ are the integral operators defined by (1.1), then the superordination condition

$$z\left(\frac{g(z,\xi)}{z}\right)^\beta \frac{zG'(z,\xi)}{G(z,\xi)} \prec\prec z\left(\frac{f(z,\xi)}{z}\right)^\beta \frac{zF'(z,\xi)}{F(z,\xi)}$$

implies that

$$z\left(\frac{I^*_{G,\beta}(g)(z,\xi)}{z}\right)^{\beta} \prec\prec z\left(\frac{I^*_{F,\beta}(f)(z,\xi)}{z}\right)^{\beta}.$$

Proof. The first part of the proof is similar to that of Theorem 2.1. As before we define the functions $H_1(z,\xi)$ and $H_2(z,\xi)$ by (2.4). From the definitions of H_1 and H_2 we obtain

$$\psi(z,\xi) = \frac{\beta-1}{\beta}H_1(z,\xi) + \frac{1}{\beta}zH_1'(z,\xi)$$

and

$$\varphi(z,\xi) = \frac{\beta-1}{\beta}H_2(z,\xi) + \frac{1}{\beta}zH_2'(z,\xi),$$

respectively. Let $q(z,\xi)$ be as in (2.5). Using the same techniques as in the proof of Theorem 2.1 we can prove that $Re\{q(z,\xi)\} > 0$ for all $z \in \mathbb{D}$ and $\xi \in \overline{\mathbb{D}}$. This means that $H_2(z,\xi)$, as a function of z, is convex (univalent) function in $\mathbb{D}$ for all $\xi \in \overline{\mathbb{D}}$. Now, we define the function $L : \mathbb{D} \times [0,+\infty) \times \overline{\mathbb{D}} \longrightarrow \mathbb{C}$ by

$$L(z,t;\xi) = \frac{\beta-1}{\beta}H_2(z,\xi) + \frac{t}{\beta}zH_2'(z,\xi).$$

As in the proof of Theorem 2.1, we see that $L(z,t;\xi)$ is a subordination chain. Therefore its definition shows that

$$L(z,t;\xi) \prec L(z,1;\xi),\ (z \in \mathbb{D},\ 0 < t \leq 1,\ \xi \in \overline{\mathbb{D}}).$$

From the last relation we obtain

$$\frac{\beta-1}{\beta}H_2(z,\xi) + \frac{t}{\beta}zH_2'(z,\xi) \in \varphi(\mathbb{D}\times\overline{\mathbb{D}}),\ (z \in \mathbb{D},\ 0 < t \leq 1,\ \xi \in \overline{\mathbb{D}}).$$

If we define the function $\psi : \mathbb{C}^2 \to \mathbb{C}$ by $\psi(r,s) = \frac{\beta-1}{\beta}r + \frac{1}{\beta}s$, then we have

$$\psi(H_2(z,\xi), tzH_2'(z,\xi)) \in \varphi(\mathbb{D}\times\overline{\mathbb{D}}),\ (z \in \mathbb{D},\ 0 < t \leq 1,\ \xi \in \overline{\mathbb{D}}).$$

Since all conditions of Lemma 1.2 are satisfied with

$$h(z,\xi) = \varphi(z,\xi),\ p(z,\xi) = H_1(z,\xi) \text{ and } q(z,\xi) = H_2(z,\xi)$$

we conclude that $H_2(z,\xi) \prec\prec H_1(z,\xi)$, and the proof is complete. □

Combining Theorems 2.1 and 2.2 we obtain the following sandwich-type result.

Corollary 2.3. *Let $g_i, G_i \in \mathcal{A}^*_{n\xi}$, $g_i(z,\xi) \neq 0$, and $G_i(z,\xi)G_i'(z,\xi) \neq 0$ for all $z \in \mathbb{D}\setminus\{0\}$, $\xi \in \overline{\mathbb{D}}$ and $i = 1,2$. Also, let $\left(z\left(\frac{I^*_{G_i,\beta}(g_i)(z,\xi)}{z}\right)^{\beta}\right)'(z,\xi) \neq 0$ for all $z \in \mathbb{D}$, $\xi \in \overline{\mathbb{D}}$ and $i = 1,2$. Suppose, also that*

$$Re\left\{1 + \frac{z\varphi_i''(z,\xi)}{\varphi_i'(z,\xi)}\right\} > -\delta,\quad (z \in \mathbb{D},\ \xi \in \overline{\mathbb{D}},\ i = 1,2) \tag{2.9}$$

where δ is given by (2.2) and

$$\varphi_i(z,\xi) = z\left(\frac{g_i(z,\xi)}{z}\right)^{\beta}\frac{zG_i'(z,\xi)}{G_i(z,\xi)},\ (i = 1,2,\ Re\beta > 1). \tag{2.10}$$

In addition, assume that

$$z\left(\frac{f(z,\xi)}{z}\right)^{\beta}\frac{zF'(z,\xi)}{F(z,\xi)}$$

as a function of z*, is univalent in* $\mathbb{D}$ *for all* $\xi \in \overline{\mathbb{D}}$ *and that* $z\left(\frac{I^*_{F,\beta}(f)(z,\xi)}{z}\right)^{\beta}$*, as a function of* z*, is univalent in* $\overline{\mathbb{D}}$ *for all* $\xi \in \overline{\mathbb{D}}$*. If* $I^*_{F,\beta}$ *and* $I^*_{G,\beta}$ *are the integral operators defined by* (1.1)*, then the condition*

$$z\left(\frac{g_1(z,\xi)}{z}\right)^{\beta}\frac{zG_1'(z,\xi)}{G_1(z,\xi)} \prec\prec z\left(\frac{f(z,\xi)}{z}\right)^{\beta}\frac{zF'(z,\xi)}{F(z,\xi)} \prec\prec z\left(\frac{g_2(z,\xi)}{z}\right)^{\beta}\frac{zG_2'(z,\xi)}{G_2(z,\xi)}$$

implies that

$$z\left(\frac{I^*_{G_1,\beta}(g_1)(z,\xi)}{z}\right)^{\beta} \prec\prec z\left(\frac{I^*_{F,\beta}(f)(z,\xi)}{z}\right)^{\beta} \prec\prec z\left(\frac{I^*_{G_2,\beta}(g_2)(z,\xi)}{z}\right)^{\beta}.$$

In Corollary 2.3 we assumed that $z\left(\frac{f(z,\xi)}{z}\right)^{\beta}\frac{zF'(z,\xi)}{F(z,\xi)}$ is univalent function of z in $\mathbb{D}$ for all $\xi \in \overline{\mathbb{D}}$ and that $z\left(\frac{I^*_{F,\beta}(f)(z,\xi)}{z}\right)^{\beta}$, as a function of z, is univalent in $\overline{\mathbb{D}}$ for all $\xi \in \overline{\mathbb{D}}$. In the following result we replace these assumptions by another condition.

Corollary 2.4. *Let* $g_i, G_i \in \mathcal{A}^*_{n\xi}$*,* $g_i(z,\xi) \neq 0$*, and* $G_i(z,\xi)G_i'(z,\xi) \neq 0$ *for all* $z \in \mathbb{D}\setminus\{0\}$*,* $\xi \in \overline{\mathbb{D}}$ *and* $i = 1,2$*. Also, let* $\left(z\left(\frac{I^*_{G_i,\beta}(g_i)(z,\xi)}{z}\right)^{\beta}\right)'(z,\xi) \neq 0$ *and* $\left(z\left(\frac{I^*_{F,\beta}(f)(z,\xi)}{z}\right)^{\beta}\right)'(z,\xi) \neq 0$ *for all* $z \in \mathbb{D}$*,* $\xi \in \overline{\mathbb{D}}$ *and* $i = 1,2$*. Suppose, also that the conditions* (2.9) *and* (2.10) *are satisfied and that*

$$Re\left\{1+\frac{z\psi''(z,\xi)}{\psi'(z,\xi)}\right\} > \frac{-\delta}{n}, \quad (z \in \mathbb{D},\ \xi \in \overline{\mathbb{D}}), \tag{2.11}$$

where δ *is given by* (2.2) *and* $\psi(z,\xi) = z\left(\frac{f(z,\xi)}{z}\right)^{\beta}\frac{zF'(z,\xi)}{F(z,\xi)}$*. If* $I^*_{F,\beta}$ *and* $I^*_{G_i,\beta}$ *are the integral operators given by* (1.1)*, then the condition*

$$z\left(\frac{g_1(z,\xi)}{z}\right)^{\beta}\frac{zG_1'(z,\xi)}{G_1(z,\xi)} \prec\prec z\left(\frac{f(z,\xi)}{z}\right)^{\beta}\frac{zF'(z,\xi)}{F(z,\xi)} \prec\prec z\left(\frac{g_2(z,\xi)}{z}\right)^{\beta}\frac{zG_2'(z,\xi)}{G_2(z,\xi)}$$

implies that

$$z\left(\frac{I^*_{G_1,\beta}(g_1)(z,\xi)}{z}\right)^{\beta} \prec\prec z\left(\frac{I^*_{F,\beta}(f)(z,\xi)}{z}\right)^{\beta} \prec\prec z\left(\frac{I^*_{G_2,\beta}(g_2)(z,\xi)}{z}\right)^{\beta}.$$

Proof. It is sufficient to show that the condition (2.11) implies the univalence of $\psi(z,\xi)$, as a function of z, in $\mathbb{D}$ and the univalence of $H_1(z,\xi) = z\left(\frac{I^*_{F,\beta}(f)(z,\xi)}{z}\right)^{\beta}$, as a function of z, in $\overline{\mathbb{D}}$ for all $\xi \in \overline{\mathbb{D}}$. Since $0 \leq \delta \leq \frac{n}{2}$, the condition (2.11) implies that $\psi(z,\xi)$, as a function of z, is close-to-convex (univalent) in $\mathbb{D}$ for all $\xi \in \overline{\mathbb{D}}$, (see Kaplan's Theorem [5]). In addition, by using the same techniques as in the proof of Theorem 2.1 we conclude that $H_1(z,\xi)$ is convex (univalent) function in $\mathbb{D}$ for all

$\xi \in \overline{\mathbb{D}}$ (In fact, without loss of generality, we can assume that $H_1(z, \xi)$ is univalent in $\overline{\mathbb{D}}$ for all $\xi \in \overline{\mathbb{D}}$). Therefore all conditions of Corollary 2.3 are satisfied and we obtain the result. □

References

[1] Antonino, J.A., *Strong differential subordination and applications to univalency conditions*, J. Korean Math. Soc., **43**(2006), 311-322.

[2] Bulboacă, T., *A class of double subordination-preserving integral operators*, Pure Math. Appl., **15**(2004), 87-106.

[3] Cho, N.E., *Sandwich-type theorems for a class of nonlinear integral operators*, Proceeding Book of the International Symposium on Geometric Function Theory and Applications, 2007, 25-38.

[4] Cho, N.E., Bulboacă, T., *Subordination and superordination properties for a class of integral operators*, Acta Math. Sinica, **26**(2010), 515-522.

[5] Duren, P.L., *Univalent Functions*, Springer-Verlag, New York, 1983.

[6] Jeyaraman, M.P., Suresh, T.K., *Strong differential subordination and superordination of analytic functions*, J. Math. Anal. Appl., **385**(2012), 854-864.

[7] Miller, S.S., Mocanu, P.T., *Differential Subordinations, Theory and Applications*, Marcel Dekker, Inc., New York, 2000.

[8] Oros, Gh., *Briot-Bouquet strong differential superordinations and sandwich theorems*, Math. Reports, **12**(62)(2010), 277-283.

[9] Oros, G.I., *On a new strong differential subordination*, Acta Univ. Apulensis, **32**(2012), 243-250.

[10] Oros, G.I., Oros, Gh., *Strong differential subordination*, Turkish Journal of Mathematics, **33**(2009), 249-257.

[11] Raina, R.K., Sharma, P., *Subordination preserving properties associated with a class of operators*, Le Matematiche, **68**(2013), 217-228.

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Stud. Univ. Babeş-Bolyai Math. 66(2021), No. 4, 677–690
DOI: 10.24193/subbmath.2021.4.07

Harmonic mappings and its directional convexity

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Abstract. For any μ_j $(\mu_j \in \mathbb{C}, |\mu_j| = 1, j = 1, 2)$, we consider the rotations f_{μ_1} and F_{μ_2} of right half-plane harmonic mappings $f, F \in S_{\mathcal{H}}$ which are CHD with the prescribed dilatations $\omega_f(z) = (a - z) / (1 - az)$ for some a $(-1 < a < 1)$ and $\omega_F(z) = e^{i\theta} z^n$ $(n \in \mathbb{N}, \theta \in \mathbb{R})$, $\omega_F(z) = (b - z) / (1 - bz)$, $\omega_F(z) = \left(b - ze^{i\phi}\right) / \left(1 - bze^{i\phi}\right)$ $(-1 < b < 1, \phi \in \mathbb{R})$, respectively. It is proved that the convolution $f_{\mu_1} * F_{\mu_2} \in S_{\mathcal{H}}$ and is convex in the direction of $\overline{\mu_1 \mu_2}$ under certain conditions on the parameters involved.

Mathematics Subject Classification (2010): 31A05, 30C45, 30C55.

Keywords: Harmonic functions, half-plane mappings, convexity in one direction, harmonic convolution, directional convexity, Rouche's theorem.

1. Introduction and preliminaries

Let $\mathcal{H}$ denotes the class of complex-valued functions $f = u + iv$ which are harmonic in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, where u and v are real-valued harmonic functions in $\mathbb{D}$. A function $f \in \mathcal{H}$ can also be expressed as $f = h + \bar{g}$, where h and g are analytic in $\mathbb{D}$, and are called the analytic and co-analytic parts of f, respectively. The Jacobian of the function $f = h + \overline{g}$ is given by $J_f(z) = |h'(z)|^2 - |g'(z)|^2$. According to the Lewy's [8], every harmonic function $f = h + \overline{g} \in \mathcal{H}$ is locally univalent and sense preserving in $\mathbb{D}$ if and only if $J_f(z) > 0$ in $\mathbb{D}$ which is equivalent to the existence of an analytic function $\omega_f(z) = g'(z)/h'(z)$ in $\mathbb{D}$ such that $|\omega_f(z)| < 1$ for all $z \in \mathbb{D}$. The function $\omega_f(z)$ is called the dilatation of the function f. The class of all univalent, sense preserving harmonic functions $f = h + \overline{g} \in \mathcal{H}$, normalized by the conditions $h(0) = 0 = g(0)$ and $h'(0) = 1$ is denoted by $S_{\mathcal{H}}$. If the function $f = h + \overline{g} \in S_{\mathcal{H}}$, then the functions h and g are of the form:

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad \text{and} \qquad g(z) = \sum_{n=1}^{\infty} b_n z^n \qquad (|b_1| < 1). \tag{1.1}$$

The subclass of functions $f = h + \overline{g} \in S_{\mathcal{H}}$ satisfying condition $g'(0) = 0$ (or equivalently $\omega_f(0) = 0$) is denoted by $S^0_{\mathcal{H}}$. Further, the subclasses of convex, close to convex functions f in $S_{\mathcal{H}}$ $\left(\text{or } S^0_{\mathcal{H}}\right)$ are denoted, respectively, by $K_{\mathcal{H}}, C_{\mathcal{H}}$ $\left(\text{or } K^0_{\mathcal{H}}, C^0_{\mathcal{H}}\right)$. The convolution of two analytic functions

$$f(z) = \sum_{n=1}^{\infty} a_n z^n \text{ and } g(z) = \sum_{n=1}^{\infty} b_n z^n$$

is defined by

$$(f * g)(z) = \sum_{n=1}^{\infty} a_n b_n z^n.$$

The convolution of two harmonic functions $f = h + \overline{g}$ and $F = H + \overline{G}$ is defined by

$$(f * F)(z) = g * G + \overline{(h * H)}.$$

A domain $\Omega \subset \mathbb{C}$ is said to be convex in the direction $e^{i\gamma}$ $(\gamma \in \mathbb{R})$, if for every $t \in \mathbb{C}$, the set $\Omega \cap \{t + re^{i\gamma} : r \in \mathbb{R}\}$ is either connected or empty. In particular, a domain is convex in the horizontal direction CHD if every line parallel to the real axis has a connected or empty intersection with Ω. Clunie and Sheil-Small [2] introduced *shear construction method* which provides a univalent harmonic function from a related analytic function and this fundamental theorem is the following:

Theorem 1.1. [2]*A locally univalent harmonic function $f = h + \overline{g}$ is a univalent mapping of $\mathbb{D}$ onto a domain convex in the direction of $e^{i\gamma}$ if and only if $h - e^{2i\gamma} g$ is a analytic univalent mapping of $\mathbb{D}$ onto a domain convex in the direction of $e^{i\gamma}$.*

We may construct a harmonic function $f = h + \overline{g} \in S_{\mathcal{H}}$, where h and g are of the form (1.1) by the shearing of a normalized analytic function $(h - g) / (1 - b_1)$ which is univalent. Throughout the paper we take $b_1 = -a$ $(-1 < a < 1)$.

Definition 1.2 (Slanted and right half-planes). The region H^a_μ for some μ $(\mu \in \mathbb{C}, |\mu| = 1)$ and for some a $(-1 < a < 1)$ defined by

$$H^a_\mu := \left\{ w \in \mathbb{C} : \Re(\mu w) > -\frac{1+a}{2} \right\} \tag{1.2}$$

is called a slanted half-plane and the region $H^a_1 =: H^a$ $(-1 < a < 1)$ is the right half-plane. When $\mu = e^{i\gamma}$ for $\gamma \in [0, 2\pi)$, we denote the region H^a_μ by $\mathcal{H}^a_\gamma$.

The class $S(H^a_\mu)$ consists of functions $f \in S_{\mathcal{H}}$ which map $\mathbb{D}$ onto a slanted half-plane H^a_μ and in particular, $S(H^a)$ denotes a class of functions $f \in S_{\mathcal{H}}$ which are the right half-plane mappings. If $\mu = e^{i\gamma}$ for $\gamma \in [0, 2\pi)$, then the class $S(H^a_\mu)$ is denoted by $S(\mathcal{H}^a_\gamma)$. Also, if $a = 0$, then the class $S(\mathcal{H}^a_\gamma)$ is denoted by $S^0(\mathcal{H}_\gamma)$ (see Dorff *et al.* [4].)

Definition 1.3 (Rotation by μ). The rotations of the function f by μ $(\mu \in \mathbb{C}, |\mu| = 1)$, denoted by f_μ is given by

$$f_\mu(z) = \overline{\mu} f(\mu z).$$

The convolution of two analytic convex mappings is convex. However the convolution of two convex harmonic functions need not be convex under convolution. Therefore, it is interesting to study convolution properties of harmonic functions.

Convolution of harmonic functions convex in the given direction has also been studied in [1, 3, 4, 5, 6, 7, 9, 10, 12, 13] . In this work rotations of right half-plane harmonic mappings which are CHD are defined and the convolution of these rotations are studied unlike the earlier studies [4, 10, 11, 13], where the convolution of slanted half-plane harmonic mappings which are convex in certain directions are studied. Indeed, we study the convolution $f_{\mu_1} * F_{\mu_2}$, where for complex number μ_j with $|\mu_j| = 1$, $j = 1, 2$, f_{μ_1} and F_{μ_2} are the rotations of right half-plane harmonic mappings $f, F \in S_{\mathcal{H}}$ which are CHD with dilations $\omega_f(z) = (a - z) / (1 - az)$, $(-1 < a < 1)$, $\omega_F(z) = e^{i\theta} z^n$, $(n \in \mathbb{N}, \theta \in \mathbb{R})$, $\omega_F(z) = (b - z) / (1 - bz)$, $(-1 < b < 1)$ and $\omega_F(z) = \left(b - ze^{i\phi}\right) / \left(1 - bze^{i\phi}\right)$, $(-1 < b < 1, \phi \in \mathbb{R})$, respectively. It is proved that the convolution $f_{\mu_1} * F_{\mu_2} \in S_{\mathcal{H}}$ and is convex in the direction of $\overline{\mu_1 \mu_2}$ under certain conditions on the parameters involved.

2. Preliminaries

We need following lemmas in proving our results.

Lemma 2.1. [14]*Let the function $f : \mathbb{D} \to \mathbb{C}$ be an analytic function with $f(0) = 0$ and $f'(0) \neq 0$. Suppose that*

$$\varphi(z) = \frac{z}{\left(1 + ze^{i\theta}\right)\left(1 + ze^{-i\theta}\right)} \quad (\theta \in \mathbb{R}; z \in \mathbb{D}). \tag{2.1}$$

If the function f satisfy

$$\Re\left(\frac{zf'(z)}{\varphi(z)}\right) > 0 \quad (z \in \mathbb{D}), \tag{2.2}$$

then the function f is convex in the direction of real axis.

Lemma 2.2. *(Cohn's rule* [15, p. 375]*) For a polynomial p given by*

$$p(z) = p_0(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 \qquad (a_n \neq 0)$$

of degree n, let p^* be an associated polynomial given by

$$p^*(z) = p_0^*(z) = z^n \overline{p\left(\frac{1}{\overline{z}}\right)} = \overline{a_n} + \overline{a_{n-1}} z + \cdots + \overline{a_1} z^{n-1} + \overline{a_0} z^n.$$

Denote by r and s the number of zeroes of the polynomial inside the unit circle and on it, respectively. If $|a_0| < |a_n|$, then the polynomial p_1 is given by

$$p_1(z) = \frac{\overline{a_n} p(z) - a_0 p^*(z)}{z}$$

is of degree $n - 1$ with $r_1 = r - 1$ and $s_1 = s$ the number of zeroes of p_1 inside the unit circle and on it, respectively.

We first mention the following result which can be proved by using the definition of rotation and [2, Theorem 1.1].

Lemma 2.3. *If the function $f = h + \overline{g} \in S_{\mathcal{H}}$ is CHD, then for any μ $(\mu \in \mathbb{C}, |\mu| = 1)$, $f_\mu = H + \overline{G} \in S_{\mathcal{H}}$ is convex in the direction of $\overline{\mu}$, where*

$$H(z) = \overline{\mu} h(\mu z) \quad \text{and } G(z) = \mu g(\mu z). \tag{2.3}$$

Lemma 2.4. *Let the function* $f = h + \overline{g} \in S(H^a)$*. Then for any* μ $(\mu \in \mathbb{C}, |\mu| = 1)$, $f_\mu = H + \overline{G} \in S(H^a_\mu)$*, where* H *and* G *are given by* (2.3) *and hence*

$$H(z) + \overline{\mu}^2 G(z) = \frac{(1+a)z}{1-\mu z} \quad (z \in \mathbb{D})\,. \tag{2.4}$$

In particular,

$$h(z) + g(z) = \frac{(1+a)z}{1-z} \quad (z \in \mathbb{D})\,. \tag{2.5}$$

The proof of Lemma 2.4 is similar to the proof of [4, Theorem 1.1]

Lemma 2.5. *Let for each* $j = 1, 2$,*let the function* $f_j = h_j + \overline{g_j} \in S(H^{a_j})$ *be CHD maps and the function* $f = f_1 * f_2 \in S_{\mathcal{H}}$*. Then for any* μ_j $(\mu_j \in \mathbb{C}, |\mu_j| = 1, j = 1, 2)$, $f_{\mu_j} \in S(H^{a_j}_{\mu_j})$ *is convex in the direction of* $\overline{\mu_j}$ *and* $f_{\mu_1} * f_{\mu_2} = f_{\mu_1\mu_2} \in S_{\mathcal{H}}$ *is convex in the direction of* $\overline{\mu_1\mu_2}$ *in* $\mathbb{D}$.

Proof. Let, for each $j = 1, 2$, the function $f_j = h_j + \overline{g_j} \in S(H^{a_j})$ be CHD maps. Then, by Theorem 1.1, $F_j := h_j - g_j$ is CHD and, from (2.5) of Lemma 2.4

$$h_j(z) + g_j(z) = \frac{(1+a_j)\,z}{1-z} \quad (z \in \mathbb{D})\,. \tag{2.6}$$

Then for any μ_j $(\mu_j \in \mathbb{C}, |\mu_j| = 1, j = 1, 2)$, $f_{\mu_j} = H_j + \overline{G_j} \in S(H^{a_j}_{\mu_j})$ and is convex in the direction of $\overline{\mu_j}$ by Lemma 2.3, where, in view of (2.3),

$$H_j(z) = \overline{\mu_j} h_j(\mu_j z) \;\text{ and } G_j(z) = \mu_j g_j(\mu_j z).$$

Since the function $f = f_1 * f_2 = h + \overline{g}$ we have

$$\begin{aligned} f_{\mu_1} * f_{\mu_2} &= H_1 * H_2 + \overline{G_1 * G_2} \\ &= \overline{\mu_1\mu_2}\,(h_1 * h_2)\,(\mu_1\mu_2 z) + \overline{\mu_1\mu_2\;(g_1 * g_2)\,(\mu_1\mu_2 z)} \\ &= \overline{\mu_1\mu_2} h\,(\mu_1\mu_2 z) + \overline{\mu_1\mu_2\; g\,(\mu_1\mu_2 z)} \\ &= f_{\mu_1\mu_2} := H + \overline{G}. \end{aligned}$$

Since $f = f_1 * f_2 \in S_{\mathcal{H}}$, by Lemma 2.3, $f_{\mu_1} * f_{\mu_2} = f_{\mu_1\mu_2} \in S_{\mathcal{H}}$. We now show that $f_{\mu_1\mu_2}$ is convex in the direction of $\overline{\mu_1\mu_2}$. In view of Lemma 2.3, it is enough to prove that $f = h + \overline{g}$ is CHD or by Theorem 1.1, $h - g$ is CHD. Since $h = h_1 * h_2$ and $g = g_1 * g_2$, we have

$$\begin{aligned} F_1 &= (h_1 - g_1) * (h_2 + g_2) \\ &= h_1 * h_2 + h_1 * g_2 - g_1 * h_2 - g_1 * g_2 \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} F_2 &= (h_2 - g_2) * (h_1 + g_1) \\ &= h_1 * h_2 + g_1 * h_2 - h_1 * g_2 - g_1 * g_2, \end{aligned} \tag{2.8}$$

where from (2.6),

$$F_1 = (h_1(z) - g_1(z)) * \frac{(1+a_2)z}{1-z} = (1+a_2)\,(h_1(z) - g_1(z)) \tag{2.9}$$

and

$$F_2 = (h_2(z) - g_2(z)) * \frac{(1+a_1)z}{1-z} = (1+a_1)(h_2(z) - g_2(z)). \tag{2.10}$$

Then from (2.7) and (2.8),

$$\frac{1}{2}(F_1 + F_2) = h_1 * h_2 - g_1 * g_2.$$

Hence, we only need to prove that $F_1 + F_2$ is CHD. We have from (2.9) and (2.10)

$$F_1 + F_2 = (1+a_2)(h_1(z) - g_1(z)) + (1+a_1)(h_2(z) - g_2(z))$$

and

$$z(F_1+F_2)'(z) = (1+a_2)z(h_1'(z) + g_1'(z))\,p_1(z) + (1+a_1)z(h_2'(z) + g_2'(z))\,p_2(z) \tag{2.11}$$

where for each $j = 1, 2$,

$$p_i(z) = \frac{h_i'(z) - g_i'(z)}{h_i'(z) + g_i'(z)}$$

Since $f_i \in S_H$, the dilatation $\omega_{f_i} = g_i/f_i$ satisfy $|\omega_{f_i}| < 1$ and hence

$$\Re(p_i(z)) = \Re\left(\frac{1-\omega_{f_i}(z)}{1+\omega_{f_i}(z)}\right) > 0.$$

Hence, on using the derivative of (2.6), in (2.11) we have

$$z(F_1+F_2)'(z) = (1+a_2)(1+a_1)\frac{z}{(1-z)^2}\left[p_1(z) + p_2(z)\right]$$

and with $\varphi(z) = \frac{z}{(1-z)^2}$, we have

$$\Re\left(\frac{z(F_1+F_2)'(z)}{\varphi(z)}\right) = (1+a_2)(1+a_1)\Re(p_1(z) + p_2(z)) > 0 \quad (z \in \mathbb{D}).$$

By Lemma 2.1 it follows that $F_1 + F_2$ is CHD and hence, its harmonic shear f is CHD and by Lemma 2.3, $f_{\mu_1\mu_2}$ is convex in the direction of $\overline{\mu_1\mu_2}$ in $\mathbb{D}$. This completes the proof. □

A equivalent form of Lemma 2.5 is as follows:

Lemma 2.6. *Let for each $j = 1, 2$, $f_j = h_j + \overline{g_j} \in S(H^{a_j})$ be CHD and let f_{γ_j} be the rotations of f_j by $e^{i\gamma_j}$. Then $f_{\gamma_j} \in S(\mathcal{H}^{a_j}_{\gamma_j})$ is convex in the direction of $e^{-i\gamma_j}$ and $f_{\gamma_1} * f_{\gamma_2} \in S_{\mathcal{H}}$ and is convex in the direction of $-(\gamma_1 + \gamma_2)$ if $f_1 * f_2 \in S_{\mathcal{H}}$.*

Our next lemma gives a formula for the dilatation of the convolution if two slanted right-half plane mapping.

Lemma 2.7. *Let the function $f = h + \overline{g} \in S(H^a)$ with the dilatation*

$$\omega_f(z) = (a-z)/(1-az) \ \ (-1 < a < 1)$$

and let the function $F = H + \overline{G} \in S(H^b)$ with a dilatation ω_F. Then the dilatation $\tilde{\omega}(z)$ of the convolution $f * F$ is given by

$$\tilde{\omega}(z) = \frac{2(a-z)\,\omega_F(z)(1+\omega_F(z)) - (1-a)z(1-z)\,\omega_F'(z)}{2(1-az)(1+\omega_F(z)) - (1-a)z(1-z)\,\omega_F'(z)}. \tag{2.12}$$

Proof. Since $f = h + \overline{g} \in S(H^a)$ and the dilatation $\omega_f(z) = (a - z)/(1 - az)$, we obtain from (2.5) that

$$h'(z) = \frac{1 - az}{(1-z)^3} \quad \text{and} \quad g'(z) = \frac{a - z}{(1-z)^3}.$$

which gives

$$h(z) = \frac{1 + a}{2}\left[\frac{z}{1 - z} + \frac{(1-a)\,z}{(1+a)\,(1-z)^2}\right] \tag{2.13}$$

and

$$g(z) = \frac{1 + a}{2}\left[\frac{z}{1 - z} - \frac{(1-a)\,z}{(1+a)\,(1-z)^2}\right]. \tag{2.14}$$

Using (2.13) and (2.14) the dilatation $\tilde{\omega}(z)$ of $f * F$ is given by

$$\tilde{\omega}(z) = \frac{(g * G)'(z)}{(h * H)'(z)} = \frac{2aG'(z) - (1-a)\,zG''(z)}{2H'(z) + (1-a)\,zH''(z)}. \tag{2.15}$$

Since $G'(z) = \omega_F(z)H'(z)$, we have $G''(z) = \omega'_F(z)H'(z) + \omega_F(z)H''(z)$ and, from (2.5),

$$H(z) + G(z) = \frac{(1+b)z}{1 - z} \quad (z \in \mathbb{D}) \tag{2.16}$$

which in turn gives

$$H'(z) = \frac{1 + b}{(1 + \omega_F(z))\,(1-z)^2} \tag{2.17}$$

and

$$H''(z) = \frac{(1+b)\,[2\,(1 + \omega_F(z)) - (1-z)\,\omega'_F(z)]}{(1 + \omega_F(z))^2\,(1-z)^3}. \tag{2.18}$$

Hence, from (2.15), we obtain

$$\tilde{\omega}(z) = \frac{\{2a\omega_F(z) - (1-a)\,z\omega'_F(z)\}\,H'(z) - (1-a)\,z\omega_F(z)H''(z)}{2H'(z) + (1-a)\,zH''(z)}$$

On using (2.17) and (2.18), the desired expression for $\tilde{\omega}(z)$ follows. □

A particular form of Lemma 2.7 is as follows:

Corollary 2.8. *Let the function $f = h + \overline{g} \in S(H^0)$ with the dilatation $\omega_f(z) = -z$ and let the function $F = H + \overline{G} \in S(H^b)$ with a dilatation $\omega_F(z)$. Then the dilatation $\tilde{\omega}(z)$ of the convolution $f * F$ is given by*

$$\tilde{\omega}(z) = \frac{-2z\omega_F(z)\,(1 + \omega_F(z)) - z\,(1-z)\,\omega'_F(z)}{2\,(1 + \omega_F(z)) - z\,(1-z)\,\omega'_F(z)}. \tag{2.19}$$

3. Main results

We now prove our first main result on convolution $f_{\mu_1} * F_{\mu_2}$, where the function $f = h + \overline{g} \in S(H^a)$ with the dilatation $\omega_f(z) = (a-z)/(1-az)$ $(-1 < a < 1)$ and $F = H + \overline{G} \in S(H^0)$ with the dilatation $\omega_F = e^{i\theta} z^n$ $(n \in \mathbb{N}, \theta \in \mathbb{R})$ which is as below:

Theorem 3.1. *Let the function $f = h + \overline{g} \in S(H^a)$ and the function*

$$F = H + \overline{G} \in S(H^0)$$

be CHD maps with the dilatations

$$\omega_f(z) = (a-z)/(1-az) \ \ (-1 < a < 1) \ \ \text{and}\ \omega_F(z) = e^{i\theta} z^n \ \ (n \in \mathbb{N},\ \ \theta \in \mathbb{R}),$$

respectively, then for any μ_j $(\mu_j \in \mathbb{C}, |\mu_j| = 1, j = 1, 2)$, the function $f_{\mu_1} * F_{\mu_2} \in S_{\mathcal{H}}$ is convex in the direction of $\overline{\mu_1 \mu_2}$ for $a \in [(n-2)/(n+2), 1)$.

Proof. In view of Lemma 2.5, it is enough to show that $f * F \in S_{\mathcal{H}}$. Let $\tilde{\omega}(z)$ be the dilatation of the convolution $f * F$. Since $\omega_F(z) = e^{i\theta} z^n$ $(n \in \mathbb{N}, \theta \in \mathbb{R})$ in (2.12) of Lemma 2.7, gives

$$\tilde{\omega}(z) = \frac{2(a-z)e^{i\theta} z^n \left(1 + e^{i\theta} z^n\right) - (1-a) z (1-z) n e^{i\theta} z^{n-1}}{2(1-az)(1 + e^{i\theta} z^n) - (1-a) z (1-z) n e^{i\theta} z^{n-1}} = -z^n e^{2i\theta} \frac{p(z)}{p^*(z)},$$

where

$$p(z) = p_0(z) = z^{n+1} - a z^n + \frac{1}{2}(2 + an - n) e^{-i\theta} z + \frac{1}{2}(n - 2a - an) e^{-i\theta} \tag{3.1}$$

and

$$p^*(z) = z^{n+1} \overline{p\left(\frac{1}{\overline{z}}\right)}.$$

Let $A_1, A_2, A_3, \cdots, A_{n+1}$ be the zeros (not necessarily distinct) of the polynomial p, so that $1/\overline{A_1}$, $1/\overline{A_2}$, $1/\overline{A_3}$, $\ldots$, $1/\overline{A_{n+1}}$ are the zeros of the polynomial p^*, Then, it follows that

$$\tilde{\omega}(z) = -z^n e^{2i\theta} \frac{(z - A_1)(z - A_2) \cdots (z - A_{n+1})}{(1 - \overline{A_1} z)(1 - \overline{A_2} z) \cdots (1 - \overline{A_{n+1}} z)}.$$

Now, we only need to prove that $|\tilde{\omega}(z)| < 1$. If $a = (n-2)/(n+2)$, then

$$p(z) = e^{-i\theta} \left[z^{n+1} e^{i\theta} - \frac{n-2}{n+2} z^n e^{i\theta} - \frac{n-2}{n+2} z + 1 \right]$$

and

$$p^*(z) = z^{n+1} e^{i\theta} - \frac{n-2}{n+2} z^n e^{i\theta} - \frac{n-2}{n+2} z + 1$$

which proves that

$$|\tilde{\omega}(z)| = \left| -z^n e^{i\theta} \right| < 1. \tag{3.2}$$

Let $(n-2)/(n+2) < a < 1$. We first show that each zero A_i of p lies inside and on the unit circle: $|A_i| \le 1$ for each $i = 1, 2, \ldots, n+1$. We apply the Cohn's rule to the polynomial p of degree $n+1$ given by (3.1). Since

$$|a_0| := \left| \frac{1}{2}(n - 2a - an) e^{-i\theta} \right| < 1 =: |a_{n+1}|$$

and

$$\frac{(1-a)(2+n)[2(1+a)-(1-a)n]}{4} > 0,$$

we see that

$$\begin{aligned} p_1(z) &= \frac{\overline{a_{n+1}}p(z) - a_0 p^*(z)}{z} \\ &= \frac{(1-a)(2+n)[2(1+a)-(1-a)n]}{4} q_1(z), \end{aligned}$$

and hence, $p_1(z)$ has same same zeros as the polynomial

$$q_1(z) = z^n - \frac{n}{n+2} z^{n-1} + \frac{2}{n+2} e^{-i\theta} \tag{3.3}$$

has. If $n = 1$, $q_1(z)$ has a zero at

$$z = \frac{1}{3} - \frac{2}{3} e^{-i\theta}$$

which lies inside or on the unit circle $|z| = 1$. Hence, $p_1(z)$ has a zero inside or on the unit circle $|z| = 1$ if $-\frac{1}{3} < a < 1$. If $n \geq 2$ and if we write Eq. (3.3) as

$$q_1(z) = q_2(z) + q_3(z), \tag{3.4}$$

where $q_2(z) = z^n$ and

$$q_3(z) = -\frac{n}{n+2} z^{n-1} + \frac{2}{n+2} e^{-i\theta},$$

then on $|z| = 1 + \epsilon \ (\epsilon > 0)$,

$$|q_2(z)| = (1+\epsilon)^n$$

and

$$|q_3(z)| \leq \frac{n}{n+2} (1+\epsilon)^{n-1} + \frac{2}{n+2}.$$

Therefore, we have

$$|q_3(z)| < |q_2(z)|$$

on $|z| = 1 + \epsilon$ if

$$\frac{n}{n+2} (1+\epsilon)^{n-1} + \frac{2}{n+2} < (1+\epsilon)^n.$$

Since $(1+\epsilon)^{n-1} > 1$,

$$\frac{n}{n+2} (1+\epsilon)^{n-1} + \frac{2}{n+2} < \frac{n+2}{n+2} (1+\epsilon)^{n-1} = (1+\epsilon)^{n-1} < (1+\epsilon)^n,$$

we have

$$|q_3(z)| < |q_2(z)| \text{ on } |z| = 1 + \epsilon$$

and hence, by well known Rouche's Theorem the polynomials q_2 and $q_2 + q_3 = q_1$ have same number of zeros inside the disk $|z| < 1 + \epsilon$. As the polynomial q_2 has n zeros inside the disk $|z| < 1 + \epsilon$, the polynomial q_1 has n zeroes in that disk. Letting $\epsilon \to 0$, we obtain that the polynomial q_1 has all of its n zeros in the disk $|z| \leq 1$. It proves consequently that all the zeros A_i of $p(z)$ lie inside or on the unit circle $|z| = 1$. This proves the result. □

Our second main result on convolution $f_{\mu_1} * F_{\mu_2}$ of rotation of $f = h + \overline{g} \in S(H^a)$ with the dilatation $\omega_f(z) = (a - z)/(1 - az)$ $(-1 < a < 1)$ and the function $F = H + \overline{G} \in S(H^b)$ with the dilatation $\omega_F(z) = (b - z)/(1 - bz)$ $(-1 < b < 1)$ is as below:

Theorem 3.2. *Let the function $f = h + \overline{g} \in S(H^a)$ and the function*

$$F = H + \overline{G} \in S(H^b)$$

be CHD maps with the dilatation

$$\omega_f(z) = (a - z)/(1 - az) \ \ (-1 < a < 1)$$

and

$$\omega_F(z) = (b - z)/(1 - bz) \ \ (-1 < b < 1)\,,$$

respectively, then for any μ_j $(\mu_j \in \mathbb{C}, |\mu_j| = 1, j = 1, 2)$, $f_{\mu_1} * F_{\mu_2} \in S_{\mathcal{H}}$ is convex in the direction of $\overline{\mu_1 \mu_2}$ provided $(a + b)/(1 + ab) \in [-1/3, 1)$.

Proof. To prove the result, in view of Lemma 2.5, we prove that $f * F \in S_{\mathcal{H}}$. Let $\tilde{\omega}(z)$ be the dilatation of $f * F$ when $\omega_F(z) = (b - z)/(1 - bz)$ $(-1 < b < 1)$. Then from Lemma 2.7 we have

$$\tilde{\omega}(z) = \frac{2\,(a - z)\,\frac{b - z}{1 - bz}\left(1 + \frac{b - z}{1 - bz}\right) - (1 - a)\,z\,(1 - z)\,\frac{b^2 - 1}{(1 - bz)^2}}{2\,(1 - az)\left(1 + \frac{b - z}{1 - bz}\right) - (1 - a)\,z\,(1 - z)\,\frac{b^2 - 1}{(1 - bz)^2}} = \frac{r(z)}{r^*(z)}, \tag{3.5}$$

where

$$r(z) = z^2 + \frac{1}{2}\,(ab - 3a - 3b + 1)\,z + ab \tag{3.6}$$

and

$$r^*(z) = z^2 \overline{r\left(\frac{1}{\overline{z}}\right)}.$$

Hence, if A and B are the zeros of $r(z)$, then $1/\overline{A}$ and $1/\overline{B}$ are the zeros of $r^*(z)$ and we write

$$\tilde{\omega}(z) = \frac{(z - A)(z - B)}{(1 - \overline{A}z)(1 - \overline{B}z)}.$$

Now we prove that $|\tilde{\omega}(z)| < 1$ or equivalently that A and B lie inside or on the unit circle $|z| = 1$. For this we apply the Cohn's rule to the polynomial r. Since $|ab| < 1$ we see that

$$r_1(z) = \frac{r(z) - ab\,r^*(z)}{z} = (1 - ab)\left[(1 + ab)\,z + \frac{1}{2}\,(ab - 3a - 3b + 1)\right]$$

has a zero at

$$z_0 = \frac{3}{2}\left(\frac{a + b}{1 + ab}\right) - \frac{1}{2}.$$

The zero z_0 lies inside or on the unit circle $|z| = 1$ if $(a + b)/(1 + ab) \in [-1/3, 1)$. This proves the result. $\square$

Taking $b = a$ in Theorem 3.2, we get following result:

Corollary 3.3. *Let $f_j \in S(H^a)$ be CHD with the dilatation $\omega_{f_j}(z) = (a-z)/(1-az)$ for each $j = 1, 2$. Then for any μ_j $(\mu_j \in \mathbb{C}, |\mu_j| = 1, j = 1, 2)$, $f_{\mu_1} * f_{\mu_2} \in S_{\mathcal{H}}$ is convex in the direction of $\overline{\mu_1\mu_2}$ for $a \in [-3+2\sqrt{2}, 1)$.*

The next result gives a condition for the directional convexity of the convolution $f_{\mu_1} * F_{\mu_2}$ when $f = h + \overline{g} \in S(H^0)$ with the dilatation $\omega_f = -z$ and $F = H + \overline{G} \in S(H^b)$ with the dilatation $\omega_F = \left(b - ze^{i\phi}\right) / \left(1 - bze^{i\phi}\right) (-1 < b < 1, \phi \in \mathbb{R})$.

Theorem 3.4. *Let the function $f = h + \overline{g} \in S(H^0)$ and the function*

$$F = H + \overline{G} \in S(H^b)$$

be CHD maps with the dilatations

$$\omega_f(z) = -z \text{ and } \omega_F(z) = (b - ze^{i\phi})/(1 - bze^{i\phi}) \ (-1 < b < 1, \phi \in \mathbb{R}),$$

respectively. Then for any μ_j $(\mu_j \in \mathbb{C}, |\mu_j| = 1, j = 1, 2)$, $f_{\mu_1} * F_{\mu_2} \in S_{\mathcal{H}}$ is convex in the direction of $\overline{\mu_1\mu_2}$ if any one of the following case holds:

(i) $\cos\phi = 1$ *and* $-1/3 \le b < 1$
(ii) $-1 \le \cos\phi < 1$ *and* $b^2 < 1/(5 + 4\cos\theta)$.

Proof. From (2.19), it follows that the dilatation $\tilde{\omega}$ of $f * F$ is given by

$$\tilde{\omega}(z) = \frac{-2z\omega_F(1+\omega_F) - z(1-z)\omega_F'}{2(1+\omega_F) - z(1-z)\omega_F'},$$

where $\omega_F(z) = (b - ze^{i\phi})/(1 - bze^{i\phi})$ and $\omega_F'(z) = (b^2-1)e^{i\phi}/(1 - be^{i\phi}z)^2$ $(-1 < b < 1$, $\phi \in \mathbb{R})$. Hence, we have

$$\tilde{\omega}(z) = -ze^{2i\phi}\frac{t(z)}{t^*(z)},$$

where

$$t(z) = z^2 - \frac{1+3b}{2}e^{-i\phi}z + \frac{[2b - (1-b)e^{i\phi}]e^{-2i\phi}}{2}$$

and

$$t^*(z) = 1 - \frac{1+3b}{2}e^{i\phi}z + \frac{[2b - (1-b)e^{-i\phi}]e^{2i\phi}}{2}.$$

We need only to show that $|\tilde{\omega}(z)| < 1$. (i) If $\cos\phi = 1$, then

$$t(z) = z^2 - \frac{1+3b}{2}z + \frac{3b-1}{2} = (z-1)\left(z - \frac{3b-1}{2}\right)$$

and

$$t^*(z) = (1-z)\left(1 - \frac{3b-1}{2}z\right).$$

Hence, the dilatation $\tilde{\omega}$ of $f * F$ becomes

$$\tilde{\omega}(z) = -z\frac{(z-1)}{(1-z)}\frac{(z - \frac{3b-1}{2})}{(1 - \frac{3b-1}{2}z)}$$

which imply that $|\tilde{\omega}(z)| < 1$ when $|(3b-1)/2| \leq 1$ or when $-1/3 \leq b < 1$. This proves the result if case (i) holds.
(ii)If $-1 \leq \cos\phi < 1$, choose

$$a_0 = \frac{[2b-(1-b)e^{i\phi}]e^{-2i\phi}}{2} \quad \text{and} \quad a_1 = -\frac{(1+3b)e^{-i\phi}}{2}.$$

Then $t(z) = z^2 + a_1 z + a_0$ and in this case, we have

$$\begin{aligned} 1-|a_0|^2 &= 1 - \frac{[2b-(1-b)e^{i\phi}][2b-(1-b)e^{-i\phi}]}{4} \\ &= \frac{1-b}{4}[b(5+4\cos\phi)+3] > 0 \end{aligned}$$

when $b > -3/(5+4\cos\phi)$. Under the condition when $b > -3/(5+4\cos\phi)$, we apply Cohn's rule to the polynomial $t(z)$. Consider

$$t_1(z) = \frac{t(z)-a_0 t^*(z)}{z} = \frac{1-b}{4}[b(5+4\cos\phi)+3](z-z_0),$$

where

$$z_0 = \frac{(1+3b)(1+2e^{-i\phi})}{3+b(5+4\cos\phi)} =: \frac{u(b)}{v(b)}.$$

We have

$$|v(b)|^2 - |u(b)|^2 = 4(1-\cos\phi)[1-b^2(5+4\cos\phi)] > 0$$

when

$$b^2 < \frac{1}{5+4\cos\phi} \quad \text{or} \quad -\frac{1}{\sqrt{5+4\cos\phi}} < b < \frac{1}{\sqrt{5+4\cos\phi}}.$$

Since

$$b^2 < \frac{1}{5+4\cos\phi} \Rightarrow b > -\frac{3}{5+4\cos\phi}$$

when $-1 \leq \cos\phi < 1$, the result is proved if case (ii) holds. □

For $b=0$, Theorem 3.4, reduces to the following simpler form:

Corollary 3.5. *Let $f, F \in S(H^0)$ with the dilatations*

$$\omega_f(z) = -z \text{ and } \omega_F(z) = -ze^{i\phi} \ (\phi \in \mathbb{R}),$$

respectively. Then for any μ_j $(\mu_j \in \mathbb{C}, |\mu_j| = 1, j = 1,2)$, $f_{\mu_1} * F_{\mu_2} \in S_{\mathcal{H}}$ is convex in the direction of $\overline{\mu_1\mu_2}$.

