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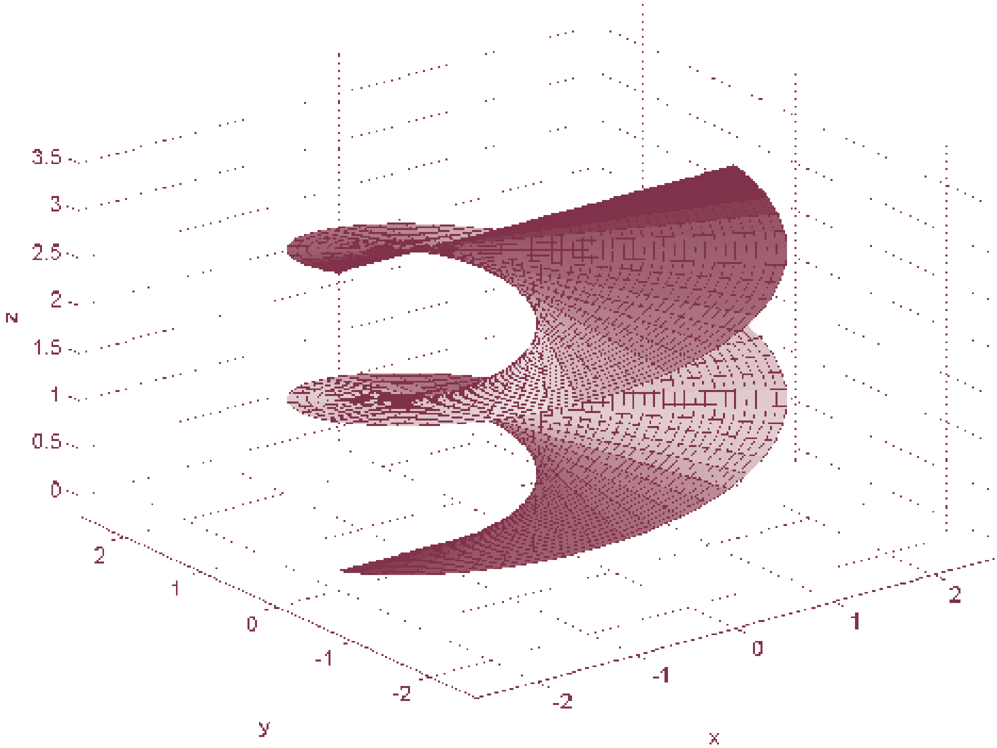

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Group graded Morita equivalences for wreath products

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Abstract. Starting with group graded Morita equivalences, we obtain Morita equivalences for tensor products and wreath products.

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1. Introduction

In this article, we continue the study done in [2], [3] and [4], and we obtain group graded Morita equivalences for tensor products (Proposition 3.3) and wreath products (Theorem 5.3). The main motivation for such constructions in the representation theory of finite groups is given by the fact that in order to prove most reduction theorems, recent results of Britta Späth, surveyed in [5], [6] and [7], show that a new character triple can be constructed via a wreath product construction of character triples ([7, Theorem 2.21]). There is a link between character triples and group graded Morita equivalences, presented in [3], so we want to prove that a similar wreath product construction can also be made for the corresponding group graded Morita equivalences.

More precisely, in Theorem 6.7 of [3], it is proved that certain character triples relations utilized by Britta Späth: the first-order relation ([7, Definition 2.1]) and the central-order relation ([7, Definition 2.7]), are consequences of a special type of group graded Morita equivalences induced by a graded bimodule over a G-graded G-acted algebra (usually denoted by $\mathcal{C}$ as in Section 2.3), where G is a finite group. More details about group graded Morita theory over $\mathcal{C}$ can be found in [4].

Another motivation comes from the fact that it is already known by [1, Theorem 5.1.21] that Morita equivalences can be extended to wreath products.

This paper is organized as follows: In Section 2, we introduce the general notations and we recall from [3] the definitions of a G-graded G-acted algebra, of a G-graded algebra over $\mathcal{C}$, of a G-graded bimodule over $\mathcal{C}$ and the notion of a G-graded Morita equivalence over $\mathcal{C}$. In Section 3, we prove that the previously recalled algebraic constructions are compatible with tensor products and the main proposition in this section, Proposition 3.3, proves that the tensor products of some group graded Morita equivalent algebras over some group graded group acted algebras remain group graded Morita equivalent over a group graded group acted algebra. In Section 4, we prove that the previously enumerated algebra types are also compatible with wreath products. Finally, in Section 5, our main result, Theorem 5.3, proves that the wreath product between a G-graded bimodule over $\mathcal{C}$ and S_n (the symmetric group of order n) is also a group graded bimodule over $\mathcal{C}^{\otimes n}$, and moreover, if this bimodule induces a G-graded Morita equivalence over $\mathcal{C}$, then its wreath product with S_n will induce a group graded Morita equivalence over $\mathcal{C}^{\otimes n}$.

2. Preliminaries

2.1. All rings in this paper are associative with identity $1 \neq 0$ and all modules are left (unless otherwise specified) unital and finitely generated. Throughout this article n will represent an arbitrary nonzero natural number, and $\mathcal{O}$ is a commutative ring.

2.2. Let G be a finite group and N a normal subgroup of G. We denote by $\bar{G} := G/N$.

Note that most results in this paper will utilize "$\bar{G}$-gradings", although this is not essential: one may consider instead the gradings to be given directly by G. The reasoning behind this particular choice is to match our notations previously used in articles [2] and [3], given that our main application for the results of this project is the strongly $\bar{G}$-graded algebra $A = b\mathcal{O}G$, where b is a $\bar{G}$-invariant block of $\mathcal{O}N$.

2.3. We recall from [3] the following definitions:

Definition 2.4. An algebra $\mathcal{C}$ is a *$\bar{G}$-graded $\bar{G}$-acted algebra* if

(1) $\mathcal{C}$ is $\bar{G}$-graded, and we write $\mathcal{C} = \bigoplus_{\bar{g}\in\bar{G}} \mathcal{C}_{\bar{g}}$;
(2) $\bar{G}$ acts on $\mathcal{C}$ (always on the left in this article);
(3) for all $\bar{g}, \bar{h} \in \bar{G}$ and for all $c \in \mathcal{C}_{\bar{h}}$ we have ${}^{\bar{g}}c \in \mathcal{C}_{\bar{g}\bar{h}}$.

Definition 2.5. Let $\mathcal{C}$ be a $\bar{G}$-graded $\bar{G}$-acted algebra. We say the A is a $\bar{G}$-graded algebra over $\mathcal{C}$ if there is a $\bar{G}$-graded $\bar{G}$-acted algebra homomorphism

$$\zeta : \mathcal{C} \to C_A(B),$$

where $B := A_1$ and $C_A(B)$ is the centralizer of B in A, i.e. for any $\bar{h} \in \bar{G}$ and $c \in \mathcal{C}_{\bar{h}}$, we have $\zeta(c) \in C_A(B)_{\bar{h}}$, and for every $\bar{g} \in \bar{G}$, $\zeta({}^{\bar{g}}c) = {}^{\bar{g}}\zeta(c)$.

Definition 2.6. Let A and A' be two $\bar{G}$-graded crossed products over a $\bar{G}$-graded $\bar{G}$-acted algebra $\mathcal{C}$, with structure maps ζ and ζ', respectively.

a) We say that $\tilde{M}$ is a $\bar{G}$-graded (A, A')-bimodule over $\mathcal{C}$ if:

(1) $\tilde{M}$ is an (A, A')-bimodule;

(2) $\tilde{M}$ has a decomposition $\tilde{M} = \bigoplus_{\bar{g}\in\bar{G}} \tilde{M}_{\bar{g}}$ such that $A_{\bar{g}}\tilde{M}_{\bar{x}}A'_{\bar{h}} \subseteq \tilde{M}_{\bar{g}\bar{x}\bar{h}}$, for all $\bar{g}, \bar{x}, \bar{h} \in \bar{G}$;

(3) $\tilde{m}_{\bar{g}}c = {}^{\bar{g}}c\tilde{m}_{\bar{g}}$, for all $c \in \mathcal{C}$, $\tilde{m}_{\bar{g}} \in \tilde{M}_{\bar{g}}$, $\bar{g} \in \bar{G}$, where $c\tilde{m} = \zeta(c)\tilde{m}$ and $\tilde{m}c = \tilde{m}\zeta'(c)$, for all $c \in \mathcal{C}$, $\tilde{m} \in \tilde{M}$.

b) $\bar{G}$-graded (A, A')-bimodules over $\mathcal{C}$ form a category, where the morphisms between $\bar{G}$-graded (A, A')-bimodules over $\mathcal{C}$ are just homomorphisms between $\bar{G}$-graded (A, A')-bimodules.

Definition 2.7. Let A and A' be two $\bar{G}$-graded crossed products over a $\bar{G}$-graded $\bar{G}$-acted algebra $\mathcal{C}$, and let $\tilde{M}$ be a $\bar{G}$-graded (A, A')-bimodule over $\mathcal{C}$. Clearly, the A-dual $\tilde{M}^* = \mathrm{Hom}_A(\tilde{M}, A)$ of $\tilde{M}$ is a $\bar{G}$-graded (A', A)-bimodule over $\mathcal{C}$. We say that $\tilde{M}$ induces a $\bar{G}$-graded Morita equivalence over $\mathcal{C}$ between A and A', if $\tilde{M} \otimes_{A'} \tilde{M}^* \simeq A$ as $\bar{G}$-graded (A, A)-bimodules over $\mathcal{C}$ and if $\tilde{M}^* \otimes_A \tilde{M} \simeq A'$ as $\bar{G}$-graded (A', A')-bimodules over $\mathcal{C}$.

3. Tensor products

3.1. Consider G_i to be a finite group, N_i to be a normal subgroup of G_i and denote by $\bar{G}_i = G_i/N_i$, for all $i \in \{1, \ldots, n\}$. We denote by

$$\bar{\mathbf{G}} := \prod_{i=1}^{n} \bar{G}_i.$$

Lemma 3.2. *Let A_i be $\bar{G}_i$-graded algebras and $\mathcal{C}_i$ be $\bar{G}_i$-graded $\bar{G}_i$-acted algebras, for all $i \in \{1, \ldots, n\}$. The following affirmations hold:*

(1) *The tensor product $\mathbf{A} := A_1 \otimes \ldots \otimes A_n$ is a $\bar{\mathbf{G}}$-graded algebra;*

(2) *If A_i are $\bar{G}_i$-graded crossed products, for all $i \in \{1, \ldots, n\}$, then $\mathbf{A}$ is a $\bar{\mathbf{G}}$-graded crossed product;*

(3) *The tensor product $\boldsymbol{\mathcal{C}} := \mathcal{C}_1 \otimes \ldots \otimes \mathcal{C}_n$ is a $\bar{\mathbf{G}}$-graded $\bar{\mathbf{G}}$-acted algebra;*

(4) *If A_i are $\bar{G}_i$-graded algebras over $\mathcal{C}_i$, for all $i \in \{1, \ldots, n\}$, then $\mathbf{A}$ is a $\bar{\mathbf{G}}$-graded algebra over $\boldsymbol{\mathcal{C}}$.*

Proof. (1) It is clear that $\mathbf{A}$ is a $\bar{\mathbf{G}}$-graded algebra, with the $(g_1, \ldots, g_n)$-component

$$\mathbf{A}_{(g_1,\ldots,g_n)} := A_{1,g_1} \otimes \ldots \otimes A_{n,g_n},$$

where A_{i,g_i} is the g_i-component of A_i, for all $g_i \in \bar{G}_i$ and for all i.

(2) Choose invertible homogeneous elements $u_{i,g}$ in each $A_{i,g}$, for all $g \in \bar{G}_i$ and for all i. Thus, for each $(g_1, \ldots, g_n) \in \bar{\mathbf{G}}$ the homogeneous element

$$u_{(g_1,\ldots,g_n)} := u_{1,g_1} \otimes \ldots \otimes u_{n,g_n} \in \mathbf{A}_{(g_1,\ldots,g_n)}$$

is clearly invertible.

(3) The $\bar{\mathbf{G}}$-grading of $\boldsymbol{\mathcal{C}}$ is a given by (1). The action of $\bar{\mathbf{G}}$ on $\boldsymbol{\mathcal{C}}$ is defined by

$${}^{(g_1,\ldots,g_n)}(a_1 \otimes \ldots \otimes a_n) := {}^{g_1}a_1 \otimes \ldots \otimes {}^{g_n}a_n,$$

for all $(g_1,\dots,g_n) \in \bar{\mathbf{G}}$ and $a_1 \otimes \dots \otimes a_n \in \boldsymbol{\mathcal{C}}$. It is easy too see that for all $(g_1,\dots,g_n),(h_1,\dots,h_n) \in \bar{\mathbf{G}}$ and for all $a_1 \otimes \dots \otimes a_n \in \boldsymbol{\mathcal{C}}_{(h_1,\dots,h_n)}$ we have

$$^{(g_1,\dots,g_n)}(a_1 \otimes \dots \otimes a_n) \in \boldsymbol{\mathcal{C}}_{(g_1,\dots,g_n)(h_1,\dots,h_n)}.$$

(4) By part (1), the identity component of $\mathbf{A}$ is $\mathbf{B} = B_1 \otimes \dots \otimes B_n$, where B_i is the identity component of A_i, for all i.

By the assumptions, we have the $\bar{G}_i$-graded $\bar{G}_i$-acted structure homomorphisms

$$\zeta_i : \mathcal{C}_i \to C_{A_i}(B_i),$$

for all i. We define $\zeta : \boldsymbol{\mathcal{C}} \to C_{\mathbf{A}}(\mathbf{B})$ by

$$\zeta(a_1 \otimes \dots \otimes a_n) = \zeta_1(a_1) \otimes \dots \otimes \zeta_n(a_n),$$

for all $a_1 \otimes \dots \otimes a_n \in \boldsymbol{\mathcal{C}}$. It is easy to prove that ζ verifies the conditions of Definition 2.5. □

Proposition 3.3. *Assume that $\mathcal{C}_i$ are $\bar{G}_i$-graded $\bar{G}_i$-acted algebras and that A_i and A_i' are $\bar{G}_i$-graded crossed products over $\mathcal{C}_i$, for all $i \in \{1,\dots,n\}$. If A_i and A_i' are $\bar{G}_i$-graded Morita equivalent over $\mathcal{C}_i$, and if $\tilde{M}_i$ is a $\bar{G}_i$-graded (A_i, A_i')-bimodule over $\mathcal{C}_i$, that induces the said equivalence, for all i, then:*

1. *$\tilde{\mathbf{M}} := \tilde{M}_1 \otimes \dots \otimes \tilde{M}_n$ is a $\bar{\mathbf{G}}$-graded $(\mathbf{A}, \mathbf{A}')$-bimodule over $\boldsymbol{\mathcal{C}}$, where $\mathbf{A} := A_1 \otimes \dots \otimes A_n$, $\mathbf{A}' := A_1' \otimes \dots \otimes A_n'$ and $\boldsymbol{\mathcal{C}} := \mathcal{C}_1 \otimes \dots \otimes \mathcal{C}_n$;*
2. *$\tilde{\mathbf{M}}$ induces a $\bar{\mathbf{G}}$-graded Morita equivalence over $\boldsymbol{\mathcal{C}}$ between $\mathbf{A}$ and $\mathbf{A}'$.*

Proof. (1) By Lemma 3.2, $\mathbf{A}$ and $\mathbf{A}'$ are $\bar{\mathbf{G}}$-graded crossed products over $\boldsymbol{\mathcal{C}}$.

Obviously, $\tilde{\mathbf{M}}$ is a $\bar{\mathbf{G}}$-graded $(\mathbf{A}, \mathbf{A}')$-bimodule with the $(g_1,\dots,g_n)$-component

$$\tilde{\mathbf{M}}_{(g_1,\dots,g_n)} := \tilde{M}_{1,g_1} \otimes \dots \otimes \tilde{M}_{n,g_n},$$

where $\tilde{M}_{i,g_i}$ is the g_i-component of $\tilde{M}_i$, for all $g_i \in \bar{G}_i$ and for all i. It is also clear that

$$(\tilde{m}_{1,g_1} \otimes \dots \otimes \tilde{m}_{n,g_n})(c_1 \otimes \dots \otimes c_n) = {}^{g}(c_1 \otimes \dots \otimes c_n)(\tilde{m}_{1,g_1} \otimes \dots \otimes \tilde{m}_{n,g_n}),$$

for all $\tilde{m}_{1,g_1} \otimes \dots \otimes \tilde{m}_{n,g_n} \in \tilde{\mathbf{M}}_{(g_1,\dots,g_n)}$ and $c_1 \otimes \dots \otimes c_n \in \boldsymbol{\mathcal{C}}$ and for all $g = (g_1,\dots,g_n) \in \bar{\mathbf{G}}$.

(2) It remains to prove that

$$\tilde{\mathbf{M}} \otimes_{\mathbf{A}'} (\tilde{\mathbf{M}})^* \simeq \mathbf{A} \text{ as } \bar{\mathbf{G}}\text{-graded } (\mathbf{A}, \mathbf{A})\text{-bimodules over } \boldsymbol{\mathcal{C}},$$

and that

$$(\tilde{\mathbf{M}})^* \otimes_{\mathbf{A}} \tilde{\mathbf{M}} \simeq \mathbf{A}' \text{ as } \bar{\mathbf{G}}\text{-graded } (\mathbf{A}', \mathbf{A}')\text{-bimodules over } \boldsymbol{\mathcal{C}}.$$

We will only check the first isomorphism:

$$\begin{aligned}
\tilde{\mathbf{M}} \otimes_{\mathbf{A}'} (\tilde{\mathbf{M}})^* &= (\tilde{M}_1 \otimes \dots \otimes \tilde{M}_n) \otimes_{\mathbf{A}'} (\tilde{M}_1 \otimes \dots \otimes \tilde{M}_n)^* \\
&\simeq (\tilde{M}_1 \otimes \dots \otimes \tilde{M}_n) \otimes_{\mathbf{A}'} (\tilde{M}_1^* \otimes \dots \otimes \tilde{M}_n^*) \\
&= (\tilde{M}_1 \otimes \dots \otimes \tilde{M}_n) \otimes_{A_1' \otimes \dots \otimes A_n'} (\tilde{M}_1^* \otimes \dots \otimes \tilde{M}_n^*) \\
&\simeq (\tilde{M}_1 \otimes_{A_1'} \tilde{M}_1^*) \otimes \dots \otimes (\tilde{M}_n \otimes_{A_n'} \tilde{M}_n^*) \\
&\simeq A_1 \otimes \dots \otimes A_n \quad = \quad \mathbf{A},
\end{aligned}$$

as $\bar{\mathbf{G}}$-graded $(\mathbf{A}, \mathbf{A})$-bimodules over $\boldsymbol{\mathcal{C}}$. □

4. Wreath products for algebras

Consider the notations from Section 2. We denote $\bar{G}^n := \bar{G} \times \ldots \times \bar{G}$ (n times). We recall the definition of a wreath product as in [7, Definition 2.19] and [1, Section 5.1.C]:

Definition 4.1. The wreath product $\bar{G} \wr S_n$ is the semidirect product $\bar{G}^n \rtimes S_n$, where S_n acts on $\bar{G}^n$ (on the left) by permuting the components:

$$^{\sigma}(g_1, \ldots, g_n) := (g_{\sigma^{-1}(1)}, \ldots, g_{\sigma^{-1}(n)})$$

More exactly, the elements of $\bar{G} \wr S_n$ are of the form $((g_1, \ldots, g_n), \sigma)$, and the multiplication is:

$$((g_1, \ldots, g_n), \sigma)((h_1, \ldots, h_n), \tau) := ((g_1, \ldots, g_n) \cdot {}^{\sigma}(h_1, \ldots, h_n), \sigma\tau),$$

for all $g_1, \ldots, g_n, h_1, \ldots, h_n \in \bar{G}$ and $\sigma, \tau \in S_n$.

Definition 4.2. Let A be an algebra. We denote by $A^{\otimes n} := A \otimes \ldots \otimes A$ (n times). The wreath product $A \wr S_n$ is

$$A \wr S_n := A^{\otimes n} \otimes \mathcal{O}S_n$$

as $\mathcal{O}$-modules, with multiplication

$$\begin{aligned}((a_1 \otimes \ldots \otimes a_n) \otimes \sigma)&((b_1 \otimes \ldots \otimes b_n) \otimes \tau)\\ &:= ((a_1 \otimes \ldots \otimes a_n) \cdot {}^{\sigma}(b_1 \otimes \ldots \otimes b_n)) \otimes (\sigma\tau),\end{aligned}$$

where

$$^{\sigma}(b_1 \otimes \ldots \otimes b_n) := b_{\sigma^{-1}(1)} \otimes \ldots \otimes b_{\sigma^{-1}(n)},$$

for all $(a_1 \otimes \ldots \otimes a_n) \otimes \sigma$, $(b_1 \otimes \ldots \otimes b_n) \otimes \tau \in A \wr S_n$.

Lemma 4.3. *Let A be a $\bar{G}$-graded algebra and $\mathcal{C}$ be a $\bar{G}$-graded $\bar{G}$-acted algebra. The following affirmations hold:*

1. *$A \wr S_n$ is a $\bar{G} \wr S_n$-graded algebra;*
2. *If A is a $\bar{G}$-graded crossed product, then $A \wr S_n$ is a $\bar{G} \wr S_n$-graded crossed product;*
3. *$\mathcal{C}^{\otimes n}$ is a $\bar{G} \wr S_n$-acted $\bar{G}^n$-graded algebra;*
4. *If A is a $\bar{G}$-graded algebra over $\mathcal{C}$, then $A \wr S_n$ is a $\bar{G} \wr S_n$-graded algebra over $\mathcal{C}^{\otimes n}$.*

Proof. (1) The $((g_1, \ldots, g_n), \sigma)$-component of $A \wr S_n$ is

$$(A \wr S_n)_{((g_1,\ldots,g_n),\sigma)} := (A_{g_1} \otimes \ldots \otimes A_{g_n}) \otimes \mathcal{O}\sigma,$$

for each $((g_1, \ldots, g_n), \sigma) \in \bar{G} \wr S_n$. Indeed,

$$\begin{aligned}(A \wr S_n)&_{((g_1,\ldots,g_n),\sigma)}(A \wr S_n)_{((h_1,\ldots,h_n),\tau)}\\ &= ((A_{g_1} \otimes \ldots \otimes A_{g_n}) \otimes \mathcal{O}\sigma)((A_{h_1} \otimes \ldots \otimes A_{h_n}) \otimes \mathcal{O}\tau)\\ &= ((A_{g_1} \otimes \ldots \otimes A_{g_n}) \cdot {}^{\sigma}(A_{h_1} \otimes \ldots \otimes A_{h_n})) \otimes (\mathcal{O}\sigma \otimes \mathcal{O}\tau)\\ &= ((A_{g_1} \otimes \ldots \otimes A_{g_n}) \cdot (A_{h_{\sigma^{-1}(1)}} \otimes \ldots \otimes A_{h_{\sigma^{-1}(n)}})) \otimes \mathcal{O}(\sigma\tau)\\ &= (A_{g_1}A_{h_{\sigma^{-1}(1)}} \otimes \ldots \otimes A_{g_n}A_{h_{\sigma^{-1}(n)}}) \otimes \mathcal{O}(\sigma\tau)\\ &\subseteq (A_{g_1h_{\sigma^{-1}(1)}} \otimes \ldots \otimes A_{g_nh_{\sigma^{-1}(n)}}) \otimes \mathcal{O}(\sigma\tau)\\ &= (A \wr S_n)_{(((g_1,\ldots,g_n)\cdot{}^{\sigma}(h_1,\ldots,h_n)),\sigma\tau)}\end{aligned}$$

$$= (A \wr S_n)_{((g_1,\ldots,g_n),\sigma)((h_1,\ldots,h_n),\tau)}.$$

(2) We choose invertible homogeneous elements $u_g \in A_g$ for all $g \in \bar{G}$. For $((g_1, \ldots, g_n), \sigma) \in \bar{G} \wr S_n$ the homogeneous element

$$u_{((g_1,\ldots,g_n),\sigma)} := (u_{g_1} \otimes \ldots \otimes u_{g_n}) \otimes \sigma,$$

is clearly invertible, with

$$u^{-1}_{((g_1,\ldots,g_n),\sigma)} := (u^{-1}_{g_{\sigma(1)}} \otimes \ldots \otimes u^{-1}_{g_{\sigma(n)}}) \otimes \sigma^{-1}.$$

(3) By Lemma 3.2, we know that $\mathcal{C}^{\otimes n}$ is a $\bar{G}^n$-graded algebra. It remains to prove that it is $\bar{G} \wr S_n$-acted and that the action is compatible with the gradings. We define the action of $\bar{G} \wr S_n$ on $\mathcal{C}^{\otimes n}$ as follows:

$$^{((g_1,\ldots,g_n),\sigma)}(c_1 \otimes \ldots \otimes c_n) := {}^{g_1}c_{\sigma^{-1}(1)} \otimes \ldots \otimes {}^{g_n}c_{\sigma^{-1}(n)},$$

where $((g_1, \ldots, g_n), \sigma) \in \bar{G} \wr S_n$ and $c_1 \otimes \ldots \otimes c_n \in \mathcal{C}^{\otimes n}$. We have:

$$\begin{aligned} {}^{((1_{\bar{G}},\ldots,1_{\bar{G}}),e)}(a_1 \otimes \ldots \otimes a_n) &= {}^{1_{\bar{G}}}a_1 \otimes \ldots \otimes {}^{1_{\bar{G}}}a_n \\ &= a_1 \otimes \ldots \otimes a_n, \end{aligned}$$

$$\begin{aligned} &{}^{(((g_1,\ldots,g_n),\sigma)((h_1,\ldots,h_n),\tau))}(a_1 \otimes \ldots \otimes a_n) \\ &= {}^{((g_1,\ldots,g_n)\cdot{}^{\sigma}(h_1,\ldots,h_n),\sigma\tau)}(a_1 \otimes \ldots \otimes a_n) \\ &= {}^{((g_1,\ldots,g_n)\cdot(h_{\sigma^{-1}(1)},\ldots,h_{\sigma^{-1}(n)}),\sigma\tau)}(a_1 \otimes \ldots \otimes a_n) \\ &= {}^{((g_1h_{\sigma^{-1}(1)},\ldots,g_nh_{\sigma^{-1}(n)}),\sigma\tau)}(a_1 \otimes \ldots \otimes a_n) \\ &= {}^{g_1h_{\sigma^{-1}(1)}}a_{(\sigma\tau)^{-1}(1)} \otimes \ldots \otimes {}^{g_nh_{\sigma^{-1}(n)}}a_{(\sigma\tau)^{-1}(n)} \\ &= {}^{g_1h_{\sigma^{-1}(1)}}a_{\tau^{-1}(\sigma^{-1}(1))} \otimes \ldots \otimes {}^{g_nh_{\sigma^{-1}(n)}}a_{\tau^{-1}(\sigma^{-1}(n))} \\ &= {}^{((g_1,\ldots,g_n),\sigma)}({}^{h_1}a_{\tau^{-1}(1)} \otimes \ldots \otimes {}^{h_n}a_{\tau^{-1}(n)}) \\ &= {}^{((g_1,\ldots,g_n),\sigma)}\Big({}^{((h_1,\ldots,h_n),\tau)}(a_1 \otimes \ldots \otimes a_n)\Big), \end{aligned}$$

$$\begin{aligned} &{}^{((g_1,\ldots,g_n),\sigma)}((a_1 \otimes \ldots \otimes a_n) \cdot (b_1 \otimes \ldots \otimes b_n)) \\ &= {}^{((g_1,\ldots,g_n),\sigma)}(a_1b_1 \otimes \ldots \otimes a_nb_n) \\ &= {}^{g_1}(a_{\sigma^{-1}(1)}b_{\sigma^{-1}(1)}) \otimes \ldots \otimes {}^{g_n}(a_{\sigma^{-1}(n)}b_{\sigma^{-1}(n)}) \\ &= {}^{g_1}a_{\sigma^{-1}(1)} \cdot {}^{g_1}b_{\sigma^{-1}(1)} \otimes \ldots \otimes {}^{g_n}a_{\sigma^{-1}(n)} \cdot {}^{g_n}b_{\sigma^{-1}(n)} \\ &= ({}^{g_1}a_{\sigma^{-1}(1)} \otimes \ldots \otimes {}^{g_n}a_{\sigma^{-1}(n)})({}^{g_1}b_{\sigma^{-1}(1)} \otimes \ldots \otimes {}^{g_n}b_{\sigma^{-1}(n)}) \\ &= {}^{((g_1,\ldots,g_n),\sigma)}(a_1 \otimes \ldots \otimes a_n) \cdot {}^{((g_1,\ldots,g_n),\sigma)}(b_1 \otimes \ldots \otimes b_n), \end{aligned}$$

and

$$\begin{aligned} {}^{((h_1,\ldots,h_n),\tau)}(c_1 \otimes \ldots \otimes c_n) &= {}^{h_1}c_{\tau^{-1}(1)} \otimes \ldots \otimes {}^{h_n}c_{\tau^{-1}(n)} \\ &\in \mathcal{C}_{{}^{h_1}g_{\tau^{-1}(1)}} \otimes \ldots \otimes \mathcal{C}_{{}^{h_n}g_{\tau^{-1}(n)}} \\ &= (\mathcal{C}^{\otimes n})_{({}^{h_1}g_{\tau^{-1}(1)},\ldots,{}^{h_n}g_{\tau^{-1}(n)})} \\ &= (\mathcal{C}^{\otimes n})_{{}^{((h_1,\ldots,h_n),\tau)}(g_1,\ldots,g_n)}, \end{aligned}$$

for all $a_1 \otimes \ldots \otimes a_n$, $b_1 \otimes \ldots \otimes b_n \in \mathcal{C}^{\otimes n}$, for all $c_1 \otimes \ldots \otimes c_n \in \mathcal{C}^{\otimes n}_{(g_1,\ldots,g_n)}$ and for all $((g_1, \ldots, g_n), \sigma)$, $((h_1, \ldots, h_n), \tau) \in \bar{G} \wr S_n$.

(4) By assumption, there exists a $\bar{G}$-graded $\bar{G}$-acted algebra homomorphism $\zeta : \mathcal{C} \to C_A(B)$, where B is the identity component of A. Henceforth, we have a $\bar{G}^n$-graded $\bar{G}^n$-acted algebra homomorphism:

$$\zeta^{\otimes n} : \mathcal{C}^{\otimes n} \to C_A(B)^{\otimes n}.$$

Now, via the identification:

$$A^{\otimes n} \ni a_1 \otimes \ldots \otimes a_n = (a_1 \otimes \ldots \otimes a_n) \otimes e \in A \wr S_n,$$

we clearly have the following inclusion:

$$C_A(B)^{\otimes n} \subseteq C_{A \wr S_n}(B^{\otimes n}) = C_{A \wr S_n}((A \wr S_n)_{((1_{\bar{G}},\ldots,1_{\bar{G}}),e)}).$$

Therefore, we obtain the required $\bar{G} \wr S_n$-graded $\bar{G} \wr S_n$-acted algebra map

$$\zeta_{\mathrm{wr}} : \mathcal{C}^{\otimes n} \to C_{A \wr S_n}((A \wr S_n)_{((1_{\bar{G}},\ldots,1_{\bar{G}}),e)}).$$

Indeed, given $((g_1, \ldots, g_n), \sigma) \in \bar{G} \wr S_n$ and $c_1 \otimes \ldots \otimes c_n \in \mathcal{C}^{\otimes n}$ we have:

$$\begin{aligned} \zeta_{\mathrm{wr}}({}^{((g_1,\ldots,g_n),\sigma)}(c_1 \otimes \ldots \otimes c_n)) &= \zeta_{\mathrm{wr}}({}^{g_1}c_{\sigma^{-1}(1)} \otimes \ldots \otimes {}^{g_n}c_{\sigma^{-1}(n)}) \\ &= \zeta({}^{g_1}c_{\sigma^{-1}(1)}) \otimes \ldots \otimes \zeta({}^{g_n}c_{\sigma^{-1}(n)}) \\ &= {}^{g_1}\zeta(c_{\sigma^{-1}(1)}) \otimes \ldots \otimes {}^{g_n}\zeta(c_{\sigma^{-1}(n)}) \\ &= {}^{((g_1,\ldots,g_n),\sigma)}(\zeta(c_1) \otimes \ldots \otimes \zeta(c_n)) \\ &= {}^{((g_1,\ldots,g_n),\sigma)}\zeta_{\mathrm{wr}}(c_1 \otimes \ldots \otimes c_n). \end{aligned}$$

Henceforth, $A \wr S_n$ is a $\bar{G} \wr S_n$-graded algebra over $\mathcal{C}^{\otimes n}$. □

5. Morita equivalences for wreath products

Consider the notations from Section 2 and Section 4. We recall the definition of a wreath product between a module and S_n.

Definition 5.1. Let A and A' be two algebras. Assume that $\tilde{M}$ is an (A, A')-bimodule. The wreath product $\tilde{M} \wr S_n$ is defined by

$$\tilde{M} \wr S_n := \tilde{M}^{\otimes n} \otimes \mathcal{O}S_n$$

as $\mathcal{O}$-modules, with operations

$$\begin{aligned} &((a_1 \otimes \ldots \otimes a_n) \otimes \sigma)((\tilde{m}_1 \otimes \ldots \otimes \tilde{m}_n) \otimes \tau) \\ &\qquad := ((a_1 \otimes \ldots \otimes a_n) \cdot {}^{\sigma}(\tilde{m}_1 \otimes \ldots \otimes \tilde{m}_n)) \otimes (\sigma\tau), \end{aligned}$$

and

$$\begin{aligned} &((\tilde{m}_1 \otimes \ldots \otimes \tilde{m}_n) \otimes \tau)((a'_1 \otimes \ldots \otimes a'_n) \otimes \pi) \\ &\qquad := ((\tilde{m}_1 \otimes \ldots \otimes \tilde{m}_n) \cdot {}^{\tau}(a'_1 \otimes \ldots \otimes a'_n)) \otimes (\tau\pi), \end{aligned}$$

where

$${}^{\sigma}(\tilde{m}_1 \otimes \ldots \otimes \tilde{m}_n) := \tilde{m}_{\sigma^{-1}(1)} \otimes \ldots \otimes \tilde{m}_{\sigma^{-1}(n)},$$

for all $(a_1 \otimes \ldots \otimes a_n) \otimes \sigma \in A \wr S_n$, $(\tilde{m}_1 \otimes \ldots \otimes \tilde{m}_n) \otimes \tau \in \tilde{M} \wr S_n$ and $(a'_1 \otimes \ldots \otimes a'_n) \otimes \pi \in A' \wr S_n$.

5.2. Let $\mathcal{C}$ be a $\bar{G}$-graded $\bar{G}$-acted algebra and A and A' be two $\bar{G}$-graded crossed products over $\mathcal{C}$, with identity components B and B' respectively.

If $\tilde{M}$ is an (A, A')-bimodule which induces a Morita equivalence between A and A', by the results of [1, Section 5.1.C], we already know that $\tilde{M} \wr S_n$ induces a Morita equivalence between $A \wr S_n$ and $A' \wr S_n$. The question that arises is whether this result can be extended to give a graded Morita equivalence over a group graded group acted algebra.

Theorem 5.3. *Let $\tilde{M}$ be a $\bar{G}$-graded (A, A')-bimodule over $\mathcal{C}$. Then, the following affirmations hold:*

1. *$\tilde{M} \wr S_n$ is a $\bar{G} \wr S_n$-graded $(A \wr S_n, A' \wr S_n)$-bimodule over $\mathcal{C}^{\otimes n}$;*
2. *$(A \wr S_n) \otimes_{B^{\otimes n}} M^{\otimes n} \simeq M^{\otimes n} \otimes_{B'^{\otimes n}} (A' \wr S_n) \simeq \tilde{M} \wr S_n$ as $\bar{G} \wr S_n$-graded $(A \wr S_n, A' \wr S_n)$-bimodules over $\mathcal{C}^{\otimes n}$, where M is the identity component of $\tilde{M}$;*
3. *If $\tilde{M}$ induces a $\bar{G}$-graded Morita equivalence over $\mathcal{C}$ between A and A', then $\tilde{M} \wr S_n$ induces a $\bar{G} \wr S_n$-graded Morita equivalence over $\mathcal{C}^{\otimes n}$ between $A \wr S_n$ and $A' \wr S_n$.*

Proof. (1) By Lemma 4.3, we know that $A \wr S_n$ and $A' \wr S_n$ are $\bar{G} \wr S_n$-graded crossed products over $\mathcal{C}^{\otimes n}$.

It is also known that $A \wr S_n$ and $A' \wr S_n$ are strongly S_n-graded algebras, and given this grading, if we denote

$$\Delta_{S_n}(A \wr S_n \otimes (A' \wr S_n)^{\mathrm{op}}) := (A \wr S_n \otimes (A' \wr S_n)^{\mathrm{op}})_{\delta(S_n)},$$

where $\delta(S_n) := \{(\sigma, \sigma^{-1}) \mid \sigma \in S_n\}$, we have the following isomorphism of algebras:

$$\Delta_{S_n}(A \wr S_n \otimes (A' \wr S_n)^{\mathrm{op}}) \simeq (A^{\otimes n} \otimes (A'^{\otimes n})^{\mathrm{op}}) \otimes \mathcal{O}S_n.$$

Moreover, [1, Lemma 5.1.19] states that $\tilde{M}^{\otimes n}$ is a left $\mathcal{O}S_n$-module with the action given by permutations. Henceforth, it is easy to see that $\tilde{M}^{\otimes n}$ is a $(A^{\otimes n} \otimes (A'^{\otimes n})^{\mathrm{op}}) \otimes \mathcal{O}S_n$-module, and thereby (the above isomorphism), $\tilde{M}^{\otimes n}$ extends to a $\Delta_{S_n}(A \wr S_n \otimes (A' \wr S_n)^{\mathrm{op}})$-module. Thus, $\tilde{M} \wr S_n := \tilde{M}^{\otimes n} \otimes \mathcal{O}S_n$ becomes an $(A \wr S_n, A' \wr S_n)$-bimodule.

Now, we will prove that $\tilde{M} \wr S_n$ is a $\bar{G} \wr S_n$-graded $(A \wr S_n, A' \wr S_n)$-bimodule. Indeed, for all $((g_1, \ldots, g_n), \sigma) \in \bar{G} \wr S_n$, the $((g_1, \ldots, g_n), \sigma)$-component of $\tilde{M} \wr S_n$ is:

$$(\tilde{M} \wr S_n)_{((g_1, \ldots, g_n), \sigma)} := (\tilde{M}_{g_1} \otimes \ldots \otimes \tilde{M}_{g_n}) \otimes \mathcal{O}\sigma.$$

The verification for this definition is straightforward, as follows. For each $((g_1, \ldots, g_n), \sigma)$, $((x_1, \ldots, x_n), \pi)$ and $((h_1, \ldots, h_n), \tau) \in \bar{G} \wr S_n$ we have:

$$\begin{aligned}
&(A \wr S_n)_{((g_1,\ldots,g_n),\sigma)} (\tilde{M} \wr S_n)_{((x_1,\ldots,x_n),\pi)} (A' \wr S_n)_{((h_1,\ldots,h_n),\tau)} \\
&= ((A_{g_1} \otimes \ldots \otimes A_{g_n}) \otimes \mathcal{O}\sigma)((\tilde{M}_{x_1} \otimes \ldots \otimes \tilde{M}_{x_n}) \otimes \mathcal{O}\pi)((A'_{h_1} \otimes \ldots \otimes A'_{h_n}) \otimes \mathcal{O}\tau) \\
&= ((A_{g_1} \tilde{M}_{x_{\sigma^{-1}(1)}} \otimes \ldots \otimes A_{g_n} \tilde{M}_{x_{\sigma^{-1}(n)}}) \otimes \mathcal{O}(\sigma\pi))((A'_{h_1} \otimes \ldots \otimes A'_{h_n}) \otimes \mathcal{O}\tau) \\
&= (A_{g_1} \tilde{M}_{x_{\sigma^{-1}(1)}} A'_{h_{(\sigma\pi)^{-1}(1)}} \otimes \ldots \otimes A_{g_n} \tilde{M}_{x_{\sigma^{-1}(n)}} A'_{h_{(\sigma\pi)^{-1}(n)}}) \otimes \mathcal{O}(\sigma\pi\tau) \\
&\subseteq (\tilde{M}_{g_1 x_{\sigma^{-1}(1)} h_{(\sigma\pi)^{-1}(1)}} \otimes \ldots \otimes \tilde{M}_{g_n x_{\sigma^{-1}(n)} h_{(\sigma\pi)^{-1}(n)}}) \otimes \mathcal{O}(\sigma\pi\tau) \\
&= (\tilde{M} \wr S_n)_{((g_1 x_{\sigma^{-1}(1)} h_{(\sigma\pi)^{-1}(1)}, \ldots, g_n x_{\sigma^{-1}(n)} h_{(\sigma\pi)^{-1}(n)}), \sigma\pi\tau)} \\
&= (\tilde{M} \wr S_n)_{((g_1 x_{\sigma^{-1}(1)}, \ldots, g_n x_{\sigma^{-1}(n)}), \sigma\pi)((h_1,\ldots,h_n),\tau)} \\
&= (\tilde{M} \wr S_n)_{((g_1,\ldots,g_n),\sigma)((x_1,\ldots,x_n),\pi)((h_1,\ldots,h_n),\tau)}.
\end{aligned}$$

Therefore, $\tilde{M} \wr S_n$ is a $\bar{G} \wr S_n$-graded $(A \wr S_n, A' \wr S_n)$-bimodule. Note that the identity component of $\tilde{M} \wr S_n$ (with respect to the $\bar{G} \wr S_n$-grading) is

$$(\tilde{M} \wr S_n)_1 = M^{\otimes n},$$

where M is the identity component of $\tilde{M}$.
Finally, we will prove that

$$\begin{aligned}((\tilde{m}_{g_1} \otimes \ldots \otimes \tilde{m}_{g_n}) \otimes \sigma)&(c_1 \otimes \ldots \otimes c_n)\\ &= {}^{((g_1,\ldots,g_n),\sigma)}(c_1 \otimes \ldots \otimes c_n)((\tilde{m}_{g_1} \otimes \ldots \otimes \tilde{m}_{g_n}) \otimes \sigma),\end{aligned}$$

for all $(\tilde{m}_{g_1} \otimes \ldots \otimes \tilde{m}_{g_n}) \otimes \sigma \in (\tilde{M} \wr S_n)_{((g_1,\ldots,g_n),\sigma)}$ and $c_1 \otimes \ldots \otimes c_n \in \mathcal{C}^{\otimes n}$ and for all $((g_1, \ldots, g_n), \sigma) \in \bar{G} \wr S_n$. Indeed,

$$\begin{aligned}((\tilde{m}_{g_1} \otimes \ldots \otimes \tilde{m}_{g_n}) \otimes \sigma)&(c_1 \otimes \ldots \otimes c_n)\\ &= (\tilde{m}_{g_1} c_{\sigma^{-1}(1)} \otimes \ldots \otimes \tilde{m}_{g_n} c_{\sigma^{-1}(n)}) \otimes \sigma\\ &= ({}^{g_1}c_{\sigma^{-1}(1)}\tilde{m}_{g_1} \otimes \ldots \otimes {}^{g_n}c_{\sigma^{-1}(n)}\tilde{m}_{g_n}) \otimes \sigma\\ &= ({}^{g_1}c_{\sigma^{-1}(1)} \otimes \ldots \otimes {}^{g_n}c_{\sigma^{-1}(n)})((\tilde{m}_{g_1} \otimes \ldots \otimes \tilde{m}_{g_n}) \otimes \sigma)\\ &= {}^{((g_1,\ldots,g_n),\sigma)}(c_1 \otimes \ldots \otimes c_n)((\tilde{m}_{g_1} \otimes \ldots \otimes \tilde{m}_{g_n}) \otimes \sigma).\end{aligned}$$

Henceforth, $\tilde{M} \wr S_n$ is a $\bar{G} \wr S_n$-graded $(A \wr S_n, A' \wr S_n)$-bimodule over $\mathcal{C}^{\otimes n}$.

(2) In this part, in order to prove our claim, we want to use a similar technique as in part (1), but with regard to the grading given by $\bar{G} \wr S_n$.

Henceforth, we regard $(A' \wr S_n)^{\mathrm{op}}$ as a $\bar{G} \wr S_n$-graded crossed product over $\mathcal{C}^{\otimes n}$, where the $((g_1, \ldots, g_n), \sigma)$-component of $(A' \wr S_n)^{\mathrm{op}}$ is:

$$(A' \wr S_n)^{\mathrm{op}}_{((g_1,\ldots,g_n),\sigma)} := ((A' \wr S_n)_{((g_1,\ldots,g_n),\sigma)^{-1}})^{\mathrm{op}},$$

for all $((g_1, \ldots, g_n), \sigma) \in \bar{G} \wr S_n$, and we consider:

$$\begin{aligned}&\Delta^{\mathcal{C}^{\otimes n}}_{\bar{G}\wr S_n}(A \wr S_n \otimes_{\mathcal{C}^{\otimes n}} (A' \wr S_n)^{\mathrm{op}}) :=\\ &\quad\bigoplus_{((g_1,\ldots,g_n),\sigma)\in\bar{G}\wr S_n}((A \wr S_n)_{((g_1,\ldots,g_n),\sigma)} \otimes_{\mathcal{C}^{\otimes n}} (A' \wr S_n)^{\mathrm{op}}_{((g_1,\ldots,g_n),\sigma)}),\end{aligned}$$

which, by [3, Lemma 2.8], is an $\mathcal{O}$-algebra.

Now, given the fact from part (1) of this theorem, that $\tilde{M} \wr S_n$ is a $\bar{G} \wr S_n$-graded $(A \wr S_n, A' \wr S_n)$-bimodule over $\mathcal{C}^{\otimes n}$, we obtain, by [3, Proposition 2.11], that $(\tilde{M} \wr S_n)_1 = M^{\otimes n}$ extends to a $\Delta^{\mathcal{C}^{\otimes n}}_{\bar{G}\wr S_n}(A \wr S_n \otimes_{\mathcal{C}^{\otimes n}} (A' \wr S_n)^{\mathrm{op}})$-module and that we have the following isomorphisms of $\bar{G} \wr S_n$-graded $(A \wr S_n, A' \wr S_n)$-bimodules over $\mathcal{C}^{\otimes n}$:

$$(A \wr S_n) \otimes_{B^{\otimes n}} M^{\otimes n} \simeq M^{\otimes n} \otimes_{B'^{\otimes n}} (A' \wr S_n) \simeq \tilde{M} \wr S_n.$$

More exactly, these isomorphisms are:

$$f : (A \wr S_n) \otimes_{B^{\otimes n}} M^{\otimes n} \to \tilde{M} \wr S_n, \quad (a \otimes \sigma) \otimes m \mapsto (a \cdot {}^{\sigma}m) \otimes \sigma,$$

and

$$g : M^{\otimes n} \otimes_{B'^{\otimes n}} (A' \wr S_n) \to \tilde{M} \wr S_n, \quad m \otimes (a' \otimes \sigma) \mapsto (m \cdot a') \otimes \sigma$$

for all $a \in A^{\otimes n}$, $a' \in A'^{\otimes n}$, $m \in M^{\otimes n}$ and $\sigma \in S_n$.

We prove that f is an isomorphism of $\bar{G} \wr S_n$-graded $(A \wr S_n, A' \wr S_n)$-bimodules over $\mathcal{C}^{\otimes n}$. The verification for g is similar. The left $A \wr S_n$-module structure of $(A \wr$

$S_n) \otimes_{B^{\otimes n}} M^{\otimes n}$ is clear. We recall and particularize from [3, Proposition 2.11] the right $A' \wr S_n$-module structure of $(A \wr S_n) \otimes_{B^{\otimes n}} M^{\otimes n}$:

$$\begin{aligned}
&((a \otimes \sigma) \otimes_{B^{\otimes n}} m) \cdot (a'_g \otimes \tau) \\
&\quad = ((a \otimes \sigma)(u_g \otimes \tau)) \otimes_{B^{\otimes n}} ((({}^{\tau^{-1}}u_g^{-1} \otimes \tau^{-1}) \otimes_{\mathcal{C}^{\otimes n}} (a'_g \otimes \tau)^{\mathrm{op}})m) \\
&\quad = (a \cdot {}^{\sigma}u_g \otimes \sigma\tau) \otimes_{B^{\otimes n}} (({}^{\tau^{-1}}u_g^{-1} \otimes \tau^{-1})(m \otimes e)(a'_g \otimes \tau)) \\
&\quad = (a \cdot {}^{\sigma}u_g \otimes \sigma\tau) \otimes_{B^{\otimes n}} (({}^{\tau^{-1}}u_g^{-1} \cdot {}^{\tau^{-1}}m \otimes \tau^{-1})(a'_g \otimes \tau)) \\
&\quad = (a \cdot {}^{\sigma}u_g \otimes \sigma\tau) \otimes_{B^{\otimes n}} ({}^{\tau^{-1}}u_g^{-1} \cdot {}^{\tau^{-1}}m \cdot {}^{\tau^{-1}}a'_g),
\end{aligned}$$

for all $a \in A^{\otimes n}$, $\sigma, \tau \in S_n$, $m \in M^{\otimes n}$, $a'_g \in A'^{\otimes n}_g$, $g \in \bar{G}^n$ and u_g is an invertible homogeneous element of $A_g^{\otimes n}$. We start by proving that f is a morphism of $(A \wr S_n, A' \wr S_n)$-bimodules:

$$\begin{aligned}
f((b \otimes \tau) \cdot ((a \otimes \sigma) \otimes_{B^{\otimes n}} m)) &= f(((b \otimes \tau) \cdot (a \otimes \sigma)) \otimes_{B^{\otimes n}} m) \\
&= f((b \cdot {}^{\tau}a \otimes \tau\sigma) \otimes_{B^{\otimes n}} m) \\
&= b \cdot {}^{\tau}a \cdot {}^{\tau\sigma}m \otimes \tau\sigma \\
&= b \cdot {}^{\tau}(a \cdot {}^{\sigma}m) \otimes \tau\sigma \\
&= (b \otimes \tau)(a \cdot {}^{\sigma}m \otimes \sigma) \\
&= (b \otimes \tau) f((a \otimes \sigma) \otimes_{B^{\otimes n}} m),
\end{aligned}$$

$$\begin{aligned}
&f(((a \otimes \sigma) \otimes_{B^{\otimes n}} m) \cdot (a'_g \otimes \tau)) \\
&\quad = f((a \cdot {}^{\sigma}u_g \otimes \sigma\tau) \otimes_{B^{\otimes n}} ({}^{\tau^{-1}}u_g^{-1} \cdot {}^{\tau^{-1}}m \cdot {}^{\tau^{-1}}a'_g)) \\
&\quad = (a \cdot {}^{\sigma}u_g \cdot {}^{\sigma\tau}({}^{\tau^{-1}}u_g^{-1} \cdot {}^{\tau^{-1}}m \cdot {}^{\tau^{-1}}a'_g)) \otimes \sigma\tau \\
&\quad = (a \cdot {}^{\sigma}u_g \cdot {}^{\sigma\tau\tau^{-1}}u_g^{-1} \cdot {}^{\sigma\tau\tau^{-1}}m \cdot {}^{\sigma\tau\tau^{-1}}a'_g) \otimes \sigma\tau \\
&\quad = (a \cdot {}^{\sigma}u_g \cdot {}^{\sigma}u_g^{-1} \cdot {}^{\sigma}m \cdot {}^{\sigma}a'_g) \otimes \sigma\tau \\
&\quad = (a \cdot {}^{\sigma}m \cdot {}^{\sigma}a'_g) \otimes \sigma\tau \\
&\quad = ((a \cdot {}^{\sigma}m) \otimes \sigma)(a'_g \otimes \tau) \\
&\quad = f((a \otimes \sigma) \otimes_{B^{\otimes n}} m)(a'_g \otimes \tau),
\end{aligned}$$

for all $a, b \in A^{\otimes n}$, $\sigma, \tau \in S_n$, $m \in M^{\otimes n}$, $a'_g \in A'^{\otimes n}_g$, $g \in \bar{G}^n$ and u_g is an invertible homogeneous element of $A_g^{\otimes n}$.

Next, we will prove that f is $\bar{G} \wr S_n$-grade preserving. We recall from [3, Proposition 2.11] that the $\bar{G} \wr S_n$-grading of $(A \wr S_n) \otimes_{B^{\otimes n}} M^{\otimes n}$ is given by $A \wr S_n$. We have:

$$\begin{aligned}
&f(((a_{g_1} \otimes \ldots \otimes a_{g_n}) \otimes \sigma) \otimes_{B^{\otimes n}} (m_1 \otimes \ldots \otimes m_n)) \\
&\quad = ((a_{g_1} m_{\sigma^{-1}(1)} \otimes \ldots \otimes a_{g_n} m_{\sigma^{-1}(n)}) \otimes \sigma \\
&\quad \in ((A_{g_1}\tilde{M}_1 \otimes \ldots \otimes A_{g_n}\tilde{M}_1) \otimes \mathcal{O}\sigma \\
&\quad \subseteq ((\tilde{M}_{g_1} \otimes \ldots \otimes \tilde{M}_{g_n}) \otimes \mathcal{O}\sigma \\
&\quad = (\tilde{M} \wr S_n)_{((g_1,\ldots,g_n),\sigma)},
\end{aligned}$$

for all $(a_{g_1} \otimes \ldots \otimes a_{g_n}) \otimes \sigma \in (A \wr S_n)_{((g_1,\ldots,g_n),\sigma)}$, $m_1 \otimes \ldots \otimes m_n \in M^{\otimes n}$ and for all $((g_1, \ldots, g_n), \sigma) \in \bar{G} \wr S_n$.

Finally, we will prove that f is bijective. Because both modules have the same $\mathcal{O}$-rank, it is enough to prove that f is surjective. If $(\tilde{m}_{g_1} \otimes \ldots \otimes \tilde{m}_{g_n}) \otimes \sigma$ is an arbitrary element of $(\tilde{M} \wr S_n)_{((g_1,\ldots,g_n),\sigma)}$, then it is clear that

$$((u^{-1}_{g_1^{-1}} \otimes \ldots \otimes u^{-1}_{g_n^{-1}}) \otimes \sigma) \otimes (u_{g^{-1}_{\sigma(1)}} \tilde{m}_{g_{\sigma(1)}} \otimes \ldots \otimes u_{g^{-1}_{\sigma(n)}} \tilde{m}_{g_{\sigma(n)}}) \in (A \wr S_n) \otimes_{B^{\otimes n}} M^{\otimes n}$$

and that

$$\begin{aligned} & f(((u^{-1}_{g_1^{-1}} \otimes \ldots \otimes u^{-1}_{g_n^{-1}}) \otimes \sigma) \otimes (u_{g^{-1}_{\sigma(1)}} \tilde{m}_{g_{\sigma(1)}} \otimes \ldots \otimes u_{g^{-1}_{\sigma(n)}} \tilde{m}_{g_{\sigma(n)}})) \\ & \quad = ((u^{-1}_{g_1^{-1}} \otimes \ldots \otimes u^{-1}_{g_n^{-1}}) \cdot {}^{\sigma}(u_{g^{-1}_{\sigma(1)}} \tilde{m}_{g_{\sigma(1)}} \otimes \ldots \otimes u_{g^{-1}_{\sigma(n)}} \tilde{m}_{g_{\sigma(n)}})) \otimes \sigma \\ & \quad = ((u^{-1}_{g_1^{-1}} \otimes \ldots \otimes u^{-1}_{g_n^{-1}}) \cdot (u_{g_1^{-1}} \tilde{m}_{g_1} \otimes \ldots \otimes u_{g_n^{-1}} \tilde{m}_{g_n})) \otimes \sigma \\ & \quad = (u^{-1}_{g_1^{-1}} u_{g_1^{-1}} \tilde{m}_{g_1} \otimes \ldots \otimes u^{-1}_{g_n^{-1}} u_{g_n^{-1}} \tilde{m}_{g_n}) \otimes \sigma \\ & \quad = (\tilde{m}_{g_1} \otimes \ldots \otimes \tilde{m}_{g_n}) \otimes \sigma, \end{aligned}$$

for all $((g_1, \ldots, g_n), \sigma) \in \bar{G} \wr S_n$, where u_g represents an invertible homogeneous element of A_g, for all $g \in \bar{G}$.