Further, for $\phi = \pi + \theta$ Theorem 3.4 takes the following form:

Corollary 3.6. *Let the function $f = h + \overline{g} \in S(H^0)$ with the dilation $\omega_f(z) = -z$ and let the function $F = H + \overline{G} \in S(H^b)$ with the dilatation*

$$\omega_F(z) = (b+ze^{i\theta})/(1+bze^{i\theta}) \ (-1 < b < 1, \ \theta \in \mathbb{R}).$$

Then for any μ_j $(\mu_j \in \mathbb{C}, |\mu_j| = 1, j = 1,2)$, $f_{\mu_1} * F_{\mu_2} \in S_{\mathcal{H}}$ is convex in the direction of $\overline{\mu_1\mu_2}$ if any one of the following case holds:

(i) $\cos\theta = -1$ *and* $-1/3 \leq b < 1$
(ii) $-1 < \cos\theta \leq 1$ *and* $b^2 < 1/(5-4\cos\theta)$.

4. Examples

In this section, we give following examples to illustrate our main results. Examples 1 and 2 are based on Theorem 3.1, Example 3 is on Theorem 3.2 and Example 4 is on Theorem 3.4.

Example 4.1. Let for some $\mu_1, \mu_2 \in \mathbb{C}$ and for some a $(-1 < a < 1)$,

$$f_{\mu_1}(z) = h(z) + \overline{g(z)} \in S(H^a),$$

where

$$h(z) = \frac{2z - (1+a)\mu_1 z^2}{2\left(1 - \mu_1 z\right)^2} \text{ and } g(z) = \frac{2az - (1+a)\mu_1 z^2}{2\left(1 - \mu_1 z\right)^2}$$

and $F_{\mu_2}(z) = H(z) + \overline{G(z)} \in S(H^0)$, where

$$H(z) = \frac{2z - \mu_2 z^2}{2\left(1 - \mu_2 z\right)^2} \text{ and } G(z) = \frac{-\mu_2 z^2}{2\left(1 - \mu_2 z\right)^2}.$$

Then $f_{\mu_1} * F_{\mu_2} \in S_{\mathcal{H}}$ if $a \in [-1/3, 1)$ and is convex in the direction of $\overline{\mu_1 \mu_2}$.

Example 4.2. Let for some $\mu_1, \mu_2 \in \mathbb{C}$ and for some a $(-1 < a < 1)$,

$$f_{\mu_1}(z) = h(z) + \overline{g(z)} \in S(H^a),$$

where

$$h(z) = \frac{2z - (1+a)\mu_1 z^2}{2\left(1 - \mu_1 z\right)^2} \text{ and } g(z) = \frac{2az - (1+a)\mu_1 z^2}{2\left(1 - \mu_1 z\right)^2}$$

and $F_{\mu_2}(z) = H(z) + \overline{G(z)} \in S(H^0)$, where

$$H(z) = \frac{\overline{\mu_2}}{8} \log \frac{1 + \mu_2 z}{1 - \mu_2 z} + \frac{3z - 2\mu_2 z^2}{4\left(1 - \mu_2 z\right)^2} \text{ and } G(z) = -\frac{\overline{\mu_2}}{8} \log \frac{1 + \mu_2 z}{1 - \mu_2 z} + \frac{z - 2\mu_2 z^2}{4\left(1 - \mu_2 z\right)^2}.$$

Then $f_{\mu_1} * F_{\mu_2} \in S_{\mathcal{H}}$ if $a \in [0, 1)$ and is convex in the direction of $\overline{\mu_1 \mu_2}$.

Example 4.3. Let for some $\mu_1, \mu_2 \in \mathbb{C}$ and for some a $(-1 < a < 1)$,

$$f_{\mu_1}(z) = h(z) + \overline{g(z)} \in S(H^a),$$

where

$$h(z) = \frac{2z - (1+a)\mu_1 z^2}{2\left(1 - \mu_1 z\right)^2} \text{ and } g(z) = \frac{2az - (1+a)\mu_1 z^2}{2\left(1 - \mu_1 z\right)^2}$$

and $F_{\mu_2}(z) = H(z) + \overline{G(z)} \in S(H^{1/2})$, where

$$H(z) = \frac{4z - 3\mu_2 z^2}{4\left(1 - \mu_2 z\right)^2} \text{ and } G(z) = \frac{2z - 3\mu_2 z^2}{4\left(1 - \mu_2 z\right)^2}.$$

Then $f_{\mu_1} * F_{\mu_2} \in S_{\mathcal{H}}$ if $a \in [-5/7, 1)$ and is convex in the direction of $\overline{\mu_1 \mu_2}$.

Example 4.4. Let for some $\mu_1, \mu_2 \in \mathbb{C}$ and for some b $(-1 < b < 1)$,

$$f_{\mu_1}(z) = h(z) + \overline{g(z)} \in S(H^0),$$

where

$$h(z) = \frac{2z - \mu_1 z^2}{2\left(1 - \mu_1 z\right)^2} \text{ and } g(z) = \frac{-\mu_1 z^2}{2\left(1 - \mu_1 z\right)^2}.$$

and $F_{\mu_2}(z) = H(z) + \overline{G(z)} \in S(H^b)$, where

$$H(z) = \frac{1-b}{4}\overline{\mu_2}\log\frac{1+\mu_2 z}{1-\mu_2 z} + \frac{1+b}{2}\frac{z}{1-\mu_2 z}$$

and

$$G(z) = -\frac{1-b}{4}\overline{\mu_2}\log\frac{1+\mu_2 z}{1-\mu_2 z} + \frac{1+b}{2}\frac{z}{1-\mu_2 z}.$$

Then $f_{\mu_1} * F_{\mu_2} \in S_{\mathcal{H}}$ and is convex in the direction of $\overline{\mu_1\mu_2}$.

Acknowledgement. The authors are thankful to the referee for suggesting to add some examples based on the technique used in our Main Results.

References

[1] Beig, S., Ravichandran, V., *Convolution and convex combination of harmonic mappings*, Bull. Iranian Math. Soc., **45**(2019), no. 5, 1467-1486.

[2] Clunie, J., Sheil-Small, T., *Harmonic univalent functions*, Ann. Acad. Sci. Fenn. Ser. A I Math., **9**(1984), 3-25.

[3] Dorff, M., *Convolutions of planar harmonic convex mappings*, Complex Variables Theory Appl., **45**(2001), no. 3, 263-271.

[4] Dorff, M., Nowak, M., Wołoszkiewicz, M., *Convolutions of harmonic convex mappings*, Complex Var. Elliptic Equ., **57**(2012), no. 5, 489-503.

[5] Dorff, M., Rolf, J.S., *Anamorphosis, mapping problems, and harmonic univalent functions*, in: Explorations in Complex Analysis, 197-269, Classr. Res. Mater. Ser, Math. Assoc. America, Washington, DC.

[6] Kumar, R., et al., *Convolution properties of some harmonic mappings in the right half-plane*, Bull. Malays. Math. Sci. Soc., **39**(2016), no. 1, 439-455.

[7] Kumar, R., Gupta, S., Singh, S., *Convolution properties of a slanted right half-plane mapping*, Mat. Vesnik, **65**(2013), no. 2, 213-221.

[8] Lewy, H., *On the non-vanishing of the Jacobian in certain one-to-one mappings*, Bull. Amer. Math. Soc., **42**(1936), no. 10, 689-692.

[9] Li, Y., Liu, Z., *Convolutions of harmonic right half-plane mappings*, Open Math., **14**(2016), no. 1, 789-800.

[10] Li, L., Ponnusamy, S., *Convolutions of slanted half-plane harmonic mappings*, Analysis (Munich), **33**(2013), no. 2, 159-176.

[11] Li, L., Ponnusamy, S., *Solution to an open problem on convolutions of harmonic mappings*, Complex Var. Elliptic Equ., **58**(2013), no. 12, 1647-1653.

[12] Li, L., Ponnusamy, S., *Note on the convolution of harmonic mappings*, Bull. Aust. Math. Soc., **99**(2019), no. 3, 421-431.

[13] Liu, Z., Ponnusamy, S., *Univalency of convolutions of univalent harmonic right half-plane mappings*, Comput. Methods Funct. Theory, **17**(2017), no. 2, 289-302.

[14] Pommerenke, C., *On starlike and close-to-convex functions*, Proc. London Math. Soc., **13**(1963), no. 3, 290-304.

[15] Rahman, Q.I., Schmeisser, G., *Analytic Theory of Polynomials*, London Mathematical Society Monographs. New Series, 26, The Clarendon Press, Oxford University Press, Oxford, 2002.

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Stud. Univ. Babeş-Bolyai Math. 66(2021), No. 4, 691–708
DOI: 10.24193/subbmath.2021.4.08

Positive solutions for fractional differential equations with non-separated type nonlocal multi-point and multi-term integral boundary conditions

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Abstract. In this paper, we investigate a class of nonlinear fractional differential equations that contain both the multi-term fractional integral boundary condition and the multi-point boundary condition. By the Krasnoselskii fixed point theorem we obtain the existence of at least one positive solution. Then, we obtain the existence of at least three positive solutions by the Legget-Williams fixed point theorem. Two examples are given to illustrate our main results.

Mathematics Subject Classification (2010): 34A08, 34B15, 34B18.

Keywords: Fractional differential equations, Riemann-Liouville fractional derivative, multi-term fractional integral boundary condition, fixed point theorems.

1. Introduction

Differential equations of fractional order are one of the fast growing area of research in the field of mathematics and have recently been proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. Indeed, one can find numerous applications of fractional order differential equations in viscoelasticity, electro-chemistry, control theory, movement through porous media, electromagnetics, and signal processing of wireless communication system, etc (see [6, 7, 9, 18, 22, 23, 26, 29, 30]). Now, there are many papers dealing with the problem for different kinds of boundary value conditions such as multi-point boundary condition (see [1, 12, 13, 14, 21, 25, 28, 31]), integral boundary condition (see [3, 4, 5, 8, 15, 24, 32, 33]), and many other boundary conditions (see [2, 11, 16, 20, 35]).

In this paper, we are dedicated to considering fractional differential equations that contain both the multi-term fractional integral boundary condition and the multi-point boundary condition:

$$\begin{cases} D^q u(t) + f(t, u(t)) = 0, \quad 1 < q \leq 2,\ 0 < t < 1, \\ u(0) = 0, \quad u(1) = \sum_{i=1}^{m} \alpha_i (I^{p_i} u)(\eta) + \sum_{i=1}^{m} \beta_i u(\xi_i), \end{cases} \tag{1.1}$$

where D^q is the standard Riemann-Liouville fractional derivative of order q, I^{p_i} is the Riemann-Liouville fractional integral of order $p_i > 0$, $i = 1, 2, ..., m$, $0 < \xi_1 < \xi_2 < ... < \xi_m < 1$, $0 < \eta < 1$, $f : [0,1] \times [0,\infty) \to [0,\infty)$ and α_i, $\beta_i \geq 0$ with $i = 1, 2, ..., m$, are real constants such that

$$\Gamma(q) \sum_{i=1}^{m} \frac{\alpha_i \eta^{p_i+q-1}}{\Gamma(p_i + q)} + \sum_{i=1}^{m} \beta_i \xi_i^{q-1} < 1.$$

Zhou and Jiang [36] considered the fractional boundary value problem

$$\begin{cases} D_{0+}^{\alpha} u(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \\ u'(0) - \beta u(\xi) = 0,\ u'(1) + \sum_{i=1}^{m-3} \gamma_i u(\eta_i) = 0 \end{cases}$$

where α is a real number with $1 < \alpha \leq 2$, $0 \leq \beta \leq 1$, $0 \leq \gamma_i \leq 1$, $i = 1, 2, ..., m-3$, $0 \leq \xi < \eta_1 < \eta_2 < ... < \eta_{m-3} \leq 1$, D_{0+}^{α} is the Caputo's derivative. The authors used the fixed point index theory and Krein-Rutman theorem to obtain the existence results.

Ji et al. [17] investigated the existence and multiplicity results of positive solutions for the following boundary value problem:

$$\begin{cases} D_{0+}^{\alpha} u(t) + f\left(t, u(t), D_{0+}^{\mu} u(t)\right) = 0, \quad 0 < t < 1, \\ u(0) = 0, \quad u(1) + D_{0+}^{\beta} u(1) = k u(\xi) + l D_{0+}^{\beta} u(\eta), \end{cases}$$

where D_{0+}^{α} is the Riemann-Liouville fractional derivative of order $1 < \alpha \leq 2$, $0 \leq \beta \leq 1$, ξ, $\eta \in (0,1)$, $0 \leq \mu < 1$, $1 \leq \alpha - \beta$, $1 \leq \alpha - \mu$, $1 - l\eta^{\alpha-\beta-1}$, and $f : [0,1] \times [0,+\infty) \times (-\infty,+\infty) \to [0,+\infty)$ is continuous. They used the Leggett-Williams fixed point theorem to obtain the existence and multiplicity results of positive solutions.

Wang et al. [34] considered the following boundary value problem

$$\begin{cases} D^{\sigma} u(t) + f(t, u(t)) = 0, \quad t \in [0,1], \\ u^{(i)}(0) = 0, \quad i = 0, 1, 2, ..., n-2, \\ u(1) = \sum_{i=1}^{m-2} \beta_i \int_0^{\eta_i} u(s)\, ds + \sum_{i=1}^{m-2} \gamma_i u(\eta_i), \end{cases}$$

where D^{σ} represents the standard Riemann-Liouville fractional derivative of order σ satisfying $n - 1 < \sigma \leq n$ with $n \geq 3$. The authors used Krasnoselkii's fixed point theorem, Schauder type fixed point theorem, Banach's contraction mapping principle and nonlinear alternative for single-valued maps to obtain the existence results.

Inspired by the above works, in this paper, we establish the existence and multiplicity of positive solutions of the boundary value problem (1.1). Our paper is organized as follows. After this section, some definitions and lemmas will be established in Section 2. In Section 3, we give our main results in Theorems 3.1 and 3.2. Finally, in Section 4, as applications, some examples are presented to illustrate our main results

2. Preliminaries

In this section, we introduce some notations and definitions of fractional calculus, which can be found in [18, 27, 30]. We also state two fixed-point theorems due to Guo–Krasnosel'skii and Leggett–Williams.

Definition 2.1. The Riemann-Liouville fractional integral of order $\alpha > 0$ for a function $f : (0, +\infty) \to \mathbb{R}$ is defined as

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds,$$

provided the right side is pointwise defined on $(0, +\infty)$ where $\Gamma(.)$ is the Gamma function.

Definition 2.2. The Riemann-Liouville fractional derivative order $\alpha > 0$ of a continuous function $u : (0, \infty) \to \mathbb{R}$ is defined by

$$D_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} u(s)\, ds,$$

where $n = \lceil \alpha \rceil + 1$, $\lceil \alpha \rceil$ denotes the integer part of number α, provided that the right side is pointwise defined on $(0, \infty)$.

Lemma 2.3. *(i) If $u \in L^p(0,1)$, $1 \leq p \leq +\infty$, $\beta > \alpha > 0$, then*

$$I_{0+}^{\alpha} I_{0+}^{\beta} u(t) = I_{0+}^{\alpha+\beta} u(t).$$

(ii) If $\alpha > 0$ and $\gamma \in (-1, +\infty)$, then

$$I_{0+}^{\alpha} t^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\alpha+\gamma}.$$

Lemma 2.4. *Let $\alpha > 0$ and for any $y \in L^1(0,1)$. Then, the general solution of the fractional differential equation $D_{0+}^{\alpha} u(t) + y(t) = 0$, $0 < t < 1$ is given by*

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\, ds + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n},$$

where $c_0, c_1, ..., c_{n-1}$ are real constants and $n = \lceil \alpha \rceil + 1$.

Definition 2.5. Let E be a real Banach space. A nonempty convex closed set $K \subset E$ is said to be a cone provided that

(i) $au \in K$ for all $u \in K$ and all $a \geq 0$, and
(ii) $u, -u \in K$ implies $u = 0$.

Definition 2.6. The map α is defined as a nonnegative continuous concave functional on a cone K of a real Banach space E provided that $\alpha : K \to [0, +\infty)$ is continuous and

$$\alpha(tx + (1-t)y) \geq t\alpha(x) + (1-t)\alpha(y)$$

for all $x, y \in K$ and $0 \leq t \leq 1$.

Lemma 2.7. *Let* $\Delta = 1 - \Gamma(q)\sum_{i=1}^{m} \frac{\alpha_i \eta^{p_i+q-1}}{\Gamma(p_i+q)} - \sum_{i=1}^{m} \beta_i \xi_i^{q-1} > 0$, α_i, $\beta_i \geq 0$, $p_i > 0$, $i = 1, 2, ...m$, *and* $h \in C[0,1]$. *The unique solution* $u \in AC[0,1]$ *of the boundary value problem*

$$D^q u(t) + h(t) = 0, \quad t \in (0,1), \ q \in (1,2] \tag{2.1}$$

$$u(0) = 0, \quad u(1) = \sum_{i=1}^{m} \alpha_i (I^{p_i} u)(\eta) + \sum_{i=1}^{m} \beta_i u(\xi_i) \tag{2.2}$$

is given by

$$u(t) = \int_0^1 G(t,s) h(s) ds, \tag{2.3}$$

where $G(t,s)$ *is the Green's function given by*

$$G(t,s) = g(t,s) + \frac{t^{q-1}}{\Delta} \sum_{i=1}^{m} \frac{\alpha_i}{\Gamma(p_i+q)} g_i(\eta,s) + \frac{t^{q-1}}{\Delta} \sum_{i=1}^{m} \beta_i g(\xi_i, s) \tag{2.4}$$

where

$$g(t,s) = \frac{1}{\Gamma(q)} \begin{cases} t^{q-1}(1-s)^{q-1} - (t-s)^{q-1}, & 0 \leq s \leq t \leq 1, \\ t^{q-1}(1-s)^{q-1}, & 0 \leq t \leq s \leq 1, \end{cases} \tag{2.5}$$

and

$$g_i(\eta,s) = \begin{cases} \eta^{p_i+q-1}(1-s)^{q-1} - (\eta-s)^{p_i+q-1}, & 0 \leq s \leq \eta < 1, \\ \eta^{p_i+q-1}(1-s)^{q-1}, & 0 < \eta \leq s \leq 1, \end{cases} \tag{2.6}$$

Proof. By Lemma 2.4, the general solution for the above equation (2.1) is

$$u(t) = -\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} h(s) ds + c_1 t^{q-1} + c_2 t^{q-2},$$

where $c_1, c_2 \in \mathbb{R}$. The first condition of (2.2) implies that $c_2 = 0$. Thus

$$u(t) = -\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} h(s) ds + c_1 t^{q-1}. \tag{2.7}$$

Taking the Riemann-Liouville fractional integral of order $p_i > 0$ for (2.7) and using Lemma 2.3, we get that

$$\begin{aligned} (I^{p_i} u)(t) &= \int_0^t \frac{(t-s)^{p_i-1}}{\Gamma(p_i)} \left(c_1 s^{q-1} - \int_0^s \frac{(s-r)^{q-1}}{\Gamma(q)} dr \right) h(s) ds \\ &= c_1 \int_0^t \frac{(t-s)^{p_i-1} s^{q-1}}{\Gamma(p_i)} ds - \int_0^t \frac{(t-s)^{p_i-1}}{\Gamma(p_i)} \int_0^s \frac{(s-r)^{q-1}}{\Gamma(q)} h(r) dsdr \\ &= c_1 \frac{t^{p_i+q-1}\Gamma(q)}{\Gamma(p_i+q)} - \frac{1}{\Gamma(p_i+q)} \int_0^t (t-s)^{p_i+q-1} h(s) ds. \end{aligned}$$

The second condition of (2.2) yields

$$c_1 - \frac{1}{\Gamma(q)} \int_0^1 (1-s)^{q-1} h(s)\,ds = c_1 \sum_{i=1}^{m} \frac{\alpha_i \eta^{p_i+q-1}\Gamma(q)}{\Gamma(p_i+q)} - \sum_{i=1}^{m} \frac{\alpha_i}{\Gamma(p_i+q)} \int_0^\eta (\eta-s)^{p_i+q-1} h(s)\,ds + c_1 \sum_{i=1}^{m} \beta_i \xi_i^{q-1} - \frac{1}{\Gamma(q)} \sum_{i=1}^{m} \beta_i \int_0^{\xi_i} (\xi_i - s)^{q-1} h(s)\,ds.$$

Then, we have that

$$c_1 = \frac{1}{\Delta} \left\{ \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} h(s)\,ds - \sum_{i=1}^{m} \frac{\alpha_i}{\Gamma(p_i+q)} \int_0^\eta (\eta-s)^{p_i+q-1} h(s)\,ds - \frac{1}{\Gamma(q)} \sum_{i=1}^{m} \beta_i \int_0^{\xi_i} (\xi_i-s)^{q-1} h(s)\,ds \right\}.$$

Hence, the solution is

$$\begin{aligned}
u(t) &= -\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} h(s)\,ds + \frac{t^{q-1}}{\Delta\Gamma(q)} \int_0^1 (1-s)^{q-1} h(s)\,ds \\
&\quad - \frac{t^{q-1}}{\Delta} \sum_{i=1}^{m} \frac{\alpha_i}{\Gamma(p_i+q)} \int_0^n (\eta-s)^{p_i+q-1} h(s)\,ds \\
&\quad - \frac{t^{q-1}}{\Delta\Gamma(q)} \sum_{i=1}^{m} \beta_i \int_0^{\xi_i} (\xi_i-s)^{q-1} h(s)\,ds \\
&= -\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} h(s)\,ds + \frac{t^{q-1}}{\Gamma(q)} \int_0^1 (1-s)^{q-1} h(s)\,ds \\
&\quad + \frac{t^{q-1}}{\Delta} \left\{ \sum_{i=1}^{m} \frac{\alpha_i \eta^{p_i+q-1}}{\Gamma(p_i+q)} + \frac{1}{\Gamma(q)} \sum_{i=1}^{m} \beta_i \xi_i^{q-1} \right\} \int_0^1 (1-s)^{q-1} h(s)\,ds \\
&\quad - \frac{t^{q-1}}{\Delta} \sum_{i=1}^{m} \frac{\alpha_i}{\Gamma(p_i+q)} \int_0^n (\eta-s)^{p_i+q-1} h(s)\,ds \\
&\quad - \frac{t^{q-1}}{\Delta\Gamma(q)} \sum_{i=1}^{m} \beta_i \int_0^{\xi_i} (\xi_i-s)^{q-1} h(s)\,ds \\
&= \int_0^1 g(t,s) h(s)\,ds + \frac{t^{q-1}}{\Delta} \sum_{i=1}^{m} \frac{\alpha_i}{\Gamma(p_i+q)} \int_0^1 g_i(\eta,s) h(s)\,ds \\
&\quad + \frac{t^{q-1}}{\Delta} \sum_{i=1}^{m} \beta_i \int_0^1 g(\xi_i,s) h(s)\,ds \\
&= \int_0^1 G(t,s) h(s)\,ds.
\end{aligned}$$

□

Lemma 2.8. *The Green's function $G(t,s)$ has the following properties:*
(P_1) $G(t,s)$ *is continuous on* $[0,1]\times[0,1]$.
(P_2) $G(t,s)\geq 0$ *for all* $0\leq s,t\leq 1$.
(P_3) $G(t,s)\leq \max\limits_{0\leq t\leq 1} G(t,s)\leq g(s,s)\left(1+\frac{\sum_{i=1}^{m}\beta_i}{\Delta}\right)+\sum\limits_{i=1}^{m}\frac{\alpha_i}{\Delta\Gamma(p_i+q)}g_i(\eta,s).$

(P_4) $\int_0^1 \max\limits_{0\leq t\leq 1} G(t,s)\,ds\leq\left(1+\frac{\sum_{i=1}^{m}\beta_i}{\Delta}\right)\frac{\Gamma(q)}{\Gamma(2q)}+\sum\limits_{i=1}^{m}\frac{\alpha_i\eta^{p_i+q-1}}{\Delta\Gamma(p_i+q)}\left(\frac{p_i+q(1-\eta)}{q(p_i+q)}\right).$

(P_5) $\min\limits_{\eta\leq t\leq 1} G(t,s)\geq\sum\limits_{i=1}^{m}\frac{\alpha_i\eta^{q-1}}{\Delta\Gamma(p_i+q)}g_i(\eta,s)+(q-1)\sum\limits_{i=1}^{m}\frac{\beta_i\left(\xi_i^{q-1}-\xi_i^{q}\right)\eta^{q-1}}{\Delta}sg(s,s)$

for $s\in[0,1]$.

Proof. It is easy to check that (P_1) holds. To prove (P_2), we will show that $g(t,s)\geq 0$ and $g_i(\eta,s)\geq 0$, $i=1,2,...,m$, for all $0\leq s,t\leq 1$. For $t\leq s$, it is clear that $G(t,s)\geq 0$, we only need to prove the case $s\leq t$.
Then

$$\begin{aligned}
g(t,s) &= \frac{1}{\Gamma(q)}\left[t^{q-1}(1-s)^{q-1}-(t-s)^{q-1}\right]\\
&= \frac{1}{\Gamma(q)}\left[(t-ts)^{q-1}-(t-s)^{q-1}\right]\\
&\geq \frac{1}{\Gamma(q)}\left[(t-s)^{q-1}-(t-s)^{q-1}\right]=0.
\end{aligned}$$

For $0\leq s\leq\eta<1$, we have

$$\begin{aligned}
g_i(\eta,s) &= \eta^{p_i+q-1}(1-s)^{q-1}-(\eta-s)^{p_i+q-1}\\
&= \eta^{p_i}(\eta-\eta s)^{q-1}-(\eta-s)^{p_i+q-1}\\
&\geq \eta^{p_i}(\eta-s)^{q-1}-(\eta-s)^{p_i+q-1}\\
&= (\eta-s)^{q-1}\left(\eta^{p_i}-(\eta-s)^{p_i}\right)\\
&\geq 0.
\end{aligned}$$

When $0<\eta\leq s\leq 1$, $g_i(\eta,s)=\eta^{p_i+q-1}(1-s)^{q-1}\geq 0$. Therefore, $g_i(\eta,s)\geq 0$, $i=1,2,...,m$ for all $0\leq s\leq 1$.
Now, we prove (P_3). For a given $s\in[0,1]$, when $0\leq s\leq t\leq 1$

$$\Gamma(q)\,g(t,s)=t^{q-1}(1-s)^{q-1}-(t-s)^{q-1}$$

and thus

$$\begin{aligned}
\Gamma(q)\frac{\partial}{\partial t}g(t,s) &= (q-1)\,t^{q-2}(1-s)^{q-1}-(q-1)(t-s)^{q-2}\\
&= (q-1)(t-ts)^{q-2}(1-s)-(q-1)(t-s)^{q-2}\\
&\leq (q-1)(t-s)^{q-2}(1-s)-(q-1)(t-s)^{q-2}\\
&= -s(q-1)(t-s)^{q-2}.
\end{aligned}$$

Hence, $g(t,s)$ is decreasing with respect to t. Then we have $g(t,s) \leq g(s,s)$ for $0 \leq s \leq t \leq 1$. For $0 \leq t \leq s \leq 1$

$$\Gamma(q)\frac{\partial}{\partial t}g(t,s) = (q-1)\,t^{q-2}(1-s)^{q-1} \geq 0,$$

which means that $g(t,s)$ is increasing with respect to t. Thus $g(t,s) \leq g(s,s)$ for $0 \leq t \leq s \leq 1$. Therefore $g(t,s) \leq g(s,s)$ for $0 \leq s,t \leq 1$.
From the above analysis, we have for $0 \leq s \leq 1$ that

$$\begin{aligned}G(t,s) \leq \max_{0\leq t\leq 1} G(t,s) &= \max_{0\leq t\leq 1}\Bigg(g(t,s)\\ &\quad+\frac{t^{q-1}}{\Delta}\sum_{i=1}^{m}\frac{\alpha_i}{\Gamma(p_i+q)}g_i(\eta,s)+\frac{t^{q-1}}{\Delta}\sum_{i=1}^{m}\beta_i g(\xi_i,s)\Bigg)\\ &\leq g(s,s)\left(1+\frac{\sum_{i=1}^{m}\beta_i}{\Delta}\right)+\sum_{i=1}^{m}\frac{\alpha_i}{\Delta\Gamma(p_i+q)}g_i(\eta,s).\end{aligned}$$

To prove (P_4), by direct integration, we have

$$\begin{aligned}\int_0^1 \max_{0\leq t\leq 1} G(t,s)\,ds &\leq \int_0^1\left[g(s,s)\left(1+\frac{\sum_{i=1}^{m}\beta_i}{\Delta}\right)+\sum_{i=1}^{m}\frac{\alpha_i}{\Delta\Gamma(p_i+q)}g_i(\eta,s)\right]ds\\ &=\left(1+\frac{\sum_{i=1}^{m}\beta_i}{\Delta}\right)\int_0^1\frac{s^{q-1}(1-s)^{q-1}}{\Gamma(q)}ds\\ &\quad+\sum_{i=1}^{m}\frac{\alpha_i}{\Delta\Gamma(p_i+q)}\left(\int_\eta^1 \eta^{p_i+q-1}(1-s)^{q-1}\right)ds\\ &\quad+\sum_{i=1}^{m}\frac{\alpha_i}{\Delta\Gamma(p_i+q)}\left(\int_0^\eta\left[\eta^{p_i+q-1}(1-s)^{q-1}-(\eta-s)^{p_i+q-1}\right]ds\right)\\ &=\left(1+\frac{\sum_{i=1}^{m}\beta_i}{\Delta}\right)\frac{\Gamma(q)}{\Gamma(2q)}+\sum_{i=1}^{m}\frac{\alpha_i\eta^{p_i+q-1}}{\Delta\Gamma(p_i+q)}\left(\frac{p_i+q(1-\eta)}{q(p_i+q)}\right).\end{aligned}$$

Now, we shall prove (P_5).
Firstly, let $k_1(\xi_i,s)=\frac{g(\xi_i,s)}{g(s,s)}$ for $0<s<\xi_i<1$, $i=1,2,...,m$, then we get

$$k_1(\xi_i,s)=\frac{(\xi_i(1-s))^{q-1}-(\xi_i-s)^{q-1}}{s^{q-1}(1-s)^{q-1}}=\frac{(q-1)\int_{\xi_i-s}^{\xi_i(1-s)}x^{q-2}dx}{s^{q-1}(1-s)^{q-1}}.$$

Since the function $x\longmapsto x^{q-2}$ is continuous and decreasing on $[\xi_i-s,\ \xi_i(1-s)]$, we have

$$\begin{aligned}k_1(\xi_i,s) &\geq \frac{(q-1)(\xi_i(1-s))^{q-2}\left[\xi_i(1-s)-(\xi_i-s)\right]}{s^{q-1}(1-s)^{q-1}}\\ &=\frac{(q-1)\,\xi_i^{q-2}(1-s)^{q-2}s(1-\xi_i)}{s^{q-1}(1-s)^{q-1}}\\ &\geq (q-1)\,\xi_1^{q-1}(1-\xi_i)\,s.\end{aligned}$$

Let

$$k_2(\xi_i, s) = \frac{g(\xi_i, s)}{g(s, s)}$$

for $0 < \xi_i \leq s < 1$, $i = 1, 2, ..., m$, then we get

$$k_2(\xi_i, s) = \frac{\xi_i^{q-1}}{s^{q-1}} \geq \frac{\xi_i^{q-1}}{s^{q-2}} = \xi_i^{q-1} s^{2-q} \geq (q-1)\, \xi_i^{q-1} (1 - \xi_i)\, s.$$

Therefore, we have

$$g(\xi_i, s) \geq (q-1)\, sg(s, s) \left(\xi_i^{q-1} - \xi_i^q\right) \quad for \quad 0 < s, \xi_i < 1 \tag{2.8}$$

Furthermore, the inequality in (2.8) is satisfied for $s \in \{0, 1\}$. Hence

$$g(\xi_i, s) \geq (q-1)\, sg(s, s) \left(\xi_i^{q-1} - \xi_i^q\right) \quad for \quad 0 \leq s, \xi_i \leq 1. \tag{2.9}$$

Secondly, from $g(t, s) \geq 0$, $g_i(\eta, s) \geq 0$, $i = 1, 2, ..., m$ and from (2.9), we have

$$\begin{aligned}
\min_{\eta \leq t \leq 1} G(t, s) &= \min_{\eta \leq t \leq 1} \left(g(t, s) + \frac{t^{q-1}}{\Delta} \sum_{i=1}^{m} \frac{\alpha_i}{\Gamma(p_i + q)} g_i(\eta, s) \right. \\
&\quad \left. + \frac{t^{q-1}}{\Delta} \sum_{i=1}^{m} \beta_i g(\xi_i, s) \right) \\
&\geq \min_{\eta \leq t \leq 1} g(t, s) + \min_{\eta \leq t \leq 1} \frac{t^{q-1}}{\Delta} \sum_{i=1}^{m} \frac{\alpha_i}{\Gamma(p_i + q)} g_i(\eta, s) \\
&\quad + \min_{\eta \leq t \leq 1} \frac{t^{q-1}}{\Delta} \sum_{i=1}^{m} \beta_i g(\xi_i, s) \\
&\geq \sum_{i=1}^{m} \frac{\alpha_i \eta^{q-1}}{\Delta \Gamma(p_i + q)} g_i(\eta, s) + (q-1) \sum_{i=1}^{m} \frac{\beta_i \left(\xi_i^{q-1} - \xi_i^q\right) \eta^{q-1}}{\Delta} sg(s, s)
\end{aligned}$$

for $0 \leq s \leq 1$. This completes the proof. □

Let $E = C([0, 1], \mathbb{R})$ be the Banach space of all continuous functions defined on $[0, 1]$ that are mapped into $\mathbb{R}$ with the norm defined as $\|u\| = \sup_{t \in [0,1]} |u(t)|$. If $u \in E$ satisfies the problem (1.1) and $u(t) \geq 0$ for any $t \in [0, 1]$, then u is called a nonnegative solution of the problem (1.1). If u is a nonnegative solution of the problem (1.1) with $\|u\| > 0$, then u is called a positive solution of the problem (1.1). Define the cone $\mathcal{K} \in E$ by

$$\mathcal{K} = \{u \in E : \ u(t) \geq 0\},$$

and the operator $A : K \to E$ by

$$Au(t) := \int_0^1 G(t, s) f(s, u(s))\, ds. \tag{2.10}$$

In view of Lemma 2.7, the nonnegative solutions of problem (1.1) are given by the operator equation $u(t) = Au(t)$

Lemma 2.9. *Suppose that* $f:[0,1]\times[0,\infty)\to[0,\infty)$ *is continuous. The operator* $A:\mathcal{K}\to\mathcal{K}$ *is completely continuous.*

Proof. Since $G(t,s)\geq 0$ for $s,t\in[0,1]$, we have $Au(t)\geq 0$ for all $u\in\mathcal{K}$. Therefore, $A:\mathcal{K}\to\mathcal{K}$.
For a constant $R>0$, we define $\Omega=\{u\in\mathcal{K}:\ \|u\|<R\}$.
Let

$$L=\max_{0\leq t\leq 1,0\leq u\leq R}|f(t,u)|. \tag{2.11}$$

Then, for $u\in\Omega$, from Lemma 2.8, we have

$$\begin{aligned}|Au(t)|&=\left|\int_0^1 G(t,s)f(s,u(s))\,ds\right|\\ &\leq L\int_0^1 G(t,s)\,ds\\ &\leq L\int_0^1\left(g(s,s)\left(1+\frac{\sum_{i=1}^m\beta_i}{\Delta}\right)+\sum_{i=1}^m\frac{\alpha_i}{\Delta\Gamma(p_i+q)}g_i(\eta,s)\right)ds\\ &\leq\left(1+\frac{\sum_{i=1}^m\beta_i}{\Delta}\right)\frac{\Gamma(q)}{\Gamma(2q)}+\sum_{i=1}^m\frac{\alpha_i\eta^{p_i+q-1}}{\Delta\Gamma(p_i+q)}\left(\frac{p_i+q(1-\eta)}{q(p_i+q)}\right).\end{aligned}$$

Hence, $\|Au\|\leq M$, and so $A(\Omega)$ is uniformly bounded. Now, we shall show that $A(\Omega)$ is equicontinuous. For $u\in\Omega$, $t_1,t_2\in[0,1]$, $t_1<t_2$, we have

$$|Au(t_2)-Au(t_1)|\leq L\int_0^1|G(t_2,s)-G(t_1,s)|\,ds,$$

where L is defined by (2.11). Since $G(t,s)$ is continuous on $[0,1]\times[0,1]$, therefore $G(t,s)$ is uniformly continuous on $[0,1]\times[0,1]$. Hence, for any $\epsilon>0$, there exists a positive constant

$$\delta=\frac{1}{2}\left[\frac{\epsilon\Gamma(q)}{L}\left(\frac{1}{\frac{1}{q}+\sum_{i=1}^m\frac{\alpha_i\eta^{p_i+q-1}}{\Delta\Gamma(p_i+q)}\left(\frac{p_i+q(1-n)}{q(p_i+q)}\right)+\frac{\Gamma(q)}{\Gamma(2q)}\frac{\sum_{i=1}^m\beta_i}{\Delta}}\right)\right]$$

whenever $|t_2-t_1|<\delta$, we have the following two cases.

Case 1. $\delta\leq t_1<t_2<1$.

Therefore,

$$\begin{aligned}
|Au(t_2) - Au(t_1)| &\leq L \int_0^1 |G(t_2,s) - G(t_1,s)|\, ds \\
&= L \left[\int_0^{t_1} |G(t_2,s) - G(t_1,s)|\, ds + \int_{t_1}^{t_2} |G(t_2,s) - G(t_1,s)|\, ds \right. \\
&\quad \left. + \int_{t_2}^1 |G(t_2,s) - G(t_1,s)|\, ds \right] \\
&\leq \frac{\left(t_2^{q-1} - t_1^{q-1}\right) L}{\Gamma(q)} \int_0^1 (1-s)^{q-1}\, ds \\
&\quad + \frac{\left(t_2^{q-1} - t_1^{q-1}\right) L}{\Delta} \int_0^1 \sum_{i=1}^m \frac{\alpha_i}{\Gamma(p_i+q)} g_i(\eta,s)\, ds \\
&\quad + \frac{\left(t_2^{q-1} - t_1^{q-1}\right) L}{\Delta} \sum_{i=1}^m \beta_i \int_0^1 g(s,,s)\, ds \\
&= \frac{\left(t_2^{q-1} - t_1^{q-1}\right) L}{\Gamma(q)} \left[\frac{1}{q} + \sum_{i=1}^m \frac{\alpha_i \eta^{p_i+q-1}}{\Delta\Gamma(p_i+q)} \left(\frac{p_i + q(1-n)}{q(p_i+q)} \right) \right. \\
&\quad \left. + \frac{\Gamma(q)}{\Gamma(2q)} \frac{\sum_{i=1}^m \beta_i}{\Delta} \right] \\
&\leq \frac{(q-1)\,\delta^{q-1} L}{\Gamma(q)} \left[\frac{1}{q} + \sum_{i=1}^m \frac{\alpha_i \eta^{p_i+q-1}}{\Delta\Gamma(p_i+q)} \left(\frac{p_i + q(1-n)}{q(p_i+q)} \right) \right. \\
&\quad \left. + \frac{\Gamma(q)}{\Gamma(2q)} \frac{\sum_{i=1}^m \beta_i}{\Delta} \right] \\
&< \epsilon.
\end{aligned}$$

Case 2. $0 \leq t_1 < 1,\ t_2 < 2\delta.$
Hence

$$\begin{aligned}
&|Au(t_2) - Au(t_1)| \leq L \int_0^1 |G(t_2,s) - G(t_1,s)|\, ds \\
&< \frac{\left(t_2^{q-1} - t_1^{q-1}\right) L}{\Gamma(q)} \left[\frac{1}{q} + \sum_{i=1}^m \frac{\alpha_i \eta^{p_i+q-1}}{\Delta\Gamma(p_i+q)} \left(\frac{p_i + q(1-n)}{q(p_i+q)} \right) + \frac{\Gamma(q)}{\Gamma(2q)} \frac{\sum_{i=1}^m \beta_i}{\Delta} \right] \\
&\leq \frac{t_2^{q-1} L}{\Gamma(q)} \left[\frac{1}{q} + \sum_{i=1}^m \frac{\alpha_i \eta^{p_i+q-1}}{\Delta\Gamma(p_i+q)} \left(\frac{p_i + q(1-n)}{q(p_i+q)} \right) + \frac{\Gamma(q)}{\Gamma(2q)} \frac{\sum_{i=1}^m \beta_i}{\Delta} \right] \\
&< \frac{(2\delta)^{q-1} L}{\Gamma(q)} \left[\frac{1}{q} + \sum_{i=1}^m \frac{\alpha_i \eta^{p_i+q-1}}{\Delta\Gamma(p_i+q)} \left(\frac{p_i + q(1-n)}{q(p_i+q)} \right) + \frac{\Gamma(q)}{\Gamma(2q)} \frac{\sum_{i=1}^m \beta_i}{\Delta} \right] \\
&= \epsilon.
\end{aligned}$$

Thus, $A(\Omega)$ is equicontinuous. In view of the Arzela-Ascoli theorem, we have that $\overline{A(\Omega)}$ is compact, which means $A : \mathcal{K} \to \mathcal{K}$ is a completely continuous operator. This completes the proof. $\square$

Theorem 2.10. [10] *Let E be a Banach space, and let $\mathcal{K} \in E$ be a cone. Assume that Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let $T : \mathcal{K} \cap \left(\overline{\Omega}_2 \setminus \Omega_1\right) \to \mathcal{K}$ be a completely continuous operator such that:*

(i) $\|Tu\| \geq \|u\|$, $u \in \mathcal{K} \cap \partial\Omega_1$, and $\|Tu\| \leq \|u\|$, $u \in \mathcal{K} \cap \partial\Omega_2$; or

(ii) $\|Tu\| \leq \|u\|$, $u \in \mathcal{K} \cap \partial\Omega_1$, and $\|Tu\| \geq \|u\|$, $u \in \mathcal{K} \cap \partial\Omega_2$. Then T has a fixed point $\mathcal{K} \cap \left(\overline{\Omega}_2 \setminus \Omega_1\right)$.

Theorem 2.11. [19] *Let $\mathcal{K}$ be a cone in the real Banach space E and $c > 0$ be a constant. Assume that there exists a concave nonnegative continuous functional θ on $\mathcal{K}$ with $\theta(u) \leq \|u\|$ for all $u \in \overline{\mathcal{K}}_c$. Let $A : \overline{\mathcal{K}}_c \to \overline{\mathcal{K}}_c$ be a completely continuous operator. Suppose that there exist constants $0 < a < b < d \leq c$ such that the following conditions hold:*
(i) $\{u \in \mathcal{K}(\theta, b, d) : \theta(u) > b\} \neq \emptyset$ and $\theta(Au) > b$ for $u \in \mathcal{K}(\theta, b, d)$;
(ii) $\|Au\| < a$ for $\|u\| \leq a$;
(iii) $\theta(Au) > b$ for $u \in \mathcal{K}(\theta, b, c)$ with $\|Au\| > d$.
Then A has at least three fixed points u_1, u_2 and u_3 in $\overline{\mathcal{K}}_c$ such that
$\|u_1\| < a$, $b < \theta(u_2)$, $a < \|u_3\|$ with $\theta(u_3) < b$.

Remark 2.12. If there holds $d = c$, then condition (i) implies condition (iii) of Theorem 2.11.

3. Main results

In this section, in order to establish some results of existence and multiplicity of positive solutions for BVP (1.1), we will impose growth conditions on f which allow us to apply Theorems 2.10 and 2.11.

For convenience, we denote

$$\Lambda_1 = \sum_{i=1}^{m} \frac{\alpha_i \eta^{p_i+2(q-1)}}{\Delta\Gamma(p_i+q)} \left(\frac{p_i+q(1-\eta)}{q(p_i+q)}\right) + (q-1)\sum_{i=1}^{m} \frac{\beta_i\left(\xi_i^{q-1}-\xi_i^{q}\right)\eta^{q-1}}{\Delta} \times \frac{\Gamma(q+1)}{\Gamma(2q+1)}$$
$$\Lambda_2 = \left(1 + \frac{\sum_{i=1}^{m}\beta_i}{\Delta}\right)\frac{\Gamma(q)}{\Gamma(2q)} + \sum_{i=1}^{m} \frac{\alpha_i \eta^{p_i+q-1}}{\Delta\Gamma(p_i+q)}\left(\frac{p_i+q(1-\eta)}{q(p_i+q)}\right)$$
$$\Lambda_3 = \sum_{i=1}^{m} \frac{\alpha_i \eta^{p_i+2(q-1)}(1-\eta)^q}{\Delta\Gamma(p_i+q)q} + (q-1)\sum_{i=1}^{m} \frac{\beta_i\left(\xi_i^{q-1}-\xi_i^{q}\right)\eta^{q-1}(1-\eta)^{2q}\Gamma(q+1)}{\Delta\Gamma(2q+1)}$$

Theorem 3.1. *Let $f : [0,1] \times [0,\infty) \to [0,\infty)$ be a continuous function. Assume that there exist constants $r_2 > r_1 > 0$, $M_1 \in \left(\Lambda_1^{-1}, \infty\right)$ and $M_2 \in \left(0, \Lambda_2^{-1}\right)$ such that:*
(H_1) $f(t,u) \geq M_1 r_1$, for $(t,u) \in [0,1] \times [0, r_1]$;
(H_2) $f(t,u) \leq M_2 r_2$, for $(t,u) \in [0,1] \times [0, r_2]$.
Then boundary value problem (1.1) has at least one positive solution u such that $r_1 \leq \|u\| \leq r_2$.

Proof. From Lemma 2.9, the operator $A : \mathcal{K} \to \mathcal{K}$ is completely continuous.
We divide the rest of the proof into two steps.

Step 1. Let $\Omega_1 = \{u \in E : \|u\| < r_1\}$, then for any $u \in \mathcal{K} \cap \Omega_1$, we have $0 \leq u(t) \leq r_1$ for all $t \in [0,1]$. From (H_1), it follows for $t \in [\eta, 1]$ that

$$\begin{aligned}
(Au)(t) &= \int_0^1 G(t,s) f(s,u(s))\,ds \\
&\geq \int_0^1 \min_{\eta \leq t \leq 1} G(t,s) f(s,u(s))\,ds \\
&\geq M_1 r_1 \left\{ \sum_{i=1}^m \frac{\alpha_i \eta^{q-1}}{\Delta \Gamma(p_i+q)} \int_0^1 g_i(\eta,s)\,ds \right. \\
&\quad \left. + (q-1) \sum_{i=1}^m \frac{\beta_i \left(\xi_i^{q-1} - \xi_i^q\right) \eta^{q-1}}{\Delta} \int_0^1 s g(s,s)\,ds \right\} \\
&= M_1 r_1 \left\{ \sum_{i=1}^m \frac{\alpha_i \eta^{q-1}}{\Delta \Gamma(p_i+q)} \left(\int_\eta^1 \eta^{p_i+q-1} (1-s)^{q-1}\,ds \right. \right. \\
&\quad \left. + \int_0^\eta \left[\eta^{p_i+q-1} (1-s)^{q-1} - (\eta-s)^{p_i+q-1} \right] ds \right) \\
&\quad \left. + (q-1) \sum_{i=1}^m \frac{\beta_i \left(\xi_i^{q-1} - \xi_i^q\right) \eta^{q-1}}{\Delta} \times \frac{\Gamma(q+1)}{\Gamma(2q+1)} \right\} \\
&= M_1 r_1 \left\{ \sum_{i=1}^m \frac{\alpha_i \eta^{p_i+2(q-1)}}{\Delta \Gamma(p_i+q)} \left(\frac{p_i + q(1-\eta)}{q(p_i+q)} \right) \right. \\
&\quad \left. + (q-1) \sum_{i=1}^m \frac{\beta_i \left(\xi_i^{q-1} - \xi_i^q\right) \eta^{q-1}}{\Delta} \times \frac{\Gamma(q+1)}{\Gamma(2q+1)} \right\} \\
&\geq r_1 = \|u\|,
\end{aligned}$$

which means that

$$\|Au\| \geq \|u\| \quad \textit{for } u \in \mathcal{K} \cap \partial\Omega_1. \tag{3.1}$$

Step 2. Let $\Omega_2 = \{u \in E : \|u\| < r_2\}$, then for any $u \in \mathcal{K} \cap \partial\Omega_2$, we have $0 \leq u(t) \leq r_2$ for all $t \in [0,1]$. It follows from (H_2) that for $t \in [0,1]$,

$$\begin{aligned}
(Au)(t) &= \int_0^1 G(t,s) f(s,u(s))\,ds \\
&\leq M_2 r_2 \left\{ \left(1 + \frac{\sum_{i=1}^m \beta_i}{\Delta} \right) \frac{\Gamma(q)}{\Gamma(2q)} + \sum_{i=1}^m \frac{\alpha_i \eta^{p_i+q-1}}{\Delta \Gamma(p_i+q)} \left(\frac{p_i + q(1-\eta)}{q(p_i+q)} \right) \right\} \\
&\leq r_2 = \|u\|,
\end{aligned}$$

which means that

$$\|Au\| \leq \|u\| \quad \textit{for any } u \in \mathcal{K} \cap \partial\Omega_2. \tag{3.2}$$

By (i) of Theorem 2.10, we get that A has a fixed point u in $\mathcal{K}$ with $r_1 \leq \|u\| \leq r_2$, which is also a positive solution of boundary value problem (1.1). □

Theorem 3.2. *Let $f : [0,1] \times [0,\infty) \to [0,\infty)$ be a continuous function. Suppose that there exist constants $0 < a < b < c$ such that the following assumptions hold:*
(H_3) $f(t,u) < \Lambda_2^{-1} a$ *for* $(t,u) \in [0,1] \times [0,a]$;
(H_4) $f(t,u) > \Lambda_3^{-1} b$ *for* $(t,u) \in [\eta,1] \times [b,c]$;
(H_5) $f(t,u) \leq \Lambda_2^{-1} c$ *for* $(t,u) \in [0,1] \times [0,c]$.
Then boundary value problem (1.1) has at least one nonnegative solution u_1 and two positive solutions u_2, u_3 in $\overline{\mathcal{K}}_c$ with

$$\|u_1\| < a, \quad b < \min_{\eta \leq t \leq 1} u_2(t) \, and \quad a < \|u_3\| \quad with \quad \min_{\eta \leq t \leq 1} u_3(t) < b.$$

Proof. We show that all the conditions of Theorem 2.11 are satisfied.
If $u \in \overline{\mathcal{K}}_c$, then $\|u\| \leq c$. Condition (H_5) implies $f(t, u(t)) \leq \Lambda_2^{-1} c$ for $t \in [0,1]$. Consequently,

$$\begin{aligned}
&(Au)(t) = \int_0^1 G(t,s) f(s,u(s))\, ds \\
&\leq \Lambda_2^{-1} c \int_0^1 \left[\left(1 + \frac{\sum_{i=1}^m \beta_i}{\Delta}\right) g(s,s) + \sum_{i=1}^m \frac{\alpha_i \eta^{p_i+q-1}}{\Delta \Gamma(p_i+q)} \left(\frac{p_i + q(1-\eta)}{q(p_i+q)}\right) g_i(\eta,s) \right] ds \\
&= \Lambda_2^{-1} c \left\{ \left(1 + \frac{\sum_{i=1}^m \beta_i}{\Delta}\right) \frac{\Gamma(q)}{\Gamma(2q)} + \sum_{i=1}^m \frac{\alpha_i \eta^{p_i+q-1}}{\Delta \Gamma(p_i+q)} \left(\frac{p_i + q(1-\eta)}{q(p_i+q)}\right) \right\} \\
&= c,
\end{aligned}$$

which implies $\|Au\| \leq c$. Hence, $A : \overline{\mathcal{K}}_c \to \overline{\mathcal{K}}_c$ is completely continuous.
If $u \in \overline{\mathcal{K}}_a$, then (H_3) yields

$$\begin{aligned}
(Au)(t) &< \Lambda_2^{-1} \int_0^1 \left[\left(1 + \frac{\sum_{i=1}^m \beta_i}{\Delta}\right) g(s,s) + \sum_{i=1}^m \frac{\alpha_i}{\Delta \Gamma(p_i+q)} g_i(\eta,s) \right] ds \\
&= \Lambda_2^{-1} a \left\{ \left(1 + \frac{\sum_{i=1}^m \beta_i}{\Delta}\right) \frac{\Gamma(q)}{\Gamma(2q)} + \sum_{i=1}^m \frac{\alpha_i \eta^{p_i+q-1}}{\Delta \Gamma(p_i+q)} \left(\frac{p_i + q(1-\eta)}{q(p_i+q)}\right) \right\} \\
&= a.
\end{aligned}$$

Thus $\|Au\| < a$. Therefore, condition (ii) of Theorem 2.11 holds.
Define a concave nonnegative continuous functional θ on $\mathcal{K}$ by

$$\theta(u) = \min_{\eta \leq t \leq 1} |u(t)|.$$

To check condition (i) of Theorem 2.11, we choose $u(t) = \frac{b+c}{2}$ for $t \in [0,1]$. It is easy to see that $u(t) \in \mathcal{K}(\theta, b, c)$ and $\theta(u) = \theta\left(\frac{b+c}{2}\right) > b$, which means that $\{\mathcal{K}(\theta, b, c) : \theta(u) > b\} \neq \emptyset$. Hence, if $u \in \mathcal{K}(\theta, b, c)$, then $b \leq u(t) \leq c$ for $t \in [\eta, 1]$.

From assumption (H_4), we have

$$\begin{aligned}\theta\left(Au\right) &= \min_{\eta\leq t\leq 1}\left|\left(Au\right)\left(t\right)\right| \\ &\geq \int_{\eta}^{1}\min_{\eta\leq t\leq 1}G\left(t,s\right)f\left(s,u\left(s\right)\right)ds \\ &> \Lambda_3^{-1}b\left\{\sum_{i=1}^{m}\frac{\alpha_i\eta^{q-1}}{\Delta\Gamma\left(p_i+q\right)}\int_{\eta}^{1}g_i\left(\eta,s\right)ds\right. \\ &\quad\left.+\left(q-1\right)\sum_{i=1}^{m}\frac{\beta_i\left(\xi_i^{q-1}-\xi_i^{q}\right)\eta^{q-1}}{\Delta}\int_{\eta}^{1}sg\left(s,s\right)ds\right\} \\ &= \Lambda_3^{-1}b\left\{\sum_{i=1}^{m}\frac{\alpha_i\eta^{p_i+2(q-1)}\left(1-\eta\right)^{q}}{\Delta\Gamma\left(p_i+q\right)q}\right. \\ &\quad\left.+\left(q-1\right)\sum_{i=1}^{m}\frac{\beta_i\left(\xi_i^{q}-\xi_i^{q}\right)\eta^{q-1}\left(1-\eta\right)^{2q}\Gamma\left(q+1\right)}{\Delta\Gamma\left(2q+1\right)}\right\} \\ &= b.\end{aligned}$$

Thus $\theta\left(Au\right)>b$ for all $u\in\mathcal{K}\left(\theta,b,c\right)$. This shows that condition (i) of Theorem 2.11 is also satisfied.

By Theorem 2.11 and Remark 2.12, boundary value problem (1.1) has at least one nonnegative solution u_1 and two positive solutions u_2, u_3, which satisfy

$$\|u_1\|<a,\qquad b<\min_{\eta\leq t\leq 1}\left|u_2\left(t\right)\right|\quad a<\|u_3\|\quad \textit{with}\ \min_{\eta\leq t\leq 1}\left|u\left(t\right)\right|<b.$$

The proof is complete. □

4. Examples

4.1. Example

Consider the fractional differential equations with boundary value as follows:

$$\begin{cases} D^{\frac{3}{2}}u\left(t\right)+f\left(t,u\left(t\right)\right)=0, \quad 0<t<1, \\ u\left(0\right)=0 \\ u\left(1\right)=2\left(I^{\frac{3}{2}}u\right)\left(\frac{1}{4}\right)+\frac{1}{2}\left(I^{\frac{\pi}{4}}u\right)\left(\frac{1}{4}\right)+\frac{4}{5}\left(I^{\frac{2}{3}}u\right)\left(\frac{1}{4}\right)+\frac{3}{15}u\left(\frac{1}{3}\right)+\frac{3}{20}u\left(\frac{1}{4}\right)+\frac{1}{4}u\left(\frac{1}{5}\right), \end{cases} \tag{4.1}$$

where

$$f\left(t,u\right)\begin{cases} u(1-u^2)+4\left(1+\frac{3}{4}t\right), \ 0\leq t\leq 1;\ 0\leq u\leq 1 \\ 4\left(1+\frac{3}{4}t\right)e^{1-u}+\sin^2\left(\pi\left(1-u\right)\right), \quad 0\leq t\leq 1;\ 1\leq u\leq 21. \end{cases}$$

Set $m=3$, $\eta=\frac{1}{4}$, $q=\frac{3}{2}$, $\alpha_1=2$, $\alpha_2=\frac{1}{2}$, $\alpha_3=\frac{4}{5}$,$p_1=\frac{3}{2}$, $p_2=\frac{\pi}{4}$, $p_3=\frac{2}{3}$, $\beta_1=\frac{1}{4}$, $\beta_2=\frac{3}{20}$, $\beta_3=\frac{3}{5}$, $\xi_1=\frac{1}{5}$, $\xi_2=\frac{1}{4}$ and $\xi_3=\frac{1}{3}$

Consequently, we can get

$$\Delta = 1 - \Gamma(q) \sum_{i=1}^{m} \frac{\alpha_i \eta^{p_i+q-1}}{\Gamma(p_i+q)} - \sum_{i=1}^{m} \beta_i \xi_i^{q-1} \approx 0.265299.$$

Then, by direct calculations, we can obtain that

$$\begin{aligned}\Lambda_1 &= \sum_{i=1}^{m} \frac{\alpha_i \eta^{p_i+2(q-1)}}{\Delta\Gamma(p_i+q)} \left(\frac{p_i + q(1-\eta)}{q(p_i+q)}\right) \\ &\quad + (q-1) \sum_{i=1}^{m} \frac{\beta_i \left(\xi_i^{q-1} - \xi_i^{q}\right) \eta^{q-1}}{\Delta} \times \frac{\Gamma(q+1)}{\Gamma(2q+1)} \\ &\approx 0.45478 \\ \Lambda_2 &= \left(1 + \frac{\sum_{i=1}^{m} \beta_i}{\Delta}\right) \frac{\Gamma(q)}{\Gamma(2q)} + \sum_{i=1}^{m} \frac{\alpha_i \eta^{p_i+q-1}}{\Delta\Gamma(p_i+q)} \left(\frac{p_i + q(1-\eta)}{q(p_i+q)}\right) \\ &\approx 2.63219.\end{aligned}$$

Choose $r_1 = 1$, $r_2 = 21$, $M_1 = 3$ and $M_2 = 0.35$, $f(t,u)$ satisfies

$$f(t,u) \geq 4 \geq 3 = M_1 r_1, \qquad \forall (t,u) \in [0,1] \times [0,1]$$

and

$$f(t,u) \leq 7 \leq 7.35 = M_2 r_2 \qquad \forall (t,u) \in [0,1] \times [0,21]$$

Thus, (H_1) and (H_2) hold. By Theorem 3.1, we have that boundary value problem (4.1) has at least one positive solution u such that $1 < \|u\| < 21$.

4.2. Example

Consider the following boundary value problem:

$$\begin{cases} D^{\frac{3}{2}} u(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \\ u(0) = 0 \\ u(1) = \frac{1}{8}\left(I^{\frac{1}{2}} u\right)\left(\frac{1}{8}\right) + \frac{1}{3}\left(I^{\frac{3}{2}} u\right)\left(\frac{1}{8}\right) + \frac{1}{4}\left(I^{\frac{5}{2}} u\right)\left(\frac{1}{8}\right) + \frac{1}{3} u\left(\frac{1}{2}\right) + \frac{1}{5} u\left(\frac{1}{8}\right) + \frac{1}{7} u\left(\frac{1}{6}\right), \end{cases} \tag{4.2}$$

where

$$f(t,u) \begin{cases} u\left(\frac{3}{4} - u\right) + \frac{3}{16}\left(t^2 + 2\right), & , 0 \leq t \leq 1,\ 0 \leq u \leq \frac{3}{4}, \\ \frac{1}{4}\left(t^2 + 2\right) \cos^2\left(\frac{2\pi}{9} u\right) + 120\left(\frac{3}{4} - u\right)^2, & 0 \leq t \leq 1,\ \frac{3}{4} \leq u \leq \frac{3}{2}, \\ \frac{1}{16}\left(t^2 + 1082\right) - 10 \sin^2\left(u - \frac{3}{2}\right)\pi, & , 0 \leq t \leq 1,\ \frac{3}{2} \leq u \leq \infty. \end{cases}$$

Set $m = 3$, $\eta = \frac{1}{8}$, $q = \frac{3}{2}$, $\alpha_1 = \frac{1}{8}$, $\alpha_2 = \frac{1}{3}$, $\alpha_3 = \frac{1}{4}$, $p_1 = \frac{1}{2}$, $p_2 = \frac{3}{2}$, $p_3 = \frac{5}{2}$, $\beta_1 = \frac{1}{3}$, $\beta_2 = \frac{1}{5}$, $\beta_3 = \frac{1}{7}$, $\xi_1 = \frac{1}{2}$, $\xi_2 = \frac{1}{4}$ and $\xi_3 = \frac{1}{6}$.
Consequently, we can get

$$\Delta = 1 - \Gamma(q) \sum_{i=1}^{m} \frac{\alpha_i \eta^{p_i+q-1}}{\Gamma(p_i+q)} - \sum_{i=1}^{m} \beta_i \xi_i^{q-1} \approx 0.589749.$$

Then, by direct calculations, we can obtain that

$$\Lambda_2 = \left(1+\frac{\sum_{i=1}^{m}\beta_i}{\Delta}\right)\frac{\Gamma(q)}{\Gamma(2q)} + \sum_{i=1}^{m}\frac{\alpha_i\eta^{p_i+q-1}}{\Delta\Gamma(p_i+q)}\left(\frac{p_i+q(1-\eta)}{q(p_i+q)}\right) \approx 0,97003,$$

$$\Lambda_3 = \sum_{i=1}^{m}\frac{\alpha_i\eta^{p_i+2(q-1)}(1-\eta)^q}{\Delta\Gamma(p_i+q)\,q} + (q-1)\sum_{i=1}^{m}\frac{\beta_i\left(\xi_i^{q-1}-\xi_i^{q}\right)\eta^{q-1}(1-\eta)^{2q}\Gamma(q+1)}{\Delta\Gamma(2q+1)}$$
$$\approx 0.02390086.$$

Choose $a=\frac{3}{4}$, $b=\frac{3}{2}$ and $c=66$, then $f(t,u)$ satisfies

$$f(t,u) \le \frac{45}{64} < 0.773175 \approx \Lambda_2^{-1}a, \quad \forall (t,u)\in[0,1]\times\left[0,\frac{3}{4}\right],$$

$$f(t,u) \ge 67.62 > 62.73 \approx \Lambda_3^{-1}b, \quad \forall (t,u)\in\left[\frac{1}{8},1\right]\times\left[\frac{3}{2},66\right]$$

and

$$f(t,u) \le 67.6875 < 68.0391 \approx \Lambda_2^{-1}c, \quad \forall (t,u)\in[0,1]\times[0,66].$$

Thus, (H_3), (H_4) and (H_5) hold. By Theorem 3.2, we have that boundary value problem (4.2) has at least one nonnegative solution u_1 and two positive solutions u_2, u_3 such that $\|u_1\| < \frac{3}{4}$, $\frac{3}{2} < \min_{\frac{1}{8}\le t\le 1} u_2(t)$ and $a < \|u_3\|$ with $\min_{\frac{1}{8}\le t\le 1} u_3(t) < \frac{3}{2}$.

References

[1] Agarwal, R.P., Alsaedi, A., Alsharif, A., Ahmad, B., *On nonlinear fractional-order boundary value problems with nonlocal multi-point conditions involving Liouville-Caputo derivatives*, Differ. Equ. Appl., **9**(2017), no. 2, 147-160.

[2] Bouteraa, N., Benaicha, S., *Existence of solutions for three-point boundary value problem for nonlinear fractional differential equations*, Bull. Transilv. Univ. Braşov Ser. III., **10**(59)(2017), no. 1.

[3] Bouteraa, N., Benaicha, S., Djourdem, H., *Positive solutions for nonlinear fractional differential equation with nonlocal boundary conditions*, Universal Journal of Mathematics and Applications, **1**(2018), 39-45.

[4] Cabada, A., Wang, G., *Positive solutions of nonlinear fractional differential equations with integral boundary value conditions*, J. Math. Anal. Appl., **389**(2012), 403-411.

[5] Cui, Y., Zou, Y., *Existence of solutions for second-order integral boundary value problems*, Nonlinear Anal. Model. Control., **21**(2016), 828-838.

[6] Delbosco, D., *Fractional calculus and function spaces*, J. Fract. Calc., **6**(1996), 45-53.

[7] Diethelm, L., Freed, K., *On the solutions of nonlinear fractional order differential equations used in the modelling of viscoplasticity*, in: Keil, F., Mackens, W., Voss, H., Werthers, J. (eds.), Scientific Computing in Chemical Engineering II - Computational Fluid Dynamics, Reaction Engineering and Molecular Properties, Springer, Heidelberg, 1999.

[8] Djourdem, H., S. Benaicha, S., *Existence of positive solutions for a nonlinear three-point boundary value problem with integral boundary conditions*, Acta Math. Univ. Comenianae, **87**(2018), no. 2, 167-177.

[9] Glockle, W.G., Nonnenmacher, T.F., *A fractional calculus approach of self-similar protein dynamics*, Biophys. J., **68**(1995), 46-53.

[10] Guo, D., Lakshmikantham, V., *Nonlinear Problems in Abstract Cones*, Math. Anal. Appl., (1988).

[11] Guo, L., Liu, L., Wu, Y., *Existence of positive solutions for singular fractional differential equations with infinite-point boundary conditions*, Nonlinear Anal. Model. Control, **21**(2016), no. 5, 635-650.

[12] He, J., *Some applications of nonlinear fractional differential equations and their approximations*, Bull. Am. Soc. Inf. Sci. Technol., **15**(1999), 86-90.

[13] Henderson, J., Luca, R., *Existence and multiplicity for positive solutions of a system of higher-order multi-point boundary value problems*, NoDEA Nonlinear Differential Equations Appl., **20**(2013), 1035-1054.

[14] Henderson, J., Luca, R., *Systems of Riemann'Liouville fractional equations with multi-point boundary conditions*, Appl. Math. Comput., **309**(2017), 303-323.

[15] Henderson, J., Luca, R., Tudorache, A., *On a system of fractionl differential equations with coupled integral boundary conditions*, Fract. Calc. Appl. Anal., **18**(2015), 361-386.

[16] Infante, G., *Positive solutions of nonlocal boundary value problems with singularities*, Discrete Contin. Dyn. Syst., (2009), Supplement, 377-384.

[17] Ji, Y.D., Guo, Y.P., Qiu, J.Q., Yang, L.Y., *Existence of positive solutions for a boundary value problem of nonlinear fractional differential equations*, Adv. Differ. Equ., **2015**(2015), Art. 13.

[18] Kilbas, A.A., Srivastava, H.M., Trujillo, J.J., *Theory and Applications of Fractional Differential Equations*, Elsevier Science B.V., Amsterdam, 2006.

[19] Leggett, R.W., Williams, L.R., *Multiple positive fixed points of nonlinear operators on ordered Banach spaces*, Indiana Univ. Math. J., **28**(1979), 673-688.

[20] Liu, L., Jiang, J., Wu, Y., *The unique solution of a class of sum mixed monotone operator equations and its application to fractional boundary value problems*, J. Nonlinear Sci. Appl., **9**(5)(2016), 2943-2958.

[21] Ma, D-X., *Positive solutions of multi-point boundary value problem of fractional differential equation*, Arab J. Math. Sci., **21**(2015), no. 2, 225-236.

[22] Mainardi, F., *Fractional Calculus: Some Basic Problems in Continuum and Statistical Mechanics*, in: Carpinteri, C.A., Mainardi, F. (eds.), Fractal and Fractional Calculus in Continuum Mechanics, Springer, Vienna, 1997.

[23] Metzler, F., Schick, W., Kilian, H.G., Nonnenmache, T.F., *Relaxation in filled polymers: a fractional calculus approach*, J. Chem. Phys., **103**(1995), 7180-7186.

[24] Nica, O., Precup, R., *On the nonlocal initial value problem for first order differential systems*, Stud. Univ. Babeş-Bolyai Math., **56**(2011), no. 3, 113-125.

[25] Nyamoradi, N., *Existence of solutions for multi-point boundary value problems for fractional differential equations*, Arab J. Math. Sci., **18**(2012), 165-175.

[26] Oldham, K.B., Spanier, J., *The fractional calculus*, Math. Anal. Appl., (1974).

[27] Podlubny, I., *Fractional differential equations*, Math. Anal. Appl., (1999).

[28] Pu, R., Zhang, X., Cui, Y., Li, P., Wang, W., *Positive solutions for singular semipositone fractional diferential equation subject to multipoint boundary conditions*, Funct. Spaces, **2017**, Art. ID 5892616, 2017, 8 pages.

[29] Ross B., (ed.), *The Fractional Calculus and its Applications*, Lecture Notes in Math, Vol. 475, Springer-Verlag, Berlin, 1975.

[30] Samko, S.G., Kilbas, A.A., Marichev, O.I., *Fractional Integral and Derivatives (Theory and Applications)*, Gordon and Breach, Switzerland, 1993.

[31] Shah, K., Zeb, S., Khan, R.A., *Existence of triple positive solutions for boundary value problem of nonlinear fractional differential equations*, Comput. Methods Differ. Equ., **5**(2017), no. 2, 158-169.

[32] Sun, Y., Zhao, M., *Positive solutions for a class of fractional differential equations with integral boundary conditions*, Appl. Math. Lett., **34**(2014), 17-21.

[33] Tariboon, J., Ntouyas, S.K., Sudsutad, W., *Positive solutions for fractional differential equations with three-point multi-term fractional integral boundary conditions*, Adv. Difference Equ., **2014**(2014), 17 pages.

[34] Wang, Y., Liang S., Wang, Q., *Multiple positive solutions of fractional-order boundary value problem with integral boundary conditions*, J. Nonlinear Sci. Appl., **10**(2017), 6333-6343.

[35] Webb, J.R.L., *Nonlocal conjugate type boundary value problems of higher order*, Nonlinear Anal., **71**(2009), 1933-1940.

[36] Zhou L., Jiang, W., *Positive solutions for fractional differential equations with multi-point boundary value problems*, Journal of Applied Mathematics and Physics, **2**(2014), 108-114.

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Stud. Univ. Babeş-Bolyai Math. 66(2021), No. 4, 709–722
DOI: 10.24193/subbmath.2021.4.09

Positive solution of Hilfer fractional differential equations with integral boundary conditions

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Abstract. In this article, we have interested the study of the existence and uniqueness of positive solutions of the first-order nonlinear Hilfer fractional differential equation

$$D_{0^+}^{\alpha,\beta} y(t) = f(t, y(t)),\ 0 < t \leq 1,$$

with the integral boundary condition

$$I_{0^+}^{1-\gamma} y(0) = \lambda \int_0^1 y(s)ds + d,$$

where $0 < \alpha \leq 1$, $0 \leq \beta \leq 1$, $\lambda \geq 0$, $d \in \mathbb{R}^+$, and $D_{0^+}^{\alpha,\beta}$, $I_{0^+}^{1-\gamma}$ are fractional operators in the Hilfer, Riemann-Liouville concepts, respectively. In this approach, we transform the given fractional differential equation into an equivalent integral equation. Then we establish sufficient conditions and employ the Schauder fixed point theorem and the method of upper and lower solutions to obtain the existence of a positive solution of a given problem. We also use the Banach contraction principle theorem to show the existence of a unique positive solution. The result of existence obtained by structure the upper and lower control functions of the nonlinear term is without any monotonous conditions. Finally, an example is presented to show the effectiveness of our main results.

Mathematics Subject Classification (2010): 34A08, 34B15, 34B18, 34A12, 47H10.

Keywords: Fractional differential equations, positive solution, upper and lower solutions, fixed point theorem, existence and uniqueness.

1. Introduction

Fractional differential equations have high significance due to their application in many fields such as applied and engineering sciences, etc. In the recent years, there has been a significant development in ordinary and partial differential equations involving fractional derivatives, see the monographs of Kilbas et al.[8], Miller and Ross [10], Podlubny[12], Hilfer [7] and reference therein. In particular, many interesting results of the existence of positive solutions of nonlinear fractional differential equations have been discussed, see [2, 4, 5, 9, 11, 13, 14] and reference therein. The integral boundary conditions have various applications in applied fields such as, underground water flow, blood flow problems, thermo-elasticity, population dynamics, chemical engineering, and so forth. Since only positive solutions are useful for many applications. For example, Abdo et al in [1] discussed the existence and uniqueness of a positive solution for the nonlinear fractional differential equations with integral boundary condition of the form

$$
\begin{aligned}
{}^{c}D^{\alpha}_{0+}y(t) &= f(t, y(t)), && t \in [0,1] \\
y(0) &= \lambda \int_0^1 y(s)ds + d, && \lambda \geq 0, d \in \mathbb{R}^+,
\end{aligned}
$$

where ${}^{c}D^{\alpha}_{0+}$ is the Caputo fractional derivative of order $\alpha \in (0,1)$, and f satisfied some appropriate assumptions.

Ardjouni and Djoudi in [3], discussed the existence and uniqueness of a positive solution for the nonlinear fractional differential equations

$$
\begin{aligned}
D^{\alpha}_{1+}x(t) &= f(t, x(t)), \quad t \in [1,e] \\
x(1) &= \lambda \int_1^e x(s)ds + d,
\end{aligned}
$$

where D^{α}_{1+} is the Caputo-Hadamard fractional derivative of order $\alpha \in (0,1)$, $\lambda \geq 0$, $d \in \mathbb{R}^+$, and f satisfies some suitable hypotheses. On the other hand, Long et al. [9] investigated some existence of positive solutions of period boundary value problems of fractional differential equations

$$
\begin{cases}
D^{\alpha,\beta}_{0+}x(t) = \lambda x(t) + f(t, x(t)), & t \in (0,b] \\
\lim\limits_{t \longrightarrow 0^+} t^{1-\gamma}x(t) = \lim\limits_{t \longrightarrow b^-} t^{1-\gamma}x(0), & \gamma = \alpha + \beta - \alpha\beta
\end{cases}
$$

where $\lambda < 0$, $D^{\alpha,\beta}_{0+}$ is the Hilfer fractional derivative of order $\alpha \in (0,1)$ and type $\beta \in [0,1]$ and f satisfied some appropriate conditions.

Motivated by the above works, in this paper, we discuss the existence and uniqueness of positive solution of the following nonlinear Hilfer fractional differential equations with integral boundary condition in a weighted space of continuous functions

$$
D^{\alpha,\beta}_{0+}y(t) = f(t, y(t)), \quad 0 < t \leq 1 \tag{1.1}
$$

$$
I^{1-\gamma}_{0+}y(0) = \lambda \int_0^1 y(s)ds + d, \tag{1.2}
$$

where $D^{\alpha,\beta}_{0+}$ is the left-sided Hilfer fractional derivative of order $\alpha \in (0,1)$ of type $\beta \in [0,1]$, $\lambda \geq 0$, $d \in \mathbb{R}^+$ and $f : [0,1] \times \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a continuous, $I^{1-\gamma}_{0+}$ is the Riemann–Liouville fractional integral of order $1-\gamma$, with $\gamma = \alpha + \beta(1-\alpha)$. The Hilfer fractional derivative can be regarded as an interpolator between the Riemann–Liouville derivative ($\beta = 0$) and Caputo derivative ($\beta = 1$). Furthermore, there are studies addressed the given problem in cases of $\beta = 0, 1$, however, to the best of our knowledge, there are no results of the Hilfer problem (1.1)-(1.2), hence, our article aims to fill this gap.

This article is constructed as follows: In Section 2, we recall some concepts which will be useful throughout this article. Section 3, contains certain sufficient conditions to establish the existence criterions of positive solution by using the Schauder fixed point theorem and the technique of upper and lower solutions. Section 4, demonstrates the uniqueness of the positive solution by using the Banach contraction principle. We are given an example in last section.

2. Preliminaries

Let $C_{1-\gamma}[0,1]$ be a weighted space of all continuous function defined on the intervel $(0,1]$, such that

$$C_{1-\gamma}[0,1]=\left\{y:(0,1]\to\mathbb{R}^+;\ t^{1-\gamma}y(t)\in C[0,1]\right\},0\leq\gamma\leq 1$$

with the norm

$$\|y\|_{C_{1-\gamma}[0,1]}=\max_{t\in[0,1]}\left|t^{1-\gamma}y(t)\right|.$$

It is clear that $C_{1-\gamma}([0,1],\mathbb{R}^+)$ is Banach space with the above norm. Define the cone $\Omega\subset C_{1-\gamma}[0,1]$ by

$$\Omega=\{y(t)\in C_{1-\gamma}[0,1]:y(t)\geq 0,\ t\in(0,1]\}.$$

Definition 2.1. [8] The left-sided Riemann-Liouville fractional integral of order $\alpha>0$ with the lower limit zero for a function $y:\mathbb{R}^+\longrightarrow\mathbb{R}$ is defined by

$$(I_{0^+}^{\alpha}y)(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}y(s)ds,$$

provided the right-hand side is pointwise on $\mathbb{R}^+$, where Γ is the gamma function.

Definition 2.2. [8] The left-sided Riemann-Liouville fractional derivative of order $0<\alpha<1$ with the lower limit zero of a function $y:\mathbb{R}^+\longrightarrow\mathbb{R}$ is defined by

$$D_{0^+}^{\alpha}y(t)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t(t-s)^{\alpha-1}y(s)ds.$$

provided the right-hand side is pointwise on $\mathbb{R}^+$.

Definition 2.3. [8] The left-sided Caputo fractional derivative of order $0<\alpha<1$ with the lower limit zero of a function $y:\mathbb{R}^+\longrightarrow\mathbb{R}$ is given by

$${}^cD_{0^+}^{\alpha}y(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t(t-s)^{\alpha-1}y'(s)ds.$$

provided the right-hand side is pointwise on $\mathbb{R}^+$.

Definition 2.4. [6] The left-sided Hilfer fractional derivative of order $0<\alpha<1$ and type $0\leq\beta\leq 1$ with the lower limit zero of a function $y:\mathbb{R}^+\longrightarrow\mathbb{R}$ is given by

$$D_{0^+}^{\alpha,\beta}y(t)=I_{0^+}^{\beta(1-\alpha)}DI_{0^+}^{(1-\beta)(1-\alpha)}y(t),$$

where $D=\frac{d}{dt}$. One has,

$$D_{0^+}^{\alpha,\beta}y(t)=I_{0^+}^{\beta(1-\alpha)}D_{0^+}^{\gamma}y(t),\tag{2.1}$$

where

$$D^{\gamma}_{0+}y(t) = DI^{1-\gamma}_{0+}y(t),\ \gamma = \alpha + \beta(1-\alpha).$$

In the forthcoming analysis, we need the following spaces:

$$C^{\alpha,\beta}_{1-\gamma}[0,1] = \left\{y \in C_{1-\gamma}[0,1] : D^{\alpha,\beta}_{0+}y \in C_{1-\gamma}[0,1]\right\},$$

and

$$C^{\gamma}_{1-\gamma}[0,1] = \left\{y \in C_{1-\gamma}[0,1] : D^{\gamma}_{0+}y \in C_{1-\gamma}[0,1]\right\}. \tag{2.2}$$

Since $D^{\alpha,\beta}_{0+}y = I^{\beta(1-\alpha)}_{0+}D^{\gamma}_{0+}y$, it is obvious that $C^{\gamma}_{1-\gamma}[0,1] \subset C^{\alpha,\beta}_{1-\gamma}[0,1]$.

Lemma 2.5. [2] *Let $\alpha > 0$, $\beta > 0$ and $\gamma = \alpha + \beta - \alpha\beta$. If $y \in C^{\gamma}_{1-\gamma}[0,1]$, then*

$$I^{\gamma}_{0+}D^{\gamma}_{0+}y = I^{\alpha}_{0+}\ D^{\alpha,\beta}_{0+}y,$$

and

$$D^{\gamma}_{0+}I^{\alpha}_{0+}y = D^{\beta(1-\alpha)}_{0+}y.$$

Theorem 2.6. [6] *Let $y \in C_{\gamma}[0,1]$, $0 < \alpha < 1$, and $0 \leq \gamma < 1$. Then*

$$D^{\alpha}_{0+}I^{\alpha}_{0+}y(t) = y(t),\ \forall t \in (0,1].$$

Moreover, if $y \in C_{\gamma}[0,1]$ and $I^{1-\beta(1-\alpha)}_{0+}y \in C^{1}_{\gamma}[0,1]$, then

$$D^{\alpha,\beta}_{0+}I^{\alpha}_{0+}y(t) = y(t),\ \textit{for a.e. } t \in (0,1].$$

Theorem 2.7. [6] *Let $\alpha, \beta \geq 0$ and $y \in C^{1}_{\gamma}[0,1]$, $0 < \alpha < 1$, and $0 \leq \gamma < 1$. Then*

$$I^{\alpha}_{0+}I^{\beta}_{0+}y(t) = I^{\alpha+\beta}_{0+}y(t).$$

Lemma 2.8. [8] *Let $\alpha \geq 0$, and $\sigma > 0$. Then*

$$I^{\alpha}_{0+}t^{\sigma-1} = \frac{\Gamma(\sigma)}{\Gamma(\alpha+\sigma)}t^{\alpha+\sigma-1},\ t > 0$$

and

$$D^{\alpha}_{0+}t^{\alpha-1} = 0, \quad 0 < \alpha < 1.$$

Lemma 2.9. [6] *Let $0 < \alpha < 1$, $0 \leq \gamma \leq 1$, if $y \in C_{\gamma}[0,1]$ and $I^{1-\alpha}_{0+}y \in C^{1}_{\gamma}[0,1]$, we have*

$$I^{\alpha}_{0+}D^{\alpha}_{0+}y(t) = y(t) - \frac{I^{1-\alpha}_{0+}y(0)}{\Gamma(\gamma)}t^{\alpha-1}, \quad \textit{for all } t \in (0,1].$$

Lemma 2.10. [6] *Let $y \in C_{\gamma}[0,1]$. If $0 \leq \gamma < \alpha < 1$, then*

$$\lim_{t \longrightarrow 0+} I^{\alpha}_{0+}y(t) = I^{\alpha}_{0+}\ y(0) = 0.$$

3. Existence of positive solution

In this section we will discuss the existence of positive solution for equation 1.1 with condition 1.2 . Befor starting in prove our result , we interduce the following conditions:

(H_1) $f:(0,1]\times[0,\infty)\longrightarrow[0,\infty)$ is continuous such that $f(\cdot,y(\cdot))\in C_{1-\gamma}[0,1]$ for any $y\in C_{1-\gamma}[0,1]$.

(H_2) There exist a positive constant L_f such that

$$|f(t,x)-f(t,y)|\le L_f\,\|x-y\|_{C_{1-\gamma}}.$$

The following lemmas are fundamental to our results.

Lemma 3.1. *If $Q(t):=\int_\tau^1(s-\tau)^{\alpha-1}ds$, for $\tau\in[0,1]$, then*

$$\frac{Q(\tau)}{\Gamma(\alpha)}<e. \tag{3.1}$$

Proof. According to the definition of gamma function with some simple computation, we obtain

$$\begin{aligned}\frac{Q(\tau)}{\Gamma(\alpha)} &= \frac{\int_\tau^1(s-\tau)^{\alpha-1}ds}{\int_0^\infty s^{\alpha-1}e^{-s}ds}\\ &= \frac{\int_0^{1-\tau}s^{\alpha-1}ds}{\int_0^\infty s^{\alpha-1}e^{-s}ds}\le\frac{e\int_0^{1-\tau}s^{\alpha-1}e^{-s}ds}{\int_0^\infty s^{\alpha-1}e^{-s}ds}<e.\end{aligned}$$

□

Lemma 3.2. *Assume that $Q(\tau):=\int_\tau^1(s-\tau)^{\alpha-1}ds$ for $\tau\in[0,1]$, $\mu:=1-\frac{\lambda}{\Gamma(\gamma+1)}\neq0$, $f\in C_{1-\gamma}[0,1]$ and $y\in C^\gamma_{1-\gamma}[0,1]$ exist. A function y is the solution of*

$$D^{\alpha,\beta}_{0^+}y(t)=f(t,y(t)),\ 0<t\le1, \tag{3.2}$$

$$I^{1-\gamma}_{0^+}y(0)=\lambda\int_0^1y(s)ds+d, \tag{3.3}$$

if and only if y satisfies the fractional integral equation

$$y(t)=\Lambda t^{\gamma-1}+\frac{\lambda t^{\gamma-1}}{\Gamma(\gamma)\mu}\int_0^1\frac{Q(\tau)}{\Gamma(\alpha)}f(\tau,y(\tau))d\tau+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,y(s))ds, \tag{3.4}$$

where $\Lambda:=\left(\frac{\lambda}{\mu\Gamma(\gamma)\Gamma(\gamma+1)}+\frac{1}{\Gamma(\gamma)}\right)d$.

Proof. First, Assume that y satisfies equation (3.2), then by applying $I^\alpha_{0^+}$ on both side of equation (3.2) and use Lemma 2.9, integral condition, we obtain

$$y(t)=\frac{\lambda t^{\gamma-1}}{\Gamma(\gamma)}\int_0^1y(s)ds+\frac{d}{\Gamma(\gamma)}t^{\gamma-1}+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,y(s))ds. \tag{3.5}$$

Set $A:=\int_0^1y(s)ds$. This the assumption with the equation (3.5) implies

$$A=\frac{d}{\mu\Gamma(\gamma+1)}+\frac{1}{\mu}\int_0^1\frac{Q(\tau)}{\Gamma(\alpha)}f(\tau,y(\tau))d\tau, \tag{3.6}$$

substituting (3.6) into (3.5), we attain

$$y(t)=\Lambda t^{\gamma-1}+\frac{\lambda t^{\gamma-1}}{\Gamma(\gamma)\mu}\int_0^1\frac{Q(\tau)}{\Gamma(\alpha)}f(\tau,y(\tau))d\tau+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,y(s))ds,$$

for all $t\in(0,1]$.
Conversely, assume that y satisfies (3.4). Applying $I_{0^+}^{1-\gamma}$ to both sides of (3.4) yields

$$\begin{aligned}I_{0^+}^{1-\gamma}y(t) &= \Lambda\Gamma(\gamma)+\frac{\lambda}{\mu}\int_0^1\frac{Q(\tau)}{\Gamma(\alpha)}f(\tau,y(\tau))d\tau\\ &\quad+\frac{1}{\Gamma(1-\gamma+\alpha)}\int_0^t(t-s)^{\alpha-\gamma}f(s,y(s))ds.\end{aligned}$$

Taking the limit at $t\longrightarrow 0^+$ of last equality and using Lemma 2.10 with $1-\gamma<1-\gamma+\alpha$, we get

$$I_{0^+}^{1-\gamma}y(0)=\Lambda\Gamma(\gamma)+\frac{\lambda}{\mu}\int_0^1\frac{Q(\tau)}{\Gamma(\alpha)}f(\tau,y(\tau))d\tau.$$

From the equation (3.6) with help of the definition of Λ, it follows that the integral boundary conditions given in (3.3) is satisfied, i.e. $I_{0^+}^{1-\gamma}y(0)=\lambda\int_0^1 y(s)ds+d$.
Next, applying $D_{0^+}^{\gamma}$ to both sides of (3.4) and using lemmas 2.5, 2.8, yields

$$D_{0^+}^{\gamma}y(t)=D_{0^+}^{\beta(1-\alpha)}f(t,y(t)) \tag{3.7}$$

since $y\in C_{1-\gamma}^{\gamma}[0,1]$, by (2.2), we have $D_{0^+}^{\gamma}y(t)\in C_{1-\gamma}[0,1]$, therefore

$$D_{0^+}^{\beta(1-\alpha)}f=DI_{0^+}^{1-\beta(1-\alpha)}f\in C_{1-\gamma}[0,1].$$

For $f\in C_{1-\gamma}[0,1]$, it is clear that $I_{0^+}^{1-\beta(1-\alpha)}f\in C_{1-\gamma}^{1}[0,1]$. Consequently, f and $I_{0^+}^{1-\beta(1-\alpha)}f$ satisfy Lemma 2.9.
Now, we apply $I_{0^+}^{\beta(1-\alpha)}$ to both side of equation (**??**), then Lemma 2.9 and definition of Hilfer operator imply that

$$D_{0^+}^{\alpha,\beta}y(t)=f(t,y(t))-\frac{I_{0^+}^{1-\beta(1-\alpha)}f(0,y(0))}{\Gamma(\beta(1-\alpha))}t^{\beta(1-\alpha)-1}$$

By virtue of Lemma 2.10, one can obtain

$$D_{0^+}^{\alpha,\beta}y(t)=f(t,y(t)).$$

This completes the proof. □

Lemma 3.3. *Assume that* (H_1) *and (*3.1*) are satisfied. Then the operator* $\Delta:\Omega\longrightarrow\Omega$ *defined by*

$$\Delta y(t)=\Lambda t^{\gamma-1}+\frac{\lambda t^{\gamma-1}}{\Gamma(\gamma)\mu}\int_0^1\frac{Q(\tau)}{\Gamma(\alpha)}f(\tau,y(\tau))d\tau+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,y(s))ds \tag{3.8}$$

is compact.

Proof. We know that the operator $\Delta : \Omega \longrightarrow \Omega$ is continuous, from fact that $f(t, y(t))$ is continuous and nonnegative. Define bounded set $B_r \subset \Omega$ as follows

$$B_r = \left\{ y \in \Omega : \|y\|_{C_{1-\gamma}} \leq r \right\}.$$

The function $f : (0,1] \times B_r \longrightarrow \mathbb{R}^+$ is bounded, then there exist $\xi > 0$ such that

$$0 < f(t, y(t)) \leq \xi.$$

In view of equation (3.8), Lemma 3.1, and for all $y \in B_r$, $t \in (0,1]$, we have

$$\begin{aligned} & \left|\Delta y(t) t^{1-\gamma}\right| \\ \leq \;& \Lambda + \frac{\lambda}{\Gamma(\gamma)\mu} \int_0^1 \frac{Q(\tau)}{\Gamma(\alpha)} \left|f(\tau, y(\tau))\right| d\tau + \frac{t^{1-\gamma}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left|f(s, y(s))\right| ds \\ \leq \;& \Lambda + \frac{\lambda e}{\Gamma(\gamma)\mu} \int_0^1 \left|f(\tau, y(\tau))\right| d\tau + \frac{t^{1-\gamma}}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left|f(s, y(s))\right| ds \\ \leq \;& \Lambda + \frac{\lambda e \xi}{\Gamma(\gamma)\mu} + \frac{t^{1-\gamma+\alpha}\xi}{\Gamma(\alpha+1)}, \end{aligned}$$

which implies

$$\|\Delta y\|_{C_{1-\gamma}} \leq \left[\Lambda + \frac{\lambda e \xi}{\Gamma(\gamma)\mu} + \frac{\xi}{\Gamma(\alpha+1)}\right].$$

Thus, $\Delta(B_r)$ is uniformly bounded.
Next, we prove that $\Delta(B_r)$ is equicontinuous. Let $y \in B_r$. Then for any $\delta, \eta \in (0,1]$ with $0 < \delta < \eta \leq 1$, we have

$$\begin{aligned} & \left|\eta^{1-\gamma}\Delta y(\eta) - \delta^{1-\gamma}\Delta y(\delta)\right| \\ = \;& \left|\frac{\eta^{1-\gamma}}{\Gamma(\alpha)} \int_0^\eta (\eta-s)^{\alpha-1} f(s, y(s)) ds - \frac{\delta^{1-\gamma}}{\Gamma(\alpha)} \int_0^\delta (\delta-s)^{\alpha-1} f(s, y(s)) ds\right| \\ \leq \;& \frac{\eta^{1-\gamma} - \delta^{1-\gamma}}{\Gamma(\alpha)} \int_0^\delta \left|(\eta-s)^{\alpha-1} - (\delta-s)^{\alpha-1}\right| \left|f(s, y(s))\right| ds \\ & + \frac{\eta^{1-\gamma}}{\Gamma(\alpha)} \int_\delta^\eta (\eta-s)^{\alpha-1} \left|f(s, y(s))\right| ds \\ \leq \;& \frac{\left[\eta^{1-\gamma} - \delta^{1-\gamma}\right]\xi}{\Gamma(\alpha)} \int_0^\delta \left((\delta-s)^{\alpha-1} - (\eta-s)^{\alpha-1}\right) ds \\ & + \frac{\eta^{1-\gamma}\xi}{\Gamma(\alpha)} \int_\delta^\eta (\eta-s)^{\alpha-1} ds \\ \leq \;& \frac{\left[\eta^{1-\gamma} - \delta^{1-\gamma}\right]\xi}{\Gamma(\alpha+1)} \left[(\delta^\alpha - \eta^\alpha) + (\eta-\delta)^\alpha\right] + \frac{\eta^{1-\gamma}\xi}{\Gamma(\alpha+1)} (\eta-\delta)^\alpha. \end{aligned} \tag{3.9}$$

By the classical Mean value Theorem, we have

$$\begin{aligned} \delta^\alpha - \eta^\alpha &= \alpha\,(\delta-\eta)\,T, \\ &\leq \alpha\,(\delta-\eta)\,. \end{aligned} \tag{3.10}$$

The last inequality with(3.9) implies

$$\begin{aligned}
&\left|\eta^{1-\gamma}\Delta y(\eta)-\delta^{1-\gamma}\Delta y(\delta)\right| \\
\leq\ & \frac{\left[\eta^{1-\gamma}-\delta^{1-\gamma}\right]\xi}{\Gamma(\alpha+1)}\left[\alpha\left(\delta-\eta\right)+(\eta-\delta)^{\alpha}\right]+\frac{\eta^{1-\gamma}\xi}{\Gamma(\alpha+1)}(\eta-\delta)^{\alpha} \\
\leq\ & \frac{\left[\eta^{1-\gamma}-\delta^{1-\gamma}\right]\xi}{\Gamma(\alpha+1)}(\eta-\delta)^{\alpha}+\frac{\eta^{1-\gamma}\xi}{\Gamma(\alpha+1)}(\eta-\delta)^{\alpha} \\
\leq\ & \frac{2\eta^{1-\gamma}\xi}{\Gamma(\alpha+1)}(\eta-\delta)^{\alpha}-\frac{\delta^{1-\gamma}\xi}{\Gamma(\alpha+1)}(\eta-\delta)^{\alpha}.
\end{aligned}$$

As $\delta\longrightarrow\eta$ the right-hand side of the preceding inequality is independent of y and tends to zero. So,

$$\left|\eta^{1-\gamma}\Delta y(\eta)-\delta^{1-\gamma}\Delta y(\delta)\right|\longrightarrow 0,\forall\left|\eta-\delta\right|\longrightarrow 0.$$

Hence, $\Delta(B_r)$ is an equicontinuous set. By the Arzela-Ascoli theorem we get that $\Delta(B_r)$ is relatively compact set, which prove that $\Delta:\Omega\longrightarrow\Omega$ is a compact operator. □

Definition 3.4. For any $y\in[a,b]\subset\mathbb{R}^{+}$, we define the upper-control function by

$$G(t,x)=\sup_{a\leq y\leq x}f(t,y),$$

and the lower-control function by

$$g(t,x)=\inf_{x\leq y\leq b}f(t,y).$$

It is obvious that these functions are nondecreasing on $[a,b]$, i.e.

$$g(t,x)\leq f(t,y)\leq G(t,x).$$

Definition 3.5. Let $\overline{y}$, $\underline{y}\in\Omega$ such that $0<\underline{y}\leq\overline{y}\leq 1$ satisfy the following Hilfer problem

$$\begin{aligned}
D_{0^+}^{\alpha,\beta}\overline{y}(t) &\geq G(t,x),\ 0<t\leq 1\\
I_{0^+}^{1-\gamma}\overline{y}(0) &\geq \lambda\int_0^1\overline{y}(s)ds+d,
\end{aligned}$$

or

$$\overline{y}(t)\geq\Lambda t^{\gamma-1}+\frac{\lambda t^{\gamma-1}}{\Gamma(\gamma)\mu}\int_0^1\frac{Q(\tau)}{\Gamma(\alpha)}G(\tau,\overline{y}(\tau))d\tau+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}G(s,\overline{y}(s))ds,$$

and

$$\begin{aligned}
D_{0^+}^{\alpha,\beta}\underline{y}(t) &\leq g(t,x),\ 0<t\leq 1\\
I_{0^+}^{1-\gamma}\underline{y}(0) &\leq \lambda\int_0^1\underline{y}(s)ds+d,
\end{aligned}$$

or

$$\underline{y}(t)\leq\Lambda t^{\gamma-1}+\frac{\lambda t^{\gamma-1}}{\Gamma(\gamma)\mu}\int_0^1\frac{Q(\tau)}{\Gamma(\alpha)}g(\tau,\underline{y}(\tau))d\tau+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}g(s,\underline{y}(s))ds.$$

Then the functions $\overline{y}(t)$ and $\underline{y}(t)$ are called the upper and lower solutions of the Hilfer problem (1.1)-(1.2).