(3) Furthermore, by Proposition 3.3, $\tilde{M}^{\otimes n}$ is a $\bar{G}^n$-graded $(A^{\otimes n}, A'^{\otimes n})$-bimodule over $\mathcal{C}^{\otimes n}$, which induces a $\bar{G}^n$-graded Morita equivalence over $\mathcal{C}^{\otimes n}$ between $A^{\otimes n}$ and $A^{\otimes n}$, thus by [1, Theorem 5.1.2] with respect to the $\bar{G}^n$-grading, we have that $(\tilde{M}^{\otimes n})_1 = M^{\otimes n}$ is a $(B^{\otimes n}, B'^{\otimes n})$-bimodule, which induces a Morita equivalence between $B^{\otimes n}$ and $B'^{\otimes n}$.

Now, by the previous statements, and by using [3, Theorem 3.3] with respect to the $\bar{G} \wr S_n$-grading, we obtain that $\tilde{M} \wr S_n$ induces a $\bar{G} \wr S_n$-graded Morita equivalence over $\mathcal{C}^{\otimes n}$ between $A \wr S_n$ and $A' \wr S_n$. □

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References

[1] Marcus, A., *Representation Theory of Group-Graded Algebras*, Nova Science Publ. Inc., 1999.

[2] Marcus, A., Minuţă, V.A., *Group graded endomorphism algebras and Morita equivalences*, Mathematica, **62(85)**(2020), no. 1, 73-80.

[3] Marcus, A., Minuţă, V.A., *Character triples and equivalences over a group graded G-algebra*, J. Algebra, **565**(2021), 98-127.

[4] Minuţă, V.A., *Graded Morita theory over a G-graded G-acted algebra*, Acta Univ. Sapientiae Math., **12**(2020), no. 1, 164-178.

[5] Späth, B., *A reduction theorem for Dade's projective conjecture*, J. Eur. Math. Soc. (JEMS), **19**(2017), no. 4, 1071-1126.

[6] Späth, B., *Inductive Conditions for Counting Conjectures via Character Triples*, in Representation Theory - Current Trends and Perspectives, (H. Krause, P. Littelmann, G. Malle, K.H. Neeb, C. Schweigert, Eds.), EMS Ser. Congr. Rep., Zürich, 2017, 665-680.

[7] Späth, B., *Reduction theorems for some global-local conjectures*, in Local Representation Theory and Simple Groups, (R. Kessar, G. Malle, D. Testerman, Eds.), EMS Ser. Lect. Math., Zürich, 2018, 23-61.

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Different type parameterized inequalities via generalized integral operators with applications

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Abstract. The authors have proved an identity for a generalized integral operator via differentiable function with parameters. By applying the established identity, the generalized trapezium, midpoint and Simpson type integral inequalities have been discovered. It is pointed out that the results of this research provide integral inequalities for almost all fractional integrals discovered in recent past decades. Various special cases have been identified. Some applications of presented results to special means and new error estimates for the trapezium and midpoint quadrature formula have been analyzed. The ideas and techniques of this paper may stimulate further research in the field of integral inequalities.

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1. Introduction

The following inequality, named Hermite-Hadamard inequality, is one of the most famous inequalities in the literature for convex functions.

Theorem 1.1. *Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function and $e_1, e_2 \in I$ with $e_1 < e_2$. Then the following inequality holds:*

$$f\left(\frac{e_1+e_2}{2}\right) \leq \frac{1}{e_2-e_1}\int_{e_1}^{e_2} f(x)dx \leq \frac{f(e_1)+f(e_2)}{2}. \tag{1.1}$$

This inequality (1.1) *is also known as trapezium inequality.*

The trapezium inequality has remained an area of great interest due to its wide applications in the field of mathematical analysis. Authors of recent decades have studied (1.1) in the premises of newly invented definitions due to motivation of convex function. Interested readers see the references [1]-[10], [12]-[17], [19]-[25].
The following inequality is well known in the literature as Simpson's inequality.

Theorem 1.2. *Let $f : [e_1, e_2] \longrightarrow \mathbb{R}$ be four time differentiable on the interval (e_1, e_2) and having the fourth derivative bounded on (e_1, e_2), that is*

$$\|f^{(4)}\|_\infty = \sup_{x\in(e_1,e_2)} |f^{(4)}| < \infty.$$

Then, we have

$$\left|\int_{e_1}^{e_2} f(x)dx - \frac{e_2 - e_1}{3}\left[\frac{f(e_1) + f(e_2)}{2} + 2f\left(\frac{e_1 + e_2}{2}\right)\right]\right|$$
$$\leq \frac{1}{2880}\|f^{(4)}\|_\infty(e_2 - e_1)^5. \tag{1.2}$$

Inequality (1.2) gives an error bound for the classical Simpson quadrature formula, which is one of the most used quadrature formulae in practical applications. In recent years, various generalizations, extensions and variants of such inequalities have been obtained. For other recent results concerning Simpson type inequalities, see [11],[18]. The aim of this paper is to establish trapezium, midpoint and Simpson type generalized integral inequalities for preinvex functions and some new error bounds for midpoint and trapezium quadrature formula. Interestingly, the special cases of presented results, are fractional integral inequalities. Therefore, it is important to summarize the study of fractional integrals.
Let us recall some special functions and evoke some basic definitions as follows:

Definition 1.3. For $k \in \mathbb{R}^+$ and $x \in \mathbb{C}$, the k-gamma function is defined by

$$\Gamma_k(x) = \lim_{n\longrightarrow\infty} \frac{n!k^n(nk)^{\frac{x}{k}-1}}{(x)_{n,k}}. \tag{1.3}$$

Its integral representation is given by

$$\Gamma_k(\alpha) = \int_0^\infty t^{\alpha-1}e^{-\frac{t^k}{k}}dt. \tag{1.4}$$

One can note that

$$\Gamma_k(\alpha + k) = \alpha\Gamma_k(\alpha). \tag{1.5}$$

For $k = 1$, (1.4) gives integral representation of gamma function.

Definition 1.4. [15] Let $f \in L[e_1, e_2]$. Then k-fractional integrals of order $\alpha, k > 0$ with $e_1 \geq 0$ are defined by

$$I^{\alpha,k}_{e_1^+}f(x) = \frac{1}{k\Gamma_k(\alpha)}\int_{e_1}^x (x - t)^{\frac{\alpha}{k}-1}f(t)dt, \quad x > e_1$$

and

$$I^{\alpha,k}_{e_2^-}f(x) = \frac{1}{k\Gamma_k(\alpha)}\int_x^{e_2} (t - x)^{\frac{\alpha}{k}-1}f(t)dt, \quad e_2 > x. \tag{1.6}$$

For $k = 1$, k-fractional integrals give Riemann-Liouville integrals. For $\alpha = k = 1$, k-fractional integrals give classical integrals.

Also, let define a function $\varphi : [0, \infty) \longrightarrow [0, \infty)$ satisfying the following conditions:

$$\int_0^1 \frac{\varphi(t)}{t} dt < \infty, \tag{1.7}$$

$$\frac{1}{A} \leq \frac{\varphi(s)}{\varphi(r)} \leq A \text{ for } \frac{1}{2} \leq \frac{s}{r} \leq 2 \tag{1.8}$$

$$\frac{\varphi(r)}{r^2} \leq B \frac{\varphi(s)}{s^2} \text{ for } s \leq r \tag{1.9}$$

$$\left| \frac{\varphi(r)}{r^2} - \frac{\varphi(s)}{s^2} \right| \leq C|r - s| \frac{\varphi(r)}{r^2} \text{ for } \frac{1}{2} \leq \frac{s}{r} \leq 2 \tag{1.10}$$

where $A, B, C > 0$ are independent of $r, s > 0$. If $\varphi(r)r^\alpha$ is increasing for some $\alpha \geq 0$ and $\frac{\varphi(r)}{r^\beta}$ is decreasing for some $\beta \geq 0$, then φ satisfies (1.7)-(1.10), see [20]. Therefore, the left-sided and right-sided generalized integral operators are defined as follows:

$${}_{e_1^+}I_\varphi f(x) = \int_{e_1}^{x} \frac{\varphi(x - t)}{x - t} f(t)dt, \quad x > e_1, \tag{1.11}$$

$${}_{e_2^-}I_\varphi f(x) = \int_{x}^{e_2} \frac{\varphi(t - x)}{t - x} f(t)dt, \quad x < e_2. \tag{1.12}$$

The most important feature of generalized integrals is that; they produce Riemann-Liouville fractional integrals, k-Riemann-Liouville fractional integrals, Katugampola fractional integrals, conformable fractional integrals, Hadamard fractional integrals, etc., see [19].

Motivated by the above literatures, the main objective of this paper is to discover in Section 2, an interesting identity in order to study some new bounds regarding trapezium, midpoint and Simpson type integral inequalities. By using the established identity as an auxiliary result, some new estimates for trapezium, midpoint and Simpson type integral inequalities via generalized integrals are obtained. It is pointed out that some new fractional integral inequalities have been deduced from main results. In Section 3, some applications to special means and new error estimates for the midpoint and trapezium quadrature formula are given. The ideas and techniques of this paper may stimulate further research in the field of integral inequalities.

2. Main results

Throughout this study, let $P = [me_1, me_1 + \eta(e_2, me_1)]$ with $e_1 < e_2$, $m \in (0, 1]$ be an invex subset with respect to $\eta : P \times P \longrightarrow \mathbb{R}$. Also, for brevity, we define

$$\Lambda_m(t) := \int_0^t \frac{\varphi\left(\eta(e_2, me_1)u\right)}{u} du < \infty, \quad \eta(e_2, me_1) > 0. \tag{2.1}$$

For establishing some new results regarding general fractional integrals we need to prove the following lemma.

Lemma 2.1. *Let $f : P \longrightarrow \mathbb{R}$ be a differentiable mapping on P° and $\gamma_1, \gamma_2 \in \mathbb{R}$. If $f' \in L(P)$ and $\lambda \in (0,1]$, then the following identity for generalized fractional integrals hold:*

$$\frac{\gamma_1 f(me_1) + \gamma_2 f(me_1 + \lambda\eta(e_2, me_1))}{2}$$
$$+ \left[\frac{2\lambda\Lambda_m\left(\frac{\lambda}{2}\right)}{\eta(e_2, me_1)} - \frac{\gamma_1 + \gamma_2}{2}\right] f\left(me_1 + \frac{\lambda}{2}\eta(e_2, me_1)\right) - \frac{\lambda}{\eta(e_2, me_1)}$$
$$\times \Big[{}_{\left(me_1 + \frac{\lambda}{2}\eta(e_2,me_1)\right)^+} I_\varphi f\left(me_1 + \lambda\eta(e_2, me_1)\right) + {}_{\left(me_1 + \frac{\lambda}{2}\eta(e_2,me_1)\right)^-} I_\varphi f\left(me_1\right)\Big]$$
$$= \frac{\lambda\eta(e_2, me_1)}{2} \tag{2.2}$$
$$\times \Bigg\{ \int_0^{\frac{1}{2}} \left(\frac{2\lambda\Lambda_m(\lambda t)}{\eta(e_2, me_1)} - \gamma_1\right) f'\left(me_1 + (\lambda t)\eta(e_2, me_1)\right) dt$$
$$- \int_{\frac{1}{2}}^1 \left(\frac{2\lambda\Lambda_m((1-t)\lambda)}{\eta(e_2, me_1)} - \gamma_2\right) f'\left(me_1 + (\lambda t)\eta(e_2, me_1)\right) dt \Bigg\}.$$

We denote

$$T_{f,\Lambda_m}(\lambda, \gamma_1, \gamma_2; e_1, e_2) := \frac{\lambda\eta(e_2, me_1)}{2} \tag{2.3}$$
$$\times \Bigg\{ \int_0^{\frac{1}{2}} \left(\frac{2\lambda\Lambda_m(\lambda t)}{\eta(e_2, me_1)} - \gamma_1\right) f'\left(me_1 + (\lambda t)\eta(e_2, me_1)\right) dt$$
$$- \int_{\frac{1}{2}}^1 \left(\frac{2\lambda\Lambda_m((1-t)\lambda)}{\eta(e_2, me_1)} - \gamma_2\right) f'\left(me_1 + (\lambda t)\eta(e_2, me_1)\right) dt \Bigg\}.$$

Proof. Integrating by parts eq. (2.3) and changing the variable of integration, we have

$$T_{f,\Lambda_m}(\lambda, \gamma_1, \gamma_2; e_1, e_2) = \frac{\lambda\eta(e_2, me_1)}{2}$$
$$\times \Bigg\{ \frac{2\lambda}{\eta(e_2, me_1)} \int_0^{\frac{1}{2}} \Lambda_m(\lambda t) f'\left(me_1 + (\lambda t)\eta(e_2, me_1)\right) dt$$
$$- \gamma_1 \int_0^{\frac{1}{2}} f'\left(me_1 + (\lambda t)\eta(e_2, me_1)\right) dt$$
$$- \frac{2\lambda}{\eta(e_2, me_1)} \int_{\frac{1}{2}}^1 \Lambda_m((1-t)\lambda) f'\left(me_1 + (\lambda t)\eta(e_2, me_1)\right) dt$$
$$+ \gamma_2 \int_{\frac{1}{2}}^1 f'\left(me_1 + (\lambda t)\eta(e_2, me_1)\right) dt \Bigg\}$$
$$= \frac{\lambda\eta(e_2, me_1)}{2} \times \Bigg\{ \frac{2\Lambda_m(\lambda t) f\left(me_1 + (\lambda t)\eta(e_2, me_1)\right)}{\eta^2(e_2, me_1)} \Bigg|_0^{\frac{1}{2}}$$
$$- \frac{2\lambda}{\eta^2(e_2, me_1)} \times \int_0^{\frac{1}{2}} \frac{\varphi\left(\eta(e_2, me_1)(\lambda t)\right)}{\lambda t} f\left(me_1 + (\lambda t)\eta(e_2, me_1)\right) dt$$

$$-\frac{\gamma_1}{\lambda\eta(e_2,me_1)}f\left(me_1+(\lambda t)\eta(e_2,me_1)\right)\Big|_0^{\frac{1}{2}}$$

$$-\frac{2\Lambda_m((1-t)\lambda)f\left(me_1+(\lambda t)\eta(e_2,me_1)\right)}{\eta^2(e_2,me_1)}\Bigg|_{\frac{1}{2}}^1$$

$$-\frac{2\lambda}{\eta^2(e_2,me_1)}\times\int_{\frac{1}{2}}^1\frac{\varphi\left(\eta(e_2,me_1)((1-t)\lambda)\right)}{(1-t)\lambda}f\left(me_1+(\lambda t)\eta(e_2,me_1)\right)dt$$

$$+\frac{\gamma_2}{\lambda\eta(e_2,me_1)}f\left(me_1+(\lambda t)\eta(e_2,me_1)\right)\Big|_{\frac{1}{2}}^1\Bigg\}$$

$$=\frac{\gamma_1 f(me_1)+\gamma_2 f(me_1+\lambda\eta(e_2,me_1))}{2}$$

$$+\left[\frac{2\lambda\Lambda_m\left(\frac{\lambda}{2}\right)}{\eta(e_2,me_1)}-\frac{\gamma_1+\gamma_2}{2}\right]f\left(me_1+\frac{\lambda}{2}\eta(e_2,me_1)\right)-\frac{\lambda}{\eta(e_2,me_1)}$$

$$\times\left[{}_{\left(me_1+\frac{\lambda}{2}\eta(e_2,me_1)\right)^+}I_\varphi f\left(me_1+\lambda\eta(e_2,me_1)\right)+{}_{\left(me_1+\frac{\lambda}{2}\eta(e_2,me_1)\right)^-}I_\varphi f\left(me_1\right)\right].$$

This completes the proof of the lemma. □

Remark 2.2. **a.** Taking $\lambda=m=1$, $\gamma_1=\gamma_2=0$, $\eta(e_2,me_1)=e_2-me_1$ and $\varphi(t)=t$ in Lemma 2.1, we get the midpoint type identity.

b. Taking $\lambda=m=1$, $\gamma_1=\gamma_2=1$, $\eta(e_2,me_1)=e_2-me_1$ and $\varphi(t)=t$ in Lemma 2.1, we get Hermite-Hadamard type identity.

c. Taking $\lambda=m=1$, $\gamma_1=\frac{1}{6}$, $\gamma_2=\frac{5}{6}$, $\eta(e_2,me_1)=e_2-me_1$ and $\varphi(t)=t$ in Lemma 2.1, we get new Simpson type identity.

Theorem 2.3. *Let $f:P\longrightarrow\mathbb{R}$ be a differentiable mapping on P° and $0\le\gamma_1,\gamma_2\le1$. If $|f'|^q$ is preinvex on P and $\lambda\in(0,1]$ for $q>1$ and $p^{-1}+q^{-1}=1$, then the following inequality for generalized fractional integrals hold:*

$$\left|T_{f,\Lambda_m}(\lambda,\gamma_1,\gamma_2;e_1,e_2)\right|\le\frac{\lambda\eta(e_2,me_1)}{2\sqrt[q]{8}} \tag{2.4}$$

$$\times\Big\{\sqrt[p]{B_{\Lambda_m}(\lambda,\gamma_1;p)}\times\sqrt[q]{(4-\lambda)|f'(me_1)|^q+\lambda|f'(e_2)|^q}$$

$$+\sqrt[p]{C_{\Lambda_m}(\lambda,\gamma_2;p)}\times\sqrt[q]{(4-3\lambda)|f'(me_1)|^q+3\lambda|f'(e_2)|^q}\Big\},$$

where

$$B_{\Lambda_m}(\lambda,\gamma_1;p):=\int_0^{\frac{1}{2}}\left|\frac{2\lambda\Lambda_m(\lambda t)}{\eta(e_2,me_1)}-\gamma_1\right|^p dt \tag{2.5}$$

and

$$C_{\Lambda_m}(\lambda,\gamma_2;p):=\int_{\frac{1}{2}}^1\left|\frac{2\lambda\Lambda_m((1-t)\lambda)}{\eta(e_2,me_1)}-\gamma_2\right|^p dt. \tag{2.6}$$

Proof. From Lemma 2.1, preinvexity of $|f'|^q$, Hölder inequality and properties of the modulus, we have

$$\begin{aligned}
\left|T_{f,\Lambda_m}(\lambda,\gamma_1,\gamma_2;e_1,e_2)\right| &\leq \frac{\lambda\eta(e_2,me_1)}{2}\\
\times\Bigg\{\int_0^{\frac{1}{2}}&\left|\frac{2\lambda\Lambda_m(\lambda t)}{\eta(e_2,me_1)}-\gamma_1\right|\left|f'\left(me_1+(\lambda t)\eta(e_2,me_1)\right)\right|dt\\
+\int_{\frac{1}{2}}^1&\left|\frac{2\lambda\Lambda_m((1-t)\lambda)}{\eta(e_2,me_1)}-\gamma_2\right|\left|f'\left(me_1+(\lambda t)\eta(e_2,me_1)\right)\right|dt\Bigg\}\\
&\leq \frac{\lambda\eta(e_2,me_1)}{2}\\
\times\Bigg\{\left(\int_0^{\frac{1}{2}}\left|\frac{2\lambda\Lambda_m(\lambda t)}{\eta(e_2,me_1)}-\gamma_1\right|^p dt\right)^{\frac{1}{p}}&\left(\int_0^{\frac{1}{2}}\left|f'\left(me_1+(\lambda t)\eta(e_2,me_1)\right)\right|^q dt\right)^{\frac{1}{q}}\\
+\left(\int_{\frac{1}{2}}^1\left|\frac{2\lambda\Lambda_m((1-t)\lambda)}{\eta(e_2,me_1)}-\gamma_2\right|^p dt\right)^{\frac{1}{p}}&\left(\int_{\frac{1}{2}}^1\left|f'\left(me_1+(\lambda t)\eta(e_2,me_1)\right)\right|^q dt\right)^{\frac{1}{q}}\Bigg\}\\
&\leq \frac{\lambda\eta(e_2,me_1)}{2}\times\Bigg\{\sqrt[p]{B_{\Lambda_m}(\lambda,\gamma_1;p)}\\
\times\left(\int_0^{\frac{1}{2}}\left[(1-\lambda t)|f'(me_1)|^q+(\lambda t)|f'(e_2)|^q\right]dt\right)^{\frac{1}{q}}&\\
&+\sqrt[p]{C_{\Lambda_m}(\lambda,\gamma_2;p)}\\
\times\left(\int_{\frac{1}{2}}^1\left[(1-\lambda t)|f'(me_1)|^q+(\lambda t)|f'(e_2)|^q\right]dt\right)^{\frac{1}{q}}\Bigg\}&\\
&=\frac{\lambda\eta(e_2,me_1)}{2\sqrt[q]{8}}\\
\times\Big\{\sqrt[p]{B_{\Lambda_m}(\lambda,\gamma_1;p)}\times\sqrt[q]{(4-\lambda)|f'(me_1)|^q+\lambda|f'(e_2)|^q}&\\
+\sqrt[p]{C_{\Lambda_m}(\lambda,\gamma_2;p)}\times\sqrt[q]{(4-3\lambda)|f'(me_1)|^q+3\lambda|f'(e_2)|^q}\Big\}.&
\end{aligned}$$

The proof of this theorem is complete. ☐

We point out some special cases of Theorem 2.3.

Corollary 2.4. *Taking $p=q=2$ in Theorem 2.3, we get*

$$\begin{aligned}
\left|T_{f,\Lambda_m}(\lambda,\gamma_1,\gamma_2;e_1,e_2)\right| &\leq \frac{\lambda\eta(e_2,me_1)}{4\sqrt{2}} \qquad (2.7)\\
\times\Big\{\sqrt{B_{\Lambda_m}(\lambda,\gamma_1;2)}\times\sqrt{(4-\lambda)|f'(me_1)|^2+\lambda|f'(e_2)|^2}&\\
+\sqrt{C_{\Lambda_m}(\lambda,\gamma_2;2)}\times\sqrt{(4-3\lambda)|f'(me_1)|^2+3\lambda|f'(e_2)|^2}\Big\}.&
\end{aligned}$$

Corollary 2.5. *Taking* $|f'| \le K$ *in Theorem* 2.3*, we get*

$$\left|T_{f,\Lambda_m}(\lambda,\gamma_1,\gamma_2;e_1,e_2)\right| \le \frac{K\lambda\eta(e_2,me_1)}{2\sqrt[q]{2}} \times\left\{\sqrt[p]{B_{\Lambda_m}(\lambda,\gamma_1;p)}+\sqrt[p]{C_{\Lambda_m}(\lambda,\gamma_2;p)}\right\}. \tag{2.8}$$

Corollary 2.6. *Taking* $\lambda = m = 1$*,* $\gamma_1 = \gamma_2 = 0$*,* $\eta(e_2, me_1) = e_2 - me_1$ *and* $\varphi(t) = t$ *in Theorem* 2.3*, we get the following midpoint type inequality:*

$$\left|T_f(1,0,0;e_1,e_2)\right| \le \frac{\sqrt[p]{2}(e_2-e_1)}{2\sqrt[q]{8}\sqrt[p]{2^{p+1}(p+1)}} \times\left\{\sqrt[q]{|f'(e_1)|^q+3|f'(e_2)|^q}+\sqrt[q]{3|f'(e_1)|^q+|f'(e_2)|^q}\right\}. \tag{2.9}$$

Corollary 2.7. *Taking* $\lambda = m = 1$*,* $\gamma_1 = \gamma_2 = 1$*,* $\eta(e_2, me_1) = e_2 - me_1$ *and* $\varphi(t) = t$ *in Theorem* 2.3*, we get the following trapezium type inequality:*

$$\left|T_f(1,1,1;e_1,e_2)\right| \le \frac{(e_2-e_1)}{2\sqrt[q]{8}\sqrt[p]{2(p+1)}} \times\left\{\sqrt[q]{|f'(e_1)|^q+3|f'(e_2)|^q}+\sqrt[q]{3|f'(e_1)|^q+|f'(e_2)|^q}\right\}. \tag{2.10}$$

Corollary 2.8. *Taking* $\lambda = m = 1$*,* $\gamma_1 = \frac{1}{6}$*,* $\gamma_2 = \frac{5}{6}$*,* $\eta(e_2, me_1) = e_2 - me_1$ *and* $\varphi(t) = t$ *in Theorem* 2.3*, we get the following Simpson type inequality:*

$$\left|T_f\left(1,\frac{1}{6},\frac{5}{6};e_1,e_2\right)\right| \le \frac{\sqrt[p]{5^{p+1}+1}(e_2-e_1)}{12\sqrt[q]{8}\sqrt[p]{12(p+1)}} \times\left\{\sqrt[q]{|f'(e_1)|^q+3|f'(e_2)|^q}+\sqrt[q]{3|f'(e_1)|^q+|f'(e_2)|^q}\right\}. \tag{2.11}$$

Theorem 2.9. *Let* $f : P \longrightarrow \mathbb{R}$ *be a differentiable mapping on* P° *and* $0 \le \gamma_1, \gamma_2 \le 1$*. If* $|f'|^q$ *is preinvex on* P *and* $\lambda \in (0, 1]$ *for* $q \ge 1$*, then the following inequality for generalized fractional integrals hold:*

$$\begin{aligned}\left|T_{f,\Lambda_m}(\lambda,\gamma_1,\gamma_2;e_1,e_2)\right| &\le \frac{\lambda\eta(e_2,me_1)}{2}\times\left\{\Big[B_{\Lambda_m}(\lambda,\gamma_1;1)\Big]^{1-\frac{1}{q}}\right.\\ &\times\sqrt[q]{\Big[B_{\Lambda_m}(\lambda,\gamma_1;1)-\lambda D_{\Lambda_m}(\lambda,\gamma_1)\Big]|f'(me_1)|^q+\lambda D_{\Lambda_m}(\lambda,\gamma_1)|f'(e_2)|^q}\\ &+\Big[C_{\Lambda_m}(\lambda,\gamma_2;1)\Big]^{1-\frac{1}{q}}\\ &\left.\times\sqrt[q]{\Big[C_{\Lambda_m}(\lambda,\gamma_2;1)-\lambda E_{\Lambda_m}(\lambda,\gamma_2)\Big]|f'(me_1)|^q+\lambda E_{\Lambda_m}(\lambda,\gamma_2)|f'(e_2)|^q}\right\},\end{aligned} \tag{2.12}$$

where

$$D_{\Lambda_m}(\lambda,\gamma_1) := \int_0^{\frac{1}{2}} t\left|\frac{2\lambda\Lambda_m(\lambda t)}{\eta(e_2,me_1)}-\gamma_1\right|dt, \tag{2.13}$$

$$E_{\Lambda_m}(\lambda,\gamma_2):=\int_{\frac{1}{2}}^{1} t\left|\frac{2\lambda\Lambda_m((1-t)\lambda)}{\eta(e_2,me_1)}-\gamma_2\right|dt \tag{2.14}$$

and $B_{\Lambda_m}(\lambda,\gamma_1;1)$, $C_{\Lambda_m}(\lambda,\gamma_2;1)$ *are defined as in Theorem* 2.3.

Proof. From Lemma 2.1, preinvexity of $|f'|^q$, power mean inequality and properties of the modulus, we have

$$\begin{aligned}
&\left|T_{f,\Lambda_m}(\lambda,\gamma_1,\gamma_2;e_1,e_2)\right| \le \frac{\lambda\eta(e_2,me_1)}{2}\\
&\times\left\{\int_0^{\frac{1}{2}}\left|\frac{2\lambda\Lambda_m(\lambda t)}{\eta(e_2,me_1)}-\gamma_1\right|\left|f'\left(me_1+(\lambda t)\eta(e_2,me_1)\right)\right|dt\right.\\
&\left.+\int_{\frac{1}{2}}^{1}\left|\frac{2\lambda\Lambda_m((1-t)\lambda)}{\eta(e_2,me_1)}-\gamma_2\right|\left|f'\left(me_1+(\lambda t)\eta(e_2,me_1)\right)\right|dt\right\}\\
&\le \frac{\lambda\eta(e_2,me_1)}{2}\times\left\{\left(\int_0^{\frac{1}{2}}\left|\frac{2\lambda\Lambda_m(\lambda t)}{\eta(e_2,me_1)}-\gamma_1\right|dt\right)^{1-\frac{1}{q}}\right.\\
&\times\left(\int_0^{\frac{1}{2}}\left|\frac{2\lambda\Lambda_m(\lambda t)}{\eta(e_2,me_1)}-\gamma_1\right|\left|f'\left(me_1+(\lambda t)\eta(e_2,me_1)\right)\right|^q dt\right)^{\frac{1}{q}}\\
&+\left(\int_{\frac{1}{2}}^{1}\left|\frac{2\lambda\Lambda_m((1-t)\lambda)}{\eta(e_2,me_1)}-\gamma_2\right|dt\right)^{1-\frac{1}{q}}\\
&\left.\times\left(\int_{\frac{1}{2}}^{1}\left|\frac{2\lambda\Lambda_m((1-t)\lambda)}{\eta(e_2,me_1)}-\gamma_2\right|\left|f'\left(me_1+(\lambda t)\eta(e_2,me_1)\right)\right|^q dt\right)^{\frac{1}{q}}\right\}\\
&\le \frac{\lambda\eta(e_2,me_1)}{2}\times\Big[B_{\Lambda_m}(\lambda,\gamma_1;1)\Big]^{1-\frac{1}{q}}\\
&\times\left\{\left(\int_0^{\frac{1}{2}}\left|\frac{2\lambda\Lambda_m(\lambda t)}{\eta(e_2,me_1)}-\gamma_1\right|\Big[(1-\lambda t)|f'(me_1)|^q+(\lambda t)|f'(e_2)|^q\Big]dt\right)^{\frac{1}{q}}\right.\\
&+\Big[C_{\Lambda_m}(\lambda,\gamma_2;1)\Big]^{1-\frac{1}{q}}\\
&\left.\times\left\{\left(\int_{\frac{1}{2}}^{1}\left|\frac{2\lambda\Lambda_m((1-t)\lambda)}{\eta(e_2,me_1)}-\gamma_2\right|\Big[(1-\lambda t)|f'(me_1)|^q+(\lambda t)|f'(e_2)|^q\Big]dt\right)^{\frac{1}{q}}\right\}\right.\\
&= \frac{\lambda\eta(e_2,me_1)}{2}\times\left\{\Big[B_{\Lambda_m}(\lambda,\gamma_1;1)\Big]^{1-\frac{1}{q}}\right.\\
&\times\sqrt[q]{\Big[B_{\Lambda_m}(\lambda,\gamma_1;1)-\lambda D_{\Lambda_m}(\lambda,\gamma_1)\Big]|f'(me_1)|^q+\lambda D_{\Lambda_m}(\lambda,\gamma_1)|f'(e_2)|^q}\\
&+\Big[C_{\Lambda_m}(\lambda,\gamma_2;1)\Big]^{1-\frac{1}{q}}
\end{aligned}$$

$$\times \sqrt[q]{\left[C_{\Lambda_m}(\lambda,\gamma_2;1)-\lambda E_{\Lambda_m}(\lambda,\gamma_2)\right]|f'(me_1)|^q+\lambda E_{\Lambda_m}(\lambda,\gamma_2)|f'(e_2)|^q}\Bigg\}.$$

The proof of this theorem is complete. □

We point out some special cases of Theorem 2.9.

Corollary 2.10. *Taking $q = 1$ in Theorem 2.9, we get*

$$\left|T_{f,\Lambda_m}(\lambda,\gamma_1,\gamma_2;e_1,e_2)\right| \le \frac{\lambda\eta(e_2,me_1)}{2} \tag{2.15}$$

$$\times\Big\{\left[B_{\Lambda_m}(\lambda,\gamma_1;1)-\lambda D_{\Lambda_m}(\lambda,\gamma_1)\right]|f'(me_1)|+\lambda D_{\Lambda_m}(\lambda,\gamma_1)|f'(e_2)|$$

$$+\left[C_{\Lambda_m}(\lambda,\gamma_2;1)-\lambda E_{\Lambda_m}(\lambda,\gamma_2)\right]|f'(me_1)|+\lambda E_{\Lambda_m}(\lambda,\gamma_2)|f'(e_2)|\Big\}.$$

Corollary 2.11. *Taking $|f'| \le K$ in Theorem 2.9, we get*

$$\left|T_{f,\Lambda_m}(\lambda,\gamma_1,\gamma_2;e_1,e_2)\right| \le \frac{K\lambda\eta(e_2,me_1)}{2} \tag{2.16}$$

$$\times\Big\{B_{\Lambda_m}(\lambda,\gamma_1;1)+C_{\Lambda_m}(\lambda,\gamma_2;1)\Big\}.$$

Corollary 2.12. *Taking $\lambda = m = 1$, $\gamma_1 = \gamma_2 = 0$, $\eta(e_2, me_1) = e_2 - me_1$ and $\varphi(t) = t$ in Theorem 2.9, we get the following midpoint type inequality:*

$$\left|T_f(1,0,0;e_1,e_2)\right| \le \frac{(e_2-e_1)}{8\sqrt[q]{3}} \tag{2.17}$$

$$\times\Big\{\sqrt[q]{|f'(e_1)|^q+2|f'(e_2)|^q}+\sqrt[q]{2|f'(e_1)|^q+|f'(e_2)|^q}\Big\}.$$

Corollary 2.13. *Taking $\lambda = m = 1$, $\gamma_1 = \gamma_2 = 1$, $\eta(e_2, me_1) = e_2 - me_1$ and $\varphi(t) = t$ in Theorem 2.9, we get the following trapezium type inequality:*

$$\left|T_f(1,1,1;e_1,e_2)\right| \le \frac{(e_2-e_1)}{8\sqrt[q]{6}} \tag{2.18}$$

$$\times\Big\{\sqrt[q]{|f'(e_1)|^q+5|f'(e_2)|^q}+\sqrt[q]{5|f'(e_1)|^q+|f'(e_2)|^q}\Big\}.$$

Corollary 2.14. *Taking $\lambda = m = 1$, $\gamma_1 = \frac{1}{6}$, $\gamma_2 = \frac{5}{6}$, $\eta(e_2, me_1) = e_2 - me_1$ and $\varphi(t) = t$ in Theorem 2.9, we get the following Simpson type inequality:*

$$\left|T_{f,\Lambda_1}\left(1,\frac{1}{6},\frac{5}{6};e_1,e_2\right)\right| \le \left(\frac{13}{72}\right)^{1-\frac{1}{q}}\frac{(e_2-e_1)}{2} \tag{2.19}$$

$$\times\left\{\frac{\sqrt[q]{305|f'(e_1)|^q+163|f'(e_2)|^q}}{\sqrt[q]{2592}}+\frac{\sqrt[q]{5938|f'(e_1)|^q+1550|f'(e_2)|^q}}{\sqrt[q]{41472}}\right\}.$$

Remark 2.15. Applying our Theorems 2.3 and 2.9 for special values of parameter λ, γ_1 and γ_2, for appropriate choices of function

$$\varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)}, \ \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}; \quad \varphi(t) = t(e_2 - t)^{\alpha-1}$$

for $\alpha \in (0,1)$; $\varphi(t) = \frac{t}{\alpha}\exp\left[\left(-\frac{1-\alpha}{\alpha}\right)t\right]$ for $\alpha \in (0,1)$, such that $|f'|^q$ to be preinvex (or convex in special case), we can deduce some new general fractional integral inequalities. The details are left to the interested reader.

3. Applications

Consider the following special means for different real numbers α, β and $\alpha\beta \neq 0$, as follows.

1. The arithmetic mean:
$$A := A(\alpha,\beta) = \frac{\alpha+\beta}{2},$$
2. The harmonic mean:
$$H := H(\alpha,\beta) = \frac{2}{\frac{1}{\alpha}+\frac{1}{\beta}},$$
3. The logarithmic mean:
$$L := L(\alpha,\beta) = \frac{\beta-\alpha}{\ln|\beta| - \ln|\alpha|},$$
4. The generalized log-mean:
$$L_r := L_r(\alpha,\beta) = \left[\frac{\beta^{r+1}-\alpha^{r+1}}{(r+1)(\beta-\alpha)}\right]^{\frac{1}{r}}; \quad r \in \mathbb{Z}\setminus\{-1,0\}.$$

It is well known that L_r is monotonic nondecreasing over $r \in \mathbb{Z}$ with $L_{-1} := L$. In particular, we have the following inequality $H \leq L \leq A$. Now, using the theory results in Section 2, we give some applications to special means for different real numbers.

Proposition 3.1. *Let $e_1, e_2 \in \mathbb{R}\setminus\{0\}$, where $e_1 < e_2$. Then for $r \in \mathbb{N}$ and $r \geq 2$, where $q > 1$ and $p^{-1} + q^{-1} = 1$, the following inequality hold:*

$$\left|A^r(e_1,e_2) - L_r^r(e_1,e_2)\right| \leq \frac{r(e_2-e_1)}{\sqrt[q]{8}\sqrt[p]{2^{p+1}(p+1)}} \tag{3.1}$$

$$\times\left\{\sqrt[q]{A\left(|e_1|^{q(r-1)}, 3|e_2|^{q(r-1)}\right)} + \sqrt[q]{A\left(3|e_1|^{q(r-1)}, |e_2|^{q(r-1)}\right)}\right\}.$$

Proof. Taking $\lambda = m = 1$, $\gamma_1 = \gamma_2 = 0$, $\eta(e_2, me_1) = e_2 - me_1$, $f(t) = t^r$ and $\varphi(t) = t$, in Theorem 2.3, one can obtain the result immediately. ☐

Proposition 3.2. *Let $e_1, e_2 \in \mathbb{R} \setminus \{0\}$, where $e_1 < e_2$. Then for $r \in \mathbb{N}$ and $r \geq 2$, where $q > 1$ and $p^{-1} + q^{-1} = 1$, the following inequality hold:*

$$\left| A(e_1^r, e_2^r) - L_r^r(e_1, e_2) \right| \leq \frac{r(e_2 - e_1)}{2\sqrt[q]{4}\sqrt[p]{2(p+1)}} \tag{3.2}$$

$$\times \left\{ \sqrt[q]{A\left(|e_1|^{q(r-1)}, 3|e_2|^{q(r-1)}\right)} + \sqrt[q]{A\left(3|e_1|^{q(r-1)}, |e_2|^{q(r-1)}\right)} \right\}.$$

Proof. Taking $\lambda = m = 1$, $\gamma_1 = \gamma_2 = 1$, $\eta(e_2, me_1) = e_2 - me_1$, $f(t) = t^r$ and $\varphi(t) = t$, in Theorem 2.3, one can obtain the result immediately. □

Proposition 3.3. *Let $e_1, e_2 \in \mathbb{R} \setminus \{0\}$, where $e_1 < e_2$. Then for $r \in \mathbb{N}$ and $r \geq 2$, where $q > 1$ and $p^{-1} + q^{-1} = 1$, the following inequality hold:*

$$\left| \frac{A(e_1^r, 5e_2^r)}{6} + \frac{A^r(e_1, e_2)}{2} - L_r^r(e_1, e_2) \right| \leq \frac{r\sqrt[p]{5^{p+1} + 1}(e_2 - e_1)}{12\sqrt[q]{4}\sqrt[p]{12(p+1)}} \tag{3.3}$$

$$\times \left\{ \sqrt[q]{A\left(|e_1|^{q(r-1)}, 3|e_2|^{q(r-1)}\right)} + \sqrt[q]{A\left(3|e_1|^{q(r-1)}, |e_2|^{q(r-1)}\right)} \right\}.$$

Proof. Taking $\lambda = m = 1$, $\gamma_1 = \frac{1}{6}$, $\gamma_2 = \frac{5}{6}$, $\eta(e_2, me_1) = e_2 - me_1$, $f(t) = t^r$ and $\varphi(t) = t$, in Theorem 2.3, one can obtain the result immediately. □

Proposition 3.4. *Let $e_1, e_2 \in \mathbb{R} \setminus \{0\}$, where $e_1 < e_2$. Then for $q > 1$ and $p^{-1} + q^{-1} = 1$, the following inequality hold:*

$$\left| \frac{1}{A(e_1, e_2)} - \frac{1}{L(e_1, e_2)} \right| \leq \sqrt[q]{\frac{3}{8}} \frac{(e_2 - e_1)}{\sqrt[p]{2^{p+1}(p+1)}} \tag{3.4}$$

$$\times \left\{ \frac{1}{\sqrt[q]{H(|e_1|^{2q}, 3|e_2|^{2q})}} + \frac{1}{\sqrt[q]{H(3|e_1|^{2q}, |e_2|^{2q})}} \right\}.$$

Proof. Taking $\lambda = m = 1$, $\gamma_1 = \gamma_2 = 0$, $\eta(e_2, me_1) = e_2 - me_1$, $f(t) = \dfrac{1}{t}$ and $\varphi(t) = t$, in Theorem 2.3, one can obtain the result immediately. □

Proposition 3.5. *Let $e_1, e_2 \in \mathbb{R} \setminus \{0\}$, where $e_1 < e_2$. Then for $q > 1$ and $p^{-1} + q^{-1} = 1$, the following inequality hold:*

$$\left| \frac{1}{H(e_1, e_2)} - \frac{1}{L(e_1, e_2)} \right| \leq \sqrt[q]{\frac{3}{4}} \frac{(e_2 - e_1)}{2\sqrt[p]{2(p+1)}} \tag{3.5}$$

$$\times \left\{ \frac{1}{\sqrt[q]{H(|e_1|^{2q}, 3|e_2|^{2q})}} + \frac{1}{\sqrt[q]{H(3|e_1|^{2q}, |e_2|^{2q})}} \right\}.$$

Proof. Taking $\lambda = m = 1$, $\gamma_1 = \gamma_2 = 1$, $\eta(e_2, me_1) = e_2 - me_1$, $f(t) = \dfrac{1}{t}$ and $\varphi(t) = t$, in Theorem 2.3, one can obtain the result immediately. □

Proposition 3.6. *Let $e_1, e_2 \in \mathbb{R}\setminus\{0\}$, where $e_1 < e_2$. Then for $q > 1$ and $p^{-1}+q^{-1} = 1$, the following inequality hold:*

$$\left|\frac{5}{6H(5e_1,e_2)}+\frac{1}{2A(e_1,e_2)}-\frac{1}{L(e_1,e_2)}\right| \leq \sqrt[q]{\frac{3}{4}}\frac{\sqrt[p]{5^{p+1}+1}(e_2-e_1)}{12\sqrt[p]{12(p+1)}} \tag{3.6}$$

$$\times\left\{\frac{1}{\sqrt[q]{H\left(|e_1|^{2q},3|e_2|^{2q}\right)}}+\frac{1}{\sqrt[q]{H\left(3|e_1|^{2q},|e_2|^{2q}\right)}}\right\}.$$

Proof. Taking $\lambda = m = 1$, $\gamma_1 = \frac{1}{5}$, $\gamma_2 = \frac{5}{6}$, $\eta(e_2, me_1) = e_2 - me_1$, $f(t) = \dfrac{1}{t}$ and $\varphi(t) = t$, in Theorem 2.3, one can obtain the result immediately. □

Proposition 3.7. *Let $e_1, e_2 \in \mathbb{R}\setminus\{0\}$, where $e_1 < e_2$. Then for $r \in \mathbb{N}$ and $r \geq 2$, where $q \geq 1$, the following inequality hold:*

$$\left|A^r(e_1,e_2)-L_r^r(e_1,e_2)\right| \leq \sqrt[q]{\frac{2}{3}}\frac{r(e_2-e_1)}{8} \tag{3.7}$$

$$\times\left\{\sqrt[q]{A\left(|e_1|^{q(r-1)},2|e_2|^{q(r-1)}\right)}+\sqrt[q]{A\left(2|e_1|^{q(r-1)},|e_2|^{q(r-1)}\right)}\right\}.$$

Proof. Taking $\lambda = m = 1$, $\gamma_1 = \gamma_2 = 0$, $\eta(e_2, me_1) = e_2 - me_1$, $f(t) = t^r$ and $\varphi(t) = t$, in Theorem 2.9, one can obtain the result immediately. □

Proposition 3.8. *Let $e_1, e_2 \in \mathbb{R}\setminus\{0\}$, where $e_1 < e_2$. Then for $r \in \mathbb{N}$ and $r \geq 2$, where $q \geq 1$, the following inequality hold:*

$$\left|A(e_1^r,e_2^r)-L_r^r(e_1,e_2)\right| \leq \frac{r(e_2-e_1)}{8\sqrt[q]{3}} \tag{3.8}$$

$$\times\left\{\sqrt[q]{A\left(|e_1|^{q(r-1)},5|e_2|^{q(r-1)}\right)}+\sqrt[q]{A\left(5|e_1|^{q(r-1)},|e_2|^{q(r-1)}\right)}\right\}.$$

Proof. Taking $\lambda = m = 1$, $\gamma_1 = \gamma_2 = 1$, $\eta(e_2, me_1) = e_2 - me_1$, $f(t) = t^r$ and $\varphi(t) = t$, in Theorem 2.9, one can obtain the result immediately. □

Proposition 3.9. *Let $e_1, e_2 \in \mathbb{R}\setminus\{0\}$, where $e_1 < e_2$. Then for $r \in \mathbb{N}$ and $r \geq 2$, where $q \geq 1$, the following inequality hold:*

$$\left|\frac{A\left(e_1^r,5e_2^r\right)}{6}+\frac{A^r(e_1,e_2)}{2}-L_r^r(e_1,e_2)\right| \leq \left(\frac{13}{72}\right)^{1-\frac{1}{q}}\frac{r(e_2-e_1)}{2} \tag{3.9}$$

$$\times\left\{\frac{\sqrt[q]{A\left(305|e_1|^{q(r-1)},163|e_2|^{q(r-1)}\right)}}{\sqrt[q]{1296}}+\frac{\sqrt[q]{A\left(5938|e_1|^{q(r-1)},1550|e_2|^{q(r-1)}\right)}}{\sqrt[q]{20736}}\right\}.$$

Proof. Taking $\lambda = m = 1$, $\gamma_1 = \frac{1}{6}$, $\gamma_2 = \frac{5}{6}$, $\eta(e_2, me_1) = e_2 - me_1$, $f(t) = t^r$ and $\varphi(t) = t$, in Theorem 2.9, one can obtain the result immediately. □

Proposition 3.10. *Let $e_1, e_2 \in \mathbb{R} \setminus \{0\}$, where $e_1 < e_2$. Then for $q \geq 1$, the following inequality hold:*

$$\left| \frac{1}{A(e_1,e_2)} - \frac{1}{L(e_1,e_2)} \right| \leq \sqrt[q]{\frac{4}{3}} \frac{(e_2-e_1)}{8} \tag{3.10}$$
$$\times \left\{ \frac{1}{\sqrt[q]{H\left(|e_1|^{2q}, 2|e_2|^{2q}\right)}} + \frac{1}{\sqrt[q]{H\left(2|e_1|^{2q}, |e_2|^{2q}\right)}} \right\}.$$

Proof. Taking $\lambda = m = 1$, $\gamma_1 = \gamma_2 = 0$, $\eta(e_2, me_1) = e_2 - me_1$, $f(t) = \dfrac{1}{t}$ and $\varphi(t) = t$, in Theorem 2.9, one can obtain the result immediately. □

Proposition 3.11. *Let $e_1, e_2 \in \mathbb{R} \setminus \{0\}$, where $e_1 < e_2$. Then for $q \geq 1$, the following inequality hold:*

$$\left| \frac{1}{H(e_1,e_2)} - \frac{1}{L(e_1,e_2)} \right| \leq \sqrt[q]{\frac{5}{3}} \frac{(e_2-e_1)}{8} \tag{3.11}$$
$$\times \left\{ \frac{1}{\sqrt[q]{H\left(|e_1|^{2q}, 5|e_2|^{2q}\right)}} + \frac{1}{\sqrt[q]{H\left(5|e_1|^{2q}, |e_2|^{2q}\right)}} \right\}.$$

Proof. Taking $\lambda = m = 1$, $\gamma_1 = \gamma_2 = 1$, $\eta(e_2, me_1) = e_2 - me_1$, $f(t) = \dfrac{1}{t}$ and $\varphi(t) = t$, in Theorem 2.9, one can obtain the result immediately. □

Proposition 3.12. *Let $e_1, e_2 \in \mathbb{R} \setminus \{0\}$, where $e_1 < e_2$. Then for $q \geq 1$, the following inequality hold:*

$$\left| \frac{5}{6H(5e_1,e_2)} + \frac{1}{2A(e_1,e_2)} - \frac{1}{L(e_1,e_2)} \right| \leq \left(\frac{13}{72}\right)^{1-\frac{1}{q}} \frac{(e_2-e_1)}{2} \tag{3.12}$$
$$\times \left\{ \sqrt[q]{\frac{49715}{1296}} \frac{1}{\sqrt[q]{H\left(163|e_1|^{2q}, 305|e_2|^{2q}\right)}} \right.$$
$$\left. + \sqrt[q]{\frac{2300975}{10368}} \frac{1}{\sqrt[q]{H\left(1550|e_1|^{2q}, 5938|e_2|^{2q}\right)}} \right\}.$$

Proof. Taking $\lambda = m = 1$, $\gamma_1 = \frac{1}{6}$, $\gamma_2 = \frac{5}{6}$, $\eta(e_2, me_1) = e_2 - me_1$, $f(t) = \dfrac{1}{t}$ and $\varphi(t) = t$, in Theorem 2.9, one can obtain the result immediately. □

Remark 3.13. Applying our Theorems 2.3 and 2.9 for special values of parameter λ, γ_1 and γ_2, for appropriate choices of function

$$\varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)}, \ \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}; \quad \varphi(t) = t(e_2 - t)^{\alpha-1}$$

for $\alpha \in (0,1)$; $\varphi(t) = \frac{t}{\alpha} \exp\left[\left(-\frac{1-\alpha}{\alpha}\right)t\right]$ for $\alpha \in (0,1)$, such that $|f'|^q$ to be convex, we can deduce some new general fractional integral inequalities using above special means. The details are left to the interested reader.

Next, we provide some new error estimates for the midpoint and trapezium quadrature formula. Let Q be the partition of the points $e_1 = x_0 < x_1 < \ldots < x_k = e_2$ of the interval $[e_1, e_2]$. Let consider the following quadrature formula:

$$\int_{e_1}^{e_2} f(x)dx = M(f,Q) + E(f,Q), \quad \int_{e_1}^{e_2} f(x)dx = T(f,Q) + E^*(f,Q)$$

where

$$M(f,Q) = \sum_{i=0}^{k-1} f\left(\frac{x_i + x_{i+1}}{2}\right)(x_{i+1} - x_i)$$

and

$$T(f,Q) = \sum_{i=0}^{k-1} \frac{f(x_i) + f(x_{i+1})}{2}(x_{i+1} - x_i)$$

are the midpoint and trapezium version and $E(f,Q)$, $E^*(f,Q)$ are denote their associated approximation errors.

Proposition 3.14. *Let $f : [e_1, e_2] \longrightarrow \mathbb{R}$ be a differentiable function on (e_1, e_2), where $e_1 < e_2$. If $|f'|^q$ is convex on $[e_1, e_2]$ for $q > 1$ and $p^{-1} + q^{-1} = 1$, then the following inequality holds:*

$$\big|E(f,Q)\big| \leq \frac{\sqrt[p]{2}}{2\sqrt[q]{8}\sqrt[p]{2^{p+1}(p+1)}} \times \sum_{i=0}^{k-1}(x_{i+1} - x_i)^2 \tag{3.13}$$
$$\times\left\{\sqrt[q]{|f'(x_i)|^q + 3|f'(x_{i+1})|^q} + \sqrt[q]{3|f'(x_i)|^q + |f'(x_{i+1})|^q}\right\}.$$

Proof. Applying Theorem 2.3 for $\lambda = m = 1$, $\gamma_1 = \gamma_2 = 0$, $\eta(e_2, me_1) = e_2 - me_1$ and $\varphi(t) = t$ on the subintervals $[x_i, x_{i+1}]$ $(i = 0, \ldots, k-1)$ of the partition Q, we have

$$\left|f\left(\frac{x_i + x_{i+1}}{2}\right) - \frac{1}{x_{i+1} - x_i}\int_{x_i}^{x_{i+1}} f(x)dx\right| \leq \frac{\sqrt[p]{2}(x_{i+1} - x_i)}{2\sqrt[q]{8}\sqrt[p]{2^{p+1}(p+1)}} \tag{3.14}$$
$$\times\left\{\sqrt[q]{|f'(x_i)|^q + 3|f'(x_{i+1})|^q} + \sqrt[q]{3|f'(x_i)|^q + |f'(x_{i+1})|^q}\right\}.$$

Hence from (3.14), we get

$$\big|E(f,Q)\big| = \left|\int_{e_1}^{e_2} f(x)dx - M(f,Q)\right|$$
$$\leq \left|\sum_{i=0}^{k-1}\left\{\int_{x_i}^{x_{i+1}} f(x)dx - f\left(\frac{x_i + x_{i+1}}{2}\right)(x_{i+1} - x_i)\right\}\right|$$
$$\leq \sum_{i=0}^{k-1}\left|\left\{\int_{x_i}^{x_{i+1}} f(x)dx - f\left(\frac{x_i + x_{i+1}}{2}\right)(x_{i+1} - x_i)\right\}\right|$$
$$\leq \frac{\sqrt[p]{2}}{2\sqrt[q]{8}\sqrt[p]{2^{p+1}(p+1)}} \times \sum_{i=0}^{k-1}(x_{i+1} - x_i)^2$$
$$\times\left\{\sqrt[q]{|f'(x_i)|^q + 3|f'(x_{i+1})|^q} + \sqrt[q]{3|f'(x_i)|^q + |f'(x_{i+1})|^q}\right\}.$$

The proof of this proposition is complete. □

Proposition 3.15. *Let $f : [e_1, e_2] \longrightarrow \mathbb{R}$ be a differentiable function on (e_1, e_2), where $e_1 < e_2$. If $|f'|^q$ is convex on $[e_1, e_2]$ for $q > 1$ and $p^{-1} + q^{-1} = 1$, then the following inequality holds:*

$$\left|E(f,Q)\right| \le \frac{1}{2\sqrt[q]{8}\sqrt[p]{2(p+1)}} \times \sum_{i=0}^{k-1}(x_{i+1}-x_i)^2 \tag{3.15}$$

$$\times\left\{\sqrt[q]{|f'(x_i)|^q + 3|f'(x_{i+1})|^q} + \sqrt[q]{3|f'(x_i)|^q + |f'(x_{i+1})|^q}\right\}.$$

Proof. The proof is analogous as to that of Proposition 3.14 taking $\lambda = m = 1$, $\gamma_1 = \gamma_2 = 1$, $\eta(e_2, me_1) = e_2 - me_1$ and $\varphi(t) = t$. □

Proposition 3.16. *Let $f : [e_1, e_2] \longrightarrow \mathbb{R}$ be a differentiable function on (e_1, e_2), where $e_1 < e_2$. If $|f'|^q$ is convex on $[e_1, e_2]$ for $q \ge 1$, then the following inequality holds:*

$$\left|E^*(f,Q)\right| \le \frac{1}{8\sqrt[q]{3}} \times \sum_{i=0}^{k-1}(x_{i+1}-x_i)^2 \tag{3.16}$$

$$\times\left\{\sqrt[q]{|f'(x_i)|^q + 2|f'(x_{i+1})|^q} + \sqrt[q]{2|f'(x_i)|^q + |f'(x_{i+1})|^q}\right\}.$$

Proof. Applying Theorem 2.9 for $\lambda = m = 1$, $\gamma_1 = \gamma_2 = 0$, $\eta(e_2, me_1) = e_2 - me_1$ and $\varphi(t) = t$ on the subintervals $[x_i, x_{i+1}]$ $(i = 0, \dots, k-1)$ of the partition Q, we have

$$\left|\frac{f(x_i)+f(x_{i+1})}{2} - \frac{1}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}} f(x)dx\right| \le \frac{(x_{i+1}-x_i)}{\sqrt[q]{3}} \tag{3.17}$$

$$\times\left\{\sqrt[q]{|f'(x_i)|^q + 2|f'(x_{i+1})|^q} + \sqrt[q]{2|f'(x_i)|^q + |f'(x_{i+1})|^q}\right\}.$$

Hence from (3.17), we get

$$\left|E^*(f,Q)\right| = \left|\int_{e_1}^{e_2} f(x)dx - T(f,Q)\right|$$

$$\le \left|\sum_{i=0}^{k-1}\left\{\int_{x_i}^{x_{i+1}} f(x)dx - \frac{f(x_i)+f(x_{i+1})}{2}(x_{i+1}-x_i)\right\}\right|$$

$$\le \sum_{i=0}^{k-1}\left|\left\{\int_{x_i}^{x_{i+1}} f(x)dx - \frac{f(x_i)+f(x_{i+1})}{2}(x_{i+1}-x_i)\right\}\right|$$

$$\le \frac{1}{8\sqrt[q]{3}} \times \sum_{i=0}^{k-1}(x_{i+1}-x_i)^2$$

$$\times\left\{\sqrt[q]{|f'(x_i)|^q + 2|f'(x_{i+1})|^q} + \sqrt[q]{2|f'(x_i)|^q + |f'(x_{i+1})|^q}\right\}.$$

The proof of this proposition is complete. □

Proposition 3.17. *Let $f : [e_1, e_2] \longrightarrow \mathbb{R}$ be a differentiable function on (e_1, e_2), where $e_1 < e_2$. If $|f'|^q$ is convex on $[e_1, e_2]$ for $q \geq 1$, then the following inequality holds:*

$$\left|E^*(f,Q)\right| \leq \frac{1}{8\sqrt[q]{6}} \times \sum_{i=0}^{k-1}(x_{i+1}-x_i)^2 \tag{3.18}$$

$$\times\left\{ \sqrt[q]{|f'(x_i)|^q + 5|f'(x_{i+1})|^q} + \sqrt[q]{5|f'(x_i)|^q + |f'(x_{i+1})|^q} \right\}.$$

Proof. The proof is analogous as to that of Proposition 3.16 taking $\lambda = m = 1$, $\gamma_1 = \gamma_2 = 1$, $\eta(e_2, me_1) = e_2 - me_1$ and $\varphi(t) = t$. □

Remark 3.18. Applying our Theorems 2.3 and 2.9, where $m = 1$, for special values of parameter λ, γ_1 and γ_2, for appropriate choices of function

$$\varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)},\ \frac{t^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)};$$

$$\varphi(t) = t(e_2 - t)^{\alpha-1}$$

for $\alpha \in (0, 1)$;

$$\varphi(t) = \frac{t}{\alpha}\exp\left[\left(-\frac{1-\alpha}{\alpha}\right)t\right]$$

for $\alpha \in (0, 1)$, such that $|f'|^q$ to be convex, we can deduce some new bounds for the midpoint and trapezium quadrature formula using above ideas and techniques. The details are left to the interested reader.

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References

[1] Aslani, S.M., Delavar, M.R., Vaezpour, S.M., *Inequalities of Fejér type related to generalized convex functions with applications*, Int. J. Anal. Appl., **16**(2018), no. 1, 38-49.

[2] Chen, F.X., Wu, S.H., *Several complementary inequalities to inequalities of Hermite-Hadamard type for s-convex functions*, J. Nonlinear Sci. Appl., **9**(2016), no. 2, 705-716.

[3] Chu, Y.M., Khan, M.A., Khan, T.U., Ali, T., *Generalizations of Hermite-Hadamard type inequalities for MT-convex functions*, J. Nonlinear Sci. Appl., **9**(2016), no. 5, 4305-4316.

[4] Delavar, M.R., Dragomir, S.S., *On η-convexity*, Math. Inequal. Appl., **20**(2017), 203-216.

[5] Delavar, M.R., De La Sen, M. *Some generalizations of Hermite-Hadamard type inequalities*, Springer Plus, **5**(2016), no. 1661.

[6] Dragomir, S.S., Agarwal, R.P., *Two inequalities for differentiable mappings and applications to special means of real numbers and trapezoidal formula*, Appl. Math. Lett., **11**(1998), no. 5, 91-95.

[7] Farid, G., Rehman, A.U., *Generalizations of some integral inequalities for fractional integrals*, Ann. Math. Sil., **31**(2017), pp. 14.

[8] Kashuri, A., Liko, R., *Some new Hermite-Hadamard type inequalities and their applications*, Stud. Sci. Math. Hung., **56**(2019), no. 1, 103-142.

[9] Khan, M.A., Chu, Y.M., Kashuri, A., Liko, R., *Hermite-Hadamard type fractional integral inequalities for $MT_{(r;g,m,\phi)}$-preinvex functions*, J. Comput. Anal. Appl., **26**(2019), no. 8, 1487-1503.

[10] Khan, M.A., Chu, Y.M., Kashuri, A., Liko, R., Ali, G., *New Hermite-Hadamard inequalities for conformable fractional integrals*, J. Funct. Spaces, (2018), Article ID 6928130, pp. 9.

[11] Liu, W.J., *Some Simpson type inequalities for h-convex and (α, m)-convex functions*, J. Comput. Anal. Appl., **16**(2014), no. 5, 1005-1012.

[12] Liu, W., Wen, W., Park, J., *Hermite-Hadamard type inequalities for MT-convex functions via classical integrals and fractional integrals*, J. Nonlinear Sci. Appl., **9**(2016), 766-777.

[13] Luo, C., Du, T.S., Khan, M.A., Kashuri, A., Shen, Y., *Some k-fractional integrals inequalities through generalized $\lambda_{\phi m}$-MT-preinvexity*, J. Comput. Anal. Appl., **27**(2019), no. 4, 690-705.

[14] Mihai, M.V., *Some Hermite-Hadamard type inequalities via Riemann-Liouville fractional calculus*, Tamkang J. Math, **44**(2013), no. 4, 411-416.

[15] Mubeen, S., Habibullah, G.M., *k-Fractional integrals and applications*, Int. J. Contemp. Math. Sci., **7**(2012), 89-94.

[16] Omotoyinbo, O., Mogbodemu, A., *Some new Hermite-Hadamard integral inequalities for convex functions*, Int. J. Sci. Innovation Tech., **1**(2014), no. 1, 1-12.

[17] Özdemir, M.E., Dragomir, S.S., Yildiz, C., *The Hadamard's inequality for convex function via fractional integrals*, Acta Math. Sci., Ser. A, Chin. Ed., **33**(2013), no. 5, 153-164.

[18] Qi, F., Xi, B.Y., *Some integral inequalities of Simpson type for $GA-\epsilon$-convex functions*, Georgian Math. J., **20**(2013), no. 5, 775-788.

[19] Sarikaya, M.Z., Ertuğral, F., *On the generalized Hermite-Hadamard inequalities*, https://www.researchgate.net/publication/321760443.

[20] Sarikaya, M.Z., Yildirim, H., *On generalization of the Riesz potential*, Indian Jour. of Math. and Mathematical Sci., **3**(2007), no. 2, 231-235.

[21] Set, E., Noor, M.A., Awan, M.U., Gözpinar, A., *Generalized Hermite-Hadamard type inequalities involving fractional integral operators*, J. Inequal. Appl., **169**(2017), 1-10.

[22] Wang, H., Du, T.S., Zhang, Y., *k-fractional integral trapezium-like inequalities through (h, m)-convex and (α, m)-convex mappings*, J. Inequal. Appl., **2017**(2017), no. 311, pp. 20.

[23] Xi, B.Y., Qi, F., *Some integral inequalities of Hermite-Hadamard type for convex functions with applications to means*, J. Funct. Spaces Appl., **2012**(2012), Article ID 980438, pp. 14.

[24] Zhang, X.M., Chu, Y.M., Zhang, X.H., *The Hermite-Hadamard type inequality of GA-convex functions and its applications*, J. Inequal. Appl., (2010), Article ID 507560, pp. 11.

[25] Zhang, Y., Du, T.S., Wang, H., Shen, Y.J., Kashuri, A., *Extensions of different type parameterized inequalities for generalized (m, h)-preinvex mappings via k-fractional integrals*, J. Inequal. Appl., **2018**(2018), no. 49, pp. 30.

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New version of generalized Ostrowski-Grüss type inequality

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Abstract. Ostrowski inequality is one of the celebrated inequalities in Mathematics. The main purpose of our study is to generalize the result of Ostrowski-Grüss type inequality for first differentiable mappings and apply it to probability density functions, composite quadrature rules and special means.

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1. Introduction

Literary, such integral inequality that measures the deviation of the integral of the product of two functions and the product of the integrals is referred to Grüss inequality [9].

In 1938, a Ukrainian mathematician A.M. Ostrowski (1893-1986) presented an inequality in his paper [15]. Since then this inequality is known in the history as Ostrowski inequality. A number of authors have written about generalizations of Ostrowski's inequality in the last few years. For example, this topic is considered in [1, 4, 5, 6, 7, 11, 12, 13, 16]. This inequality has been proved to be an exalted and applicable tool for the development of various branches of Mathematics. Integral inequalities that create bounds on the physical quantities are of great importance in the sense that these types of inequalities are not only applicable in integral operator theory, statistics, probability theory, numerical integration, nonlinear analysis, information theory, stochastic analysis and approximation theory but also we can find its applications in different areas of biological sciences, physics and technology.

S.S. Dragomir and S. Wang [7], in the year 1997, gave a proof of the following Ostrowski-Grüss type inequality:

Theorem 1.1. *Let $\phi : I \to \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an interval, be a mapping differentiable in the interior I° of I, and let $b_1, b_2 \in I^\circ$ with $b_1 < b_2$. If $\alpha \leq \phi'(\eta) \leq \lambda$, $\eta \in [b_1, b_2]$ for some constants $\alpha, \lambda \in \mathbb{R}$, then*

$$\left|\phi(\eta) - \frac{1}{b_2 - b_1}\int_{b_1}^{b_2} \phi(\xi)d\xi - \frac{\phi(b_2) - \phi(b_1)}{b_2 - b_1}\left(\eta - \frac{b_1 + b_2}{2}\right)\right| \leq \frac{1}{4}(b_2 - b_1)(\lambda - \alpha), \tag{1.1}$$

$\forall\ \eta \in [b_1, b_2]$.

The above inequality gives a relation between the Ostrowski inequality [15] and the Grüss inequality [14].

In the year 2000, by the use of pre-Grüss inequality, N. Ujević, M. Matić and J. E. Pečarić [11] had improved the factor of the right membership of (1.1) with $\frac{1}{4\sqrt{3}}$ as follows:

Theorem 1.2. *Let $\phi : I \to \mathbb{R}$, where I is an interval such that, $I \subseteq \mathbb{R}$, be a mapping differentiable in the interior I° of I, and let $b_1, b_2 \in I^\circ$ with $b_1 < b_2$. If $\alpha \leq \phi'(\eta) \leq \lambda$, $\eta \in [b_1, b_2]$ for some constants $\alpha,\ \lambda \in \mathbb{R}$, then*

$$\left|\phi(\eta) - \frac{1}{b_2 - b_1}\int_{b_1}^{b_2} \phi(\xi)d\xi - \frac{\phi(b_2) - \phi(b_1)}{b_2 - b_1}\left(\eta - \frac{b_1 + b_2}{2}\right)\right| \leq \frac{1}{4\sqrt{3}}(b_2 - b_1)(\lambda - \alpha), \tag{1.2}$$

$\forall\ \eta \in [b_1, b_2]$.

In the year 2000, by the use of Čebyšev functional, N. S. Barnett et al. [2] improved the result given by N. Ujević, M. Matić and J. E. Pečarić by proving first membership of the right side of (2.1) in terms of Euclidean norm as follows:

Theorem 1.3. *Let $\phi : [b_1, b_2] \to \mathbb{R}$ be an absolutely continuous function whose first derivative $\phi' \in L_2[b_1, b_2]$. Then we have the following inequality*

$$\begin{aligned}&\left|\phi(\eta) - \frac{1}{b_2 - b_1}\int_{b_1}^{b_2} \phi(\xi)d\xi - \frac{\phi(b_2) - \phi(b_1)}{b_2 - b_1}\left(\eta - \frac{b_1 + b_2}{2}\right)\right| \\ &\leq \frac{(b_2 - b_1)}{2\sqrt{3}}\left[\frac{1}{b_2 - b_1}\|\phi'\|_2^2 - \left(\frac{\phi(b_2) - \phi(b_1)}{b_2 - b_1}\right)^2\right]^{\frac{1}{2}} \\ &\leq \frac{1}{4\sqrt{3}}(b_2 - b_1)(\lambda - \alpha), \end{aligned}\tag{1.3}$$

if $\alpha \leq \phi'(\xi) \leq \lambda$ a.e for ξ on $[b_1, b_2]$ $\forall\ \eta \in [b_1, b_2]$.

In [2], we can evaluate the pre-Grüss inequality as follows:

$$T^2(\phi, \psi) \leq T(\phi, \phi)\,T(\psi, \psi),$$

where $T(\phi,\psi)$ is the Čebyšev functional as defined in [3] and $\phi,\psi\in L_2[b_1,b_2]$.

In the next section, we provide a generalization of (1.3) and then use it to probability density functions, composite quadrature rules and special means.

2. Main result

Theorem 2.1. *Let $\phi:[b_1,b_2]\to\mathbb{R}$ be an absolutely continuous function whose first derivative $\phi'\in L_2[b_1,b_2]$, we have*

$$\begin{aligned}
&\Bigg|(1-h)\left[\phi(\eta)-\frac{\phi(b_2)-\phi(b_1)}{b_2-b_1}\left(\eta-\frac{b_1+b_2}{2}\right)\right]+h\frac{\phi(b_1)+\phi(b_2)}{2}\\
&+\frac{\phi(b_1+b_2-\eta)-\phi(\eta)}{2}+\left(\eta-\frac{b_1+b_2}{2}\right)\left(\frac{\phi(b_2)-\phi(b_1)}{b_2-b_1}\right)\\
&-\frac{1}{b_2-b_1}\int_{b_1}^{b_2}\phi(\xi)d\xi\Bigg|\\
\le\ &\left[\frac{(b_2-b_1)^2}{12}(3h^2-3h+1)+(1-2h)(\eta-b_1)\left(\eta-\frac{b_1+b_2}{2}\right)\right.\\
&\left.+h(1-h)\left(\eta-\frac{b_1+b_2}{2}\right)^2\right]^{\frac{1}{2}}\left[\frac{1}{b_2-b_1}\|\phi'\|_2^2-\left(\frac{\phi(b_2)-\phi(b_1)}{b_2-b_1}\right)^2\right]^{\frac{1}{2}}\\
\le\ &\frac{1}{2}(\lambda-\alpha)\left[\frac{(b_2-b_1)^2}{12}(3h^2-3h+1)+(1-2h)(\eta-b_1)\left(\eta-\frac{b_1+b_2}{2}\right)\right.\\
&\left.+h(1-h)\left(\eta-\frac{b_1+b_2}{2}\right)^2\right]^{\frac{1}{2}}, \qquad (2.1)
\end{aligned}$$

if $\alpha\le\phi'(\xi)\le\lambda$ a.e for ξ on $[b_1,b_2]$, $\forall\,\eta\in\left[b_1+h\frac{b_2-b_1}{2},\frac{b_1+b_2}{2}\right]$ and $h\in[0,1]$.

Proof. Consider the following kernel defined in [8] $p:[b_1,b_2]^2\to\mathbb{R}$

$$p(\eta,\xi)=\begin{cases}\xi-\left(b_1+h\frac{b_2-b_1}{2}\right), & if\ \xi\in[b_1,\eta],\\ \xi-\left(\frac{b_1+b_2}{2}-h\frac{b_2-b_1}{2}\right), & if\ \xi\in(\eta,b_1+b_2-\eta],\\ \xi-\left(b_2-h\frac{b_2-b_1}{2}\right), & if\ \xi\in(b_1+b_2-\eta,b_2].\end{cases}$$

By replacing $\phi(\xi)$ with $p(\eta,\xi)$ and $\psi(\xi)$ with $\phi'(\xi)$ in Korkine's identity defined as:

$$T(\phi,\psi):=\frac{1}{2(b_2-b_1)^2}\int_{b_1}^{b_2}\int_{b_1}^{b_2}(\phi(\xi)-\phi(s))(\psi(\xi)-\psi(s))d\xi ds,$$

we get

$$\begin{aligned}
&\frac{1}{b_2-b_1}\int_{b_1}^{b_2} p(\eta,\xi)\phi'(\xi)d\xi - \frac{1}{b_2-b_1}\int_{b_1}^{b_2} p(\eta,\xi)d\xi \frac{1}{b_2-b_1}\int_{b_1}^{b_2}\phi'(\xi)d\xi \\
= \; & \frac{1}{2(b_2-b_1)^2}\int_{b_1}^{b_2}\int_{b_1}^{b_2}(p(\eta,\xi)-p(\eta,s))(\phi'(\xi)-\phi'(s))d\xi ds. \qquad (2.2)
\end{aligned}$$

We have,

$$\begin{aligned}
&\frac{1}{b_2-b_1}\int_{b_1}^{b_2} p(\eta,\xi)\phi'(\xi)d\xi \\
= \; & \frac{(1-2h)}{2}\phi(\eta) + \frac{h}{2}[\phi(b_1)+\phi(b_2)] - \frac{1}{b_2-b_1}\int_{b_1}^{b_2}\phi(\xi)d\xi + \frac{\phi(b_1+b_2-\eta)}{2},
\end{aligned}$$

$$\frac{1}{b_2-b_1}\int_{b_1}^{b_2} p(\eta,\xi)d\xi = h\left(\frac{b_1+b_2}{2}-\eta\right)$$

and

$$\frac{1}{b_2-b_1}\int_{b_1}^{b_2}\phi'(\xi)d\xi = \frac{\phi(b_2)-\phi(b_1)}{b_2-b_1}.$$

Identity (2.2) becomes,

$$\begin{aligned}
&(1-h)\left[\phi(\eta) - \frac{\phi(b_2)-\phi(b_1)}{b_2-b_1}\left(\eta-\frac{b_1+b_2}{2}\right)\right] + h\frac{\phi(b_1)+\phi(b_2)}{2} \\
&+\frac{\phi(b_1+b_2-\eta)-\phi(\eta)}{2} + \left(\eta-\frac{b_1+b_2}{2}\right)\left(\frac{\phi(b_2)-\phi(b_1)}{b_2-b_1}\right) \\
&-\frac{1}{b_2-b_1}\int_{b_1}^{b_2}\phi(\xi)d\xi \\
= \; & \frac{1}{2(b_2-b_1)^2}\int_{b_1}^{b_2}\int_{b_1}^{b_2}(p(\eta,\xi)-p(\eta,s))(\phi'(\xi)-\phi'(s))d\xi ds. \qquad (2.3)
\end{aligned}$$

$\forall\ \eta \in \left[b_1 + h\frac{b_2-b_1}{2}, \frac{b_1+b_2}{2}\right]$ and $h \in [0,1]$.

By using the Cauchy-Schwartz inequality in terms of double integrals, we can write

$$\frac{1}{2(b_2-b_1)^2}\left|\int_{b_1}^{b_2}\int_{b_1}^{b_2}(p(\eta,\xi)-p(\eta,s))(\phi'(\xi)-\phi'(s))d\xi ds\right|$$

$$\leq \left(\frac{1}{2(b_2-b_1)^2}\int_{b_1}^{b_2}\int_{b_1}^{b_2}(p(\eta,\xi)-p(\eta,s))^2 d\xi ds\right)^{\frac{1}{2}}$$

$$\times\left(\frac{1}{2(b_2-b_1)^2}\int_{b_1}^{b_2}\int_{b_1}^{b_2}(\phi'(\xi)-\phi'(s))^2 d\xi ds\right)^{\frac{1}{2}}. \tag{2.4}$$

However,

$$\frac{1}{2(b_2-b_1)^2}\int_{b_1}^{b_2}\int_{b_1}^{b_2}(p(\eta,\xi)-p(\eta,s))^2 d\xi ds$$

$$= \frac{1}{b_2-b_1}\int_{b_1}^{b_2}p^2(\eta,\xi)d\xi-\left(\frac{1}{b_2-b_1}\int_{b_1}^{b_2}p(\eta,\xi)d\xi\right)^2$$

$$= \frac{(b_2-b_1)^2}{12}(3h^2-3h+1)+h(1-h)\left(\eta-\frac{b_1+b_2}{2}\right)^2$$

$$+(1-2h)(\eta-b_1)\left(\eta-\frac{b_1+b_2}{2}\right) \tag{2.5}$$

and

$$\frac{1}{2(b_2-b_1)^2}\int_{b_1}^{b_2}\int_{b_1}^{b_2}(\phi'(\xi)-\phi'(s))^2 d\xi ds$$

$$= \frac{1}{b_2-b_1}\|\phi'\|_2^2-\left(\frac{\phi(b_2)-\phi(b_1)}{b_2-b_1}\right)^2. \tag{2.6}$$

By using (2.3) – (2.6), we evaluate the first inequality of (2.1).
By using the following Grüss inequality, we proved the second inequality of (2.1)

$$0\leq\frac{1}{b_2-b_1}\int_{b_1}^{b_2}\left(\phi'(\xi)\right)^2d\xi-\left(\frac{1}{b_2-b_1}\int_{b_1}^{b_2}\phi'(\xi)d\xi\right)^2\leq\frac{1}{4}(\lambda-\alpha)^2,$$

where $\alpha\leq\phi'(\xi)\leq\lambda$ *a.e* for ξ on $[b_1,b_2]$. □

Remark 2.2. Since

$$3h^2-3h+1\leq 1, \forall\ h\in[0,1]$$

and is minimum for $h=\frac{1}{2}$.

Thus, (2.1) shows an overall improvement in the inequality obtained by Barnett et al. in [2].

We have some remarks of (2.1) in the form of special cases.

Remark 2.3. Under the assumptions of Theorem 2.1 we can get different special cases by putting different values of h and η.

Special Case 1. For any value of h and $\eta = b_1$ or $h = 1$ and $\eta = \frac{b_1+b_2}{2}$ or $h = \frac{1}{2}$ and $\eta = b_2$, (2.1) gives trapezoid inequality [17],

$$\begin{aligned} &\left|(b_2-b_1)\frac{\phi(b_1)+\phi(b_2)}{2} - \int_{b_1}^{b_2} \phi(\xi)d\xi\right| \\ \leq\ & \frac{(b_2-b_1)^2}{2\sqrt{3}}\left[\frac{1}{b_2-b_1}\|\phi'\|_2^2 - \left(\frac{\phi(b_2)-\phi(b_1)}{b_2-b_1}\right)^2\right]^{\frac{1}{2}} \\ \leq\ & \frac{1}{4\sqrt{3}}(\lambda-\alpha)(b_2-b_1)^2. \end{aligned} \tag{2.7}$$

Special Case 2. If we take $h = 0$ and $\eta = \frac{b_1+b_2}{2}$, (2.1) becomes mid-point inequality [17],

$$\begin{aligned} &\left|(b_2-b_1)\phi\left(\frac{b_1+b_2}{2}\right) - \int_{b_1}^{b_2} \phi(\xi)d\xi\right| \\ \leq\ & \frac{(b_2-b_1)^2}{2\sqrt{3}}\left[\frac{1}{b_2-b_1}\|\phi'\|_2^2 - \left(\frac{\phi(b_2)-\phi(b_1)}{b_2-b_1}\right)^2\right]^{\frac{1}{2}} \\ \leq\ & \frac{1}{4\sqrt{3}}(\lambda-\alpha)(b_2-b_1)^2. \end{aligned} \tag{2.8}$$

Special Case 3. If $h = \frac{1}{2}$ and $\eta = \frac{b_1+b_2}{2}$, (2.1) becomes an averaged mid-point and trapezoid inequality [17],

$$\begin{aligned} &\left|\frac{\phi(b_1)+2\phi\left(\frac{b_1+b_2}{2}\right)+\phi(b_2)}{4} - \frac{1}{b_2-b_1}\int_{b_1}^{b_2} \phi(\xi)d\xi\right| \\ \leq\ & \frac{(b_2-b_1)^2}{4\sqrt{3}}\left[\frac{1}{b_2-b_1}\|\phi'\|_2^2 - \left(\frac{\phi(b_2)-\phi(b_1)}{b_2-b_1}\right)^2\right]^{\frac{1}{2}} \\ \leq\ & \frac{1}{8\sqrt{3}}(\lambda-\alpha)(b_2-b_1)^2. \end{aligned} \tag{2.9}$$

Special Case 4. If $h = \frac{1}{3}$ and $\eta = \frac{b_1+b_2}{2}$, (2.1) becomes $\frac{1}{3}$ Simpson's inequality for differentiable function ϕ [17],

$$\begin{aligned}
&\left|\frac{(b_2-b_1)}{6}\left[\phi(b_1)+4\phi\left(\frac{b_1+b_2}{2}\right)+\phi(b_2)\right]-\int_{b_1}^{b_2}\phi(t)dt\right| \\
\leq\;& \frac{(b_2-b_1)^2}{6}\left[\frac{1}{b_2-b_1}\|\phi'\|_2^2-\left(\frac{\phi(b_2)-\phi(b_1)}{b_2-b_1}\right)^2\right]^{\frac{1}{2}} \\
\leq\;& \frac{1}{12}(\lambda-\alpha)(b_2-b_1)^2. \qquad (2.10)
\end{aligned}$$

3. Applications

3.1. For probability density functions

Let X, ϕ and Φ be a continuous random variable, the probability density function and the cumulative distribution function, respectively such that $\phi : [b_1, b_2] \to \mathbb{R}_+$ and $\Phi : [b_1, b_2] \to [0, 1]$, defined as,

$$\Phi(\eta) = \int_{b_1}^{\eta} \phi(\xi)d\xi,\ \eta \in \left[b_1 + h\frac{b_2-b_1}{2}, \frac{b_1+b_2}{2}\right] \subset [b_1, b_2],$$

and the expectation of the random variable X on $[b_1, b_2]$ is defined as,

$$E(X) = \int_{b_1}^{b_2} \xi\ \phi(\xi)\ d\xi.$$

Then, we have:

Theorem 3.1. *By using above assumptions and if the probability density function $\phi \in L_2[b_1, b_2]$, we have*

$$\begin{aligned}
&\left|(1-h)\left[\Phi(\eta)-\frac{1}{b_2-b_1}\left(\eta-\frac{b_1+b_2}{2}\right)\right]+\frac{h}{2}-\frac{b_2-E(X)}{b_2-b_1}\right. \\
&\left.+\frac{\Phi(b_1+b_2-\eta)-\Phi(\eta)}{2}+\frac{1}{b_2-b_1}\left(\eta-\frac{b_1+b_2}{2}\right)\right| \\
\leq\;& \frac{1}{b_2-b_1}\left[\frac{(b_2-b_1)^2}{12}(3h^2-3h+1)+h(1-h)\left(\eta-\frac{b_1+b_2}{2}\right)^2\right. \\
&\left.+(1-2h)(\eta-b_1)\left(\eta-\frac{b_1+b_2}{2}\right)\right]^{\frac{1}{2}}\ \left[(b_2-b_1)\|\phi\|_2^2-1\right]^{\frac{1}{2}},
\end{aligned}$$

$$\leq \frac{(M-m)}{2}\left[\frac{(b_2-b_1)^2}{12}(3h^2-3h+1)+h(1-h)\left(\eta-\frac{b_1+b_2}{2}\right)^2\right.$$
$$\left.+(1-2h)(\eta-b_1)\left(\eta-\frac{b_1+b_2}{2}\right)\right]^{\frac{1}{2}}, \quad (3.1)$$

where $m \leq \phi \leq M$ *a.e on* $[b_1, b_2]$, $\forall\ \eta \in \left[b_1 + h\frac{b_2-b_1}{2}, \frac{b_1+b_2}{2}\right]$.

Proof. By putting $\phi = \Phi$ in (2.1), we obtain (3.1). □

Corollary 3.2. *By using the assumptions of Theorem* 3.1 *we have,*

$$\left|(1-h)Pr\left(X \leq \frac{b_1+b_2}{2}\right)+\frac{h}{2}-\frac{b_2-E(X)}{b_2-b_1}\right|$$
$$\leq \frac{1}{2\sqrt{3}}(3h^2-3h+1)^{\frac{1}{2}}[(b_2-b_1)\|\phi\|_2^2-1]^{\frac{1}{2}}$$
$$\leq \frac{1}{4\sqrt{3}}(3h^2-3h+1)^{\frac{1}{2}}(b_2-b_1)(M-m). \quad (3.2)$$

3.2. For composite quadrature rules

To obtain the estimates of composite quadrature rules, we may use (2.1),

Theorem 3.3. *Let* I_n*:* $b_1 = u_0 < u_1 < \cdots < u_{n-1} < u_n = b_2$ *be a partition of the interval* $[b_1, b_2]$, $\Delta u_j = u_{j+1} - u_j$, $h \in [0,1]$, $u_j + h\frac{\Delta u_j}{2} \leq \xi_j \leq \frac{u_j+u_{j+1}}{2}$, $j = 0, \ldots, n-1$. *Then,*

$$\int_{b_1}^{b_2} \phi(\xi)d\xi = S(\phi, I_n, \xi, h) + R(\phi, I_n, \xi, h).$$

where

$$S(\phi, I_n, \xi, h) = \sum_{j=0}^{n-1}\left[h\left(\frac{\phi(u_j)+\phi(u_{j+1})}{2}\right)+\frac{\phi(u_j+u_{j+1}-\xi_j)-\phi(\xi_j)}{2}\right.$$
$$+(1-h)\left\{\phi(\xi_j)-\frac{\phi(u_{j+1})-\phi(u_j)}{\Delta u_j}\left(\xi_j-\frac{u_j+u_{j+1}}{2}\right)\right\}$$
$$\left.+\left(\xi_j-\frac{u_j+u_{j+1}}{2}\right)\left(\frac{\phi(u_{j+1})-\phi(u_j)}{\Delta u_j}\right)\right]\Delta u_j. \quad (3.3)$$

and

$$\begin{aligned}
&|R(\phi, I_n, \xi, h)| \\
\leq\ & \sum_{j=0}^{n-1} \left[\frac{\Delta u_j^2}{12}(3h^2 - 3h + 1) + (1-2h)(\xi_j - u_j)\left(\xi_j - \frac{u_j + u_{j+1}}{2}\right) \right. \\
& \left. + h(1-h)\left(\xi_j - \frac{u_j + u_{j+1}}{2}\right)^2 \right]^{\frac{1}{2}} \left[\Delta u_j \|\phi'\|_2^2 - (\phi(u_{j+1}) - \phi(u_j))\right]^{\frac{1}{2}} \\
\leq\ & \frac{1}{2}(\lambda - \alpha) \sum_{j=0}^{n-1} \Delta u_j \left[\frac{\Delta u_j^2}{12}(3h^2 - 3h + 1) + h(1-h)\left(\xi_j - \frac{u_j + u_{j+1}}{2}\right)^2 \right. \\
& \left. + (1-2h)(\xi_j - u_j)\left(\xi_j - \frac{u_j + u_{j+1}}{2}\right) \right]^{\frac{1}{2}}. \qquad (3.4)
\end{aligned}$$

Proof. Applying inequality (2.1) on $\xi_j \in \left[u_j + h\frac{\Delta u_j}{2}, \frac{u_j + u_{j+1}}{2}\right]$ and summing over j from 0 to $n-1$ and using triangular inequality we get (3.4). □

Special Case 1. If $h = 0$ in (3.3) and (3.4), $(j = 0, \ldots, n-1)$ we have,

$$S(\phi, I_n, \xi, h) = \frac{1}{2} \sum_{j=0}^{n-1} [\phi(\xi_j) + \phi(u_j + u_{j+1} - \xi_j)] \Delta u_j.$$

and

$$\begin{aligned}
&|R(\phi, I_n, \xi, h)| \\
\leq\ & \sum_{j=0}^{n-1} \left[\frac{\Delta u_j^2}{12} + (\xi_j - u_j)\left(\xi_j - \frac{u_j + u_{j+1}}{2}\right) \right]^{\frac{1}{2}} \\
& \times \left[\Delta u_j \|\phi'\|_2^2 - (\phi(u_{j+1}) - \phi(u_j))^2\right]^{\frac{1}{2}} \\
\leq\ & \frac{1}{2}(\lambda - \alpha) \sum_{j=0}^{n-1} \Delta u_j^2 \left[\frac{\Delta u_j^2}{12} + (\xi_j - u_j)\left(\xi_j - \frac{u_j + u_{j+1}}{2}\right) \right]^{\frac{1}{2}}.
\end{aligned}$$

Special Case 2. If $\xi_j = \frac{u_j + u_{j+1}}{2}$ in (3.3) and (3.4), $(j = 0, \ldots, n-1)$, we have a perturbed composite mid point and trapezoidal quadrature rule.

$$\begin{aligned}
S(\phi, I_n, h) = \sum_{j=0}^{n-1} & \left[(1-h)\phi\left(\frac{u_j + u_{j+1}}{2}\right) \right. \\
& \left. + h\left(\frac{\phi(u_j) + \phi(u_{j+1})}{2}\right) \right] \Delta u_j \qquad (3.5)
\end{aligned}$$

and

$$
\begin{aligned}
&|R(\phi, I_n, h)| \\
\leq\ & \frac{1}{2\sqrt{3}}(3h^2 - 3h + 1)^{\frac{1}{2}} \sum_{j=0}^{n-1} \Delta u_j \left[\Delta u_j \|\phi'\|_2^2 - (\phi(u_{j+1}) - \phi(u_j))^2\right]^{\frac{1}{2}} \\
\leq\ & \frac{1}{4\sqrt{3}}(\lambda - \alpha)(3h^2 - 3h + 1)^{\frac{1}{2}} \sum_{j=0}^{n-1} \Delta u_j. \qquad (3.6)
\end{aligned}
$$

Special Case 3. If $h = 0$ in (3.5) and (3.6), $(j = 0, \ldots, n-1)$, then we have composite midpoint quadrature rule.

$$
S(\phi, I_n) = \sum_{j=0}^{n-1} \Delta u_j \phi\left(\frac{u_j + u_{j+1}}{2}\right). \qquad (3.7)
$$

and

$$
\begin{aligned}
&|R(\phi, I_n)| \\
\leq\ & \frac{1}{2\sqrt{3}} \sum_{j=0}^{n-1} \Delta u_j \left[\Delta u_j \|\phi'\|_2^2 - (\phi(u_{j+1}) - \phi(u_j))^2\right]^{\frac{1}{2}} \\
\leq\ & \frac{1}{4\sqrt{3}}(\lambda - \alpha) \sum_{j=0}^{n-1} \Delta u_j^2. \qquad (3.8)
\end{aligned}
$$

Special Case 4. If $h = \frac{3}{10}$ in (3.5) and (3.6), $(j = 0, \ldots, n-1)$, then we have a composite mid point and trapezoidal quadrature rule.

$$
\begin{aligned}
S(\phi, I_n) = \frac{1}{10} \sum_{j=0}^{n-1} & \left[7\phi\left(\frac{u_j + u_{j+1}}{2}\right)\right. \\
& \left. +3\left(\frac{\phi(u_j) + \phi(u_{j+1})}{2}\right)\right] \Delta u_j \qquad (3.9)
\end{aligned}
$$

and

$$
\begin{aligned}
&|R(\phi, I_n)| \\
\leq\ & \frac{\sqrt{37}}{20\sqrt{3}} \sum_{j=0}^{n-1} \Delta u_j \left[\Delta u_j \|\phi'\|_2^2 - (\phi(u_{j+1}) - \phi(u_j))^2\right]^{\frac{1}{2}} \\
\leq\ & \frac{\sqrt{37}}{40\sqrt{3}}(\lambda - \alpha) \sum_{j=0}^{n-1} \Delta u_j^2. \qquad (3.10)
\end{aligned}
$$

Special Case 5. If $h = 1$ in (3.5) and (3.6), for $j = 0, \ldots, n-1$, we have a composite trapezoidal rule.

$$
S(\phi, I_n) = \frac{1}{2} \sum_{j=0}^{n-1} (\phi(u_j) + \phi(u_{j+1})) \Delta u_j \qquad (3.11)
$$

and

$$\begin{aligned}
&|R(\phi, I_n)| \\
\leq\ & \frac{1}{2\sqrt{3}} \sum_{j=0}^{n-1} \Delta u_j \left[\Delta u_j \|\phi'\|_2^2 - (\phi(u_{j+1}) - \phi(u_j))^2\right]^{\frac{1}{2}} \\
\leq\ & \frac{1}{4\sqrt{3}} (\lambda - \alpha) \sum_{j=0}^{n-1} \Delta u_j^2. \qquad (3.12)
\end{aligned}$$

3.3. For special means

Throughout this section A, G, H, L, I and L_p stands for Arithmetic, Geometric, Harmonic, Logarithmic, Identric and $p-$Logarithmic means, respectively, for definitions we refer the readers to [10].

Example 3.4. Let the function ϕ be defined by $\phi(\eta) = \eta^p, p \in \mathbb{R}\backslash\{-1, 0\}$. Then we have,

$$\frac{1}{b_2 - b_1} \int_{b_1}^{b_2} \phi(\xi) d\xi = L_p^p,$$

$$\frac{\phi(b_2) - \phi(b_1)}{b_2 - b_1} = p L_{p-1}^{p-1},$$

$$\frac{\phi(b_1) + \phi(b_2)}{2} = \frac{b_1^p + b_2^p}{2} = A(b_1^p, b_2^p),$$

$$\frac{1}{b_2 - b_1} \|\phi'\|_2^2 = \frac{1}{b_2 - b_1} \int_{b_1}^{b_2} |\phi'(\xi)|^2 d\xi = p^2 L_{2(p-1)}^{2(p-1)}$$

and

$$\frac{\phi(b_1 + b_2 - \eta)}{2} = \frac{(2A - \eta)^p - \eta^p}{2}.$$

Thus, (2.1) becomes,

$$\begin{aligned}
&\left|(1-h)\left[\eta^p - p L_{p-1}^{p-1}(\eta - A)\right] + hA(b_1^p, b_2^p) - L_p^p + \frac{(2A-\eta)^p - \eta^p}{2} \right. \\
&\left. + p L_{p-1}^{p-1}(\eta - A)\right| \\
\leq\ & |p| \left[\frac{(b_2 - b_1)^2}{12}(3h^2 - 3h + 1) + h(1-h)(\eta - A)^2 \right. \\
&\left. + (1 - 2h)(\eta - b_1)(\eta - A)\right]^{\frac{1}{2}} \times \left[L_{2(p-1)}^{2(p-1)} - L_{p-1}^{2(p-1)}\right]^{\frac{1}{2}}. \qquad (3.13)
\end{aligned}$$

By taking $\eta = A$ in (3.13), we obtain,

$$\begin{aligned}
&\left|(1-h)A^p + hA(b_1^p, b_2^p) - L_p^p\right| \\
\leq\ & |p| \frac{(b_2 - b_1)}{2\sqrt{3}} (3h^2 - 3h + 1)^{\frac{1}{2}} \left[L_{2(p-1)}^{2(p-1)} - L_{p-1}^{2(p-1)}\right]^{\frac{1}{2}},
\end{aligned}$$

which is minimum for $h=\frac{1}{2}$.
By taking $h=1$, we obtain,

$$\left|A(b_1^p,b_2^p)-L_p^p\right|\leq \frac{(b_2-b_1)}{2\sqrt{3}}|p|\left[L_{2(p-1)}^{2(p-1)}-L_{p-1}^{2(p-1)}\right]^{\frac{1}{2}}.$$

Example 3.5. Let the function ϕ be defined by $\phi(\eta)=\frac{1}{\eta}$, $(\eta\in\left[b_1+h\frac{b_2-b_1}{2},\frac{b_1+b_2}{2}\right]\subset(0,\infty))$. Then,

$$\frac{1}{b_2-b_1}\int_{b_1}^{b_2}\phi(\xi)d\xi=\frac{1}{L},$$

$$\frac{\phi(b_2)-\phi(b_1)}{b_2-b_1}=-\frac{1}{G^2},$$

$$\frac{\phi(b_1)+\phi(b_2)}{2}=\frac{A}{G^2},$$

$$\frac{1}{b_2-b_1}\|\phi'\|_2^2=\frac{1}{b_2-b_1}\int_{b_1}^{b_2}|\phi'(\xi)|^2d\xi=\frac{b_1^2+b_1b_2+b_2^2}{3G^6},$$

$$\frac{1}{b_2-b_1}\int_{b_1}^{b_2}|\phi'(\xi)|^2d\xi-\left(\frac{\phi(b_2)-\phi(b_1)}{b_2-b_1}\right)^2=\frac{(b_2-b_1)^2}{3G^6}$$

and

$$\frac{\phi(b_1+b_2-\eta)}{2}=\frac{\eta-A}{\eta(2A-\eta)}.$$

Thus, (2.1) takes the form,

$$\begin{aligned}&\left|(1-h)\left[\frac{1}{\eta}+\frac{1}{G^2}(\eta-A)\right]+\frac{hA}{G^2}-\frac{1}{L}+\frac{\eta-A}{\eta(2A-\eta)}-\frac{(\eta-A)}{G^2}\right|\\ \leq\ &\left[\frac{(b_2-b_1)^2}{12}(3h^2-3h+1)+h(1-h)(\eta-A)^2\right.\\ &\left.+(1-2h)(\eta-b_1)(\eta-A)\right]^{\frac{1}{2}}\times\frac{(b_2-b_1)}{\sqrt{3}G^3}. \qquad (3.14)\end{aligned}$$

Put $\eta=A$ in (3.14), we obtain,

$$\left|(1-h)\frac{1}{A}+\frac{hA}{G^2}-\frac{1}{L}\right|\leq\frac{(b_2-b_1)^2}{6G^3}(3h^2-3h+1)^{\frac{1}{2}}.$$

By taking $\eta = L$ in (3.14), we obtain,

$$\left|\frac{2hA}{G^2} - \frac{hL}{G^2} - \frac{h}{L} + \frac{L-A}{L(2A-L)}\right|$$
$$\leq \left[\frac{(b_2-b_1)^2}{12}(3h^2-3h+1) + h(1-h)(L-A)^2 + (1-2h)(L-b_1)(L-A)\right]^{\frac{1}{2}} \times \frac{(b_2-b_1)}{\sqrt{3}G^3}.$$

Example 3.6. Let the function ϕ be defined by $\phi(\eta) = \ln \eta$, $(\eta \in \left[b_1 + h\frac{b_2-b_1}{2}, \frac{b_1+b_2}{2}\right] \subset (0,\infty))$. Then

$$\frac{1}{b_2-b_1}\int_{b_1}^{b_2} \phi(\xi)d\xi = \ln I,$$

$$\frac{\phi(b_2)-\phi(b_1)}{b_2-b_1} = \frac{1}{L},$$

$$\frac{\phi(b_1)+\phi(b_2)}{2} = \ln G,$$

$$\frac{1}{b_2-b_1}\int_{b_1}^{b_2} |\phi'(\xi)|^2 d\xi = \frac{1}{G^2},$$

$$\frac{\phi(b_1+b_2-\eta)-\phi(\eta)}{2} = \ln\left(\frac{2A-\eta}{\eta}\right)^{\frac{1}{2}}$$

and

$$\frac{1}{b_2-b_1}\int_{b_1}^{b_2} |\phi'(\xi)|^2 d\xi - \left(\frac{\phi(b_2)-\phi(b_1)}{b_2-b_1}\right)^2 = \frac{L^2-G^2}{L^2G^2}.$$

Therefore, (2.1) becomes,

$$\left|\ln\frac{\eta^{(1-h)}G^h}{I} + h\frac{(\eta-A)}{L} + \ln\left(\frac{2A-\eta}{\eta}\right)^{\frac{1}{2}}\right|$$
$$\leq \left[\frac{(b_2-b_1)^2}{12}(3h^2-3h+1) + h(1-h)(\eta-A)^2 + (1-2h)(\eta-b_1)(\eta-A)\right]^{\frac{1}{2}} \times \frac{[L^2-G^2]^{\frac{1}{2}}}{LG}. \quad (3.15)$$

Choose $\eta = A$ in (3.15), we obtain,

$$\left|\ln \frac{A^{(1-h)}G^h}{I}\right| \leq \frac{(b_2-b_1)}{2\sqrt{3}}(3h^2-3h+1)^{\frac{1}{2}}\frac{[L^2-G^2]^{\frac{1}{2}}}{LG}.$$

At $h = 1$, (3.15) becomes,

$$\left|\ln \frac{G}{I}\right| \leq \frac{(b_2-b_1)(L^2-G^2)^{\frac{1}{2}}}{2\sqrt{3}LG}.$$

By taking $\eta = I$ in (3.15), we obtain,

$$\left|\ln \frac{G^h}{I^h} + h\frac{(I-A)}{L} + \ln\left(\frac{2A-I}{I}\right)^{\frac{1}{2}}\right|$$
$$\leq \left[\frac{(b_2-b_1)^2}{12}(3h^2-3h+1) + h(1-h)(I-A)^2 \right.$$
$$\left. +(1-2h)(I-b_1)(I-A)\right]^{\frac{1}{2}} \times \frac{[L^2-G^2]^{\frac{1}{2}}}{LG}.$$

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References

[1] Anastassiou, G.A., *Multivariate Ostrowski type inequalities*, Acta Math. Hungar., **76**(1997), 267-278.

[2] Barnett, N.S., Dragomir, S.S., Sofo, A., *Better bounds for an inequality of Ostrowski type with applications*, Demonstratio Math., **34**(2001), no. 3, 533-542. Preprint RGMIA Res. Rep. Coll., **3**(2000), no. 1, Article 11.

[3] Cebyšev, P.L., *Sur les expressions approximatives des intégrales définies par les autres prises entre les mêmes limites*, Proc. Math. Soc. Charkov, **2**(1882), 93-98.

[4] Cerone, P., Dragomir, S.S., Roumeliotis, J., *A new generalization of Ostrowski integral inequality for mappings whose derivatives are bounded and applications in numerical integration and for special means*, Appl. Math. Lett., **13**(2000), no. 1, 19-25.

[5] Cheng, X.L., *Improvement of some Ostrowski-Grüss type inequalities*, Comput. Math. Appl., **42**(2001), no. 1/2, 109-114.

[6] Dragomir, S.S., Sofo, A., *An integral inequality for twice differentiable mappings and applications*, Tamkang J. Math., **31**(2000), no. 4, 257-266. Preprint RGMIA Res. Rep. Coll., **2**(1999), no. 2.

[7] Dragomir, S.S., Wang, S., *An inequality of Ostrowski-Grüss type and its applications to the estimation of error bounds for some special means and for some numerical quadrature rules*, Comput. Math. Appl., **33**(1997), no. 11, 16-20.

[8] Ghazi, W., Qayyum, A., *A note on new Ostrowski type inequalities using a generalized kernel*, Bulletin of Mathematical Analysis and Application, **9**(2017), 74-91.

[9] Grüss, G., *Über das Maximum des absoluten Betrages von*, Math. Z., **39**(1935), no. 1, 215-226.

[10] Irshad, N., Khan, A.R., Shaikh, M.A., Generalized Ostrowski type inequality with applications in numerical integration, probability theory and special means, In Dr. Manuel Alberto M. Ferreira (Eds.), *Current Topics in Mathematics and Computer Science*, Vol. 9 (74-91). B P International, India-United Kingdom, 2021.

[11] Matić, M., Pečarić, J.E., Ujević, N., *Improvement and further generalization of some inequalities of Ostrowski-Grüss type*, Comput. Math. Appl., **39**(2000), no. 3/4, 161-175.

[12] Milovanović, G.V., Pečarić, J.E., *On generalization of the inequality of A. Ostrowski and some related applications*, Univ. Beograd Publ., Elektrotehn. Fak. Ser. Mat. Fiz. No. 544-576, (1976), 155-158.

[13] Mitrinović, D.S., Pečarić, J.E., Fink, A.M., *Inequalities for Functions and their Integrals and Derivatives*, Kluwer Academic Publishers, 1991.

[14] Mitrinović, D.S., Pečarić, J.E., Fink, A.M., *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht, 1991.

[15] Ostrowski, A., *Über die Absolutabweichung einer differentienbaren Funktionen von ihren Integralmittelwert*, Comment. Math. Helv., **10**(1938), 226-227.

[16] Ujevic, N., *New bounds for the first inequality of Ostrowski-Gruss type and applications*, Computers and Mathematics with Applications, **46**(2003), 421-427.

[17] Zafar, F., Mir, N.A., *A generalization of Ostrowski-Grüss type inequality for first differentiable mappings*, Tamsui Oxford J. Math. Sci., **26**(2010), no. 1, 61-76.

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Differential subordination for Janowski functions with positive real part

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Abstract. Theory of differential subordination provides techniques to reduce differential subordination problems into verifying some simple algebraic condition called admissibility condition. We exploit the first order differential subordination theory to get several sufficient conditions for function satisfying several differential subordinations to be a Janowski function with positive real part. As applications, we obtain sufficient conditions for normalized analytic functions to be Janowski starlike functions.

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1. Motivation

The class of all analytic functions defined on the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ that fixes the origin and has derivative 1 at the origin is denoted by $\mathcal{A}$. An analytic function p is *subordinate* to the analytic function q, written $p \prec q$, if $p = q \circ \omega$ for some analytic function $\omega : \mathbb{D} \to \mathbb{D}$ with $\omega(0) = 0$. If the function q is univalent in $\mathbb{D}$, then $p \prec q$ if and only if $p(0) = q(0)$ and $p(\mathbb{D}) \subseteq q(\mathbb{D})$. The class $\mathcal{P}$ consists of Carathéodory functions $p : \mathbb{D} \to \mathbb{C}$ of the form $p(z) = 1 + c_1 z + c_2 z + \cdots$ that maps the unit disk $\mathbb{D}$ into a region on the right half plane. For arbitrary fixed numbers A and B satisfying $-1 \leq B < A \leq 1$, denote by $\mathcal{P}[A, B]$ the class of analytic functions $p \in \mathcal{P}$ satisfy the subordination $p(z) \prec (1 + Az)/(1 + Bz)$. We call the functions in $\mathcal{P}[A, B]$ as *Janowski functions with positive real part.* The class $\mathcal{S}^*[A,\, B]$ consists of functions $f \in \mathcal{A}$ such that $zf'(z)/f(z) \in \mathcal{P}[A,\, B]$ for $z \in \mathbb{D}$. The functions in the class $\mathcal{S}^*[A,\, B]$ are called the *Janowski starlike functions*, introduced by Janowski [12]. In particular, $S^*[1 - 2\alpha, -1] = \mathcal{S}^*(\alpha)$ is the class of starlike functions of order α, see [11, 23].

Nunokawa [21] proved that if $1 + zp'(z) \in \mathcal{P}[1, 0]$, then $p \in \mathcal{P}[1, 0]$. In 2007, Ali *et al.* [3] determined the conditions on β and numbers $A, B, D, E \in [-1, 1]$ so that $p \in \mathcal{P}[A,B]$ whenever $1+\beta zp'(z)$ or $1+\beta zp'(z)/p(z)$ or $1+\beta zp'(z)/p^2(z)$ is in the class $\mathcal{P}[D,E]$. In 2018, authors [17] obtained the sharp lower bound on β so that the function $p(z)$ is subordinate to the functions e^z and $(1+Az)/(1+Bz)$ whenever $1+\beta zp'(z)/p^j(z)$, $(j=0,1,2)$ is subordinate to the functions with positive real part like $\sqrt{1+z}$, $(1+Az)/(1+Bz)$. Recently, Ahuja *et al.* [1] computed sharp estimates for β so that a Carathéodory function is subordinate to a starlike function with positive real part whenever $1+\beta zp'(z)/p^j(z)$, $(j=0,1,2)$ is subordinate to lemniscate starlike function. For more details, see [6, 8, 21, 25, 26]. Motivated by work done in [1, 3, 5, 7, 9, 10, 17], by using admissibility condition technique, a condition on β is established so that $p \in \mathcal{P}[A,B]$ when $1+\beta p'(z)/p^k(z)$ with $k \in \mathbb{N}\cup\{0\}$, $p(z)+\beta zp'(z)/p^2(z)$, $1+\beta(zp'(z))^2/p^k(z)$ and $1/p(z)-\beta zp'(z)/p^k(z)$ for $k \in \mathbb{N}\cup\{0\}$ are in the class $\mathcal{P}[D,E]$. We compute a condition on α and β for $p \in \mathcal{P}[A,B]$ whenever $(1-\alpha)p(z)+\alpha p^2(z)+\beta zp'(z)/p^k(z) \in \mathcal{P}[D,E]$ for $k=0,1$ as well. Additionally, a condition on β and γ is determined in a Briot-Bouquet differential type subordination relation: $p(z)+zp'(z)/(\beta p(z)+\gamma)^2 \in \mathcal{P}[D,E]$ implies $p \in \mathcal{P}[A,B]$. As an application, we obtained some sufficient conditions for a normalized analytic function f in $\mathcal{S}^*[A,B]$. Kanas [14] described the admissibility condition for the function to map $\mathbb{D}$ on to region bounded by parabola and hyperbola. We prove our result by using the corresponding admissibility conditions for the Janowski functions with positive real part.