Theorem 3.6. *Assume that* (H_1) *and* (3.1) *hold. Then there exists at least one positive solution* $y(t) \in C_{1-\gamma}[0,1]$ *of the Hilfer problem* (1.1),(1.2), *such that*

$$\underline{y}(t) \leq y(t) \leq \overline{y}(t), \qquad 0 < t \leq 1.$$

where $\overline{y}(t)$ *and* $\underline{y}(t)$ *are upper and lower solutions of Hilfer problem* (1.1),(1.2) *respectively.*

Proof. In view of Lemma (3.2), the solution of problem (1.1)-(1.2)is given by

$$y(t) = \Lambda t^{\gamma-1} + \frac{\lambda t^{\gamma-1}}{\Gamma(\gamma)\mu}\int_0^1 \frac{Q(\tau)}{\Gamma(\alpha)} f(\tau, y(\tau))d\tau + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, y(s))ds$$

Define

$$\Upsilon = \left\{x(t) : x(t) \in \Omega,\ \underline{y}(t) \leq x(t) \leq \overline{y}(t),\ 0 < t \leq 1\right\}$$

endowed with the norm $\|x\| = \max_{t\in(0,1]} |x(t)|$, then we have $\|x\| \leq b$. Hence, Υ is a convex, bounded, and closed subset of the Banach space $C_{1-\gamma}[0,1]$. Now, to apply the Schauder fixed point theorem, we divide the proof into several steps as follows:
Step 1. We need to prove that $\Delta : \Omega \longrightarrow \Omega$ is compact .

According to Lemma 3.3, the operator $\Delta : \Omega \longrightarrow \Omega$ is compact. Since $\Upsilon \subset \Omega$, the operator $\Delta : \Upsilon \longrightarrow \Upsilon$ is compact too.

Step 2. We need to prove that $\Delta : \Upsilon \longrightarrow \Upsilon$. Indeed, by the definitions 3.4, 3.5, then for any $x(t) \in \Upsilon$, we have

$$\begin{aligned}
\Delta x(t) &= \Lambda t^{\gamma-1} + \frac{\lambda t^{\gamma-1}}{\Gamma(\gamma)\mu}\int_0^1 \frac{Q(\tau)}{\Gamma(\alpha)} f(\tau, x(\tau))d\tau + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, x(s))ds \\
&\leq \Lambda t^{\gamma-1} + \frac{\lambda t^{\gamma-1}}{\Gamma(\gamma)\mu}\int_0^1 \frac{Q(\tau)}{\Gamma(\alpha)} G(\tau, x(\tau))d\tau + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} G(s, x(s))ds \\
&\leq \Lambda t^{\gamma-1} + \frac{\lambda t^{\gamma-1}}{\Gamma(\gamma)\mu}\int_0^1 \frac{Q(\tau)}{\Gamma(\alpha)} G(\tau, \overline{y}(\tau))d\tau + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} G(s, \overline{y}(s))ds \\
&\leq \overline{y}(t). \qquad (3.11)
\end{aligned}$$

Also

$$\begin{aligned}
\Delta x(t) &= \Lambda t^{\gamma-1} + \frac{\lambda t^{\gamma-1}}{\Gamma(\gamma)\mu}\int_0^1 \frac{Q(\tau)}{\Gamma(\alpha)} f(\tau, y(\tau))d\tau + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s, y(s))ds \\
&\geq \Lambda t^{\gamma-1} + \frac{\lambda t^{\gamma-1}}{\Gamma(\gamma)\mu}\int_0^1 \frac{Q(\tau)}{\Gamma(\alpha)} g(\tau, x(\tau))d\tau + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s, x(s))ds \\
&\geq \Lambda t^{\gamma-1} + \frac{\lambda t^{\gamma-1}}{\Gamma(\gamma)\mu}\int_0^1 \frac{Q(\tau)}{\Gamma(\alpha)} g(\tau, \underline{y}(\tau))d\tau + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s, \underline{y}(s))ds \\
&\geq \underline{y}(t). \qquad (3.12)
\end{aligned}$$

From (3.11) and (3.12), we conclude that $\underline{y}(t) \leq \Delta x(t) \leq \overline{y}(t)$, and hence $\Delta x(t) \in \Upsilon$, for $0 < t \leq 1$ i. e. $\Delta : \Upsilon \longrightarrow \Upsilon$.

In view of the above steps and Schauder fixed point theorem, the problem (1.1)-(1.2) has at least one positive solution $y(t) \in \Upsilon$. □

Corollary 3.7. *Assume that* $f : (0,1] \times [0,\infty) \longrightarrow [0,\infty)$ *is continuous, and there exist* $A_1, A_2 > 0$ *such that*

$$A_1 \leq f(t,y) \leq A_2, \quad (t,y) \in (0,1] \times \mathbb{R}^+. \tag{3.13}$$

Then the Hilfer problem (1.1)*-*(1.2) *has at least one positive solution* $y(t) \in \Upsilon$*. Moreover,*

$$\frac{d}{\Gamma(\gamma)}t^{\gamma-1} + \frac{A_1}{\Gamma(\alpha+1)}t^{\alpha} \leq y(t) \leq \frac{d}{\Gamma(\gamma)}t^{\gamma-1} + \frac{A_2}{\Gamma(\alpha+1)}t^{\alpha}. \tag{3.14}$$

Proof. From the Definition 3.4 and equation (3.13), we have

$$A_1 \leq g(t,y) \leq G(t,y) \leq A_2. \tag{3.15}$$

Now, we consider the following Hilfer problem

$$D_{0^+}^{\alpha,\beta}\overline{y}(t) = A_2, \qquad I_{0^+}^{1-\gamma}\overline{y}(0) = d. \tag{3.16}$$

Then, the Hilfer problem (3.16) has a positive solution

$$\begin{aligned} \overline{y}(t) &= \frac{t^{\gamma-1}}{\Gamma(\gamma)} I_{0^+}^{1-\gamma}\overline{y}(0) + I_{0^+}^{\alpha} A_2 \\ &= \frac{d}{\Gamma(\gamma)}t^{\gamma-1} + \frac{A_2}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} ds \\ &= \frac{d}{\Gamma(\gamma)}t^{\gamma-1} + \frac{A_2}{\Gamma(\alpha+1)}t^{\alpha}. \end{aligned}$$

By (3.15) we conclude that

$$\overline{y}(t) = \frac{d}{\Gamma(\gamma)}t^{\gamma-1} + \frac{A_2}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} ds \geq \frac{d}{\Gamma(\gamma)}t^{\gamma-1} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} G(s,\overline{y}) ds.$$

Thus, the function $\overline{y}(t)$ is the upper solution of the Hilfer problem (1.1)-(1.2). In the similar way, if the Hilfer problem of the type

$$D_{0^+}^{\alpha,\beta}\underline{y}(t) = A_1, \qquad I_{0^+}^{1-\gamma}\underline{y}(0) = d. \tag{3.17}$$

Obviously, the Hilfer problem (3.17) has also a positive solution

$$\begin{aligned} \underline{y}(t) &= \frac{t^{\gamma-1}}{\Gamma(\gamma)} I_{0^+}^{1-\gamma}\underline{y}(0) + I_{0^+}^{\alpha} A_1 \\ &= \frac{d}{\Gamma(\gamma)}t^{\gamma-1} + \frac{A_1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} ds \\ &= \frac{d}{\Gamma(\gamma)}t^{\gamma-1} + \frac{A_1}{\Gamma(\alpha+1)}t^{\alpha}. \end{aligned}$$

Similarly, by (3.15) we infer that

$$\underline{y}(t) = \frac{d}{\Gamma(\gamma)}t^{\gamma-1} + \frac{A_1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} ds \leq \frac{d}{\Gamma(\gamma)}t^{\gamma-1} + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s,\overline{y}) ds.$$

Hence, the function $\underline{y}(t)$ is the lower solution of the Hilfer problem (1.1)-(1.2).
By Theorem (3.6), we deduce that the problem (1.1)-(1.2) has at least one positive solution $y(t) \in \Omega$, which verifies the inequality (3.14). □

4. Uniqueness of positive solution

In this section, we will demonstrate the uniqueness of the positive solution using the Banach contraction principle.

Theorem 4.1. *Assume that* $f : (0,1] \times [0,\infty) \longrightarrow [0,\infty)$ *is continuous, the condition* (H_2) *and the inequality* (3.1) *hold. If*

$$\left(\frac{\lambda e}{\Gamma(\gamma)\mu} + \frac{1}{\Gamma(\alpha+1)}\right) L_f < 1. \tag{4.1}$$

Then the problem (1.1)*-*(1.2) *has a unique positive solution in* Υ.

Proof. According to Theorem (3.6), the problem (1.1)-(1.2) has at least one positive solution in Υ as the form

$$\begin{aligned} y(t) \quad \longrightarrow \quad \Delta y(t) &= \Lambda t^{\gamma-1} + \frac{\lambda t^{\gamma-1}}{\Gamma(\gamma)\mu}\int_0^1 \frac{Q(\tau)}{\Gamma(\alpha)} f(\tau, y(\tau))d\tau \\ &+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,y(s))ds. \end{aligned}$$

Now, we need only to proof that the operator Δ is contraction mapping on Υ. Indeed, for any $y_1, y_2 \in \Upsilon$ and $t \in (0,1]$, we have

$$\begin{aligned} & \left|t^{1-\gamma}\Delta y_1(t) - t^{1-\gamma}\Delta y_2(t)\right| \\ \leq\ & \frac{\lambda}{\Gamma(\gamma)\mu}\int_0^1 \frac{Q(\tau)}{\Gamma(\alpha)} \left|f(\tau, y_1(\tau)) - f(\tau, y_2(\tau)\right| d\tau \\ & + \frac{t^{1-\gamma}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|f(s,y_1(s)) - f(s,y_2(s))\right| ds \\ \leq\ & \frac{\lambda e}{\Gamma(\gamma)\mu}\int_0^1 \left|f(\tau, y_1(\tau)) - f(\tau, y_2(\tau)\right| d\tau \\ & + \frac{t^{1-\gamma}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|f(s,y_1(s)) - f(s,y_2(s))\right| ds \\ \leq\ & \frac{\lambda e}{\Gamma(\gamma)\mu}\int_0^1 L_f \left\|y_1 - y_2\right\|_{C_{1-\gamma}} d\tau + \frac{t^{1-\gamma}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} L_f \left\|y_1 - y_2\right\|_{C_{1-\gamma}} ds \\ \leq\ & \frac{\lambda e L_f}{\Gamma(\gamma)\mu}\left\|y_1 - y_2\right\|_{C_{1-\gamma}} + \frac{t^{1-\gamma+\alpha}}{\Gamma(\alpha+1)} L_f \left\|y_1 - y_2\right\|_{C_{1-\gamma}} \\ \leq\ & \left(\frac{\lambda e}{\Gamma(\gamma)\mu} + \frac{1}{\Gamma(\alpha+1)}\right) L_f \left\|y_1 - y_2\right\|_{C_{1-\gamma}} \end{aligned}$$

The hypothesis (4.1) shows that Δ is a contraction mapping. The conclusion from the Banach contraction principle that the Hilfer problem (1.1)-(1.2) has a unique positive solution $u(t) \in C_{1-\gamma}[0,1]$. □

5. An example

In this part, we present an example to illustrate our result.

Example 5.1. Consider the following nonlinear Hilfer fractional problem

$$\begin{aligned} D_{0^+}^{\frac{1}{2},\frac{1}{3}} y(t) &= \frac{3}{5}\left(t^{\frac{1}{2}}\left|\sin y(t)\right| + 1\right), \qquad t \in (0,1] \\ I_{0^+}^{\frac{2}{3}} y(0) &= 1 \end{aligned} \tag{5.1}$$

where $\alpha = \frac{1}{2}$, $\beta = \frac{1}{3}$, $\gamma = \frac{2}{3}$, $\lambda = 0$, $d = 1$ and $f(t, y(t)) = \frac{3}{5}\left(t^{\frac{1}{2}}\left|\sin y(t)\right| + 1\right)$. It is easy to see that

$$t^{\frac{1}{3}} f(t, y(t)) = \frac{3}{5}\left(t^{\frac{5}{6}}\left|\sin y(t)\right| + t^{\frac{1}{3}}\right) \in C[0,1].$$

Hence $f(t, y(t)) \in C_{\frac{1}{3}}[0,1]$, which means that f satisfies (H_1). Next, we show that f satisfies (H_2). In fact, for any $y_1, y_2 \in C_{\frac{1}{3}}[0,1]$ and $t \in (0,1]$, we have

$$\begin{aligned} \left|f(t, y_1(t)) - f(t, y_2(t))\right| &\leq \left|\frac{3}{5}t^{\frac{1}{2}}\sin y_1(t) - \frac{3}{5}t^{\frac{1}{2}}\sin y_2(t)\right| \\ &\leq \frac{3}{5}t^{\frac{1}{6}}\left|t^{\frac{1}{3}}\sin y_1(t) - t^{\frac{1}{3}}\sin y_2(t)\right| \\ &\leq \frac{3}{5}\left\|y_1 - y_2\right\|_{C_{\frac{1}{3}}} = L_f\left\|y_1 - y_2\right\|_{C_{\frac{1}{3}}}. \end{aligned}$$

Since f is continuous and

$$\frac{3}{5} \leq f(t, y) \leq \frac{6}{5}, \qquad (t, y) \in (0,1] \times \mathbb{R}^+.$$

Then the Hilfer problem (5.1) has a positive solution which verifies $\underline{y}(t) \leq y(t) \leq \overline{y}(t)$ where

$$\overline{y}(t) = \frac{1}{\Gamma(\frac{2}{3})}t^{\frac{-1}{3}} + \frac{6}{5\Gamma(\frac{3}{2})}t^{\frac{1}{2}},$$

and

$$\underline{y}(t) = \frac{1}{\Gamma(\frac{2}{3})}t^{\frac{-1}{3}} + \frac{3}{5\Gamma(\frac{3}{2})}t^{\frac{1}{2}}.$$

are respectively the upper and lower solutions of Hilfer problem (5.1). Furthermore, by simple computations, the condition (4.1) also is satisfied, i.e.

$$\left(\frac{\lambda e}{\Gamma(\gamma)\mu} + \frac{1}{\Gamma(\alpha+1)}\right)L_f = \frac{1}{\Gamma(\frac{3}{2})}\frac{3}{5} \simeq 0.7 < 1.$$

Thus, since all the hypotheses in Theorems 3.6, 4.1 are fulfilled, our results can be applied to the Hilfer problem.

6. Conclusion

This paper studies the existence and uniqueness of positive solution of the nonlinear fractional differential equation with integral boundary condition and Hilfer fractional derivative operator. The proof of the main results relies on the Schauder fixed point theorem, Banach contraction mapping principle and technique of upper and lower solution.

The method of constructing a pair of upper and lower control functions with respect to the nonlinear term without monotone demand provides a new efficient technique to handle the nonlinear structure. This method is a tremendous tool for solving nonlinear differential equations in applied mathematics. The obtained results extend some known results in the literature. An example is introduced to illustrate the main results of this paper.

References

[1] Abdo, M.S., Wahash, H.A., Panchal, S.K., *Positive solution of a fractional differential equation with integral boundary conditions*, Journal of Applied Mathematics and Computational Mechanics, **17**(2018), no. 3.

[2] Almalahi, M.A., Abdo, M.S., Panchal, S.K., *Existence and Ulam-Hyers-Mittag-Leffler stability results of ψ-Hilfer nonlocal Cauchy problem*, Rend. Circ. Mat. Palermo, II. Ser (2020).

[3] Ardjouni, A., *Positive solutions for nonlinear Hadamard fractional differential equations with integral boundary conditions*, AIMS Mathematics, **4**(2019), 1101-1113.

[4] Ardjouni, A., Djoudi, A., *Existence and uniqueness of positive solutions for first-order nonlinear Liouville–Caputo fractional differential equations*, São Paulo Journal of Mathematical Sciences, (2019), 1-10.

[5] Boulares, H., Ardjouni, A., Laskri, Y., *Positive solutions for nonlinear fractional differential equations*, Positivity, **21**(3)(2017), 1201-1212.

[6] Furati, K.M., Kassim, M.D., *Existence and uniqueness for a problem involving Hilfer fractional derivative*, Comput. Math. Applic., **64**(2012), 1616-1626.

[7] Hilfer R., *Applications of Fractional Calculus in Physics*, World Scientific Publishing Co., Inc., River 27 Edge, NJ, Singapore, 2000.

[8] Kilbas, A.A., Srivastava, H.M., Trujillo, J.J., *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, Elsevier, Amsterdam, **207**, 2006.

[9] Long, T., Li, C., He, J., *Existence of positive solutions for period BVPs with Hilfer derivative*, Journal of Applied Mathematics and Computing, **60**(2019), no. 1-2, 223-236.

[10] Miller, K.S., Ross, B., *An Introduction to the Fractional Calculus and Differential Equations*, New York, John Wiley, 1993.

[11] Nan, L.I., Changyou, W.A.N.G., *New existence results of positive solution for a class of nonlinear fractional differential equations*, Acta Mathematica Scientia, **33**(2013), no. 3, 847-854.

[12] Podlubny, I., *Fractional differential equations: an introduction to fractional derivatives*, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications, Elsevier, **198**, 1998.

[13] Wang, F., Liu, L., Kong, D., Wu, Y., *Existence and uniqueness of positive solutions for a class of nonlinear fractional differential equations with mixed-type boundary value conditions*, Nonlinear Analysis: Modelling and Control, **24**(2019), no. 1, 73-94.

[14] Zhang, S., *The existence of a positive solution for a nonlinear fractional differential equation*, Journal of Mathematical Analysis and Applications, **252**(2000), no. 2, 804-812.

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Stud. Univ. Babeş-Bolyai Math. 66(2021), No. 4, 723–738
DOI: 10.24193/subbmath.2021.4.10

Existence of positive solutions for a class of BVPs in Banach spaces

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Abstract. In this work, we use index fixed point theory for perturbation of expansive mappings by ℓ-set contractions to study the existence of bounded positive solutions for a class of two-point boundary value problem (BVP) associated to second-order nonlinear differential equation on the positive half-line. The nonlinearity, which may exhibit a singularity at the origin, is written as a sum of two functions which behave differently. These functions, depend on the solution and its derivative, take values in a general Banach space and have at most polynomial growth. An example to illustrate the main results is given.

Mathematics Subject Classification (2010): 34B15, 34B18, 34B40, 47H08, 47H10.

Keywords: Boundary value problem, Green's function, unbounded interval, measure of noncompactness, fixed point index, sum operator.

1. Introduction

The theory of ordinary differential equations in Banach space is a rapidly growing area of research, it is developed for example in the books by Guo *et al.* [12], Guo and Lakshmikantham [11], Lakshmikantham and Leela [14], Deimling [2], and Zeidler [19] or in the papers by P. Li *et al.* [15] and by Y. Liu [16].

In the past decades, the study of BVPs defined on compact intervals has been considered by many authors with application of a huge variety of methods and techniques. However, BVPs defined on unbounded intervals are scarce, as they require other types of techniques to overcome the lack of compactness. Historically, these problems began at the end of nineteenth century with A. Kneser [13]. In this work, the lack of compactness is overcome with some techniques and specific tools.

Let $\mathcal{P}$ be a cone in some Banach space E, that is a closed convex subset such that $\alpha\mathcal{P} \subset \mathcal{P}$ for all positive real number α and $\mathcal{P} \cap (-\mathcal{P}) = \{0\}$.

Notice that E is partially ordered by cone $\mathcal{P}$, i.e. $x \leq y$ if and only if $y - x \in \mathcal{P}$. For details on cone theory see [11].

Throughout this paper, $(E, \|.\|)$ denotes a Banach space and $\mathcal{P}$ is a cone in E. Being given a positive real parameter k and $f\colon \mathbb{R}^+ \times \mathcal{P} \times E \to \mathcal{P}$ a continuous function, we are interested in the study of the existence of bounded positive solutions to the second-order boundary value problem:

$$\begin{cases} -x''(t) + k^2 x(t) = m(t) f(t, x(t), x'(t)), \quad t \in (0, +\infty). \\ x(0) = 0, \quad \lim_{t\to+\infty} x(t) = 0, \end{cases} \tag{1.1}$$

where the coefficient $m \in \mathcal{C}((0,+\infty), \mathbb{R}^+) \cap L^1((0,+\infty), \mathbb{R}^+)$ may be singular at $t = 0$ and it does not vanish identically on any subinterval of $(0, +\infty)$.
Also, we consider the problem

$$\begin{cases} -y'' + cy' + \lambda y &= m(t) g(t, y(t), y'(t)), \quad t \in (0, +\infty) \\ y(0) = \lim_{t\to+\infty} y(t) &= 0 \end{cases} \tag{1.2}$$

where c, λ are positive constants and $g\colon \mathbb{R}^+ \times \mathcal{P} \times E \to \mathcal{P}$ is a continuous function. Letting $k = \sqrt{\lambda + \frac{c^2}{4}}$ and $x(t) = y(t)e^{-\frac{c}{2}t}$, the problem (1.2) leads to the problem (1.1) for the new unknown x and modified nonlinear term

$$f(t, x(t), x'(t)) = e^{\frac{-c}{2}t} g\left(t, e^{\frac{c}{2}t} x(t), e^{\frac{c}{2}t} x'(t) + \frac{c}{2} e^{\frac{c}{2}t} x(t)\right).$$

Notice that the problems (1.1) and (1.2) arise in many applications in physics, combustion theory and epidemiology (see [4, 8, 9, 17, 18] and the references therein). We will list some papers which provide a motivation for the introduction of this work. In [5], using the Krasnosels'kii fixed point theorem in cones for strict set-contractions, Djebali et al. investigated the existence of single and twin positive solutions to the following two-point boundary value problem of second-order nonlinear differential equations posed on the positive half-line:

$$\begin{cases} -x''(t) + k^2 x(t) = m(t) f(t, x(t)), \quad t \in (0, +\infty), \\ x(0) = 0, \quad \lim_{t\to+\infty} x(t) = 0, \end{cases}$$

where the nonlinearity $f \in \mathcal{C}(\mathbb{R}^+ \times \mathcal{P}, \mathcal{P})$ satisfies a general polynomial growth condition. Motivated by the results obtained in the scalar case $E = \mathbb{R}$ in [7], the main purpose of this work is to discuss some existence results for the problem as that of [5], when f depends also on the derivative. For this purpose, we employ the generalized fixed point index for the sum of an expansive mapping and a ℓ-set contraction developed by Djebali and Mebarki in [6].

Now we describe in more details the structure of this work. This paper is devised in three sections. The first one is devoted to the preliminaries, recalling some basic concepts, and developing a new non compactness result that is needed for our purposes. The main results are presented in section 2. We conclude with an example of application in section 3.

2. Preliminaries

2.1. Measure of noncompactness and set-contraction

In this paper, the concept of set contraction is related to Kuratowski's measure of noncompactness (α-MNC for short) [3, 10]. Recall that the Kuratowski measure of noncompactness $\alpha(V)$ of a bounded subset V of a Banach space E is the infimum of positive numbers δ such that there exist finitely many sets of diameter at most δ which cover V.

Let $J \subset \mathbb{R}^+$. The Kuratowski measures of noncompactness of a bounded set in the spaces E, $\mathcal{C}(J,E)$, $\mathcal{C}^1(J,E)$ and X are denoted by $\alpha_E(.)$, $\alpha_{\mathcal{C}}(.)$, $\alpha_{\mathcal{C}^1}(.)$ and $\alpha_X(.)$ respectively.

The following known results are used in this work.

Lemma 2.1. [10, Theorem 1.2.6]. *Let $J = [a,b]$. If $H \subset \mathcal{C}^1(J,E)$ is bounded, H and H' are equi-continuous, then*

$$\alpha_{\mathcal{C}^1}(H) = \max\left(\sup_{t\in J}\alpha_E(H(t)), \sup_{t\in J}\alpha_E(H'(t))\right),$$

where $H(t) = \{u(t) \mid u \in H\}, t \in J$.

Lemma 2.2. [10, Theorem 1.2.2] *If $H \subset \mathcal{C}(J,E)$ is bounded and equicontinuous, then $\alpha(H(.))$ is continuous on J,*

$$\alpha_{\mathcal{C}(J,E)}(H) = \sup_{t\in J}\alpha(H(t)),$$

and

$$\alpha\left(\int_J x(t)dt \mid x \in H\right) \leq \int_J \alpha(H(t))dt,$$

Let $A\colon D \subset E \to E$ be a continuous operator. The operator A is said to be *bounded* if it maps bounded sets into bounded sets, *completely continuous* if it maps bounded sets into relatively compact sets, and *compact* if the set $A(D)$ is relatively compact. The operator A is said to be a *ℓ-set contraction*, for some number $\ell \geq 0$, if it is bounded and $\alpha(A(V)) \leq \ell\alpha(V)$ for every bounded set $V \subset D$. If $\ell < 1$, we say that A is a *strict set contraction.*

We finish this part by giving the definition of an expansive mapping, let (X,d) is a metric space. A mapping $T : D \subset X \to X$ is said to be *expansive* if there exists a constant $h > 1$ such that

$$d(Tx,Ty) \geq h\, d(x,y) \quad \text{for all} \quad x,y \in D.$$

2.2. The Green's function

The following lemmas are concerned with the linear problem associated to (1.1). They provide useful estimates of the kernel G and their proofs are omitted.

Lemma 2.3. *Let v be a function such that $v \in \mathcal{C}((0,+\infty),E)$ and $\int_0^{+\infty}\|v(t)\|dt$ exists. Then the problem*

$$\begin{cases} -x''(t) + k^2x(t) = v(t), & t \in (0,+\infty), \\ x(0) = 0, \ \lim\limits_{t\to+\infty} x(t) = 0 \end{cases}$$

has a unique solution x given by

$$x(t) = \int_0^{+\infty} G(t,s)v(s)ds,$$

where G is the Green function of the problem, namely

$$G(t,s) = \frac{1}{2k}\begin{cases} e^{-ks}(e^{kt} - e^{-kt}), & if \quad 0 \le t \le s < \infty, \\ e^{-kt}(e^{ks} - e^{-ks}), & if \quad 0 \le s \le t < \infty. \end{cases} \tag{2.1}$$

Throughout this work, $0 < \gamma < \delta$ will denote some fixed numbers. The interval $[\gamma, \delta]$ will play a key role in estimating the solutions of the problem (1.1). Let

$$\begin{aligned} \Lambda_0 &= \min(e^{-k\delta}, e^{k\gamma} - e^{-k\gamma}), \\ \Lambda_1 &= \min\left(\tfrac{1-k}{1+k}\, e^{-k\delta}, e^{k\gamma} + \tfrac{k-1}{k+1}\, e^{-k\gamma}\right), \\ \Lambda_2 &= \tfrac{k}{k+1}\, e^{-k\delta}. \end{aligned} \tag{2.2}$$

Obviously, these constants are less than 1. Some fundamental properties of the kernel G are given hereafter. The proofs are omitted.

Lemma 2.4. *The Green's function G satisfies the following estimates:*

$$\begin{array}{ll} (a) & G(t,s) \ge 0, \quad \forall\, t,s \in \mathbb{R}^+. \\ (b) & G(t,s) \le G(s,s) \le \frac{1}{2k}, \quad \forall\, t,s \in \mathbb{R}^+. \\ (c) & G(t,s)e^{-\mu t} \le G(s,s)e^{-ks}, \quad \forall\, t,s \in \mathbb{R}^+,\ \forall\, \mu \ge k. \\ (d) & G(t,s) \ge \Lambda_0 G(s,s)e^{-ks}, \quad \forall\, t \in [\gamma,\delta],\ \forall\, s \in \mathbb{R}^+. \end{array}$$

Remark 2.5. The problem (1.2) is equivalent to the integral equation:

$$y(x) = \int_0^{+\infty} e^{\frac{c}{2}(x-s)} m(s)G(x,s)f(s,y(s),y'(s))\, ds. \tag{2.3}$$

The boundary conditions $y(0) = y(+\infty) = 0$ follow from $G(0,s) = 0,\ \forall\, s \ge 0$, and $\lim\limits_{x\to+\infty} e^{\frac{c}{2}x}G(x,s) = 0,\ \forall\, s \ge 0$, since $k > \frac{c}{2}$, where G is given by (2.1).

To show our existence results we will use the following lemma which contains some recent results of the fixed point index theory on the cones of Banach spaces for the sum of two operators (see [6]). Let X be a real Banach space and $\mathcal{K} \subset X$ a cone.

Lemma 2.6. *Let U be a bounded open subset of $\mathcal{K}$ and W be a subset of $\mathcal{K}$ such that $0 \in U \cap W$. Assume that $T : W \to E$ is an expansive mapping with constant $h > 1$, $F : \overline{U} \to E$ be a ℓ-set contraction with $0 \le \ell < h - 1$, and $F(\overline{U}) \subset (I - T)(W)$. Thus we have the following: if*

$$\|Fx + T0\| \le (h-1)\|x\| \ \ and \ \ Tx + Fx \ne x \ \ for\ all \ \ x \in \partial U \cap W,$$

then $i\,(T + F, U \cap W, \mathcal{K}) = 1$.

3. Main results

We begin by a new representation formula for the measure of noncompact- ness in the space X.

Let $p\colon \mathbb{R}^+ \to (0,+\infty)$ be a continuous function. Denote by X the space consisting of all weighted functions y, continuously differentiable on $\mathbb{R}^+$ which satisfy

$$\sup_{x\in\mathbb{R}^+}([\|y(x)\| + \|y'(x)\|]p(x)) < \infty.$$

Equipped with a Bielecki's type norm $\|y\|_p = \sup\limits_{x\in\mathbb{R}^+}([\|y(x)\| + \|y'(x)\|]p(x))$, it is a Banach space.

In the following, we develop a new non-compactness result in order to use it to show that an operator is ℓ-set contraction in the space X.

Lemma 3.1. *Let $B \subset X$ be such that the functions belonging in the sets*

$$pB = \{z \mid z(t) = y(t)\,p(t),\ y \in B\},$$
$$pB' = \{z \mid z(t) = y'(t)\,p(t),\ y \in B\},$$

are almost equicontinuous on $\mathbb{R}^+$ and B is a bounded set in the sense of the norm

$$\|y\|_q = \sup_{x\in\mathbb{R}^+}([\|y(t)\| + \|y'(t)\|]q(t)),$$

where the function q is positive, continuous on $\mathbb{R}^+$ and satisfies

$$\lim_{t\to+\infty}\frac{p(t)}{q(t)} = 0.$$

Then

$$\alpha_X(B) = \max\left(\sup_{t\in\mathbb{R}^+}\alpha_E\left(B(t)p(t)\right),\ \sup_{t\in\mathbb{R}^+}\alpha_E\left((B)'(t)p(t)\right)\right), \tag{3.1}$$

where $B(t) = \{u(t) \mid u \in B\}$ for $t \in \mathbb{R}^+$.

Proof. Let $B \subset X$ be bounded in the sense of the norm

$$\|y\|_q = \sup_{t\in\mathbb{R}^+}([\|y(t)\| + \|y'(t)\|]q(t)).$$

Thus there exists $r > 0$ such that $\|y\|_q \le r$ for all $y \in B$. Since the function q is positive on $\mathbb{R}^+$ and satisfies $\lim\limits_{t\to+\infty}\frac{p(t)}{q(t)} = 0$, for any $\varepsilon > 0$, there exists $T > 0$ such that

$$\begin{aligned}
&\|y(t_1)\,p(t_1) - y(t_2)\,p(t_2)\| \\
\le\ & \tfrac{p(t_1)}{q(t_1)}\|y(t_1)\|\,q(t_1) + \tfrac{p(t_2)}{q(t_2)}\|y(t_2)\|\,q(t_2) \\
\le\ & \tfrac{p(t_1)}{q(t_1)}\left(\|y(t_1)\| + \|y'(t_1)\|\right)q(t_1) + \tfrac{p(t_2)}{q(t_2)}\left(\|y(t_2)\| + \|y'(t_2)\|\right)q(t_2) < \varepsilon,
\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}
&\|y'(t_1)\,p(t_1) - y'(t_2)\,p(t_2)\| \\
\le\ & \tfrac{p(t_1)}{q(t_1)}\|y'(t_1)\|\,q(t_1) + \tfrac{p(t_2)}{q(t_2)}\|y'(t_2)\|\,q(t_2) \\
\le\ & \tfrac{p(t_1)}{q(t_1)}\left(\|y(t_1)\| + \|y'(t_1)\|\right)q(t_1) + \tfrac{p(t_2)}{q(t_2)}\left(\|y(t_2)\| + \|y'(t_2)\|\right)q(t_2) < \varepsilon,
\end{aligned} \tag{3.3}$$

uniformly with respect to $y \in B$ as $t_1, t_2 \ge T$.

We first claim that

$$\alpha_X(B) \leq \max\left(\sup_{t\in\mathbb{R}^+} \alpha_E\left(B(t)p(t)\right), \sup_{t\in\mathbb{R}^+} \alpha_E\left(B'(t)p(t)\right)\right).$$

Denote by $B|_{[0,T]}$ and $B'|_{[0,T]}$ the restriction of B and B' on $[0,T]$. Since the sets $B(t)p(t)$ and $B'(t)p(t)$ are equi-continuous on $[0,T]$, Lemma 2.1 ensures that

$$\begin{aligned}\alpha_{\mathcal{C}^1}(Bp|_{[0,T]}) &= \max\left(\sup_{t\in[0,T]} \alpha_E(B(t)p(t)), \sup_{t\in[0,T]} \alpha_E\left(B'(t)p(t)\right)\right) \\ &\leq \max\left(\sup_{t\in\mathbb{R}^+} \alpha_E(B(t)p(t)), \sup_{t\in\mathbb{R}^+} \alpha_E(B'(t)p(t))\right),\end{aligned}$$

where $Bp|_{[0,T]} = \{y(t)p(t) : y \in B,\, t \in [0,T]\}$.

By the definition of the MNC $\alpha_{\mathcal{C}^1}$, there exists $\{B_i\}_{i=1}^n$ such that $B = \bigcup_{i=1}^n B_i$ and for $i = 1, ..., n$,

$$\operatorname{diam}_{\mathcal{C}^1}(B_i p|_{[0,T]}) \leq \max\left(\sup_{t\in\mathbb{R}^+} \alpha_E(B(t)p(t)), \sup_{t\in\mathbb{R}^+} \alpha_E(B'(t)p(t))\right) + \varepsilon, \tag{3.4}$$

where $\operatorname{diam}_{\mathcal{C}^1}(.)$ denotes the diameter of the bounded subsets of $\mathcal{C}^1([0,T],E)$.
Furthermore, for $i = 1, ..., n$, fixed, for all $y_1,\ y_2 \in B_i$ and $t \geq T$, we deduce from (3.2)-(3.4), for $i = 1, \ldots, n$ the following estimates:

$$\begin{aligned} & \|(y_1(t) - y_2(t))\|\, p(t) \\ \leq\ & \left(\|(y_1(t) + y_1'(t))\| + \|(y_1'(t) - y_2'(t))\| + \|(y_2(t) + y_2'(t))\|\right) p(t) \\ \leq\ & \left(\|(y_1(t) + y_1'(t))\| q(t) + \|(y_2(t) + y_2'(t))\| q(t)\right) \tfrac{p(t)}{q(t)} + \|(y_1'(t) - y_2'(t))\| p(t) \\ \leq\ & \varepsilon + \|y_1'(t)p(t) - y_1'(T)p(T)\| + \|y_1'(T)p(T) - y_2'(T)p(T)\| \\ & + \|y_2'(T)p(T) - y_2'(t)p(t)\| \\ \leq\ & 2\varepsilon + \max\left(\sup_{t\in\mathbb{R}^+} \alpha_E(B(t)p(t)), \sup_{t\in\mathbb{R}^+} \alpha_E(B'(t)p(t))\right) + \varepsilon + \varepsilon, \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} & \|(y_1'(t) - y_2'(t))\|\, p(t) \\ \leq\ & \left(\|(y_1(t) + y_1'(t))\| + \|(y_1(t) - y_2(t))\| + \|(y_2(t) + y_2'(t))\|\right) p(t) \\ \leq\ & \left(\|(y_1(t) + y_1'(t))\| q(t) + \|(y_2(t) + y_2'(t))\| q(t)\right) \tfrac{p(t)}{q(t)} + \|(y_1(t) - y_2(t))\| p(t) \\ \leq\ & \varepsilon + \|y_1(t)p(t) - y_1(T)p(T)\| + \|y_1(T)p(T) - y_2(T)p(T)\| \\ & + \|y_2(T)p(T) - y_2(t)p(t)\| \\ \leq\ & 2\varepsilon + \max\left(\sup_{t\in\mathbb{R}^+} \alpha_E(B(t)p(t)), \sup_{t\in\mathbb{R}^+} \alpha_E(B'(t)p(t))\right) + \varepsilon + \varepsilon. \end{aligned} \tag{3.6}$$

Therefore (3.4), (3.5) and (3.6) guarantee that

$$\operatorname{diam}_X(B_i) \leq \max\left(\sup_{t\in\mathbb{R}^+} \alpha_E(B(t)p(t)), \sup_{t\in\mathbb{R}^+} \alpha_E(B'(t)p(t))\right) + 4\varepsilon$$

Noting that $B = \bigcup\limits_{i=1}^{n} B_i$, we infer

$$\alpha_X(B) \leq \max\left(\sup_{t\in\mathbb{R}^+} \alpha_E(B(t)p(t)), \sup_{t\in\mathbb{R}^+} \alpha_E(B'(t)p(t))\right) + 4\varepsilon,$$

and ε being arbitrary, we deduce that

$$\alpha_X(B) \leq \max\left(\sup_{t\in\mathbb{R}^+} \alpha_E(B(t)p(t)), \sup_{t\in\mathbb{R}^+} \alpha_E(B'(t)p(t))\right).$$

Conversely, we prove that $\max\left(\sup\limits_{t\in\mathbb{R}^+} \alpha_E(B(t)p(t)), \sup\limits_{t\in\mathbb{R}^+} \alpha_E(B'(t)p(t))\right) \leq \alpha_X(B)$.
Given $\varepsilon > 0$, there exists $\{B_i\}_{i=1}^n$ such that $B = \bigcup\limits_{i=1}^{n} B_i$ and $\mathrm{diam}_X(B_i) \leq \alpha_X(B) + \varepsilon$.
Thus, for fixed i, for every $t \in \mathbb{R}^+$ and all $y_1,\ y_2 \in B_i$, we have

$$\|(y_1(t) - y_2(t))\|\, p(t)\| \leq \|y_1 - y_2\|_\omega < \alpha_X(B) + \varepsilon,$$

and

$$\|(y_1'(t) - y_2'(t))\|\, p(t)\| \leq \|y_1 - y_2\|_\omega < \alpha_X(B) + \varepsilon.$$

Since $B(t) = \bigcup\limits_{i=1}^{n} B_i(t)$, we have $\alpha_E(B(t)p(t)) \leq \alpha_X(B) + \varepsilon$.
Now ε being arbitrary, we deduce that $\sup\limits_{t\in\mathbb{R}^+} \alpha_E(B(t)p(t)) \leq \alpha_X(B)$.
In accordance with $B'(t) = \bigcup\limits_{i=1}^{n} B_i'(t)$, we get $\sup\limits_{t\in\mathbb{R}^+} \alpha_E(B'(t)p(t)) \leq \alpha_X(B)$, where $H'(t) = \{y'(t)|\ y \in H\},\ t \in \mathbb{R}^+$, whence the reversed inequality and then the desired result. □

Let $\omega > 0$ be a given real parameter. Consider the Banach space with weight function $e^{-\omega t}$

$$X = \left\{x \in \mathcal{C}^1(\mathbb{R}^+, E) : \sup_{t\in\mathbb{R}^+} \left((\|x(t)\| + \|x'(t)\|)\, e^{-\omega t}\right) < \infty\right\},$$

endowed with the norm

$$\|x\|_\omega = \sup_{t\in\mathbb{R}^+} \left((\|x(t)\| + \|x'(t)\|)\, e^{-\omega t}\right).$$

Define the cone

$$\widetilde{\mathcal{K}} = \{x \in X : x \geq 0 \quad \text{on} \quad \mathbb{R}^+\}.$$

With $\mathcal{K}$ we denote the set of all equi-continuous families in $\widetilde{\mathcal{K}}$.
Take $\varepsilon \in (0,1)$ and $p, q > 0$ arbitrarily. Let $A, B_1, B_2, B_3, R, \tau$ be positive constants such that

$$G(t,s) + |G_t(t,s)| \leq A, \quad t, s \in [0,\infty),$$

and

$$0 < \tau < \frac{1}{4}, \quad A\,(B_1 + B_2 R^p + B_3 R^q) < \min\{\tau, R\}.$$

Define the conical shell

$$\mathcal{K}_R = \{x \in \mathcal{K} : \|x\|_\omega < R\}.$$

Assume that

(H1). $f \in \mathcal{C}(\mathbb{R}^+ \times \mathcal{P} \times E, \mathcal{P})$ be such that

$$\|f(t,x,y)\| \leq a_0(t) + a_1(t)\|x\|^p + a_2(t)\|y\|^q$$

for any $(t,x,y) \in \mathbb{R}^+ \times \mathcal{P} \times E$, where $a_i \in \mathcal{C}(\mathbb{R}^+, \mathbb{R}^+)$, $i \in \{0,1,2\}$,

$$\int_0^\infty m(s)a_0(s)ds \leq B_1, \quad \int_0^\infty m(s)a_1(s)e^{wsp}ds \leq B_2, \quad \int_0^\infty m(s)a_2(s)e^{\omega sq}ds \leq B_3.$$

Theorem 3.2. *Assume* $(H1)$. *Then the problem* (1.1) *has at least one positive solution* x *in* $\mathcal{K}$ *such that*

$$\sup_{t\in\mathbb{R}^+} \left((\|x(t)\| + \|x'(t)\|)\, e^{-\omega t} \right) \leq R.$$

Proof. For $x \in X$ define the operators

$$\begin{aligned} Tx(t) &= (1+\varepsilon)x(t), \\ Fx(t) &= -\varepsilon \int_0^\infty G(t,s)m(s)f(s,x(s),x'(s))ds, \quad t \in (0,\infty). \end{aligned}$$

1. Note that $T : \mathcal{K}_R \to X$ is an $(1+\varepsilon)$-expansive operator.

2. Now we will prove that the operator $F : \overline{\mathcal{K}_R} \to X$ is continuous. From the assumption $(H1)$, we can show that

$$\sup_{t\in\mathbb{R}^+} e^{-\omega t} \left(\|Fx(t)\| + \|(Fx)'(t)\| \right) < \infty,$$

which imply that

$$F(\overline{\mathcal{K}_R}) \subset X.$$

Let $\{x_n\}_{n\in\mathbb{N}}, \{x\} \subset \overline{\mathcal{K}_R}$ with $\|x_n - x\|_w \to 0$, as $n \to \infty$. Hence, $\{x_n\}_{n\in\mathbb{N}}$ is bounded in $\overline{\mathcal{K}_R}$. Then there exists a positive constant r such that $\max\{\|x_n\|_\omega, n \in \mathbb{N}, \|x\|_\omega\} \leq r$. We have

$$\begin{aligned} &\int_0^\infty e^{-\omega t}G(t,s)m(s)\|f(s,x_n(s),x_n'(s)) - f(s,x(s),x'(s))\|ds \\ &\leq \int_0^\infty e^{-\omega t}G(t,s)m(s)\left(\|f(s,x_n(s),x_n'(s))\| + \|f(s,x(s),x'(s))\|\right)ds \\ &\leq \int_0^\infty e^{-\omega t}G(t,s)m(s)\Big(2a_0(s) + a_1(s)\left(\|x_n(s)\|^p + \|x(s)\|^p\right) \\ &\quad + a_2(s)\left(\|x_n'(s)\|^q + \|x'(s)\|^q\right)\Big)ds \\ &\leq 2B_1A + AB_2\left(\|x_n\|_\omega^p + \|x\|_\omega^p\right) + AB_3\left(\|x_n\|_\omega^q + \|x\|_\omega^q\right) \\ &\leq 2A\left(B_1 + B_2r^p + B_3r^q\right), \quad t \in (0,\infty), \end{aligned}$$

and

$$\begin{aligned}
&\int_0^\infty e^{-\omega t}|G_t(t,s)|m(s)\|f(s,x_n(s),x_n'(s))-f(s,x(s),x'(s))\|ds\\
&\leq \int_0^\infty e^{-\omega t}|G_t(t,s)|m(s)\left(\|f(s,x_n(s),x_n'(s))\|+\|f(s,x(s),x'(s))\|\right)ds\\
&\leq \int_0^\infty e^{-\omega t}|G_t(t,s)|m(s)\Big(2a_0(s)+a_1(s)\left(\|x_n(s)\|^p+\|x(s)\|^p\right)\\
&\quad +a_2(s)\left(\|x_n'(s)\|^q+\|x'(s)\|^q\right)\Big)ds\\
&\leq 2B_1A+AB_2\left(\|x_n\|_\omega^p+\|x\|_\omega^p\right)+AB_3\left(\|x_n\|_\omega^q+\|x\|_\omega^q\right)\\
&\leq 2A\left(B_1+B_2r^p+B_3r^q\right),\quad t\in(0,\infty).
\end{aligned}$$

Thus, the Lebesgue dominated convergence theorem both with the continuity of f imply

$$\sup_{t\in\mathbb{R}^+}\left(e^{-\omega t}\|(Fx_n)(t)-(Fx)(t)\|\right)\to 0,\quad \text{as}\quad n\to\infty,$$

and

$$\sup_{t\in\mathbb{R}^+}\left(e^{-\omega t}\|(Fx_n)'(t)-(Fx)'(t)\|\right)\to 0,\quad \text{as}\quad n\to\infty.$$

As a result,

$$\|Fx_n-Fx\|_\omega\to 0,\quad \text{as}\quad n\to\infty,$$

i.e., the operator F is continuous.

3. We have $F:\overline{\mathcal{K}_R}\to X$ and for $x\in\overline{\mathcal{K}_R}$ we get

$$\begin{aligned}
&\left(\|Fx(t)\|+\|(Fx)'(t)\|\right)e^{-\omega t}\\
&\leq \varepsilon e^{-\omega t}\int_0^\infty\left(G(t,s)+|G_t(t,s)|\right)m(s)\|f(s,x(s),x'(s))\|ds\\
&\leq \varepsilon e^{-\omega t}A\int_0^\infty m(s)\left(a_0(s)+a_1(s)\|x(s)\|^p+a_2(s)\|x'(s)\|^q\right)ds\\
&\leq \varepsilon e^{-\omega t}A\int_0^\infty m(s)\left(a_0(s)+a_1(s)e^{\omega ps}R^p+a_2(s)e^{\omega qs}R^q\right)ds\\
&\leq \varepsilon e^{-\omega t}A\left(B_1+B_2R^p+B_3R^q\right)\\
&\leq \varepsilon A\left(B_1+B_2R^p+B_3R^q\right)\\
&\leq \tau\varepsilon\\
&< \frac{\varepsilon}{4},\quad t\in(0,\infty).
\end{aligned}$$

Hence,

$$\|Fx\|_\omega\leq\frac{\varepsilon}{4}.$$

Therefore $F(\overline{\mathcal{K}_R})$ is uniformly bounded. Since $F:\overline{\mathcal{K}_R}\to X$ is continuous, we have that $F(\overline{\mathcal{K}_R})$ is equi-continuous. Consequently $F:\overline{\mathcal{K}_R}\to X$ is a 0-set contraction.