2. Janowski functions

Let $\psi(r,s,t;z)\colon \mathbb{C}^3\times\mathbb{D}\to\mathbb{C}$ be a function and let h be univalent in $\mathbb{D}$. An analytic function p satisfying the second-order differential subordination

$$\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z), \tag{2.1}$$

is known as its *solution.* The univalent function q is a dominant of the solutions of the differential subordination (2.1) if $p \prec q$ for all p satisfying (2.1). A dominant $\tilde{q}$ which satisfies $\tilde{q} \prec q$ for all dominant q of (2.1) is known as *best dominant* of (2.1) and it is unique up to a rotation. Let $\mathcal{Q}$ be the class consisting of all analytic and injective functions q on $\overline{\mathbb{D}}\setminus\mathbf{E}(q)$, where $\mathbf{E}(q)=\{\xi\in\partial\mathbb{D} : \lim_{z\to\xi} q(z)=\infty\}$ such that $q'(\xi)\neq 0$ for $\xi\in\overline{\mathbb{D}}\setminus\mathbf{E}(q)$. Let Ω be a set in $\mathbb{C}$, $q\in\mathcal{Q}$ and n be a positive integer. The class $\Psi_n[\Omega,q]$ of admissible functions $\psi:\mathbb{C}^3\times\mathbb{D}\to\mathbb{C}$ that satisfy the admissibility condition:

$$\psi(r,s,t;z)\notin\Omega \tag{2.2}$$

whenever

$$r=q(\xi),\ s=m\ \xi\ q'(\xi) \text{ and } \operatorname{Re}\left(\frac{t}{s}+1\right) \ \geq\ m\operatorname{Re}\left(\frac{\xi q''(\xi)}{q'(\xi)}+1\right)$$

for $z\in\mathbb{D}, \xi\in\overline{\mathbb{D}}\setminus\mathbf{E}(q)$ and $m\geq n\geq 1$. In particular, let $\Psi_1[\Omega,q]=\Psi[\Omega,q]$. For more details, see [4, 13, 15, 16, 19, 24]. For this class $\Psi_n[\Omega,q]$, the following result is well-known.

Theorem 2.1. [20, Theorem 2.3b, p. 28] *Let the function* $\psi \in \Psi_n[\Omega, q]$ *with* $q(0) = a$. *If the function* $p \in \mathcal{H}[a, n]$ *satisfies*

$$\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega, \tag{2.3}$$

then $p(z) \prec q(z)$.

We begin by describing the class of admissible function $\Psi_n[\Omega,\, q]$ when $q : \mathbb{D} \to \mathbb{C}$ is the function given by $q(z) = (1 + Az)/(1 + Bz)$ where $-1 \le B < A \le 1$. Note that $q(0) = 1$ and $E(q) \subset \{-1\}$. Clearly, the function q is univalent in $\overline{\mathbb{D}} \setminus \mathbf{E}(q)$. Therefore $q \in \mathcal{Q}$ and the domain $q(\mathbb{D})$ is

$$\Delta\; = q(\mathbb{D}) = \left\{ w \in \mathbb{C} : \left| \frac{w-1}{A-Bw} \right| < 1 \right\}.$$

For $\varsigma = e^{i\theta}$ and $0 < \theta < 2\pi$, we have

$$q(\varsigma) = \frac{1 + Ae^{i\theta}}{1 + Be^{i\theta}},\; q'(\varsigma) = \frac{A - B}{(1 + Be^{i\theta})^2} \text{ and } q''(\varsigma) = \frac{-2B(A - B)}{(1 + Be^{i\theta})^3}.$$

and a simple calculation yields

$$\operatorname{Re}\left(\frac{\varsigma q''(\varsigma)}{q'(\varsigma)} + 1 \right) = \frac{1 - B^2}{1 + B^2 + 2B\cos\theta}.$$

Thus we get the following **condition of admissibility**: $\psi(r, s, t; z) \notin \Omega$ whenever $(r, s, t; z) \in \operatorname{dom}\psi$ and

$$r = \frac{1 + Ae^{i\theta}}{1 + Be^{i\theta}},\; s = \frac{m(A - B)e^{i\theta}}{(1 + Be^{i\theta})^2} \text{ and } \operatorname{Re}\left(\frac{t}{s} + 1 \right) \ge \frac{(1 - B^2)m}{1 + B^2 + 2B\cos\theta} \tag{2.4}$$

where $0 < \theta < 2\pi$ and $m \ge n \ge 1$ and the class of all such functions ψ satisfying the admissibility condition is denoted by $\Psi(\Omega; A, B)$.

When $q(z) = (1 + Az)/(1 + Bz)$, Theorem 2.1 specializes to the following first order differential subordination result:

Theorem 2.2. *Let* $p \in \mathcal{H}[1,\, n]$ *with* $n \in \mathbb{N}$. *Let* Ω *be a subset of* $\mathbb{C}$ *and* $\psi : \mathbb{C}^2 \times \mathbb{D} \to \mathbb{C}$ *with domain* D *satisfy* $\psi(r, s; z) \notin \Omega$ *for all* $z \in \mathbb{D}$, *where* r *and* s *are given by* (2.4). *If* $(p(z), zp'(z); z) \in D$ *and* $\psi(p(z), zp'(z); z) \in \Omega$ *for* $z \in \mathbb{D}$, *then* $p \in \mathcal{P}[A, B]$.

We investigate functions that naturally arise in the investigation of univalent functions to be admissible. In the first result, we show that $\psi(r, s; z) = 1 + \beta s/r^k$ is an admissible function.

Theorem 2.3. *Let* $\beta \ne 0$, $-1 \le B < A \le 1$ *and* $-1 \le E < D \le 1$ *satisfy the condition*

(i) $|\beta|(A - B) \ge (D - E)(1 + |A|)^k(1 + |B|)^{2-k} + |E\beta(A - B)|$; $(k = 0, 1, 2)$ *or*

(ii) $|\beta|(A - B)(1 - |B|)^{k-2} \ge (D - E)(1 + |A|)^k + |E\beta(A - B)|(1 + |B|)^{k-2}$; $(k > 2)$.

If p *is analytic in* $\mathbb{D}$ *and*

$$1 + \beta \frac{zp'(z)}{p^k(z)} \in \mathcal{P}[D, E]; \quad k \in \mathbb{N} \cup \{0\},$$

then $p \in \mathcal{P}[A, B]$.

Proof. Let $\Omega = \{w \in \mathbb{C} : |(w-1)/(D-Ew)| < 1\}$.
The function $\psi : (\mathbb{C} \setminus \{0\}) \times \mathbb{C} \times \mathbb{D} \to \mathbb{C}$ is defined as

$$\psi(r, s; z) = 1 + \beta \frac{s}{r^k}$$

where k is a non-negative integer. Using the values of r, s from (2.4), we have

$$\psi(r, s; z) = 1 + \beta \frac{m(A-B)e^{i\theta}(1+Be^{i\theta})^k}{(1+Ae^{i\theta})^k(1+Be^{i\theta})^2}.$$

By making use of Theorem 2.2, the desired subordination is showed if we prove $\psi \in \Psi[\Omega; A, B]$. For this purpose, set

$$\boldsymbol{\chi}(w,\, D,\, E) = \left| \frac{w-1}{D-Ew} \right|.$$

(i) When $k = 0, 1, 2$. A simple calculation gives

$$\begin{aligned}
|\boldsymbol{\chi}(\psi(r, s; z),\, D,\, E)| &= \left| \frac{\beta m(A-B)e^{i\theta}}{(D-E)(1+Ae^{i\theta})^k(1+Be^{i\theta})^{2-k} - E\beta m e^{i\theta}(A-B)} \right| \\
&\geq \frac{|\beta| m(A-B)}{(D-E)|(1+Ae^{i\theta})^k||(1+Be^{i\theta})^{2-k}| + |E\beta m(A-B)|} \\
&\geq \frac{|\beta| m(A-B)}{(D-E)(1+|A|)^k(1+|B|)^{2-k} + m|E\beta(A-B)|} \\
&=: \phi(m).
\end{aligned}$$

Observe that the function $\phi(m)$ is an increasing function for $m \geq 1$ by first derivative test. Hence the minimum value of $\phi(m)$ occurs at $m = 1$. Thus, the last inequality becomes

$$|\boldsymbol{\chi}(\psi(r, s; z),\, D,\, E)| \geq \phi(1) \geq 1$$

if the inequality

$$|\beta|(A-B) \geq (D-E)(1+|A|)^k(1+|B|)^{2-k} + |E\beta(A-B)|$$

holds. Therefore, $\psi(r, s; z) \notin \Omega$ which implies $\psi \in \Psi(\Omega; A, B)$ and we get the desired $p \prec q$.

(ii) When $k > 2$, we note that

$$\begin{aligned}
|\boldsymbol{\chi}(\psi(r, s; z),\, D,\, E)| &= \left| \frac{\beta m e^{i\theta}(A-B)(1+Be^{i\theta})^{k-2}}{(D-E)(1+Ae^{i\theta})^k - E\beta m e^{i\theta}(A-B)(1+Be^{i\theta})^{k-2}} \right| \\
&\geq \frac{|\beta| m(A-B)(1-|B|)^{k-2}}{(D-E)(1+|A|)^k + m|E\beta(A-B)|(1+|B|)^{k-2}} \\
&=: \phi(m).
\end{aligned}$$

As previous case, note that $\phi(m) \geq \phi(1)$. Hence the last inequality is written as

$$|\boldsymbol{\chi}(\psi(r, s; z),\, D,\, E)| \geq 1$$

provided

$$|\beta|(A-B)(1-|B|)^{k-2} \geq (D-E)(1+|A|)^k + |E\beta(A-B)|(1+|B|)^{k-2}.$$

Therefore, we get $p \prec q$. ∎

Remark 2.4. When $k = 0$ and 1, Theorem 2.3 reduces to [3, Lemma 2.1, p. 2] and [3, Lemma 2.10, p. 6] respectively. When $k = 2$ and $\beta = 1$, Theorem 2.3 simplifies to [3, Lemma 2.6, p. 5].

For a positive integer k, next theorem gives a conditions on β so that the differential subordination

$$1 + \beta(zp'(z))^2/p^k(z) \in \mathcal{P}[D, E]$$

implies $p \in \mathcal{P}[A, B]$.

Theorem 2.5. *Suppose k is a non-negative integer, $\beta \neq 0$, $-1 \leq B < A \leq 1$ and $-1 \leq E < 0 < D \leq 1$ satisfy either*

(i) for $0 \leq k < 4$,

$$|\beta|(A - B)^2 \geq (D - E)(1 + |A|)^k(1 + |B|)^{4-k} + |E\beta(A - B)^2|, \tag{2.5}$$

or

(ii) for $k \geq 4$,

$$|\beta|(A - B)^2(1 - |B|)^{k-4} \geq (D - E)(1 + |A|)^k + |E\beta(A - B)^2|(1 + |B|)^{k-4}. \tag{2.6}$$

If p is analytic in $\mathbb{D}$ and $1 + \beta(zp'(z))^2/p^k(z) \in \mathcal{P}[D, E]$, then $p \in \mathcal{P}[A, B]$.

Proof. By considering the domain Ω as in Theorem 2.3 and the analytic function

$$\psi(r, s; z) = 1 + \beta s^2/r^k$$

where k is non-negative integer, we need to show $\psi \in \Psi[\Omega, A, B]$.

(i) Let $0 \leq k < 4$. In view of (2.4), we note that

$$\psi(r, s; z) = 1 + \beta\frac{m^2(A - B)^2e^{2i\theta}(1 + Be^{i\theta})^k}{(1 + Be^{i\theta})^4(1 + Ae^{i\theta})^k}$$

so that

$$\begin{aligned}|\boldsymbol{\chi}(\psi(r, s; z), D, E)| &= \left|\frac{\beta m^2(A - B)^2e^{2i\theta}}{(D - E)(1 + Ae^{i\theta})^k(1 + Be^{i\theta})^{4-k} - E\beta m^2(A - B)^2e^{2i\theta}}\right| \\ &= \frac{|\beta|m^2(A - B)^2}{|(D - E)(1 + Ae^{i\theta})^k(1 + Be^{i\theta})^{4-k} - E\beta m^2(A - B)^2e^{2i\theta}|} \\ &\geq \frac{|\beta|m^2(A - B)^2}{(D - E)(1 + |A|)^k(1 + |B|)^{4-k} + m^2|E\beta(A - B)^2|} \\ &=: \phi(m).\end{aligned}$$

A calculation shows that $\phi'(m) > 0$ for $m \geq 1$. Therefore $\phi(m) \geq \phi(1)$. The last inequality simplifies to

$$|\boldsymbol{\chi}(\psi(r, s; z), D, E)| \geq 1$$

whenever the inequality (2.5) holds. As a conclusion it is noted that $\psi(r, s; z) \notin \Omega$. Thus we get the required subordination.

(ii) Let $k \geq 4$. Proceeding as in (i), we have

$$\psi(r,s,t;z) = 1 + \frac{\beta m^2 (A-B)^2 e^{2i\theta}(1+Be^{i\theta})^{k-4}}{(1+Ae^{i\theta})^k}.$$

so that

$$\begin{aligned}
|\boldsymbol{\chi}(\psi(r,s;z),\, D,\, E)| &= \left| \frac{\beta m^2 (A-B)^2 e^{2i\theta}(1+Be^{i\theta})^{k-4}}{(D-E)(1+Ae^{i\theta})^k - E\beta m^2 (A-B)^2 (1+Be^{i\theta})^{k-4} e^{2i\theta}} \right| \\
&\geq \frac{|\beta| m^2 (A-B)^2 |(1+Be^{i\theta})^{k-4}|}{|(1+Ae^{i\theta})^k (D-E)| + m^2 |E\beta(A-B)^2 e^{2i\theta}|\, |(1+Be^{i\theta})^{k-4}|} \\
&\geq \frac{|\beta| m^2 (A-B)^2 (1-|B|)^{k-4}}{(1+|A|)^k (D-E) + m^2 |E\beta(A-B)^2| (1+|B|)^{k-4}} \\
&=: \phi(m).
\end{aligned}$$

A calculation shows that $\phi(m)$ is an increasing function for $m \geq 1$ and thus has minimum value at $m = 1$. As similar analysis of previous case, we get $p \in \mathcal{P}[A,B]$. ☐

In [18], a lower bound on β is determined such that

$$p(z) + \beta z p'(z)/p^2(z) \prec \sqrt{1+z}$$

implies

$$p(z) \prec \sqrt{1+z}.$$

Recently, Sharma and Ravichandran [22] established similar type subordination for analytic functions associated to Cardioid. Motivated by this work, the condition on β is computed so that $p(z) + \beta z p'(z)/p^2(z) \in \mathcal{P}[D,E]$ implies $p \in \mathcal{P}[A,B]$.

Theorem 2.6. *Suppose* $-1 \leq B < A \leq 1$ *and* $-1 \leq E < D \leq 1$ *satisfy*

$$\begin{aligned}
(A-B)(|\beta|(1-|B|) - (1+|A|)^2) \geq\ & (1+|A|)^2((D-E) + |DB - EA|) \\
& + |E\beta(A-B)|(1+|B|).
\end{aligned} \tag{2.7}$$

If p *is analytic in* $\mathbb{D}$ *and* $p(z) + \beta z p'(z)/p^2(z) \in \mathcal{P}[D,E]$*, then* $p \in \mathcal{P}[A,B]$.

Proof. Consider the domain Ω as in Theorem 2.3.
The analytic function $\psi : \mathbb{C} \setminus \{0\} \times \mathbb{C} \times \mathbb{D} \to \mathbb{D}$ is defined as

$$\psi(r,s;z) = r + \beta s / r^2.$$

For required subordination, we need to show $\psi(r,s,t,z) \notin \Omega$. For the values of r, s in (2.4), we have

$$\psi(r,s;z) = \frac{(1+Ae^{i\theta})^3 + \beta m e^{i\theta}(A-B)(1+Be^{i\theta})}{(1+Ae^{i\theta})^2 (1+Be^{i\theta})}$$

so that

$$\begin{aligned}
|\boldsymbol{\chi}(\psi(r,s;z),\, D,\, E)| &= \frac{(A-B)|\beta m(1+Be^{i\theta})+(1+Ae^{i\theta})^2|}{\begin{array}{r}|(1+Ae^{i\theta})^2(D(1+Be^{i\theta})-E(1+Ae^{i\theta}))\\ -E\beta m e^{i\theta}(A-B)(1+Be^{i\theta})|\end{array}}\\
&= \frac{(A-B)|\beta m(1+Be^{i\theta})+(1+Ae^{i\theta})^2|}{\begin{array}{r}|(1+Ae^{i\theta})^2((D-E)+(DB-EA)e^{i\theta})\\ -E\beta m e^{i\theta}(A-B)(1+Be^{i\theta})|\end{array}}\\
&\geq \frac{(A-B)|\beta m(1+Be^{i\theta})|-|(1+Ae^{i\theta})^2|}{\begin{array}{r}|(1+Ae^{i\theta})^2((D-E)+(DB-EA)e^{i\theta})|\\ +|E\beta m e^{i\theta}(A-B)(1+Be^{i\theta})|\end{array}}\\
&\geq \frac{(A-B)(|\beta|m(1-|B|)-(1+|A|)^2)}{\begin{array}{r}(1+|A|)^2((D-E)+|DB-EA|)\\ +m|E\beta(A-B)|(1+|B|)\end{array}}\\
&=: \phi(m).
\end{aligned}$$

The function $\phi(m)$ is an increasing for $m \geq 1$. So the function $\phi(m)$ attains its minimum value at $m=1$. Then $|\boldsymbol{\chi}(\psi(r,s;z),\, D,\, E)| \geq \phi(1) \geq 1$ provided the inequality (2.7) holds. By Theorem 2.2, we have $\psi \in \Psi(\Omega; A, B)$ and this proves the result. □

In [25], authors derived condition on α and β so that subordination

$$(1-\alpha)p(z)+\alpha p^2(z)+\beta z p'(z)/p^k(z) \prec 1+\frac{4}{3}z+\frac{2}{3}z^2 \quad (k=0,\,1)$$

implies

$$p(z) \prec 1+\frac{4}{3}z+\frac{2}{3}z^2.$$

In view of this work, next two theorems give a relation between α and β so that

$$(1-\alpha)p(z)+\alpha p^2(z)+\beta z p'(z)/p^k(z) \in \mathcal{P}[D,E]$$

(where $k=0,1$) implies $p \in \mathcal{P}[A,B]$.

Theorem 2.7. *Let $-1 \leq B < A \leq 1$, $-1 \leq E < 0 < D \leq 1$, $\beta \neq 0$ and $0 \leq \alpha \leq 1$. Assume that*

$$\begin{aligned}
(A-B)(|\beta|-(1+|B|)-\alpha(1+|A|)) \geq (1+|B|)(D(1+|B|)&\\
-E(1-\alpha)(1+|A|))-E\alpha(1+|A|)^2&\\
+|E\beta(A-B)|.&
\end{aligned} \tag{2.8}$$

If p is analytic in $\mathbb{D}$ and $(1-\alpha)p(z)+\alpha p^2(z)+\beta z p'(z) \in \mathcal{P}[D,E]$, then $p \in \mathcal{P}[A,B]$.

Proof. Consider the domain Ω as in Theorem 2.3. The analytic function $\psi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{D}$ is defined as

$$\psi(r,s;z) = (1-\alpha)r+\alpha r^2+\beta s.$$

To show $\psi \in \Psi[\Omega,\, A,\, B]$, it suffices to prove $|\boldsymbol{\chi}(\psi(r,s;z),\, D,\, E)| \geq 1$. It is easy to deduce that

$$\psi(r,s;z) = \frac{(1-\alpha)(1+Ae^{i\theta})(1+Be^{i\theta})+\alpha(1+Ae^{i\theta})^2+\beta m(A-B)e^{i\theta}}{(1+Be^{i\theta})^2}$$

such that

$$\begin{aligned}
|\boldsymbol{\chi}(\psi(r,s;z),\, D,\, E)| &= \frac{(A-B)|\beta m+(1+Be^{i\theta})+\alpha(1+Ae^{i\theta})|}{\begin{array}{r}|(1+Be^{i\theta})(D(1+Be^{i\theta})-E(1-\alpha)(1+Ae^{i\theta}))\\ -E\alpha(1+Ae^{i\theta})^2-E\beta m(A-B)e^{i\theta}|\end{array}}\\
&\geq \frac{(A-B)(|\beta|m-(1+|B|)-\alpha(1+|A|))}{\begin{array}{r}|(1+Be^{i\theta})||(D(1+Be^{i\theta})-E(1-\alpha)(1+Ae^{i\theta}))|\\ +|E\alpha(1+Ae^{i\theta})^2|+|E\beta m(A-B)e^{i\theta}|\end{array}}\\
&\geq \frac{(A-B)(|\beta|m-(1+|B|)-\alpha(1+|A|)}{\begin{array}{r}(1+|B|)(D(1+|B|)-E(1-\alpha)(1+|A|))\\ -E\alpha(1+|A|)^2+m|E\beta(A-B)|\end{array}}\\
&=: \phi(m).
\end{aligned}$$

Note that $\phi(m)\geq\phi(1)$ for $m\geq 1$ and therefore $|\boldsymbol{\chi}(\psi(r,s;z),\, D,\, E)|\geq 1$ whenever the inequality (2.8) holds. Thus $\psi(r,s;z)\notin\Omega$ and Theorem 2.2 yields the desired subordination. □

As an implication of Theorems 2.5-2.7, each of following is sufficient condition for function $f\in\mathcal{S}^*[A,B]$:

(a) $\dfrac{zf'(z)}{f(z)}\left(1+\beta\left(\dfrac{zf'(z)}{f(z)}\right)^{-1}\left(1+\dfrac{zf''(z)}{f'(z)}-\dfrac{zf'(z)}{f(z)}\right)^2\right)\in\mathcal{P}[D,E]$

where $-1\leq B<A\leq 1$, $-1\leq E<0<D\leq 1$ and β satisfies following inequality

$$|\beta|(A-B)^2\geq (D-E)(1+|A|)^2(1+|B|)^2+|E\beta(A-B)^2|,$$

(b) $\dfrac{zf'(z)}{f(z)}\left(1+\beta\left(\dfrac{zf'(z)}{f(z)}\right)^{-2}\left(1+\dfrac{zf''(z)}{f'(z)}-\dfrac{zf'(z)}{f(z)}\right)\right)\in\mathcal{P}[D,E]$,

where $-1\leq B<A\leq 1$, $-1\leq E<D\leq 1$ and β satisfies an inequality (2.7).

(c) $(1-\alpha+\beta)\frac{zf'(z)}{f(z)}+(\alpha-\beta)\left(\frac{zf'(z)}{f(z)}\right)^2+\beta\frac{z^2f''(z)}{f'(z)}\in\mathcal{P}[D,E]$

whenever $\beta\neq 0$, $-1\leq B<A\leq 1$, $-1\leq E<0<D\leq 1$ and the inequality (2.8) holds.

Corollary 2.8. *Let $p\in\mathcal{P}$. For $-1\leq B<A\leq 1$, $-1\leq E<0<D\leq 1$ and $\beta\neq 0$. We assume that*

$$\begin{aligned}(A-B)(|\beta|-(1+|B|))&\geq (D-E)+|2BD-E(A+B)|\\ &\quad+|E\beta(A-B)|+|DB^2-EAB|.\end{aligned}\tag{2.9}$$

If $p(z)+\beta zp'(z)\in\mathcal{P}[D,E]$, then $p\in\mathcal{P}[A,B]$.

Corollary 2.9. *Let $p\in\mathcal{P}$, $\beta\neq 0$, $-1\leq B<A\leq 1$ and $-1\leq E<0<D\leq 1$. We assume the following inequality*

$$\begin{aligned}(A-B)(|\beta|-(1+|B|)-(1+|A|))\geq D(1+|B|)^2-E(1+|A|)^2\\ +|E\beta(A-B)|.\end{aligned}$$

If $p^2(z)+\beta zp'(z)\in\mathcal{P}[D,E]$, then $p\in\mathcal{P}[A,B]$.

Theorem 2.10. *Let $p \in \mathcal{P}$, $-1 \le B < A \le 1$, $-1 \le E < 0 < D \le 1$, $G = 1 + |A|$ and $0 \le \alpha \le 1$. Assume that*

$$\begin{aligned}&(A-B)(|\beta| - G - \alpha(G)^2(1-|B|)^{-1})\\ &\quad \ge (G)(D(1+|B|) - E(1-\alpha)(G)) - E\alpha(G)^3(1-|B|)^{-1} + |E\beta(A-B)|\end{aligned} \tag{2.10}$$

If $(1-\alpha)p(z) + \alpha p^2(z) + \beta z p'(z)/p(z) \in \mathcal{P}[D,E]$, then $p \in \mathcal{P}[A,B]$.

Proof. By considering Ω be as in Theorem 2.3 and the analytic function

$$\psi(r,s,t;z) = (1-\alpha)r + \alpha r^2 + \beta s/r,$$

it is enough to prove $|\boldsymbol{\chi}(\psi(r,s;z)| \ge 1$. Using (2.4), we have

$$\psi(r,s;z) = \frac{(1-\alpha)(1+Ae^{i\theta})^2(1+Be^{i\theta}) + \alpha(1+Ae^{i\theta})^3 + \beta m(A-B)(1+Be^{i\theta})e^{i\theta}}{(1+Ae^{i\theta})(1+Be^{i\theta})^2}.$$

A simple computation yields

$$\begin{aligned}|\boldsymbol{\chi}(\psi(r,s;z)| &= \left| \frac{\begin{array}{c}(1+Ae^{i\theta})^2(1+Be^{i\theta}) + \alpha(1+Ae^{i\theta})^2(A-B)e^{i\theta} + \\ \beta m(A-B)(1+Be^{i\theta})e^{i\theta} - (1+Ae^{i\theta})(1+Be^{i\theta})^2\end{array}}{\begin{array}{c}D(1+Ae^{i\theta})(1+Be^{i\theta})^2 - E((1-\alpha)(1+Ae^{i\theta})^2 \\ (1+Be^{i\theta}) + \alpha(1+Ae^{i\theta})^3 + \beta m(A-B)(1+Be^{i\theta})e^{i\theta})\end{array}} \right| \\
&= \frac{(A-B)|\beta m + (1+Ae^{i\theta}) + \alpha(1+Ae^{i\theta})^2(1+Be^{i\theta})^{-1}|}{\begin{array}{c}|(1+Ae^{i\theta})(D(1+Be^{i\theta}) - E(1-\alpha)(1+Ae^{i\theta})) - \\ E\alpha(1+Ae^{i\theta})^3(1+Be^{i\theta})^{-1} - E\beta m e^{i\theta}(A-B)|\end{array}} \\
&\ge \frac{(A-B)(|\beta|m - (1+|A|) - \alpha(1+|A|)^2(1-|B|)^{-1})}{\begin{array}{c}|(1+Ae^{i\theta})(D(1+Be^{i\theta}) - E(1-\alpha)(1+Ae^{i\theta}))| + \\ |E\alpha(1+Ae^{i\theta})^3(1+Be^{i\theta})^{-1}| + |E\beta e^{i\theta} m(A-B)|\end{array}} \\
&\ge \frac{(A-B)(|\beta|m - (1+|A|) - \alpha(1+|A|)^2(1-|B|)^{-1})}{\begin{array}{c}(1+|A|)(D(1+|B|) - E(1-\alpha)(1+|A|)) - \\ E\alpha(1+|A|)^3(1-|B|)^{-1} + m|E\beta(A-B)|\end{array}} =: \phi(m).\end{aligned}$$

It is observed that $\phi'(m) > 0$ for all $m \ge 1$. As computation done in the previous theorem, we get the required subordination result. ☐

Corollary 2.11. *Let $p \in \mathcal{P}$, $\beta \ne 0$, $-1 \le B < A \le 1$ and $-1 \le E < 0 < D \le 1$. Suppose that*

$$\begin{aligned}(A-B)(|\beta| - (1+|A|)) \ge (D-E) + |D(B+A) - 2EA| + |E\beta(A-B)|\\ + |DBA - EA^2|.\end{aligned} \tag{2.11}$$

If $p(z) + \beta z p'(z)/p(z) \in \mathcal{P}[D,E]$, then $p \in \mathcal{P}[A,B]$.

For a positive integer k, the condition on β is determined so that

$$(1/p(z)) - \beta z p'(z)/p^k(z) \in \mathcal{P}[D,E]$$

implies $p \in \mathcal{P}[A,B]$.

Theorem 2.12. *Let $p \in \mathcal{P}$, $-1 \le B < A \le 1$ and $-1 \le E < 0 < D \le 1$. Then $p \in \mathcal{P}[A,B]$ for each of the following subordination conditions:*

(a) $(1/p(z)) - \beta z p'(z) \in \mathcal{P}[D,E]$ *where* β *satisfies*

$$\begin{aligned}&(A-B)(|\beta|(1-|A|)-(1+|B|)^2)\\ &\ge (1+|B|)^2(D(1+|A|)-E(1+|B|))+|E\beta(A-B)|(1+|A|)\end{aligned} \tag{2.12}$$

(b) $(1/p(z)) - \beta z p'(z)/p^k(z) \in \mathcal{P}[D,E]$ *for* $k=1,2$ *where* β *satisfies*

$$\begin{aligned}&(A-B)(|\beta|-(1+|A|)^{k-1}(1+|B|)^{2-k})\\ &\ge (1+|A|)^{k-1}(1+|B|)^{2-k}(D(1+|A|)-E(1+|B|))\\ &\quad +|E\beta(A-B)|.\end{aligned} \tag{2.13}$$

(c) $(1/p(z)) - \beta z p'(z)/p^k(z) \in \mathcal{P}[D,E]$ *for* $k>2$ *where* β *satisfies*

$$\begin{aligned}&(A-B)(|\beta|(1-|B|)^{k-2}-(1+|A|)^{k-1})\\ &\ge (1+|A|)^{k-1}(D(1+|A|)-E(1+|B|))\\ &\quad +|E\beta(A-B)|(1+|B|)^{k-2}.\end{aligned} \tag{2.14}$$

Proof. For $k=0,1,2,3,\ldots$, let Ω as in Theorem 2.3 and the function ψ be defined as

$$\psi(r,s;z)=\frac{1}{r}-\beta\frac{s}{r^k}.$$

In view of (2.4), the function ψ takes the following shape:

$$\psi(r,s;z)=\frac{1+Be^{i\theta}}{1+Ae^{i\theta}}-\frac{\beta m e^{i\theta}(A-B)(1+Be^{i\theta})^k}{(1+Be^{i\theta})^2(1+Ae^{i\theta})^k}. \tag{2.15}$$

(a) For $k=0$, we have

$$\begin{aligned}|\boldsymbol{\chi}(\psi(r,s;z),\,D,\,E)| &= \frac{|(A-B)((1+Be^{i\theta})^2+\beta m(1+Ae^{i\theta}))|}{\begin{aligned}&|(1+Be^{i\theta})^2(D(1+Ae^{i\theta})-E(1+Be^{i\theta}))\\ &\quad +E\beta m(A-B)e^{i\theta}(1+Ae^{i\theta})|\end{aligned}}\\
&\ge \frac{(A-B)(|\beta|m|1+Ae^{i\theta}|-|(1+Be^{i\theta})^2|)}{\begin{aligned}&|(1+Be^{i\theta})^2(D(1+Ae^{i\theta})-E(1+Be^{i\theta}))|\\ &\quad +|E\beta m(A-B)e^{i\theta}(1+Ae^{i\theta})|\end{aligned}}\\
&\ge \frac{(A-B)(|\beta|m(1-|A|)-(1+|B|)^2)}{\begin{aligned}&|(1+Be^{i\theta})^2||(D(1+Ae^{i\theta})-E(1+Be^{i\theta}))|\\ &\quad +|E\beta m(A-B)(1+Ae^{i\theta})|\end{aligned}}\\
&\ge \frac{(A-B)(|\beta|m(1-|A|)-(1+|B|)^2)}{\begin{aligned}&(1+|B|)^2(|D(1+Ae^{i\theta})|+|E(1+Be^{i\theta})|)\\ &\quad +|E\beta m(A-B)||(1+Ae^{i\theta})|\end{aligned}}\\
&\ge \frac{(A-B)(|\beta|m(1-|A|)-(1+|B|)^2)}{\begin{aligned}&(1+|B|)^2(D(1+|A|)|-E(1+|B|))\\ &\quad +m|E\beta(A-B)|(1+|A|)\end{aligned}} =: \phi(m).\end{aligned}$$

Using first derivative test we note that ϕ is an increasing function for $m \geq 1$. Thus the function $\phi(m)$ has minimum value at $m = 1$. Therefore $|\boldsymbol{\chi}(\psi(r, s; z),\, D,\, E)| \geq 1$ whenever the inequality (2.12) holds. Thus Theorem 2.2 complete the desired proof. Part (b) and (c) can be proved as part (a). We are omitting further details here. $\square$

Let $\beta \neq 0$, $-1 \leq B < A \leq 1$ and $-1 \leq E < 0 < D \leq 1$. If one of the following subordination holds for $f \in \mathcal{A}$:

(i) For $(A-B)(|\beta| - (1+|B|)) \geq (1+|B|)(D(1+|A|) - E(1+|B|)) + |E\beta(A-B)|$,

$$\left(\frac{zf'(z)}{f(z)}\right)^{-1}\left(-\beta\frac{zf'(z)}{f(z)}\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right)\right) \in \mathcal{P}[D, E];$$

(ii) For $(A-B)(|\beta| - (1+|A|)) \geq (1+|A|)(D(1+|A|) - E(1+|B|)) + |E\beta(A-B)|$,

$$\left(\frac{zf'(z)}{f(z)}\right)^{-1}\left(1 - \beta\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right)\right) \in \mathcal{P}[D, E];$$

then $f \in \mathcal{S}^*[A, B]$.

Motivated by the work in [2], we obtain the conditions on A, B, D, E for a general Briot-Bouquet differential subordination in the following theorem.

Theorem 2.13. *Let $-1 \leq B < A \leq 1$, $-1 \leq E < D \leq 1$ and $\beta\gamma > 0$ satisfy*

$$\begin{aligned}&(A-B)((1-|B|) - (\beta(1+|A|) + \gamma(1+|B|))^2)\\ &\geq (\beta(1+|A|) + \gamma(1+|B|))^2(D - E + |DB - EA|)|E|(A-B)(1+|B|).\end{aligned} \tag{2.16}$$

If $p \in \mathcal{P}$satifies the following subordination

$$p(z) + \frac{zp'(z)}{(\beta p(z) + \gamma)^2} \in \mathcal{P}[D, E],$$

then $p \in \mathcal{P}[A, B]$.

Proof. Let Ω be defined as in Theorem 2.3. Consider the analytic function

$$\psi(r, s; z) = r + \frac{s}{(\beta r + \gamma)^2}.$$

The required subordination is obtained if we show $\psi \in \Psi[\Omega,\, A,\, B]$ by making use of Theorem 2.2. Using (2.4), the function $\psi(r, s; z)$ takes the following form

$$\psi(r, s; z) = \frac{(1 + Ae^{i\theta})(\beta(1 + Ae^{i\theta}) + \gamma(1 + Be^{i\theta}))^2 + m(A - B)(1 + Be^{i\theta})e^{i\theta}}{(1 + Be^{i\theta})(\beta(1 + Ae^{i\theta}) + \gamma(1 + Be^{i\theta}))^2}$$

so that

$$
\begin{aligned}
|\boldsymbol{\chi}(\psi(r,s;z),\, D,\, E)| &= \frac{\begin{array}{r}|(A-B)((\beta(1+Ae^{i\theta})+\gamma(1+Be^{i\theta}))^2\\ +m(1+Be^{i\theta}))|\end{array}}{\begin{array}{r}|(D(1+Be^{i\theta})-E(1+Ae^{i\theta}))(\beta(1+Ae^{i\theta})\\ +\gamma(1+Be^{i\theta}))^2-Eme^{i\theta}(A-B)(1+Be^{i\theta})|\end{array}}\\
&\geq \frac{\begin{array}{r}(A-B)(m|1+Be^{i\theta}|-|(\beta(1+Ae^{i\theta})\\ +\gamma(1+Be^{i\theta}))^2|)\end{array}}{\begin{array}{r}|(D-E)(\beta(1+Ae^{i\theta})+\gamma(1+Be^{i\theta}))^2|\\ +|e^{i\theta}(DB-EA)(\beta(1+Ae^{i\theta})+\gamma(1+Be^{i\theta}))^2|\\ +|Eme^{i\theta}(A-B)(1+Be^{i\theta})|\end{array}}\\
&\geq \frac{\begin{array}{r}(A-B)(m(1-|B|)-(|\beta(1+Ae^{i\theta})|\\ +|\gamma(1+Be^{i\theta})|)^2)\end{array}}{\begin{array}{r}|(D-E)(\beta(1+Ae^{i\theta})+\gamma(1+Be^{i\theta}))^2|\\ +|(DB-EA)(\beta(1+Ae^{i\theta})+\gamma(1+Be^{i\theta}))^2|\\ +|Em(A-B)(1+Be^{i\theta})|\end{array}}\\
&\geq \frac{(A-B)(m(1-|B|)-(\beta(1+|A|)+\gamma(1+|B|))^2)}{\begin{array}{r}(D-E)(\beta(1+|A|)+\gamma(1+|B|))^2\\ +|(DB-EA)(\beta(1+|A|)+\gamma(1+|B|))^2|\\ +m|E(A-B)|(1+|B|)\end{array}}\\
&=: \phi(m).
\end{aligned}
$$

A computation shows that $\phi'(m) > 0$. Thus for $m \geq 1$, $\phi(m) \geq \phi(1)$ and therefore $|\boldsymbol{\chi}(\psi(r,s;z),\, D,\, E)| \geq 1$ whenever the inequality (2.16) holds. This implies that $\psi(r,s;z) \notin \Omega$. Hence the desired subordination is obtained. □

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References

[1] Ahuja, O.P., Kumar, S., Ravichandran, V., *Applications of first order differential subordination for functions with positive real part*, Stud. Univ. Babeş-Bolyai Math., **63**(2018), no. 3, 303-311.

[2] Ali, R.M., Ravichandran, V., Seenivasagan, N., *On Bernardi's integral operator and the Briot-Bouquet differential subordination*, J. Math. Anal. Appl., **324**(2006), 663-668.

[3] Ali, R.M., Ravichandran, V., Seenivasagan, N., *Sufficient conditions for Janowski starlikeness*, Int. J. Math. Math. Sci., **2007**, Art. ID 62925, 7 pp.

[4] Anand, S., Kumar, S., Ravichandran, V., *First-order differential subordinations for Janowski starlikeness*, in "Mathematical Analysis. I. Approximation Theory", 185-196, Springer Proc. Math. Stat., 306, Springer, Singapore, 2020.

[5] Bohra, N., Kumar, S., Ravichandran, V., *Some special differential subordinations*, Hacet. J. Math. Stat., **48**(2019), no. 4, 1017-1034.

[6] Bulboacă, T., *Differential Subordinations and Superordinations*, Recent Results, House of Scientific Book Publ., Cluj-Napoca, 2005.

[7] Cho, N.E., Kumar, S., Kumar, V., Ravichandran, V., *Differential subordination and radius estimates for starlike functions associated with the Booth lemniscate*, Turkish J. Math., **42**(2018), no. 3, 1380-1399.

[8] Cho, N.E., Kumar, S., Kumar, V., Ravichandran, V., Srivastava, H.M., *Starlike functions related to the Bell numbers*, Symmetry, **11**(2019), no. 2, Article 219, 17 pp.

[9] Chojnacka, O., Lecko, A., *Differential subordination of a harmonic mean to a linear function*, Rocky Mountain J. Math., **48**(2018), no. 5, 1475-1484.

[10] Gandhi, S., Kumar, S., Ravichandran, V., *First order differential subordinations for Caratheodory functions*, Kyungpook Math. J., **58**(2018), 257-270.

[11] Goodman, A.W., *Univalent Functions*, Vol. II, Mariner Publishing Co., Inc., Tampa, FL, 1983.

[12] Janowski, W., *Extremal problems for a family of functions with positive real part and for some related families*, Ann. Polon. Math., **23**(1970/1971), 159-177.

[13] Kanas, S., *Techniques of the differential subordination for domains bounded by conic sections*, Int. J. Math. Math. Sci., **2003**, no. 38, 2389-2400.

[14] Kanas, S., *Differential subordination related to conic sections*, J. Math. Anal. Appl., **317**(2006), no. 2, 650-658.

[15] Kanas, S., Lecko, A., *Differential subordination for domains bounded by hyperbolas*, Zeszyty Nauk. Politech. Rzeszowskiej Mat., **175**(1999), no. 23, 61-70.

[16] Kim, I.H., Sim, Y.J., Cho, N.E., *New criteria for Carathéodory functions*, J. Inequal. Appl., **2019**, 2019:13.

[17] Kumar, S., Ravichandran, V., *Subordinations for functions with positive real part*, Complex Anal. Oper. Theory, **12**(2018), no. 5, 1179-1191.

[18] Kumar, S.S., Kumar, V., Ravichandran, V., Cho, N.E., *Sufficient conditions for starlike functions associated with the lemniscate of Bernoulli*, J. Inequal. Appl. **2013**(2013), 176, 13 pp.

[19] Miller, S.S., Mocanu, P.T., *On some classes of first-order differential subordinations*, Michigan Math. J., **32**(1985), no. 2, 185-195.

[20] Miller, S.S., Mocanu, P.T., *Differential Subordinations*, Monographs and Textbooks in Pure and Applied Mathematics, 225, Marcel Dekker, Inc., New York, 2000.

[21] Nunokawa, M., Obradović, M., Owa, S., *One criterion for univalency*, Proc. Amer. Math. Soc., **106**(1989), no. 4, 1035-1037.

[22] Ravichandran, V., Sharma, K., *Sufficient conditions for starlikeness*, J. Korean Math. Soc., **52**(2015), no. 4, 727-749.

[23] Robertson, M.S., *Certain classes of starlike functions*, Michigan Math. J., **32**(1985), no. 2, 135-140.

[24] Seoudy, T.M., Aouf, M.K., *Classes of admissible functions associated with certain integral operators applied to meromorphic functions*, Bull. Iranian Math. Soc., **41**(2015), no. 4, 793-804.

[25] Sharma, K., Ravichandran, V., *Applications of subordination theory to starlike functions*, Bull. Iranian Math. Soc., **42**(2016), no. 3, 761-777.

[26] Tuneski, N., Bulboacă, T., *Sufficient conditions for bounded turning of analytic functions*, Ukraïn. Mat. Zh., **70**(2018), no. 8, 1118-1127.

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Geometric properties of mixed operator involving Ruscheweyh derivative and Sălăgean operator

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Abstract. Operator theory is a magnificent tool for studying the geometric behaviors of holomorphic functions in the open unit disk. Recently, a combination between two well known differential operators, Ruscheweyh derivative and Sălăgean operator are suggested by Lupas in [10]. In this effort, we shall follow the same principle, to formulate a generalized differential-difference operator. We deliver a new class of analytic functions containing the generalized operator. Applications are illustrated in the sequel concerning some differential subordinations of the operator.

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1. Introduction

Differential operators in a complex domain play a significant role in functions theory and its information. They have used to describe the geometric interpolation of analytic functions in a complex domain. Also, they have utilized to generate new formulas of holomorphic functions. Lately, Lupas [10] presented a amalgamation of two well-known differential operators prearranged by Ruscheweyh [12] and Sălăgean [13]. Later, these operators are investigated by researchers considering different classes and formulas of analytic functions [5, 8].

In this note, we consider a special class of functions in the open unit disk

$$\sqcup = \{\xi \in \mathbb{C} | \, |\xi| < 1\}$$

denoting by Σ and having the series

$$\varphi(\xi) = \xi + \sum_{n=2}^{\infty} \varphi_n \xi^n, \quad \xi \in \sqcup.$$

Let $\varphi \in \Sigma$, then the Ruscheweyh formula is indicated by the structure formula

$$\Phi^m \varphi(\xi) = \xi + \sum_{n=2}^{\infty} C^m_{m+n-1} \varphi_n \xi^n.$$

While, the Sălăgean operator admits the construction

$$\Psi^m \varphi(\xi) = \xi + \sum_{n=2}^{\infty} n^m \varphi_n \xi^n.$$

Lupas operator is formulated by the structure

$$\lambda^m_\sigma \varphi(\xi) = \xi + \sum_{n=2}^{\infty} [\sigma n^m + (1-\sigma) C^m_{m+n-1}]\varphi_n \xi^n, \quad \xi \in \sqcup, \sigma \in [0,1].$$

Newly, Ibrahim and Darus [7] considered the next differential operator

$$\begin{aligned}
&\Theta^0_\kappa \varphi(\xi) = \varphi(\xi)\\
&\Theta^1_\kappa \varphi(\xi) = \xi\, \varphi(\xi)' + \frac{\kappa}{2}\left(\varphi(\xi) - \varphi(-\xi) - 2\xi\right), \quad \kappa \in \mathbb{R}\\
&\vdots\\
&\Theta^m_\kappa \varphi(\xi) = \Theta_\kappa(\Theta^{m-1}_\kappa \varphi(\xi))\\
&\qquad\qquad = \xi + \sum_{n=2}^{\infty}[n + \frac{\kappa}{2}(1+(-1)^{n+1})]^m \varphi_n \xi^n.
\end{aligned}$$

When $\kappa = 0$,we have $\Psi^m \varphi(\xi)$ In addition, it is a modified formula of the well-known Dunkl operator [2], where κ is known as the Dunkl order. Proceeding, we define a generalized formula of λ^m_σ, as follows:

$$\begin{aligned}
J^m_{\sigma,\kappa}\varphi(\xi) &= (1-\sigma)\Phi^m \varphi(\xi) + \sigma \Theta^m_\kappa \varphi(\xi)\\
&= \xi + \sum_{n=2}^{\infty}[(1-\sigma)C^m_{m+n-1} + \sigma\big(n + \frac{\kappa}{2}(1+(-1)^{n+1})\big)^m]\, \varphi_n \xi^n.
\end{aligned} \tag{1.1}$$

Clearly, the operator $J^m_{\sigma,\kappa}\varphi(\xi) \in \Sigma$.

Remark 1.1.

- $m = 0 \Longrightarrow J^0_{\sigma,\kappa}\varphi(\xi) = \varphi(\xi)$;
- $\kappa = 0 \Longrightarrow J^m_{\sigma,0}\varphi(\xi) = \lambda^m_\sigma \varphi(\xi)$;
- $\sigma = 0 \Longrightarrow J^m_{0,\kappa}\varphi(\xi) = \Phi^m \varphi(\xi)$;
- $\sigma = 1 \Longrightarrow J^m_{1,\kappa}\varphi(\xi) = \Theta^m_\kappa \varphi(\xi)$;
- $\kappa = 0, \sigma = 1 \Longrightarrow J^m_{1,0}\varphi(\xi) = \Psi^m \varphi(\xi)$.

Definition 1.2. Consider the following data $\epsilon \in [0,1), \sigma \in [0,1], \kappa \geq 0$, and $m \in \mathbb{N}$. Then a function $\varphi \in \Sigma$ belongs to the set $\top_m(\sigma,\kappa,\epsilon)$ if and only if

$$\Re\Big((J^m_{\sigma,\kappa}\varphi(\xi))'\Big) > \epsilon, \quad \xi \in \sqcup.$$

Observe that the set $\top_m(\sigma,\kappa,\epsilon)$ is an extension of the well known class of bounded turning functions (see [1]-[14]). Next results are requested to prove our results depending on the subordination concept (see [11]).

Lemma 1.3. *Suppose that $\hbar$ is convex function such that $\hbar(0)=\flat$, and there is a complex number with a positive real part μ. If $\flat\in\mathfrak{H}[\flat,n]$, where*

$$\mathfrak{H}[\flat,n]=\{\flat\in\mathfrak{H}:\flat(\xi)=\flat+\flat_n\xi^n+\flat_{n+1}\xi^{n+1}+...\}$$

(the space of holomorphic functions) and

$$\flat(\xi)+\frac{1}{\mu}\xi\flat'(\xi)\prec\hbar(\xi),\quad \xi\in\sqcup,$$

then

$$\flat(\xi)\prec\iota(\xi)\prec\hbar(\xi),$$

with

$$\iota(\xi)=\frac{\mu}{n\,\xi^{\mu/n}}\int_0^{\xi}\hbar(\tau)\tau^{\overline{\frac{\mu}{(n-1)}}}d\tau,\quad \xi\in\sqcup.$$

Lemma 1.4. *Suppose that the convex function $\flat(\xi)$ satisfies the functional*

$$\hbar(\xi)=\flat(\xi)+n\mu(\xi\,\flat'(\xi))$$

for $\mu>0$ and n is a positive integer. If $\flat\in\mathfrak{H}[\hbar(0),n]$, and $\flat(\xi)+\mu\xi\flat'(\xi)\prec\hbar(\xi)$, $\xi\in\sqcup$ then $\flat(\xi)\prec\hbar(\xi)$, and this outcome is sharp.

Lemma 1.5. *(i) If $\lambda>0,\gamma>0$, $\beta=\beta(\gamma,\lambda,n)$ and $\flat\in\mathfrak{H}[1,n]$ then*

$$\flat(\xi)+\lambda\xi\flat'(\xi)\prec\left[\frac{1+\xi}{1-\xi}\right]^{\beta}\Rightarrow\flat(\xi)\prec\left[\frac{1+\xi}{1-\xi}\right]^{\gamma}.$$

(ii) If $\epsilon\in[0,1)$, $\lambda=\lambda(\epsilon,n)$ and $\flat\in\mathfrak{H}[1,n]$ then

$$\Re\Big(\flat^2(\xi)+2\flat(\xi).\xi\flat'(\xi)\Big)>\epsilon\Rightarrow\Re(\flat(\xi))>\lambda.$$

2. Results

In this section, we investigate some geometric conducts of the operator (1.1).

Theorem 2.1. *The set $\top_m(\sigma,\kappa,\epsilon)$ is convex.*

Proof. Suppose that $\varphi_i, i=1,2$ are two functions belonging to $\top_m(\sigma,\kappa,\epsilon)$ satisfying

$$\varphi_1(\xi)=\xi+\sum_{n=2}^{\infty}\varphi_n\xi^n$$

and

$$\varphi_2(\xi)=\xi+\sum_{n=2}^{\infty}\phi_n\xi^n.$$

It is sufficient to prove that the function

$$\Pi(x_1)=\wp_1\varphi_1(\xi)+\wp_2\varphi_2(\xi),\quad \xi\in\sqcup$$

is in $\top_m(\sigma,\kappa,\epsilon)$, where $\wp_1>0, \wp_2>0$ and $\wp_1+\wp_2=1$. The formula of $\Pi(z)$ yields

$$\Pi(\xi)=\xi+\sum_{n=2}^{\infty}(\wp_1\varphi_n+\wp_2\phi_n)\xi^n.$$

Thus, under the operator (1.1), we get

$$J^m_{\sigma,\kappa}\Pi(\xi)=\xi+\sum_{n=2}^{\infty}(\wp_1\varphi_n+\wp_2\phi_n)[(1-\sigma)C^m_{m+n-1}+\sigma\big(n+\frac{\kappa}{2}(1+(-1)^{n+1})\big)^m]\xi^n.$$

By making a differentiation, we obtain

$$\begin{aligned}&\Re\{(J^m_{\alpha,\kappa}\Pi(\xi))'\}\\ &=1+\wp_1\Re\left\{\sum_{n=2}^{\infty}n[(1-\sigma)C^m_{m+n-1}+\sigma\left(n+\frac{\kappa}{2}(1+(-1)^{n+1})\right)^m]\varphi_n\xi^{n-1}\right\}\\ &+\wp_2\Re\left\{\sum_{n=2}^{\infty}n[(1-\sigma)C^m_{m+n-1}+\sigma\left(n+\frac{\kappa}{2}(1+(-1)^{n+1})\right)^m]\phi_n\,\xi^{n-1}\right\}=\epsilon.\end{aligned}$$

□

Theorem 2.2. *Define the following functions: $\varphi\in\top_m(\sigma,\kappa,\epsilon)$, ϕ be convex and*

$$F(\xi)=\frac{2+c}{\xi^{1+c}}\int_0^{\xi}t^c\,\varphi(t)dt,\quad \xi\in\sqcup.$$

Then

$$\left(J^m_{\alpha,\kappa}\varphi(\xi)\right)'\prec\phi(\xi)+\frac{(\xi\,\phi'(\xi))}{2+c},\quad c>0,$$

yields

$$\left(J^m_{\sigma,\kappa}F(\xi)\right)'\prec\phi(\xi),$$

and this outcome is sharp.

Proof. By the assumptions, we have

$$\left(J^m_{\sigma,\kappa}F(\xi)\right)'+\frac{\left(J^m_{\sigma,\kappa}F(\xi)\right)''}{2+c}=\left(J^m_{\sigma,\kappa}\varphi(\xi)\right)'.$$

Consequently, we get

$$\left(J^m_{\sigma,\kappa}F(\xi)\right)'+\frac{\left(J^m_{\sigma,\kappa}F(\xi)\right)''}{2+c}\prec\phi(\xi)+\frac{(\xi\phi'(\xi))}{2+c}.$$

Assuming

$$\flat(\xi):=\left(J^m_{\sigma,\kappa}F(\xi)\right)',$$

one can find

$$\flat(\xi)+\frac{(\xi\flat'(\xi))}{2+c}\prec\phi(\xi)+\frac{(\xi\phi'(\xi))}{2+c}.$$

In virtue of Lemma 1.3, we have

$$\left(J^m_{\sigma,\kappa}F(\xi)\right)'\prec\phi(\xi),$$

and ϕ is the best dominant. □

Theorem 2.3. *Assume the convex function ϕ achieving $\phi(0) = 1$ and for $\varphi \in \Sigma$*

$$\Big(J^m_{\sigma,\kappa}\varphi(\xi)\Big)' \prec \phi(\xi) + \xi\,\phi'(\xi), \quad \xi \in \sqcup,$$

then

$$\frac{J^m_{\sigma,\kappa}\varphi(\xi)}{\xi} \prec \phi(\xi),$$

and this outcome is sharp.

Proof. Formulate the next functional

$$\flat(z) := \frac{J^m_{\sigma,\kappa}\varphi(\xi)}{\xi} \in \mathfrak{H}[1,1] \tag{2.1}$$

Consequently, we get

$$J^m_{\sigma,\kappa}\varphi(\xi) = \xi\,\flat(\xi) \Longrightarrow \Big(J^m_{\sigma,\kappa}\varphi(\xi)\Big)' = \flat(\xi) + \xi\flat'(\xi).$$

Therefore, we obtain the inequality

$$\flat(\xi) + \xi\flat'(\xi) \prec \phi(\xi) + \xi\phi'(\xi).$$

According to Lemma 1.4, we attain

$$\frac{J^m_{\sigma,\kappa}\varphi(\xi)}{\xi} \prec \phi(\xi),$$

and ϕ is the best dominant. □

Theorem 2.4. *For $\varphi \in \Sigma$ if the inequality*

$$(J^m_{\sigma,\kappa}\varphi(\xi))' \prec \left(\frac{1+\xi}{1-\xi}\right)^{\beta}, \quad \xi \in \sqcup,\ \beta > 0,$$

achieves then

$$\Re\Big(\frac{J^m_{\sigma,\kappa}\varphi(\xi)}{\xi}\Big) > \epsilon$$

for some $\epsilon \in [0,1)$.

Proof. For the function $\flat(\xi)$ in (2.1), we have

$$(J^m_{\sigma,\kappa}\varphi(\xi))' = \xi\flat'(\xi) + \flat(\xi) \prec \left(\frac{1+\xi}{1-\xi}\right)^{\beta}.$$

According to Lemma 1.5.i, there occurs a constant $\gamma > 0$ with $\beta = \beta(\gamma)$ with

$$\frac{J^m_{\sigma,\kappa}\varphi(\xi)}{\xi} \prec \left(\frac{1+\xi}{1-\xi}\right)^{\gamma}.$$

This yields $\Re(J^m_{\sigma,\kappa}\varphi(\xi)/\xi) > \epsilon$, for some $\epsilon \in [0,1)$. □

Theorem 2.5. *Assume that $\varphi \in \Sigma$ achieves the inequality*

$$\Re\Big((J^m_{\sigma,\kappa}\varphi(\xi))'\frac{J^m_{\sigma,\kappa}\varphi(\xi)}{\xi}\Big) > \frac{\sigma}{2}, \quad \xi \in \sqcup, \sigma \in [0,1).$$

Then $J^m_{\sigma,\kappa}\varphi(\xi) \in \top_m(\sigma,\kappa,\epsilon)$ for some $\epsilon \in [0,1)$. In addition, it is univalent of bounded turning in $\sqcup$.

Proof. Assume the function $\flat(\xi)$ as in (2.1). A Calculation implies that

$$\Re\Big(\flat^2(\xi) + 2\flat(\xi).\xi\flat'(\xi)\Big) = 2\Re\Big((J^m_{\sigma,\kappa}\varphi(\xi))'\frac{J^m_{\sigma,\kappa}\varphi(\xi)}{\xi}\Big) \ > \ \sigma. \tag{2.2}$$

Lemma 1.5.ii, implies that there occurs a constant $\lambda(\sigma)$ satisfying $\Re(\flat(\xi)) > \lambda(\sigma)$. Thus, we obtain $\Re(\flat(\xi)\ ) > \epsilon$ for some $\epsilon \in [0,1)$. It yields from (2.2) that $\Re\Big(J^m_{\sigma,\kappa}\varphi(\xi))'\Big) > \epsilon$ and by Noshiro-Warschawski and Kaplan Theorems (see [3]), we have that $J^m_{\sigma,\kappa}\varphi(\xi)$ is univalent and of bounded turning in $\sqcup$. □

References

[1] Darus, M., Ibrahim, R.W., *Partial sums of analytic functions of bounded turning with applications*, J. Comput. Appl. Math., **29**(2010), 81-88.

[2] Dunkl, C.F., *Differential-difference operators associated with reflections groups*, Trans. Am. Math. Soc., **311**(1989), 167-183.

[3] Duren, P., *Univalent Functions*, Grundlehren der Mathematischen Wissenschaften, Springer-Verlag New York Inc., 1983.

[4] Ibrahim, R.W., *Geometric properties of the differential shift plus complex Volterra operator,* Asian-Eur. J. Math., **11.01**(2018), 1850013.

[5] Ibrahim, R.W., *Generalized Briot-Bouquet differential equation based on new differential operator with complex connections*, Gen. Math., **28**(2020), 105-114.

[6] Ibrahim, R.W., Darus M., *Extremal bounds for functions of bounded turning,* Int. Math. Forum, **6**(2011), 1623-1630.

[7] Ibrahim, R.W., Darus M., *Subordination inequalities of a new Sălăgean-difference operator*, Int. J. Math. Comput. Sci., **14**(2019), 573-582.

[8] Ibrahim, R.W., Elobaid, R.M., Obaiys, S.J., *Generalized Briot-Bouquet differential equation based on new differential operator with complex connections*, Axioms, **9**(2020), 1-13.

[9] Krishna, D., et al., *Third Hankel determinant for bounded turning functions of order alpha*, J. Nigerian Math. Soc., **34.2**(2015), 121-127.

[10] Lupaş Alb, A., *On special differential subordinations using Sălăgean and Ruscheweyh operators*, Math. Inequal. Appl., **12**(2009), 781-790.

[11] Miller, S.S., Mocanu, P.T., *Differential Subordinations: Theory and Applications*, CRC Press, 2000.

[12] Ruscheweyh, St., *New criteria for univalent functions*, Proc. Amer. Math. Soc., **49**(1975), 109-115.

[13] Sălăgean, G. St., *Subclasses of univalent functions*, Lecture Notes in Math., Springer Verlag, Berlin, **1013**(1983), 362-372.

[14] Tuneski, N., Bulboacă, T., *Sufficient conditions for bounded turning of analytic functions*, Ukr. Math. J., **70.8**(2019), 1288-1299.

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On some classes of holomorphic functions whose derivatives have positive real part

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Abstract. In this paper we discuss about normalized holomorphic functions whose derivatives have positive real part. For this class of functions, denoted R, we present a general distortion result (some upper bounds for the modulus of the k-th derivative of a function). We present also some remarks on the functions whose derivatives have positive real part of order α, $\alpha \in (0, 1)$. More details about these classes of functions can be found in [6], [8], [7, Chapter 4] and [4]. In the last part of this paper we present two new subclasses of normalized holomorphic functions whose derivatives have positive real part which generalize the classes R and $R(\alpha)$. For these classes we present some general results and examples.

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1. Introduction

In this paper we denote $U = U(0, 1)$ the open unit disc in the complex plane, $\mathcal{H}(U)$ the family of all holomorphic functions on the unit disc and S the family of all univalent normalized ($f(0) = 0$ and $f'(0) = 1$) functions on the unit disc. Also, let us denote

$$\mathcal{P} = \left\{p \in \mathcal{H}(U) : p(0) = 1 \text{ and } \operatorname{Re}[p(z)] > 0, \quad z \in U\right\}$$

the *Carathéodory class* and

$$R = \{f \in \mathcal{H}(U) : f(0) = 0, f'(0) = 1 \text{ and } \operatorname{Re}[f'(z)] > 0, \quad z \in U\}$$

the *class of normalized functions whose derivative has positive real part.* For more details about these classes, one may consult [1], [2, Chapter 7], [3, Chapter 2] or [7, Chapter 3].

Remark 1.1. Notice that, according to a result due to Noshiro and Warschawski (see [1, Theorem 2.16], [6] or [7, Theorem 4.5.1]), we have that each function from R is also univalent on the unit disc U. Hence, $R \subseteq S$.

Remark 1.2. Another important result (see [7, p. 87]) says that $f \in R$ if and only if $f' \in \mathcal{P}$.

Remark 1.3. During this paper, we use the following notations for the series expansions of $p \in \mathcal{P}$ and $f \in S$:

$$p(z) = 1 + p_1 z + p_2 z^2 + ... + p_n z^n + ... \tag{1.1}$$

and

$$f(z) = z + a_2 z^2 + a_3 z^3 + ... + a_n z^n + ..., \tag{1.2}$$

for all $z \in U$.

2. Preliminaries

First, we present some classical results regarding to the coefficient estimations and distortion results for the Carathéodory class $\mathcal{P}$. For details and proofs, one may consult [2, Chapter 7], [3, Chapter 2], [6, Lemma 1] or [7, Chapter 3].

Proposition 2.1. *Let $p \in \mathcal{P}$. Then*

$$|p_n| \leq 2, \quad n \geq 1, \tag{2.1}$$

$$\frac{1-|z|}{1+|z|} \leq \mathrm{Re}[p(z)] \leq |p(z)| \leq \frac{1+|z|}{1-|z|} \tag{2.2}$$

and

$$|p'(z)| \leq \frac{2}{(1-|z|)^2}, \tag{2.3}$$

for all $z \in U$. These estimates are sharp. The extremal function is $p : U \to \mathbb{C}$ given by

$$p(z) = \frac{1+z}{1-z}, \quad z \in U. \tag{2.4}$$

The next result is another important result regarding to the coefficient estimations and distortion results for the class R. For more details and proofs, one may consult [6, Theorem 1], [7, Chapter 4] or [8, Theorem A].

Proposition 2.2. *Let $f \in R$. Then*

$$|a_n| \leq \frac{2}{n}, \quad n \geq 2, \tag{2.5}$$

$$\frac{1-|z|}{1+|z|} \leq \mathrm{Re}\big[f'(z)\big] \leq |f'(z)| \leq \frac{1+|z|}{1-|z|}. \tag{2.6}$$

and

$$-|z| + 2\log(1+|z|) \leq |f(z)| \leq -|z| - 2\log(1-|z|). \tag{2.7}$$

for all $z \in U$*. These estimates are sharp. The extremal function is* $f : U \to \mathbb{C}$ *given by*

$$f(z) = -z - \frac{2}{\lambda}\log(1-\lambda z), \quad |\lambda| = 1, \quad z \in U. \tag{2.8}$$

Remark 2.3. Let $r = |z| < 1$. Then, for every $k \in \mathbb{N}^*$, the following relation hold

$$T_k = \frac{1}{(1-r)^k} = \sum_{p=0}^{\infty} \frac{(k+p-1)! \cdot r^p}{p! \cdot (k-1)!}. \tag{2.9}$$

This remark will be used in the next section as part of the proofs of the main results.

Proof. Let us consider the following Taylor series expansion

$$\frac{1}{1-r} = 1 + r + r^2 + ... + r^n + ..., \quad -1 < r < 1.$$

Then

$$\frac{1}{(1-r)^2} = \frac{\partial}{\partial r}\left[\frac{1}{1-r}\right] = 1 + 2r + 3r^2 + ... + nr^{n-1} + ..., \; -1 < r < 1.$$

It is easy to prove relation (2.9) using mathematical induction. For this, let us consider

$$P(k) : \frac{1}{(1-r)^k} = \sum_{p=0}^{\infty} \frac{(k+p-1)! \cdot r^p}{p! \cdot (k-1)!}, \quad k \geq 1.$$

Assume that $P(k)$ is true and let us prove that $P(k+1)$ is also true, where

$$P(k+1) : \frac{1}{(1-r)^{k+1}} = \sum_{p=0}^{\infty} \frac{(k+p)! \cdot r^p}{p! \cdot k!}.$$

Indeed,

$$\frac{k}{(1-r)^{k+1}} = \frac{\partial}{\partial r}\left[\frac{1}{(1-r)^k}\right] = \frac{\partial}{\partial r}\left[\sum_{p=0}^{\infty} \frac{(k+p-1)! \cdot r^p}{p! \cdot (k-1)!}\right]$$
$$= \sum_{p=1}^{\infty} \frac{(k+p-1)! \cdot p \cdot r^{p-1}}{p! \cdot (k-1)!} = \sum_{p=0}^{\infty} \frac{(k+p)! \cdot r^p}{p! \cdot (k-1)!}$$

and then

$$\frac{1}{(1-r)^{k+1}} = \sum_{p=0}^{\infty} \frac{(k+p)! \cdot r^p}{p! \cdot k!}, \; r > 1.$$

Hence, $P(k)$ is true for all $k \geq 1$ and the relation (2.9) holds. □

3. General distortion result for the class R

Starting from the previous proposition, we give a general distortion result (some upper bounds for the modulus of the k-th derivative) for the frunction from the class R.

Theorem 3.1. *If* $f \in R$*, then the following estimate hold:*

$$|f^{(k)}(z)| \leq \frac{2(k-1)!}{(1-|z|)^k}, \quad z \in U, \quad k \geq 1.$$

Proof. It is clear that R is a subclass of class S. Then the k-th derivative of a function $f \in R$ has the form

$$f^{(k)}(z) = \sum_{n=0}^{\infty} \frac{(k+n)!}{n!} a_{k+n} z^n, \quad z \in U. \tag{3.1}$$

Let $|z| \leq r < 1$. In view of relations (2.5) and (3.1) we obtain that

$$\begin{aligned} |f^{(k)}(z)| &= \left| \sum_{n=0}^{\infty} \frac{(k+n)!}{n!} a_{k+n} z^n \right| \leq \sum_{n=0}^{\infty} \frac{(k+n)!}{n!} |a_{k+n}| \cdot |z^n| \\ &\leq \sum_{n=0}^{\infty} \frac{(k+n)!}{n!} \cdot \frac{2}{k+n} r^n = 2 \cdot \sum_{n=0}^{\infty} \frac{(k+n-1)! r^n}{n!} \\ &= 2(k-1)! \cdot \sum_{n=0}^{\infty} \frac{(k+n-1)! r^n}{n!(k-1)!} \\ &= 2(k-1)! \cdot \frac{1}{(1-r)^k} = \frac{2(k-1)!}{(1-r)^k}. \end{aligned}$$

Hence, we obtain that

$$|f^{(k)}(z)| \leq \frac{2(k-1)!}{(1-r)^k}, \quad k \in \mathbb{N}^*, \quad |z| \leq r < 1. \qquad \square$$

Remark 3.2. Notice that the above result is not sharp for $k = 1$ (in view of relation (2.6)), but it is sharp for $k \geq 2$ and the extremal function is given by (2.8).

4. Some remarks on the class $R(\alpha)$

Let $\alpha \in [0, 1)$. Then

$$R(\alpha) = \{f \in \mathcal{H}(U) : f(0) = 0, f'(0) = 1, \mathrm{Re}\big[f'(z)\big] > \alpha, z \in U\}$$

denotes the *class of functions whose derivative has positive real part of order* α. For more details about this class, one may consult [4] and [5].

Remark 4.1. It is easy to prove that $f \in R(\alpha)$ if and only if $g \in \mathcal{P}$, where $g : U \to \mathbb{C}$ is given by

$$g(z) = \frac{1}{1-\alpha}\Big(f'(z) - \alpha\Big), \quad z \in U. \tag{4.1}$$

Proposition 4.2. *Let $\alpha \in [0, 1)$ and $f \in R(\alpha)$. Then*

$$|a_n| \leq \frac{2(1-\alpha)}{n}, \quad n \geq 2, \tag{4.2}$$

and these estimates are sharp. The equality holds for the function $f : U \to \mathbb{C}$ given by

$$f(z) = \frac{(2\alpha - 1)\lambda z - 2(1-\alpha)\log(1-\lambda z)}{\lambda} \tag{4.3}$$

with $|\lambda| = 1$.

Proof. Let $f \in R(\alpha)$ be of the form (1.2). Then

$$f'(z) = 1 + \sum_{n=1}^{\infty}(n+1)a_{n+1}z^n, \quad z \in U.$$

Let us consider the function $g : U \to \mathbb{C}$ given by

$$g(z) = \frac{1}{1-\alpha}\Big(f'(z) - \alpha\Big), \quad z \in U.$$

Then $g \in \mathcal{P}$ and

$$g(z) = \frac{f'(z) - \alpha}{1-\alpha} = \frac{1 - \alpha + \sum_{n=1}^{\infty}(n+1)a_{n+1}z^n}{1-\alpha} = 1 + \sum_{n=1}^{\infty}\frac{(n+1)}{1-\alpha}a_{n+1}z^n$$

or, equivalent

$$g(z) = 1 + \sum_{n=1}^{\infty} p_n z^n, \quad \text{where} \quad p_n = \frac{n+1}{1-\alpha}a_{n+1}. \tag{4.4}$$

Taking into account the relations (2.1) and (4.4) we obtain that

$$\left|\frac{n+1}{1-\alpha}a_{n+1}\right| \le 2 \Leftrightarrow |a_{n+1}| \le \frac{2(1-\alpha)}{n+1}, \quad \forall\ n \ge 1.$$

So we obtain that

$$|a_n| \le \frac{2(1-\alpha)}{n}, \quad \forall\ n \ge 2.$$

The function given by relation (4.3) is obtained from the extremal function of the Carathédory class. We have the following Taylor expansion

$$f(z) = z + (1-\alpha)\lambda z^2 + \frac{2}{3}(1-\alpha)\lambda^2 z^3 + \ldots$$

leading to the estimates

$$|a_2| = \big|(1-\alpha)\lambda\big| = 1 - \alpha$$

$$|a_3| = \left|\frac{2}{3}(1-\alpha)\lambda\right| = \frac{2(1-\alpha)}{3}$$

and the equalities hold for every $n \ge 2$. □

Remark 4.3. The previous result can be found also in [5, Theorem 3.5] with another version of the proof.

Next, we present a growth and distortion result for the class $R(\alpha)$. Starting from this theorem we give also a general distortion result (some upper bounds for the modulus of the k-th derivative) for the class $R(\alpha)$.

Theorem 4.4. *Let $\alpha \in [0, 1)$ and $f \in R(\alpha)$. Then*

$$|f(z)| \le (2\alpha - 1)|z| + 2(\alpha - 1)\log(1 - |z|), \tag{4.5}$$

$$|f(z)| \ge -|z| - 2(\alpha - 1)\log(1 + |z|) \tag{4.6}$$

and

$$\frac{1-2\alpha-|z|}{1+|z|} \le |f'(z)| \le \frac{1+(1-2\alpha)|z|}{1-|z|}, \tag{4.7}$$

for all $z \in U$. These estimates are sharp. The extremal function is $f: U \to \mathbb{C}$ given by

$$f(z) = (2\alpha-1)z - \frac{2(1-\alpha)\log(1-\lambda z)}{\lambda}, \quad |\lambda| = 1, \quad z \in U. \tag{4.8}$$

Proof. Let $\alpha \in [0,1)$ and $f \in R(\alpha)$. In view of Remark 4.1 and Proposition 2.1, we obtain that

$$\left|\frac{1}{1-\alpha}\big[f'(z)-\alpha\big]\right| \le \frac{1+|z|}{1-|z|}$$

$$\big|f'(z)-\alpha\big| \le \frac{(1-\alpha)(1+|z|)}{1-|z|}$$

Then

$$|f'(z)| \le \frac{(1-\alpha)(1+|z|)}{1-|z|} + \alpha = \frac{1+(1-2\alpha)|z|}{1-|z|}$$

On the other hand,

$$\left|\frac{1}{1-\alpha}\big[f'(z)-\alpha\big]\right| \ge \frac{1-|z|}{1+|z|}$$

$$\big|f'(z)-\alpha\big| \ge \frac{(1-\alpha)(1-|z|)}{1+|z|}$$

Then

$$|f'(z)| \ge \frac{(1-\alpha)(1-|z|)}{1+|z|} - \alpha = \frac{1-2\alpha-|z|}{1+|z|}$$

Hence, we obtain relations (4.7). Finally, to obtain the relations (4.5) and (4.6), it is enough to integrate the relation (4.7). □

Theorem 4.5. *Let $\alpha \in [0,1)$ and $f \in R(\alpha)$. Then the following estimate hold:*

$$|f^{(k)}(z)| \le \frac{2(1-\alpha)(k-1)!}{(1-|z|)^k}, \quad z \in U, \quad k \ge 1.$$

Proof. Let $\alpha \in [0,1)$. It is clear that $R(\alpha)$ is a subclass of class S. Then the k-th derivative of a function $f \in R(\alpha)$ has the form

$$f^{(k)}(z) = \sum_{n=0}^{\infty} \frac{(k+n)!}{n!} a_{k+n} z^n, \quad z \in U. \tag{4.9}$$

Let $|z| \le r < 1$. According to the relations (4.2) and (4.9) we obtain that

$$\begin{aligned}
|f^{(k)}(z)| &= \left|\sum_{n=0}^{\infty} \frac{(k+n)!}{n!} a_{k+n} z^n\right| \le \sum_{n=0}^{\infty} \frac{(k+n)!}{n!} |a_{k+n}| \cdot |z^n| \\
&\le \sum_{n=0}^{\infty} \frac{(k+n)!}{n!} \cdot \frac{2(1-\alpha)}{k+n} r^n = 2(1-\alpha) \cdot \sum_{n=0}^{\infty} \frac{(k+n-1)! r^n}{n!} \\
&= 2(1-\alpha)(k-1)! \cdot \sum_{n=0}^{\infty} \frac{(k+n-1)! r^n}{n!(k-1)!} = \frac{2(1-\alpha)(k-1)!}{(1-r)^k},
\end{aligned}$$

Hence, we obtain that

$$|f^{(k)}(z)| \le \frac{2(1-\alpha)(k-1)!}{(1-r)^k}, \quad k \in \mathbb{N}^*, \quad |z| \le r < 1. \qquad \square$$

Remark 4.6. Notice that, for $k = 1$, the previous result is not sharp. The sharpness is obtained if $k \ge 2$ for the function f defined by (4.8).

Remark 4.7. It is clear that if $\alpha = 0$, then $R(0) = R$ and we obtain the classical results from the previous section.

5. The class R_p

Let $p \in \mathbb{N}^*$. Starting from the well-known class R, we define

$$R_p = \{f \in \mathcal{H}(U) : f(0) = 0, f'(0) = 1, f^{(p)}(0) = 1, \mathrm{Re}\big[f^{(p)}(z)\big] > 0, z \in U\}$$

the *class of normalized functions whose p-th derivative has positive real part*. This is the natural extension of the class R (extension which preserves the connection with the Carathéodory class). We present for this class some important results, a few examples and structure formulas (in the particular cases $p = 2$ and $p = 3$). It is clear that if $p = 1$, then $R_1 = R$.

Remark 5.1. In previous definition we have the following equivalent conditions

$$f^{(p)}(0) = 1 \Leftrightarrow a_p = \frac{1}{p!}, \tag{5.1}$$

for $p \in \mathbb{N}^*$ arbitrary fixed. Indeed, if $f \in R_p$, then

$$f^{(p)}(z) = \sum_{n=0}^{\infty} \frac{(n+p)!}{n!} a_{n+p} z^n = p! \cdot a_p + \frac{(p+1)!}{1!} a_{p+1} z + \frac{(p+2)!}{2!} a_{p+2} z^2 + \ldots$$

For $z = 0$ we obtain

$$f^{(p)}(0) = p! \cdot a_p.$$

Hence

$$f^{(p)}(0) = 1 \Leftrightarrow p! \cdot a_p = 1 \Leftrightarrow a_p = \frac{1}{p!}, \quad p \ge 1.$$

Remark 5.2. Let $p \in \mathbb{N}^*$ be arbitrary fixed. In view of above definition we deduce that

$$f \in R_p \Leftrightarrow f^{(p)} \in \mathcal{P},$$

so we can use the properties of Carathéodory class $\mathcal{P}$ to describe the function $f^{(p)}$ and then we can obtain some properties for $f \in R_p$.

Proposition 5.3. *Let $p \in \mathbb{N}^*$ and $f \in R_p$. Then the following relation hold:*

$$|a_n| \le \frac{2(n-p)!}{n!}, \quad n \ge p, \tag{5.2}$$

Proof. Let $f \in R_p$. Then

$$f^{(p)}(z) = \sum_{n=0}^{\infty} \frac{(n+p)!}{n!} a_{n+p} z^n, \quad z \in U.$$

Taking into account Remark 5.2 and Proposition 2.1 we have that

$$f^{(p)} \in \mathcal{P},$$

and

$$\left| \frac{(n+p)!}{n!} a_{n+p} \right| \leq 2, \quad \forall\, n \geq 2.$$

In view of above relations we obtain

$$|a_{n+p}| \leq \frac{2 \cdot n!}{(n+p)!}$$

or, an equivalent form

$$|a_n| \leq \frac{2(n-p)!}{n!}, \quad \forall\, n \geq p.$$

□

Theorem 5.4. *Let $p \in \mathbb{N}^*$ and $f \in R_p$. Then the following estimate hold:*

$$|f^{(k)}(z)| \leq \frac{2(k-p)!}{(1-|z|)^{k-p+1}}, \quad z \in U, \quad k \geq p. \tag{5.3}$$

Proof. Let $f \in R_p$. Then

$$f^{(k)}(z) = \sum_{n=0}^{\infty} \frac{(k+n)!}{n!} a_{n+k} z^n, \quad z \in U. \tag{5.4}$$

Let $|z| \leq r < 1$. Using relations (5.2) and (5.4) we obtain

$$\begin{aligned} |f^{(k)}(z)| &= \left| \sum_{n=0}^{\infty} \frac{(k+n)!}{n!} a_{k+n} z^n \right| \leq \sum_{n=0}^{\infty} \frac{(k+n)!}{n!} |a_{k+n}| \cdot |z^n| \\ &\leq \sum_{n=0}^{\infty} \frac{(k+n)!}{n!} \cdot \frac{2(n+k-p)!}{(k+n)!} r^n = 2 \cdot \sum_{n=0}^{\infty} \frac{(n+k-p)! r^n}{n!} \\ &= 2(k-p)! \cdot \sum_{n=0}^{\infty} \frac{(k+n-p)! r^n}{n!(k-p)!} = \frac{2(k-p)!}{(1-r)^{k-p+1}}. \end{aligned}$$

Hence,

$$|f^{(k)}(z)| \leq \frac{2(k-p)!}{(1-|z|)^{k-p+1}}, \quad z \in U, \quad k \geq p.$$

□

Remark 5.5. In estimates (5.3) we have the following existence condition:

$$\forall\, k, p \in \mathbb{N}^* : \quad k \geq p.$$

In other words, for $p \in \mathbb{N}^*$ arbitrary fixed we can estimate the derivatives of order k with $k \geq p$ (the derivatives of order at least p). In particular, for $p = 1$ (i.e. for the class R) we can estimate all derivatives of order at least 1.

Remark 5.6. For the bounds of the modulus of the first $(p-1)$ derivatives of a function $f \in R_p$ we can apply the following argument

$$\forall\, j \in \{0, ..., p-1\} : \quad |f^{(j)}(z)| \leq \underbrace{\int_0^r ... \int_0^r}_{(p-j)\text{ times}} \left[\frac{1+\rho}{1-\rho}\right] d\rho \tag{5.5}$$

In particular,

$$|f^{(p-1)}(z)| \leq -|z| - 2\log(1-|z|)$$

and

$$|f^{(p-2)}(z)| \leq \frac{-|z|(|z|-4)}{2} - 2(|z|-1)\log(1-|z|).$$

Hence, for $f \in R_p$ we obtain general upper bounds, as follows:

- if $0 \leq k < p$, we use relation (5.3);
- if $k \geq p$, we use relation (5.5).

Remark 5.7. If $p = 1$, then $R_1 = R$ and we obtain the result (*general result of distortion*) from Theorem 3.1.

In following results we discuss about the relation between two consecutive classes of order p, respectively $p+1$, for $p \in \mathbb{N}^*$ arbitrary choosen.

Proposition 5.8. *Let $p \in \mathbb{N}^*$. Then $R_p \cap R_{p+1} \neq \emptyset$.*

For $p \in \mathbb{N}^*$ we can find a function f which belongs to both class R_p and R_{p+1}. We present two examples to illustrate this proposition (first for the case $p = 1$ and second for the general case $p \geq 2$).

Example 5.9. Let $f : U \to \mathbb{C}$ be given by $f(z) = \frac{1}{2}z^2 + z$, $z \in U$. Then $f \in R_1 \cap R_2$.

Proof. Indeed, we have

$$f(0) = 0$$
$$f'(z) = z + 1$$
$$f''(z) = 1, \quad z \in U.$$

For $z = 0$ we obtain

$$f'(0) = f''(0) = 1 \quad \text{and} \quad \text{Re}f''(z) = 1 > 0, \quad \forall\, z \in U.$$

Then, in view of definition, $f \in R_2$. On the other hand,

$$f'(0) = 1 \quad \text{and} \quad \text{Re}f'(z) = \text{Re}(z+1) = 1 + \text{Re}z > 0, \quad \forall\, z \in U,$$

and this means that $f \in R_1$. □

Example 5.10. Let $p \geq 2$ and let $f : U \to \mathbb{C}$ be given by

$$f(z) = z + \frac{1}{p!}z^p + \frac{1}{(p+1)!}z^{p+1}, \quad z \in U.$$

Then $f \in R_p \cap R_{p+1}$.

Proposition 5.11. *Let $p \in \mathbb{N}^*$. In general, $R_p \not\subseteq R_{p+1}$.*

For $p \in \mathbb{N}^*$ we can find a function f which belongs to the class R_p, but does not belong to the class R_{p+1}. We present two examples to illustrate this statement.

Example 5.12. Let $f : U \to \mathbb{C}$ be given by $f(z) = z$, $z \in U$. Then $f \in R = R_1$, but $f \notin R_2$.

Example 5.13. Let $p \geq 2$ and let $f : U \to \mathbb{C}$ be given by $f(z) = z + \frac{1}{p!}z^p$, $z \in U$. Then $f \in R_p$, but $f \notin R_{p+1}$.

Remark 5.14. The above example can be generalized by adding the terms between z and $\frac{1}{p!}z^p$. We can consider the function $f : U \to \mathbb{C}$ given by

$$f(z) = z + \sum_{n=2}^{p-1} a_n z^n + \frac{1}{p!}z^p, \quad z \in U.$$

For $n \in \{2, 3, ..., p-1\}$ the coefficients a_n can be real or complex numbers, but $a_1 = 1$ and $a_p = \dfrac{1}{p!} \in \mathbb{R}$.

Proposition 5.15. *Let $p \in \mathbb{N}^*$. In general, $R_{p+1} \not\subseteq R_p$.*

For $p \in \mathbb{N}^*$ we can find a function f which belongs to the class R_{p+1}, but does not belong to the class R_p. We present also two examples to illustrate this statement.

Example 5.16. Let $f : U \to \mathbb{C}$ be given by $f(z) = z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3$, $z \in U$. Then $f \in R_2$, but $f \notin R_1$.

Proof. Indeed, we have

$$f(0) = 0, \quad f'(z) = 1 + z + \frac{z^2}{2} \quad \text{and} \quad f''(z) = 1 + z, \quad z \in U.$$

Then

$$f'(0) = f''(0) = 1 \quad \text{and} \quad \text{Re}f''(z) = 1 + \text{Re}z > 0, \quad z \in U.$$

Hence, in view of definition, $f \in R_2$. But,

$$\text{Re}f'(z) = 1 + \text{Re}z + \frac{1}{2}\text{Re}z^2 > -\frac{1}{2}, \quad z \in U.$$

Then $\text{Re}f'(z) \not> 0$, $z \in U$ and hence $f \notin R_1$. □

Example 5.17. Let $p \geq 2$ and let $f : U \to \mathbb{C}$ be given by $f(z) = z + \frac{1}{(p+1)!}z^{p+1}$, $z \in U$. Then $f \in R_{p+1}$, but $f \notin R_p$.

Remark 5.18. Let $p \in \mathbb{N}^*$. Then

1. $R_p \not\subseteq R_{p+1}$;
2. $R_p \not\supseteq R_{p+1}$;
3. $R_p \cap R_{p+1} \neq \emptyset$.

Remark 5.19. Let $p \geq 2$ and consider the polynomial

$$q(z) = z + a_2 z^2 + a_3 z^3 + ... + a_{p-1} z^{p-1} + a_p z^p, \quad z \in U.$$

Then $q \in R_p$ if and only if $a_p = \dfrac{1}{p!}$.

5.1. Structure formula for $p = 2$ and $p = 3$

Proposition 5.20. *Let $f : U \to \mathbb{C}$. Then $f \in R_2$ if and only if there exists a function μ measurable on $[0, 2\pi]$ such that*

$$f(z) = -\frac{z^2}{2} - 2 \cdot \int_0^{2\pi} e^{it}\big[(z - e^{it})\log(1 - ze^{-it}) - z\big]d\mu(t),$$

where $\log 1 = 0$.

Proof. According to Remark 5.2 we have that $f'' \in \mathcal{P}$. Hence, in view of Herglotz formula we obtain that

$$f''(z) = \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} d\mu(t), \quad \mu \in [0, 2\pi].$$

Then,

$$f(z) = \int_0^z \left(\int_0^z \int_0^{2\pi} \frac{e^{it} + s}{e^{it} - s} d\mu(t) ds \right) ds = \int_0^z \left[\int_0^{2\pi} \left(\int_0^z \frac{e^{it} + s}{e^{it} - s} ds \right) d\mu(t) \right] ds.$$

Using [7, Theorem 3.2.2] we know that

$$f(z) = \int_0^z \big[-\zeta - 2\int_0^{2\pi} e^{it}\log(1 - \zeta e^{-it}) d\mu(t)\big] d\zeta,$$

so we obtain

$$f(z) = -\frac{z^2}{2} - 2 \cdot \int_0^{2\pi} e^{it}\big[(z - e^{it})\log(1 - ze^{-it}) - z\big]d\mu(t). \qquad \square$$

Remark 5.21. It is possible to obtain a structure formula for the case $p = 3$:

$$f(z) = -\frac{z^3}{6} - 2 \cdot \int_0^{2\pi} e^{it}\left[\left(\frac{z^2}{2} + e^{-it} - e^{it}(z - e^{it}) \right) \log(1 - ze^{-it}) - 2z - \frac{z^2}{2} \right] d\mu(t),$$

where $\log 1 = 0$.

6. The class $R_p(\alpha)$

Let $\alpha \in [0, 1)$ and $p \in \mathbb{N}^*$. Then we define

$$R_p(\alpha) = \{ f \in \mathcal{H}(U) : f(0) = 0, f'(0) = 1, f^{(p)}(0) = 1, \mathrm{Re}\big[f^{(p)}(z)\big] > \alpha, z \in U \}.$$

the *class of normalized functions whose p-th derivative has positive real part of order* α.

Remark 6.1. Let $\alpha \in [0, 1)$ and $p \in \mathbb{N}^*$. Then $f \in R_p(\alpha)$ if and only if $g \in \mathcal{P}$, where $g : U \to \mathbb{C}$ is given by

$$g(z) = \frac{f^{(p)}(z) - \alpha}{1 - \alpha}, \quad z \in U.$$

Proposition 6.2. *Let $\alpha \in [0, 1)$ and $p \in \mathbb{N}^*$. If $f \in R_p(\alpha)$, then the following relation hold:*

$$|a_n| \le \frac{2(1 - \alpha)(n - p)!}{n!}, \quad n \ge p, \tag{6.1}$$

Proof. Similar to the proof of Proposition 4.2. $\square$

Theorem 6.3. *Let $\alpha \in [0,1)$ and $p \in \mathbb{N}^*$. If $f \in R_p(\alpha)$, then the following estimate hold for all $k \in \mathbb{N}^*$ with $k \geq p$:*

$$|f^{(k)}(z)| \leq \frac{2(1-\alpha)(k-p)!}{(1-|z|)^{k-p+1}}, \quad z \in U. \tag{6.2}$$

Proof. Similar to the proof of Theorem 4.5. □

Remark 6.4. If $\alpha = 0$, then $R_p(0) = R_p$ and we obtain Proposition 5.3 and Theorem 5.4 from previous section. If, in addition, $p = 1$, then $R_1(0) = R$ and we obtain the coefficient estimates, respectively the growth and distortion result regarded to the class R.

References

[1] Duren, P.L., *Univalent Functions*, Springer-Verlag, Berlin and New York, 1983.

[2] Goodman, A.W., *Univalent Functions*, Vols. I and II, Mariner Publ. Co., Tampa, Florida, 1983.

[3] Graham, I., Kohr, G., *Geometric Function Theory in One and Higher Dimensions*, Marcel Deker Inc., New York, 2003.

[4] Krishna, D.V., RamReddy, T., *Coefficient inequality for a function whose derivative has a positive real part of order α*, Math. Bohem., **140**(2015), 43-52.

[5] Krishna, D.V., Venkateswarlu, B., RamReddy, T., *Third Hankel determinant for bounded turning functions of order alpha*, J. Nigerian Math. Soc., **34**(2015), 121-127.

[6] MacGregor, T.H., *Functions whose derivative has a positive real part*, Trans. Amer. Math. Soc., **104**(1962), 532-537.

[7] Mocanu, P.T., Bulboacă, T., Sălăgean, G.Ş., *Geometric Theory of Univalent Functions*, (in romanian), House of the Book of Science, Cluj-Napoca, 2006.

[8] Thomas, D.K., *On functions whose derivative has positive real part*, Proc. Amer. Math. Soc., **98**(1986), 68-70.

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Certain class of m-fold functions by applying Faber polynomial expansions

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Abstract. In this paper, we introduce new class $\Sigma_m(\mu, \lambda, \gamma, \beta)$ of m-fold symmetric bi-univalent functions. Furthermore, we use the Faber polynomial expansions to find upper bounds for the general coefficients $|a_{mk+1}| (k \geqq 2)$ of functions in the class $\Sigma_m(\mu, \lambda, \gamma, \beta)$. Moreover, we obtain estimates for the initial coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ for functions in this class. The results presented in this paper would generalize and improve some recent works.

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1. Introduction

Let $\mathcal{A}$ be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : \ |z| < 1\}$.

We let $\mathcal{S}$ to denote the class of functions $f \in \mathcal{A}$ which are univalent in $\mathbb{U}$ (see details [5, 7]).

Every function $f \in \mathcal{S}$ has an inverse f^{-1}, which is defined by

$$f^{-1}(f(z)) = z \ (z \in \mathbb{U})$$

and

$$f(f^{-1}(w)) = w \ \left(|w| < r_0(f), \ r_0(f) \geqq \frac{1}{4} \right).$$

In fact, the inverse function f^{-1} is given by

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots . \tag{1.2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$, if both f and f^{-1} are univalent in $\mathbb{U}$. Let $\sigma_{\mathcal{B}}$ denote the class of bi-univalent functions in $\mathbb{U}$. In fact that this widely-cited work by Srivastava et al. [18] actually revived the study of analytic and bi-univalent functions in recent years and that it has led to a flood of papers on the subject by (for example) Srivastava et al. [16, 17, 18, 21, 22] and others [6, 23] .

Also the coefficients of $g = f^{-1}$, the inverse map of $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}$, are given by the Faber polynomial [9] (see also [1, 2]):

$$g(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, \cdots, a_n) w^n, \tag{1.3}$$

where

$$\begin{aligned} K_{n-1}^{-n} = & \frac{(-n)!}{(-2n+1)!(n-1)!} a_2^{n-1} + \frac{(-n)!}{(2(-n+1))!(n-3)!} a_2^{n-3} a_3 \\ & + \frac{(-n)!}{(-2n+3)!(n-4)!} a_2^{n-4} a_4 + \frac{(-n)!}{(2(-n+2))!(n-5)!} a_2^{n-5} \\ \times [a_5 + (-n+2) a_3^2] & + \frac{(-n)!}{(-2n+5)!(n-6)!} a_2^{n-6} [a_6 + (-2n+5) a_3 a_4] \\ & + \sum_{j \geqq 7} a_2^{n-j} V_j, \end{aligned}$$

such that V_j with $7 \leqq j \leqq n$ is a homogeneous polynomial in the variables $a_2, a_3, \cdots, a_n$.

In particular, the first three terms of K_{n-1}^{-n} are

$$\frac{1}{2} K_1^{-2} = -a_2, \ \frac{1}{3} K_2^{-3} = 2a_2^2 - a_3, \ \frac{1}{4} K_3^{-4} = -(5a_2^3 - 5a_2 a_3 + a_4).$$

In general, for $n \geq 1$ and for any $\mu \in \mathbb{R}$, an expansion of K_n^{μ} is (see for details [1, 20] or [2])

$$K_n^{\mu}(a_2, \ldots, a_{n+1}) = \mu a_{n+1} + \frac{\mu(\mu-1)}{2} D_n^2 + \frac{\mu!}{(\mu-3)!3!} D_n^3 + \cdots + \frac{\mu!}{(\mu-n)!n!} D_n^n \tag{1.4}$$

where

$$D_n^m = D_n^m(a_2, a_3, \ldots, a_{n+1}) = \sum_{n=1}^{\infty} \frac{m!(a_2)^{\nu_1} \cdots (a_{n+1})^{\nu_n}}{\nu_1! \cdots \nu_n!}, \tag{1.5}$$

the sum is taken over all non negative integers $\nu_1, \nu_2, \cdots, \nu_n$ satisfying

$$\begin{cases} \nu_1 + \nu_2 + \cdots + \nu_n = m, \\ \nu_1 + 2\nu_2 + \cdots + n\nu_n = n \end{cases}$$

The polynomials D_n^m proved by Todorov [20].
It is clear that $D_n^n(a_2, a_3, \cdots, a_{n+1}) = a_2^n$ $(n \geq 1)$, [20, Page 2].

For each function $f \in \mathcal{S}$ function, the function

$$h(z) = \sqrt[m]{f(z^m)} \tag{1.6}$$

is univalent and maps the unit disk $\mathbb{U}$ into a region with m-fold symmetry. A function is said to be m-fold symmetric (see [14, 15]) if it has the following normalized form

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1} \ (z \in \mathbb{U}, m \in \mathbb{N}). \tag{1.7}$$

We denote by $\mathcal{S}_m$ the class of m-fold symmetric univalent functions in $\mathbb{U}$.

The functions in the class $\mathcal{S}$ are said to be one-fold symmetric. The normalized form of f is given as in (1.7) and the series expansion for f^{-1}, which has been recently proven by Srivastava et al. [19], is given as follows:

$$f^{-1}(w) = w + \sum_{k=1}^{\infty} A_{mk+1} w^{mk+1} \tag{1.8}$$

$$\begin{aligned} &= w - a_{m+1}w^{m+1} + [(m+1)a_{m+1}^2 - a_{2m+1}]w^{2m+1} \\ &-[\frac{1}{2}(m+1)(3m+2)a_{m+1}^3 - (3m+2)a_{m+1}a_{2m+1} + a_{3m+1}]w^{3m+1} + \cdots . \end{aligned} \tag{1.9}$$

We denote by Σ_m the class of m-fold symmetric bi-univalent functions in $\mathbb{U}$. Thus, when $m = 1$, the formula (1.9) coincides with the formula (1.2). Some examples of m-fold symmetric bi-univalent functions are given as follows:

$$\left(\frac{z^m}{1-z^m}\right)^{\frac{1}{m}}, \ \left[\frac{1}{2}\log\left(\frac{1+z^m}{1-z^m}\right)^{\frac{1}{m}}\right] \ \textit{and} \ [-\log(1-z^m)]^{\frac{1}{m}}$$

with the corresponding inverse functions

$$\left(\frac{w^m}{1+w^m}\right)^{\frac{1}{m}}, \ \left(\frac{e^{2w^m}-1}{e^{2w^m}+1}\right)^{\frac{1}{m}} \ \textit{and} \ \left(\frac{e^{w^m}-1}{e^{w^m}}\right)^{\frac{1}{m}},$$

respectively.

In this work, we introduce new class $\Sigma_m(\mu, \lambda, \gamma, \beta)$ of m-fold symmetric bi-univalent functions defined on $\mathbb{U}$ and use the Faber polynomial expansions to obtain the general coefficients $a_{mk+1}(k \geqq 2)$ of m-fold bi-univalent functions in this class. Also, we gain estimates for the general coefficients and early coefficients of functions belonging to this class. We show that the results would improve some of the previouse works like Hamidi and Jahangiri [11, 12, 13], Eker [8], Srivastava et al. [18, 19], Çağlar et al. [6], Frasin and Aouf [10] and Altinkaya and Yalçin [3].

2. Preliminary results

For finding the coefficients for functions belonging to the class $\Sigma_m(\mu, \lambda, \gamma, \beta)$, we need the following lemmas and remarks.

Lemma 2.1. [1, 2] *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}$. *Then for any* $\mu \in \mathbb{R}$, *there are the polynomials* K_n^μ, *such that*

$$\left(\frac{f(z)}{z}\right)^\mu = 1 + \sum_{n=1}^{\infty} K_n^\mu(a_2, \cdots, a_{n+1}) z^n,$$

where K_n^μ *given by* (1.4).

In particular

$$K_1^\mu(a_2) = \mu a_2, \quad K_2^\mu(a_2, a_3) = \mu a_3 + \frac{\mu(\mu-1)}{2} a_2^2$$

and

$$K_3^\mu(a_2, a_3, a_4) = \mu a_4 + \mu(\alpha - 1) a_2 a_3 + \frac{\mu(\alpha-1)(\mu-2)}{3!} a_2^3.$$

Remark 2.2. Let $f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1} \in \mathcal{S}_m$. Then for any $\mu \in \mathbb{R}$, there are the polynomials K_k^μ, such that

$$\left(\frac{f(z)}{z}\right)^\mu = 1 + \sum_{k=1}^{\infty} K_k^\mu(a_{m+1}, \cdots, a_{mk+1}) z^{mk}.$$

Proof. The proof has been satisfied from $f(z) \in \mathcal{S}$, and Lemma 2.1. □

Case 2.3. *In special case, if* $a_{m+1} = \cdots = a_{m(k-1)+1} = 0$, *then*

$$K_i^\mu(a_{m+1}, \cdots, a_{mi+1}) = 0 \ ; \ 1 \leqq i \leqq k-1$$

and

$$K_k^\mu(a_{m+1}, \cdots, a_{mk+1}) = \mu a_{mk+1}.$$

Lemma 2.4. [4] *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathcal{S}$. *Then*

$$\frac{zf'(z)}{f(z)} = 1 - \sum_{k=1}^{\infty} F_k(a_2, \cdots, a_{k+1}) z^k,$$

where $F_k(a_2, a_3, \cdots, a_{k+1})$ *is a Faber polynomial of degree* k,

$$F_k(a_2, a_3, \cdots, a_{k+1}) = \sum_{i_1 + 2i_2 + \cdots + ki_k = k} A_{(i_1, i_2, \cdots, i_k)} a_2^{i_1} a_3^{i_2} \cdots a_{k+1}^{i_k} \tag{2.1}$$

and

$$A_{(i_1, i_2, \cdots, i_k)} := (-1)^{k + 2i_1 + 3i_2 + \cdots + (k+1)i_k} \frac{(i_1 + i_2 + \cdots + i_k - 1)! k}{i_1! i_2! \cdots i_k!}. \tag{2.2}$$

The first Faber polynomials $F_k(a_2, a_3, \cdots, a_{k+1})$ *are given by:*

$$F_1(a_2) = -a_2, \ F_2(a_2, a_3) = a_2^2 - 2a_3 \text{ and } F_3(a_2, a_3, a_4) = -a_2^3 + 3a_2a_3 - 3a_4.$$

Lemma 2.5. *Let* $f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1} \in \mathcal{S}_m$*. Then we can write,*

$$\frac{zf'(z)}{f(z)} = 1 - \sum_{k=1}^{\infty} T_{mk}(a_{m+1}, \cdots, a_{mk+1}) z^{mk},$$

where

$$T_{mk}(a_{m+1}, \cdots, a_{mk+1}) = F_{mk}(\underbrace{0, \cdots, 0, a_{m+1}, 0, \cdots, 0, a_{mk+1}}_{mk})$$

$$= \sum_{mi_m + 2mi_{2m} + \cdots + mki_{mk} = mk} A_{(i_1, i_2, \ldots, i_{mk})} a_{m+1}^{i_m} a_{2m+1}^{i_{2m}} \cdots a_{mk+1}^{i_{mk}}.$$

Proof. By using Lemma 2.4 for function $f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1} \in \mathcal{S}_m$, we have

$$\frac{zf'(z)}{f(z)} = 1 - \sum_{k \geqq 1} F_k z^k.$$

Suppose that $t, j \in \mathbb{N}$ and $1 \leqq j \leq m - 1$. We consider three cases for k.

(i) If $1 \leqq k \leq m - 1$, then $F_k(\underbrace{0, \cdots, 0}_{k}) = 0$.

(ii) If $k = tm$, then, we have

$$F_{tm}(\underbrace{0, \cdots, 0, a_{m+1}, 0, \cdots, 0, a_{2m+1}, 0, \cdots, 0, a_{tm+1}}_{tm})$$

$$= \sum_{mi_m + 2mi_{2m} + \cdots + tmi_{tm} = tm} A_{(i_1, i_2, \ldots, i_{tm})} a_{m+1}^{i_m} a_{2m+1}^{i_{2m}} \cdots a_{tm+1}^{i_{tm}}.$$

(iii) If $k = tm + j$, then

$$F_{tm+j}(\underbrace{0, \cdots, 0, a_{m+1}, 0, \cdots, 0, a_{2m+1}, 0, \cdots, 0, a_{tm+1}, \underbrace{0, \cdots, 0}_{j}}_{tm+j})$$

$$= \sum_{mi_m + 2mi_{2m} + \cdots + tmi_{tm} = tm+j} A_{(i_1, i_2, \ldots, i_{tm+j})} a_{m+1}^{i_m} a_{2m+1}^{i_{2m}} \cdots a_{tm+1}^{i_{tm}}.$$

Since the equation

$$mi_m + 2mi_{2m} + \cdots + tmi_{tm} = tm + j,$$

doesn't have positive integer solution, so

$$F_{tm+j}(0, \cdots, 0, a_{m+1}, 0, \cdots, 0, a_{2m+1}, 0, \cdots, 0, a_{tm+1}, 0, \cdots, 0) = 0. \qquad \square$$

Case 2.6. *In special case, if* $a_{m+1} = \cdots = a_{m(k-1)+1} = 0$*, then*

$$T_{mi}(a_{m+1}, \cdots, a_{mi+1}) = 0 \ ; \ 1 \leqq i \leqq k - 1$$

and

$$T_{mk}(a_{m+1}, \cdots, a_{mk+1}) = -mka_{mk+1}.$$

Lemma 2.7. *Let* $f(z)=z+\sum_{k=1}^{\infty}a_{mk+1}z^{mk+1}\in\mathcal{S}_m$*, then for every* $\mu\geqq 0$*, we have the following expansion,*

$$(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu}+\lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}=1+\sum_{k=1}^{\infty}L_k(a_{m+1},\cdots,a_{mk+1})z^{mk}$$

$$=\begin{cases}\left(\frac{f(z)}{z}\right)^{\mu}+\frac{\lambda z}{\mu}\frac{d}{dz}\left(\frac{f(z)}{z}\right)^{\mu}=1+\sum_{k=1}^{\infty}\frac{\mu+\lambda mk}{\mu}K_k^{\mu}(a_{m+1},\cdots,a_{mk+1})z^{mk};\mu>0\\ \\ \lambda\frac{zf'(z)}{f(z)}+1-\lambda=1-\sum_{k=1}^{\infty}\lambda T_{mk}(a_{m+1},\cdots,a_{mk+1})z^{mk}\ ;\ \mu=0.\end{cases}$$

Proof. The proof has been satisfied from Remark 2.2, and Lemma 2.5. □

Lemma 2.8. [15] *If* $p\in\mathcal{P}$*, then* $|c_k|\leqq 2$ *for each* k*, where* $\mathcal{P}$ *is the family of all functions* p *analytic in* $\mathbb{U}$ *for which* $Re(p(z))>0$ *where*

$$p(z)=1+c_1z+c_2z^2+c_3z^3+\cdots.$$

3. Class $\Sigma_m(\mu,\lambda,\gamma,\beta)$

In this section, we introduce and investigate class $\Sigma_m(\mu,\lambda,\gamma,\beta)$ of m-fold symmetric bi-univalent functions defined on $\mathbb{U}$.

Definition 3.1. A function f given by (1.7) is said to be in the class $\Sigma_m\ (\mu,\lambda,\gamma,\beta)$ $(\mu\geqq 0,\ \lambda\geqq 1,\ \gamma\in\mathbb{C}-\{0\},\ 0\leqq\beta<1)$, if the following conditions are satisfied:

$$f\in\Sigma_m,\ \Re\left(1+\frac{1}{\gamma}\left[(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu}+\lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}-1\right]\right)>\beta\ (z\in\mathbb{U})$$

and

$$\Re\left(1+\frac{1}{\gamma}\left[(1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu}+\lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1}-1\right]\right)>\beta\ (w\in\mathbb{U}),$$

where the function g is the inverse of f given by (1.8).

Remark 3.2. There are some options of the parameters γ,λ and μ which would provide interesting classes of m-fold symmetric bi-univalent functions. For example,

(I) By putting $\gamma=1$; the class $\Sigma_m(\mu,\lambda,\gamma,\beta)$ reduces to the class $\mathcal{N}^{\mu}_{\Sigma_m}(\beta,\lambda)$, which was considered by Altinkaya and Yalçn [3].

(II) By putting $\gamma=1$ and $\lambda=1$ and $\mu=0$; the class $\Sigma_m(\mu,\lambda,\gamma,\beta)$ reduces to the class of m-fold symmetric bi-starlike of order β, which was considered by Jahangiri and Hamidi [11].

(III) By putting $\gamma=1$ and $\mu=1$; the class $\Sigma_m(\mu,\lambda,\gamma,\beta)$ reduces to the class $\mathcal{A}^{\lambda}_{\Sigma,m}(\beta)$, which was considered by Eker [8].

(IV) By putting $\gamma=1$, $\mu=1$ and $\lambda=1$; the class $\Sigma_m(\mu,\lambda,\gamma,\beta)$ reduces to the class $\mathcal{H}_{\Sigma,m}(\beta)$, which was considered by Srivastava et al. [19].

Remark 3.3. For one-fold symmetric bi-univalent functions, we denote the class $\Sigma_1(\mu,\lambda,\gamma,\beta)=\Sigma(\mu,\lambda,\gamma,\beta)$. Special cases of the parameters γ,λ and μ which would provide interesting classes of bi-univalent functions as follows:

(I) By putting $\gamma=1$; the class $\Sigma(\mu,\lambda,\gamma,\beta)$ reduces to the class $\mathcal{N}_{\Sigma}^{\mu}(\beta,\lambda)$, which was considered by Çağlar et al. [6].
(II) By putting $\gamma=1$, $\lambda=1$ and $\mu=0$; the class $\Sigma(\mu,\lambda,\gamma,\beta)$ reduces to the class $\mathcal{S}_{\Sigma}^{*}(\beta)$, which was studied by Brannan and Taha [5].
(III) By putting $\gamma=1$ and $\mu=1$; the class $\Sigma(\mu,\lambda,\gamma,\beta)$ reduces to the class $\mathcal{B}_{\Sigma}(\beta,\lambda)$, which was studied by Frasin, Aouf [10] and Hamidi, Jahangiri [13].
(IV) By putting $\gamma=1$, $\lambda=1$ and $\mu=1$; the class $\Sigma(\mu,\lambda,\gamma,\beta)$ reduces to the class $\mathcal{H}_{\Sigma}(\beta)$, which was studied by Srivastava et al. [18].
(V) By putting $\gamma=1$, $\lambda=1$ and $0\leqq\mu<1$; the class $\Sigma(\mu,\lambda,\gamma,\beta)$ reduces to the class of bi-Bazilevic of order β and type μ, which was studied by Jahangiri and Hamidi [12].