4. Let $y \in \overline{\mathcal{K}_R}$ be arbitrarily chosen. Set

$$z(t) = \int_0^\infty G(t,s)m(s)f(s,y(s),y'(s))ds, \quad t \in (0,\infty).$$

We have that $z \in \mathcal{K}$ and using the above computations, we have

$$\begin{aligned}
(\|z(t)\| + \|z'(t)\|)\, e^{-\omega t} &\leq e^{-\omega t} \int_0^\infty \left(G(t,s) + |G_t(t,s)|\right) m(s) \|f(s,y(s),y'(s))\| ds \\
&\leq A\left(B_1 + B_2 R^p + B_3 R^q\right) \\
&\leq A\left(B_1 + B_2 R^p + B_3 R^q\right) \\
&\leq R,
\end{aligned}$$

so, $\|z\|_\omega \leq R$. Therefore $z \in \mathcal{K}_R$. Also,

$$\begin{aligned}
(I-T)z(t) &= -\varepsilon z(t) \\
&= -\varepsilon \int_0^\infty G(t,s)m(s)f(s,y(s),y'(s))ds \\
&= Fy(t), \quad t \in (0,\infty).
\end{aligned}$$

Thus

$$F\left(\overline{\mathcal{K}_R}\right) \subset (I-T)(\mathcal{K}).$$

5. Note that $0 \in \mathcal{K}_R$ and for any $x \in \partial \mathcal{K}_R$ we have

$$\begin{aligned}
\|Fx + T0\|_\omega &= \|Fx\|_\omega \\
&\leq \varepsilon A\left(B_1 + B_2 R^p + B_3 R^q\right) \\
&\leq \varepsilon R \\
&= \varepsilon \|x\|_\omega.
\end{aligned}$$

Assume that there exists an $x \in \partial \mathcal{K}_R$ such that

$$Fx + Tx = x.$$

Then

$$\begin{aligned}
&(\|x(t)\| + \|x'(t)\|)\, e^{-\omega t} \\
\geq\ & (1+\varepsilon)e^{-\omega t} \left(\|x(t)\| + \|x'(t)\|\right) \\
&-\varepsilon e^{-\omega t} \int_0^\infty \left(G(t,s) + |G_t(t,s)|\right) m(s) \|f(s,x(s),x'(s))\| ds \\
\geq\ & (1+\varepsilon)e^{-\omega t} \left(\|x(t)\| + \|x'(t)\|\right) \\
&-\varepsilon A\left(B_1 + B_2 R^p + B_3 R^q\right), \quad t \in (0,\infty),
\end{aligned}$$

or

$$\varepsilon A\left(B_1 + B_2 R^p + B_3 R^q\right) \geq \varepsilon e^{-\omega t} \left(\|x(t)\| + \|x'(t)\|\right), \quad t \in (0,\infty),$$

whereupon

$$A\left(B_1 + B_2 R^p + B_3 R^q\right) \geq R.$$

This is a contradiction.

By 1, 2, 3, 4, 5 and Lemma 2.6, we conclude that the operator $T+F$ has a fixed point $x \in \mathcal{K}_R$, which is a solution of the problem (1.1). This completes the proof. □

Remark 3.3. A discussion on the existence of a positive real R that verifies an inequality of the type $A\left(B_1 + B_2 R^p + B_3 R^q\right) < R$, with respect to p and q, is given in [7, Remark 3.2]. So that the same constant R checks the inequality $A\left(B_1 + B_2 R^p + B_3 R^q\right) < \tau < \frac{1}{4}$ it is necessary that $AB_1 < \tau$.

In the case when the last inequality does not hold, or in a general way the inequality $A\left(B_1 + B_2 R^p + B_3 R^q\right) < \tau < \frac{1}{4}$ is not satisfied, additional conditions on the nonlinearity f are needed to show that the operator F is a ℓ-set contraction.

To overcome the problem pointed out in the remark we consider the conditions (H2) and (H3).

(H2). For every $r > 0$ and all subinterval $[a,b] \subset \mathbb{R}^+$, the nonlinearity f is uniformly continuous on $[a,b] \times B_E(0,r) \times B_E(0,r)$, where $B_E(0,r) = \{x \in E : \|x\| \leq r\}$.

(H3). There exist a positive functions $l_1, l_2 \in L^1(\mathbb{R}^+)$ such that

$$\alpha(f(t,B_1,B_2)) \leq l_1(t)\alpha(B_1) + l_2(t)\alpha(B_2), \quad t \in \mathbb{R}^+,$$

for every bounded subsets $B_1, B_2 \subset E$, where

$$A\int_0^{+\infty} m(t)(l_1(t) + l_2(t))dt < 1.$$

And we present the following theorem.

Theorem 3.4. *Assume* $(H1) - (H3)$. *Then the problem* (1.1) *has at least one positive solution* x *in* $\mathcal{K}$ *such that*

$$\sup_{t\in\mathbb{R}^+} \left(\left(\|x(t)\| + \|x'(t)\|\right) e^{-\omega t}\right) \leq R.$$

Proof. The proof of this theorem is similar to that of Theorem 3.2, we will only show how the operator F is a ℓ-set contraction with $\ell < \varepsilon$ under conditions $(H2)$ and $(H3)$. Firstly, using Lemma 3.1 for $p(t) = e^{-\omega t}$ and $q(t) = e^{-\mu t}$ with $\mu < \omega$, we get the following result.

Lemma 3.5. *Assume that* $(H1)$ *holds. If* V *be a bounded subset of* $\overline{\mathcal{K}_R}$, *then*

$$\alpha_X(FV) = \max\left(\sup_{t\in\mathbb{R}^+} \alpha_E\left(e^{-\omega t} FV(t)\right),\ \sup_{t\in\mathbb{R}^+} \alpha_E\left(e^{-\omega t}(FV)'(t)\right)\right).$$

Proof. Let $V \subset \overline{\mathcal{K}_R}$ be arbitrary.

(a) $F(V) \subset X$ is a uniformly bounded set with respect to the norm $\|.\|_\mu$. Indeed, as in Theorem 3.2, we obtain

$$\|x\|_\mu \leq \varepsilon\, A(B_1 + B_2\|x\|_\mu^p + B_3\|x\|_\mu^q),\ \forall x \in V.$$

(b) The families $\{e^{-\omega t}(FV(t))\}_{t\in\mathbb{R}^+}$ and $\{e^{-\omega t}(FV)'(t))\}_{t\in\mathbb{R}^+}$ are almost equicontinuous on $\mathbb{R}^+$. The proof is similar to the one in [5, Lemma 1.3.3]. □

Now, Suppose that $V \subset \overline{\mathcal{K}_R}$; we prove that there exists a constant $0 \leq \ell < \varepsilon$ such that $\alpha_X(FV) \leq \ell\alpha_X(V)$. Lemma 3.5 tells us that it is enough to verify that

$$\max\left(\sup_{t\in\mathbb{R}^+} \alpha_E\left(e^{-\omega t} FV(t)\right),\ \sup_{t\in\mathbb{R}^+} \alpha_E\left(e^{-\omega t}(FV)'(t)\right)\right) \leq \ell\alpha_X(V). \tag{3.7}$$

Let $x \in V$. we introduce, for each $n \geq 1$, the approximating operator F_n by

$$F_n x(t) = -\varepsilon \int_0^n G(t,s)m(s)f(s,x(s),x'(s))ds.$$

Step 1. From $(H1)$and $(H3)$, for every $t \in (0,\infty)$, we have that

$$\begin{aligned} & e^{-\omega t}\|Fx(t) - F_n x(t)\| \\ \leq \ & \int_n^{+\infty} e^{-\omega t}G(t,s)m(s)\|f(s,x(s),x'(s))\|ds \\ \leq \ & \varepsilon\, A(\int_n^{+\infty} m(s)a_0(s)ds + \|x\|_\omega^p \int_n^{+\infty} e^{p\omega s}m(s)a_1(s)ds \\ & + \|x\|_\omega^q \int_n^{+\infty} e^{q\omega s}m(s)a_2(s)ds). \end{aligned}$$

Similarly, we also have

$$\begin{aligned} & e^{-\omega t}\|(F_n x)'(t) - (Fx)'(t)\| \\ \leq \ & \varepsilon\, A(\int_n^{+\infty} m(s)a_0(s)ds + \|x\|_\omega^p \int_n^{+\infty} e^{p\omega s}m(s)a_1(s)ds \\ & + \|x\|_\omega^q \int_n^{+\infty} e^{q\omega s}m(s)a_2(s)ds). \end{aligned}$$

As a consequence, we get

$$\begin{aligned} & \|Fx - F_n x\|_\omega \\ = \ & \sup_{t\in\mathbb{R}^+} \{e^{-\omega t}\left(\|Fx(t) - F_n x(t)\| + \|(F_n x)'(t) - (Fx)'(t)\|\right)\} \\ \leq \ & 2\varepsilon\, A(\int_n^{+\infty} m(s)a_0(s)ds + \|x\|_\omega^p \int_n^{+\infty} e^{p\omega s}m(s)a_1(s)ds \\ & + \|x\|_\omega^q \int_n^{+\infty} e^{q\omega s}m(s)a_2(s)ds). \end{aligned}$$

The convergence of the integrals guarantee that

$$\begin{aligned} \lim_{n\to+\infty} \int_n^{+\infty} m(s)a_0(s)ds = 0, \\ \lim_{n\to+\infty} \int_n^{+\infty} e^{p\omega s}m(s)a_1(s)ds = 0, \\ \lim_{n\to+\infty} \int_n^{+\infty} e^{q\omega s}m(s)a_2(s)ds = 0. \end{aligned}$$

Then, for all $x \in V$ and $t \in (0,\infty)$, we have

$$\begin{aligned} & d(e^{-\omega t}(F_n x)(t), e^{-\omega t}(FV)(t)) \\ = \ & \inf_{y\in B}\{e^{-\omega t}\left(\|F_n x(t) - Fy(t)\| + \|(F_n x)'(t) - (Fx)'(t)\|\right)\} \\ \leq \ & e^{-\omega t}\left(\|F_n x(t) - Fx(t)\| + \|(F_n x)'(t) - (Fx)'(t)\|\right) \\ & \to 0, \text{ as } n \to \infty, \end{aligned}$$

hence for every $t \in (0,\infty)$

$$\sup_{x\in V} d(e^{-\omega t}(F_n x)(t), e^{-\omega t}(FV)(t)) \to 0, \text{ as } n \to \infty.$$

Similarly, for every $t \in (0,\infty)$

$$\sup_{x\in V} d(e^{-\omega t}(F_n V)(t), e^{-\omega t}(Fx)(t)) \to 0, \text{ as } n \to \infty.$$

Then the Hausdorff distance

$$H_d(e^{-\omega t}FV(t), e^{-\omega t}F_n V(t))$$

tends to 0, as n tends to $+\infty$ for all t in $(0,\infty)$. The Lipschitz property of the MNC α guarantees

$$\lim_{n\to+\infty} \alpha\left(e^{-\omega t}F_nV(t)\right) = \alpha\left(e^{-\omega t}FV(t)\right),\ \forall\, t\in(0,+\infty), \tag{3.8}$$

and

$$\lim_{n\to+\infty} \alpha\left(e^{-\omega t}(F_nV)'(t)\right) = \alpha\left(e^{-\omega t}(FV)'(t)\right),\ \forall\, t\in(0,+\infty), \tag{3.9}$$

Step 2. In what follows, we estimate $\alpha\left(e^{-\omega t}F_nV(t)\right)$. Using Assumption $(H3)$, Lemma 3.1 and the properties of the Green function lead to estimations:

$$\begin{aligned}
\alpha\left(e^{-\omega t}FV(t)\right) &= \lim_{n\to+\infty}\alpha\left(e^{-\omega t}(F_nV)(t)\right)\\
&= \varepsilon \lim_{n\to+\infty}\alpha\left(\{e^{-\omega t}\textstyle\int_0^n G(t,s)m(s)f(s,x(s),x'(s))ds\},\ x\in V\right)\\
&\le \varepsilon A \lim_{n\to+\infty}\textstyle\int_0^n m(s)\alpha\left(f(s,x(s),x'(s)),\ x\in V\right)ds\\
&\le \varepsilon A \lim_{n\to+\infty}\textstyle\int_0^n m(s)\left(l_1(s)\alpha\left(e^{-\omega s}V(s)\right)+l_2(s)\alpha\left(e^{-\omega s}V'(s)\right)\right)ds\\
&\le \alpha_X(V)\varepsilon A \lim_{n\to+\infty}\textstyle\int_0^n m(s)(l_1(s)+l_2(s))ds\\
&\le \alpha_X(V)\varepsilon A \textstyle\int_0^{+\infty} m(s)(l_1(s)+l_2(s))ds.
\end{aligned}$$

Since t is arbitrary

$$\sup_{t\in\mathbb{R}^+}\alpha\left(e^{-\omega t}(FV)(t)\right)\le \ell\alpha_X(V),$$

Similarly, we find that

$$\sup_{t\in\mathbb{R}^+}\alpha\left(e^{-\omega t}(FV)'(t)\right)\le \ell\alpha_X(V),$$

where, $\ell=\varepsilon A\int_0^{+\infty} m(s)(l_1(s)+l_2(s))ds$.
From Lemma 3.5, we immediately deduce that

$$\alpha_X(FV)\le \ell\,\alpha_X(V),$$

meaning that $F:\overline{\mathcal{K}_R}\to X$ is a ℓ-set contraction with $\ell<\varepsilon$. □

4. An example

Consider the following nonlinear boundary value problem for system of n scalar differential equations in the Banach space $E=\mathbb{R}^n$ with the Euclidean norm

$$\|x\| = \left(\sum_{i=1}^n x_i^2\right)^{\frac{1}{2}}$$

with $x=(x_1,\dots,x_n)\mid x_i\in\mathbb{R},\ i=1,\dots,n$ and let $0<p,q<1$:

$$\begin{cases} -x_i''(t)+k^2x_i(t)=\frac{e^{-p\omega t}}{t^2\sqrt{t}}\frac{1-\cos t}{t+1}(1+(x_i(t))^p+(x_i'(t))^q),\ t>0\\ x_i(0)=0,\ \ \lim\limits_{t\to+\infty}x_i(t)=0,\ \ i=1,2,\dots,n. \end{cases} \tag{4.1}$$

Let $\mathcal{P}=\{x=(x_1,x_2,\dots,x_n)\in\mathbb{R}^n\mid x_i\ge 0,\ i=1,2,\dots,n\}$. Then $\mathcal{P}$ is a cone in $\mathbb{R}^n$ and clearly System (4.1) can be rewritten in the form (1.1) in E. In this case,

$$x=(x_1,\dots,x_n),\ y=(y_1,\dots,y_n),\quad f=(f^{(1)},\dots,f^{(n)})$$

where for any $i \in \{1, \dots, n\}$, $f^{(i)}$ is defined by

$$f^{(i)}(t, x_1, \dots, x_n, y_1, \dots, y_n) \quad = \quad \tfrac{1-\cos\, t}{t+1}(1 + x_i^p + y_i^q), \quad \text{for } t \geq 0.$$

Then, we have $f^{(i)}$ is uniformly continuous on $[a, b] \times B_E(0, r)$, for all $[a, b] \subset I$ and $r > 0$. The singular coefficient is given by $m(t) = \frac{e^{-p\omega t}}{t^2\sqrt{t}}$ for $t > 0$. Then

$$\begin{aligned}
\|f(t, x, y)\|^2 &= \sum_{i=1}^{n}(f^{(i)}(t, x, y))^2 \\
&\leq 4\tfrac{(1-\cos t)^2}{(t+1)^2}\left(n + \sum_{i=1}^{n} x_i^{2p} + \sum_{i=1}^{n} y_i^{2q}\right) \\
&\leq 4\tfrac{(1-\cos t)^2}{(t+1)^2}\left(n + (\sum_{i=1}^{n} x_i^2)^p + (\sum_{i=1}^{n} y_i^2)^q\right) \\
&\leq 4\tfrac{(1-\cos t)^2}{(t+1)^2}\left(n + \|x\|^{2p} + \|y\|^{2q}\right).
\end{aligned}$$

Hence,

$$\|f(t, x, y)\| \leq a_1(t)\|x\|^p + a_2(t)\|y\|^q + a_0(t),$$

where $a_1(t) = a_2(t) = 2\,\frac{1-\cos t}{t+1}$ and $a_2(t) = 2\sqrt{n}\,\frac{1-\cos t}{t+1}$.
Moreover, since in the vicinity the origin, $\frac{1-\cos\, s}{s^2\sqrt{s}} \sim \frac{1}{2\sqrt{s}}$ and for any $\alpha > 0$,

$$\int_0^{+\infty} \frac{e^{-\alpha s}}{(s+1)\sqrt{s}}\, dx < \infty,$$

we deduce the convergence of the integrals

$$\int_0^{+\infty} e^{p\omega s} m(s) a_1(s) ds = 2\int_0^{+\infty} \frac{1-\cos s}{s^2\sqrt{s}(s+1)} ds,$$

$$\int_0^{+\infty} m(s) a_0(s) ds = 2\sqrt{n}\int_0^{+\infty} \frac{(1-\cos s)e^{-p\omega s}}{s^2\sqrt{s}(s+1)} ds.$$

Also, the integral

$$\int_0^{+\infty} e^{q\omega s} m(s) a_2(s) ds = 2\int_0^{+\infty} \frac{(1-\cos s)e^{(q-p)\omega s}}{s^2\sqrt{s}(s+1)} ds,$$

is converge provided $p > q$.
Here the real numbers p, q satisfy $0 < p, q < 1$, then there exists $R > 0$ such that $A\left(B_1 + B_2 R^p + B_3 R^q\right) < R$ (see [7, Remark 3.2]).
Finally, for every bounded subsets $D_1, D_2 \subset E$ and for all $t \in \mathbb{R}^+$, $x \in D_1$, $y \in D_2$, we have

$$\|f(t, x, y)\| \leq 2\frac{1-\cos t}{t+1}\left(n + \|x\|^{2p} + \|y\|^{2q}\right) \leq 4\left(n + \|x\|^{2p} + \|y\|^{2q}\right).$$

Moreover, for all $0 < t_1 < t_2 < +\infty$, $x \in D_1$, and $y \in D_2$, we have

$$\begin{aligned}
&\lim_{t_1 \to t_2} |f^{(i)}(t_1, x, y) - f^{(i)}(t_2, x, y)| \\
\leq\ &\lim_{t_1 \to t_2} \left|\tfrac{1-\cos t_1}{t_1+1}(1 + x_i^p + y_i^q) - \tfrac{1-\cos t_2}{t_2+1}(1 + x_i^p + y_i^q)\right| \\
\leq\ &\lim_{t_1 \to t_2} (1 + \|x\|_\infty^p + \|x\|_\infty^q)\left|\tfrac{1-\cos t_1}{t_1+1} - \tfrac{1-\cos t_2}{t_2+1})\right| = 0,\ \forall\, i = 1, \dots, n.
\end{aligned}$$

Then $\lim_{t_1 \to t_2} \|f(t_1, x, y) - f(t_2, x, y)\| = 0$ and

$$\begin{aligned} & \lim_{t\to+\infty} |f^{(i)}(t,x,y) - \lim_{s\to+\infty} f^{(i)}(s,x,y)| \\ \leq \quad & \lim_{t\to+\infty} \left| \tfrac{1-\cos t}{t+1}(1 + x_i^p + y_i^q) - 0 \right| = 0, \ \forall\, i = 1,\ldots,n. \end{aligned}$$

Hence, $\lim_{t\to+\infty} \|f^{(i)}(t,x,y) - \lim_{s\to+\infty} f^{(i)}(s,x,y)\| = 0.$

As a consequence, Corduneanu's compactness criterion ([1], p. 62) ensures that $f(t, D_1, D_2)$ is relatively compact in $\mathbb{R}^n$. So, $\alpha(f(t, D_1, D_2)) = 0$, for all $t \in \mathbb{R}^+$ and all bounded subset $D_1, D_2 \subset E$.

Theorem 3.4 ensures the sub-linear singular problem (4.1) has a bounded positive solution for every constants k and all $0 < p,\, q < 1$.

Acknowledgments. We would like to thank the anonymous referees for their careful reading and helpful suggestions which led to a substantial improvement of the original manuscript.

References

[1] Corduneanu, C., *Integral Equations and Stability of Feedback Systems*, Academic Press, New York, 1973.

[2] Deimling, K., *Ordinary Differential Equations in Banach Spaces*, Springer, Berlin, Germany, 1977.

[3] Deimling, K., *Nonlinear Functional Analysis*, Springer-Verlag, Berlin, Heidelberg, 1985.

[4] Djebali, S., *Travelling wave solutions to a reaction-diffusion system arising in epidemiology*, Nonlinear Anal. Real World Appl., **2**(2001), no. 4, 417-442.

[5] Djebali, S., Madjidi, F., Mebarki, K., *Existence results for singular boundary value problems on unbounded domains in Banach spaces*, Mediterr. J. Math., **11**(2014), 45-74.

[6] Djebali, S., Mebarki, K., *Fixed point index theory for perturbation of expansive mappings by k-set contractions*, Topol. Methods Nonlinear Anal., **54**(2019), no. 2, 613-640.

[7] Djebali, S., Mebarki, K., *On the singular generalized Fisher-like equation with derivative depending nonlinearity*, Appl. Math. Comput., **205**(2008), no. 1, 336-351.

[8] Djebali, S., Moussaoui, T., *A class of second order bvps on infinite intervals*, Electron. J. Qual. Theory Differ. Equ., **4**(2006), 1-19.

[9] Djebali, S., Saifi, O., Baoqiang, Y., *Positive solutions for singular BVPs on the positive half-Line arising from epidemiology and combustion theory*, Acta Math. Sci. Ser. B, **32**(2012), no. 2, 672-694.

[10] Guo, D., Cho, Y.I., Zhu, J., *Partial Ordering Methods in Nonlinear Problems*, Shangdon Science and Technology Publishing Press, Shangdon, 1985.

[11] Guo, D., Lakshmikantham, V., *Nonlinear Problems in Abstract Cones*, Notes and Reports in Mathematics in Science and Engineering, **vol. 5**, Academic Press, Boston, Mass, USA, 1988.

[12] Guo, D., Lakshmikantham, V., Liu, X., *Nonlinear Integral Equations in Abstract Spaces*, Math. and Appl., **vol. 373**, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1996.

[13] Kneser, A., *Untersuchung und asymptotische darstellung der integrale gewisser differentialgleichungen bei grossen werthen des arguments*, J. Reine Angew. Math., **116**(1896), 178-212.

[14] Lakshmikantham, V., Leela, S., *Nonlinear Differential Equations in Abstract Spaces*, Pergamon, Oxford, UK, 1981.

[15] Li, P., Chen, H., Wu, Y., *Existence of solutions of n-point boundary value problems on the half-line in Banach spaces*, Acta Appl. Math., **110**(2010), 785-795.

[16] Liu, Y., *Boundary value problems for second order differential equations on unbounded domains in a Banach space*, Appl. Math. Comput., **135**(2003), 569-583.

[17] Murray, J.D., *Mathematical Biology, Biomathematics Texts*, Springer-Verlag, 1989.

[18] Potter, A.J.B., *A fixed point theorem for positive k-set contractions*, Proc. Edinb. Math. Soc., **19**(1974), 93-102.

[19] Zeidler, E., *Nonlinear Functional Analysis and its Applications*, Vol. I: *Fixed Point Theorems*, Springer-Verlag, New York, 1986.

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Stud. Univ. Babeş-Bolyai Math. 66(2021), No. 4, 739–756
DOI: 10.24193/subbmath.2021.4.11

Baskakov-Kantorovich operators reproducing affine functions

Jorge Bustamante

Abstract. We present a new Kantorovich modification of Baskakov operators which reproduce affine functions. We present an upper estimate for the rate of convergence of the new operators in polynomial weighted spaces and characterize all functions for which there is convergence in the weighted norm.

Mathematics Subject Classification (2010): 41A35, 41A36.

Keywords: Baskakov-Kantorovich operators, polynomial weighted spaces, rate of convergence.

1. Introduction

Let $C[0,\infty)$ be the family of all real continuous functions on the semiaxis. Denote $e_k(t) = t^k$, $k \geq 0$. For a function $g : [0,\infty) \to \mathbb{R}$ we set $\|g\| = \sup\{| f(x) | : x \geq 0\}$. Throughout the paper, we fix $b \geq 0$ and set

$$\varrho(x) = 1/(1+x)^b \quad \text{and} \quad \varphi(x) = \sqrt{x(1+x)}.$$

For $\lambda > 1$ and $f : [0,\infty) \to \mathbb{R}$, the Baskakov operator is defined by

$$V_\lambda(f,x) = \sum_{k=0}^{\infty} f\left(\frac{k}{\lambda}\right) v_{\lambda,k}(x), \quad v_{\lambda,k}(x) = \binom{\lambda+k-1}{k} \frac{x^k}{(1+x)^{\lambda+k}}, \quad x \geq 0.$$

Here we present Kantorovich-Baskakov operators reproducing linear functions. For the modification we use the notations

$$I_{\lambda,k} = [k/\lambda, (k+1)/\lambda] \qquad \text{and} \qquad a_k = 2k/(2k+1), \quad k \in \mathbb{N}_0.$$

Now, for $f \in C[0,\infty)$ define (whenever the series converges)

$$M_\lambda(f,x) = \lambda \sum_{k=0}^{\infty} Q_{\lambda,k}(f) v_{\lambda,k}(x), \quad Q_{\lambda,k}(f) = \int_{I_{\lambda,k}} f(a_k t) dt.$$

Other authors have introduced Kantarovich-type modifications of some operators defined in subspaces of $C[0,\infty)$ to obtain new ones reproducing affine functions. But these modifications are non-positive or the convergence is only proved in specific proper subintervals of the original domain (see [1] and the references given there).

We will study approximation properties of the operators M_λ in some weighted spaces. The following notations are used

$$C_{\varrho,\beta}[0,\infty) = \{h \in C[0,\infty) :\ h(0) = 0,\ \|\varrho\varphi^{2\beta}h\| < \infty\}, \quad 0 \le \beta \le 1.$$

In $C_{\varrho,\beta}[0,\infty)$ we consider the norm $\|f\|_{\varrho,\beta} = \|\varrho\varphi^{2\beta}h\|$.

Section 2 is devoted to study the moments and derivatives of the operators, while some auxiliary results are included in Section 3. In Section 4 we identify all functions in $C_{\varrho,0}[0,\infty)$ for which $\|\varrho(f - M_\lambda(f))\| \to 0$ as $\lambda \to \infty$. The main result is presented in Section 5, where we obtain an estimate of the error $\|\varrho\varphi^{2\beta}(f - M_\lambda(f))\|$ in terms of a K-functional. Analogous results for Kantorovich-Szász-Mirakjan operators will appear in another paper.

In what follows C and C_i will denote absolute constants. They may be different on each occurrence. We remark that our arguments allow to obtain bounds for the constants.

2. Moments and derivatives

For $m \in \mathbb{N}$, the central moment of order m of the operators V_λ and M_λ are defined by $V_{\lambda,m}(x) = V_\lambda((t-x)^m, x)$ and $M_{\lambda,m}(x) = M_\lambda((t-x)^m, x)$ respectively. We also set

$$\theta_\lambda(t) = \frac{1}{\lambda^2}\frac{(2\lambda t)^2}{(1+2\lambda t)^2}. \tag{2.1}$$

Proposition 2.1. (see [2, page 94]) *For each $\lambda > 1$ and $x \ge 0$, $V_\lambda(e_0,x) = 1$, $V_\lambda(e_1,x) = x$,*

$$V_\lambda(e_2,x) = x^2 + \frac{x(1+x)}{\lambda}, \quad \textit{and} \quad V_{\lambda,4}(x) = \frac{\varphi^4(x)}{\lambda^2}\left(3 + \frac{2}{\lambda} + \frac{(1+2x)^2}{\lambda\varphi^2(x)}\right).$$

Proposition 2.2. *For each $\lambda > 1$ and $x \ge 0$, $M_\lambda(e_0,x) = 1$, $M_\lambda(e_1,x) = x$,*

$$M_{\lambda,2}(x) = \frac{\varphi^2(x)}{\lambda} + \frac{1}{12}V_\lambda\big(\theta_\lambda, x\big),$$

and

$$M_{\lambda,4}(x) = V_{\lambda,4}(x) + \frac{1}{2}V_\lambda\big(\theta_\lambda(t)(t-x)^2, x\big) + \frac{1}{80}V_\lambda\big(\theta_\lambda^2, x\big).$$

Proof. The first two identities follows from Proposition 2.1, since

$$\lambda\int_{I_{\lambda,k}} dt = 1 \quad \text{and} \quad \lambda\int_{I_{\lambda,k}} (a_k t)dt = \frac{\lambda k}{2k+1}\Big(\frac{(k+1)^2}{\lambda^2} - \frac{k^2}{\lambda^2}\Big) = \frac{k}{\lambda},$$

For the third equation one has (recall $a_k = (2k)/(2k+1)$)

$$M_\lambda(e_1^2, x) = \frac{\lambda}{3}\sum_{k=0}^{\infty} v_{\lambda,k}(x) a_k^2 \Big(\Big(\frac{k+1}{\lambda}\Big)^3 - \Big(\frac{k}{\lambda}\Big)^3\Big)$$
$$= \sum_{k=0}^{\infty} v_{\lambda,k}(x)\Big(\frac{k^2}{\lambda^2} + a_k^2 \frac{1}{12\lambda^2}\Big)$$
$$= x^2 + \frac{\varphi^2(x)}{n} + \frac{1}{12} V_\lambda\big(\theta_\lambda, x\big),$$

and

$$M_{\lambda,2}(x) = M_\lambda(e_1^2, x) - 2x^2 + x^2 = \frac{\varphi^2(x)}{\lambda} + \frac{1}{12} V_\lambda\big(\theta_\lambda, x\big).$$

For the third moment, first, a straightforward calculus shows that

$$\lambda a_k \int_{I_{\lambda,k}} t(a_k t - x)^2 dt = \frac{k}{\lambda}\Big(\frac{k}{\lambda} - x\Big)^2 + \frac{a_k^2}{4\lambda^2}\Big(\frac{k}{\lambda} - x\Big) + \frac{a_k^2}{12\lambda^2} x.$$

Therefore

$$M_\lambda(e_1(e_1 - x)^2, x) = V_\lambda((e_1(e_1 - x)^2, x) + \frac{1}{4} V_\lambda\Big(\theta_\lambda(t)(t - x), x\Big) + \frac{x}{12} V_\lambda\big(\theta_\lambda, x\big),$$

and

$$M_{\lambda,3}(x) = V_\lambda((e_1(e_1 - x)^2, x) + V_\lambda\big(\theta_\lambda(t)(t - x), x\big)/4 - x^2(1 + x)/\lambda$$
$$= V_{\lambda,3}(x) + x V_{\lambda,2}(x) + \frac{1}{4} V_\lambda\big(\theta_\lambda(t)(t - x), x\big) - x^2(1 + x)/\lambda$$
$$= V_{\lambda,3}(x) + \frac{1}{4} V_\lambda\big(\theta_\lambda(t)(t - x), x\big).$$

The same idea can be used to obtain the other equation. In fact, since

$$\lambda a_k \int_{I_{\lambda,k}} t(a_k t - x)^3 dt = \frac{k}{\lambda}\Big(\frac{k}{\lambda} - x\Big)^3 + \frac{a_k^2}{2\lambda^2}\Big(\Big(\frac{k}{\lambda} - x\Big)^2 + \frac{x}{2}\Big(\frac{k}{\lambda} - x\Big)\Big) + \frac{1}{80\lambda^4} a_k^4,$$

one has

$$M_\lambda(e_1(e_1 - x)^3, x) = V_\lambda((e_1(e_1 - x)^3, x) + \frac{1}{2} V_\lambda\Big(\theta_\lambda(t)(t - x)^2, x\Big)$$
$$+ \frac{x}{4} V_\lambda\Big(\theta_\lambda(t)(t - x), x\Big) + \frac{1}{80} V_\lambda\big(\theta_\lambda^2, x\big),$$
$$= V_{\lambda,4}(x) + x V_{\lambda,3}(x) + \frac{1}{2} V_\lambda\Big(\theta_\lambda(t)(t - x)^2, x\Big)$$
$$+ \frac{x}{4} V_\lambda\Big(\theta_\lambda(t)(t - x), x\Big) + \frac{1}{80} V_\lambda\big(\theta_\lambda^2, x\big),$$

and

$$M_{\lambda,4}(x) = M_\lambda(t(t - x)^3, x) - x M_\lambda((t - x)^3, x)$$
$$= M_\lambda(t(t - x)^3, x) - x V_{\lambda,3}(x) - \frac{x}{4} V_\lambda\big(\theta_\lambda(t)(t - x), x\big)$$
$$= V_{\lambda,4}(x) + \frac{1}{2} V_\lambda\Big(\theta_\lambda(t)(t - x)^2, x\Big) + \frac{1}{80} V_\lambda\big(\theta_\lambda^2, x\big). \qquad \square$$

Corollary 2.3. *For each* $\lambda > 1$ *and* $x \geq 0$, *one has*

$$M_{\lambda,2}(x) \leq \frac{13}{12}\frac{\varphi^2(x)}{\lambda} \quad \text{and} \quad M_\lambda(|\, t-x \,|, x) \leq \sqrt{\frac{13}{12}}\frac{\varphi(x)}{\sqrt{\lambda}}.$$

Moreover, if $x \geq 1/(2(\lambda+1))$, *then* $M_{\lambda,4}(x) \leq 16\varphi^4(x)/\lambda^2$.

Proof. Since $4\lambda t \leq (1+2\lambda t)^2$, one has (see (2.1))

$$\theta_\lambda(t) = \frac{1}{\lambda^2}\frac{(2\lambda t)^2}{(1+2\lambda t)^2} = \frac{\lambda t}{\lambda^2}\frac{4\lambda t}{(1+2\lambda t)^2} \leq \frac{\lambda t}{\lambda^2} = \frac{t}{\lambda}.$$

Later we need also the estimate

$$\theta_\lambda(t) = \frac{1}{\lambda^2}\frac{(2\lambda t)^2}{(1+2\lambda t)^2} \leq \frac{1}{\lambda^2}\frac{(1+2\lambda t)^2}{(1+2\lambda t)^2} = \frac{1}{\lambda^2}. \tag{2.2}$$

Therefore (see Proposition 2.2)

$$\begin{aligned} M_{\lambda,2}(x) &= \frac{\varphi^2(x)}{\lambda} + \frac{1}{12}V_\lambda\big(\theta_\lambda, x\big) \leq \frac{\varphi^2(x)}{\lambda} + \frac{1}{12\lambda}V_\lambda(t,x) \\ &= \frac{\varphi^2(x)}{\lambda} + \frac{x}{12\lambda} = \frac{\varphi^2(x)}{\lambda} + \frac{\sqrt{x^2}}{12\lambda} \leq \frac{13}{12}\frac{\varphi^2(x)}{\lambda}. \end{aligned}$$

The second inequality follows from the first one by Hölder's inequality. In fact, since M_λ is a positive operator

$$M_\lambda(|\, t-x \,|, x) \leq \sqrt{M_\lambda(1,x)M_\lambda((t-x)^2, x)}.$$

If we set $H(x) = (1+2x)^2/\varphi^2(x)$, then

$$\begin{aligned} H'(x) &= \frac{4(1+2x)(x+x^2) - (1+2x)^2(1+2x)}{(x+x^2)^2} \\ &= (1+2x)\frac{4x+4x^2-1-4x-4x^2}{(x+x^2)^2} < 0. \end{aligned}$$

Thus, if $x \geq 1/(2(\lambda+1))$,

$$H(x) \leq H\Big(\frac{1}{2(1+\lambda)}\Big) = \frac{4(1+\lambda)}{(3+2\lambda)}\Big(\frac{2+\lambda}{1+\lambda}\Big)^2 = \frac{4(2+\lambda)^2}{(3+2\lambda)}$$

On the other hand, since the function

$$\frac{2}{\lambda} + \frac{4(2+\lambda)^2}{\lambda(3+2\lambda)}$$

decreases, for $\lambda > 1$ and $x \geq 1/(2(1+\lambda))$, one has

$$\frac{2}{\lambda} + \frac{(1+2x)^2}{\lambda\varphi^2(x)} \leq \frac{2}{\lambda} + \frac{4(2+\lambda)^2}{\lambda(3+2\lambda)} \leq 2 + \frac{36}{5} < 10.$$

Thus, if $x \geq 1/(2(\lambda+1))$, then (see Proposition 2.1)

$$V_\lambda((t-x)^4, x) = \frac{\varphi^4(x)}{\lambda^2}\left(3 + \frac{2}{\lambda} + \frac{(1+2x)^2}{\lambda\varphi^2(x)}\right) \leq 13\frac{\varphi^4(x)}{\lambda^2}.$$

Also notice that

$$1 \le \frac{4(1+\lambda)^2}{3+2\lambda}x(1+x) \le 2(1+\lambda)x(1+x) \le 4\lambda x(1+x).$$

Therefore, from Proposition 2.1 one has

$$\frac{1}{2}V_\lambda\big(\theta_\lambda(t)(t-x)^2, x\big) \le \frac{1}{2\lambda^2}V_\lambda\big((t-x)^2, x\big) = \frac{\varphi^2(x)}{2\lambda^3} \le 2\frac{\varphi^4(x)}{\lambda^2}.$$

Since $2+2\lambda \le 4\lambda$,

$$\frac{1}{4\lambda} \le \frac{1}{2(1+\lambda)} \le x,$$

hence (see (2.2))

$$\frac{1}{80}V_\lambda\big(\theta_\lambda^2, x\big) \le \frac{1}{80\lambda^4} \le \frac{1}{5}\frac{x^2}{\lambda^2} \le \frac{1}{5}\frac{\varphi^4(x)}{\lambda^2}.$$

Therefore

$$M_{\lambda,4}(x) = V_{\lambda,4}(x) + \frac{1}{2}V_\lambda\Big(\theta_\lambda(t)(t-x)^2, x\Big) + \frac{1}{80}V_\lambda\big(\theta_\lambda^2, x\big) \le 16\frac{\varphi^4(x)}{\lambda^2}. \qquad \square$$

Proposition 2.4. *For each $f \in C_\varrho[0,\infty)$, $\lambda > 1$ and $x \ge 0$, one has*

$$\frac{\varphi^2(x)}{\lambda^2}M'_\lambda(f,x) = \sum_{k=0}^\infty Q_{\lambda,k}(f)\Big(\frac{k}{\lambda} - x\Big)v_{\lambda,k}(x),$$

and

$$M'_\lambda(f,x) = \lambda^2\sum_{k=0}^\infty \Big(Q_{\lambda,k+1}(f) - Q_{\lambda,k}(f)\Big)v_{\lambda+1,k}(x).$$

If $f(0) = 0$, the term corresponding to $k = 0$ should be omitted.

Proof. Since, $v_{\lambda,k}(x)$ satisfies (we set $v_{\lambda,-1} = 0$)

$$v'_{\lambda,k}(x) = \frac{k-\lambda x}{x(1+x)}v_{\lambda,k}(x) = \lambda\big(v_{\lambda+1,k-1}(x) - v_{\lambda+1,k}(x))\big),$$

we have

$$\begin{aligned} M'_\lambda(f,x) &= \lambda\sum_{k=0}^\infty Q_{\lambda,k}(f)\frac{k-\lambda x}{x(1+x)}v_{\lambda,k}(x) \\ &= \frac{\lambda^2}{x(1+x)}\sum_{k=0}^\infty Q_{\lambda,k}(f)\Big(\frac{k}{\lambda} - x\Big)v_{\lambda,k}(x) \\ &= \lambda^2\sum_{k=0}^\infty Q_{\lambda,k+1}(f)v_{\lambda+1,k}(x) - \lambda^2\sum_{k=0}^\infty Q_{\lambda,k}(f)v_{\lambda+1,k}(x) \\ &= \lambda^2\sum_{k=0}^\infty \Big(Q_{\lambda,k+1}(f) - Q_{\lambda,k}(f)\Big)v_{\lambda+1,k}(x). \qquad \square \end{aligned}$$

Proposition 2.5. *For $\lambda > 1$ and $x \ge 0$,*

$$M_\lambda\Big(\frac{1}{1+t}, x\Big) \le \frac{2\lambda}{(\lambda-1)(1+x)}.$$

Proof. For $k \geq 1$, $4k \geq 1+2k$. Therefore, if $k \geq 0$,

$$1+\frac{k}{\lambda} \leq 1+\frac{k}{\lambda}\frac{4k}{(1+2k)} \qquad \text{and} \qquad \frac{1}{1+a_k\,k/\lambda} \leq \frac{2}{1+k/\lambda}.$$

Hence

$$M_\lambda\Big(\frac{1}{1+t},x\Big) \leq \sum_{k=0}^{\infty}\frac{v_{\lambda,k}(x)}{1+a_k(k/\lambda)} \leq 2\sum_{k=0}^{\infty}\frac{\lambda}{\lambda+k}v_{\lambda,k}(x)$$

$$\leq \frac{2}{(1+x)^\lambda}\Big(1+\sum_{k=0}^{\infty}\frac{(\lambda-1)\lambda\cdots(\lambda+(k-2))}{k!(1+x)^{\lambda+k-1}}x^k\Big) \leq 2\frac{\lambda}{(\lambda-1)(1+x)}. \qquad \square$$

3. Auxiliary results

Proposition 3.1. *Assume $\gamma \in [0,2)$ and set*

$$I_\gamma(x,t) = \left|\int_x^t \frac{t-u}{\varphi^{2\gamma}(u)\varrho(u)}du\right|.$$

(i) *If $t > x > 0$ and $1/2 \leq \gamma \leq b$, then*

$$I_\gamma(x,t) \leq 2\mid t-x\mid^{3/2}(1+t)^{b-\gamma}x^{1/2-\gamma}.$$

(ii) *If $t > x > 0$, $0 \leq \gamma < 1/2$ and $b \geq \gamma$, then*

$$I_\gamma(x,t) \leq \frac{2(t-x)^{2-\gamma}}{(1+x)(1+t)^{\gamma-b-1}}.$$

(iii) *If $t > 0$, $x > 0$, $0 \leq \gamma < 2$ and $\gamma \leq b$, then*

$$I_\gamma(x,t) \leq \frac{(t-x)^2}{2-\gamma}\Big(\frac{1}{\varrho(x)\varphi^{2\gamma}(x)}+\frac{(1+t)^{b-\gamma}}{x^\gamma}\Big).$$

Proof. (i) If $t > x$ and $1/2 \leq \gamma \leq b$, then

$$\left|\int_x^t \frac{(t-u)}{u^\gamma(1+u)^{\gamma-b}}du\right| \leq \frac{(t-x)(1+t)^{b-\gamma}}{x^{\gamma-1/2}}\int_x^t \frac{1}{u^{1/2}}du$$

$$= 2\frac{(1+t)^{b-\gamma}(t-x)(\sqrt{t}-\sqrt{x})}{x^{\gamma-1/2}} \leq \frac{2(1+t)^{b-\gamma}\mid t-x\mid^{3/2}}{x^{\gamma-1/2}}.$$

(ii) If $t > x$ and $0 \leq \gamma < 1/2$, since the function $(t-u)/(1+u)$ decreases in the interval $[x,t]$, then

$$\int_x^t \frac{(t-u)}{u^\gamma(1+u)^{\gamma-b}}du = \int_x^t \frac{t-u}{(1+u)}\frac{1}{u^\gamma(1+u)^{\gamma-b-1}}du$$

$$\leq \frac{(t-x)}{(1+x)(1+t)^{\gamma-b-1}}\int_x^t \frac{du}{u^\gamma} \leq \frac{2(t-x)^{2-\gamma}}{(1+x)(1+t)^{\gamma-b-1}}.$$

(iii) First note that

$$I_\gamma(x,t) = \left|\int_x^t \frac{(t-u)}{u^\gamma}(1+u)^{b-\gamma}du\right|$$
$$\leq \left((1+x)^{b-\gamma} + (1+t)^{b-\gamma}\right)\left|\int_x^t \frac{|t-u|}{u^\gamma}du\right|.$$

Now we estimate the last integral in the expression above. If $t < x$, then, putting $u - t = \tau(x-t)$, we have

$$\left|\int_x^t \frac{|t-u|}{u^\gamma}du\right| = \int_t^x \frac{(u-t)}{u^\gamma}du = (x-t)^2\int_0^1 \frac{\tau}{\left((1-\tau)t+\tau x\right)^\gamma}d\tau$$
$$\leq \frac{(x-t)^2}{x^\gamma}\int_0^1 \tau^{1-\gamma}d\tau = \frac{(x-t)^2}{(2-\gamma)x^\gamma}.$$

If $x < t$, then

$$\left|\int_x^t \frac{|t-u|}{u^\gamma}du\right| = \int_x^t \frac{(t-u)}{u^\gamma}du \leq \frac{1}{x^\gamma}\int_x^t (t-u)du = \frac{(t-x)^2}{2x^\gamma}.$$

Hence, the inequality in (iii) holds. □

Proposition 3.2. *For each $b \geq 0$, $x \geq 0$, and $\lambda \geq 2(1+b)$, one has*

$$V_\lambda((1+t)^b, x) \leq 2^b(1+x)^b \qquad \text{and} \qquad M_\lambda((1+t)^b, x) \leq 2^{2b}(1+x)^b.$$

Moreover, for each $b \geq 0$, $x \geq 0$, and $\lambda \geq 1$, one has

$$V_\lambda((1+t)^b, x) \leq (2+b)^b(1+x)^b.$$

Proof. The case $b = 0$ is trivial, because $V_\lambda(1,x) = M_\lambda(1,x) = 1$. Thus we will assume that $b > 0$.

(i) First we prove that, for $m \in \mathbb{N}$

$$(1+k/\lambda)^m\, v_{\lambda,k}(x) \leq 2^{m-1}(1+x)^m v_{\lambda+m,k}(x). \tag{3.1}$$

If $1 \leq k < m$, from the definition of $v_{\lambda,k}(x)$, we have

$$\frac{v_{\lambda+m,k}(x)}{v_{\lambda,k}(x)} = \frac{1}{(1+x)^m}\frac{(\lambda+m)(\lambda+m+1)\cdots(\lambda+k+m-1)}{\lambda(\lambda+1)\cdots(\lambda+k-1)}$$
$$= \frac{1}{(1+x)^m}\prod_{j=0}^{k-1}\left(1+\frac{m}{\lambda+j}\right)$$

But, for $1 \leq k < m$, $1 + k/\lambda \leq 1 + m/\lambda \leq 2$ and

$$\frac{1}{2^{m-1}}\left(1+\frac{k}{\lambda}\right)^m \leq 1+\frac{k}{\lambda} \leq 1+\frac{m}{\lambda} \leq \prod_{j=0}^{k-1}\left(1+\frac{m}{\lambda+j}\right). \tag{3.2}$$

Here the condition $\lambda \geq m$ is needed.
Hence

$$\frac{1}{2^{m-1}}\left(1+\frac{k}{\lambda}\right)^m v_{\lambda,k}(x) \leq \prod_{j=0}^{k-1}\left(1+\frac{m}{\lambda+j}\right)v_{\lambda,k}(x) = (1+x)^m v_{\lambda+m,k}(x).$$

For $k \geq m$, one has the equality

$$\frac{v_{\lambda+m,k}(x)}{v_{\lambda,k}(x)} = \frac{1}{(1+x)^m} \frac{(\lambda+k)(\lambda+k+1)\cdots(\lambda+k+m-1)}{\lambda(\lambda+1)\cdots(\lambda+m-1)}$$

$$= \frac{1}{(1+x)^m} \prod_{j=0}^{m-1} \Big(1+\frac{k}{\lambda+j}\Big).$$

But for $j = 1, \cdots, m-1$,

$$\frac{1}{2}\Big(1+\frac{k}{\lambda}\Big) \leq \frac{\lambda}{\lambda+j}\Big(1+\frac{k}{\lambda}\Big) \leq \Big(1+\frac{k}{\lambda+j}\Big).$$

Therefore Hence

$$\Big(1+\frac{k}{\lambda}\Big)^m v_{\lambda,k}(x) \leq 2^{m-1} \prod_{j=0}^{m-1} \Big(1+\frac{k}{\lambda+j}\Big) v_{\lambda,k}(x) = 2^{m-1}(1+x)^m v_{\lambda+m,k}(x).$$

We have proved (3.1) and it is sufficient to verify the first inequality in Proposition 3.2 in the case of an integer b and $\lambda \geq b$.

If $b > 0$ is not an integer, let $\lceil b \rceil$ be the ceiling of b, that is the least integer greater than b. Note that $\lambda > 2(1+b) \geq 2\lceil b \rceil$. Applying Hölder's inequality we obtain

$$V_\lambda\big((1+t)^b, x\big) \leq \Big(V_\lambda\big((1+t)^{\lceil b \rceil}, x\big)\Big)^{b/\lceil b \rceil} \leq 2^{(\lceil b \rceil - 1)b/\lceil b \rceil}(1+x)^b \leq 2^b(1+x)^b.$$

(ii) Since $0 \leq a_k \leq 1$, for $k/\lambda \leq t \leq (k+1)/\lambda$, one has

$$1 + a_k t \leq 1 + (k+1)/\lambda \leq 2 + k/\lambda \leq 2(1+k/\lambda).$$

Therefore $M_\lambda((1+t)^b, x) \leq 2^b V_\lambda((1+t)^b, x)$.