Theorem 3.4. *Let f given by* (1.7) *be in the class* $\Sigma_m(\mu,\lambda,\gamma,\beta)$ $(\mu\geqq 0,\ \lambda\geqq 1,\ \gamma\in\mathbb{C}-\{0\},\ 0\leqq\beta<1)$. *If $a_{m+1}=\cdots=a_{m(k-1)+1}=0$, then*

$$|a_{mk+1}|\leqq\frac{2|\gamma|(1-\beta)}{\mu+\lambda mk}\ ;\ (k\geqq 2).$$

Proof. By using Lemma 2.7, for m-fold symmetric bi-univalent functions f of the form (1.7), we have:

$$\begin{aligned}1+\frac{1}{\gamma}&\left[(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu}+\lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}-1\right]\\&=1+\sum_{k=1}^{\infty}\frac{L_k(a_{m+1},\cdots,a_{mk+1})}{\gamma}z^{mk}.\end{aligned}\tag{3.1}$$

Similarly for its inverse map, $g(w)=f^{-1}(w)=w+\sum_{k=1}^{\infty}A_{mk+1}w^{mk+1}$, we have:

$$\begin{aligned}1+\frac{1}{\gamma}&\left[(1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu}+\lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1}-1\right]\\&=1+\sum_{k=1}^{\infty}\frac{L_k(A_{m+1},\cdots,A_{mk+1})}{\gamma}w^{mk}.\end{aligned}\tag{3.2}$$

Furthermore, since $f\in\Sigma_m(\mu,\lambda,\gamma,\beta)$, by definition, there exist two positive real-part functions

$$p(z)=1+\sum_{k=1}^{\infty}c_{mk}z^{mk}$$

and

$$q(w)=1+\sum_{k=1}^{\infty}d_{mk}w^{mk},$$

where $Re\ p(z) > 0$ and $Re\ q(w) > 0$ in $\mathbb{U}$ so that:

$$\begin{aligned} 1 + \frac{1}{\gamma}\left[(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu} + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} - 1\right] \\ = 1 + (1-\beta)\sum_{k=1}^{\infty} K_k^1(c_m, \cdots, c_{mk}) z^{mk} \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} 1 + \frac{1}{\gamma}\left[(1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu} + \lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1} - 1\right] \\ = 1 + (1-\beta)\sum_{k=1}^{\infty} K_k^1(d_m, \cdots, d_{mk}) w^{mk}. \end{aligned} \tag{3.4}$$

Equating the corresponding coefficients of (3.1) and (3.3), we have:

$$\frac{L_k(a_{m+1}, \cdots, a_{mk+1})}{\gamma} = (1-\beta)K_k^1(c_m, \cdots, c_{mk}). \tag{3.5}$$

Similarly, from (3.2) and (3.4) we obtain:

$$\frac{L_k(A_{m+1}, \cdots, A_{mk+1})}{\gamma} = (1-\beta)K_k^1(d_m, \cdots, d_{mk}). \tag{3.6}$$

Note that for $a_{mi+1} = 0$ $(1 \leqq i \leqq k-1)$, we have $A_{mi+1} = 0$ $(1 \leqq i \leqq k-1)$ and $A_{mk+1} = -a_{mk+1}$.

For $\mu > 0$, by using Case 2.3 and Lemma 2.7 the equalities (3.5), (3.6) can be written as follows:

$$\frac{\mu + \lambda mk}{\gamma} a_{mk+1} = (1-\beta)c_{mk},$$

$$-\frac{\mu + \lambda mk}{\gamma} a_{mk+1} = (1-\beta)d_{mk}.$$

By getting the absolute values of either of the above two equations and applying the Lemma 2.8, we get:

$$|a_{mk+1}| = \frac{|\gamma|(1-\beta)|c_{mk}|}{\mu + \lambda mk} = \frac{|\gamma|(1-\beta)|d_{mk}|}{\mu + \lambda mk} \leqq \frac{2|\gamma|(1-\beta)}{\mu + \lambda mk}.$$

For $\mu = 0$, by using Case 2.6 and Lemma 2.7 the equalities (3.5), (3.6) can be written as follows:

$$\frac{\lambda mk}{\gamma} a_{mk+1} = (1-\beta)c_{mk},$$

$$-\frac{\lambda mk}{\gamma} a_{mk+1} = (1-\beta)d_{mk}.$$

By getting the absolute values of either of the above two equations and applying the Lemma 2.8, we get:

$$|a_{mk+1}| = \frac{|\gamma|(1-\beta)|c_{mk}|}{\lambda mk} = \frac{|\gamma|(1-\beta)|d_{mk}|}{\lambda mk} \leqq \frac{2|\gamma|(1-\beta)}{\lambda mk}. \qquad \square$$

We obtain estimates for the initial coefficients of functions $f \in \Sigma_m(\mu, \lambda, \gamma, \beta)$.

Theorem 3.5. *Let f given by (1.7) be in the class $\Sigma_m(\mu,\lambda,\gamma,\beta)$. Then*

$$|a_{m+1}| \leqq \min\left\{\frac{2|\gamma|(1-\beta)}{\mu+\lambda m}, 2\sqrt{\frac{|\gamma|(1-\beta)}{(\mu+2\lambda m)(m+\mu)}}\right\}$$

and

$$|a_{2m+1}| \leqq \left\{\frac{2|\gamma|^2(1-\beta)^2(m+1)}{(\mu+\lambda m)^2} + \frac{2|\gamma|(1-\beta)}{\mu+2\lambda m}, \frac{|\gamma|(1-\beta)}{\mu+2\lambda m}\left[\frac{2m+\mu+1+|1-\mu|}{m+\mu}\right]\right\}.$$

Proof. For $f(z) = z + a_{m+1}z^{m+1} + a_{2m+1}z^{2m+1} + \cdots$, we get

$$1+\frac{1}{\gamma}\left[(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu} + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1} - 1\right] \tag{3.7}$$

$$= 1 + \frac{(\mu+\lambda m)}{\gamma}a_{m+1}z^{m+1} + \frac{(\mu+2\lambda m)}{\gamma}\left(a_{2m+1} + \frac{(\mu-1)}{2}\right)a_{m+1}z^{2m+1} + \cdots$$

and for

$$g(w) = f^{-1}(w) = w - a_{m+1}w^{m+1} + [(m+1)a_{m+1}^2 - a_{2m+1}]w^{2m+1} + \cdots,$$

we get

$$1+\frac{1}{\gamma}\left[(1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu} + \lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1} - 1\right] = 1 - \frac{(\mu+\lambda m)}{\gamma}a_{m+1}w^{m+1}$$

$$+ \frac{(\mu+2\lambda m)}{\gamma}\left(-a_{2m+1} + \frac{(2m+\mu+1)}{2}a_{m+1}^2\right)w^{2m+1} + \cdots. \tag{3.8}$$

Comparing the corresponding coefficients of (3.3) and (3.7), we have

$$(\mu+\lambda m)a_{m+1} = \gamma(1-\beta)c_m, \tag{3.9}$$

$$(\mu+2\lambda m)\left(a_{2m+1} + \frac{\mu-1}{2}a_{m+1}^2\right) = \gamma(1-\beta)c_{2m}. \tag{3.10}$$

Similarly, by comparing the corresponding coefficients of (3.4) and (3.8), we have

$$-(\mu+\lambda m)a_{m+1} = \gamma(1-\beta)d_m, \tag{3.11}$$

$$(\mu+2\lambda m)\left(-a_{2m+1} + \frac{(2m+\mu+1)}{2}a_{m+1}^2\right) = \gamma(1-\beta)d_{2m}. \tag{3.12}$$

From (3.9) and (3.11), we get

$$c_m = -d_m \tag{3.13}$$

and

$$a_{m+1}^2 = \frac{\gamma^2(1-\beta)^2(c_m^2+d_m^2)}{2(\mu+\lambda m)^2}. \tag{3.14}$$

Adding (3.10) and (3.12), we get

$$a_{m+1}^2 = \frac{\gamma(1-\beta)(c_{2m}+d_{2m})}{(\mu+2\lambda m)(m+\mu)}. \tag{3.15}$$

Therefore, we find from the equations (3.14), (3.15) and Lemma 2.8 that

$$|a_{m+1}| \leqq \frac{2|\gamma|(1-\beta)}{\mu+\lambda m}$$

and

$$|a_{m+1}| \leqq 2\sqrt{\frac{|\gamma|(1-\beta)}{(\mu+2\lambda m)(m+\mu)}}.$$

respectively. So we get the desired estimate on the coefficient $|a_{m+1}|$.
Next, in order to find the bound on the coefficient $|a_{2m+1}|$, we subtract (3.12) from (3.10). We thus get

$$a_{2m+1} = \frac{(m+1)}{2}a_{m+1}^2 + \frac{\gamma(1-\beta)(c_{2m}-d_{2m})}{2(\mu+2\lambda m)}. \tag{3.16}$$

Upon substituting the value of a_{m+1}^2 from (3.14) into (3.16), it follows that

$$a_{2m+1} = \frac{\gamma^2(1-\beta)^2(m+1)(c_m^2+d_m^2)}{4(\mu+\lambda m)^2} + \frac{\gamma(1-\beta)(c_{2m}-d_{2m})}{2(\mu+2\lambda m)}. \tag{3.17}$$

We thus find that

$$|a_{2m+1}| \leqq \frac{2|\gamma|^2(1-\beta)^2(m+1)}{(\mu+\lambda m)^2} + \frac{2|\gamma|(1-\beta)}{\mu+2\lambda m}.$$

On the other hand, upon substituting the value of a_{m+1}^2 from (3.15) into (3.16), it follows that

$$a_{2m+1} = \frac{\gamma(1-\beta)}{2(\mu+2\lambda m)}\left[\frac{(2m+\mu+1)c_{2m}+(1-\mu)d_{2m}}{m+\mu}\right]. \tag{3.18}$$

Consequently, we have

$$|a_{2m+1}| \leqq \frac{|\gamma|(1-\beta)}{\mu+2\lambda m}\left[\frac{2m+\mu+1+|1-\mu|}{m+\mu}\right].$$

This evidently completes the proof of Theorem 3.5. □

4. Corollaries and consequences

By setting $\gamma = 1$ in Theorem 3.4, we conclude the following result.

Corollary 4.1. *Let f given by* (1.7) *be in the class* $\mathcal{N}_{\Sigma_m}^{\mu}(\beta,\lambda)$.
If $a_{m+1} = \cdots = a_{m(k-1)+1} = 0$, *then*

$$|a_{mk+1}| \leqq \frac{2(1-\beta)}{\mu+\lambda mk}\ ;\ (k \geqq 2).$$

By setting $m = 1$ in Corollary 4.1, we conclude the following result.

Corollary 4.2. *Let f given by* (1.1) *be in the class* $\mathcal{N}_{\Sigma}^{\mu}(\beta,\lambda)$. *If* $a_2 = \cdots = a_k = 0$, *then*

$$|a_{k+1}| \leqq \frac{2(1-\beta)}{\mu+\lambda k}\ ;\ (k \geqq 2).$$

By setting $\lambda = 1$ and $\mu = 0$ in Corollary 4.1, we conclude the following result.

Corollary 4.3. *Let f given by (1.7) be m-fold symmetric bi-starlike of order β. If $a_{m+1} = \cdots = a_{m(k-1)+1} = 0$, then*

$$|a_{mk+1}| \leqq \frac{2(1-\beta)}{mk} \; ; \; (k \geqq 2).$$

By setting $m = 1$ in Corollary 4.3, we conclude the following result.

Corollary 4.4. *Let f given by (1.1) be in the class $\mathcal{S}^*_\Sigma(\beta)$. If $a_2 = \cdots = a_k = 0$, then*

$$|a_{k+1}| \leqq \frac{2(1-\beta)}{k} \; ; \; (k \geqq 2).$$

By setting $\mu = 1$ in Corollary 4.1, we conclude the following result.

Corollary 4.5. *Let f given by (1.7) be in the class $\mathcal{A}^\lambda_{\Sigma,m}(\beta)$. If $a_{m+1} = \cdots = a_{m(k-1)+1} = 0$, then*

$$|a_{mk+1}| \leqq \frac{2(1-\beta)}{1+\lambda mk} \; ; \; (k \geqq 2).$$

By setting $m = 1$ in Corollary 4.5, we conclude the following result.

Corollary 4.6. *Let f given by (1.1) be in the class $\mathcal{B}_\Sigma(\beta, \lambda)$. If $a_2 = \cdots = a_k = 0$, then*

$$|a_{k+1}| \leqq \frac{2(1-\beta)}{1+\lambda k} \; ; \; (k \geqq 2).$$

By setting $\lambda = 1$ in Corollary 4.5, we conclude the following result.

Corollary 4.7. *Let f given by (1.7) be in the class $\mathcal{H}_{\Sigma,m}(\beta)$. If $a_{m+1} = \cdots = a_{m(k-1)+1} = 0$, then*

$$|a_{mk+1}| \leqq \frac{2(1-\beta)}{1+mk} \; ; \; (k \geqq 2).$$

By setting $m = 1$ in Corollary 4.7, we conclude the following result.

Corollary 4.8. *Let f given by (1.1) be in the class $\mathcal{H}_\Sigma(\beta)$. If $a_2 = \cdots = a_k = 0$, then*

$$|a_{k+1}| \leqq \frac{2(1-\beta)}{1+k} \; ; \; (k \geqq 2).$$

By setting $m = 1$, $\lambda = 1$ and $0 \leqq \mu < 1$ in Corollary 4.1, we conclude the following result.

Corollary 4.9. *Let f given by (1.1) be bi-Bazilevic of order β and type μ. If $a_2 = \cdots = a_k = 0$, then*

$$|a_{k+1}| \leqq \frac{2(1-\beta)}{\mu+k} \; ; \; (k \geqq 2).$$

By setting $\gamma = 1$ in Theorem 3.5, we conclude the following result.

Corollary 4.10. *Let f given by (1.7) be in the class $\mathcal{N}_{\Sigma_m}^{\mu}(\beta,\lambda)$. Then*

$$|a_{m+1}| \leqq \min\left\{\frac{2(1-\beta)}{\mu+\lambda m}, 2\sqrt{\frac{1-\beta}{(\mu+2\lambda m)(m+\mu)}}\right\}$$

and

$$|a_{2m+1}| \leqq \left\{\frac{2(1-\beta)^2(m+1)}{(\mu+\lambda m)^2}+\frac{2(1-\beta)}{\mu+2\lambda m}, \frac{(1-\beta)}{\mu+2\lambda m}\left[\frac{2m+\mu+1+|1-\mu|}{m+\mu}\right]\right\}.$$

By setting $m=1$ in Corollary 4.10, we conclude the following result.

Corollary 4.11. *Let f given by (1.1) be in the class $\mathcal{N}_{\Sigma}^{\mu}(\beta,\lambda)$. Then*

$$|a_2| \leqq \min\left\{\frac{2(1-\beta)}{\mu+\lambda}, 2\sqrt{\frac{1-\beta}{(\mu+2\lambda)(1+\mu)}}\right\}$$

and

$$|a_3| \leqq \left\{\frac{4(1-\beta)^2}{(\mu+\lambda)^2}+\frac{2(1-\beta)}{\mu+2\lambda}, \frac{(1-\beta)}{\mu+2\lambda}\left[\frac{3+\mu+|1-\mu|}{1+\mu}\right]\right\}.$$

By setting $\lambda=1$ and $\mu=0$ in Corollary 4.10, we conclude the following result.

Corollary 4.12. *Let f given by (1.7) be m-fold symmetric bi-starlike of order β. Then*

$$|a_{m+1}| \leqq \begin{cases} \frac{1}{m}\sqrt{2(1-\beta)}\ ;\ 0 \leqq \beta \leqq \frac{1}{2} \\ \\ \frac{2(1-\beta)}{m}\ ;\ \frac{1}{2} \leqq \beta < 1 \end{cases}$$

and

$$|a_{2m+1}| \leqq \begin{cases} \frac{(1-\beta)(m+1)}{m^2}\ ;\ 0 \leqq \beta \leqq \frac{2m+1}{2(m+1)} \\ \\ \frac{2(1-\beta)^2(m+1)}{m^2}+\frac{1-\beta}{m}\ ;\ \frac{2m+1}{2(m+1)} \leqq \beta < 1. \end{cases}$$

By setting $m=1$ in Corollary 4.12, we conclude the following result.

Corollary 4.13. *Let f given by (1.1) be in the class $\mathcal{S}_{\Sigma}^{*}(\beta)$, then*

$$|a_2| \leqq \begin{cases} \sqrt{2(1-\beta)}\ ;\ 0 \leqq \beta \leqq \frac{1}{2} \\ \\ 2(1-\beta)\ ;\ \frac{1}{2} \leqq \beta < 1 \end{cases}$$

and

$$|a_3| \leqq \begin{cases} 2(1-\beta)\ ;\ 0 \leqq \beta \leqq \frac{3}{4} \\ \\ (1-\beta)(5-4\beta)\ ;\ \frac{3}{4} \leqq \beta < 1. \end{cases}$$

By setting $\mu=1$ in Corollary 4.10, we conclude the following result.

Corollary 4.14. *Let f given by (1.7) be in the class $\mathcal{A}_{\Sigma,m}^{\lambda}(\beta)$, then*

$$|a_{m+1}| \leqq \begin{cases} 2\sqrt{\frac{1-\beta}{(1+2\lambda m)(1+m)}}\ ;\ 0 \leqq \beta \leqq 1-\frac{(1+\lambda m)^2}{(1+2\lambda m)(1+m)} \\ \\ \frac{2(1-\beta)}{1+\lambda m}\ ;\ 1-\frac{(1+\lambda m)^2}{(1+2\lambda m)(1+m)} \leqq \beta < 1 \end{cases}$$

and

$$|a_{2m+1}| \leqq \min\left\{\frac{2(1-\beta)^2(m+1)}{(1+\lambda m)^2}+\frac{2(1-\beta)}{1+2\lambda m}, \frac{2(1-\beta)}{1+2\lambda m}\right\} = \frac{2(1-\beta)}{1+2\lambda m}.$$

Remark 4.15. The bounds on $|a_{m+1}|$ and $|a_{2m+1}|$ given in Corollary 4.14 are better than those given by Eker [8, Theorem 2].

By setting $m = 1$ in Corollary 4.14, we conclude the following result.

Corollary 4.16. *Let f given by* (1.1) *be in the class $\mathcal{B}_\Sigma(\beta,\lambda)$, then*

$$|a_2| \leqq \begin{cases} \sqrt{\frac{2(1-\beta)}{1+2\lambda}} \ ; \ 0 \leqq \beta \leqq \frac{1+2\lambda-\lambda^2}{2(1+2\lambda)} \\ \\ \frac{2(1-\beta)}{1+\lambda} \ ; \ \frac{1+2\lambda-\lambda^2}{2(1+2\lambda)} \leqq \beta < 1 \end{cases}$$

and

$$|a_3| \leqq \min\left\{\frac{4(1-\beta)^2}{(1+\lambda)^2}+\frac{2(1-\beta)}{1+2\lambda}, \frac{2(1-\beta)}{1+2\lambda}\right\} = \frac{2(1-\beta)}{1+2\lambda}.$$

Remark 4.17. The bounds on $|a_2|$ and $|a_3|$ given in Corollary 4.16 are better than those given by Frasin and Aouf [10, Theorem 3.2].

By setting $\lambda = 1$ in Corollary 4.14, we conclude the following result.

Corollary 4.18. *Let f given by* (1.7) *be in the class $\mathcal{H}_{\Sigma,m}(\beta)$, then*

$$|a_{m+1}| \leqq \begin{cases} 2\sqrt{\frac{1-\beta}{(1+2m)(1+m)}} \ ; \ 0 \leqq \beta \leqq \frac{m}{2m+1} \\ \\ \frac{2(1-\beta)}{1+m} \ ; \ \frac{m}{2m+1} \leqq \beta < 1 \end{cases}$$

and

$$|a_{2m+1}| \leqq \min\left\{\frac{2(1-\beta)^2}{1+m}+\frac{2(1-\beta)}{1+2m}, \frac{2(1-\beta)}{1+2m}\right\} = \frac{2(1-\beta)}{1+2m}.$$

Remark 4.19. The bounds on $|a_{m+1}|$ and $|a_{2m+1}|$ given in Corollary 4.18 are better than those given by Srivastava et al. [19, Theorem 3].

By setting $m = 1$ in Corollary 4.18, we conclude the following result.

Corollary 4.20. *Let f given by* (1.1) *be in the class $\mathcal{H}_\Sigma(\beta)$, then*

$$|a_2| \leqq \begin{cases} \sqrt{\frac{2(1-\beta)}{3}} \ ; \ 0 \leqq \beta \leqq \frac{1}{3} \\ \\ 1-\beta \ ; \ \frac{1}{3} \leqq \beta < 1 \end{cases}$$

and

$$|a_3| \leqq \min\left\{\frac{(1-\beta)(5-3\beta)}{3}, \frac{2(1-\beta)}{3}\right\} = \frac{2(1-\beta)}{3}.$$

Remark 4.21. The bounds on $|a_2|$ and $|a_3|$ given in Corollary 4.20 are better than those given by Srivastava et al. [18, Theorem 2].

By setting $\lambda = 1$ and $0 \leqq \mu < 1$ in Corollary 4.11, we conclude the following result.

Corollary 4.22. *Let f given by (1.1) be bi-Bazilevic of order β and type μ. Then*

$$|a_2| \leqq \begin{cases} 2\sqrt{\frac{1-\beta}{(\mu+2)(1+\mu)}} \ ; \ 0 \leqq \beta \leqq \frac{1}{2+\mu} \\ \\ \frac{2(1-\beta)}{\mu+1} \ ; \ \frac{1}{2+\mu} \leqq \beta < 1 \end{cases}$$

and

$$|a_3| \leqq \begin{cases} \frac{4(1-\beta)}{(\mu+2)(1+\mu)} \ ; \ 0 \leqq \beta \leqq \frac{1}{2+\mu} \\ \\ \frac{4(1-\beta)^2}{(\mu+1)^2} + \frac{2(1-\beta)}{\mu+2} \ ; \ \frac{1}{2+\mu} \leqq \beta < 1. \end{cases}$$

References

[1] Airault, H., Bouali, A., *Differential calculus on the Faber polynomials*, Bull. Sci. Math., **130**(2006), 179-222.

[2] Airault, H., Ren, J., *An algebra of differential operators and generating functions on the set of univalent functions*, Bull. Sci. Math. **26**(2002), no. 5, 343-367.

[3] Altinkaya, Ş., Yalçin, S., *Coefficient bounds for two new subclasses of m-fold symmetric bi-univalent functions*, Serdica Math. J., **42**(2016), 175-186.

[4] Bouali, A., *Faber polynomials, Cayley-Hamilton equation and Newton symmetric functions*, Bull. Sci. Math., **130**(2006), no. 1, 49-70.

[5] Brannan, D.A., Taha, T.S., *On Some classes of bi-univalent functions*, Stud. Univ. Babeş-Bolyai Math., **31**(1986), no. 2, 70-77.

[6] Çağlar, M., Orhan, H., Yağmur, N., *Coefficient bounds for new subclasses of bi-univalent functions*, Filomat, **27**(2013), no. 7, 1165-1171.

[7] Duren, P.L. , *Univalent Functions*, Grundlehren der Mathematischen Wissenschaften, Band 259, Springer-Verlag, New York, Berlin, Heidelberg and Tokyo, 1983.

[8] Eker, S.S., *Coefficient bounds for subclasses of m-fold symmetric bi-univalent functions*, Turkish J. Math., **40**(2016), no. 3, 641-646.

[9] Faber, G., *Über polynomische Entwickelungen*, Math. Ann., **57**(1903), no. 3, 389-408.

[10] Frasin, B.A., Aouf, M.K., *New subclasses of bi-univalent functions*, Appl. Math. Lett., **24**(2011), 1569-1573.

[11] Hamidi, S.G., Jahangiri, J.M., *Unpredictability of the coefficients of m-fold symmetric bi-starlike functions*, Internat. J. Math., **25**(2014), no. 7, Art. ID 1450064, 8 pages.

[12] Jahangiri, J.M., Hamidi, S.G., *Faber polynomial coefficient estimates for analytic bi-Bazleviç functions*, Mat. Vesnik, **67**(2015), no. 2, 123-129.

[13] Jahangiri, J.M., Hamidi, S.G., *Coefficient estimates for certain classes of bi-univalent functions*, Int. J. Math. Math. Sci., **2013**, Art. ID 190560, 4 pages.

[14] Koepf, W., *Coefficients of symmetric functions of bounded boundary rotation*, Proc. Amer. Math. Soc., **105**(1989), no. 2, 324-329.

[15] Pommerenke, Ch., *Univalent Functions*, Vandenhoeck and Ruprecht, Gottingen, 1975.

[16] Srivastava, H.M., Bansal, D., *Coefficient estimates for a subclass of analytic and bi-univalent functions*, J. Egyptian Math. Soc., **23**(2015), 242-246.

[17] Srivastava, H.M., Gaboury, S., Ghanim, F., *Coefficient estimates for some general subclasses of analytic and bi-univalent functions*, Afr. Mat., **28**(2017), 693-706.

[18] Srivastava, H.M., Mishra, A.K., Gochhayat, P., *Certain subclasses of analytic and bi-univalent functions*, Appl. Math. Lett., **23**(2010), 1188-1192.

[19] Srivastava, H.M., Sivasubramanian, S., Sivakumar, R., *Initial coefficient bounds for a subclass of m-fold symmetric bi-univalent functions*, Tbilisi Math. J., **7**(2014), 1-10.

[20] Todorov, P.G., *On the Faber polynomials of the univalent functions of class* Σ, J. Math. Anal. Appl., **162**(1991), no. 1, 268-276.

[21] Xu, Q.-H., Gui, Y.-C., Srivastava, H.M., *Coefficient estimates for a Certain subclass of analytic and bi-univalent functions*, Appl. Math. Lett., **25**(2012), 990-994.

[22] Xu, Q.-H., Xiao, H.-G., Srivastava, H.M., *A certain general subclass of analytic and bi-univalent functions and associated coefficient estimate problems*, Appl. Math. Comput., **218**(2012), no. 23, 11461-11465.

[23] Zireh, A., Salehian, S., *On the certain subclass of analytic and bi-univalent functions defined by convolution*, Acta Univ. Apulensis Math. Inform., **44**(2015), 9-19.

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Equipolar meromorphic functions sharing a set

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Abstract. Two meromorphic functions f and g having the same set of poles are known as equipolar. In this paper we study some uniqueness results of equi-polar meromorphic functions sharing a finite set and improve some recent results of Bhoosnurmath-Dyavanal [4] and Banerjee-Mallick [3] by removing some unnecessary conditions on ramification indices as well as relaxing the condition on the nature of sharing of the value ∞ by f and g from counting multiplicity to ignoring multiplicity.

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1. Introduction, definitions and results

Let f and g and be two non-constant meromorphic functions defined in the open complex plane $\mathbb{C}$. If for some $a \in \mathbb{C} \cup \{\infty\}$, f and g have the same set of a-points with the same multiplicities, we say that f and g share the value a CM (Counting Multiplicities) and if we do not consider the multiplicities, then f and g are said to share the value a IM (Ignoring Multiplicities). We do not explain the standard notations and definitions of the value distribution theory as these are available in [7].

Let $a \in \mathbb{C} \cup \{\infty\}$. We denote by $N(r, a; f \mid= 1)$, the counting function of the zeros of $f - a$ of multiplicity one. We also denote by $N(r, a; f \mid\geq l)$, the counting function of those a-points of f whose multiplicities are $\geq l$. Similarly we denote by $\overline{N}(r, a; f \mid\geq l)$ the reduced counting function of the a-points of f of multiplicity $\geq l$. We put $N_2(r, a; f) = \overline{N}(r, a; f) + \overline{N}(r, a; f \mid\geq 2)$. We put

$$\Theta(a; f) = 1 - \limsup_{r\to\infty} \frac{\overline{N}(r, a; f)}{T(r, f)};$$

$$\delta_2(a; f) = 1 - \limsup_{r\to\infty} \frac{N_2(r, a; f)}{T(r, f)}$$

and
$$\delta_{(2}(a;f) = 1 - \limsup_{r\to\infty} \frac{N(r,a;f\mid\geq 2)}{T(r,f)}.$$

Let S be a set of distinct elements of $\mathbb{C}\cup\{\infty\}$ and
$$E_f(S) = \bigcup_{a\in S}\{(z,p)\in\mathbb{C}\times\mathbb{N} : z \text{ is an } a\text{-point of } f \text{ of multiplicity } p\},$$
and
$$\overline{E}_f(S) = \bigcup_{a\in S}\{(z,1)\in\mathbb{C}\times\mathbb{N} : z \text{ is an } a\text{-point of } f\}.$$
If $E_f(S) = E_g(S)$, we say that f and g share the set S CM (Counting Multiplicity). On the other hand if $\overline{E}_f(S) = \overline{E}_g(S)$, we say that f and g share the set S IM (Ignoring Multiplicity).

It will be convenient to denote by E, any subset of nonnegative real numbers of finite measure not necessary the same in each of its occurrence.

In 1976, Gross [6] considered the uniqueness problem of meromorphic functions when the functions under consideration share sets instead of values. In this direction Gross raised the following question:
Can one find finite sets S_j, $j = 1, 2$ such that any two non-constant entire functions f and g satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2$ must be identical ?

To answer the Question of Gross [6], in 1995, Yi [13] obtained the following results.

Theorem A. [13] *Let $S = \{z : z^n + az^{n-m} + b = 0\}$, where n and m are two positive integers such that $m \geq 2$, $n \geq 2m+7$, with m and n having no common factor, a and b be two nonzero constants such that $z^n + az^{n-m} + b = 0$ has no multiple root. If f and g be two non-constant meromorphic functions having no simple poles such that $E_f(S) = E_g(S)$ and $E_f(\{\infty\}) = E_g(\{\infty\})$, then $f \equiv g$.*

Theorem B. [13] *Let $S = \{z : z^n + az^{n-1} + b = 0\}$, where $n(\geq 9)$ be an integer and a and b be two nonzero constants such that $z^n + az^{n-1} + b = 0$ has no multiple root. If f and g be two non-constant meromorphic functions such that $E_f(S) = E_g(S)$ and $E_f(\{\infty\}) = E_g(\{\infty\})$, then either $f \equiv g$ or*
$$f \equiv \frac{-ah(h^{n-1}-1)}{h^n - 1} \quad \text{and} \quad g \equiv \frac{-a(h^{n-1}-1)}{h^n-1},$$
where h is a non-constant meromorphic function.

Lahiri [8], in an attempt to investigate under which situation, $f \equiv g$, proved the following result.

Theorem C. [8] *Let S be defined as in Theorem B and $n(\geq 8)$ be an integer. If f and g be two non-constant meromorphic functions having no simple poles such that $E_f(S) = E_g(S)$ and $E_f(\{\infty\}) = E_g(\{\infty\})$, then $f \equiv g$.*

Fang and Lahiri [5], improved Theorem C by reducing the cardinality of the same range set in the following result.

Theorem D. [5] *Let $S = \{z : z^n + az^{n-1} + b = 0\}$, where $n(\geq 7)$ be an integer and a and b be two nonzero constants such that $z^n + az^{n-1} + b = 0$ has no multiple root. If f and g be two non-constant meromorphic functions having no simple poles such that $E_f(S) = E_g(S)$ and $E_f(\{\infty\}) = E_g(\{\infty\})$, then $f \equiv g$.*

Below we give the definition of weighted sharing which will be required in the sequel.

Definition 1.1. [9, 10] *Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$, we denote by $E_k(a; f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \leq k$ and $k + 1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f and g share the value a with weight k.*

The definition implies that if f, g share a value a with weight k, then z_0 is a zero of $f - a$ with multiplicity $m(\leq k)$ if and only if it is a zero of $g - a$ with multiplicity $m(\leq k)$ and z_0 is a zero of $f - a$ of multiplicity $m(> k)$ if and only if it is a zero of $g - a$ with multiplicity $n(> k)$ where m is not necessarily equal to n.

We write f, g share (a, k) to mean f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for all integers p , $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

Definition 1.2. [10] *Let $S \subset \mathbb{C} \cup \{\infty\}$ and k be a positive integer or ∞. We denote by $E_f(S, k)$ the set $\bigcup_{a \in S} E_k(a; f)$.*

Recently Bhoosnurmath-Dyavanal [4] proved the following result as an improvement of the above results by reducing the cardinality of the shared set S as well as weakening the condition on ramification indices.

Theorem E. [4] *Let $S = \{z : z^n + az^{n-1} + b = 0\}$, where $n(\geq 5)$ be an integer and a and b be two nonzero constants such that $z^n + az^{n-1} + b = 0$ has no multiple root. If f and g be two non-constant meromorphic functions such that $E_f(S, \infty) = E_g(S, \infty)$ and $E_f(\{\infty\}, \infty) = E_g(\{\infty\}, \infty)$. Also $N(r, 0; f \mid= 1) = S(r, f)$ and $N(r, 0; g \mid= 1) = S(r, g)$ and $\Theta(\infty; f) > \frac{2}{n-1}$ and $\Theta(\infty; g) > \frac{2}{n-1}$, then $f \equiv g$.*

With the aid of weighted sharing Banerjee-Mallick [3] improved Theorem E as follows.

Theorem F. [3] *Let $S = \{z : z^n + az^{n-1} + b = 0\}$, where $n(\geq 5)$ be an integer and a and b be two nonzero constants such that $z^n + az^{n-1} + b = 0$ has no multiple root. If f and g be two non-constant meromorphic functions satisfying $E_f(S, m) = E_g(S, m)$ and $E_f(\{\infty\}, \infty) = E_g(\{\infty\}, \infty)$. Also $N(r, 0; f \mid= 1) = S(r, f)$ and $N(r, 0; g \mid= 1) = S(r, g)$ and $\Theta_f + \Theta_g > \frac{4}{n-1}$. If*

(i) *$m \geq 2$ and $n \geq 5$;*

(ii) *or $m = 1$ and $n \geq 6$;*

(iii) *or $m = 0$ and $n \geq 10$,*

then $f \equiv g$, where $\Theta_f = \delta_{(2}(0; f) + \Theta(\infty; f) + \Theta(-a\frac{n-1}{n}; f)$ and Θ_g is defined similarly.

In this paper we give two-fold improvements to Theorem F as follows. Firstly we show that we can reach the conclusion of Theorem F without assuming the condition

$$\Theta_f + \Theta_g > \frac{4}{n-1}.$$

Secondly, we prove our theorem merely assuming that f and g share the value ∞ with weight 0. That is we reduce the CM sharing of ∞ by f and g to IM sharing. We also show that the cardinality of the shared set S can be reduced to 9 from 10 when $m = 0$. We state below our theorem.

Theorem 1.1. *Let $S = \{z : z^n + az^{n-1} + b = 0\}$, where $n(\geq 5)$ be an integer and a and b be two nonzero constants such that $z^n + az^{n-1} + b = 0$ has no multiple root. Let f and g be two non-constant meromorphic functions satisfying $E_f(S, m) = E_g(S, m)$, $E_f(\{\infty\}, 0) = E_g(\{\infty\}, 0)$ and $N(r, 0; f \mid= 1) = S(r, f)$ and $N(r, 0; g \mid= 1) = S(r, g)$. Then, $f \equiv g$, if any one of the following holds.*

(i) $m = 2$, $n \geq 5$;

(ii) $m = 1$, $n \geq 6$;

(iii) $m = 0$, $n \geq 9$.

Definition 1.3. [10] *Let f and g be two non-constant meromorphic functions such that f and g share $(a, 0)$ for $a \in \mathbb{C} \cup \{\infty\}$. Let z_0 be an a-point of f with multiplicity p, and an a-point of g of multiplicity q. We denote by $\overline{N}_L(r, a; f)(\overline{N}_L(r, a; g))$ the reduced counting function of those a-points of f and g where $p > q(q > p)$. We denote by $\overline{N}_*(r, a; f, g)$ the reduced counting function of those a-points of f whose multiplicities differ from the corresponding a-points of g. Clearly $\overline{N}_*(r, a; f, g) = \overline{N}_*(r, a; g, f)$ and $\overline{N}_*(r, a; f, g) = \overline{N}_L(r, a; f) + \overline{N}_L(r, a; g)$. We also denote by $N_E^{1)}(r, a; f)$ the counting function of those a-points of f and g where $p = q = 1$. similarly we denote by $\overline{N}_E^{(2}(r, a; f)$, the reduced counting function of those a-points of f such that $p = q \geq 2$.*

2. Lemmas

In this section we present some lemmas which will be required to establish our results. Let f and g be two nonconstant meromorphic functions and we define

$$F = \frac{f^{n-1}(f+a)}{-b}, \quad G = \frac{g^{n-1}(g+a)}{-b}. \tag{2.1}$$

In the lemmas several times we use the function H defined by

$$H = \frac{F''}{F'} - \frac{2F'}{F-1} - \frac{G''}{G'} + \frac{2G'}{G-1}.$$

Lemma 2.1. [12] *Let f be a non-constant meromorphic function and let*

$$R(f) = \frac{\sum_{k=0}^{n} a_k f^k}{\sum_{j=0}^{m} b_j f^j}$$

be an irreducible rational function in f with constant coefficients $\{a_k\}$ and $\{b_j\}$, where $a_n \neq 0$, $b_m \neq 0$. Then $T(r, R(f)) = dT(r, f) + S(r, f)$, where $d = max\{m, n\}$.

Lemma 2.2. [14] *If F, G be two non-constant meromorphic functions such that they share $(1, 0)$ and $H \not\equiv 0$ then,*
$N_E^{1)}(r, 1; F \mid= 1) = N_E^{1)}(r, 1; G \mid= 1) \leq N(r, H) + S(r, F) + S(r, G)$.

Lemma 2.3. [2] *Let f and g be two nonconstant meromorphic functions sharing $(1, m)$, $0 \leq m < \infty$. Then*
$\overline{N}(r, 1; f) + \overline{N}(r, 1; g) - N_E^{1)}(r, 1; f) + \left(m - \frac{1}{2}\right) \overline{N}_*(r, 1; f, g) \leq \frac{1}{2}[N(r, 1; f) + N(r, 1; g)]$.

Lemma 2.4. *Let $H \not\equiv 0$ and $E_f(S, 0) = E_g(S, 0)$ and $E_f(\{\infty\}, 0) = E_g(\{\infty\}, 0)$. Then, if F and G be given by (2.1),*

$$\begin{aligned} & N(r, H) \\ \leq \; & \overline{N}(r, 0; F \mid\geq 2) + \overline{N}(r, 0; G \mid\geq 2) + \overline{N}(r, c; F \mid\geq 2) + \overline{N}(r, c; G \mid\geq 2) \\ + \; & \overline{N}_*(r, 1; F, G) + \overline{N}_*(r, \infty; F, G) + \overline{N}_0(r, 0; F') + \overline{N}_0(r, 0; G') + S(r, F) \\ + \; & S(r, G), \end{aligned}$$

for $c \in \mathbb{C} \backslash \{0, 1\}$. Here, $\overline{N}_0(r, 0; F')$, denotes the reduced counting function of the zeros of F', which are not the zeros of $F(F-1)(F-c)$. Similarly we define $\overline{N}_0(r, 0; G')$.

Proof. From the definition of H, it follows that that the poles of H occur at the
(i) multiple zeros of F and G;
(ii) poles of F and G of different multiplicities;
(iii) 1-points of F and G of different multiplicities;
(iv) multiple c-points of F and G;
(v) the zeros of F' which are not the zeros of $F(F-1)(F-c)$;
(vi) the zeros of G' which are not the zeros of $G(G-1)(G-c)$.
Since the poles of H are all simple, the lemma follows easily. □

Lemma 2.5. [11] *If two non-constant meromorphic functions f and g share $(\infty, 0)$. Then $f^{n-1}(f+a)g^{n-1}(g+a) \not\equiv b^2$, for $n \geq 2$.*

Lemma 2.6. *Let f and g be two non-constant meromorphic functions such that $f^{n-1}(f+a) \equiv g^{n-1}(g+a)$, where $n \geq 5$ is an integer. If $N(r, 1; f \mid= 1) = S(r, f)$ and $N(r, 1; g \mid= 1) = S(r, g)$, then $f \equiv g$.*

Proof. Let

$$f^{n-1}(f+a) \equiv g^{n-1}(g+a). \tag{2.2}$$

Clearly (2.2) implies that f and g share (∞, ∞). Suppose $f \not\equiv g$. Let $y = \frac{g}{f}$. Then (2.2) implies that $y \not\equiv 1$, $y^{n-1} \not\equiv 1$, $y^n \not\equiv 1$ and

$$f \equiv -a\frac{1-y^{n-1}}{1-y^n} \tag{2.3}$$

$$\begin{aligned} \equiv \; & a\left(\frac{y^{n-1}}{1+y+y^2+\ldots+y^{n-1}} - 1\right) \\ = \; & -a\frac{1+y+y^2+\ldots+y^{n-2}}{1+y+y^2+\ldots+y^{n-1}}. \end{aligned}$$

Case 1. Let $y=\frac{g}{f}$ =constant, then it follows from (2.3) that f is constant, which is impossible.
Case 2. Let $y=\frac{g}{f}$ be non-constant.

Using Lemma 2.1, we note from (2.2), $T(r,f)=T(r,g)+O(1)$ and hence $S(r,f)=S(r,g)=S(r)$, say.

Let z_0 be a zero of $f+a$. Then in view of (2.2), z_0 must be a zero of either $g+a$ or g. If possible suppose that z_0 is a zero of $g+a$. Then $y(z_0)=1$ and from (2.3) we obtain $f(z_0)=-a(\frac{n-1}{n})\neq -a$, that is $f(z_0)+a=-a(\frac{n-1}{n})\neq 0$ which is a contradiction to our assumption. Therefore z_0 must be a zero of g. Thus we have

$$\{z: f(z)+a=0\}\subseteq\{z: g(z)=0\}. \tag{2.4}$$

Suppose z_0 be a zero of $f+a$ of multiplicity p and a zero of g of multiplicity q. Then in view of (2.2), $p=(n-1)q$. Thus $p=n-1$, if $q=1$ or $p\geq 2(n-1)$, when $q\geq 2$. Thus the least multiplicity of a zero of $f+a$ is $n-1$ and $f+a$ has no zero of multiplicity m such that $n-1<m<2(n-1)$.

We agree to denote by $\overline{N}(r,0;f+a\mid g_{=1}=0)$, the reduced counting function of the zeros of $f+a$ which are the zeros of g of multiplicity =1 and by $\overline{N}(r,0;f+a\mid g_{\geq 2}=0)$, the reduced counting function of the zeros of $f+a$ which are the zeros of g of multiplicity ≥ 2. Also we denote by $\overline{N}(r,0;f+a\mid g=0)$ the reduced counting function of the zeros of $f+a$, which are the zeros of g.

Now since $N(r,0;g\mid=1)=S(r,g)$, we have from (2.4) and above analysis,

$$\begin{aligned}
& \overline{N}(r,0;f+a)\\
=\ & \overline{N}(r,0;f+a\mid g=0)\\
=\ & \overline{N}(r,0;f+a\mid g_{=1}=0)+\overline{N}(r,0;f+a\mid g_{\geq 2}=0)\\
=\ & S(r,g)+\overline{N}(r,0;f+a\mid\geq 2(n-1))\\
=\ & S(r,f)+\overline{N}(r,0;f+a\mid\geq 2(n-1)).
\end{aligned}$$

Hence

$$(2n-2)\overline{N}(r,0;f+a)\leq T(r,f)+S(r,f).$$

From (2.3) we observe that $T(r,f)=(n-1)T(r,y)+S(r,y)$. Also

$$\begin{aligned}
& f+a\frac{n-1}{n} \qquad (2.5)\\
=\ & -a\frac{1-y^{n-1}}{1-y^n}+a\frac{n-1}{n}\\
=\ & -a\frac{(n-1)y^n-ny^{n-1}+1}{n(1-y^n)}.
\end{aligned}$$

If we put $p(y)=(n-1)y^n-ny^{n-1}+1$, then $p(0)\neq 0$ and
$p'(y)=n(n-1)y^{n-2}\{y-1\}$ and $p''(y)=n(n-1)y^{n-3}\{(n-3)y-n+2\}$.
Thus $p(1)=p'(1)=0$. Hence $p(y)=0$ has only one repeated root at $y=1$.

Thus from (2.5) we obtain

$$\sum_{i=1}^{n-1}\overline{N}(r,u_i;y)\leq\overline{N}(r,-a\frac{n-1}{n};f),$$

where u_i, $i=1,\ldots,n-1$ are the distinct zeros of $p(y)$.
Also from (2.3) we have

$$\sum_{j=1}^{n-1}\overline{N}(r,v_j;y)\leq\overline{N}(r,\infty;f)\leq T(r,f).$$

Since by our assumption $N(r,0;f\mid=1)=S(r,f)$, we have

$$\overline{N}(r,0;f)=N(r,0;f\mid=1)+\overline{N}(r,0;f\mid\geq 2)\leq S(r,f)+\frac{1}{2}T(r,f).$$

Thus we have

$$\begin{aligned}&\sum_{j=1}^{n-2}\overline{N}(r,w_j;y)+\overline{N}(r,\infty;y)\\ \leq\ &\overline{N}(r,0;f)=N(r,0;f\mid=1)+\overline{N}(r,0;f\mid\geq 2)\leq\frac{1}{2}T(r,f)+S(r,f),\end{aligned}$$

where v_js, $j=1,2,\ldots,n-1$ are the distinct roots of $1+y+y^2+\ldots+y^{n-1}=0$ and w_js, $j=1,2,\ldots,n-2$ are the distinct roots of $1+y+y^2+\ldots+y^{n-2}=0$.
From (2.2) and (2.3) we note that the zeros of y occur at those zeros of g which are the zeros of $f+a$. Hence $\overline{N}(r,0;y)\leq\overline{N}(r,0;f+a)$.
Also we have obtained $(2n-2)\overline{N}(r,0;f+a)\leq T(r,f)+S(r,f)$. Thus, we obtain by the second main theorem,

$$\begin{aligned}&(3n-4)T(r,y)\\ \leq\ &\sum_{j=1}^{n-1}\overline{N}(r,v_j;y)+\sum_{j=1}^{n-2}\overline{N}(r,w_j;y)+\sum_{i=1}^{n-1}\overline{N}(r,u_i;y)+\overline{N}(r,0;y)\\ +\ &\overline{N}(r,\infty;y)+S(r,y)\\ \leq\ &\overline{N}(r,\infty;f)+\overline{N}(r,0;f)+\overline{N}\left(r,-a\frac{n-1}{n};f\right)+\overline{N}(r,0;f+a)\\ +\ &S(r,f)\\ \leq\ &\left\{1+\frac{1}{2}+1+\frac{1}{2n-2}\right\}T(r,f)+S(r,f)\\ \leq\ &\left(\frac{5}{2}+\frac{1}{2n-2}\right)(n-1)T(r,y)+S(r,y),\end{aligned}$$

which leads to a contradiction for $n\geq 5$. This completes the proof of the Lemma. □

Lemma 2.7. *Let $S=\{z:z^n+az^{n-1}+b=0\}$, where $n(\geq 4)$ be an integer and a and b be two nonzero constants such that $z^n+az^{n-1}+b=0$ has no multiple root. If F and G are given by (2.1), then there exists an $\alpha\in\mathbb{C}\setminus\{0,a,b\}$, satisfying $N_2(r,\alpha;F)\leq(n-1)T(r,f)+S(r,f)$, $N_2(r,\alpha;G)\leq(n-1)T(r,g)+S(r,g)$, where*

$|\alpha| = \frac{(n-1)^{n-1}}{n^n}.\frac{|a|^n}{|b|}$, $\arg\alpha = \arg(\frac{a^n}{b})$ *or* $\arg\alpha = \arg(-\frac{a^n}{b})$, *according as* n *is even or odd. Here* $\arg z$ *denotes the principal argument of* z *for any* $z \in \mathbb{C} \setminus \{0\}$.

Proof. Let $p(z) = z^n + az^{n-1} + b$. Then $p'(z) = z^{n-2}\{nz + a(n-1)\}$. Thus $p'(z) = 0$ has roots at $z = 0$ and at $z = -\frac{a(n-1)}{n}$. Thus $p(z) = 0$ will have a repeated root at $-\frac{a(n-1)}{n}$ provided $p\left(-\frac{a(n-1)}{n}\right) = 0$ and this yields $b = (-1)^n \left(\frac{a}{n}\right)^n (n-1)^{n-1}$. Note that $p''\left(-\dfrac{a(n-1)}{n}\right) \neq 0$.

Thus $p(z) = 0$ has a repeated root at $-\frac{a(n-1)}{n}$ and hence only $n-1$ distinct roots provided $b = (-1)^n(\frac{a}{n})^n(n-1)^{n-1}$.
Let α be a nonzero complex number. Then

$$F - \alpha = \frac{f^{n-1}(f+a)}{-b} - \alpha = \frac{f^n + af^{n-1} + \alpha b}{-b}.$$

We choose α in such a manner that the equation $z^n + az^{n-1} + \alpha b = 0$ has repeated roots. It is clear from the above discussion that in this case we must have

$$\alpha b = (-1)^n \left(\frac{a}{n}\right)^n (n-1)^{n-1}.$$

This implies $|\alpha| = \frac{(n-1)^{n-1}}{n^n} \cdot \frac{|a|^n}{|b|}$, $\arg\alpha = \arg(\frac{a^n}{b})$ or $\arg\alpha = \arg(-\frac{a^n}{b})$, according as n is even or odd. If $w_1, w_2, \ldots, w_{n-1}$, be the distinct roots of $z^n + az^{n-1} + \alpha b = 0$, then we have

$$\begin{aligned}
& N_2(r,\alpha;F) \\
= \;& \overline{N}(r,\alpha;F) + \overline{N}(r,\alpha;F \mid\geq 2) \\
\leq \;& \sum_{i=1}^{n-1} \overline{N}(r,w_i;f) + \sum_{i=1}^{n-1} \overline{N}(r,w_i;f \mid\geq 2) + S(r,f) \\
= \;& \sum_{i=1}^{n-1} \{\overline{N}(r,w_i;f) + \overline{N}(r,w_i;f \mid\geq 2)\} + S(r,f) \\
= \;& \sum_{i=1}^{n-1} N_2(r,w_i;f) + S(r,f) \\
\leq \;& (n-1)T(r,f) + S(r,f).
\end{aligned}$$

This completes the proof. □

Lemma 2.8. *Let* F, G *be given by (2.1) and* $V = (\frac{F'}{F-1} - \frac{F'}{F}) - (\frac{G'}{G-1} - \frac{G'}{G}) \not\equiv 0$. *If* $\overline{N}(r,0;f \mid= 1) = S(r,f)$ *and* $\overline{N}(r,0;g \mid= 1) = S(r,g)$ *and* f, g *share* $(\infty,0)$; F, G, *share* $(1,0)$, *then*

$$\{n-1\}\overline{N}(r,\infty;f) \leq \left\{\frac{1}{2} + 1\right\}\{T(r,f) + T(r,g)\} + \overline{N}_*(r,1;F,G) + S(r,f) + S(r,g).$$

Proof. Let z_0 be a pole of f and g of respective multiplicities p and q. Then from (2.1), around z_0, we have

$$F = \frac{A(z)}{(z-z_0)^{np}}, \quad G = \frac{B(z)}{(z-z_0)^{nq}}. \tag{2.6}$$

Where $A(z)$ and $B(z)$ are analytic at z_0, and $A(z_0) \neq 0$, $B(z_0) \neq 0$.
Thus

$$\frac{F'}{F-1} = \frac{A'}{A-(z-z_0)^{np}} - \frac{npA}{(z-z_0)[A-(z-z_0)^{np}]}$$

and

$$\frac{F'}{F} = \frac{A'}{A} - \frac{np}{z-z_0}.$$

Therefore a simple calculation yields,

$$\begin{aligned} \frac{F'}{F-1} - \frac{F'}{F} &= (z-z_0)^{np-1}\left\{\frac{A'}{A}.\frac{z-z_0}{A-(z-z_0)^{np}} - \frac{np}{A-(z-z_0)^{np}}\right\} \\ &= (z-z_0)^{np-1}\phi(z), \end{aligned}$$

say, where $\phi(z)$ is analytic at z_0 and $\phi(z_0) \neq 0$. Similarly we obtain,

$$\begin{aligned} \frac{G'}{G-1} - \frac{G'}{G} &= (z-z_0)^{nq-1}\left\{\frac{B'}{B}.\frac{z-z_0}{B-(z-z_0)^{nq}} - \frac{nq}{B-(z-z_0)^{nq}}\right\} \\ &= (z-z_0)^{nq-1}\psi(z), \end{aligned}$$

say, where $\psi(z)$ is analytic at z_0 and $\psi(z_0) \neq 0$. Therefore, around z_0,

$$V = (z-z_0)^{np-1}\phi(z) - (z-z_0)^{nq-1}\psi(z).$$

Thus V has a zero at z_0, of order at least $n-1$.
We note by Millux's theorem

$$\begin{aligned} & m(r,V) \\ = \; & m\left(r, \left(\frac{F'}{F-1} - \frac{F'}{F}\right) - \left(\frac{G'}{G-1} - \frac{G'}{G}\right)\right) \\ \leq \; & m\left(r, \frac{F'}{F-1}\right) + m\left(r, \frac{G'}{G-1}\right) + m\left(r, \frac{F'}{F}\right) + m\left(r, \frac{G'}{G}\right) \\ = \; & S(r,F) + S(r,G) = S(r,f) + S(r,g). \end{aligned}$$

Hence from above analysis and by the first fundamental theorem, we have

$$\begin{aligned} & \{n-1\}\overline{N}(r,\infty;f) \\ \leq \; & N(r,0;V) \\ \leq \; & T(r,V) + O(1) \\ \leq \; & N(r,\infty;V) + S(r,f) + S(r,g) \\ \leq \; & \overline{N}(r,0;f) + \overline{N}(r,0;g) + \overline{N}(r,0;f+a) + \overline{N}(r,0;g+a) \\ + \; & \overline{N}_*(r,1;F,G) + S(r,f) + S(r,g). \end{aligned}$$

Now since $\overline{N}(r,0;f\mid=1)=S(r,f)$ and $\overline{N}(r,0;g\mid=1)=S(r,g)$, we have

$$\overline{N}(r,0;f)\le\frac{1}{2}T(r,f)+S(r,f)$$

and

$$\overline{N}(r,0;g)\le\frac{1}{2}T(r,g)+S(r,g).$$

Therefore from above, we have

$$\begin{aligned}&\{n-1\}\overline{N}(r,\infty;f)\\ \le\ &\left\{\frac{1}{2}+1\right\}\{T(r,f)+T(r,g\}+\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g).\end{aligned}$$

This completes the proof. □

Lemma 2.9. [1] *Let F and G be defined by (2.1) and F and G share $(1,m)$, $0\le m<\infty$. Also let $w_1,\dots,w_n$ be the distinct roots of the equation $z^n+az^{n-1}+b=0$, where $b\neq(-1)^n(\frac{a}{n})^n(n-1)^{n-1}$, $n\ge3$. Then*

$$\overline{N}_L(r,1;F)\le\frac{1}{m+1}\left\{\overline{N}(r,0;f)+\overline{N}(r,\infty;f)\right\}-N_{\odot}(r,0;f')+S(r,f),$$

where $N_{\odot}(r,0;f')=N(r,0;f'\mid f\neq0,w_1,\dots,w_n)$. Similar inequality holds for $\overline{N}_L(r,1;G)$.

Lemma 2.10. *Let F and G be defined by (2.1) and F and G share $(1,m)$, $0\le m<\infty$. Also let $\overline{N}(r,0;f\mid=1)=S(r,f)$ and $\overline{N}(r,0;g\mid=1)=S(r,g)$. Then*

$$\begin{aligned}&\overline{N}_*(r,1;F,G)\\ \le\ &\frac{1}{m+1}\left\{\frac{1}{2}[T(r,f)+T(r,g)]+\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\right\}+S(r,f)+S(r,g).\end{aligned}$$

Proof. Since $\overline{N}_*(r,1;F,G)=\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)$ and from the condition of the Lemma it follows that

$$\overline{N}(r,0;f)\le\frac{1}{2}T(r,f)+S(r,f)$$

and

$$\overline{N}(r,0;g)\le\frac{1}{2}T(r,g)+S(r,g),$$

the Lemma follows from Lemma 2.9. □

Lemma 2.11. Let F and G be defined by (2.1) and F and G share $(1,m)$, $0\le m<\infty$. Also let $\overline{N}(r,0;f\mid=1)=S(r,f)$ and $\overline{N}(r,0;g\mid=1)=S(r,g)$ and f and g share $(\infty,0)$. Then

$$\begin{aligned}&\left[n-1-\frac{2}{m+1}\right]\overline{N}(r,\infty;f)\\ \le\ &\left[\frac{3}{2}+\frac{1}{2(m+1)}\right]\{T(r,f)+T(r,g)\}\\ +\ &S(r,f)+S(r,g).\end{aligned}$$

Proof. From Lemmas 2.8 and 2.10, we have

$$\begin{aligned}
& \{n-1\}\overline{N}(r,\infty;f) \\
\leq\ & \left\{\frac{1}{2}+1+\frac{1}{2(m+1)}\right\}\{T(r,f)+T(r,g\}+\frac{2}{m+1}\overline{N}(r,\infty;f) \\
+\ & S(r,f)+S(r,g).
\end{aligned}$$

The lemma follows easily from above. □

3. Proof of theorem

Proof of Theorem 1.1. **Case 1.** $H \not\equiv 0$. By Lemma 2.1, we obtain from the definitions of F and G, $T(r,F)=nT(r,f)+S(r,f)$, $T(r,G)=nT(r,g)+S(r,g)$.

We denote by $N_0(r,0;F')$, the counting function of the zeros of F' which are not the zeros of $F(F-1)(F-c)$, for some $c\in\mathbb{C}\setminus\{0,1\}$. Similarly we define $N_0(r,0;G')$. Now applying the second main theorem to F and G, we obtain for some $c\in\mathbb{C}\setminus\{0,1\}$,

$$\begin{aligned}
& 2\{T(r,F)+T(r,G)\} \\
\leq\ & \overline{N}(r,0;F)+\overline{N}(r,c;F)+\overline{N}(r,1;F)+\overline{N}(r,\infty;F)+\overline{N}(r,0;G)+\overline{N}(r,c;G) \\
+\ & \overline{N}(r,1;G)+\overline{N}(r,\infty;G)-N_0(r,0;F')-N_0(r,0;G')+S(r,f)+S(r,g),
\end{aligned}$$

and hence

$$\begin{aligned}
& 2n\{T(r,f)+T(r,g)\} \\
\leq\ & \overline{N}(r,0;F)+\overline{N}(r,c;F)+\overline{N}(r,1;F)+\overline{N}(r,\infty;F)+\overline{N}(r,0;G)+\overline{N}(r,c;G) \\
+\ & \overline{N}(r,1;G)+\overline{N}(r,\infty;G)-N_0(r,0;F')-N_0(r,0;G')+S(r,f)+S(r,g).
\end{aligned}$$

Using Lemma 2.2, Lemma 2.3 and 2.4 and 2.9 we have from above,

$$\begin{aligned}
& 2n\{T(r,f)+T(r,g)\} \qquad (3.1)\\
\leq\ & N_2(r,0;F)+N_2(r,c;F)+3\overline{N}(r,\infty;f)+N_2(r,0;G)+N_2(r,c;G) \\
+\ & \frac{n}{2}\{T(r,f)+T(r,g)\}+\left(\frac{3}{2}-m\right)\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g) \\
\leq\ & 2\overline{N}(r,0;f)+N_2(r,0;f+a)+(n-1)T(r,f)+3\overline{N}(r,\infty;f) \\
+\ & 2\overline{N}(r,0;g)+N_2(r,0;g+a)+(n-1)T(r,g)+\frac{n}{2}\{T(r,f)+T(r,g)\} \\
+\ & \left(\frac{3}{2}-m\right)\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g).
\end{aligned}$$

Subcase 1.1. $m = 2$. We obtain from (3.1) using Lemma 2.8,

$$\begin{aligned}
& \left(\frac{n}{2}-1\right)\{T(r,f)+T(r,g)\} \qquad (3.2)\\
\leq\ & \overline{N}(r,\infty;f)+\frac{2.3}{2(n-1)}\{T(r,f)+T(r,g)\}+\left(\frac{2}{n-1}-\frac{1}{2}\right)\overline{N}_*(r,1;F,G)\\
+\ & S(r,f)+S(r,g).\\
\leq\ & \frac{1}{2}\{T(r,f)+T(r,g)\}+\frac{2.3}{2(n-1)}\{T(r,f)+T(r,g)\}\\
+\ & \left(\frac{2}{n-1}-\frac{1}{2}\right)\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g).
\end{aligned}$$

But this leads to a contradiction for $n \geq 5$.

Subcase 1.2. $m = 1$. Then proceeding as in Subcase 1.1, the Lemma 2.2 with $m = 1$ and Lemma 2.3, yield the following.

$$\begin{aligned}
& 2n\{T(r,f)+T(r,g)\}\\
\leq\ & 2\{T(r,f)+T(r,g\}+(n-1)\{T(r,f)+T(r,g\}+\frac{n}{2}\{T(r,f)+T(r,g\}\\
+\ & 3\overline{N}(r,\infty;f)+\frac{1}{2}\overline{N}_*(r,1;F,G)+S(r,f)+S(r,g).
\end{aligned}$$

Using the Lemma 2.10 we obtain from above,

$$\begin{aligned}
& \left(\frac{n}{2}-1\right)\{T(r,f)+T(r,g\}\\
\leq\ & 3\overline{N}(r,\infty;f)+\frac{1}{2}.\frac{1}{1+1}\left[\frac{1}{2}T(r,f)+\frac{1}{2}T(r,g)+\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\right]\\
+\ & S(r,f)+S(r,g)\\
=\ & \left\{\frac{3}{2}+\frac{1}{4}\right\}[\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)]+\frac{1}{8}\{T(r,f)+T(r,g\}\\
+\ & S(r,f)+S(r,g)\\
\leq\ & \left\{\frac{3}{2}+\frac{1}{4}+\frac{1}{8}\right\}\{T(r,f)+T(r,g\}+S(r,f)+S(r,g).
\end{aligned}$$

This leads to a contradiction for $n \geq 6$.

Subcase 1.3. $m = 0$. Proceeding as in Subcase 1.2., we obtain using Lemmas 2.10 and 2.11 with $m = 0$,

$$\begin{aligned}
& \left\{\frac{n}{2} - 1\right\}\{T(r,f) + T(r,g\}\\
\leq\ & 3\overline{N}(r,\infty;f) + \frac{3}{2}\overline{N}_*(r,1;F,G)\\
\leq\ & 3\overline{N}(r,\infty;f) + \frac{3}{2}\left\{\frac{1}{2}T(r,f) + \frac{1}{2}T(r,f) + \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g)\right\}\\
+\ & S(r,f) + S(r,g)\\
=\ & 6.\frac{2}{n-3}\{T(r,f) + T(r,g)\} + \frac{3}{4}\{T(r,f) + T(r,g)\} + S(r,f) + S(r,g)\\
=\ & \left(\frac{12}{n-3} + \frac{3}{4}\right)\{T(r,f) + T(r,g)\} + S(r,f) + S(r,g),
\end{aligned}$$

this leads to a contradiction for $n \geq 9$.

Case 2. $H \equiv 0$. We have

$$F \equiv \frac{AG + B}{CG + D}, \tag{3.3}$$

where $AD - BC \neq 0$. Clearly from above and the definitions of F and G we have $T(r,F) = T(r,G) + O(1)$ and $T(r,f) = T(r,g) + O(1)$.

Subcase 2.1. $AC \neq 0$. Since f and g share $\{\infty\}$, it follows from (3.2) that ∞ is an exceptional value of f and g. So by the second main theorem we get,

$$\begin{aligned}
& nT(r,f)\\
\leq\ & \overline{N}(r,0;F) + \overline{N}(r,\infty;F) + \overline{N}(r,\frac{A}{C};F) + S(r,f)\\
\leq\ & \overline{N}(r,0;f) + \overline{N}(r,0;f+a) + \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) + S(r,f)\\
\leq\ & 2T(r,f) + S(r,f),
\end{aligned}$$

which leads to a contradiction for $n \geq 5$.

Subcase 2.2. Let $A \neq 0$ and $C = 0$. Then $F = \gamma G + \beta$, where $\gamma = \frac{A}{D} \neq 0$ and $\beta = \frac{B}{D}$. It is obvious that F and G cannot omit the value 1. For if F omits the value 1, then f(and g as well) omits the distinct roots of the equation $z^n + az^{n-1} + b = 0$, which certainly leads to a contradiction for $n \geq 3$.

Thus F and G assume the value 1 and we have from above

$$F = \gamma G + (1 - \gamma). \tag{3.4}$$

If $\gamma = 1$ we have $F \equiv G$ and by Lemma 2.6, we have $f \equiv g$.

So let $\gamma \neq 1$. Since $N(r,0;f \mid= 1) = S(r,f)$ and $N(r,0;g \mid= 1) = S(r,g)$, we have from (3.4) using the second main theorem,

$$\begin{aligned}
& nT(r,f)\\
\leq\ & \overline{N}(r,0;F) + \overline{N}(r,1-\gamma;F) + \overline{N}(r,\infty;F) + S(r,f)\\
\leq\ & \frac{1}{2}T(r,f) + \overline{N}(r,0;f+a) + \frac{1}{2}T(r,g) + \overline{N}(r,0;g+a) + \overline{N}(r,\infty;f) + S(r,f)\\
\leq\ & 4T(r,f) + S(r,f).
\end{aligned}$$

This leads to a contradiction for $n \geq 5$.
Subcase 2.3. $A = 0$, $C \neq 0$. Then clearly $B \neq 0$. Hence, $F \equiv \frac{1}{\zeta G+\eta}$. We can show as before that F and G cannot omit the value 1 and hence $F \equiv \frac{1}{\zeta G+1-\zeta}$. Let $\zeta = 1$. Then $FG \equiv 1$. This is a contradiction by Lemma 2.5.

So $\zeta \neq 1$. Now since f and g share ∞, the relation $F \equiv \frac{1}{\zeta G+1-\zeta}$, at once implies F cannot assume the values ∞ and 0, and therefore f cannot assume the values ∞, 0 and $-a$. This is impossible. This completes the proof of the theorem. □

References

[1] Banerjee, A., *Some uniqueness results on meromorphic functions sharing three sets,* Ann. Polon. Math., **92**(2007), 261-274.

[2] Banerjee, A., *On the uniqueness of meromorphic functions sharing two sets,* Comm. Math. Anal., **10**(2010), no. 10, 1-10.

[3] Banerjee, A., Mallick, S., *Uniqueness of meromorphic functions sharing two sets having deficient values,* Matematychni Stud., **41**(2014), no.2, 168-178.

[4] Bhoosnurmath, S.S., Dyavanal, R. , *Uniqueness of meromorphic functions sharing sets,* Bull. Math. Anal. Appl., **3**(2011), no. 3, 200-208.

[5] Fang, M., Lahiri, I., *Unique range set for certain meromorphic functions,* Indian J. Math., **45**(2003), no. 2, 141-150.

[6] Gross, F., *Factorization of meromorphic functions and some open problems,* Proc. Conf. Univ. Kentucky, Lexington, Ky (1976) Lecture Notes in Math., Springer (Berlin), **599**(1977), 51-69.

[7] Hayman, W.K., *Meromorphic Functions,* Clarendon Press, Oxford 1964.

[8] Lahiri, I., *The range set of meromorphic derivatives,* Northeast Math. J., **14**(1998), no. 3, 353-360.

[9] Lahiri, I., *Weighted sharing and uniqueness of meromorphic functions,* Nagoya Math. J., **161**(2001), 193-206.

[10] Lahiri, I., *Weighted value sharing and uniqueness of meromorphic functions,* Complex Var. Theory Appl., **46**(2001), 241-253.

[11] Lahiri, I., *On a question of Hong Xun Yi,* Arch Math. (Brno), **38**(2002), 119-128.

[12] Mohon'ko, A.Z., *On the Nevanlinna characteristics of some meromorphic functions,* Theory of Funct. Funct. Anal. Appl., **14**(1971), 83-87.

[13] Yi, H.X., *Unicity theorems for meromorphic or entire functions II,* Bull. Austral. Math. Soc., **53**(1995), 71-82.

[14] Yi, H.X., *Meromorphic functions that share two sets*, (Chinese), Acta Math. Sinica, **45**(2002), no. 1, 75-82.

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Existence of solution for Hilfer fractional differential problem with nonlocal boundary condition in Banach spaces

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Abstract. This paper is devoted to study the existence of a solution to Hilfer fractional differential equation with nonlocal boundary condition in Banach spaces. We use the equivalent integral equation to study the considered Hilfer differential problem with nonlocal boundary condition. The Mönch type fixed point theorem and the measure of the noncompactness technique are the main tools in this study. We demonstrate the existence of a solution with a suitable illustrative example.

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1. Introduction

The calculus of arbitrary order has been extensively studied in the last four decades. It has been proved to be an adequate tool in almost all branches of science and engineering. Because of its widespread applications, fractional calculus is becoming an integral part of applied mathematics research. Indeed, fractional differential equations have been found useful to describe abundant phenomena in physics and engineering, and the modest amount of work in this direction has taken place, see [1, 4, 9] and references therein. For basic development and theoretical applications of fractional differential equations, see [15, 17].

In the past two decades, the fractional differential equations are extensively studied for existence, uniqueness, continuous dependence and stability of the solution. For some fundamental results in existence theory of various fractional differential problems with initial and boundary conditions, see survey papers [1, 4], the monograph [17], the research papers [2, 3, 7, 5, 6, 8, 9, 10, 11, 12, 16, 20, 22] and references therein.

Recently, in [22], Wang and Zhang obtained some existence of the solutions of IVP for the class of Hilfer FDEs:

$$D_{0^+}^{\mu,\nu} z(t) = f(t, z(t)), \quad 0 < \mu < 1,\, 0 \leq \nu \leq 1,\; t \in (a, b] \tag{1.1}$$

$$I_{a^+}^{1-\gamma} z(a^+) = \sum_{k=1}^{m} \lambda_k z(\tau_k),\; \tau_k \in (a, b],\; \mu \leq \gamma = \mu + \nu(1-\mu), \tag{1.2}$$

by using fixed point theorems of Krasnoselskii and Schauder.

In the year 2018, Thabet et al. [19] investigated the existence of a solution to BVP for Hilfer FDEs:

$$D_{a^+}^{\mu,\nu} z(t) = f\left(t, z(t), Sz(t)\right), 0 < \mu < 1, 0 \leq \nu \leq 1, \qquad t \in (a, b], \tag{1.3}$$

$$I_{a^+}^{1-\gamma}\left[uz(a^+) + vz(b^-)\right] = w, \quad \mu \leq \gamma = \mu + \nu(1-\mu),\; u, v, w \in \mathbb{R}, \tag{1.4}$$

by using the Mönch fixed point theorem.

Motivated by works cited above, in this paper, we consider the nonlocal boundary value problem for a class of Hilfer fractional differential equations (HNBVP):

$$D_{a^+}^{\mu,\nu} z(t) = f(t, z(t)), \quad 0 < \mu < 1,\, 0 \leq \nu \leq 1, t \in (a, b], \tag{1.5}$$

$$I_{a^+}^{1-\gamma} cz(a^+) + I_{a^+}^{1-\gamma} dz(b^-) = \sum_{k=1}^{m} \lambda_k z(\tau_k), \tau_k \in (a, b],\; \mu \leq \gamma = \mu + \nu(1-\mu), \tag{1.6}$$

where $D_{a^+}^{\mu,\nu}$ is the Hilfer fractional derivative of order μ and type ν, $I_{a^+}^{1-\gamma}$ is the Riemann-Liouville fractional integral of order $1-\gamma$, $f : (a, b] \times E \to E$ be a function such that $f(t, z) \in C_{1-\gamma}([a, b], E)$ for any $z \in C_{1-\gamma}([a, b], E)$, E is a Banach space, $c, d \in \mathbb{R}$, and τ_k $(k = 1, 2, ..., m)$ are prefixed points satisfying $a < \tau_1 < \tau_2 < ... < \tau_m < b$, λ_k are real numbers.
The measure of noncompactness technique and a fixed point theorem of Monch type are the main tools in this analysis.

The paper is organized as follows: Some preliminary concepts related to our problem are listed in Section 2 which will be useful in the sequel. In Section 3, we first establish an equivalent integral equation of BVP and then we present the existence of its solution. An illustrative example is provided in the last section.

2. Preliminaries

In this section, we present some definitions, lemmas and weighted spaces which are useful in further development of this paper.

Let $J_1 = [a, b]$ and $J_2 = (a, b](\infty < a < b < +\infty)$. Let $C(J_1, E)$, be the Banach spaces of all continuous function $g : J_1 \to E$ with the norm $\|g\|_\infty = \sup\{|g(t)|; t \in J_1\}$. Here $L^p(J_1, E)$, $p > 1$, is the Banach space of measurable functions on J_1 with the L^p norm where

$$\|g\|_{L^p} = \left(\int_a^b |g(s)|^p \, ds\right)^{\frac{1}{p}} < \infty.$$

Let $L^\infty(J_1, E)$ be the Banach space of measurable functions $z : J_1 \longrightarrow E$ which are bounded and equipped with the norm $\|z\|_{L^\infty} = \inf\{e > 0 : \|z\| \leq e, \text{ a.e } t \in J_1\}$. Moreover, for a given set $\mathcal{V}$ of functions $v : J_1 \longrightarrow E$ let us denote by

$$\mathcal{V}(t) = \{v(t) : v \in \mathcal{V}; t \in J_1\},$$

$$\mathcal{V}(J_1) = \{v(t) : v \in \mathcal{V}; t \in J_1\}.$$

Definition 2.1. [17] Let $\mu > 0$. The left sided Riemann-Liouville fractional integral of order μ of $g \in L^1(J_1, E)$ is defined by

$$I^{\mu}_{a^+} g(t) = \frac{1}{\Gamma(\mu)} \int_a^t (t-s)^{\mu-1} g(s) ds, \quad t > a, \tag{2.1}$$

where $\Gamma(\cdot)$ is the Euler's Gamma function and $a \in \mathbb{R}$.

Definition 2.2. [17] Let $n-1 < \mu < n$. The left sided Riemann-Liouville and Caputo fractional derivatives of order μ of $g \in L^1(J_1, E)$ are defined by

$$D^{\mu}_{a^+} g(t) = \frac{1}{\Gamma(n-\mu)} \frac{d^n}{dt^n} \int_a^t (t-s)^{n-\mu-1} g(s) ds, \ t > a, \tag{2.2}$$

and

$${}^C D^{\mu}_{a^+} g(t) = \frac{1}{\Gamma(n-\mu)} \int_a^t (t-s)^{n-\mu-1} g^{(n)}(s) ds, \ t > a,$$

respectively, where $n = [\mu] + 1$, and $[\mu]$ denotes the integer part of μ.

Definition 2.3. [15] The left sided Hilfer fractional derivative of function $g \in L^1(J_1, E)$ of order $0 < \mu < 1$ and type $0 \leq \nu \leq 1$ is denoted as $D^{\mu,\nu}_{a^+}$ and defined by

$$D^{\mu,\nu}_{a^+} g(t) = I^{\nu(1-\mu)}_{a^+} D I^{(1-\nu)(1-\mu)}_{a^+} g(t), \ D = \frac{d}{dt}. \tag{2.3}$$

where $I^{\mu}_{a^+}$ and $D^{\mu}_{a^+}$ are Riemann-Liouville fractional integral and derivative defined by (2.1) and (2.2), respectively.

Remark 2.4. From Definition 2.3, we observe that:

(i) The operator $D^{\mu,\nu}_{a^+}$ can be written as

$$D^{\mu,\nu}_{a^+} = I^{\nu(1-\mu)}_{a^+} D I^{(1-\gamma)}_{a^+} = I^{\nu(1-\mu)}_{a^+} D^{\gamma}, \ \gamma = \mu + \nu(1-\mu).$$

(ii) The Hilfer fractional derivative can be regarded as an interpolator between the Riemann-Liouville derivative ($\nu = 0$) and Caputo derivative ($\nu = 1$) as

$$D^{\mu,\nu}_{a^+} = \begin{cases} D I^{(1-\mu)}_{a^+} = D^{\mu}_{a^+}, & if \ \nu = 0; \\ I^{(1-\mu)}_{a^+} D = {}^C D^{\mu}_{a^+}, & if \ \nu = 1. \end{cases}$$

(iii) In particular, if $\gamma = \mu + \nu(1-\mu)$, then

$$(D^{\mu,\nu}_{a^+} g)(t) = \left(I^{\nu(1-\mu)}_{a^+}\left(D^{\gamma}_{a^+} g\right)\right)(t),$$

where $\left(D^{\gamma}_{a^+} g\right)(t) = \frac{d}{dt}\left(I^{(1-\nu)(1-\mu)}_{a^+} g\right)(t)$.

Definition 2.5. [17] Let $0 \leq \gamma < 1$. The weighted spaces $C_\gamma(J_1, E)$ and $C^n_{1-\gamma}(J_1, E)$ are defined by

$$C_\gamma(J_1, E) = \{g : J_2 \to E : (t-a)^\gamma g(t) \in C(J_1, E)\},$$

and

$$C^n_\gamma(J_1, E) = \{g : J_2 \to E,\ g \in C^{n-1}(J_1, E) : g^{(n)}(t) \in C_\gamma(J_1, E)\},\ n \in \mathbb{N}$$

with the norms

$$\|g\|_{C_\gamma} = \|(t-a)^\gamma g\|_C = \sup\{|(t-a)^\gamma g(t)| : t \in J_1\},$$

and

$$\|g\|_{C^n_{1-\gamma}} = \sum_{k=0}^{n-1} \|g^{(k)}\|_C + \|g^{(n)}\|_{C_{1-\gamma}}, \tag{2.4}$$

respectively. Furthermore we recall following weighted spaces

$$C^{\mu,\nu}_{1-\gamma}(J_1, E) = \left\{g \in C_{1-\gamma}(J_1, E) : D^{\mu,\nu}_{a^+} g \in C_{1-\gamma}(J_1, E)\right\}, \quad \gamma = \mu + \nu(1-\mu) \tag{2.5}$$

and

$$C^\gamma_{1-\gamma}(J_1, E) = \left\{g \in C_{1-\gamma}(J_1, E) : D^\gamma_{a^+} g \in C_{1-\gamma}(J_1, E)\right\}, \quad \gamma = \mu + \nu(1-\mu).$$

Let $0 < \mu < 1, 0 \leq \nu \leq 1$ and $\gamma = \mu + \nu(1-\mu)$. Clearly, $D^{\mu,\nu}_{a^+} g = I^{\nu(1-\mu)}_{a^+} D^\gamma_{a^+} g$ and $C^\gamma_{1-\gamma}(J_1, E) \subset C^{\mu,\nu}_{1-\gamma}(J_1, E)$.

Lemma 2.6. [9] *If $\mu > 0$, $\nu > 0$ and $g \in L^1(J_1, E)$ for $t \in [a, b]$, then the following properties hold:*

$$\left(I^\mu_{a^+} I^\nu_{a^+} g\right)(t) = \left(I^{\mu+\nu}_{a^+} g\right)(t) \text{ and } \left(D^\mu_{a^+} I^\nu_{a^+} g\right)(t) = g(t).$$

In particular, if $g \in C_\gamma(J_1, E)$ or $g \in C(J_1, E)$, then the above properties hold for each $t \in J_2$ or $t \in J_1$ respectively.

Lemma 2.7. [17] *Let $\mu > 0$ and $\delta > 0$. Then for $t > a$, we have*

(i). $I^\mu_{a^+}(t-a)^{\delta-1} = \frac{\Gamma(\delta)}{\Gamma(\delta+\mu)}(t-a)^{\delta+\mu-1}$,

(ii). $D^\mu_{a^+}(t-a)^{\mu-1} = 0, \quad \mu \in (0, 1)$.

Lemma 2.8. [15] *Let $\mu > 0$, $\nu > 0$ and $\gamma = \mu + \nu(1-\mu)$. If $g \in C^\gamma_{1-\gamma}(J_1, E)$, then*

$$I^\gamma_{a^+} D^\gamma_{a^+} g = I^\mu_{a^+} D^{\mu,\nu}_{a^+} g,\ D^\gamma_{a^+} I^\mu_{a^+} g = D^{\nu(1-\mu)}_{a^+} g.$$

Lemma 2.9. [15] *Let $0 < \mu < 1$, $0 \leq \nu \leq 1$ and $g \in C_{1-\gamma}(J_1, E)$. Then*

$$I^\mu_{a^+} D^{\mu,\nu}_{a^+} g(t) = g(t) - \frac{I^{(1-\nu)(1-\mu)}_{a^+} g(a)}{\Gamma(\mu + \nu(1-\mu))}(t-a)^{\mu+\nu(1-\mu)-1}.$$

Moreover, if $\gamma = \mu + \nu(1-\mu)$, $g \in C_{1-\gamma}(J_1, E)$ and $I^{1-\gamma}_{a^+} g \in C^1_{1-\gamma}(J_1, E)$, then

$$I^\gamma_{a^+} D^\gamma_{a^+} g(t) = g(t) - \frac{I^{1-\gamma}_{a^+} g(a)}{\Gamma(\gamma)}(t-a)^{\gamma-1}.$$

Lemma 2.10. [16] *If* $0 < \mu \le \gamma < 1$ *and* $g \in C_\gamma(J_1, E)$*, then*

$$(I^{\mu}_{a^+} g)(a) = \lim_{t \to a^+} I^{\mu}_{a^+} g(t) = 0.$$

Lemma 2.11. [18] *Let* E *be a Banach space and let* Υ_E *be the bounded subsets of* E*. The Kuratowski measure of noncompactness is the map* $\alpha : \Upsilon_E \longrightarrow [0, \infty)$ *defined by*

$$\alpha(\mathcal{S}) = \inf\{\varepsilon > 0 : \mathcal{S} \subset \cup_{i=1}^{m} \mathcal{S}_i \textit{ and the diam } (\mathcal{S}_i) \le \varepsilon\}; \mathcal{S} \subset \Upsilon_E.$$

Lemma 2.12. [14, 13] *For all nonempty subsets* $\mathcal{S}_1, \mathcal{S}_2 \subset E$*. The Kuratowski measure of noncompactness* $\alpha(\cdot)$ *satisfies the following properties:*

1. $\alpha(\mathcal{S}) = 0 \Longleftrightarrow \overline{\mathcal{S}}$ is compact ($\mathcal{S}$ is relatively compact);
2. $\alpha(\mathcal{S}) = \alpha(\overline{\mathcal{S}}) = \alpha(conv\mathcal{S})$, where where $\overline{\mathcal{S}}$ and $conv\mathcal{S}$ denote the closure and convex hull of the bounded set $\mathcal{S}$ respectively;
3. $\mathcal{S}_1 \subset \mathcal{S}_2 \Longrightarrow \alpha(\mathcal{S}_1) \le \alpha(\mathcal{S}_2)$;
4. $\alpha(\mathcal{S}_1 + \mathcal{S}_2) \le \alpha(\mathcal{S}_1) + \alpha(\mathcal{S}_2)$, where $\mathcal{S}_1 + \mathcal{S}_2 = \{s_1 + s_2 : s \in \mathcal{S}_1, s \in \mathcal{S}_2\}$;
5. $\alpha(\kappa\mathcal{S}) = |\kappa|\, \alpha(\overline{\mathcal{S}})$, $\kappa \in \mathbb{R}$;

Lemma 2.13. [18] *Let* $\mathbb{B}$ *be a bounded, closed and convex subset of a Banach space* E *such that* $0 \in \mathbb{B}$*, and let* $\mathcal{T}$ *be a continuous mapping of* $\mathbb{B}$ *into itself. If for every subset* $\mathcal{V}$ *of* $\mathbb{B}$

$$\mathcal{V} = \overline{co}\mathcal{T}(\mathcal{V}) \textit{ or } \mathcal{V} = \mathcal{T}(\mathcal{V}) \cup \{0\} \Longrightarrow \alpha(\mathcal{V}) = 0$$

holds. Then $\mathcal{T}$ *has a fixed point.*

Lemma 2.14. [21] *Let* $\mathbb{B}$ *be a bounded, closed and convex subset of a Banach space* $C(J_1, E)$*,* F *is a continuous function on* $J_1 \times J_1$*; and a function* $f : J_1 \times E \longrightarrow E$ *satisfying the Carathéodory conditions, and assume there exists* $\rho \in L^P(J_1, \mathbb{R}^+)$ *such that, for each* $t \in J_1$ *and each bounded set* $\mathbb{B}^* \subset E$*; one has*

$$\lim_{r \longrightarrow 0^+} \alpha(f(J_{t,r} \times \mathbb{B}^*)) \le \rho(t)\alpha(\mathbb{B}^*), \textit{where } J_{t,r} \in [t - r, t] \cap J_1.$$

If $\mathcal{V}$ *is an equicontinuous subset of* $\mathbb{B}$*; then*

$$\alpha\left(\left\{ \int_{J_1} F(t,s) f(s, z(s)) ds : z \in \mathcal{V} \right\}\right) \le \int_{J_1} \|F(t,s)\| \, \rho(s) \alpha(\mathcal{V}(s)) ds.$$

Lemma 2.15. [10] *Let* $\gamma = \mu + \nu(1 - \mu)$ *where* $0 < \mu < 1$ *and* $0 \le \nu \le 1$*. Let* $f : (a, b] \times \mathbb{R} \to \mathbb{R}$ *be a function such that* $f(t, z) \in C_{1-\gamma}[a, b]$ *for any* $z \in C_{1-\gamma}[a, b]$*. If* $z \in C^{\gamma}_{1-\gamma}[a, b]$*, then* z *satisfies IVP*

$$D^{\mu,\nu}_{a^+} z(t) = f(t, z(t)), \quad 0 < \mu < 1, \, 0 \le \nu \le 1, \; t \in [a, b],$$

$$I^{1-\gamma}_{a^+} z(0^+) = z_a, \quad \mu \le \gamma$$

if and only if z *satisfies the Volterra integral equation*

$$z(t) = \frac{z_a}{\Gamma(\gamma)}(t - a)^{\gamma - 1} + \frac{1}{\Gamma(\mu)} \int_a^t (t - s)^{\mu - 1} f(s, z(s)) ds, \quad t > a. \tag{2.6}$$

3. Main results

Now we prove the existence of solution of HNBVP (1.5)-(1.6) in $C^{\gamma}_{1-\gamma}(J_1,E) \subset C^{\mu,\nu}_{1-\gamma}(J_1,E)$ under measure of noncompactness technique and a fixed point theorem of Mönch type.

Definition 3.1. A function $z \in C^{\gamma}_{1-\gamma}(J_1,E)$ is said to be a solution of HNBVP (1.5)-(1.6) if z satisfies the fractional differential equation $D^{\mu,\nu}_{a^+} z(t) = f(t,z(t))$ on J_2, and the nonlocal boundary condition $I^{1-\gamma}_{a^+}\left[cz(a^+) + dz(b^-)\right] = \sum_{k=1}^{m} \lambda_k z(\tau_k)$.

In the beginning, we need the following axiom lemma:

Lemma 3.2. *Let $0<\mu<1$, $0 \le \nu \le 1$ where $\gamma = \mu+\nu(1-\mu)$, and $f : J_2 \times E \to E$ be a function such that $f(t,z) \in C_{1-\gamma}(J_1,E)$ for any $z \in C_{1-\gamma}(J_1,E)$. If $z \in C^{\gamma}_{1-\gamma}(J_1,E)$, then z satisfies HNBVP* (1.5)-(1.6) *if and only if z satisfies the following integral equation*

$$\begin{aligned} z(t) &= \frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)} \frac{1}{(c+d-A)} \sum_{k=1}^{m} \frac{\lambda_k}{\Gamma(\mu)} \int_a^{\tau_k} (\tau_k - s)^{\mu-1} f(s,z(s))ds \\ & - \frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)} \frac{d}{(c+d-A)} \frac{1}{\Gamma(1-\gamma+\mu)} \int_a^{b} (b-s)^{-\gamma+\mu} f(s,z(s))ds \\ & + \frac{1}{\Gamma(\mu)} \int_a^{t} (t-s)^{\mu-1} f(s,z(s))ds, \end{aligned} \tag{3.1}$$

where $A = \sum_{k=1}^{m} \lambda_k \frac{(\tau_k - a)^{\gamma-1}}{\Gamma(\gamma)}$, *and* $c+d \neq A$.

Proof. In view of Lemma 2.15, the solution of (1.5) can be written as

$$z(t) = \frac{I^{1-\gamma}_{a^+} z(a^+)}{\Gamma(\gamma)} (t-a)^{\gamma-1} + \frac{1}{\Gamma(\mu)} \int_a^{t} (t-s)^{\mu-1} f(s,z(s))ds, \quad t>a. \tag{3.2}$$

Applying $I^{1-\gamma}_{a^+}$ on both sides of (3.2) and taking the limit $t \to b^-$, we obtain

$$I^{1-\gamma}_{a^+} z(b^-) = I^{1-\gamma}_{a^+} z(a^+) + \frac{1}{\Gamma(1-\gamma+\mu)} \int_a^{b} (b-s)^{-\gamma+\mu} f(s,z(s))ds. \tag{3.3}$$

Now, we substitute $t = \tau_k$ in (3.2) and multiply by λ_k to obtain

$$\lambda_k z(\tau_k) = \lambda_k \left[\frac{I^{1-\gamma}_{a^+} z(a^+)}{\Gamma(\gamma)} (\tau_k - a)^{\gamma-1} + \frac{1}{\Gamma(\mu)} \int_a^{\tau_k} (\tau_k - s)^{\mu-1} f(s,z(s))ds \right]. \tag{3.4}$$

Using the nonlocal boundary condition (1.6) with (3.3) and (3.4), we have

$$\begin{aligned} I^{1-\gamma}_{a^+} z(a^+) &= \frac{1}{c} \sum_{k=1}^{m} \lambda_k z(\tau_k) - \frac{d}{c} I^{1-\gamma}_{a^+} z(a^+) \\ & + \frac{d}{c\Gamma(1-\gamma+\mu)} \int_a^{b} (b-s)^{-\gamma+\mu} f(s,z(s))ds. \end{aligned}$$

Therefore, by (3.4), we have

$$\begin{aligned}
I_{a^+}^{1-\gamma} z(a^+) &= \frac{1}{c}\sum_{k=1}^{m}\lambda_k \frac{I_{a^+}^{1-\gamma} z(a^+)}{\Gamma(\gamma)}(\tau_k - a)^{\gamma-1} \\
&+\frac{1}{c}\sum_{k=1}^{m}\frac{\lambda_k}{\Gamma(\mu)}\int_a^{\tau_k}(\tau_k - s)^{\mu-1} f(s,z(s))ds \\
&-\frac{d}{c} I_{a^+}^{1-\gamma} z(a^+) - \frac{d}{c}\frac{1}{\Gamma(1-\gamma+\mu)}\int_a^b (b-s)^{-\gamma+\mu} f(s,z(s))ds. \\
&= \frac{1}{(c+d-A)}\sum_{k=1}^{m}\frac{\lambda_k}{\Gamma(\mu)}\int_a^{\tau_k}(\tau_k - s)^{\mu-1} f(s,z(s))ds \\
&-\frac{d}{(c+d-A)}\frac{1}{\Gamma(1-\gamma+\mu)}\int_a^b (b-s)^{-\gamma+\mu} f(s,z(s))ds. \qquad (3.5)
\end{aligned}$$

Submitting (3.5) into (3.2), we obtain

$$\begin{aligned}
z(t) &= \frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)}\frac{1}{(c+d-A)}\sum_{k=1}^{m}\frac{\lambda_k}{\Gamma(\mu)}\int_a^{\tau_k}(\tau_k - s)^{\mu-1} f(s,z(s))ds \\
&-\frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)}\frac{d}{(c+d-A)}\frac{1}{\Gamma(1-\gamma+\mu)}\int_a^b (b-s)^{-\gamma+\mu} f(s,z(s))ds \\
&+\frac{1}{\Gamma(\mu)}\int_a^t (t-s)^{\mu-1} f(s,z(s))ds. \qquad (3.6)
\end{aligned}$$

Conversely, applying $I_{a^+}^{1-\gamma}$ on both sides of (3.1), then it follows from Lemmas 2.6, 2.7, and some simple computations that

$$\begin{aligned}
&I_{a^+}^{1-\gamma}\big(cz(a^+) + dz(b^-)\big) \\
&= \frac{c}{(c+d-A)}\sum_{k=1}^{m}\frac{\lambda_k}{\Gamma(\mu)}\int_a^{\tau_k}(\tau_k - s)^{\mu-1} f(s,z(s))ds \\
&-\frac{cd}{(c+d-A)}\frac{1}{\Gamma(1-\gamma+\mu)}\int_a^b (b-s)^{-\gamma+\mu} f(s,z(s))ds \\
&+\frac{d}{(c+d-A)}\sum_{k=1}^{m}\frac{\lambda_k}{\Gamma(\mu)}\int_a^{\tau_k}(\tau_k - s)^{\mu-1} f(s,z(s))ds \\
&-\frac{d^2}{(c+d-A)}\frac{1}{\Gamma(1-\gamma+\mu)}\int_a^b (b-s)^{-\gamma+\mu} f(s,z(s))ds \\
&+\frac{d}{\Gamma(1-\gamma+\mu)}\int_a^b (b-s)^{-\gamma+\mu} f(s,z(s))ds.
\end{aligned}$$

Which implies

$$\begin{aligned} & I_{a^+}^{1-\gamma}\big(cz(a^+)+dz(b^-)\big) \\ = & \left(\frac{c}{(c+d-A)}+\frac{d}{(c+d-A)}\right)\sum_{k=1}^{m}\frac{\lambda_k}{\Gamma(\mu)}\int_a^{\tau_k}(\tau_k-s)^{\mu-1}f(s,z(s))ds \\ & -\left(d-\frac{cd}{(c+d-A)}-\frac{d^2}{(c+d-A)}\right)\int_a^b\frac{(b-s)^{-\gamma+\mu}}{\Gamma(1-\gamma+\mu)}f(s,z(s))ds \\ = & \frac{c+d}{(c+d-A)}\sum_{k=1}^{m}\frac{\lambda_k}{\Gamma(\mu)}\int_a^{\tau_k}(\tau_k-s)^{\mu-1}f(s,z(s))ds \\ & -\frac{Ad}{(c+d-A)}\frac{1}{\Gamma(1-\gamma+\mu)}\int_a^b(b-s)^{-\gamma+\mu}f(s,z(s))ds. \end{aligned}$$

From (3.4) and (3.5), we conclude that

$$I_{a^+}^{1-\gamma}\big(cz(a^+)+dz(b^-)\big)=\sum_{k=1}^{m}\lambda_k z(\tau_k),$$

which shows that the nonlocal boundary condition (1.6) is satisfied.
Next, applying $D_{a^+}^{\gamma}$ on both sides of (3.1) and using Lemmas 2.7 and 2.8, we have

$$D_{a^+}^{\gamma}z(t)=D_{a^+}^{\nu(1-\mu)}f\big(t,z(t)\big). \tag{3.7}$$

Since $z\in C_{1-\gamma}^{\gamma}(J_1,E)$ and by definition of $C_{1-\gamma}^{\gamma}(J_1,E)$, we have $D_{a^+}^{\gamma}z\in C_{1-\gamma}(J_1,E)$, therefore, $D_{a^+}^{\nu(1-\mu)}f=DI_{a^+}^{1-\nu(1-\mu)}f\in C_{1-\gamma}(J_1,E)$. For $f\in C_{1-\gamma}(J_1,E)$, it is clear that $I_{a^+}^{1-\nu(1-\mu)}f\in C_{1-\gamma}(J_1,E)$. Hence f and $I_{a^+}^{1-\nu(1-\mu)}f$ satisfy the hypothesis of Lemma 2.9.
Now, by applying $I_{a^+}^{\nu(1-\mu)}$ on both sides of (3.7), we have

$$I_{a^+}^{\nu(1-\mu)}D_{a^+}^{\gamma}z(t)=I_{a^+}^{\nu(1-\mu)}D_{a^+}^{\nu(1-\mu)}f\big(t,z(t)\big).$$

Using Remark 2.4 (i), relation (3.7) and Lemma 2.9, we get

$$D_{a^+}^{\mu,\nu}z(t)=f\big(t,z(t)\big)-\frac{I_{a^+}^{1-\nu(1-\mu)}f\big(a,z(a)\big)}{\Gamma(\nu(1-\mu))}(t-a)^{\nu(1-\mu)-1},\ \text{for all } t\in J_2.$$

By Lemma 2.10, we have $I_{a^+}^{1-\nu(1-\mu)}f\big(a,z(a)\big)=0$. Therefore $D_{a^+}^{\mu,\nu}z(t)=f\big(t,z(t)\big)$. This completes the proof.
To prove the existence of solutions for the problem at hand, let us make the following hypotheses.

(H1) The function $f:J_2\times E\to E$ satisfies the Carathèodory conditions.

(H2) $f:J_2\times E\to E$ is a function such that $f(\cdot,z(\cdot))\in C_{1-\gamma}^{\nu(1-\mu)}(J_1,E)$ for any $z\in C_{1-\gamma}(J_1,E)$ and there exists $\rho\in L^p(J_1,\mathbb{R}^+)$ with $p>\frac{1}{\mu}$ and $p>\frac{1}{\gamma}$ such that

$$\left\|f\big(t,z\big)\right\|\le\rho(t)\left\|z\right\|,$$

for each $t\in J_2$, and all $z\in E$.

(H3) The following inequalities

$$\begin{aligned}\mathcal{G} \ : \ &=\Bigg(\frac{1}{\Gamma(\gamma)}\frac{(\Lambda_{q,\mu,\gamma})^{\frac{1}{q}}}{(c+d-A)}\sum_{k=1}^{m}\frac{\lambda_k}{\Gamma(\mu)}(\tau_k-a)^{\gamma+\mu-1}\\ &+(\frac{1}{\Gamma(\gamma)}\left|\frac{d}{(c+d-A)}\right|\frac{(\Delta_{q,\mu,\gamma})^{\frac{1}{q}}}{\Gamma(1-\gamma+\mu)}+\frac{(\Lambda_{q,\mu,\gamma})^{\frac{1}{q}}}{\Gamma(\mu)})(b-a)^{\mu}\Bigg)\|\rho\|_{L^p}<1,\end{aligned}$$

and

$$\begin{aligned}L^* \ : \ &=\Bigg(\frac{m}{\Gamma(\gamma)}\frac{(b-a)^{\gamma-1}}{(c+d-A)}\sum_{k=1}^{m}\frac{\lambda_k(\tau_k-a)^{\mu}}{\Gamma(\mu+1)}\\ &+(\frac{1}{\Gamma(\gamma)}\left|\frac{d}{(c+d-A)}\right|\frac{1}{\Gamma(-\gamma+\mu)}+\frac{1}{\Gamma(\mu+1)})(b-a)^{\mu}\Bigg)\|\rho\|_{L^p}<1\end{aligned}$$

hold, where $q>1$, $\frac{1}{p}+\frac{1}{q}=1$ and

$$\Lambda_{q,\mu,\gamma}:=\frac{\Gamma(q(\mu-1)+1)\Gamma(q(\gamma-1)+1)}{\Gamma(q(\mu+\gamma-2)+2)},$$

$$\Delta_{q,\mu,\gamma}:=\frac{\Gamma(q(\mu-\gamma)+1)\Gamma(q(\gamma-1)+1)}{\Gamma(q(\mu-1)+2)}.$$

Now, we are ready to prove the existence of solutions for the HNBVP (1.5)-(1.6), which is based on fixed point theorem of Mönch's type.

Theorem 3.3. *Assume that (H1)-(H3) are satisfied. Then HNBVP* (1.5)-(1.6) *has at least one solution in* $C^{\gamma}_{1-\gamma}(J_1,E)\subset C^{\mu,\nu}_{1-\gamma}(J_1,E)$.

Proof. Transform the problem (1.5)-(1.6) into a fixed point problem. Define the operator $\mathcal{T}:C_{1-\gamma}(J_1,E)\longrightarrow C_{1-\gamma}(J_1,E)$ as

$$\begin{aligned}\mathcal{T}z(t) \ &= \ z(t)=\frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)}\frac{1}{(c+d-A)}\sum_{k=1}^{m}\frac{\lambda_k}{\Gamma(\mu)}\int_a^{\tau_k}(\tau_k-s)^{\mu-1}f(s,z(s))ds\\ &-\frac{(t-a)^{\gamma-1}}{\Gamma(\gamma)}\frac{d}{(c+d-A)}\frac{1}{\Gamma(1-\gamma+\mu)}\int_a^b(b-s)^{-\gamma+\mu}f(s,z(s))ds\\ &+\frac{1}{\Gamma(\mu)}\int_a^t(t-s)^{\mu-1}f(s,z(s))ds. \qquad (3.8)\end{aligned}$$

Clearly, from Lemma 3.2, the fixed points of $\mathcal{T}$ are solutions to (1.5)-(1.6). Let

$$\mathbb{B}_R=\left\{z\in C_{1-\gamma}(J_1,E):\|z\|_{C_{1-\gamma}}\leq R\right\}.$$

We shall show that $\mathcal{T}$ satisfies the conditions of Mönch's fixed point theorem. The proof will be given in the following four steps:

Step 1. We show that $\mathcal{T}(\mathbb{B}_R) \subset \mathbb{B}_R$. From the hypothesis (H_2) and Hölder's inequality, we have

$$\begin{aligned}
&\left|(\mathcal{T}z)(t)(t-a)^{1-\gamma}\right| \\
= &\frac{1}{\Gamma(\gamma)}\frac{1}{(c+d-A)}\sum_{k=1}^{m}\frac{\lambda_k}{\Gamma(\mu)}\int_a^{\tau_k}(\tau_k-s)^{\mu-1}\left|f(s,z(s))\right|ds \\
&+\frac{1}{\Gamma(\gamma)}\left|\frac{d}{(c+d-A)}\right|\frac{1}{\Gamma(1-\gamma+\mu)}\int_a^b(b-s)^{-\gamma+\mu}\left|f(s,z(s))\right|ds \\
&+\frac{(t-a)^{1-\gamma}}{\Gamma(\mu)}\int_a^t(t-s)^{\mu-1}\left|f(s,z(s))\right|ds \\
\leq &\frac{1}{\Gamma(\gamma)}\frac{1}{(c+d-A)}\sum_{k=1}^{m}\frac{\lambda_k}{\Gamma(\mu)}\int_a^{\tau_k}(\tau_k-s)^{\mu-1}(s-a)^{\gamma-1}\rho(s)\left\|z\right\|_{C_{1-\gamma}}ds \\
&+\frac{1}{\Gamma(\gamma)}\left|\frac{d}{(c+d-A)}\right|\int_a^b\frac{(b-s)^{-\gamma+\mu}}{\Gamma(1-\gamma+\mu)}(s-a)^{\gamma-1}\rho(s)\left\|z\right\|_{C_{1-\gamma}}ds \\
&+\frac{(t-a)^{1-\gamma}}{\Gamma(\mu)}\int_a^t(t-s)^{\mu-1}(s-a)^{\gamma-1}\rho(s)\left\|z\right\|_{C_{1-\gamma}}ds \\
\leq &\frac{1}{\Gamma(\gamma)}\sum_{k=1}^{m}\frac{\lambda_k}{\Gamma(\mu)}\left(\int_a^{\tau_k}\frac{(\tau_k-s)^{(\mu-1)q}}{(c+d-A)}(s-a)^{(\gamma-1)q}ds\right)^{\frac{1}{q}}\left\|\rho\right\|_{L^p}\left\|z\right\|_{C_{1-\gamma}} \\
&+\frac{1}{\Gamma(\gamma)}\left|\frac{d}{(c+d-A)}\right|\left(\int_a^b\frac{(b-s)^{(-\gamma+\mu)q}}{\Gamma(1-\gamma+\mu)}(s-a)^{(\gamma-1)q}ds\right)^{\frac{1}{q}} \\
&\times\left\|\rho\right\|_{L^p}\left\|z\right\|_{C_{1-\gamma}}+\frac{(t-a)^{1-\gamma}}{\Gamma(\mu)} \\
&\times\left(\int_a^t(t-s)^{(\mu-1)q}(s-a)^{(\gamma-1)q}ds\right)^{\frac{1}{q}}\left\|\rho\right\|_{L^p}\left\|z\right\|_{C_{1-\gamma}}.
\end{aligned} \tag{3.9}$$

Since $q>1$, $p>\frac{1}{\mu}$ and $\frac{1}{p}+\frac{1}{q}=1$, the change of variable $s=a-u(\tau_k-a)$ yields

$$\left(\int_a^{\tau_k}(\tau_k-s)^{(\mu-1)q}(s-a)^{(\gamma-1)q}ds\right)^{\frac{1}{q}} \leq \left(\Lambda_{q,\mu,\gamma}\right)^{\frac{1}{q}}(\tau_k-a)^{\gamma+\mu-1}, \tag{3.10}$$

the change of variable $s=a-u(b-a)$ gives

$$\left(\int_a^b(b-s)^{(-\gamma+\mu)q}(s-a)^{(\gamma-1)q}ds\right)^{\frac{1}{q}} \leq \left(\Delta_{q,\mu,\gamma}\right)^{\frac{1}{q}}(b-a)^{\mu}, \tag{3.11}$$

and the change of variable $s=a-u(t-a)$ gives us

$$\left(\int_a^t(t-s)^{(\mu-1)q}(s-a)^{(\gamma-1)q}ds\right)^{\frac{1}{q}} \leq \left(\Lambda_{q,\mu,\gamma}\right)^{\frac{1}{q}}(t-a)^{\gamma+\mu-1}. \tag{3.12}$$

Substitution of (3.10),(3.11) and (3.12) into (3.9) leads

$$
\begin{aligned}
&\left|(\mathcal{T}z)(t)(t-a)^{1-\gamma}\right| \\
\leq\ & \frac{1}{\Gamma(\gamma)}\frac{1}{(c+d-A)}\sum_{k=1}^{m}\frac{\lambda_k}{\Gamma(\mu)}\left(\Lambda_{q,\mu,\gamma}\right)^{\frac{1}{q}}(\tau_k-a)^{\gamma+\mu-1}\|\rho\|_{L^p}\|z\|_{C_{1-\gamma}} \\
&+\frac{1}{\Gamma(\gamma)}\left|\frac{d}{(c+d-A)}\right|\frac{1}{\Gamma(1-\gamma+\mu)}\left(\Delta_{q,\mu,\gamma}\right)^{\frac{1}{q}}(b-a)^{\mu}\|\rho\|_{L^p}\|z\|_{C_{1-\gamma}} \\
&+\frac{(t-a)^{1-\gamma}}{\Gamma(\mu)}\left(\Lambda_{q,\mu,\gamma}\right)^{\frac{1}{q}}(t-a)^{\gamma+\mu-1}\|\rho\|_{L^p}\|z\|_{C_{1-\gamma}}.
\end{aligned}
$$

For any $z\in\mathbb{B}_R$, we obtain

$$
\begin{aligned}
\|\mathcal{T}z\|_{C_{1-\gamma}} \leq \Bigg(&\frac{1}{\Gamma(\gamma)}\frac{\left(\Lambda_{q,\mu,\gamma}\right)^{\frac{1}{q}}}{(c+d-A)}\sum_{k=1}^{m}\frac{\lambda_k}{\Gamma(\mu)}(\tau_k-a)^{\gamma+\mu-1} \\
&+\Big(\frac{1}{\Gamma(\gamma)}\left|\frac{d}{(c+d-A)}\right|\frac{\left(\Delta_{q,\mu,\gamma}\right)^{\frac{1}{q}}}{\Gamma(1-\gamma+\mu)}+\frac{\left(\Lambda_{q,\mu,\gamma}\right)^{\frac{1}{q}}}{\Gamma(\mu)}\Big)(b-a)^{\mu}\Bigg)\|\rho\|_{L^p}R.
\end{aligned}
$$

By (H3), we have $\|\mathcal{T}z\|_{C_{1-\gamma}}\leq\mathcal{G}R\leq R$, that is, $\mathcal{T}(\mathbb{B}_R)\subset\mathbb{B}_R$.
Step 2. We shall prove that $\mathcal{T}$ is completely continuous.
The operator $\mathcal{T}$ is continuous. Let $\{z_n\}_{n\in\mathbb{N}}$ is a sequence such that $z_n\to z$ in $\mathbb{B}_R$. Then for each $t\in J_2$, we have

$$
\begin{aligned}
&\left|\big((\mathcal{T}z_n)(t)-(\mathcal{T}z)(t)\big)(t-a)^{1-\gamma}\right| \\
=\ & \frac{1}{\Gamma(\gamma)}\frac{1}{(c+d-A)}\sum_{k=1}^{m}\frac{\lambda_k}{\Gamma(\mu)}\int_a^{\tau_k}(\tau_k-s)^{\mu-1}\left|f(s,z_n(s))-f(s,z(s))\right|ds \\
&+\frac{1}{\Gamma(\gamma)}\left|\frac{d}{(c+d-A)}\right|\int_a^b\frac{(b-s)^{-\gamma+\mu}}{\Gamma(1-\gamma+\mu)}\left|f(s,z_n(s))-f(s,z(s))\right|dds \\
&+\frac{(t-a)^{1-\gamma}}{\Gamma(\mu)}\int_a^t(t-s)^{\mu-1}\left|f(s,z_n(s))-f(s,z(s))\right|dds \\
\leq\ & \frac{1}{\Gamma(\gamma)}\frac{1}{(c+d-A)}\sum_{k=1}^{m}\frac{\lambda_k}{\Gamma(\mu)}\int_a^{\tau_k}(\tau_k-s)^{\mu-1}(s-a)^{\gamma-1}ds \\
&\times\left\|f\big(\cdot,z_n(\cdot)\big)-f\big(\cdot,z(\cdot)\big)\right\|_{C_{1-\gamma}} \\
&+\frac{1}{\Gamma(\gamma)}\left|\frac{d}{(c+d-A)}\right|\frac{1}{\Gamma(1-\gamma+\mu)}\int_a^b(b-s)^{-\gamma+\mu}(s-a)^{\gamma-1}ds \\
&\times\left\|f\big(\cdot,z_n(\cdot)\big)-f\big(\cdot,z(\cdot)\big)\right\|_{C_{1-\gamma}} \\
&+\frac{(t-a)^{1-\gamma}}{\Gamma(\mu)}\int_a^t(t-s)^{\mu-1}(s-a)^{\gamma-1}ds\left\|f\big(\cdot,z_n(\cdot)\big)-f\big(\cdot,z(\cdot)\big)\right\|_{C_{1-\gamma}}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\left|\big((\mathcal{T}z_n)(t)-(\mathcal{T}z)(t)\big)(t-a)^{1-\gamma}\right| \\
\leq\ & \frac{1}{(c+d-A)}\frac{\mathcal{B}(\gamma,\mu)}{\Gamma(\mu)\Gamma(\gamma)}\sum_{k=1}^{m}\frac{\lambda_k(\tau_k-a)^{\gamma-1+\mu}}{\Gamma(\mu)}\left\|f\big(\cdot,z_n(\cdot)\big)-f\big(\cdot,z(\cdot)\big)\right\|_{C_{1-\gamma}} \\
&+\left|\frac{d}{(c+d-A)}\right|\frac{(b-a)^{\mu}}{\Gamma(\mu+1)}\left\|f\big(\cdot,z_n(\cdot)\big)-f\big(\cdot,z(\cdot)\big)\right\|_{C_{1-\gamma}} \\
&+\frac{(b-a)^{\mu}}{\Gamma(\mu)}\frac{\mathcal{B}(\gamma,\mu)}{\Gamma(\mu)}\left\|f\big(\cdot,z_n(\cdot)\big)-f\big(\cdot,z(\cdot)\big)\right\|_{C_{1-\gamma}}.
\end{aligned}
$$

By (H1) and the Lebesgue dominated convergence theorem, we have

$$
\|(\mathcal{T}z_n-\mathcal{T}z)\|_{C_{1-\gamma}}\longrightarrow 0 \ as\ n\longrightarrow\infty,
$$

which means that operator $\mathcal{T}$ is continuous on $\mathbb{B}_R$.