(iii) The condition $\lambda \geq 2(1+b)$ was first used to prove equation (3.2). This restriction can be omitted if, for $1 \leq k < m$, we consider the inequalities

$$\Big(1+\frac{k}{\lambda}\Big)^m \leq \Big(1+\frac{m}{\lambda}\Big)^{m-1}\Big(1+\frac{m}{\lambda}\Big) \leq (1+m)^{m-1} \prod_{j=0}^{k-1} \Big(1+\frac{m}{\lambda+j}\Big).$$

Thus for an integer m equation (3.1) is replaced by

$$(1+k/\lambda)^m \, v_{\lambda,k}(x) \leq (1+m)^{m-1}(1+x)^m v_{\lambda+m,k}(x)$$

Moreover, if $b > 0$ is not an integer,

$$V_\lambda\big((1+t)^b, x\big) \leq \Big(V_\lambda\big((1+t)^{\lceil b \rceil}, x\big)\Big)^{b/\lceil b \rceil}$$

$$\leq \Big((1+\lceil b \rceil)^{\lceil b \rceil - 1}\Big)^{b/\lceil b \rceil}(1+x)^b \leq (2+b)^b(1+x)^b. \qquad \square$$

Proposition 3.3. *For $\beta \in (0, 2]$, $x > 0$ and any real λ satisfying $\lambda \geq 4$, one has*

$$\sum_{k=1}^{\infty} \Big(\frac{\lambda}{k}\Big)^\beta v_{\lambda,k}(x) \leq \frac{4^\beta}{x^\beta}. \tag{3.3}$$

Proof. (i) First note that, for $x > 0$,

$$x\Big(\frac{\lambda}{k}\Big)\frac{v_{\lambda,k}(x)}{v_{\lambda-1,k+1}(x)} = \Big(\frac{\lambda}{k}\Big)\frac{(k+1)!}{k!}\frac{\lambda(\lambda+1)\cdots(\lambda+k-1)}{(\lambda-1)(\lambda)\cdots(\lambda-1+k))}$$

$$= \Big(\frac{\lambda}{k}\Big)\frac{(k+1)!}{k!}\frac{1}{(\lambda-1)} = \frac{k+1}{k}\frac{\lambda}{\lambda-1} \le 4.$$

With similar arguments, we prove that

$$x^2\Big(\frac{\lambda}{k}\Big)^2\frac{v_{\lambda,k}(x)}{v_{\lambda-2,k+2}(x)} = \Big(\frac{\lambda}{k}\Big)^2\frac{(k+2)!}{k!}\frac{1}{(\lambda-1)(\lambda-2)} = \frac{k+1}{k}\frac{\lambda}{\lambda-1} \le 4^2.$$

Therefore

$$\sum_{k=1}^{\infty}\Big(\frac{\lambda}{k}\Big)v_{\lambda,k}(x) \le \frac{4}{x}\sum_{k=1}^{\infty}v_{\lambda-1,k+1}(x) \le \frac{4}{x} \tag{3.4}$$

and

$$\sum_{k=1}^{\infty}\Big(\frac{\lambda}{k}\Big)^2 v_{\lambda,k}(x) \le \frac{4^2}{x^2}\sum_{k=1}^{\infty}v_{\lambda-2,k+2}(x) \le \frac{4^2}{x^2}$$

(ii) Finally if $0 < \beta < 1$, using Hölder's inequality, we have

$$\sum_{k=1}^{\infty}\Big(\frac{\lambda}{k}\Big)^{\beta} v_{\lambda,k}(x) \le \Big(\sum_{k=1}^{\infty}\Big(\frac{\lambda}{k}\Big)v_{\lambda,k}(x)\Big)^{\beta} \le \frac{4^{\beta}}{x^{\beta}},$$

and, if $1 < \beta < 2$, then

$$\sum_{k=1}^{\infty}\Big(\frac{\lambda}{k}\Big)^{\beta} v_{\lambda,k}(x) \le \Big(\sum_{k=1}^{\infty}\Big(\frac{\lambda}{k}\Big)^2 v_{\lambda,k}(x)\Big)^{\beta/2} \le \frac{4^{\beta}}{x^{\beta}},$$

and this proves the result. □

Proposition 3.4. *If $\gamma \in [0,2)$, $b \ge \gamma$, $\lambda > 1$ and $k > 0$, then*

$$\int_{k/\lambda}^{(k+1)/\lambda}\frac{dt}{\varrho(a_k t)\varphi^{2\gamma}(a_k t)} \le \frac{2^b}{\lambda}\frac{1}{\varrho(k/\lambda)\varphi^{2\gamma}(k/\lambda)}.$$

When $\gamma = 0$ the inequality also holds for $k = 0$.

Proof. If $k > 0$, since $1/2 \le a_k \le 1$, then

$$\int_{k/\lambda}^{(k+1)/\lambda}\frac{(1+a_k u)^{b-\gamma}}{a_k^{\gamma}u^{\gamma}}du \le \frac{C_2}{\lambda}\frac{(1+k/n)^b}{\varphi^{\gamma}(k/n)}.$$ □

Lemma 3.5. *Assume $\gamma \in [0,2)$, $b \ge 2$ and $\lambda > 2(1+b)$. There exists a constant C such that*

$$A_{\lambda,\gamma}(x) := \sum_{k=1}^{\infty}v_{\lambda,k}(x)\int_x^{k/\lambda}\frac{du}{\varphi^{2\gamma}(u)\varrho(u)} \le C\frac{\varphi^{2(1-\gamma)}(x)}{\varrho(x)}.$$

and

$$B_{\lambda,\gamma}(x) := \sum_{k=1}^{\infty}\frac{v_{\lambda,k}(x)}{\varphi^{2\gamma}(k/\lambda)\varrho(k/\lambda)} \le C\lambda\frac{\varphi^{2(1-\gamma)}(x)}{\varrho(x)},$$

Proof. Notice that $1+k/\lambda \leq 4(1+(k-1)/(\lambda+1))$ for $k\geq 1$.

(i) If $\gamma=0$, the result follows from Proposition 3.2 taking into account that

$$A_{\lambda,0}(x) \leq \sum_{k=1}^{\infty} \frac{k}{\lambda}(1+k/\lambda)^b v_{\lambda,k}(x)$$

$$\leq 4^b x \sum_{k=1}^{\infty} \frac{(\lambda+1)\cdots(\lambda+k-1)}{\varrho((k-1)/(\lambda+1))} \frac{1}{(k-1)!} \frac{x^{k-1}}{(1+x)^{\lambda+1+k-1}}$$

$$= 4^b x V_{\lambda+1}((1+t)^b, x) \leq \frac{C_1 \varphi^2(x)}{\varrho(x)}.$$

(ii) Assume $0<\gamma<1$. Notice that

$$\int_x^{k/\lambda} \frac{du}{\varphi^{2\gamma}(u)\varrho(u)} \leq (1+k/\lambda)^{b-\gamma} \int_0^{k/\lambda} \frac{u}{u^{\gamma}} = \frac{(1+k/\lambda)^{b-\gamma}}{1-\gamma}(k/\lambda)^{1-\gamma}.$$

Hence, by Hölder inequality we obtain

$$A_{\lambda,\gamma}(x) \leq C_2 \sum_{k=1}^{\infty} \Big(\frac{k}{\lambda}\Big)^{1-\gamma} (1+k/\lambda)^{b-\gamma} v_{\lambda,k}(x)$$

$$\leq C_2 (V_\lambda(t,x))^{1-\gamma} \big(V_\lambda((1+t)^{(b-\gamma)/\gamma}, x\big)^{\gamma} = C_2 x^{1-\gamma} \big(V_\lambda((1+t)^{(b-\gamma)/\gamma}, x\big)^{\gamma},$$

where we use Proposition 2.1. It follows from the last inequality in Proposition 3.2 that

$$\big(V_\lambda((1+t)^{(b-\gamma)/\gamma}, x\big)^{\gamma} \leq C(b)(1+x)^{b-\gamma} \leq C(b)(1+x)^{b+1-\gamma}.$$

(iii) If $\gamma=1$, then

$$A_{\lambda,1}(x) \leq \frac{1}{\sqrt{x}} \sum_{k=1}^{\infty} \Big(\int_0^{k/\lambda} \frac{du}{\sqrt{u}}\Big) \frac{v_{\lambda,k}(x)}{(1+k/\lambda)^{-b}} \leq \frac{C_4}{\sqrt{x}} \sum_{k=1}^{\infty} \sqrt{\frac{k}{\lambda}} \frac{v_{\lambda,k}(x)}{(1+k/\lambda)^{-b}}$$

$$\leq \frac{C_4}{\sqrt{x}} \sqrt{V_\lambda(t,x) V_\lambda((1+t)^{2b}, x)} \leq \frac{C_5}{\varrho(x)}.$$

(iv) If $1<\gamma<2$, then

$$V_{\lambda,\gamma}(x) \leq \frac{1}{x} \sum_{k=1}^{\infty} \frac{v_{\lambda,k}(x)}{\varrho(k/\lambda)} \int_0^{k/\lambda} \frac{du}{u^{\gamma-1}} = \frac{C_6}{x} \sum_{k=1}^{\infty} \Big(\frac{k}{\lambda}\Big)^{2-\gamma} \frac{v_{\lambda,k}(x)}{\varrho(k/\lambda)}$$

$$\leq \frac{C_6}{x} \Big(\sum_{k=1}^{\infty} \Big(\frac{k}{\lambda}\Big) v_{\lambda,k}(x)\Big)^{2-\gamma} \Big(\sum_{k=1}^{\infty} \frac{v_{\lambda,k}(x)}{\varrho^{\gamma-1}(k/\lambda)}\Big)^{\gamma-1} \leq \frac{C_7 x^{1-\gamma}}{\varrho(x)}.$$

For the second inequality we also consider several cases.

(a) If $\gamma=0$, the results follows from Proposition 3.2 taking into account that

$$B_{\lambda,0}(x) = \lambda x \sum_{k=1}^{\infty} \frac{(\lambda+1)\cdots(\lambda+k-1)}{k!(1+k/\lambda)^{-b}(1+x)^{k-1+\lambda+1}} x^{k-1} \leq \lambda x V_{\lambda+1}((1+t)^b, x).$$

(b) If $0<\gamma<1$, we use Hölder inequality, Propositions 2.1 and 3.2 to obtain

$$\begin{aligned}B_{\lambda,\gamma}(x) &= \lambda\sum_{k=1}^{\infty}\frac{1}{\lambda}\left(\frac{k}{\lambda}\right)^{-\gamma}\frac{v_{\lambda,k}(x)}{(1+k/\lambda)^{\gamma-b}(x)} \le \lambda\sum_{k=1}^{\infty}\left(\frac{k}{n}\right)^{1-\gamma}\frac{v_{\lambda,k}(x)}{(1+k/\lambda)^{\gamma-b}}\\ &\le \lambda\big(V_\lambda(t,x)\big)^{1-\gamma}\big(V_\lambda((1+t)^{(b-\gamma)/\gamma},x)\big)^{\gamma} \le C_2\lambda\, x^{1-\gamma}(1+x)^{b-\gamma}.\end{aligned}$$

(c) If $\gamma=1$, the proof is simpler, because $B_{\lambda,1}(x)\le \lambda V_\lambda((1+t)^{b-1},x)$.
(d) If $1<\gamma<2$, then

$$\begin{aligned}B_{\lambda,\gamma}(x) &= \sum_{k=1}^{\infty}\frac{\lambda}{k}\Big(\frac{\lambda}{k}\Big)^{-1+\gamma}\frac{v_{\lambda,k}(x)}{(1+k/\lambda)^{\gamma-b}(x)}\\ &\le 2\lambda\sum_{k=1}^{\infty}\Big(\frac{k}{\lambda}\Big)^{1-\gamma}\frac{\lambda(\lambda+1)\cdots(\lambda+k-1)}{(1+k/\lambda)^{\gamma-b}}\frac{1}{(k+1)k!}\frac{x^k}{(1+x)^{\lambda+k}}\\ &\le 2\lambda\frac{1+x}{x}\sum_{k=1}^{\infty}\frac{1}{\lambda+k}\Big(\frac{k}{\lambda}\Big)^{1-\gamma}\frac{v_{\lambda,k+1}(x)}{(1+k/\lambda)^{\gamma-b}}\\ &\le C_1\lambda\frac{1+x}{x}\sum_{k=1}^{\infty}\Big(\frac{k+1}{\lambda}\Big)^{2-\gamma}\frac{v_{\lambda,k+1}(x)}{(1+(k+1)/\lambda)^{\gamma-b}}\\ &\le C_2\lambda\frac{1+x}{x}x^{2-\gamma}(1+x)^{b-\gamma} = C_2\lambda\frac{\varphi^{2(1-\gamma)}(x)}{\varrho(x)}.\end{aligned}$$

□

Lemma 3.6. *Assume $\gamma\in[0,2)$, $b\ge\gamma$ and $\lambda>2(1+b)$. There exists a constant C such that, if $0<x\le 1/(2\lambda)$, then*

$$C_{\lambda,\gamma}(x) := \sum_{k=1}^{\infty}v_{\lambda,k}(x)\int_{k/\lambda}^{(k+1)/\lambda}\frac{((k+1)/\lambda-u)}{\varphi^{2\gamma}(u)\varrho(u)}du \le \frac{C}{\lambda}\frac{\varphi^{2(1-\gamma)}(x)}{\varrho(x)},$$

and, if $\alpha,\beta\in[0,1]$, $\alpha+\beta\le b$, then

$$D_{\lambda,\alpha+\beta}(x) := \sum_{k=1}^{\infty}v_{n,k}(x)\int_{x}^{k/\lambda}\frac{(k/\lambda-s)ds}{\varrho(s)\varphi^{2(\alpha+\beta)}(s)} \le \frac{C\varphi^{2(1-\alpha)}(x)}{\lambda\, x^{\beta}\varrho(x)}.$$

Proof. Since the functions $\varphi^{2\gamma}(x)$ and $1/\varrho(x)$ increase,

$$\begin{aligned}C_{\lambda,\gamma}(x) &\le C_1\sum_{k=1}^{\infty}\frac{1}{\varphi^{2\gamma}(k/\lambda)\varrho(k/\lambda)}\int_{k/\lambda}^{(k+1)/\lambda}((k+1)/\lambda-u)du\, v_{\lambda,k}(x)\\ &\le \frac{C_2}{\lambda^2}\sum_{k=1}^{\infty}\frac{v_{\lambda,k}(x)}{\varphi^{2\gamma}(k/\lambda)\varrho(k/\lambda)} = \frac{C_2}{\lambda^2}B_{\lambda,\gamma}(x) \le \frac{C_3}{\lambda}\frac{\varphi^{2(1-\gamma)}(x)}{\varrho(x)}.\end{aligned}$$

If $0<x\le 1/(2\lambda)$ and $1/2\le\alpha+\beta\le b$, then

$$\begin{aligned}D_{\lambda,\alpha+\beta}(x) &\le \frac{1}{x^{\alpha+\beta-1/2}}\sum_{k=1}^{\infty}v_{n,k}(x)\frac{\mid k/\lambda-x\mid}{(1+k/n)^{\alpha+\beta-b}}\int_{x}^{k/\lambda}\frac{ds}{\sqrt{s}}\\ &\le \frac{C_1}{x^{\alpha+\beta-1/2}}\sum_{k=1}^{\infty}v_{n,k}(x)\frac{\mid k/\lambda-x\mid^{3/2}}{(1+k/n)^{\alpha+\beta-b}}\end{aligned}$$

$$\leq \frac{C_1}{x^{\alpha+\beta-1/2}(1+x)^{\alpha+\beta-b}}\big(V_\lambda(t-x)^2,x)\big)^{3/4} \leq C_2\frac{x^{3/4}}{\lambda^{3/4}x^{\alpha-1/2}x^{\beta}\varrho(x)}$$
$$= C_2\frac{x}{\lambda^{3/4}x^{\alpha}x^{\beta}\varrho(x)}x^{1/4} \leq \frac{C_2\varphi^{2(1-\alpha)}(x)}{\lambda\, x^{\beta}\varrho(x)}.$$

If $0 < x \leq 1/(2\lambda)$ and $\alpha+\beta < 1/2 \leq b$, from (ii) in Proposition 3.1 and Hölder inequality we obtain

$$D_{\lambda,\alpha+\beta}(x) \leq \frac{2}{(1+x)}\sum_{k=1}^{\infty} v_{n,k}(x)\frac{(k/\lambda-x)^{2-\alpha-\beta}}{(1+k/n)^{\alpha+\beta-b-1}}$$
$$\leq \frac{C_3}{(1+x)}\Big(\frac{x}{\lambda}\Big)^{(2-\alpha-\beta)/2}\frac{1}{(1+x)^{\alpha+\beta-b-1}} \leq C_3\frac{x^{\beta/2}\lambda^{\beta/2}}{\lambda^{1-\alpha/2}}\frac{x^{1-\alpha}x^{\alpha/2}}{(1+x)^{\alpha}x^{\beta}\varrho(x)}. \qquad \square$$

Proposition 3.7. *Assume $\alpha,\beta \in [0,1]$, $\alpha+\beta < 2 \leq b$. There exists a constant C such that, for $\lambda > 2(1+b)$ and $x > 0$, one has*

$$x^{\beta}\varrho(x)M_\lambda(|\, I_{\alpha+\beta}(x,t)\,|,x) \leq C\frac{\varphi^{2(1-\alpha)}(x)}{\lambda}.$$

Proof. (a) First we consider the case $0 \leq x \leq 1/(2\lambda)$. When $k = 0$, since $a_k = 0$,

$$x^{\beta}\varrho(x)\left|\lambda v_{\lambda,0}(x)\int_0^{1/\lambda}\int_0^x \frac{s}{\varphi^{2(\alpha+\beta)}(s)\varrho(u)}dsdt\right|$$
$$= x^{\beta}\int_0^x \frac{s^{1-\alpha-\beta}(1+s)^{b-\alpha-\beta}}{(1+x)^{\lambda+b}}\,ds$$
$$\leq \frac{x^{\beta}}{(1+x)^{\alpha+\beta}}\int_0^x s^{1-\alpha-\beta}ds \leq C_1\frac{x^{2-\alpha}}{(1+x)^{\alpha}} \leq C_2\frac{1}{\lambda}\varphi^{2(1-\alpha)}(x).$$

If $k > 0$ and $x \leq 1/(2\lambda)$, then $x < a_k t \leq t$, for $t \in I_{\lambda,k}$. Hence

$$\lambda\int_{I_{\lambda,k}}\Big|\int_x^{a_k t}\frac{a_k t-u}{\varphi^{2\gamma}(u)\varrho(u)}du\Big|dt \leq \int_x^{(k+1)/\lambda}\frac{((k+1)/\lambda-u)}{\varphi^{2\gamma}(u)\varrho(u)}du$$
$$= \int_{k/\lambda}^{(k+1)/\lambda}\frac{(k+1)/\lambda-u)du}{\varphi^{2\gamma}(u)\varrho(u)} + \frac{1}{\lambda}\int_x^{k/\lambda}\frac{du}{\varphi^{2\gamma}(u)\varrho(u)} + \int_x^{k/\lambda}\frac{(k/\lambda-u)du}{\varphi^{2\gamma}(u)\varrho(u)}.$$

If $R^*_{\lambda,\alpha,\beta}(x) = x^{\beta}\varrho(x)M^*_\lambda(|\, I_\lambda(x,t)\,|,x)$ (M^*_λ means that we omit the term corresponding to $k=0$), then

$$R^*_{n,\alpha,\beta}(x) \leq x^{\beta}\varrho(x)\Big(\frac{1}{\lambda}A_{\lambda,\alpha+\beta}(x) + C_{\lambda,\alpha+\beta}(x) + D_{\lambda,\alpha+\beta}(x)\Big),$$

and the result follows from the estimates given in Lemmas 3.5 and 3.6, with $\gamma = \alpha+\beta$.

(b) Now assume $x > 1/(2\lambda)$. By (iii) in Proposition 3.1 and Corollary 2.3, one has

$$x^{\beta}\varrho(x)M_\lambda\Big(\Big|\int_x^t \frac{t-u}{\varphi^{2(\alpha+\beta)}(u)\varrho(u)}du\Big|,x\Big)$$
$$\leq C_1 x^{\beta}\frac{M_\lambda((t-x)^2,x)}{\varphi^{2(\alpha+\beta)}(x)} + C_2\frac{\varrho(x)}{x^{\alpha}}M_\lambda\Big((t-x)^2(1+t)^{b-\alpha-\beta},x\Big)$$

$$\leq C_3 \frac{\varphi^{2(1-\alpha)}(x)}{\lambda} + C_2 \frac{\varrho(x)}{x^\alpha} \sqrt{M_\lambda((t-x)^4, x) M_\lambda((1+t)^{2(b-\alpha-\beta)} x)}$$
$$\leq C_4 \Big(\frac{\varphi^{2(1-\alpha)}(x)}{\lambda} + \frac{\varrho(x)}{x^\alpha} \sqrt{\frac{\varphi^4(x)}{\lambda^2}} (1+x)^{b-\alpha} \Big) \leq C_5 \frac{\varphi^{2(1-\alpha)}(x)}{\lambda}. \quad \square$$

4. Which functions can be approximated?

For $\phi(x) = \ln(1+x)$ set

$$q_\lambda = \sup_{x \geq 0} M_\lambda(|\phi(t) - \phi(x)|, x) \text{and} \quad r_\lambda = \sup_{x \geq 0} \varrho(x) M_\lambda(| \, 1/\varrho(t) - 1/\varrho(x) \, |, x).$$

Proposition 4.1. *For $b \geq 1$ there exists a constant C such that, for $\lambda > 2$,*

$$q_\lambda \leq \frac{2}{\sqrt{\lambda - 1}} \quad \text{and} \quad r_\lambda \leq \frac{C}{\sqrt{\lambda}}.$$

Proof. (i) For any $x, t \in [0, \infty)$, using the inequality $| \ln c - \ln d | \leq | c - d | / \sqrt{cd}$ for $c, d > 0$ (see [6, page 40]), we obtain

$$| \ln(1+x) - \ln(1+t) | \leq | x - t | / (\sqrt{(1+t)(1+x)}).$$

Hence (see Corollary 2.3 and Proposition 2.5)

$$M_\lambda(| \phi(t) - \phi(x)|, x) \leq \frac{1}{\sqrt{1+x}} \sqrt{M_\lambda((t-x)^2, x) M_\lambda\big((1+t)^{-1}, x\big)}$$
$$\leq \frac{1}{\sqrt{1+x}} \sqrt{2 \frac{\varphi^2(x)}{\lambda} \frac{2\lambda}{(\lambda-1)(1+x)}} = \frac{2}{\sqrt{1+x}} \sqrt{\frac{x}{\lambda - 1}}.$$

This provides the estimate for q_λ.
Note that, for $t \in I_{\lambda,k}$ and $x \geq 0$,

$$| \, t - x \, | \leq \max\{| \, k/\lambda - x \, |, | \, (k+1)/\lambda - x \, |\} \leq 1/\lambda + | \, x - k/\lambda \, |$$

and

$$\frac{(1-a_k)t}{1+a_k t} = \frac{1}{2k+1} \frac{t}{1+a_k t} \leq \frac{1}{(2k+1)} \frac{k+1}{\lambda} \leq \frac{1}{\lambda}.$$

Hence taking into account Proposition 2.1, one has

$$M_\lambda\Big(\Big|\frac{1}{1+t} - \frac{1}{1+x}\Big|, x\Big) = \lambda \sum_{k=0}^\infty v_{\lambda,k}(x) \int_{I_{\lambda,k}} \Big|\frac{1}{1+a_k t} - \frac{1}{1+x}\Big| dt$$
$$= \frac{\lambda}{1+x} \sum_{k=0}^\infty v_{\lambda,k}(x) \Big(\int_{I_{\lambda,k}} \Big| \frac{(a_k - 1)t}{1+a_k t} + \frac{t-x}{1+a_k t} \Big| dt \Big)$$
$$\leq \frac{1}{1+x} \sum_{k=0}^\infty \Big(\frac{2}{\lambda} + \Big| \frac{k}{\lambda} - x \Big| \Big) v_{\lambda,k}(x) \leq \frac{2}{\lambda(1+x)} + \frac{\sqrt{x(1+x)}}{\sqrt{\lambda}(1+x)} \leq \frac{3}{\sqrt{\lambda}}.$$

Finally, if $x \geq 0$ and $t \in I_{\lambda,k}$, using the mean value theorem, we know that there exists a point between x and $a_k t$ such that

$$| \, (1 + a_k t)^b - (1+x)^b \, | \leq b(1+\theta)^{b-1} \, | \, a_k t - x \, |$$

$$\le b\big((1+(k+1)/\lambda)^{b-1}+(1+x)^{b-1}\big)\big((1-a_k)t+\mid t-x\mid\big)$$
$$\le C_1\big((1+k/\lambda)^{b-1}+(1+x)^{b-1}\big)\big(2/\lambda+\mid x-k/\lambda\mid\big).$$

Hence, from Propositions 2.1 and 3.2,

$$M_\lambda(\mid 1/\varrho(t)-1/\varrho(x)\mid,x)\le \frac{C_2}{\lambda}V_\lambda((1+t)^{b-1},x)+\frac{2}{\lambda}(1+x)^{b-1}$$
$$+C_1V_\lambda((1+t)^{b-1}\mid t-x\mid,x)+C_1(1+x)^{b-1}V_\lambda(\mid t-x\mid,x)$$
$$\le C_3\Big(\frac{1}{\lambda}+\sqrt{V_\lambda((t-x)^2,x)}\Big)(1+x)^{b-1}$$
$$\le C_4\Big(\frac{1}{\sqrt{\lambda}}+\sqrt{x(1+x)}\Big)\frac{1}{\sqrt{\lambda}}(1+x)^{b-1}\le \frac{C_5(1+x)^b}{\sqrt{\lambda}}. \qquad \square$$

For a continuous bounded function $f:[0,\infty)\to\mathbb{R}$ and $\phi(x)=\ln(1+x)$ define

$$\Omega_\phi(f,\delta)=\sup_{t,x\in[0,\infty),|\phi(t)-\phi(x)|\le\delta}\mid f(t)-f(x)\mid.$$

Proposition 4.2. (see [5]) *Let $f\in C[0,\infty)$ be a bounded function, $\phi(x)=\ln(1+x)$ and ϕ^{-1} be the inverse function.*

(i) *For any $\delta>0$, $\Omega_\phi(f,\delta)=\omega(f\circ\phi^{-1},\delta)$, where ω is the usual first modulus of continuity.*

(ii) *The function $f\circ\phi^{-1}$ is uniformly continuous on $[0,\infty)$ if and only if for any sequence $\delta_n\to 0$ of positive numbers one has $\Omega_\phi(f,\delta_n)\to 0$.*

(iii) *For any $\delta>0$ and $x,t\in[0,\infty)$,*

$$|f(t)-f(x)|\le\Big(1+\frac{\mid\phi(t)-\phi(x)\mid}{\delta}\Big)\Omega_\phi(f,\delta).$$

Theorem 4.3. *Assume $b\ge 1$, $\phi(x)=\ln(1+x)$, and ϕ^{-1} is the inverse function. For a function $f\in C_{\varrho,0}[0,\infty)$ one has $\lim_{\lambda\to\infty}\|\varrho(M_\lambda(f)-f)\|=0$ if and only if the function $(\varrho f)\circ\phi^{-1}$ is uniformly continuous on $[0,\infty)$.*

Proof. Let q_n and r_n be given as above. Assume $(\varrho f)\circ\phi^{-1}$ is uniformly continuous. From (iii) in Proposition 4.2 we know that, for any $\delta>0$,

$$|f(t)-f(x)|\le\mid(\varrho f)(t)\mid\Big|\frac{1}{\varrho(t)}-\frac{1}{\varrho(x)}\Big|+\frac{1}{\varrho(x)}\cdot\mid(\varrho f)(t)-(\varrho f)(x)\mid\,|$$
$$\le\|\varrho f\|\cdot\Big|\frac{1}{\varrho(t)}-\frac{1}{\varrho(x)}\Big|+\frac{1}{\varrho(x)}\Big(1+\frac{\mid\phi(x)-\phi(t)\mid}{\delta}\Big)\Omega_\phi(\varrho f,\delta).$$

Therefore

$$\varrho(x)\mid M_\lambda(f,x)-f(x)\mid\le\|\varrho f\|\,\varrho(x)\,M_\lambda\Big(\Big|\frac{1}{\varrho(t)}-\frac{1}{\varrho(x)}\Big|,x\Big)$$
$$+\Big(1+\frac{1}{q_\lambda}M_\lambda(\mid\phi(x)-\phi(t)\mid,x)\Big)\Omega_\phi(\varrho f,q_\lambda)\le r_\lambda\|\varrho f\|+2\Omega_\phi(\varrho f,q_\lambda).$$

Since $r_\lambda,q_\lambda\to 0$ as $\lambda\to\infty$ (Proposition 4.1), if we assume that $(\varrho f)\circ\phi^{-1}$ is uniformly continuous on $[0,\infty)$, it follows from Proposition 4.2 that $\Omega_\phi(\varrho f,q_\lambda)\to 0$ as $\lambda\to\infty$. We have proved that $\|\varrho(M_\lambda(f)-f)\|\to 0$.

Now assume $\lim_{\lambda\to\infty} \|\varrho(M_\lambda(f) - f)\| = 0$. Taking into account Proposition 4.2, it is sufficient to prove that $\Omega_\phi(\varrho f, 1/\lambda^2) \to 0$, as $\lambda \to \infty$. By using the properties of the first modulus of continuity one has

$$\Omega_\phi\big(\varrho f, 1/\lambda^2\big) \le 2\|\varrho(f - M_\lambda(f))\| + \Omega_\phi\big(\varrho M_\lambda(f), 1/\lambda^2\big).$$

It remains to prove that the second term goes to zero. By definition we should estimate the difference $\mid \varrho(x)M_\lambda(f,x) - \varrho(y)M_\lambda(f,y) \mid$ for all point x, y satisfying

$$\mid \phi(x) - \phi(y) \mid \le 1/\lambda^2.$$

Case 1. If $0 \le y < x \le 1/\lambda$ and $\mid \phi(x) - \phi(y) \mid \le 1/\lambda^2$, there exists a point θ between x and y such that

$$\mid x - y \mid /2 \le \mid x - y \mid /(1+\theta) \le \mid \ln(1+x) - \ln(1+y) \mid \le 1/\lambda^2.$$

Therefore

$$\begin{aligned}
\mid \varrho(x)M_\lambda(f,x) - \varrho(y)M_\lambda(f,y) \mid &\le 2\|\varrho(f - M_\lambda(f)\| + \mid (\varrho f)(x) - (\varrho f)(y) \mid \\
&\le 2\|\varrho(f - M_\lambda(f)\| + \|\varrho\|_{[0,1]} \mid f(x) - f(y) \mid + \|f\|_{[0,1]} \mid \varrho(x) - \varrho(y) \mid \\
&\le 2\|\varrho(f - M_\lambda(f)\| + \|\varrho\|_{[0,1]}\omega(f, x-y)_{[0,1]} + \|f\|_{[0,1]}\omega(\varrho, x-y)_{[0,1]} \\
&\le 2\|\varrho(f - M_\lambda(f)\| + \|\varrho\|_{[0,1]}\omega\Big(f, \frac{2}{\lambda^2}\Big)_{[0,1]} + \|f\|_{[0,1]}\omega\Big(\varrho, \frac{2}{\lambda^2}\Big)_{[0,1]}, \qquad (4.1)
\end{aligned}$$

where the usual modulus of continuity is computed in the interval $[0,1]$.
Case 2. If $0 \le y < 1/\lambda < x$, we consider the inequality

$$\mid (\varrho f)(x) - (\varrho f)(y) \mid \le \mid (\varrho f)(y) - (\varrho f)(1/\lambda) \mid + \mid (\varrho f)(1/\lambda) - (\varrho f)(x) \mid . \qquad (4.2)$$

The first term was estimated in Case 1, the second one will be considered in Case 3.
Case 3. Assume that $1/\lambda \le y < x$. From the Cauchy mean value theorem, for any point $x, y \in (0,\infty)$, there is z between x and y, such that

$$\frac{\varrho(y)M_\lambda(f,y) - \varrho(x)M_\lambda(f,x)}{\phi(y) - \phi(x)} = (1+z)\Big(\varrho'(z)M_\lambda(f,z) + \varrho(z)M'_\lambda(f,z)\Big).$$

It is easy to see that (see Proposition 3.4)

$$(1+z) \mid \varrho'(z)M_\lambda(f,z) \mid \le C_3\varrho(z)\|\varrho f\| \sum_{k=0}^{\infty} \frac{v_{\lambda+1,k}(z)}{(1 + k/(\lambda+1))^{-b}} \le C_4\|\varrho f\|.$$

On the other hand, from Propositions 2.4 and 3.4 we obtain

$$\begin{aligned}
z(1+z) \mid M'_\lambda(f,z) \mid &= \lambda^2 \Big| \sum_{k=0}^{\infty} \Big(\frac{k}{\lambda} - z\Big) v_{\lambda,k}(z) \int_{I_{\lambda,k}} f(a_k t)dt \Big| \\
&\le C_5\lambda\|\varrho f\|_\infty \sum_{k=0}^{\infty} \Big|\frac{k}{\lambda} - z\Big| \frac{1}{\varrho(k/n)} v_{\lambda,k}(z) \\
&\le C_6\lambda\|\varrho f\|_\infty \frac{\varphi(z)}{\sqrt{\lambda}} \frac{1}{\varrho(z)}.
\end{aligned}$$

Thus if $z \ge 1$, then

$$(1+z) \mid \varrho(z)M'_\lambda(f,z) \mid \le C_6\|\varrho f\| \sqrt{(1+z)/z}\sqrt{\lambda} \le C_7\sqrt{\lambda}\|\varrho f\|.$$

On the other hand, from the other equation in Proposition 2.4.

$$(1+z) \mid \varrho(z)M'_\lambda(f,z) \mid \leq C\lambda\|\varrho f\|\varrho(z)\sum_{k=0}^{\infty}\frac{v_{\lambda+1,k}(z)}{\varrho(k/(\lambda+1))}.$$

Therefore, for $1/\lambda \leq z \leq 1$, $(1+z) \mid \varrho(z)M'_\lambda(f,z) \mid \leq C_4\,\lambda\|\varrho f\|$.
We have proved that, for $\lambda > 1$ and $1/\lambda \leq z$,

$$(1+z) \mid \varrho(z)M'_\lambda(f,z) \mid \leq C_8\lambda\|\varrho f\|.$$

Thus if $\mid \phi(y)-\phi(x) \mid \leq 1/\lambda^2$ and $1/\lambda \leq y < x$, then

$$\mid \varrho(y)M_\lambda(f,y)-\varrho(x)M_\lambda(f,x) \mid \leq \frac{C_9(1+\lambda)}{\lambda^2}\|\varrho f\| \leq \frac{C_{10}}{\lambda}\|\varrho f\|. \qquad (4.3)$$

From (4.1)-(4.3) we know that if $x,y \geq 0$ and $\mid \phi(x)-\phi(y) \mid \leq 1/\lambda^2$, then

$$\mid \varrho(y)M_\lambda(f,y)-\varrho(x)M_\lambda(f,x) \mid$$

$$\leq C(f,\varrho)\Big(\|\varrho(f-M_\lambda(f)\| + \omega\Big(f,\frac{2}{\lambda^2}\Big)_{[0,1]} + \omega\Big(\varrho,\frac{2}{\lambda^2}\Big)_{[0,1]} + \frac{2}{\lambda}\|\varrho f\|\Big). \qquad \square$$

5. Main results

In Theorem 5.1 we estimate the norm of the operator M_λ.

Theorem 5.1. *If $\beta \in [0,1]$ and $b \geq \beta$, there exists a constant C such that, for all $\lambda > 2(1+b)$ and every $f \in C_{\varrho,\beta}[0,\infty)$, one has $\|\varrho\varphi^{2\beta}M_\lambda(f)\| \leq C\|\varrho\varphi^{2\beta}f\|$.*

Proof. First we consider the case $0 < \beta \leq 1$. If $f \in C_{\varrho,\beta}[0,\infty)$, and $x > 0$, we use Proposition 3.4 to obtain

$$(\varrho\varphi^{2\beta})(x) \mid M_\lambda(f,x) \mid \leq (\varrho\varphi^{2\beta})(x)\|\varrho\varphi^{2\beta}f\|\lambda\sum_{k=1}^{\infty}\int_{I_{\lambda,k}}\frac{dt}{\varrho(a_kt)\varphi^{2\beta}(a_kt)}\,v_{\lambda,k}(x)$$

$$\leq C_1(\varrho\varphi^{2\beta})(x)\|\varrho\varphi^{2\beta}f\|\sum_{k=1}^{\infty}\frac{v_{\lambda,k}(x)}{\varrho(k/\lambda)\varphi^{2\beta}(k/\lambda)}$$

$$\leq C_1(\varrho\varphi^{2\beta})(x)\|\varrho\varphi^{2\beta}f\|\Big(\sum_{k=1}^{\infty}(\lambda/k)^{2\beta}v_{\lambda,k}(x)\Big)^{1/2}\Big(\sum_{k=1}^{\infty}(1+k/\lambda)^{2(b-\beta)}v_{\lambda,k}(x)\Big)^{1/2}$$

$$\leq C_2(\varrho\varphi^{2\beta})(x)\|\varrho\varphi^{2\beta}f\|\frac{(1+x)^{b-\beta}}{x^\beta} \leq C_2\|\varrho\varphi^{2\beta}f\|,$$

where we use Propositions 3.3 and 3.2.
The case $\beta = 0$ follows analogously (we do not need to use Proposition 3.3. $\square$
In the main result we use the following notations. For $\alpha,\beta \in [0,1]$ set

$$K_{\alpha,\beta}(f,t)_\varrho = \inf\big\{\|\varrho\varphi^{2\beta}(f-g)\| + t\|\varrho\varphi^{2(\alpha+\beta)}g''\| : g \in D(\alpha,\beta,\varrho)\big\},$$

where $D(\alpha,\beta,\varrho) = \{g \in C_{\varrho,\beta} : g' \in AC_{loc} : \|\varrho\varphi^{2(\alpha+\beta)}g''\| < \infty\}$.

Theorem 5.2. *If $\alpha, \beta \in [0,1]$, $\alpha+\beta < 2 \leq b$, then there exists a constant C such that, for all $\lambda > 2(1+b)$, every $f \in C_{\varrho,\beta}[0,\infty)$ and $x > 0$,*

$$\varrho(x)\varphi^{2\beta}(x) \mid M_\lambda(f,x) - f(x) \mid \leq CK_{\alpha,\beta}\Big(f, \frac{\varphi^{2(1-\alpha)}(x)}{n}\Big)_\varrho.$$

Proof. We know that the operators $M_\lambda : C_{\varrho,\beta}[0,\infty) \to C_{\varrho,\beta}[0,\infty)$ are uniformly bounded. If $x > 0$ and $g \in C^2_\varrho[0,\infty)$, use the representation

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t g''(u)(t-u)du.$$

Therefore, by setting $W(g) = \|\varrho\varphi^{2(\alpha+\beta)}g''\|$, it follows from Proposition 3.7 that

$$\begin{aligned}
&(\varrho\varphi^{2\beta})(x) \mid M_\lambda(g,x) - g(x) \mid \\
&\quad \leq (\varrho\varphi^{2\beta})(x)\lambda \sum_{k=0}^{\infty} v_{\lambda,k}(x) \int_{I_{\lambda,k}} \left| \int_x^{a_k t} g''(u)(a_k t - u)du \right| dt \\
&\quad \leq (\varrho\varphi^{2\beta})(x)W(g) \sum_{k=0}^{\infty} v_{\lambda,k}(x) \int_{I_{\lambda,k}} \left| \int_x^{a_k t} \frac{a_k t - u}{\varrho(u)\varphi^{2(\alpha+\beta)}(u)} du \right| dt \\
&\quad = (\varrho\varphi^{2\beta})(x)W(g)\, M_\lambda\left(\left| \int_x^t \frac{\mid t-u \mid du}{\varphi^{2(\alpha+\beta)}(u)\varrho(u)} \right|, x \right) \leq CW(g)\frac{\varphi^{2(1-\alpha)}(x)}{\lambda}.
\end{aligned}$$

By the definition of the K-functional we obtain

$$\varrho(x)\varphi^{2\beta}(x) \mid M_\lambda(f,x) - f(x) \mid \leq CK_{\alpha,\beta}\big(f, \frac{\varphi^{2(1-\alpha)}(x)}{\lambda}\big)_\varrho. \qquad \square$$

Remark 5.3. Theorem 5.2 combine pointwise estimates ($\alpha \in [0,1)$) with norm estimates ($\alpha = 1$). When $\beta = 0$, we pass to usual approximation in polynomial-type weighted spaces, in such a case, taking into account Theorem 6.1.1 of [3], the result can be written as: There exists a constant C such that, for all $\lambda > 2(1+b)$, every $f \in C_\varrho[0,\infty)$, and $x \geq 0$,

$$\varrho(x) \mid f(x) - M_\lambda(f,x) \mid \leq C\omega_{\varphi^\alpha}\big(f, \varphi^{(1-\alpha)}(x)/\sqrt{\lambda}\big)_\varrho,$$

where $\omega_{\varphi^\alpha}(f,t)_\varrho = \sup\{ \mid \varrho(x)\Delta^2_{h\varphi^\alpha(x)} f(x) \mid;\ 0 < h \leq t,\ x \geq h\varphi^\alpha(x)\}$.

In particular, if we chose $\alpha = 1$, then $\|\varrho(M_\lambda(f) - f)\| \to 0$, as $n \to \infty$, if $\omega_{\varphi^\alpha}(f,t)_\varrho \to 0$, as $t \to 0$.

Acknowledgment. The author would like to express their gratitude to the referee(s) for their careful reading of the paper.

References

[1] Agratini, O., *Kantorovich-type operators preserving affine functions*, Hacet. J. Math. Stat., **45**(6)(2016), 1657-1663.

[2] Bustamante, J., Carrillo-Zentella, A., Quesada, J.M., *Direct and strong converse theorems for a general sequence of positive linear operators*, Acta Math. Hungar., **136**(2012), no. 1-2, 90-106.

[3] Ditzian, Z., Totik, V., *Moduli of Smoothness*, Springer, New York, 1987.

[4] Feng, G., *Direct and inverse approximation theorems for Baskakov operators with the Jacobi-type weight*, Abstr. Appl. Anal., (2011), Art. ID 101852, 13 pages.

[5] Holhoş, A., *Uniform weighted approximation by positive linear operators*, Stud. Univ. Babeş-Bolyai Math., **56**(2011), no. 3, 135-146.

[6] Mitrinović, D.S., Pečarić, J., Fink, A.M., *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht, 1993.

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Stud. Univ. Babeş-Bolyai Math. 66(2021), No. 4, 757–768
DOI: 10.24193/subbmath.2021.4.12

Extended local convergence for Newton-type solver under weak conditions

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Abstract. We present the local convergence of a Newton-type solver for equations involving Banach space valued operators. The eighth order of convergence was shown earlier in the special case of the $k-$dimensional Euclidean space, using hypotheses up to the eighth derivative although these derivatives do not appear in the method. We show convergence using only the first derivative. This way we extend the applicability of the methods. Numerical examples are used to show the convergence conditions. Finally, the basins of attraction of the method, on some test problems are presented.

Mathematics Subject Classification (2010): 65F08, 37F50, 65N12.

Keywords: Banach space, Newton-type, local convergence, Fréchet derivative.

1. Introduction

Let $\Omega \subset \mathcal{B}_1$ be nonempty, open, and $\mathcal{B}_1, \mathcal{B}_2$ be Banach spaces.

$$\mathcal{B}(\mathcal{B}_1, \mathcal{B}_2) = \{G : \mathcal{B}_1 \longrightarrow \mathcal{B}_2 \text{ be bounded and linear}\},$$

$$T(x, d) = \{y \in \mathcal{B}_1 : \|y - x\| < d; d > 0\}$$

and

$$\bar{T}(x, d) = \{y \in \mathcal{B}_1 : \|y - x\| \leq d; d > 0\}.$$

One of the greatest challenges in Computational Mathematics is to find a solution x_* of the equation

$$\mathcal{F}(x) = 0, \tag{1.1}$$

where $\mathcal{F} : \Omega \longrightarrow \mathcal{B}_2$ is Fréchet differentiable operator. Notice that a plethora of applications from Mathematics, Science and Engineering are reduced to a form as (1.1) by utilizing Mathematical modeling [1-19]. The solution x_* is sought in closed form, but this can be achieved only in some cases. Hence, researchers develop iterative methods, generating a sequence approximating x_* under certain initial conditions.

Newton's is clearly the most popular method converging quadratically to x_*, and given as

$$x_{n+1} = x_n - \mathcal{F}'(x_n)^{-1}\mathcal{F}(x) \text{ for all } n = 0, 1, 2, \dots, \tag{1.2}$$

with $x_0 \in \Omega$. Chebyshev, Halley methods have been studied extensively which are of order three but use the expensive $\mathcal{F}''$ at each step as well as $\mathcal{F}'(x_n)^{-1}$. That is why researchers have introduced methods using divided differences of order one such as Secant, Steffensen, Kurchatov, and Aitken methods [2, 3, 4, 10, 13, 14].

In particular, we are concerned with the local convergence of the Newton-type method given as

$$\begin{aligned} x_0 \in \Omega,\ y_n &= x_n - \mathcal{F}'(x_n)^{-1}\mathcal{F}(x_n) \\ z_n &= y_n - [\frac{13}{4}I - Q(x_n)(\frac{7}{2}I - \frac{5}{4}Q(x_n))]\mathcal{F}'(x_n)^{-1}\mathcal{F}(y_n) \\ x_{n+1} &= z_n - [\frac{7}{2}I - Q(x_n)(4I - \frac{3}{2}Q(x_n))]\mathcal{F}'(x_n)^{-1}\mathcal{F}(z_n), \end{aligned} \tag{1.3}$$

where $Q(x_n) = \mathcal{F}'(x_n)^{-1}\mathcal{F}(y_n)$. Method (1.3) was studied in [17], but for the case $\mathcal{B}_1 = \mathcal{B}_2 = \mathbb{R}^k$ (k a natural number). The convergence order was established by conditions on high order derivative, and Taylor series, although these derivatives do not appear in the method (1.3). Therefore, these hypotheses limit the usage of the method (1.3).

As an academic example: Let $\mathcal{B}_1 = \mathcal{B}_2 = \mathbb{R}$, $\Omega = [-\frac{1}{2}, \frac{3}{2}]$. Define $\mathcal{F}$ on Ω by

$$\mathcal{F}(x) = x^3 \log x^2 + x^5 - x^4$$

Then, we have $x_* = 1$, and

$$\mathcal{F}'(x) = 3x^2 \log x^2 + 5x^4 - 4x^3 + 2x^2,$$

$$\mathcal{F}''(x) = 6x \log x^2 + 20x^3 - 12x^2 + 10x,$$

$$\mathcal{F}'''(x) = 6 \log x^2 + 60x^2 - 24x + 22.$$

Obviously $\mathcal{F}'''(x)$ is not bounded on Ω. So, the convergence of method (1.3) not guaranteed by the analysis in [15], [17], [18].

Other problems with the usage of the method (1.3) are: no information on how to choose the initial point x_0; bounds on $\|x_n - x_*\|$ and information on the location of x_*. All these are addressed in this paper by only using conditions on the first derivative, and in the more general setting of Banach space valued operators. That is how, we expand the applicability of the method (1.3). To avoid the usage of Taylor series and high convergence order derivatives, we rely on the computational order of convergence (COC) or the approximate computational order of convergence (ACOC) [2, 3, 4].

The layout of the rest of the paper includes: the local convergence analysis in Section 2, some numerical examples in Section 3 and the basins of attraction in Section 4.

2. Local convergence analysis

We shall introduce some scalar functions and parameters that appear in the local convergence analysis of the method (1.3). Let $\varphi_0 : [0,\infty) \longrightarrow [0,\infty)$ be an increasing and continuous function satisfying $\varphi_0(0) = 0$.

Suppose that the equation

$$\varphi_0(t) = 1 \tag{2.1}$$

has at least one positive solution. Denote by ρ_0 the smallest such solution. Let $\varphi : [0,\rho_0) \longrightarrow [0,\infty)$ and $\varphi_1 : [0,\rho_0) \longrightarrow [0,\infty)$ be increasing and continuous functions with $\varphi(0) = 0$. Define functions ψ_1 and $\bar{\psi}_1$ on the interval $[0,\rho_0)$ by

$$\psi_1(t) = \frac{\int_0^1 \varphi((1-\theta)t)d\theta}{1-\varphi_0(t)}$$

and

$$\bar{\psi}_1(t) = \psi_1(t) - 1.$$

We have $\bar{\psi}_1(0) = -1$ and $\bar{\psi}_1(t) \longrightarrow \infty$ as $t \longrightarrow \rho_0^-$. The intermediate value theorem assures that the equation $\bar{\psi}_1(t) = 0$ has at least one solution in $(0,\rho_0)$. Denote by r_1 the smallest such solution.

Suppose that the equation

$$\varphi_0(\psi_1(t)t) = 1 \tag{2.2}$$

has at least one positive solution. Denote by ρ_1 the smallest such solution. Set $\rho_2 = \min\{\rho_0, \rho_1\}$. Define the functions ψ_2 and $\bar{\psi}_2$ on $[0,\rho_2)$ by

$$\begin{aligned} \psi_2(t) = \Bigg\{ & \frac{\int_0^1 \varphi((1-\theta)\psi_1(t)t)d\theta}{1-\varphi_0(\psi_1(t)t)} \\ & + \frac{(\varphi_0(\psi_1(t)t) + \varphi_0(t)) \int_0^1 \varphi_1(\theta\psi_1(t)t)d\theta}{(1-\varphi_0(\psi_1(t)t))(1-\varphi_0(t))} \\ & + \frac{1}{4}\left[\frac{9(\varphi_0(\psi_1(t)t) + \varphi_0(t))}{1-\varphi_0(t)} \right. \\ & \left. + \frac{5\varphi_1(\psi_1(t)t)(\varphi_0(\psi_1(t)t) + \varphi_0(t))}{(1-\varphi_0(t))^2}\right] \\ & \times \frac{\int_0^1 \varphi_1(\theta\psi_1(t)t)d\theta}{1-\varphi_0(t)} \Bigg\} \psi_1(t), \end{aligned}$$

and $\bar{\psi}_2(t) = \psi_2(t) - 1$. We get $\bar{\psi}_2(0) = -1$ and $\bar{\psi}_2(t) \longrightarrow \infty$ as $t \longrightarrow \rho_2^-$. Denote by r_2 the smallest solution of equation $\bar{\psi}_2(t) = 0$ in $(0,\rho_2)$.

Suppose that the equation

$$\varphi_0(\psi_2(t)t) = 1 \tag{2.3}$$

has at least one positive solution. Denote by ρ_3 the smallest such solution. Set $\rho = \min\{\rho_2, \rho_3\}$. Define the functions ψ_3 and $\bar{\psi}_3$ on the interval $[0, \rho)$ by

$$\begin{aligned}\psi_3(t) &= \Bigg\{ \frac{\int_0^1 \varphi((1-\theta)\psi_2(t)t)d\theta}{1-\varphi_0(\psi_2(t)t)} \\ &+ \frac{(\varphi_0(\psi_2(t)t)+\varphi_0(t))\int_0^1 \varphi_1(\theta\psi_2(t)t)d\theta}{(1-\varphi_0(\psi_2(t)t))(1-\varphi_0(t))} \\ &+ \frac{1}{2}\Bigg[\frac{5(\varphi_0(\psi_2(t)t)+\varphi_0(t))}{1-\varphi_0(t)} \\ &+ \frac{3(\varphi_1(\psi_1(t)t)(\varphi_0(\psi_2(t)t)+\varphi_0(t))}{(1-\varphi_0(t))^2}\Bigg] \\ &\times \frac{\int_0^1 \varphi_1(\theta\psi_2(t)t)d\theta}{1-\varphi_0(t)}\Bigg\}\psi_2(t)\end{aligned}$$

and $\bar{\psi}_3(t) = \psi_3(t) - 1$. We get $\bar{\psi}_3(0) = -1$ and $\bar{\psi}_3(t) \longrightarrow \infty$ as $t \longrightarrow \rho^-$. Denote by r_3 the smallest solution of the equation $\bar{\psi}_3(t) = 0$ in $(0, \rho)$. Define a radius of convergence r by

$$r = \min\{r_i\}, i = 1, 2, 3. \tag{2.4}$$

It follows that for each $t \in [0, r)$

$$0 \leq \varphi_0(t) < 1, \tag{2.5}$$
$$0 \leq \varphi_0(\psi_1(t)t) < 1, \tag{2.6}$$
$$0 \leq \varphi_0(\psi_2(t)t) < 1 \tag{2.7}$$

and

$$0 \leq \psi_i(t) < 1, \; i = 1, 2, 3. \tag{2.8}$$

The local convergence analysis of the method (1.3) use the hypotheses (H):

(h1) $\mathcal{F} : \Omega \longrightarrow \mathcal{B}_2$ a continuously differentiable operator in the sense of Fréchet and there exists $x_* \in \Omega$ such that $\mathcal{F}(x_*) = 0$, and $\mathcal{F}'(x_*)^{-1} \in \mathcal{B}(\mathcal{B}_2, \mathcal{B}_1)$.

(h2) There exists function $\varphi_0 : [0, \infty) \longrightarrow [0, \infty)$ continuous, and increasing with $\varphi_0(0) = 0$ such that for each $x \in \Omega$

$$\|\mathcal{F}'(x_*)^{-1}(F'(x) - F'(x_*))\| \leq \varphi_0(\|x - x_*\|)$$

and ρ_0 given by (2.1) exists. Set $\Omega_0 = \Omega \cap T(x_*, \rho_0)$.

(h3) There exist functions $\varphi : [0, \rho_0) \longrightarrow [0, \infty), \varphi_1 : [0, \rho_0) \longrightarrow [0, \infty)$ continuous, and increasing such that for each $x, y \in \Omega_0$

$$\|\mathcal{F}'(x_*)^{-1}(\mathcal{F}(y)] - \mathcal{F}'(x))\| \leq \varphi(\|y - x\|$$

and

$$\|\mathcal{F}'(x_*)^{-1}\mathcal{F}'(x))\| \leq \varphi_1(\|x - x_*\|.$$

(h4) $\bar{T}(x_*, r) \subseteq \Omega$ and ρ_1, ρ_2 exist and are given by (2.2), (2.3), respectively and r is defined in (2.4).

(h5) There exists $r_* \geq r$ such that $\int_0^1 \varphi_0(\theta r_*)d\theta < 1$.

Set $\Omega_1 = \Omega \cap \bar{T}(x_*, r_*)$.

Next, we present the local convergence analysis of the method (1.3) using preceding notation and the hypotheses (H).

Theorem 2.1. *Suppose that the hypotheses (H) hold, and choose* $x_0 \in T(x_*, r_*) - \{x_*\}$*. Then, the sequence* $\{x_n\}$ *starting at* x_0 *and generated by the method (1.3) is well defined, remains in* $T(x_*, r)$ *for each* $n = 0, 1, 2, \ldots$ *and converges to* x_**. Moreover the following error bounds hold*

$$\|y_n - x_*\| \leq \psi_1(\|x_n - x_*\|)\|x_n - x_*\| \leq \|x_n - x_*\| < r, \tag{2.9}$$

$$\|z_n - x_*\| \leq \psi_2(\|x_n - x_*\|)\|x_n - x_*\| \leq \|x_n - x_*\|, \tag{2.10}$$

$$\|x_{n+1} - x_*\| \leq \psi_3(\|x_n - x_*\|)\|x_n - x_*\| \leq \|x_n - x_*\|, \tag{2.11}$$

where functions ψ_i *are given previously and* r *is defined in (2.4). Furthermore, the limit point* x_* *is the only solution of equation* $\mathcal{F}(x) = 0$ *in the set* Ω_1 *given in (h5).*

Proof. Estimates (2.9)-(2.11) shall be shown using mathematical induction. Let

$$x \in T(x_*, r) - \{x_*\}.$$

By (2.4), (h1) and (h2), we have in turn that

$$\|\mathcal{F}'(x_*)^{-1}(\mathcal{F}'(x) - \mathcal{F}'(x_*))\| \leq \varphi_0(\|x - x_*\|) < \varphi_0(r) < 1. \tag{2.12}$$

Estimate (2.12) and the Banach lemma on invertible operators [2, 14] assure that $\mathcal{F}'(x)^{-1} \in \mathcal{B}(\mathcal{B}_2, \mathcal{B}_1)$ and

$$\|\mathcal{F}'(x)^{-1}\mathcal{F}'(x_*)\| \leq \frac{1}{1 - \varphi_0(\|x - x_*\|)}. \tag{2.13}$$

It also follows that, for $x = x_0$, iterates y_0, z_0, x_1 are well defined by the method (1.3) for $n = 0$. We get from the first substep of the method (1.3) for $n = 0$ and (h1) that

$$y_0 - x_* = x_0 - x_* - \mathcal{F}'(x_0)^{-1}\mathcal{F}(x_0), \tag{2.14}$$

so by (2.4), (2.8) (for $i = 1$), (h3), (2.13) and (2.14), we obtain in turn that

$$\begin{aligned}
\|y_0 - x_*\| &\leq \|\mathcal{F}'(x_0)^{-1}\mathcal{F}'(x_*)\| \\
&\quad \times \left\| \int_0^1 \mathcal{F}'(x_*)^{-1}(\mathcal{F}'(x_* + \theta(x_0 - x_*)) - \mathcal{F}'(x_0))(x_0 - x_*)d\theta \right\| \\
&\leq \frac{\int_0^1 \varphi((1-\theta)\|x_0 - x_*\|)d\theta \|x_0 - x_*\|}{1 - \varphi_0(\|x_0 - x_*\|)} \\
&= \psi_1(\|x_0 - x_*\|\|x_0 - x_*\| \leq \|x_0 - x_*\| < r, \qquad (2.15)
\end{aligned}$$

which shows (2.9) for $n = 0$ and $y_0 \in T(x_*, r)$. The second substep of the method (1.3) can be written as

$$\begin{aligned}
z_0 - x_* &= (y_0 - x_* - \mathcal{F}'(y_0)^{-1}\mathcal{F}(y_0)) + (\mathcal{F}'(y_0)^{-1} - \mathcal{F}'(x_0)^{-1})\mathcal{F}(y_0) \\
&\quad - \frac{1}{4}[9(I - Q(x_0)) - 5Q(x_0)(I - Q(x_0))]Q(x_0). \qquad (2.16)
\end{aligned}$$

Then, by (2.4), (2.8) (for $i = 2$), (2.13) (for $i = 2$), (2.15) and (2.16), we get in turn that

$$\begin{aligned}
\|z_0 - x_*\| &= \|y_0 - x_* - \mathcal{F}'(y_0)^{-1}\mathcal{F}(y_0)\| \\
&\quad + \|\mathcal{F}'(y_0)^{-1}\mathcal{F}'(x_*)\|[\|\mathcal{F}'(x_*)^{-1}(\mathcal{F}'(y_0) - \mathcal{F}'(x_*))\| \\
&\quad + \|\mathcal{F}'(x_*)^{-1}(\mathcal{F}'(x_0) - \mathcal{F}'(x_*))\|] \\
&\quad + \frac{1}{4}[9\|I - Q(x_0)\| + 5\|Q(x_0)\|\|I - Q(x_0)\|]\|Q(x_0)\| \\
&\le \Bigg\{ \frac{\int_0^1 \varphi((1-\theta)\|y_0 - x_*\|)d\theta}{1 - \varphi_0(\|y_0 - x_*\|)} \\
&\quad + \frac{(\varphi_0(\|y_0 - x_*\|) + \varphi_0(\|x_0 - x_*\|)) \int_0^1 \varphi_1(\theta\|y_0 - x_*\|)d\theta}{(1 - \varphi_0(\|y_0 - x_*\|)(1 - \varphi_0(\|x_0 - x_*\|))} \\
&\quad + \frac{1}{4}\Bigg[\frac{9(\varphi_0(\|x_0 - x_*\|) + \varphi_0(\|y_0 - x_*\|))}{1 - \varphi_0(\|x_0 - x_*\|)} \\
&\quad + \frac{5\varphi_1(\|y_0 - x_*\|)(\varphi_0(\|x_0 - x_*\|) + \varphi_0(\|y_0 - x_*\|))}{(1 - \varphi_0(\|x_0 - x_*\|))^2} \Bigg] \\
&\quad \times \frac{\int_0^1 \varphi_1(\theta\|y_0 - x_*\|)d\theta}{1 - \varphi_0(\|x_0 - x_*\|)} \Bigg\} \|y_0 - x_*\| \\
&\le \psi_2(\|x_0 - x_*\|)\|x_0 - x_*\| \le \|x_0 - x_*\| < r, \qquad (2.17)
\end{aligned}$$

which shows (2.12) for $n = 0$, and $z_0 \in T(x_*, r)$. Using the third substep of the method (1.3) for $n = 0$, we can write

$$\begin{aligned}
x_1 - x_* &= (z_0 - x_* - \mathcal{F}'(z_0)^{-1}\mathcal{F}(z_0)) + (\mathcal{F}'(z_0)^{-1} - \mathcal{F}'(x_0)^{-1})\mathcal{F}(z_0) \\
&\quad - \left[\frac{5}{2}I - Q(x_0)(4I - \frac{3}{2}Q(x_0))\right] \mathcal{F}'(x_0)^{-1}\mathcal{F}(z_0) \\
&= (z_0 - x_* - \mathcal{F}'(z_0)^{-1}\mathcal{F}(z_0)) \\
&\quad \mathcal{F}'(z_0)^{-1}[(\mathcal{F}'(x_0) - \mathcal{F}'(x_*)) + (\mathcal{F}'(x_*) - \mathcal{F}'(z_0))]F(z_0) \\
&\quad - \frac{1}{2}[5(I - Q(x_0)) - 3Q(x_0)(I - Q(x_0))]\mathcal{F}'(x_0)^{-1}\mathcal{F}(z_0). \qquad (2.18)
\end{aligned}$$

Then, using (2.4), (2.8) (for $i = 3$), (2.13) (for $x = z_0$), (2.15), (2.17) and (2.18), we have in turn as in (2.17) that

$$\begin{aligned}
\|x_1 - x_*\| &\le \Bigg\{ \frac{\int_0^1 \varphi((1-\theta)\|z_0 - x_*\|)d\theta}{1 - \varphi_0(\|z_0 - x_*\|)} \\
&\quad + \frac{(\varphi_0(\|z_0 - x_*\|) + \varphi_0(\|x_0 - x_*\|)) \int_0^1 \varphi_1(\theta\|z_0 - x_*\|)d\theta}{(1 - \varphi_0(\|z_0 - x_*\|)(1 - \varphi_0(\|x_0 - x_*\|))}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\left[\frac{5(\varphi_0(\|x_0-x_*\|)+\varphi_0(\|y_0-x_*\|))}{1-\varphi_0(\|x_0-x_*\|)}\right.\\
&\left.\frac{3\varphi_1(\|y_0-x_*\|)(\varphi_0(\|x_0-x_*\|)+\varphi_0(\|y_0-x_*\|))}{(1-\varphi_0(\|x_0-x_*\|))^2}\right]\\
&\times\left.\frac{\int_0^1\varphi_1(\theta\|z_0-x_*\|)d\theta}{1-\varphi_0(\|x_0-x_*\|)}\right\}\|z_0-x_*\|\\
\leq\quad & \psi_3(\|x_0-x_*\|)\|x_0-x_*\|\leq\|x_0-x_*\|<r, \qquad (2.19)
\end{aligned}
$$

which shows (2.12) for $n=0$ and $x_1\in T(x_*,r)$. The induction for (2.9)-(2.12) is finished, by simply replacing x_0,y_0,z_0,x_1 by x_m,y_m,z_m,x_{m+1} in the preceding estimates. Then, using the estimate

$$\|x_{m+1}-x_*\|\leq c\|x_m-x_*\|\leq\|x_m-x_*\|<r, \qquad (2.20)$$

where $c=\psi_3(\|x_0-x_*\|)\in[0,1)$, we deduce that $x_{m+1}\in T(x_*,r)$, and

$$\lim_{m\longrightarrow\infty}x_m=x_*.$$

Finally, to show the uniqueness part, let $y_*\in\Omega_1$ with $\mathcal{F}(y_*)=0$. Define

$$G=\int_0^1\mathcal{F}'(x_*+\theta(y_*-x_*))d\theta.$$

Then, using (h2) and (h5), we get in turn that

$$\|\mathcal{F}'(p_*)^{-1}(G-\mathcal{F}'(x_*))\|\quad\leq\quad\int_0^1\varphi_0(\theta\|y_*-x_*\|)d\theta\leq\int_0^1\varphi_0(\theta r_*)d\theta<1,$$

so G^{-1} exists, and from

$$0=\mathcal{F}(x_*)-\mathcal{F}(y_*)=G(x_*-y_*)$$

we derive $x_*=y_*$. □

Remark 2.2. (a) In the case when $\varphi_0(t)=L_0t,\varphi(t)=Lt$ and $\Omega_0=\Omega$, the radius

$$\rho_A=\frac{2}{2L_0+L}$$

was obtained by Argyros et al. in [4] as the convergence radius for Newton's method under condition (2.7)-(2.9). Notice that the convergence radius for Newton's method, given independently by Rheinboldt [15] and Traub [19] is given by

$$\rho_{TR}=\frac{2}{3L}<\rho_A.$$

As an example, let us consider the function $F(x)=e^x-1$. Then $\alpha^*=0$. Set

$$\Omega=B(0,1).$$

Then, we have that $L_0=e-1<L=e$, so

$$\rho_{TR}=0.24252961<\rho_A=0.324947231.$$

(b) The local results can be used for projection methods such as Arnoldi's method, the generalized minimum residual method (GMREM), the generalized conjugate method (GCM) for combined Newton/finite projection methods and in connection to the mesh independence principle in order to develop the cheapest and most efficient mesh refinement strategy [2, 3, 4, 10].

(c) The results can also be used to solve equations where the operator F' satisfies the autonomous differential equation [2, 3, 4, 10]:

$$F'(x) = P(F(x)),$$

where $P : \mathcal{B}_2 \longrightarrow \mathcal{B}_2$ is a known continuous operator. Since

$$F'(x^*) = P(F(x^*)) = P(0),$$

we can apply the results without actually knowing the solution x^*. Consider as an example $F(x) = e^x - 1$. Then, we can choose $P(x) = x + 1$ and $x^* = 0$.

(d) It is worth noticing that the method (1.3) does not change when we use the conditions of the preceding Theorem instead of the stronger conditions used in [17]. Moreover, we can compute the computational order of convergence (COC) which is defined as

$$\xi = \ln\left(\frac{\|x_{n+1} - x_*\|}{\|x_n - x_*\|}\right) / \ln\left(\frac{\|x_n - x_*\|}{\|x_{n-1} - x_*\|}\right)$$

or the approximate computational order of convergence (ACOC)

$$\xi_1 = \ln\left(\frac{\|x_{n+1} - x_n\|}{\|x_n - x_{n-1}\|}\right) / \ln\left(\frac{\|x_n - x_{n-1}\|}{\|x_{n-1} - x_{n-2}\|}\right).$$

This way we obtain in practice, the order of convergence, in which higher order derivatives are not used.

3. Numerical example

We present the following example to test the convergence criteria.

Example 3.1. Let $\mathcal{B}_1 = \mathcal{B}_1 = \mathbb{R}^3$, $\Omega = U(0,1)$, $x_* = (0,0,0)^T$ and define $\mathcal{F}$ on Ω by

$$\mathcal{F}(x) = \mathcal{F}(u_1, u_2, u_3) = (e^{u_1} - 1, \frac{e-1}{2}{u_2}^2 + u_2, u_3)^T. \tag{3.1}$$

For the points $u = (u_1, u_2, u_3)^T$, the Fréchet derivative is given by

$$\mathcal{F}'(u) = \begin{pmatrix} e^{u_1} & 0 & 0 \\ 0 & (e-1)u_2 + 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Using the norm of the maximum of the rows $x_* = (0,0,0)^T$ and since

$$\mathcal{F}'(x_*) = diag(1,1,1),$$

we get by conditions (H), $\varphi_0(t) = (e-1)t$, $\varphi(t) = e^{\frac{1}{e-1}}t, \varphi_1(t) = e^{\frac{1}{e-1}}$, and

$$r_1 = 0.38269191223238574 47,$$
$$r_2 = 0.12773571026178562 3265,$$
$$r_3 = 0.08935465235314027 3657 = r.$$

Example 3.2. Let $\mathcal{B}_1 = \mathcal{B}_2 = C[0,1], \Omega = \bar{U}(0,1)$. Define function F on Ω by

$$F(\varphi)(x) = \varphi(x) - 5\int_0^1 x\theta\varphi(\theta)^3 d\theta.$$

Then, the Fréchet-derivative is given by

$$F'(\varphi(\xi))(x) = \xi(x) - 15\int_0^1 x\theta\varphi(\theta)^2\xi(\theta)d\theta, \text{ for each } \xi \in \Omega.$$

Then, we have that $x^* = 0, \varphi_0(t) = L_0 t, \varphi(t) = Lt, \varphi_1(t) = 2, L_0 = 7.5 < L = 15$. Then, the radius of convergence are given by

$$r_1 = 0.0666666666666666666,$$
$$r_2 = 0.0112691816233373796191,$$
$$r_3 = 0.00599134019182119646057 5 = r.$$

Example 3.3. Returning to the motivational example given in the introduction of this study, we can choose $\varphi_0(t) = \varphi(t) = 97$ and $\varphi_1(t) = 1 + \varphi_0(t)$. Then, the radius of convergence are given by

$$r_1 = 0.0068728522336769759,$$
$$r_2 = 0.0005865188569803861,$$
$$r_3 = 1 = 0.00018953869019822886 5 = r.$$

4. Basins of attraction

As in [12] (also see references in [9]), we analyse the basins of attraction of the method (1.3). Recall that the basins of attraction of an iterative method are the collection of all initial points from which the iterative method converges to a solution of an equation [9]. The following test problems which are systems of polynomials in two variables are considered.

Example 4.1. $\begin{cases} x^3 - y = 0 \\ y^3 - x = 0 \end{cases}$
with solutions $\{\ (-1,-1), (0,0), (1,1)\}$.

Example 4.2. $\begin{cases} 3x^2y - y^3 = 0 \\ x^3 - 3xy^2 - 1 = 0 \end{cases}$
with solutions $\{\ (-\frac{1}{2}, -\frac{\sqrt{3}}{2}), (-\frac{1}{2}, \frac{\sqrt{3}}{2}), (1,0)\}$.

Example 4.3. $\begin{cases} x^2 + y^2 - 4 = 0 \\ 3x^2 + 7y^2 - 16 = 0 \end{cases}$
with solutions $\{\ (\sqrt{3},1), (-\sqrt{3},1), (\sqrt{3},-1), (-\sqrt{3},-1)\}$.

For generating basins of attraction associated with roots of system of nonlinear equations, we consider a rectangular domain

$$\mathcal{R} = \{(x, y) \in \mathbb{R}^2 : -2 \leq x \leq 2, -2 \leq y \leq 2\}$$

of 401×401 equidistant grid points which contains all the roots of the system. Each such point $(x_0, y_0) \in \mathcal{R}$ is assigned a color in accordance with the root at which the corresponding iterative method starting from (x_0, y_0) converges. The point is marked black if either the method converges to infinity or it does not converge, with a tolerance of 10^{-8} in a maximum of 50 iterations. In this way, we distinguish the basins of attraction by their respective colors for different methods.

The basins of attraction, for the considered examples employing Newton's method (1.2) and the three-step Newton-like method (1.3), have been displayed in Fig. 1. It can be observed in Fig.1 that the basins of attraction generated by method (1.3) are smaller in size as compared to that generated by Newton's method. Therefore, the black points, which are considered as the bad initial points, are more in number in case of the former method. This phenomenon is observed because, the method (1.3) has order of convergence eight, in comparison to the quadratically convergent Newton's method. The figures presented in this work are performed in a 4-core 64-bit Windows machine with Intel Core i7-3770 processor using MATLAB programming language.

References

[1] Argyros, I.K., Ezquerro, J.A., Gutiérrez, J.M., Herńandez, M.A., Hilout, S., *On the semilocal convergence of efficient Chebyshev-Secant-type methods*, J. Comput. Appl. Math., **235**(2011), 3195-3206.

[2] Argyros, I.K., George, S., Thapa, N., *Mathematical Modeling for the Solution of Equations and Systems of Equations with Applications*, Vol. I, Nova Publishes, NY, 2018.

[3] Argyros, I.K., George, S., Thapa, N., *Mathematical Modeling for the Solution of Equations and Systems of Equations with Applications*, Vol. II, Nova Publishes, NY, 2018.

[4] Argyros, I.K., Hilout, S., *Weaker conditions for the convergence of Newtons method*, J. Complexity, **28**(2012), no. 3, 364-387.

[5] Cordero, A., Hueso, J. L., Martínez, E., Torregrosa, J.R., *A modified Newton-Jarratt's composition*, Numer. Algorithms, **55**(2010), 87-99.

[6] Grau-Sánchez, M., Noguera, M., Amat, S., *On the approximation of derivatives using divided difference operators preserving the local convergence order of iterative methods*, J. Comput. Appl. Math., **237**(2013), 363-372.

[7] Lotfi, T., Bakhtiari, P., Cordero, A., Mahdiani, K., Torregrosa, J.R., *Some new efficient multipoint iterative methods for solving nonlinear systems of equations*, Int. J. Comput. Math., **92**(2015), 1921-1934.

[8] Madhu, K., Babajee, D.K.R., Jayaraman, J., *An improvement to double-step Newton method and its multi-step version for solving system of nonlinear equations and its applications*, Numer. Algorithms, **74**(2017), 593-607.

[9] Magreñán, A.A., Argyros, I.K., *Two-step Newton methods*, J. Complexity, **30**(2014), no. 4, 533-553.

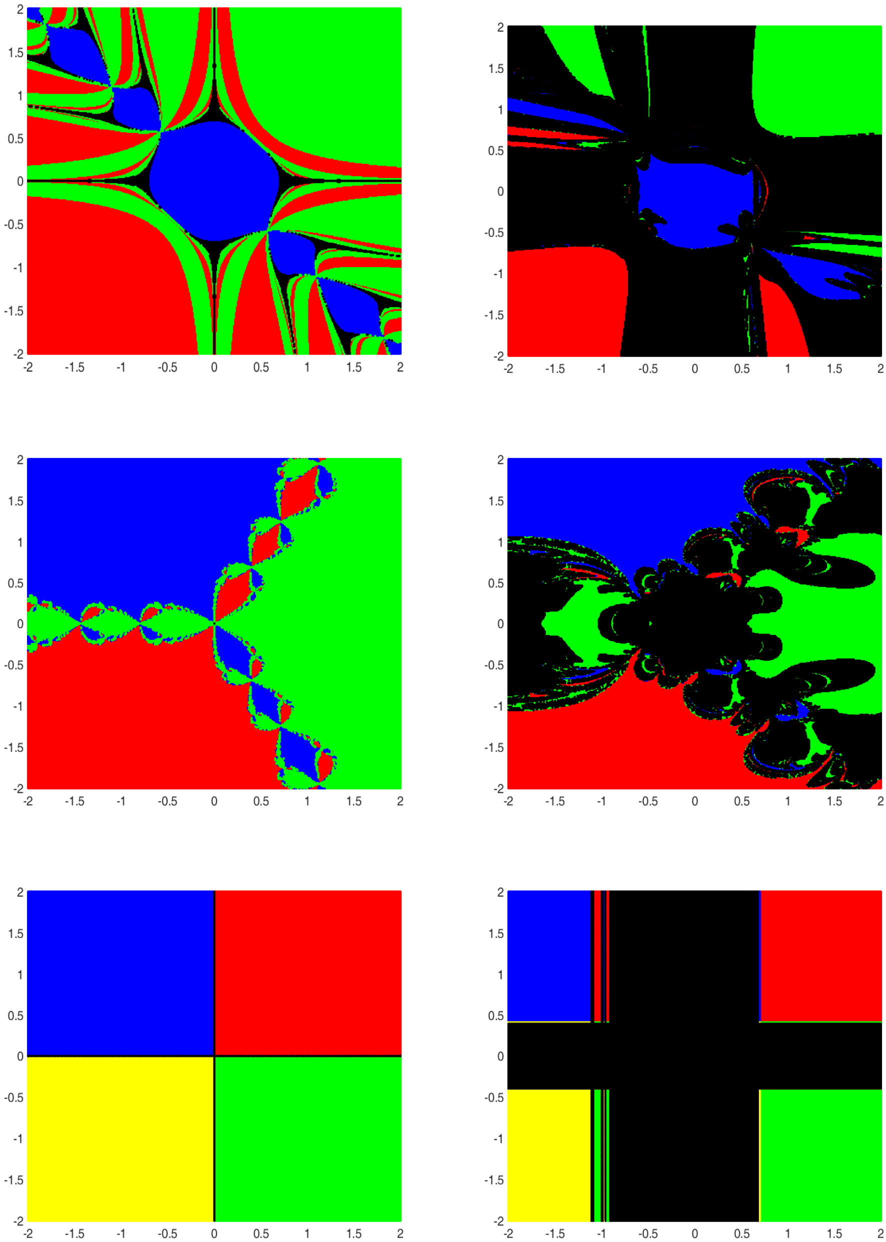

FIGURE 1. Basins of attraction of Example 4.1, 4.2, 4.3 using Newton's method (1.2) (1st col., top to bottom) and using the Newton-type method (1.3) (2nd col., top to bottom).

[10] Magreñán, A.A., Argyros, I.K., *Improved convergence analysis for Newton-like methods*, Numer. Algorithms, **71**(2016), no. 4, 811-826.

[11] Magreñán, A.A., Cordero, A., Gutiérrez, J.M., Torregrosa, J.R., *Real qualitative behavior of a fourth-order family of iterative methods by using the convergence plane*, Math. Comput. Simulation, **105**(2014), 49-61.

[12] Narang, M., Bhatia, S., Alshomrani, A.S., Kanwar, V., *General efficient class of Steffensen type methods with memory for solving systems of nonlinear equations*, J. Comput. Appl. Math., **352**(2019), 23-39.

[13] Osrowski, A.M., *Solution of Equations and Systems of Equations*, Academic Press, New York, 1960.

[14] Potra, F.A., Pták, V., *Nondiscrete Induction and Iterative Processes*, vol. 103, Pitman Advanced Publishing Program, 1984.

[15] Rheinboldt, W.C., *An adaptive continuation process for solving systems of nonlinear equations*, in: Mathematical models and numerical methods (A.N. Tikhonov et al. eds.), Banach Center, Warsaw Poland, pub. 3, 1977, 129-142.

[16] Sharma, J.R., Argyros, I.K., Kumar, S., *Ball convergence of an efficient eighth order iterative method under weak conditions*, Mathematics, **6**(2018), no. 11, 260, https://doi.org/10.3390/math6110260.

[17] Sharma, J.R., Arora, H., *Improved Newton-like methods for solving systems of nonlinear equations*, SeMA J., **74**(2017), no. 2, 147-163.

[18] Sharma, J.R., Arora, H., *A novel derivative free algorithm with seventh order convergence for solving systems of nonlinear equations*, Numer. Algorithms, **67**(2014), 917-933.

[19] Traub, J.F., *Iterative Methods for the Solution of Equations*, AMS Chelsea Publishing, 1982.

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Stud. Univ. Babeş-Bolyai Math. 66(2021), No. 4, 769–781
DOI: 10.24193/subbmath.2021.4.13

A dynamic electroviscoelastic problem with thermal effects

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Abstract. We consider a mathematical model which describes the dynamic process of contact between a piezoelectric body and an electrically conductive foundation. We model the material's behavior with a nonlinear electro-viscoelastic constitutive law with thermal effects. Contact is described with the Signorini condition, a version of Coulomb's law of dry friction. A variational formulation of the model is derived, and the existence of a unique weak solution is proved. The proofs are based on the classical result of nonlinear first order evolution inequalities, the equations with monotone operators, and the fixed point arguments.

Mathematics Subject Classification (2010): 74M15, 74M10, 74F05, 49J40.

Keywords: Piezoelectric, frictional contact, thermo-elasto-viscoplastic, fixed point, dynamic process, Coulomb's friction law, evolution inequality.

1. Introduction

Piezoelectricity is the ability of certain crystals, like the quartz, to produce a voltage when they are subjected to mechanical stress. On a nanoscopic scale, the piezoelectric phenomenon arises from a nonuniform charge distribution within a crystal unit cells, and the piezoelectricity is then perceived as the electrical polarization due to mechanical input. Different models have been developed to describe the interaction between the electric and mechanical fields (see, e.g.[12, 13, 15]). Therefore there is a need to extend the results on models for contact with deformable bodies which include coupling between mechanical and electrical properties. General models for elastic materials with piezoelectric effects can be found in [13, 17] and more recently in [2], viscoelastic piezoelectric materials in [2, 17] or elasto-viscoplastic piezoelectric materials have been studied in [9].

In this paper, we consider a general model for the dynamic process of frictional contact between a deformable body and a rigid obstacle. The material obeys an electro-viscoelastic constitutive law with piezoelectric and thermal effects. Moreover, the contact and friction are modelled by Signorini's conditions and a non local

Coulomb's friction law. We derive a variational formulation of the model, which is set as a system coupling a variational second order evolution inequality. We establish the existence of a unique weak solution of the model. The idea is to reduce the second order evolution inequality of the system to first order evolution inequality. Then adopting fixed point methods frequently we prove an existence and uniqueness of displacement and temperature fields, using monotonicity and convexity properties. The importance of this paper is to make the coupling of an electro-viscoelastic problem with thermal effects. The paper is structured as follows. In Section 2 we describe the mechanical problem and provide comments on the contact boundary conditions. In Section 3 we list the assumptions on the data and derive the variational formulation. In Sections 4, we present our main existence and uniqueness results, which state the unique weak solvability of the Signorini's contact electro-visco -elastic problem with non local Coulomb's friction lawn conditions.

2. Problem statement

We consider a body made of a piezoelectric material which occupies the domain $\Omega \subset \mathbb{R}^d$ $(d \leq 3)$ with a Lipschitz boundary Γ. The body is modelled with an electro-visco-elastic constitutive law, allowing piezoelectric effects. Let $[0.T]$ be the time interval where $T > 0$, let Γ be split into three measurable parts Γ_1, Γ_2 and Γ_3 such that $meas\ \Gamma_1 > 0$. We assume that the body is fixed on Γ_1 and surface tractions of density h act on Γ_2. On Γ_3, the body may come into contact with a rigid obstacle. In other hand, Γ be split into two measurable sets Γ_a and Γ_b such that $meas\ \Gamma_b > 0$ and $\Gamma_3 \subset \Gamma_b$. We assume that the electrical potential q_0 act on Γ_a and a surface electric charge of density q_2 act on Γ_b, we assume that the problem is quasistatic. The piezoelectric effect is the apportion of electric charges on surfaces of particular crystals after deformation. We denote by $\mathbb{S}^d$ the space of second order symmetric tensors on the space $\mathbb{R}^d$ and use $\cdot$ and $|.|$ for the inner product and the Euclidean norm on the space $\mathbb{R}^d$ (respectively; $\mathbb{S}^d$). Also ν represents the unit outward normal on Γ, the classical formulation of the electro-visco-elastic contact friction problem is described by:

Problem P. Find a displacement field $u : \Omega \times [0.T] \to \mathbb{R}^d$, a stress field $\sigma : \Omega \times [0.T] \to \mathbb{S}^d$, an electric potential field $\varphi : \Omega \times [0.T] \to \mathbb{R}$, an electric displacement field $D : \Omega \times [0.T] \to \mathbb{R}^d$ and a temperature field $\theta : \Omega \times [0,T] \to \mathbb{R}_+$ such that:

$$\sigma = \mathcal{A}\varepsilon\left(\dot{u}\right) + \mathcal{G}\varepsilon\left(u\right) - \xi^* E\left(\varphi\right) - \theta M_e \text{ in } \Omega \times [0.T]\,, \tag{2.1}$$

$$D = \beta E\left(\varphi\right) + \xi\varepsilon\left(u\right) \text{ in } \Omega \times [0.T]\,, \tag{2.2}$$

$$\rho\ddot{u} = Div\ \sigma + f_0 \text{ in } \Omega \times [0.T]\,, \tag{2.3}$$

$$\text{div}\ D = q_0 \text{ in } \Omega \times [0.T]\,, \tag{2.4}$$

$$\dot{\theta} - \text{div}\left(k\nabla\theta\right) = -M\nabla\dot{u} + q_e \text{ in } \Omega \times [0.T]\,, \tag{2.5}$$

$$-k_{ij}\frac{\partial\theta}{\partial\upsilon}\upsilon_j = k_e\left(\theta - \theta_R\right) \text{ on } \Gamma_3 \times [0.T]\,, \tag{2.6}$$

$$\theta = 0 \text{ on } \Gamma_1 \cup \Gamma_2 \times (0,T), \tag{2.7}$$

$$u = 0 \text{ on } \Gamma_1 \times [0.T]\,, \tag{2.8}$$

$$\sigma_\nu = h \text{ on } \Gamma_2 \ \times [0.T]\,, \tag{2.9}$$

$$u_\nu \leq 0 \ , \ \sigma_\nu \leq 0 \ , u_\nu \ \sigma_\nu = 0 \text{ on } \Gamma_3 \times [0.T]\,, \tag{2.10}$$

$$\left\{ \begin{array}{l} |\sigma_\tau| \leq \mu p \, |R \ \sigma_\nu| \\ |\sigma_\tau| < \mu p \, |R \ \sigma_\nu| \Longrightarrow \dot{u}_\tau = 0 \\ |\sigma_\tau| = \mu p \, |R \ \sigma_\nu| \Longrightarrow \exists \ \lambda \geq 0 \text{ such that } \sigma_\tau = -\lambda \dot{u}_\tau \end{array} \right. \quad \text{on } \Gamma_3 \times [0.T]\,, \tag{2.11}$$

$$\varphi = 0 \text{ on } \Gamma_a \times [0.T]\,, \tag{2.12}$$

$$D\nu = q_2 \text{ on } \Gamma_b \times [0.T]\,, \tag{2.13}$$

$$u\,(0) = u_0 \ , \dot{u}\,(0) = v_0 \text{ and } \theta\,(0) = \theta_0 \text{ in } \Omega \times [0.T]\,, \tag{2.14}$$

where (2.1), (2.2) are the thermo-electro -visco-elastic constitutive law of the material, we denote $\varepsilon\,(u)$ (respectively; $E\,(\varphi) = -\nabla\varphi$, $\mathcal{A}, \mathcal{G}$, ξ, ξ^*, β) the linearized strain tensor (respectively; electric field, the viscosity nonlinear tensor, the elasticity tensor, the third order piezoelectric tensor and its transpose, the electric permittivity tensor), θ represent the temperature, $M_e := (m_{ij})$ represents the thermal expansion tansor, (2.3) represents the equation of motion where ρ represents the mass density, (2.4) represents the equilibrium equation, Equation (2.5) describes the evolution of the temperature field, where $k := (k_{ij})$ represents the thermal conductivity tensor, q_e the density of volume heat sources. The associated temperature boundary condition is

given by (2.6) , where θ_r is the temperature of the foundation, and k_e is the exchange coefficient between the body and obstacle. Equation (2.7) means that the temperature vanishes on $\Gamma_1 \cup \Gamma_2 \times (0,T)$. We mention that $Div\sigma$, $divD$ are the divergence operators, (2.8) and (2.9) are the displacement and traction boundary conditions, (2.10), (2.11) the Signorini's contact with a non local Coulomb's friction law conditions. u $_\nu$ and u_τ(respectively; σ_ν and σ_τ) denote the normal displacement and the tangential displacement (respectively; the normal stress and the tangential stress). R will represent a normal regularization operator that is a linear and continuous operator $R : H^{-\frac{1}{2}}\,(\Gamma) \to \mathbb{L}^2\,(\Gamma)$. We shall need it to regularize the normal trace of the stress which is too rough on Γ. p is a non-negative function, the so-called friction bound, $\mu \geq 0$ is the coefficient of friction. The friction law was used in some studies with $p(r) = r_+$ where $r_+ = max\ \{0, r\}$. Recently, from thermodynamic considerations, a new version of Coulomb's law is proposed, it consists to take:

$$p(r) = r(1 - \alpha r)_+, \tag{2.15}$$

where α is a small positive coefficient related to the hardness and the wear of the contact surface. (2.12), (2.13) represent the electric boundary conditions. Finally, in (2.14) u_0 is the given initial displacement, v_0 is the given initial velocity and θ_0 is the initial temperature.

3. Variational formulation and preliminaries

For a weak formulation of the problem, first we introduce some notation. The indices i, j, k, l range from 1 to d and summation over repeated indices is implied. An index that follows a comma represents the partial derivative with respect to the corresponding component of the spatial variable, e. g: $u_{i.j} = \frac{\partial u_i}{\partial x_j}$. We also use the following notations:

$$H = \mathbb{L}^2(\Omega)^d = \{u = (u_i)/u_i \in \mathbb{L}^2(\Omega)\},\ \mathcal{H} = \{\sigma = (\sigma_{ij})/\sigma_{ij} = \sigma_{ji} \in \mathbb{L}^2(\Omega)\},$$
$$H_1 = \{u = (u_i)/\varepsilon(u) \in \mathcal{H}\},\ \mathcal{H}_1 = \{\sigma \in \mathcal{H}/Div\sigma \in H\},$$

The operators of deformation ε and divergence Div are defined by

$$\varepsilon(u) = (\varepsilon_{ij}(u)), \varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i}), Div\sigma = (\sigma_{ij,j}).$$

The spaces $H, \mathcal{H}, H_1$, and $\mathcal{H}_1$ are real Hilbert spaces endowed with the canonical inner products.
We denote by $|\ .\ |_H$ (respectively; $|\ .\ |_{\mathcal{H}}$, $|\ .\ |_{H_1}$, and $|\ .\ |_{\mathcal{H}1}$) the associated norm on the space H (respectively; $\mathcal{H}$, H_1, and $\mathcal{H}_1$).
We use standard notation for the $\mathbb{L}^p$ and the Sobolev spaces associated with Ω and Γ and, for a function $\psi \in H^1(\Omega)$ we still write ψ to denote it trace on Γ. We recall that the summation convention applies to a repeated index.
For the electric displacement field we use two Hilbert spaces:

$$\mathcal{W} = \mathbb{L}^2(\Omega)^d, \quad \mathcal{W}_1 = \left\{D \in \mathcal{W}, \operatorname{div} D \in \mathbb{L}^2(\Omega)\right\}$$

endowed with the inner products:

$$(D, E)_{\mathcal{W}} = \textstyle\int_\Omega D_i E_i dx, \quad (D, E)_{\mathcal{W}_1} = (D, E)_{\mathcal{W}} + (\operatorname{div} D, \operatorname{div} E)_{\mathbb{L}^2(\Omega)}.$$

And the associated norm $|.|_{\mathcal{W}}$(respectively; $|.|_{\mathcal{W}_1}$). The electric potential field is to be found in:

$$W = \left\{\psi \in H^1(\Omega),\ \psi = 0 \text{ on } \Gamma_a\right\}.$$

Since $meas\ (\Gamma_a) > 0$, the following Friedrichs-Poincaré's inequality holds, thus:

$$|\nabla\psi|_{\mathcal{W}} \geq c_F\, |\psi|_{H^1(\Omega)} \quad \forall\psi \in W, \tag{3.1}$$

where $c_F > 0$ is a constant which depends only on Ω and Γ_a. On W, we use the inner product given by:

$$(\varphi, \psi)_W = (\nabla\varphi, \nabla\psi)_{\mathcal{W}},$$

and let $|.|_W$ be the associated norm. It follows from (3.1) that $|.|_{H^1(\Omega)}$ and $|.|_W$ are equivalent norms on W and therefore $(W, |.|_W)$ is a real Hilbert space.
Moreover, by the Sobolev trace Theorem, there exists a constant $\widetilde{c}_0$, depending only on Ω, Γ_a and Γ_3 such that:

$$|\psi|_{L^2(\Gamma_3)} \leq \widetilde{c}_0\, |\psi|_W \quad \forall\psi \in W. \tag{3.2}$$

We recall that when $D \in \mathcal{W}_1$ is a sufficiently regular function, the Green's type formula holds:

$$(D, \nabla\psi)_{\mathcal{W}} + (\operatorname{div} D, \psi)_{\mathbb{L}^2(\Omega)} = \int_\Gamma D\nu.\psi da. \tag{3.3}$$

When σ is a regular function, the following Green's type formula holds:

$$(\sigma, \varepsilon(v))_{\mathcal{H}} + (Div\sigma, v)_H = \int_\Gamma \sigma\nu.vda \quad \forall v \in H_1. \tag{3.4}$$

Next, we define the space:

$$V = \{u \in H_1 / \ u = 0 \ \text{ on } \Gamma_1\}.$$

Since $meas\ (\Gamma_1) > 0$, the following Korn's inequality holds:

$$|\varepsilon(u)|_{\mathcal{H}} \geq c_K |v|_{H_1} \quad \forall v \in V, \tag{3.5}$$

where $c_K > 0$ is a constant which depends only on Ω and Γ_1. On the space V we use the inner product:

$$(u, v)_V = (\varepsilon(u), \varepsilon(v))_{\mathcal{H}},$$

let $|.|_V$ be the associated norm. It follows by (3.5) that the norms $|.|_{H_1}$ and $|.|_V$ are equivalent norms on V and therefore, $(V, |.|_V)$ is a real Hilbert space. Moreover, by the Sobolev trace Theorem, there exists a constant c_0 depending only on the domain Ω, $\Gamma_1$and Γ_3 such that:

$$|v|_{L^2(\Gamma_3)^d} \leq c_0 |v|_V \quad \forall v \in V. \tag{3.6}$$

In what follows, we assume the following assumptions on the problem P.

$$\begin{cases} (a) : \mathcal{A} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d, \\ (b) : \exists\ M_{\mathcal{A}} > 0 \text{ such that } : |\mathcal{A}(x, \varepsilon_1) - \mathcal{A}(x, \varepsilon_2)| \leq M_{\mathcal{A}} |\varepsilon_1 - \varepsilon_2| \\ \qquad \forall\ \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d,\ a.\ e.\ x \in \Omega, \\ (c) : \exists\ m_{\mathcal{A}} > 0 \text{ such that } : |\mathcal{A}(x, \varepsilon_1) - \mathcal{A}(x, \varepsilon_2), \varepsilon_1 - \varepsilon_2| \geq m_{\mathcal{A}} |\varepsilon_1 - \varepsilon_2|^2 \\ \qquad \forall\ \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d,\ a.\ e.\ x \in \Omega, \\ (d) : \text{the mapping } x \to \mathcal{A}(x, \varepsilon) \text{ is lebesgue measurable in } \Omega \text{ for all } \varepsilon \in \mathbb{S}^d, \\ (e) : \text{the mapping } x \to \mathcal{A}(x, 0) \in \mathcal{H}, \end{cases} \tag{3.7}$$

$$\begin{cases} (a) : \mathcal{G} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d, \\ (b) : \exists\ M_{\mathcal{G}} > 0 \text{ such that } : |\mathcal{G}(x, \xi_1) - \mathcal{G}(x, \xi_2)| \leq M_{\mathcal{G}} |\xi_1 - \xi_2| \\ \qquad \forall\ \xi_1, \xi_2 \in \mathbb{S}^d,\ a.\ e.\ x \in \Omega, \\ (d) : \text{the mapping } x \to \mathcal{G}(x, \xi) \text{ is lebesgue measurable in } \Omega \text{ for all } \xi \in \mathbb{S}^d, \\ (e) : \text{the mapping } x \to \mathcal{G}(x, 0) \in \mathcal{H}, \end{cases} \tag{3.8}$$

$$\begin{cases} (a) : \xi = (e_{ijk}) : \Omega \times \mathbb{S}^d \to \mathbb{R}^d, \\ (b) : \xi(x, \tau) = (e_{ijk}(x)\, \tau_{jk}) \ \ \forall \tau = (\tau_{ij}) \in \mathbb{S}^d, a.\ e.\ x \in \Omega, \\ (c) : e_{ijk} = e_{ikj} \in \mathbb{L}^\infty(\Omega), \end{cases} \tag{3.9}$$

$$\begin{cases} (a) : \beta = (\beta_{ij}) : \Omega \times \mathbb{R}^d \to \mathbb{R}^d, \\ (b) : \beta(x, E) = (b_{ij}(x)\, E_j) \ \ \forall E = (E_i) \in \mathbb{R}^d, a.e.x \in \Omega, \\ (c) : b_{ij} = b_{ji} \in \mathbb{L}^\infty(\Omega), \\ (d) : \exists\ m_\beta > 0 \text{ such that } : b_{ij}(x)\, E_i E_j \geq m_\beta |E|^2 \\ \forall E = (E_i) \in \mathbb{R}^d,\ x \in \Omega. \end{cases} \tag{3.10}$$

From the assumptions (3.9) and (3.10),we deduce that the piezoelectric operator ξ(respectively; the electric permittivity operator β) is linear, has measurable bounded

component denoted e_{ijk} (respectively; b_{ij}) and moreover, β is symmetric and positive definite.
Recall also that the transposed operator ξ^* is given by $\xi^* = (e^*_{ijk})$ where $e^*_{ijk} = e_{kij}$ and the following equality holds:

$$\xi\sigma.v = \sigma.\xi^* v \quad \forall\sigma \in \mathbb{S}^d, \qquad v \in \mathbb{R}^d.$$

The friction function satisfies:

$$\begin{cases} p : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+ \text{ verifies:} \\ (a) : \exists\ M > 0 \text{ such that } : |p(x, r_1) - p(x, r_2)| \leq M\,|r_1 - r_2| \\ \text{For every } r_1, r_2 \in \mathbb{R}, a.\ e.\ x \in \Gamma_3, \\ (b) : \text{the mapping} : x \to p(x, r) \text{ is measurable on } \Gamma_3, \text{ for every } r \in \mathbb{R}, \\ (c) : p(x, 0) = 0\ , a.\ e.\ x \in \Gamma_3. \end{cases} \tag{3.11}$$

We note that (3.11) is satisfied in the case in which p given by (2.12).
We also assume that the body forces and surface tractions have the regularity:

$$f_0 \in \mathbb{L}^2(0.T; H), \quad h \in \mathbb{L}^2\left(0.T; \mathbb{L}^2(\Gamma_2)^d\right), \tag{3.12}$$

The thermal tensors and the heat source density satisfy

$$M = (m_{ij}), m_{ij} = m_{ji} \in L^\infty(\Omega),\ q_e \in L^2(0, T; L^2(\Omega)), \tag{3.13}$$

and for some $c_k > 0$, for all $(\zeta_i) \in R_d$:

$$k\ = (k_{i,j}), k_{ij} = k_{ji} \in L^\infty(\Omega),\ k_{ij}\zeta_i\zeta_j \geq c_k\zeta_i\zeta_j \tag{3.14}$$

as well as the densities of electric charges satisfy:

$$q_0 \in \mathbb{L}^2\left(0.T; \mathbb{L}^2(\Omega)\right), \quad q_2 \in \mathbb{L}^2\left(0.T; \mathbb{L}^2(\Gamma_b)\right). \tag{3.15}$$

We define the function $f : [0.T] \to V$ and $q : [0.T] \to W$ by:

$$(f(t), v)_V = \int_\Omega f_0(t)\, v dx + \int_{\Gamma_2} h(t)\, v da \quad \forall v \in V, \quad t \in [0.T], \tag{3.16}$$

$$(q(t), \psi)_W = -\int_\Omega q_0(t)\,\psi dx + \int_{\Gamma_b} q_2(t)\,\psi da \quad \forall\psi \in W, \quad t \in [0.T]. \tag{3.17}$$

for all u, $v \in V$, $\psi \in W$ and $t \in [0.T]$, and note that conditions (3.14) and (3.15) imply that

$$f \in \mathbb{L}^2(0.T; V'), \quad q \in \mathbb{L}^2(0.T; W), \tag{3.18}$$

while the friction coefficient μ, the mass density ρ satisfies

$$\begin{aligned} &\mu \in \mathbb{L}^\infty(\Gamma_3),\ \mu(x) \geq 0, a.\ e.\ \text{on } \Gamma_3, \\ &\rho \in \mathbb{L}^\infty(\Omega) \text{ there exists } \rho^* > 0 \text{ such that } \rho(x) \geq \rho^*, a.e.x \in \Omega. \end{aligned} \tag{3.19}$$

$$u_0 \in V, \quad v_0 \in H,\ \theta_0 \in E,\ \theta_R \in W^{1,2}(0, T; L^2(\Gamma_3)), k_e \in L^\infty(\Omega, \mathbb{R}_+)\ , \tag{3.20}$$

The function $r : V \to L_2(\Omega)$ satisfies that there exists a constant $L_r > 0$ such that

$$|r(v_1) - r(v_2)|_{L2(\Omega)} \leq L_r|v_1 - v_2|_V, \forall v_1, v_2 \in V \tag{3.21}$$

We denote by the frictionfunctional $j : \mathcal{H} \times V \to \mathbb{R}$

$$j(\sigma, v) = \int_{\Gamma_3} \mu p\,|R\ \sigma_\nu|\,|v_\tau|\,da. \tag{3.22}$$

We denote by U the convex subset of admissible displacements fields given by

$$U = \{v \in H_1/v = 0 \text{ on } \Gamma_1,\ v_\nu \leq 0 \text{ on } \Gamma_3\}, \tag{3.23}$$

By a standard procedure based on Green's formula, we obtain the following formulation of the mechanical problem (2.1) – (2.14).

Problem PV. Find a displacement field $u : \Omega \times [0.T] \to \mathbb{R}^d$,an electric potentiel field $\varphi : \Omega \times [0.T] \to \mathbb{R}$, an electric displacement field $D : \Omega \times [0.T] \to \mathbb{R}^d$ such that and a temprature field $\theta : \Omega \times [0,T] \to \mathbb{R}_+$ such that:

$$\begin{aligned}(\ddot{u}, w - \dot{u})_{V' \times V} + (\sigma(t), \varepsilon(w - \dot{u}(t)))_{\mathcal{H}} + j(\sigma, w) - j(\sigma, \dot{u}(t)) \\ \geq (f(t), w - \dot{u}(t))\ \forall u, w \in V\end{aligned} \tag{3.24}$$

$$(D(t), \nabla\psi)_{\mathbb{L}^2(\Omega)^d} + (q\,(t)\,, \psi)_W = 0 \qquad \forall\psi \in W \tag{3.25}$$

$$(\dot{\theta}\,(t) + K\theta\,(t) = R_e \dot{u}\,(t) + Q\,(t) \quad t \in (0,T), \qquad \forall\psi \in W \tag{3.26}$$

$$u\,(0) = u_0\ ,\ \dot{u}\,(0) = v_0 \text{ and } \theta(0) = \theta_0 \tag{3.27}$$

where $Q : [0,T] \to E'$, $K : E \to E'$, $R : V \to E'$ are given by

$$(Q(t), \mu)_W = \int_{\Gamma_3} k_e(u_\nu)\theta_R \mu da + \int_\Omega q\mu dx, \tag{3.28}$$

$$\begin{aligned}(K\tau, \mu)_{E' \times E} &= \sum_{i,j=1}^{d} \int_\Omega k_{ij} \frac{\partial\tau}{\partial x_j} \frac{\partial\mu}{\partial x_i} dx + \int_{\Gamma_3} k_e \tau\mu da,\ (R_e\mu, v)_{E' \times E} \\ &= -\int_\Omega (M\nabla v)\, dx,\end{aligned} \tag{3.29}$$

for all $v \in V, \mu, \tau \in E$.

4. Existence and uniqueness result

Our main result which states the unique solvability of Problem are the following.

Theorem 4.1. *Let the assumptions* (3.7)−(3.20) *hold. Then, Problem PV has a unique solution* $\{u, \varphi, D, \theta\}$ *which satisfies*

$$u \in C^1\,(0.T; H) \cap W^{1.2}\,(0.T; V) \cap W^{2.2}\,(0.T; V') \tag{4.1}$$

$$\varphi \in W^{1.2}\,(0.T; W) \tag{4.2}$$

$$\sigma \in \mathbb{L}^2(0.T; \mathcal{H}), Div\sigma \in \mathbb{L}^2(0.T; V') \tag{4.3}$$

$$D \in W^{1.2}(0.T; \mathcal{W}_1) \tag{4.4}$$

$$\theta \in W^{1,2}(0,T; E') \cap L^2(0,T; E) \cap C(0,T; L^2\,(\Omega)) \tag{4.5}$$

We conclude that under the assumptions (3.7)−(3.21), the mechanical problem (2.1)−(2.14) has a unique weak solution with the regularity (4.1) – (4.5).The proof of this theorem will be carried out in several steps. It is based on arguments of first order evolution nonlinear inequalities (see Refs. [5,7-9]), evolution equations (see Ref. [2]), and fixed point arguments.

Let $G \in \mathbb{L}^2(0.T;\mathcal{H})$ and $\eta \in \mathbb{L}^2(0.T;V')$ are given, we deduce a variational formulation of Problem PV.

Problem $PV_{G\eta}$: Find a displacement field $u_{G\eta} : [0.T] \to V$ such that

$$\left\{\begin{array}{l} u_{G\eta}(t) \in U \quad (\ddot{u}_{G\eta}, w - \dot{u}_{G\eta})_{V'\times V} + (\mathcal{A}\varepsilon(\dot{u}_{G\eta}(t)), \varepsilon(w - \dot{u}_{G\eta}(t))_{\mathcal{H}} + \\ \left(\eta, w - \dot{u}_{G\eta}(t)\right)_{V'\times V} + j(G,w) - j(G,\dot{u}_{G\eta}(t)) \geq (f(t), w - \dot{u}_{G\eta}(t)) \\ \qquad\qquad \forall w \in V \end{array}\right. \tag{4.6}$$

$$\dot{u}_{G\eta}(0) = v(0) = v_0 \tag{4.7}$$

We define $f_\eta(t) \in V$ for $a.e.t \in [0.T]$ by

$$(f_\eta(t), w)_{V'\times V} = (f(t) - \eta(t), w)_{V'\times V}, \forall w \in V. \tag{4.8}$$

From (3.18), we deduce that:

$$f_\eta \in \mathbb{L}^2(0.T;V') \tag{4.9}$$

Let now $u_{G\eta} : [0.T] \to V$ be the function defined by

$$u_{G\eta}(t) = \int_0^t v_{G\eta}(s)\, ds + u_0, \quad \forall t \in [0.T] \tag{4.10}$$

We define the operator $A : V' \to V$ by

$$(Av, w)_{V'\times V} = (\mathcal{A}\varepsilon(v), \varepsilon(w))_{\mathcal{H}}, \forall v, w \in V. \tag{4.11}$$

Lemma 4.2. *For all $G \in \mathbb{L}^2(0.T;\mathcal{H})$ and $\eta \in \mathbb{L}^2(0.T;V')$, $PV_{G\eta}$ has a unique solution with the regularity:*

$$v_{G\eta} \in C(0.T;H) \cap \mathbb{L}^2(0.T;V) \text{ and} \dot{v}_{G\eta} \in \mathbb{L}^2(0.T;V'). \tag{4.12}$$

Proof. The proof from nonlinear first order evolution inequalities, given in Refs ([8]). □

In the second step, we use the displacement field $u_{G\eta}$ to consider the following variational problem.

Problem $PV1_{G\eta}$: Find an electric potential field $\varphi_{G_\eta} : \Omega \times [0.T] \to W$ such that:

$$(\beta\nabla\varphi_{G\eta}(t), \nabla\psi)_{\mathbb{L}^2(\Omega)^d} - (\xi\varepsilon(u_{G\eta}(t)), \nabla\psi)_{\mathbb{L}^2(\Omega)^d} = (q(t), \psi)_W \quad \forall \psi \in W, t \in [0.T] \tag{4.13}$$

We have the following result for$PV1_{G\eta}$:

Lemma 4.3. *There exists a unique solution $\varphi_{G\eta} \in W^{1.2}(0.T;W)$ satisfies* (4.13), *moreover if φ_1 and φ_2 are two solutions to* (4.13). *Then, there exists a constants $c > 0$ sach that:*

$$|\varphi_1(t) - \varphi_2(t)|_W \leq c\,|u_1(t) - u_2(t)|_V \;\; \forall t \in [0.T]. \tag{4.14}$$

Proof. See [16]. □

In the third step, we use the displacement field u_η obtained in Lemma 4.2 to consider the following variational problem.
Problem $PV1_{\theta\eta}$: Find $\theta_\eta : [0,T] \to E'$ satisfying a.e. $t \in (0,T)$

$$\dot{\theta_\eta}(t) + K\theta_\eta(t) = R_e\dot{u_\eta}(t) + Q(t) \quad t \in (0,T), \text{ in } E', \tag{4.15}$$

$$\theta_\eta(0) = \theta_0. \tag{4.16}$$

Lemma 4.4. *Problem $PV1_{\theta\eta}$ has a unique solution, for all $\eta \in \mathcal{W}$,*

$$\theta_\eta \in W^{1,2}(0;T;E^{'}) \cap L^2(0;T;E) \cap C(0;T;L^2(\Omega)), \quad C > 0, \ \forall \eta \in L^2(\bar{I};V')$$

satisfying

$$|\theta_{\eta_1} - \theta_{\eta_2}|^2_{L^2(\Omega)} \leq C\int_0^T |v_1(s) - v_2(s)|^2_V ds \qquad \forall\, t \in (0,T). \tag{4.17}$$

Proof. The existence and uniqueness result verifying (4.15) follows from classical result on first order evolution equation, applied to the Gelfand evolution triple

$$E \subset F \equiv F^{'} \subset E^{'}$$

We verify that the operator K is linear, continuous, strongly monotone, and from the expression of the operator R, $v_\eta \in W^{1,2}(0,T;V) \Rightarrow Rv_\eta \in W^{1,2}(0,T;F)$, as $Q \in W^{1,2}(0,T;E)$ then $Rv_\eta + Q \in W^{1,2}(0,T;E)$. We deduce (4.17) see [1]. □

We consider the operator

$$\begin{aligned} &\Lambda : \mathbb{L}^2(0.T;\mathcal{H}\times V') \to \mathbb{L}^2(0.T;\mathcal{H}\times V') \text{ be defined as} \\ &\Lambda(G,\ \eta) = (\Lambda_1(G), \Lambda_2(\eta)), \forall G \in \mathbb{L}^2(0.T;\mathcal{H}), \forall \eta \in \mathbb{L}^2(0.T;V'), \\ &|\Lambda(G_2,\ \eta_2) - \Lambda(G_1,\ \eta_2)|^2 = |(\Lambda_1(G_2), \Lambda_2(\eta_2)) - (\Lambda_1(G_1), \Lambda_2(\eta_1))|^2, \\ &|\Lambda_1(G_2) - \Lambda_1(G_1), \Lambda_2(\eta_2) - \Lambda_2(\eta_1)|^2 = |\Lambda_1(G_2) - \Lambda_1(G_1)|^2 \\ &\qquad\qquad + |\Lambda_2(\eta_2) - \Lambda_2(\eta_1)|^2. \end{aligned} \tag{4.18}$$

We show that Λ has a unique fixed point.

Lemma 4.5.

$$\Lambda(G^*,\ \eta^*) = (G^*,\ \eta^*). \tag{4.19}$$

Proof. Let (G_i, η_i) are functions in $\mathbb{L}^2(0.T;\mathcal{H}\times V')$ and denote by $(u_i,\ \varphi_i, \theta_i)$ the functions obtained in Lemma 4.2, Lemma 4.3 and Lemma 4.4,
for$(G,\ \eta) = (G_i,\ \eta_i)$ $i = 1.2$. Let $t \in [0.T]$. From (2.1) it results

$$\begin{aligned} |G_2 - G_1|^2_{\mathcal{H}} \leq c\Big(&|v_2(t) - v_1(t)|^2_V + |\varphi_2(t) - \varphi_1(t)|^2_W \\ &+ |u_2(t) - u_1(t)|^2_V + |\theta_{\eta_1} - \theta_{\eta_2}|^2_{L^2(\Omega)}\Big) \end{aligned} \tag{4.20}$$

Therefore (4.14) and (4.17) yields

$$|G_2 - G_1|^2_{\mathcal{H}} \leq c\left(|v_2(t) - v_1(t)|^2_V + |u_2(t) - u_1(t)|^2_V + \int_0^T |v_1(s) - v_2(s)|^2_V ds\right). \tag{4.21}$$

Using (4.6),we find

$$\begin{aligned}&(\dot{v}_2(t)-\dot{v}_1(t),v_2(t)-v_1(t))+(\mathcal{A}\varepsilon(v_2(t))-\mathcal{A}\varepsilon(v_1(t)),v_2(t)-v_1(t))+\\&(\eta_2(t)-\eta_1(t),v_2(t)-v_1(t))+j(G_2,v_2(t))-j(G_2,v_1(t))\\&-j(G_1,v_2(t))+j(G_1,v_1(t))\leq 0\end{aligned} \tag{4.22}$$

And, we have

$$\begin{aligned}&j(G_2,v_2(t))-j(G_2,v_1(t))-j(G_1,v_2(t))+j(G_1,v_1(t))\\&\leq\int_{\Gamma_3}\mu p\,|R\,G_{2\nu}|\,|v_{2\tau}|\,da-\int_{\Gamma_3}\mu p\,|R\,G_{2\nu}|\,|v_{1\tau}|\,da\\&\quad-\int_{\Gamma_3}\mu p\,|RG_{1\nu}|\,|v_{2\tau}|\,da+\int_{\Gamma_3}\mu p\,|R\,G_{1\nu}|\,|v_{1\tau}|\,da.\end{aligned} \tag{4.23}$$

Moreover, from (3.11), (3.19) and using the properties of R , we find

$$j(G_2,v_2(t))-j(G_2,v_1(t))-j(G_1,v_2(t))+j(G_1,v_1(t))\leq c\,|G_2-G_1|_{\mathcal{H}}\,|v_2-v_1|_V \tag{4.24}$$

So, (4.22) will be

$$\begin{aligned}&(\dot{v}_2(t)-\dot{v}_1(t),v_2(t)-v_1(t))_{V'\times V}+(\mathcal{A}\varepsilon(v_2(t))-\mathcal{A}\varepsilon(v_1(t)),v_2(t)-v_1(t))\\&+(\eta_2(t)-\eta_1(t),v_2(t)-v_1(t))\leq c\,|G_2-G_1|_{\mathcal{H}}\,|v_2-v_1|_V\end{aligned} \tag{4.25}$$

We integrate this equality with respect to time.
We use the initial conditions $v_1(0)=v_2(0)=v_0$, the relation (3.7) and Cauchy-Schwarz's inequality. for all $t\in[0,T]$. Then, using the inequality

$$2ab\leq\frac{a^2}{m_{\mathcal{A}}}+m_{\mathcal{A}}b^2,$$

we obtain

$$\begin{aligned}&\frac{1}{2}\,|v_2(t)-v_1(t)|_V^2+\frac{m_{\mathcal{A}}}{2}\int_0^t|v_2(s)-v_1(s)|_V^2\,ds\\&\leq\frac{1}{2m_{\mathcal{A}}}\int_0^t|\eta_2(s)-\eta_1(s)|_{V'}^2+\frac{m_{\mathcal{A}}}{2}\int_0^t|v_2(s)-v_1(s)|_V^2\,ds\\&+c\left(\int_0^t|G_2(s)-G_1(s)|_{\mathcal{H}}^2+\int_0^t|v_2(s)-v_1(s)|_V^2\,ds.\right)\end{aligned} \tag{4.26}$$

We apply Gronwall's inequality to obtain

$$|v_2(t)-v_1(t)|_V^2\leq c\left(\int_0^t|G_2(s)-G_1(s)|_{\mathcal{H}}^2\,ds+\int_0^t|\eta_2(s)-\eta_1(s)|_{V'}^2\right). \tag{4.27}$$

In other hand

$$|\eta_2(t)-\eta_1(t)|_{V'}^2\leq c\left(|\varphi_2(t)-\varphi_1(t)|_W^2+|u_2(t)-u_1(t)|_V^2+|\theta_{\eta_1}-\theta_{\eta_2}|_{L^2(\Omega)}^2\right) \tag{4.28}$$

Therefore (4.14) and (4.17) yields

$$|\eta_2(t)-\eta_1(t)|_{V'}^2\leq c\left(\int_0^T|v_1(s)-v_2(s)|_V^2ds\right) \tag{4.29}$$

Using (4.6), we find

$$\begin{aligned}&(\dot{v}_2(t)-\dot{v}_1(t),v_2(t)-v_1(t))+(\mathcal{A}\varepsilon(v_2(t))-\mathcal{A}\varepsilon(v_1(t)),v_2(t)-v_1(t))\\&+(\eta_2(t)-\eta_1(t),v_2(t)-v_1(t))+j(G_2,v_2(t))-j(G_2,v_1(t))\\&-j(G_1,v_2(t))+j(G_1,v_1(t))\le 0\end{aligned}\tag{4.30}$$

We integrate this equality with respect to time.
We use the initial conditions $v_1(0)=v_2(0)=v_0$, the relations (3.7), (4.24) and Cauchy-Schwarz's inequality, for all $t\in[0,T]$. Then, using the inequality

$$ab\le c\left(a^2+b^2\right),$$

we obtain

$$\int_0^t |v_2(s)-v_1(s)|_V^2\,ds\le c\left(\int_0^t|\eta_2-\eta_1|_{V'}^2\,ds+\int_0^t|G_2-G_1|_{\mathcal{H}}^2\,ds\right)\tag{4.31}$$

Applying the inequality (4.10) in (4.31). So (4.21) will be

$$|G_2(t)-G_1(t)|_{\mathcal{H}}^2\le c\left(\int_0^t|\eta_2(s)-\eta_1(s)|_{V'}^2\,ds+\int_0^t|G_2(s)-G_1(s)|_{\mathcal{H}}^2\,ds\right).\tag{4.32}$$

From (4.10), (4.29) and (4.31) we find

$$|\eta_2-\eta_1|_{V'}^2\le c\left(\int_0^t|\eta_2-\eta_1|_{V'}^2\,ds+\int_0^t|G_2-G_1|_{\mathcal{H}}^2\,ds\right).\tag{4.33}$$

Using (4.16),to see that

$$|\Lambda(G_2,\ \eta_2)-\Lambda(G_1,\ \eta_1)|^2\le c\int_0^t|(G_2,\ \eta_2)-(G_1,\ \eta_1)|_{\mathcal{H}\times V'}^2\,ds.\tag{4.34}$$

And denoting by p the powers of operator Λ, (4.32) imply by recurrence that

$$\begin{aligned}&|\Lambda^p(G_2,\ \eta_2)-\Lambda^p(G_1,\ \eta_1)|_{L^2(0.T;\mathcal{H}\times V')}^2\\&\qquad\le\frac{(ct)^p}{p!}|(G_2,\ \eta_2)-(G_1,\ \eta_1)|_{L^2(0.T;\mathcal{H}\times V')}^2.\end{aligned}\tag{4.35}$$

This inequality shows that for a sufficiently large p the operator Λ^p is a contraction on the Banach space $\mathbb{L}^2(0.T;\mathcal{H}\times V')$ and therefor, there exists a unique element: $(G^*,\ \eta^*)\in\mathbb{L}^2(0.T;\mathcal{H}\times V')$such that

$$\Lambda(G^*,\ \eta^*)=(G^*,\ \eta^*).\tag{4.36}$$

From (4.18), we find

$$(G^*,\ \eta^*)=(\ \sigma_{G^*\eta^*},\ \xi^*\nabla\varphi_{G^*\eta^*}+\mathcal{G}\varepsilon(u_{G^*\eta^*})-\theta_{G^*\eta^*}M_e).\tag{4.37}$$

□

Now, we have all the ingredients to provide the proof of Theorem 4.1.

Proof of Theorem 4.1. Existence. Let $(G^*,\ \eta^*) \in \mathbb{L}^2(0.T; \mathcal{H} \times V')$ be the fixed point of $PV_{G\eta}$ and let $(u^*,\ \varphi^*, \theta^*)$ be the solution to Problems $PV_{G\eta}$, $PV1_{G\eta}$ and $PV1_{\theta\eta}$ for $(G,\ \eta) = (G^*,\ \eta^*)$, that is, $u^* = u_{G^*\eta^*}$, $\varphi^* = \varphi_{G^*\eta^*}$ and $\theta^* = \theta_{G^*\eta^*}$. It results from(3.24), (3.25) and (3.26) that $(u^*,\ \varphi^*, \theta^*)$ is a solution of Problem PV. Property (4.1) (4.2) and (4.5) follows from Lemmas 4.2, 4.3 and 4.4.
Uniqueness. The uniqueness of the solution is a consequence of the uniqueness of the fixed point of operator defined by (4.18). □

Acknowledgment. This work has been realized thanks to the: Direction générale de la Recherche Scientifique et du Développement Technologique "DGRSDT". MESRS Algeria. And Research Project under code: PRFUCOOL03UN1901200180004.

References

[1] Adly, S., Chau, O., *On some dynamic thermal non clamped contact problems*, Math. Programm., Ser. B, (2013), no. 139, 5-26.

[2] Barboteu, M., Fernández, J.R., Ouafik, Y., *Numerical analysis of a frictionless viscoelastic piezoelectric contact problem*, M2AN Math. Model. Numer. Anal., (in press).

[3] Barbu, V., *Optimal Control of Variational Inequalities*, Pitman, Boston, 1984.

[4] Batra, R.C., Yang,J.S., *Saint Venant's principle in linear piezoelectricity, Journal of Elasticity*, **38**(1995), 209-218.

[5] Brézis, H., *Analyse Fonctionnelle, Théorie et Application,* Masson, Paris, 1987.

[6] Chau, O., *On a class of second order evolution inéquality and application*, Int. J. of Appl. Math. and Mech., **4**(2008), no. 1, 24-48.

[7] Drabla, S., Zellagui, Z., *Analysis of a electro-elastic contact problem with friction and adhesion*, Stud. Univ. Babeş-Bolyai Math., **54**(2009), 75-99.

[8] Duvaut G., Lions, J.L., *Inequalities in Mechanics and Physics*, Springer-Verlag, Berlin, 1988.

[9] Han, W., Sofonea, M., Kazmi, K., *Analysis and numerical solution of a frictionless contact problem for electro-elastic-visco-plastic materials*, Comput. Methods Appl. Mech. Engrg., **196**(2007), 3915-3926.

[10] Jarusek, J., Sofonea, M. *On the solvability of dynamic elastic-visco-plastic contact problems*, Z. Angew. Math. Mech., **88**(2008), 3-22.

[11] Mindlin, R.D., *Polarisation gradient in elastic dielectrics*, Int. J. Solids Structures, **4**(1968), 637-663.

[12] Mindlin, R.D., *Continuum and lattice theories of influence of electromechanical coupling on capaci- tance of thin dielectric films*, Int. J. Solids, **4**(1969), 1197-1213.

[13] Mindlin, R.D., *Elasticity, Piezoelectricity and Cristal lattice dynamics*, J. of Elasticity, **4** (1972), 217-280.

[14] Rochdi, M., Shillor, M., Sofonea, M., *A quasistatic contact problem with directional friction and damped response*, Appl. Anal., **68**(1998), 409-422.

[15] Selmani, L., Bensebaa, N., *An electro-viscoelastic contact problem with adhesion and damage*, Rend. Semin. Mat. Univ. Politec. Torino, **66**(2008), no. 2.

[16] Selmani, M., Selmani, L., *Frictional contact problem for elastic-viscoplastic materials with thermal effect*, Appl. Math. Mech. (English Ed.), **34**(2013), no. 6, 761-776.

[17] Sofonea, M., Essoufi, E.-H., *Quasistatic frictional contact of a viscoelastic piezoelectric body*, Adv. Math. Sci. Appl., **14**(2004), no. 1, 25-40.

[18] Touzaline, A., *A viscoelastic frictionless contact problem with adhesion*, Boll. Pol. Acad. Sci. Math., **63**(2015), 53-66.

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Stud. Univ. Babeş-Bolyai Math. 66(2021), No. 4, 783–792
DOI: 10.24193/subbmath.2021.4.14

A weighted logarithmic barrier interior-point method for linearly constrained optimization

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Abstract. In this paper, a weighted logarithmic barrier interior-point method for solving the linearly convex constrained optimization problems is presented. Unlike the classical central-path, the barrier parameter associated with the perturbed barrier problems, is not a scalar but is a weighted positive vector. This modification gives a theoretical flexibility on its convergence and its numerical performance. In addition, this method is of a Newton descent direction and the computation of the step-size along this direction is based on a new efficient technique called the tangent method. The practical efficiency of our approach is shown by giving some numerical results.

Mathematics Subject Classification (2010): 90C30, 90C25, 90C51.

Keywords: Logarithmic barrier method, linearly constrained convex optimization, interior-point methods.

1. Introduction

In this paper, we consider the linearly convex constrained optimization (LCCO) problem:

$$\bar{p} = \min f(x) \text{ subject to } x \in \mathcal{F}, \tag{P}$$

where the objective function $f : \mathbb{R}^n \to \mathbb{R}$ is twice differentiable and convex over the feasible set $\mathcal{F} = \{x \in \mathbb{R}^n : x \geq 0,\ Ax = b\}$, A is a given $(m \times n)$ matrix with full rank row m and $b \in \mathbb{R}^m$.

This problem has many important applications in theory as well in practice. In particular, it includes linear and quadratic optimization. Feasible logarithmic barrier interior-point methods gained much more attention than others. Their derived algorithms enjoy some interesting results such as polynomial complexity and numerical efficiency. However, these algorithms require that the starting point must be strictly feasible and close to the central-path.This is a hard practical task to release and even impossible. On the other hand, at each iteration, they compute a descent direction and determine a step-size on this direction. It is known that computing this latter is very expensive while using classical line search methods. In order to overcome these two difficulties, we suggest for the first, a weighted-path (see [1], [2], [3], [8], [10], [12]) where a relaxation parameter associated with perturbed problems is introduced in order to give more flexibility on the numerical aspects. Beside, we propose a new numerical efficient procedure called the tangent method for determining this displacement. Across these two modifications the numerical results obtained by our algorithm are totally improved with respect to the classical logarithmic barrier interior-point approach (see [10], [13]). The paper is organized as follows. In section 2, perturbed relaxation problems based on the weighted barrier penalization are given where the convergence to the original problem is studied. The computation of the direction and of the step-size are stated. Finally, a weighted-path interior-point algorithm is presented. In section 3, some numerical results are given to show the efficiency of our approach. Finally a conclusion and remarks end the section 4.

2. The weighted barrier penalization

Throughout the paper, we assume that the following assumptions hold.

1. There exit a strictly feasible point $x_0 > 0$ such that $Ax_0 = b$.
2. The set of optimal solutions of P is non empty bounded set.

It follows from the second hypothesis that

$$\{d \in \mathbb{R}^n : f_\infty(d) \le 0,\, d \ge 0,\, Ad = 0\} = \{0\},$$

where f_∞ denote the recession function of f. We deduce from the optimality conditions that x^* is a solution of P if and only if there exists an $y^* \in \mathbb{R}^m$ and $z^* \in \mathbb{R}^n$ such that

$$\nabla f(x^*) + A^T y^* = z^* \ge 0, \quad Ax^* = b, \quad \langle z^*, x^* \rangle = 0, x^* \ge 0. \tag{2.1}$$

2.1. The weighted perturbed problems

Let us define the function $\theta : \mathbb{R} \times \mathbb{R} \to (-\infty, +\infty]$ by

$$\theta(t,w) = \begin{cases} t(\log t - \log w) & \text{if} \quad t > 0,\ w > 0, \\ 0 & \text{if} \quad t = 0,\ w \ge 0, \\ +\infty & \text{otherwise.} \end{cases}$$

The function θ is convex, lower semi-continuous and proper. We consider now the following function defined on $\mathbb{R}^n_+ \times \mathbb{R}^n_+$ by

$$\varphi(\mu r, x) = \begin{cases} f(x) + \sum_{i=1}^{n} \theta(\mu r_i, x_i) & \text{if } x \in \mathcal{F}, \\ +\infty & \text{otherwise,} \end{cases}$$

where $\mu > 0$ is the barrier parameter and $r = (r_1, r_2, \ldots, r_n)^T \in \mathbb{R}^n_+$, the vector of the weight associated with the barrier function.

Finally, we introduce the function p^r defined by

$$p^r(\mu) = \inf_x \left[\varphi^r_\mu(x) = \varphi(\mu r, x) : x \in \mathbb{R}^n \right]. \tag{P^r_μ}$$

The function p^r is convex since φ_μ^r is convex. By construction, P_0^r is only the problem P with $\bar{p} = p^r(0)$. The function φ_μ^r is convex, lower semi-continuous and proper, its recession function is given by

$$(\varphi_\mu^r)_\infty(d) = \lim_{\alpha \to +\infty} \frac{\varphi_\mu^r(x_0 + \alpha d) - \varphi_\mu^r(x_0)}{\alpha}.$$

We obtain

$$(\varphi_\mu^r)_\infty(d) = \begin{cases} f_\infty(d) & \text{if} \quad d \geq 0,\ Ad = 0, \\ +\infty & \text{otherwise.} \end{cases}$$

Then

$$\{d \in \mathbb{R}^n \,:\, (\varphi_\mu^r)_\infty(d) \leq 0\} = \{d \in \mathbb{R}^n \,:\, f_\infty(d) \leq 0,\ d \geq 0,\ Ad = 0\},$$

where d is the descent direction and α is the step-size.
Since this set is reduced to $\{0\}$ then the problem P_μ^r admits an optimal solution for each $\mu > 0$. The function φ_μ^r is strictly convex for all $\mu \geq 0$ and $r \geq 0$, then P_μ^r has an unique optimal solution denoted by x_μ^r.

2.2. Convergence of the weighted perturbed solutions to the optimal solution of P_μ^r

The necessary and sufficient optimality conditions of (P_μ^r) imply that there exits $y_\mu^r = y(\mu, r) \in \mathbb{R}^m$, such that

$$\begin{aligned} \nabla f(x_\mu^r) - \mu X^{-1} r + A^T y_\mu^r &= 0, \quad (2.2) \\ A x_\mu^r &= b, \quad (2.3) \end{aligned}$$

where $X = Diag(x_\mu^r)$.
Note that y_μ^r is uniquely defined since A is of full rank row. In fact, the couple (x_μ^r, y_μ^r) is the solution of the system $H(x, y) = 0$ where

$$H(x,y) = \begin{pmatrix} \nabla f(x) - \mu X^{-1} r + A^T y \\ Ax - b \end{pmatrix}.$$

By the implicit function theorem, the functions $\mu \mapsto x(\mu, r) = x_\mu^r$ and $\mu \mapsto y(\mu, r) = y_\mu^r$ are differentiable on $(0, \infty)$ and we have,

$$\begin{pmatrix} \nabla^2 f(x_\mu^r) + \mu R X^{-2} & A^T \\ A & 0 \end{pmatrix} \begin{pmatrix} x'(\mu, r) \\ y'(\mu, r) \end{pmatrix} = \begin{pmatrix} X^{-1} r \\ 0 \end{pmatrix}, \quad (2.4)$$

where R=Diag(r), it follows that the function p^r is differentiable on $(0, \infty)$. Recall that

$$p^r(\mu) = f(x_\mu^r) + \mu \sum_{i=1}^n r_i (\ln \mu r_i - \ln(x_i)_\mu^r),$$

and then

$$(p^r(\mu))' = \sum_{i=1}^n r_i (1 + \ln \mu r_i - \ln(x_i)_\mu^r) + \langle \nabla f(x_\mu^r) - \mu X^{-1} r, x'(\mu, r) \rangle.$$

In view of (2.2) and (2.4)

$$\begin{aligned}(p^r(\mu))' &= \sum_{i=1}^{n} r_i(1+\ln \mu r_i - \ln(x_i)_\mu^r) - \langle A^T y_\mu^r, x'(\mu,r)\rangle,\\ &= \sum_{i=1}^{n} r_i(1+\ln \mu r_i - \ln(x_i)_\mu^r) - \langle y_\mu^r, Ax'(\mu,r)\rangle,\\ &= \sum_{i=1}^{n} r_i(1+\ln \mu r_i - \ln(x_i)_\mu^r).\end{aligned}$$

Since $x_\mu^r \in \mathcal{F}$ and p^r is convex we obtain:

$$f(x_\mu^r) \geq \bar{p} = p^r(0) \geq p^r(\mu) + (0-\mu)(p^r(\mu))' = f(x_\mu^r) - \mu\|r\|_1.$$

Consequently, we have

$$\bar{p} \leq f(x_\mu^r) \leq \bar{p} + \mu\|r\|_1.$$

Then if

$$\mu \mapsto 0, f(x_\mu^r) = \bar{p}.$$

Now, we interested to the weighted-path of $\{x_\mu^r\}$ when $\mu \mapsto 0$.
i) Case where f is strongly convex with coefficient $\tau > 0$. Hence P has a unique optimal solution x^*, and we have

$$\mu\|r\|_1 \geq f(x_\mu^r) - f(x^*) \geq \langle \nabla f(x^*), x_\mu^r - x^*\rangle + \frac{\tau}{2}\|x_\mu^r - x^*\|^2.$$

In view of (2.1), we deduce

$$\mu\|r\|_1 \geq \langle z^*, x_\mu^r\rangle + \frac{\tau}{2}\|x_\mu^r - x^*\|^2 \geq \frac{\tau}{2}\|x_\mu^r - x^*\|^2,$$

$$\|x_\mu^r - x^*\| \leq \sqrt{\frac{2\mu\|r\|_1}{\tau}}.$$

ii) For the case where f is only convex is more complex. Note first that for $\mu \leq 1$,

$$x_\mu^r \in \{x : x \geq 0,\ Ax = b,\ f(x) \leq \|r\|_1 + \bar{p}\}.$$

This set is closed convex non empty. Its recession cone is

$$\{d \in \mathbb{R}^n : f_\infty(d) \leq 0,\ d \geq 0,\ Ad = 0\} = \{0\}.$$

By the second assumption the set of optimal solutions of P is bounded which implies that each adherence value of $\{x_\mu^r\}$ when $\mu \mapsto 0$ is an optimal solution of P.

Remark 2.1. If $r = e$, where e is the vector of ones, then the weighted-path coincides with the classical path (see[6]).

2.3. The description of the method

Letting $\mathcal{F}^0 = \{x \in \mathbb{R}^n : x > 0, Ax = b\}$ the set of strictly feasible points. The principle of the method is as follows: Let $(\mu_k r, x_k) \in \mathbb{R}^n_+ \times \mathcal{F}^0$, the current iterate.

1. We make an approximated minimization of the weighted perturbed $P^r_{\mu_k}$ which gives a new point x_{k+1} such that $\varphi(\mu_{k+1} r, x_{k+1}) < \varphi(\mu_k r, x_k)$.

2. We take $\mu_{k+1} < \mu_k$.

We iterated until we obtained an approximated optimal solution of the original problem. The weighted perturbed problem is defined by

$$\min_x \varphi^r_\mu(x) = \min_x [\, f(x) + \sum_{i=1}^n \theta(\mu r_i, x_i) \ : x \in \mathcal{F}]. \qquad (P^r_\mu)$$

2.4. The Newton descent direction

At $x \in \mathcal{F}^0$, the Newton descent direction d is given by solving the following quadratic convex program:

$$\min_d [\, \langle \nabla \varphi^r_\mu(x), d\rangle + \frac{1}{2}\langle \nabla^2 \varphi^r_\mu(x) d, d\rangle : Ad = 0\,].$$

It suffices to solve the linear system with $n + m$ equations

$$\begin{pmatrix} \nabla^2 f(x) + \mu R X^{-2} & A^T \\ A & 0 \end{pmatrix} \begin{pmatrix} d \\ s \end{pmatrix} = \begin{pmatrix} \mu X^{-1} r - \nabla f(x) \\ 0 \end{pmatrix}, \qquad (2.5)$$

where $s \in \mathbb{R}^m$.
It easy to prove that the linear system (2.5) has a unique solution. The descent direction being thus obtained, it is now question of minimizing a function of one real variable to obtain the step-size α.

$$\gamma^r(\alpha) = \varphi^r_\mu(x + \alpha d) - \varphi^r_\mu(x) = f(x + \alpha d) - f(x) \ - \mu \sum_{i=1}^n r_i \ln(1 + \alpha t_i),$$

where $t = X^{-1}d$, the function γ^r is convex.
Next task, we propose a new method to determine the step-size.

2.5. A tangent method for determining the step-size

Our approach is try out a sequence of candidate values for α, when the condition $(\gamma^r(\alpha))' \leq \epsilon$ is satisfied, stopping and accept this value. We can say that this technique is done in two phases.

1. The first phase finds an interval containing the required step-size, the choice of the bounds of the interval is similar to the bisection method, when we restrict the value of α until we find the required value.
2. The second phase computes the optimal step-size within this interval, in this phase we determine the tangents T_1 and T_2 in the bounds of the interval and we select the value corresponding to the intersection of the tangents T_1 and T_2.

The upper bound on the step-size α is given by

$$\alpha_{max} = \min\left\{-\frac{x_i}{d_i}; \quad i \in \hat{I}\right\},$$

where

$$\hat{I} = \{i : \quad d_i < 0\}.$$

Because the convexity of the function $\gamma^r(\alpha)$, this technique will be more efficiency in practice, the next figures shows clearly this idea:

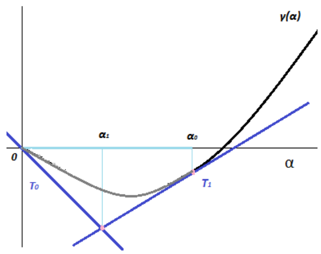

FIGURE 1

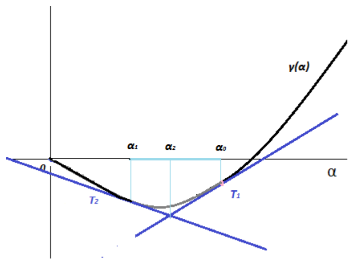

FIGURE 2

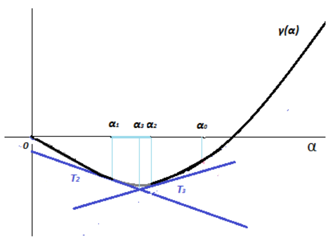

FIGURE 3

The tangent algorithm for determining the step-size is follows.

Algorithm

Input

- An accuracy parameter $\epsilon > 0$;
- A threshold parameter $0 < \beta < 1$;

$a = 0, \quad b = \beta\alpha_{max}$, such that $(\gamma^r(\alpha_{max}))' > 0$;

$\alpha = \frac{b}{2}$;

While $|(\gamma^r(\alpha))'| > \epsilon$ **do**

if $(\gamma^r(\alpha))' > 0$ then

$b = \alpha$;

if not

$a = \alpha$;

end if

$\alpha = \frac{-(\gamma^r(b))'b + \gamma^r(b) + (\gamma^r(a))'a - \gamma^r(a)}{(\gamma^r(a))' - (\gamma^r(b))'}$;

End While

We are now ready to state the generic algorithm for solving LCCO.

The generic algorithm

Threshold parameters $\epsilon > 0, \bar{\mu} > 0$ and $\lambda \in [0, 1]$, are given;

Start with $x_0 \in \mathcal{F}^0$, $\mu > \bar{\mu}$ and a weight vector $r > 0$;

1) Solve the linear system (2.5) to obtain d;

2) Take $t = X^{-1}d$;

If $\|t\| \geq \epsilon$

- Determinate $\tilde{\alpha}$ with tangent method ;
- Update $x_{k+1} = x_k + \tilde{\alpha}d_k$, $\mu_{k+1} = \lambda\mu_k$ and return to 1;

If $\|t\| \leq \epsilon$

Case 1. $\mu_k \leq \bar{\mu}$

STOP we have obtained a good approximation of the optimal solution of P;

Case 2. $\mu_k > \bar{\mu}$

We have obtained a good approximation of $p^r(\mu)$, do $\mu_{k+1} = \lambda\mu_k$ and go to 1;

3. Numerical results

In the following section, we apply our algorithm on some different examples of LCCO. A comparative numerical tests with a classical line search are presented, Our implementation is done by the Scilab 5.4.1. We use in the sequel the following notation.

Method 1: the first alternative uses the tangent technique.

Method 2: the second alternative uses the Wolfe method.

Outer: the number of outer iterations.
Inner: the number of inner iterations.
Objective: the optimal value of the objective function $\bar{p}$.
Time: the time measured in seconds. Our tolerance is $\epsilon = 10^{-6}$ in all our testing examples.
The below examples are taken from literature [10], the numerical obtained results with different values of r such as $r_1 = (0.9,\ 1,\ 0.03)^T$, $r_2 = (2,\ 1,\ 3,\ 1,\ 4)^T$, $r_3 = (1,\ 1,\ 0.4,\ 1,\ 0.4,\ 1,\ 0.4,\ 0.4,\ 1,\ 0.96)^T$ and

$$r_4 = \begin{pmatrix} 0.1,\ 0.1,\ 0.1,\ 0.1,\ 0.1,\ 0.1,\ 0.1, 0.1,\ 0.1,\ 0.1 \\ 0.3,\ 0.3,\ 0.3,\ 0.3,\ 0.3,\ 0.3,\ 0.3,\ 0.3,\ 0.3,\ 0.3 \end{pmatrix},$$

are summarized in table 1.

		Method 1		Method 2	
Example	Size (m,n)	Inner	Outer	Inner	Outer
1	(2,3)	7	7	162	7
2	(3,5)	24	6	46	8
3	(3,10)	20	7	40	8
4	(10,20)	25	7	83	7

Table 1.

In the following, we compare our approach with the classical path method (non weighted case).
Example with variable size. We consider the following LCCO problem:

$$\bar{p} = \min[f(x) : x \geq 0,\ Ax = b],$$

where $f(x) = \sum_{i=1}^{n} x_i \ln x_i$, $b_i = 1$ and

$$A[i,j] = \left\{ \begin{array}{ll} 1 & \text{if } i = j \text{ or } \ j = i + m), \\ 0 & \text{else,} \end{array} \right\}, \text{with}\, n = 2m.$$

The strictly feasible starting point is:

$$x^0 = \begin{pmatrix} 0.7, & \ldots, & 0.7, & 0.3, & \ldots, & 0.3 \end{pmatrix}^T.$$

The exact solution is:

$$x^* = \begin{pmatrix} 0.5, & 0.5, & \ldots & ,0.5 \end{pmatrix}^T.$$

The optimal values with different size of n are:

n	20	400	900
Objective	-6.9314718	-138.62944	-311.911623

The obtained numerical results with different size of n and barrier parameter μ are stated in tables 2 and 3.

Weighted case

The weigh vector is $r = (0.011, \dots, 0.011, 0.022 \dots, 0.022)^T$.

	$n = 20$		$n = 400$		$n = 900$	
μ	Outer	Time	Outer	Time	Outer	Time
0.01	2	0.296×10^{-3}	2	0.17160	2	23.476212
0.25	4	0.316×10^{-3}	4	0.28762	4	32.40515
1	5	0.499×10^{-3}	5	0.355702	5	41.84014
5	6	0.665×10^{-3}	6	0.400961	6	52.96016

Table 2.

Non weighted case

	$n = 20$		$n = 400$		$n = 900$	
μ	Outer	Time	Outer	Time	Outer	Time
0.01	4	0.325×10^{-3}	4	0. 2915518	4	33.147112
0.25	6	0.482×10^{-3}	6	0.3832547	6	40.40114
1	7	0.591×10^{-3}	7	0.4512415	7	49.88745
5	8	0.835×10^{-3}	8	0.6001489	8	58.91456

Table 3.

4. Conclusion and remarks

In this paper we have introduced a relaxation of the classical path of the perturbed LCCO problem and we have presented a new technique for determining the step-size. These have a great influence on the acceleration of the convergence of the algorithm i.e., the number of iterations and the time produced are reduced significantly. This analysis may be extended to inducing a general weight vector $w > 0$ as the barrier parameter instead of the form μr with $r > 0$.

References

[1] Achache, M., *A weighted-path-following method for the linear complementarity problem*, Stud. Univ. Babeş-Bolyai Informatica, **49**(2004), no. 1, 61-73.

[2] Achache, M., *A polynomial-time weighted path-following interior-point algorithm for linear optimization*, Asian-Eur. J. Math., **13**(2020), no. 1, 1-9.

[3] Achache, M., Khebchache, R., *A full-Newton step fessible weighted primal-dual interior point algorithm for monotone LCP*, Afr. Mat., **27**(2016), no. 3, 591-601.

[4] Achache, M., Khebchache, R., *A full-Newton step feasible weighted primal-dual interior point algorithm for monotone LCP*, Afr. Mat., **27**(2016), no. 3, 591-601.

[5] Alzalg, B., *A logarithmic barrier interior-point method based on majorant functions for second-order cone programming*, Department of Mathematics, The University of Jordan, Amman 11942, Jordan, (2017).

[6] Bachir Cherif, B., Merikhi, B., *A penalty method for nonlinear programming*, RAIRO Oper. Res., **53**(2019), no. 1, 29-38.

[7] Crouzeix, J.-P., Seeger, A., *New bounds for the extreme values of a finite sample of real numbers*, J. Math. Anal. Appl., **197**(1996), 411-426.

[8] Crouzeix, J.-P., Merikhi, B., *A logarithm barrier method for semi-definite programming*, RAIRO Oper. Res., **42**(2008), no. 2, 123-139.

[9] Darvay, Zs., *A weighted-path-following method for linear optimization*, Stud. Univ. Babeş Bolyai Informatica, **47**(2002), no. 1, 3-12.

[10] Goutali, M., *Complexité et implimentation numérique d'une méthode de points interieurs pour la programmation convexe*, Mémoire de Doctorat, Dept. Math. Univ. Sétif, 2018.

[11] Kebbiche, Z., Benterki,D., *A weighted-path-following method for linearly constrained convex programming*, Rev. Roumaine Math. Pures Appl., **57**(2012), no. 3, 245-256.

[12] Kettab, S., Benterki, D., *A relaxed logarithmic barrier method for semidefinite programming*, RAIRO Oper. Res., **49**(2015), 555-568.

[13] Menniche, L., Benterki, D., *Alogarithmic barrier approach for linear programming*, J. Comput. Appl. Math., **312**(2017), 267-275.

[14] Zhang, M., Bai, Y., Wang, G., *A new primal-dual path-following interior-point algorithm for linearly constrained convex optimization*, J. Shanghai Univ., **12**(2008), no. 6, 475-480.

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