Step 3. $\mathcal{T}(\mathbb{B}_R)$ is relatively compact.
From Step 1, we have $\mathcal{T}(\mathbb{B}_R)\subset\mathbb{B}_R$. It follows that $\mathcal{T}(\mathbb{B}_R)$ is uniformly bounded i.e. $\mathcal{T}$ maps $\mathbb{B}_R$ into itself. Moreover, we show that operator $\mathcal{T}$ is equicontinuous on $\mathbb{B}_R$. Indeed, for any $a<t_1<t_2<b$ and $z\in\mathbb{B}_R$, we get

$$
\begin{aligned}
&\left|(t_2-a)^{1-\gamma}\big(\mathcal{T}z\big)(t_2)-(t_1-a)^{1-\gamma}\big(\mathcal{T}z\big)(t_1)\right| \\
\leq\ & \frac{1}{\Gamma(\mu)}\Bigg|(t_2-a)^{1-\gamma}\int_a^{t_2}(t_2-s)^{\mu-1}f\big(s,z(s)\big)ds \\
&-(t_1-a)^{1-\gamma}\int_a^{t_1}(t_1-s)^{\mu-1}f\big(s,z(s)\big)ds\Bigg| \\
\leq\ & \frac{\|f\|_{C_{1-\gamma}}}{\Gamma(\mu)}\Bigg|(t_2-a)^{1-\gamma}\int_a^{t_2}(t_2-s)^{\mu-1}(s-a)^{\gamma-1}ds \\
&-(t_1-a)^{1-\gamma}\int_a^{t_1}(t_1-s)^{\mu-1}(s-a)^{\gamma-1}ds\Bigg| \\
\leq\ & \|f\|_{C_{1-\gamma}}\frac{\mathcal{B}(\gamma,\mu)}{\Gamma(\mu)}\left|(t_2-a)^{\mu}-(t_1-a)^{\mu}\right|,
\end{aligned}
$$

which tends to zero as $t_2\to t_1$, independent of $z\in\mathbb{B}_R$, where $\mathcal{B}(\cdot,\cdot)$ is a Beta function. Thus we conclude that $\mathcal{T}(\mathbb{B}_R)$ is equicontinuous on $\mathbb{B}_r$ and therefore is relatively compact. As a consequence of Steps 1 to 3 together with Arzela-Ascoli theorem, we conclude that $\mathcal{T}:\mathbb{B}_R\to\mathbb{B}_R$ is completely continuous operator.

Step 4. The Mönch condition is satisfied.
Let $\mathcal{V}$ be a subset of $\mathbb{B}_R$ such that $\mathcal{V}\subset\overline{co}\left(\mathcal{T}(\mathcal{V})\cup\{0\}\right)$. $\mathcal{V}$ is bounded and equicontinuous, and therefore the function $t\longrightarrow\alpha(\mathcal{V}(t))$ is continuous on J_1. By (H2)-(H3),

Lemma 2.6, and the properties of the measure α, for each $t \in J_2$

$$\begin{aligned}
\alpha(\mathcal{V}(t)) &\leq \alpha(\mathcal{T}(\mathcal{V})(t) \cup \{0\}) \leq \alpha(\mathcal{T}(\mathcal{V})(t)) \\
&\leq \frac{1}{\Gamma(\gamma)} \frac{(t-a)^{\gamma-1}}{(c+d-A)} \sum_{k=1}^{m} \frac{\lambda_k}{\Gamma(\mu)} \int_a^{\tau_k} (\tau_k - s)^{\mu-1} \rho(s) \alpha(\mathcal{V}(s)) ds \\
&+ \frac{1}{\Gamma(\gamma)} \left| \frac{d(t-a)^{\gamma-1}}{(c+d-A)} \right| \frac{1}{\Gamma(1-\gamma+\mu)} \int_a^b (b-s)^{-\gamma+\mu} \rho(s) \alpha(\mathcal{V}(s)) ds \\
&+ \frac{1}{\Gamma(\mu)} \int_a^t (t-s)^{\mu-1} \rho(s) \alpha(\mathcal{V}(s)) ds \\
&\leq \frac{1}{\Gamma(\gamma)} \frac{(b-a)^{\gamma-1}}{(c+d-A)} \sum_{k=1}^{m} \frac{\lambda_k}{\Gamma(\mu)} \left(\int_a^{\tau_k} (\tau_k - s)^{(\mu-1)q} ds \right)^{\frac{1}{q}} \|\rho\|_{L^p} m \alpha(\mathcal{V}(b)) \\
&+ \frac{1}{\Gamma(\gamma)} \left| \frac{d(b-a)^{\gamma-1}}{(c+d-A)} \right| \frac{1}{\Gamma(1-\gamma+\mu)} \left(\int_a^b (b-s)^{(-\gamma+\mu)q} ds \right)^{\frac{1}{q}} \\
&\times \|\rho\|_{L^p} \alpha(\mathcal{V}(b)) + \frac{1}{\Gamma(\mu)} \left(\int_a^t (t-s)^{(\mu-1)q} ds \right)^{\frac{1}{q}} \|\rho\|_{L^p} \alpha(\mathcal{V}(b)).
\end{aligned}$$

From the facts

$$\frac{1}{q} < 1 \Longrightarrow \frac{1}{(\mu-1)q+1} < \frac{1}{\mu},\ 1-\mu \neq \frac{1}{q},$$

and

$$\frac{1}{q} < 1 \Longrightarrow \frac{1}{(-\gamma+\mu)q+1} < \frac{1}{-\gamma+\mu+1},\ \gamma-\mu \neq \frac{1}{q},$$

we get

$$\begin{aligned}
\alpha(\mathcal{V}(t)) \leq & \left(\frac{m}{\Gamma(\gamma)} \frac{(b-a)^{\gamma-1}}{(c+d-A)} \sum_{k=1}^{m} \frac{\lambda_k (\tau_k - a)^{\mu}}{\Gamma(\mu+1)} \right. \\
& \left. + \frac{1}{\Gamma(\gamma)} \left| \frac{d}{(c+d-A)} \right| \frac{(b-a)^{\mu}}{\Gamma(-\gamma+\mu)} + \frac{(t-a)^{\mu}}{\Gamma(\mu+1)} \right) \|\rho\|_{L^p} \alpha(\mathcal{V}(b)).
\end{aligned}$$

It follows that

$$\|\alpha(\mathcal{V})\|_{L^\infty} (1 - L^*) \leq 0.$$

This means $\|\alpha(\mathcal{V})\|_{L^\infty} = 0$, i.e. $\alpha(\mathcal{V}(t)) = 0$ for all $t \in J_2$. Thus $\mathcal{V}(t)$ is relatively compact in E. In view of Arzela-Ascoli theorem, $\mathcal{V}$ is relatively compact in $\mathbb{B}_R$. An application of Lemma 2.13 shows that $\mathcal{T}$ has a fixed point which is a solution of HNBVP (1.5)-(1.6).

Finally, we show that such a solution is indeed in $C^{\gamma}_{1-\gamma}(J_1, E)$. We apply $D^{\gamma}_{a^+}$ on both sides of (3.8), and using Lemmas 2.7, 2.8, to get

$$D^{\gamma}_{a^+} z(t) = D^{\gamma}_{a^+} I^{\mu}_{a^+} f(t, z(t)) = D^{\nu(1-\mu)}_{a^+} f(t, z(t)).$$

Since $f(\cdot, z(\cdot)) \in C^{\nu(1-\mu)}_{1-\gamma}(J_1, E)$, it follows by definition of the space $C^{\nu(1-\mu)}_{1-\gamma}(J_1, E)$ that $D^{\gamma}_{a^+} z(t) \in C_{1-\gamma}(J_1, E)$ which implies that $z(t) \in C^{\gamma}_{1-\gamma}(J_1, E)$. The proof is complete. □

4. An example

We consider the Hilfer fractional differential equation with nonlocal boundary condition

$$\begin{cases} D_{0^+}^{\mu,\nu} z(t) = f\big(t, z(t)\big), \quad t \in (0,1],\ 0 < \mu < 1,\ 0 \le \nu \le 1, \\ I_{0^+}^{1-\gamma}\big[\frac{1}{4}z(0^+) + \frac{3}{4}z(1^-)\big] = \frac{2}{5}z(\frac{2}{3}), \quad \mu \le \gamma = \mu + \nu(1-\mu), \end{cases} \tag{4.1}$$

where $f\big(t, z(t)\big) = \frac{1}{16} t \sin|z(t)|$, $\mu = \frac{1}{3}$, $\nu = \frac{1}{4}$, $\gamma = \frac{1}{2}$, $c = \frac{1}{4}$, $d = \frac{3}{4}$, $\lambda_1 = \frac{2}{5}$ and $m = 1$, $\tau_1 = \frac{2}{3}$. Let $E = \mathbb{R}^+$ and $J_2 = (0,1]$.
Clearly we can see that

$$\sqrt{t} f\big(t, z\big) = \frac{1}{16}\sqrt[3]{t}\sin z \in C([0,1], \mathbb{R}^+),$$

and hence $f\big(t, z\big) \in C_{\frac{1}{2}}([0,1], \mathbb{R}^+)$. Also, observe that, for $t \in (0,1]$ and for any $z \in C_{\frac{1}{2}}([0,1], \mathbb{R}^+)$,

$$\big\| f\big(t, z\big) \big\| \le \frac{1}{16} t \, \|z\| .$$

Therefore, the conditions (H1) and (H2) is satisfied with $\rho(t) = \frac{1}{16}t \in L^p(0,1)$. Select $p = 4$, we have

$$\|\rho\|_{L^4} = \left(\int_0^1 \left| \frac{1}{16} s \right|^4 ds \right)^{\frac{1}{4}} = \frac{327\,680^{\frac{3}{4}}}{327\,680}.$$

It is easy to check that conditions in (H3) are satisfied too. Indeed, by some simple computations with $q = \frac{4}{3}$, we get

$$\Lambda_{q,\mu,\gamma} = \frac{\Gamma(q(\mu-1)+1)\Gamma(q(\gamma-1)+1)}{\Gamma(q(\mu+\gamma-2)+2)} = \frac{\Gamma(\frac{1}{9})\Gamma(\frac{1}{3})}{\Gamma(\frac{4}{9})},$$

and

$$\Delta_{q,\mu,\gamma} = \frac{\Gamma(q(\mu-\gamma)+1)\Gamma(q(\gamma-1)+1)}{\Gamma(q(\mu-1)+2)} = \frac{\Gamma(\frac{7}{9})\Gamma(\frac{1}{3})}{\Gamma(\frac{10}{9})},$$

also, we have

$$A = \lambda_1 \frac{(\tau_1)^{\gamma-1}}{\Gamma(\gamma)} = \frac{\sqrt{\frac{6}{\pi}}}{5}.$$

It follows that $\mathcal{G} \simeq 0.35 < 1$, and $L^* \simeq 0.06 < 1, (m = 1)$. An application of Theorem 3.3 implies that problem (4.1) has a solution in $C_{\frac{1}{2}}^{\frac{1}{2}}([0,1], \mathbb{R}^+)$.

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References

[1] Abbas, S., Benchohra, M., Lazreg, J.E., Zhou, Y., *A survey on Hadamard and Hilfer fractional differential equations: analysis and stability*, Chaos Solitons Fractals, **102**(2017), 47-71.

[2] Abdo, M.S., Panchal, S.K., *Fractional integro-differential equations involving ψ-Hilfer fractional derivative*, Adv. Appl. Math. Mech., **11**(2019), no. 2, 338-359.

[3] Abdo, M.S., Panchal, S.K., Saeed, A.M., *Fractional Boundary value problem with $\psi-$Caputo fractional derivative*, Proc. Indian Acad. Sci. (Math. Sci.), **129**(2019), no. 5, 65.

[4] Agarwal, R.P., Benchohra, M., Hamani, S., *A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions*, Acta Appl. Math., **109**(2010), no. 3, 973-1033.

[5] Bhairat, S.P., *Existence and continuation of solution of Hilfer fractional differential equations*, J. Math. Model., **7**(2018), no. 1, 1-20.

[6] Bhairat, S.P., Dhaigude, D.B., *Local existence and uniqueness of solutions for Hilfer-Hadamard fractional differential problem*, Nonlinear Dyn. Syst. Theory, **18**(2018), no. 2, 144-153.

[7] Bhairat, S.P., Dhaigude, D.B., *Existence of solutions of generalized fractional differential equation with with nonlocal initial condition*, Math. Bohem., **144**(2019), no. 2, 203-220.

[8] Dhaigude, D.B., Bhairat, S.P., *Existence and uniqueness of solution of Cauchy-type problem for Hilfer fractional differential equations*, Commun. Appl. Anal., **22**(2018), no. 1, 121-134.

[9] Diethelm, K., *The analysis of fractional differential equations*, J. Math. Anal. Appl., **265**(2004), 229-248.

[10] Furati, K.M., Kassim, M.D., *Existence and uniqueness for a problem involving Hilfer fractional derivative*, Comput. Math. Appl., **64**(2012), no. 6, 1616-1626.

[11] Furati, K.M., Tatar, N.E., *An existence result for a nonlocal fractional differential problem*, J. Fract. Calc. Appl., **26**(2004), 43-54.

[12] Gaafar, F.M., *Continuous and integrable solutions of a nonlinear Cauchy problem of fractional order with nonlocal coditions*, J. Egyptian Math. Soc., **22**(2014), no. 3, 341-347.

[13] Gonzalez, C., Melado, A.J., Fuster, E.L., *A Monch type fixed point theorem under the interior condition,* J. Math. Anal. Appl., **352**(2009), no. 2, 816-821.

[14] Granas, A., Dugundji, J., *Fixed Point Theory*, Springer Monographs in Mathematics, Springer-Verlag, New York, 2003.

[15] Hilfer, R., *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000.

[16] Hilfer, R., Luchko, Y., Tomovski, Z., *Operational method for the solution of fractional differential equations with generalized Riemann-Lioville fractional derivative*, Fract. Calc. Appl. Anal., **12**(2009), no. 3, 289-318.

[17] Kilbas, A.A., Srivastava, H.M., Trujillo, J.J., *Theory and Applications of Fractional Differential Equations*, North-Holland Math. Stud., 204 Elsevier, Amsterdam 2006.

[18] Mönch, H., *Boundary value problem for nonlinear ordinary differential equations of second order in Banach spaces,* Nonlinear Anal., **75**(1980) no. 5, 985-999.

[19] Sabri, T.M., Ahmed, B., Agarwal, R.P., *On abstract Hilfer fractional integrodifferential equations with boundary conditions*, Arab J. Math. Sci., (2019).

[20] Vivek, D., Kanagarajan, K., Elsayed, E.M., *Some existence and stability results for Hilfer-fractional implicit differential equations with nonlocal conditions*, Mediterr. J. Math., **15**(2018), no. 1, 1-15.

[21] Szufla, S., *On the application of measure of noncompactness to existence theorems,* Rend. Semin. Mat. Univ. Padova, **75**(1986), 1-14.

[22] Wang, J., Zhang, Y., *Nonlocal initial value problems for differential equations with Hilfer fractional derivative*, Appl. Math. Comput., **266**(2015), 850-859.

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Study of a mixed problem for a nonlinear elasticity system by topological degree

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Abstract. In this paper, we consider a mixed problem for a nonlinear elasticity system with laws of general behavior. The coefficients of elasticity depends on x meanwhile the density of the volumetric forces depends on the displacement. The main aim of this paper is to apply the Schauder's fixed point theorem and the techniques of topological degree to prove a theorem of the existence and the uniqueness of the solution of the corresponding variational problem.

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1. Introduction

This work consists in solving the mixed problem for the nonlinear elasticity system, by means of two methods, namely, the theorem of Schauder and the techniques of the topological degree [7].

First, we introduce the following notations needed in this paper. Let Ω be a connected open bounded domain of $\mathbb{R}^N, (N=3)$ with Lipschitz boundary Γ. Let Γ_0 a part of Γ of strictly positive superficial measure, and let Γ_1 be the complement of Γ_0 in Γ. For a given field of displacement u, we associate a linearized displacement tensor $\varepsilon(u)$ defined by

$$\varepsilon\left(\nabla u(x)\right) = \frac{1}{2}\left(\nabla^T u + \nabla u\right),$$

whose components are

$$\varepsilon_{ij}\left(u(x)\right) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), 1 \le i, j \le 3. \tag{1.1}$$

The corresponding constraints tensor $\sigma(u)$ is given by

$$\sigma_{ij}(u(x)) = \sum_{k,h=1}^{3} a_{ijkh}(x)\varepsilon_{kh}(u(x)),\ 1 \leq i,j \leq 3. \tag{1.2}$$

Equation (1.2) describes a linear relation between the stress tensor $\{\sigma_{ij}\}$ and the deformation tensor $\{\varepsilon_{ij}\}$. The elasticity coefficients a_{ijkh} satisfy the following properties:

1. Properties of symmetry:
$$a_{ijkh} = a_{jikh} = a_{ijhk},\ \forall 1 \leq i,j,k,h \leq 3; \tag{1.3}$$
2. Property of ellipticity:
$$\exists \alpha > 0,\ \forall \{\xi_{ij}\} \in \mathbb{R}^{N^2},\ \sum_{k,h=1}^{3} a_{ijkh}\xi_{ij}\xi_{kh} \geq \alpha \sum_{i,j=1}^{3} \xi_{ij}^2. \tag{1.4}$$

2. Position of problem

We consider a fundamental example of a nonlinear elliptic problem derived from the Mechanics of Solids, namely, the nonlinear elasticity system. Let f be such that $f(x,u(x)) = (f_1(x,u(x)), f_2(x,u(x)), f_3(x,u(x)))$ of $(L^2(\Omega))^3$ and $g = (g_1, g_2, g_3)$ of $(L^2(\Gamma_1))^3$, the problem is to find a function $u = (u_1, u_2, u_3)$ solution of the nonlinear elliptic problem:

$$-\sum_{j=1}^{3} \frac{\partial}{\partial x_j}\sigma_{ij}(u) = f_i(x,u) \quad \text{in} \quad \Omega; \quad \forall\ 1 \leq i \leq 3; \tag{2.1}$$

$$u_i = 0 \quad \text{on} \quad \Gamma_0; \quad \forall\ 1 \leq i \leq 3; \tag{2.2}$$

$$\sum_{j=1}^{3} \sigma_{ij}(u)\,\eta_j = g_i \quad \text{on} \quad \Gamma_1; \quad \forall\ 1 \leq i \leq 3. \tag{2.3}$$

Equations (2.1), (2.2) and (2.3) describe the small displacements u from the natural state of a non-homogeneous elastic solid subjected to a volume density of forces f in Ω, and to a superficial density of forces g on Γ_1, the displacements u being fixed by zero on Γ_0, i.e., $\gamma u \mid_{\Gamma_0} = 0$.

Several authors studied the system of elasticity with laws of particular behavior and using various techniques for example in [1], Ciarlet used the implicit function theorem to show the existence and uniqueness of a solution. Dautry-Lions [2], studied the linear problem in a regular boundary domain. Later on, Merouani in [6], [4], [5], studied the Lamé (elasticity) system in a polygonal boundary domain.

The bibliography quoted here does not claim to be exhaustive and the deficiency must be attributed to the author's ignorance and not to the author's ill will.

The tensor of the constraints considered here is linear and grouped, as special cases, some models used in Ciarlet [1], Lions [3] and Dautry-Lions [2]. Let us cite by the way the examples:

1. The problem of pure displacement for a homogeneous or heterogeneous material of St Vennan-Kirchhoff where:
   - The applied volumetric forces f are dead (does not depend on u),
   - The tensor of stress is in the form (material of St Vennan-Kirchhoff) where

$$\begin{cases} \sigma_{ij}(u(x)) = \lambda(trE_{ij}(\nabla u(x))) + 2\mu E_{ij}(\nabla u(x)), \\ 1 \leq i,j \leq 3, \lambda > 0, \mu > 0. \end{cases}$$

2. The coefficients of elasticity have the form:

$$a_{ijpq} = \lambda\delta_{ij}\delta_{pq} + \mu(\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp}), 1 \leq i,\ j,\ p,\ q \leq 3$$

   with, λ and μ depend on x or not.
3. The applied volumetric forces f have the form

$$f(\xi) = |\xi|^{p-1}\xi,\ 1 < p < \infty.$$

The material is not homogeneous, we assume that the functions a_{ijkh} belong to $L^\infty(\Omega)$, $1 \leq i,j,k,h \leq N$ and the elliptic property is uniform, there exist a constant $\alpha > 0$, independent of x, such that (1.4) is verified almost everywhere on Ω.

3. Weak formulation

We suppose that the solution u of (2.1) – (2.3) exists and belongs to $\left(H^2(\Omega)\right)^3$. Multiply the equation (2.1) by $v \in V$, and integrate on Ω, we obtain:

$$-\int_\Omega \sum_{j=1}^{3} \frac{\partial}{\partial x_j}\sigma_{ij}(u)\, v_i(x)\, dx = \int_\Omega f_i(x, u(x))\, v_i(x)\, dx,$$

where

$$V = \left\{ v \in \left(H^1(\Omega)\right)^3;\ v = 0 \text{ in } \Gamma_0 \right\},$$

is a closed vector subspace of $\left(H^1(\Omega)\right)^3$, equipped with the norm $\|.\|_V = \|.\|_{(H^1(\Omega))^3}$. By Green's formula, we have

$$\int_\Omega \sum_{i,j=1}^{3} \sigma_{ij}(u)\frac{\partial v_i}{\partial x_j} dx - \int_\Gamma \sum_{i,j=1}^{3} \sigma_{ij}(u)\,\eta_j v_i d\Gamma = \sum_{i=1}^{3}\int_\Omega f_i(x, u(x))\, v_i(x) dx,$$

which implies

$$\int_\Omega \sum_{i,j=1}^{3}\sum_{k,h=1}^{3} a_{ijkh}(x)\,\varepsilon_{kh}(u)\varepsilon_{ij}(v) dx = \int_\Omega f(x, u(x))\, v(x)\, dx + \int_{\Gamma_1} g(x)\, v(x)\, d\Gamma,\ \forall v \in V.$$

4. Existence theorem

4.1. Existence with Schauder's theorem

Let us first recall the notion of Caratheodory function.

Definition 4.1. (Function of Caratheodory): Let N, p, $q \in \mathbb{N}^*$ and Ω an open set of $\mathbb{R}^N$. Let a be an application of $\Omega \times \mathbb{R}^p$ to $\mathbb{R}^q$. We say that a is a Caratheodory function if $a(\,;\, s)$ is a Borel function for all s of $\mathbb{R}^p$ and $a(x;\,)$ is continuous for almost all x of Ω.

In this section, we need the following assumptions:

$$\begin{cases} \Omega \text{ is a connected open bounded domain of } \mathbb{R}^N, \\ \text{ with Lipschitz boundary } \Gamma; \\ \exists \alpha > 0 \text{ and } \beta > 0 \text{ such as } \alpha \leq a_{ijkh}(s) \leq \beta \, \text{a.e. for all} \, s \in \mathbb{R}; \\ f \in \left(L^\infty(\Omega \times \mathbb{R})\right)^3; \\ \varepsilon_{ij} \text{ is a continuous function}, \forall 1 \leqslant i,j \leqslant 3. \end{cases} \tag{4.1}$$

Under the assumptions (4.1), we try to show the existence of u, the solution of the following nonlinear problem:

$$\begin{cases} u \in V \\ \displaystyle\int_\Omega \sum_{i,j=1}^{3} \sum_{k,h=1}^{3} a_{ijkh}(x) \varepsilon_{kh}(u(x)) \varepsilon_{ij}(v(x)) dx \\ = \displaystyle\int_\Omega f(x, u(x)) v(x) dx + \int_{\Gamma_1} g\,(x)\, v\,(x)\, d\Gamma, \, \forall v \in V \end{cases} \tag{P}$$

Theorem 4.2. *Under the assumptions* (4.1)*, there exist a solution* u *of the problem* (P).

Proof. For $\overline{u} \in \left(L^2(\Omega)\right)^3$, we have the existence and the uniqueness of the solution u of the following problem:

$$\begin{cases} u \in V \\ \displaystyle\int_\Omega \sum_{i,j=1}^{3} \sum_{k,h=1}^{3} a_{ijkh}(x) \varepsilon_{kh}(u(x)) \varepsilon_{ij}(v(x)) dx \\ = \displaystyle\int_\Omega f(x, \overline{u}(x)) v(x) dx + \int_{\Gamma_1} g\,(x)\, v\,(x)\, d\Gamma, \, \forall v \in V. \end{cases} \tag{P1}$$

More precisely, to show the existence and uniqueness of u, the solution of $(P1)$, we apply the Lax-Milgram Lemma [1]. Let $T(\overline{u}) = u$, where T is an application of E in E with

$$E = \left(L^2(\Omega)\right)^3.$$

A fixed point of T is a solution of the problem (P). To prove the existence of such a fixed point, we apply the Schauder's fixed point theorem. First, we will show that the image of T lies in a bounded of V. Using α cited in hypothesis (4.1), we have

$$\begin{cases} \alpha \int\limits_{\Omega} \sum_{i,j=1}^{3} \sum_{k,h=1}^{3} \varepsilon_{kh}(u(x))\varepsilon_{ij}(v(x))dx \\ \leq \int\limits_{\Omega} \sum_{i,j=1}^{3} \sum_{k,h=1}^{3} a_{ijkh}(x)\varepsilon_{kh}(u(x))\varepsilon_{ij}(v(x))dx \\ = \int\limits_{\Omega} f(x,\overline{u}(x))v(x)dx + \int_{\Gamma_1} g(x)\, v(x)\, d\Gamma. \end{cases} \tag{4.2}$$

Taking $v = u$ in (4.2), and using Korn's inequality [8], we obtain

$$\alpha C \left\| u \right\|_{(H^1(\Omega))^3}^2 \leq \int\limits_{\Omega} f(x,\overline{u}(x))u(x)dx + \int_{\Gamma_1} g(x)\, u(x)\, d\Gamma.$$

By Cauchy-Schwartz inequality, the bound L^{∞} of f and the trace theorem, we get,

$$\begin{aligned} \alpha C \left\| u \right\|_{(H^1(\Omega))^3}^2 &\leq C_1 \left\| u \right\|_{(L^2(\Omega))^3} + \left\| g \right\|_{(L^2(\Gamma_1))^3} \left\| u \right\|_{(L^2(\Gamma_1))^3} \\ \alpha C \left\| u \right\|_{(H^1(\Omega))^3}^2 &\leq C_1 \left\| u \right\|_{(H^1(\Omega))^3} + C_2 C_3 \left\| u \right\|_{(H^1(\Omega))^3}, \end{aligned}$$

which implies

$$\left\| u \right\|_V = \left\| u \right\|_{(H^1(\Omega))^3} \leq \frac{C_1 + C_2 C_3}{\alpha C} = R,$$

thus

$$\left\| u \right\|_{(L^2(\Omega))^3} \leq R,$$

so

$$u \in B_R = \left\{ u \in \left(L^2(\Omega)\right)^3 / \ \left\| u \right\|_{(L^2(\Omega))^3} \leq R \right\}.$$

And as a result, the image of T is in a bounded of $V \subset \left(H^1(\Omega)\right)^3$. By Rellich's theorem the image of T is in a compact of $\left(L^2(\Omega)\right)^3$. Taking R large enough, therefore, the application T sends B_R in B_R and $\{T(\overline{u}), \overline{u} \in B_R\}$ is relatively compact in $\left(L^2(\Omega)\right)^3$. To apply Schauder's fixed point theorem, it remains to show the continuity of T. Let $(\overline{u}_n)_{n\in\mathbb{N}}$ a sequence of $\left(L^2(\Omega)\right)^3$ such as $\overline{u}_n \to \overline{u}$ in $\left(L^2(\Omega)\right)^3$, when $n \to +\infty$. Letting $u_n = T(\overline{u}_n)$. After extracting a subsequence, we can assume that $\overline{u}_n \to \overline{u}$ a.e., and that there exist $w \in V$ such that $u_n \to w$ weakly in V and so $u_n \to w$ strongly in $\left(L^2(\Omega)\right)^3$). Now, we will show that w is the solution of the problem $(P1)$. Indeed, let $v \in V$, we have

$$\int\limits_{\Omega} \sum_{i,j=1}^{3} \sum_{k,h=1}^{3} a_{ijkh}(x)\varepsilon_{kh}(u_n(x))\varepsilon_{ij}(v(x))dx = \int\limits_{\Omega} f(x,\overline{u}_n)v(x)dx + \int_{\Gamma_1} g(x)\, v(x)\, d\Gamma, \ \forall v \in V$$

Passing to the limit when $n \to +\infty$ (using Dominated Convergence Theorem), we will have

$$\int_\Omega \sum_{i,j=1}^{3} \sum_{k,h=1}^{3} a_{ijkh}(x)\varepsilon_{kh}(w)\varepsilon_{ij}(v(x))dx = \int_\Omega f(x,\overline{u}(x))v(x)dx + \int_{\Gamma_1} g(x)\,v(x)\,d\Gamma,\ \forall v \in V.$$

This proves that $w = T(\overline{u}) = u$. We have thus proved, after extraction of a subsequence, that $T(\overline{u}_n) \to T(\overline{u})$ in $\left(L^2(\Omega)\right)^3$. By the absurd one can show that this convergence remains true without extraction of subsequence. Thus, we have proved the continuity of T. Therefore, we can apply the Schauder's fixed point theorem and to conclude that there is a fixed point of T, which ends the proof. □

4.2. Existence by topological degree

We take again the same previous problem:

$$\begin{cases} u \in V \\ \displaystyle\int_\Omega \sum_{i,j=1}^{3} \sum_{k,h=1}^{3} a_{ijkh}(x)\varepsilon_{kh}(u(x))\varepsilon_{ij}(v(x))dx \\ \displaystyle = \int_\Omega f(x,u(x))v(x)dx + \int_{\Gamma_1} g(x)\,v(x)\,d\Gamma,\ \forall v \in V \end{cases} \tag{P}$$

which is the weak formulation of the problem (2.1)-(2.3).
The following assumptions are made.

$$\begin{cases} (i)\ \Omega \text{ is a connected open bounded domain of } \mathbb{R}^N, \\ \quad \text{with Lipchitez boundary } \Gamma, \\ (ii)\ \varepsilon_{ij} \text{ is a continuous function } \forall 1 \leqslant i,j \leqslant 3, \\ (iii)\ \exists \alpha \text{ and } \beta > 0; \text{ such that } \alpha \le a_{ijkh}(x) \le \beta \text{ a.e. on } \Omega, \\ (iv)\ f \text{ is a Carathéodory function, and } \exists C_2 \ge 0 \text{ and } d \in (L^2(\Omega))^3; \\ |f(x,s)| \le d(x) + C_2\,|s|\,, \\ (v) \displaystyle\lim_{s\to\infty} \frac{f(x,s)}{s} = 0. \end{cases} \tag{4.3}$$

Theorem 4.3. *Under the assumptions* (4.3)*, there exist a solution of the problem* (P)*. In addition, if f does not depend to u, then the solution is unique.*

Proof. The method of the topological degree requires a priory estimates, i.e., the estimates on u, without knowing its existence. We therefore suppose that u is a solution of (P). The great advantage of considering (P) rather than $(P1)$ is to have only u, not u and $\overline{u}$, and this greatly simplifies the estimates.

We rewrite the problem (P) under the following form:

$$\begin{cases} u \in V \\ \int\limits_{\Omega} \sum_{i,j=1}^{3} \sum_{k,h=1}^{3} a_{ijkh}(x)\varepsilon_{kh}(u(x))\varepsilon_{ij}(v(x))dx = \langle F(u), v\rangle_{V',V} \end{cases}$$

where $F(u)$ is, for $u \in \left(L^2(\Omega)\right)^3$, the element of V' defined by

$$\langle F(u), v\rangle_{V',V} = \int\limits_{\Omega} f(x, u(x))v(x)dx + \int_{\Gamma_1} g(x)\, v(x)\, d\Gamma.$$

According to the hypothesis (iv), the Cauchy-Schwartz inequality and the trace theorem, we have

$$\begin{aligned} |\langle F(u), v\rangle| &\leq \int\limits_{\Omega} |f(x, u(x))| \,.\, |v(x)|\, dx + \int_{\Gamma_1} |g(x)| \,.\, |v(x)|\, d\Gamma \\ &\leq \int\limits_{\Omega} |d(x) + C_2 |u|| \,.\, |v|\, dx + \int_{\Gamma_1} |g(x)| \,.\, |v(x)|\, d\Gamma \\ &\leq \int\limits_{\Omega} |d(x)| \,.\, |v|\, dx + C_2 \int\limits_{\Omega} |u|\, |v|\, dx + \int_{\Gamma_1} |g(x)|\, |v(x)|\, d\Gamma \\ &\leq \|d\|_{L^2(\Omega)^3} \|v\|_{L^2(\Omega)^3} + C_2 \|u\|_{L^2(\Omega)^3} \|v\|_{L^2(\Omega)^3} + \|g\|_{L^2(\Gamma_1)^3} \|v\|_{L^2(\Gamma_1)^3} \\ &\leq \|d\|_{(L^2(\Omega))^3} \|v\|_{(H^1(\Omega))^3} + C_2 \|u\|_{(L^2(\Omega))^3} \|v\|_{(H^1(\Omega))^3} + C_0 C \|v\|_{(H^1(\Omega))^3} \\ &\leq \left[\|d\|_{(L^2(\Omega))^3} + C_2 R + C C_0\right] \|v\|_V \,. \end{aligned}$$

Then

$$\|F(u)\|_{V'} \leq \|d\|_{(L^2(\Omega))^3} + C_2 R + C C_0. \tag{4.4}$$

We deduce that $F(u)$ is an element of V', for any $u \in (L^2(\Omega))^3$.
We will show that

$$F : \left(L^2(\Omega)\right)^3 \to V'$$

$$u \longmapsto F(u)$$

is continuous. For this, we need the Lebesgue Dominated Convergence Theorem.
Let $u, \widetilde{u} \in \left(L^2(\Omega)\right)^3$; we have

$$\begin{aligned} \langle F(u), v\rangle &= \int\limits_{\Omega} f(x, u(x))v(x)dx + \int_{\Gamma_1} g(x)\, v(x)\, d\Gamma \\ \langle F(\widetilde{u}), v\rangle &= \int\limits_{\Omega} f(x, \widetilde{u}(x))v(x)dx + \int_{\Gamma_1} g(x)\, v(x)\, d\Gamma, \end{aligned}$$

so

$$\begin{aligned}
\|F(u)-F(\widetilde{u})\|_{V'} &= \sup_{\substack{v\in V\\ \|v\|=1}} \langle F(u)-F(\widetilde{u}),v\rangle_{V',V} \\
&= \sup_{\substack{v\in V\\ \|v\|=1}} \left[\int_\Omega (f(u)-f(\widetilde{u})).vdx\right] \\
&\leq \sup_{\substack{v\in V\\ \|v\|=1}} \left[\|f(u)-f(\widetilde{u})\|_{(L^2(\Omega))^3}\cdot\|v\|_{(L^2(\Omega))^3}\right] \\
&\leq \sup_{\substack{v\in V\\ \|v\|=1}} \left[\|f(u)-f(\widetilde{u})\|_{(L^2(\Omega))^3}\|v\|_V\right] \\
&\leq \|f(u)-f(\widetilde{u})\|_{(L^2(\Omega))^3}.
\end{aligned}$$

So, if $(u_n)_{n\in\mathbb{N}}$ is a sequence of $(L^2(\Omega))^3$ such that $u_n\to\widetilde{u}$ in $(L^2(\Omega))^3$, we have

$$\|F(u_n)-F(\widetilde{u})\|_{V'}\leq\|f(u_n)-f(\widetilde{u})\|_{(L^2(\Omega))^3}.$$

So, $\exists\,(u_n)$ subsequence such that

$$u_n\to\widetilde{u}(x) \text{ almost everywhere in } \Omega$$

and $\exists H\in(L^2(\Omega))^3$ such that

$$|u_n|\leq H\text{almost everywhere in } \Omega.$$

Then, we notice that $f(u_n)\to f(\widetilde{u})$ because f is continuous a.e. in Ω. According to the hypothesis (iv), $|f(u_n)|\leq d(x)+C_2\,|u_n|$, and as $|u_n|\leq H$ we find

$$|f(u_n)|\leq d(x)+C_2.H \text{ almost everywhere in } \Omega.$$

So, by the Lebesgue Dominated Convergence Theorem , we obtain

$$\|f(u_n)-f(\widetilde{u})\|_{(L^2(\Omega))^3}\to 0 \text{ when } n\to\infty$$

and consequently,

$$\|F(u_n)-F(\widetilde{u})\|_{V'}\to 0\,\text{when}\,n\to\infty$$

hence the continuity of F. For $S\in V'$, the linear problem

$$\left\{\begin{array}{c} w\in V \\ \displaystyle\int_\Omega\sum_{i,j=1}^{3}\sum_{k,h=1}^{3}a_{ijkh}(x)\varepsilon_{kh}(w(x))\varepsilon_{ij}(v(x))dx=\langle S,v\rangle_{V',V}, \end{array}\right. \tag{4.5}$$

admits a unique solution $w\in V$ (see [1]). We denote by B_u the operator which to S in V' associates w solution of (4.5). The operator B_u is linear continuous from V' into V and V is injected compactly into $\left(L^2(\Omega)\right)^3$ (because the boundary Γ is lipschitzian). We deduce that the operator B_u is compact from V' in $\left(L^2(\Omega)\right)^3$. The problem (P)

is equivalent to solving the fixed point problem $u = B_u(F(u))$. We will show, using the topological degree techniques, that the following problem admits a solution

$$\begin{cases} u \in \left(L^2(\Omega)\right)^3, \\ u = B_u(F(u)). \end{cases}$$

For $t \in [0,1]$, we put the application h such that:

$$h : [0,1] \times \left(L^2(\Omega)\right)^3 \to \left(L^2(\Omega)\right)^3$$
$$(t,u) \longmapsto h(t,u) = B_u(tF(u))$$

For $R > 0$, we put $B_R = \left\{u \in \left(L^2(\Omega)\right)^3 \text{such that } \|u\|_{(L^2(\Omega))^3} < R\right\}$. We will show

(1)$-$ $\exists R > 0;$ $\left.\begin{cases} u - h(t,u) = 0 \\ t \in [0,1], u \in \left(L^2(\Omega)\right)^3 \end{cases}\right\} \Longrightarrow \|u\|_{(L^2(\Omega))^3} < R;$

(2)$-$ h is continuous from $[0,1] \times \overline{B}_R$ into $\overline{B}_R$;

(3)$-$ $\left\{h(t,u),\ t \in [0,1],\ u \in \overline{B}_R\right\}$ is relatively compact in $\left(L^2(\Omega)\right)^3$.

If we suppose that we have proved the statements (1), (2) and (3), we have no solution to the equation $u - h(t,u) = 0$ on the boundary of the ball B_R, and we can thus define the degree $d(Id - h(t,.), B_R, 0)$. This degree does not depend of t, so we have

$$\begin{aligned} d(I_d - h(t,.), B_R, 0) &= d(Id - h(0,.), B_R, 0) \\ &= d(Id, B_R, 0) = 1. \end{aligned}$$

We deduce the existence of $u \in B_R$ such that $u - h(1,u) = 0$, that is to say

$$u = B_u(F(u)).$$

Thus u is solution of (P). Now, it remains to show the statements (1), (2) and (3). Let us begin with the proof of (3) (for every $R > 0$). We suppose $\|u\|_{(L^2(\Omega))^3} \leq R$. We have

$$F(u) \in V', \text{ and } \langle F(u), v\rangle_{V',V} = \int_\Omega f(x,u(x))v(x)dx + \int_{\Gamma_1} g(x)\, v(x)\, d\Gamma.$$

We have

$$\|F(u)\|_{V'} \leq \|d\|_{(L^2(\Omega))^3} + C_2 R + CC_0.$$

So

$$t\,\|F(u)\|_{V'} \leq \|d\|_{(L^2(\Omega))^3} + C_2 R + CC_0 = \widetilde{R},\ \forall t \in [0,1]$$

We put $h(t,u) = B_u(tF(u)) = w$ and show that there exists $\overline{R}$ depending only of R, C_0, C, C_2, and α such that

$$\|h(t,u)\|_V \leq \overline{R} \Longleftrightarrow \|w\|_V \leq \overline{R}$$

By definition, w is solution of

$$\begin{cases} w \in V \\ \displaystyle\int_\Omega \sum_{i,j=1}^{3} \sum_{k,h=1}^{3} a_{ijkh}(x)\varepsilon_{kh}(w(x))\varepsilon_{ij}(v(x))dx \\ = \langle tF(u), v\rangle_{V',V},\ \forall v \in V \end{cases} \tag{4.6}$$

Taking $v = w$ in (4.6), by Korn's inequality, we obtain,

$$\alpha k \|w\|^2_{(H^1(\Omega))^3} \leq \|tF(u)\|_{V'} \|w\|_V$$

$$\Longleftrightarrow \alpha k \|w\|^2_{(H^1(\Omega))^3} \leq \|tF(u)\|_{V'} \|w\|_V \leq \widetilde{R} \|w\|_V$$

which implies:

$$\|h(t,u)\|_V = \|w\|_V \leq \overline{R}$$

with

$$\overline{R} = \frac{\widetilde{R}}{\alpha k} = \frac{\|d\|_{(L^2(\Omega))^3} + C_2 R + CC_0}{\alpha k}.$$

From Rellich's Theorem, we deduce that the set $\left\{h(t,u), t \in [0,1], u \in \overline{B}_R\right\}$ is relatively compact in $\left(L^2(\Omega)\right)^3$, which shows (3). Let us show now the point (2). Let $(t_n)_{n\in\mathbb{N}} \subset [0,1]$ such that $t_n \to t$ when $n \to +\infty$ and $(u_n)_{n\in\mathbb{N}} \subset \left(L^2(\Omega)\right)^3$ with $u_n \to u$ in $\left(L^2(\Omega)\right)^3$. We want to show that $h(t_n,u_n) \to h(t,u)$ in $\left(L^2(\Omega)\right)^3$. Let $w_n = h(t_n,u_n)$ and $w = h(t,u)$. To show that $w_n \to w$ in $\left(L^2(\Omega)\right)^3$. We take the limit on the following problem,

$$\left\{\begin{array}{l} w_n \in V \\ \displaystyle\int_\Omega \sum_{i,j=1}^{3} \sum_{k,h=1}^{3} a_{ijkh}(x)\varepsilon_{kh}(w_n)\varepsilon_{ij}(v)dx \\ \displaystyle = t_n \int_\Omega f(u_n)v(x)\,dx + t_n \int_{\Gamma_1} g(x)\,v(x)\,d\Gamma \end{array}\right. \tag{4.7}$$

We already know that $(w_n)_{n\in\mathbb{N}}$ is bounded in V, because the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in $\left(L^2(\Omega)\right)^3$ (this is what was shown in the previous step: if $\|u_n\|_{(L^2(\Omega))^3} \leq R$ then $\|w_n\|_V \leq \overline{R}$). The sequence $(w_n)_{n\in\mathbb{N}}$ is bounded in V, and to a subsequence, we have

$$\begin{array}{rcl} w_n & \to & \overline{w} \text{ in } V \text{ weakly and } w_n \to \overline{w} \text{ in } \left(L^2(\Omega)\right)^3, \\ u_n & \to & u \text{ a.e. and } \exists H \in \left(L^2(\Omega)\right)^3;\ |u_n| \leq H \text{ a.e..} \end{array}$$

Since $w_n \to \overline{w}$ in $\left(L^2(\Omega)\right)^3$, then there exist a subsequence denoted again w_n such that

$$w_n \to \overline{w} \text{ a.e. and } \exists K \in \left(L^2(\Omega)\right)^3;\ |w_n| \leq K \text{ a.e.}$$

Let $v \in V$ and as ε_{kh} is continuous then $\varepsilon_{kh}(w_n) \to \varepsilon_{kh}(\overline{w})$ a.e. and so

$$\sum_{i,j=1}^{3} \sum_{k,h=1}^{3} a_{ijkh}(x)\varepsilon_{kh}(w_n)\varepsilon_{ij}(v) \to \sum_{i,j=1}^{3} \sum_{k,h=1}^{3} a_{ijkh}(x)\varepsilon_{kh}(\overline{w})\varepsilon_{ij}(v) \text{ a.e.}$$

we have also

$$\left|\sum_{i,j=1}^{3} \sum_{k,h=1}^{3} a_{ijkh}(x)\varepsilon_{kh}(w_n)\varepsilon_{ij}(v)\right| \leq \beta \sum_{k,h=1}^{3} |\varepsilon_{kh}(w_n)| \sum_{i,j=1}^{3} |\varepsilon_{ij}(v)| \in L^1(\Omega),$$

by Dominated Convergence Theorem, we have

$$\int_\Omega \sum_{i,j=1}^{3}\sum_{k,h=1}^{3} a_{ijkh}(x)\varepsilon_{kh}(w_n)\varepsilon_{ij}(v)dx \to \int_\Omega \sum_{i,j=1}^{3}\sum_{k,h=1}^{3} a_{ijkh}(x)\varepsilon_{kh}(\overline{w})\varepsilon_{ij}(v)dx,\ n\to+\infty.$$

As $f(u_n)\to f(u)$ a.e. and $|f(u_n)|\le |d|+C_2\,|H|$.
By the Lebesgue Dominated Convergence Theorem we have $f(u_n)\to f(u)$ in $\left(L^2(\Omega)\right)^3$ and consequently

$$\int_\Omega f(u_n)vdx \to \int_\Omega f(u)vdx$$

when $n\to+\infty$. Passing to the limit in (4.7), we obtain,

$$\int_\Omega \sum_{i,j=1}^{3}\sum_{k,h=1}^{3} a_{ijkh}(x)\varepsilon_{kh}(\overline{w})\varepsilon_{ij}(v)dx = t\int_\Omega f(u)vdx + t\int_{\Gamma_1} g\left(x\right)v\left(x\right)d\Gamma,$$

and so $\overline{w}=h(t,u)=w$. By the absurd argument, we show that $w_n\to w$ in V weakly and $w_n\to w$ in $\left(L^2(\Omega)\right)^3$, where $w_n=h(t_n,u_n)$ and $w=h(t,u)$; the application h is continuous and consequently (2) holds. It remains now to demonstrate (1). We want to show that:

$$\exists R>0;\left\{\begin{array}{c} u-h(t,u)=0 \\ t\in[0,1]\,,\ u\in\left(L^2(\Omega)\right)^3 \end{array}\right\}\Longrightarrow \|u\|_{(L^2(\Omega))^3}<R.$$

Let $t\in[0,1]$ and $u=h(t,u)=tB_u(F(u))$, that is to say

$$\left\{\begin{array}{l} \displaystyle\int_\Omega \sum_{i,j=1}^{3}\sum_{k,h=1}^{3} a_{ijkh}(x)\varepsilon_{kh}(u)\varepsilon_{ij}(v)dx \\ t=\displaystyle\int_\Omega f(u)vdx + t\int_{\Gamma_1} g\left(x\right)v\left(x\right)d\Gamma,\ \forall v\in V \\ u\in V \end{array}\right. \tag{4.8}$$

We choose $v=u$ in (4.8). By the hypotheses (4.3), and Korn's inequality, we have

$$\alpha k\left\|u\right\|^2_{(H^1(\Omega))^3}\le\int_\Omega |f(u)u|\,dx+\int_{\Gamma_1}|g\left(x\right)|\,|u\left(x\right)|\,d\Gamma.$$

We are going to deduce from this inequality the existence of $R>0$ such that

$$\|u\|_{(L^2(\Omega))^3}<R.$$

Here, we use the hypothesis (v), i.e.,

$$\lim_{s\to\pm\infty}\frac{f(x,s)}{s}=0.$$

We argue by the absurd. Let us suppose that a such R does not exist. Then there exist a sequence $(u_n)_{n\in\mathbb{N}^*}$ of elements of V such that

$$\|u_n\|_{(L^2(\Omega))^3}\ge n \text{ and } \alpha k\left\|u_n\right\|^2_{(H^1(\Omega))^3}\le\int_\Omega |f(u_n)u_n|\,dx+\int_{\Gamma_1}|g\left(x\right)|\,|u_n\left(x\right)|\,d\Gamma.$$

Let us show that this is impossible. Letting $v_n = \dfrac{u_n}{\|u_n\|_V}$. We have $\|v_n\|_V = 1$ and

$$\alpha k \|v_n\|^2_{(H^1(\Omega))^3} \leq \int_\Omega \left| \frac{f(u_n)}{\|u_n\|_{(H^1(\Omega))^3}} v_n \right| dx + \int_{\Gamma_1} \left| \frac{g(x)}{\|u_n\|_{(H^1(\Omega))^3}} v_n \right| d\Gamma.$$

Or, according to hypothesis (iv) i.e., $|f(x,s)| \leq d(x) + C_2 |s|$ and the trace theorem we have

$$\begin{aligned}
\alpha k \|v_n\|^2_{(H^1(\Omega))^3} &\leq \int_\Omega \frac{|d| + C_2 |u_n|}{\|u_n\|_{(H^1(\Omega))^3}} |v_n| \, dx + \frac{\|g(x)\|_{(L^2(\Gamma_1))^3} \|v_n\|_{(L^2(\Gamma_1))^3}}{\|u_n\|_{(H^1(\Omega))^3}} \\
&\leq \int_\Omega \frac{|d| \, |v_n|}{\|u_n\|_{(H^1(\Omega))^3}} dx + C_2 \int_\Omega |v_n|^2 \, dx + \frac{\|g(x)\|_{(L^2(\Gamma_1))^3} C \|v_n\|_{(H^1(\Omega))^3}}{\|u_n\|_{(H^1(\Omega))^3}} \\
&\leq \frac{\|d\|_{(L^2(\Omega))^3} \|v_n\|_{(L^2(\Omega))^3}}{\|u_n\|_{(H^1(\Omega))^3}} + C_2 \|v_n\|^2_{(L^2(\Omega))^3} + \frac{C_0 C}{\|u_n\|_{(H^1(\Omega))^3}} \\
&\leq \frac{\|d\|_{(L^2(\Omega))^3} \|v_n\|_{(L^2(\Omega))^3}}{\|u_n\|_{(L^2(\Omega))^3}} + C_2 \|v_n\|^2_{(H^1(\Omega))^3} + \frac{C_0 C}{\|u_n\|_{(L^2(\Omega))^3}} \\
&\leq \|d\|_{(L^2(\Omega))^3} \|v_n\|_{(L^2(\Omega))^3} + C_2 + C_0 C \\
&\leq \|d\|_{(L^2(\Omega))^3} + C_2 + C_0 C,
\end{aligned}$$

which implies

$$\|v_n\|^2_V \leq \frac{\|d\|_{(L^2(\Omega))^3} + C_2 + C_0 C}{\alpha k},$$

so, $(v_n)_{n \in \mathbb{N}^*}$ is bounded in V, and hence there exist a subsequence, $v_n \to v$ in $(L^2(\Omega))^3$. We also have

$$\begin{aligned}
v_n &\to v \text{ a.e. on } \Omega, \\
|v_n| &\leq H \text{ with } H \in \left(L^2(\Omega)\right)^3.
\end{aligned}$$

As $\|v_n\|_V = 1$, we have

$$\alpha k \leq \int_\Omega \left| \frac{f(u_n)}{\|u_n\|_{(H^1(\Omega))^3}} v_n \right| dx + \int_{\Gamma_1} \left| \frac{g(x)}{\|u_n\|_{(H^1(\Omega))^3}} v_n \right| d\Gamma.$$

Letting

$$X_n = \int_\Omega \left| \frac{f(u_n)}{\|u_n\|_{(H^1(\Omega))^3}} v_n \right| dx + \int_{\Gamma_1} \left| \frac{g(x)}{\|u_n\|_{(H^1(\Omega))^3}} v_n \right| d\Gamma.$$

Now, we show that $X_n \to 0$ when $n \to +\infty$, which is impossible since X_n is reduced by the constant αk which is strictly positive.

- Let us show that $\frac{|f(u_n)|\,|v_n|}{\|u_n\|_{(H^1(\Omega))^3}} \to 0$ a.e. with domination, we shall have then by the Dominated Convergence Theorem that $\int\limits_\Omega \left|\frac{f(u_n)}{\|u_n\|_{(H^1(\Omega))^3}} v_n\right| dx \to 0$ when $n \to +\infty$.

We show first of all the domination. We have

$$\begin{aligned}\frac{|f(u_n)|}{\|u_n\|_{(H^1(\Omega))^3}} &\leq \frac{|d| + C_2\,|u_n|}{\|u_n\|_{(H^1(\Omega))^3}} \leq \frac{|d|}{\|u_n\|_{(H^1(\Omega))^3}} + C_2\,|v_n| \\ &\leq \frac{|d|}{\|u_n\|_{(L^2(\Omega))^3}} + C_2\,|v_n| \\ &\leq |d| + C_2 H\end{aligned}$$

so

$$\left|\frac{f(u_n)}{\|u_n\|_{(H^1(\Omega))^3}} v_n\right| \leq (|d| + C_2 H)\, H \in \left(L^1(\Omega)\right)^3.$$

Now, we show that the convergence is almost everywhere. We have $v_n \to v$ a.e. so $\exists A$; $mes(A^c) = 0$ and $v_n(x) \to v(x)\, \forall x \in A$.

Case 1. If $v(x) > 0$; $v_n(x) \to v(x)$, but

$$\lim_{n\to+\infty} \|u_n\|_{(L^2(\Omega))^3} = +\infty$$

so

$$u_n(x) = v_n(x)\, \|u_n\|_V \to +\infty.$$

$$\frac{f(u_n(x))}{\|u_n\|_{(H^1(\Omega))^3}} v_n(x) = \frac{f(u_n(x)) u_n(x)}{u_n(x)\, \|u_n\|_{(H^1(\Omega))^3}} v_n(x) = \frac{f(u_n(x))}{u_n(x)} \left(v_n(x)\right)^2 \to 0,\ n \to +\infty.$$

We used here $\lim\limits_{s\to+\infty} f(s)/s = 0$.

Case 2. If $v(x) < 0$; we have the same

$$\lim_{n\to+\infty} \frac{f(u_n(x))}{\|u_n\|_{(H^1(\Omega))^3}} v_n(x) = 0$$

because

$$\lim_{s\to-\infty} f(s)/s = 0.$$

Case 3. If $v(x) = 0$;

$$\begin{aligned}\left|\frac{f(u_n(x))}{\|u_n\|_{(H^1(\Omega))^3}} v_n(x)\right| &\leq \frac{|d(x)| + C_2\,|u_n(x)|}{\|u_n\|_{(H^1(\Omega))^3}}\, |v_n(x)| \\ &\leq (|d(x)| + C_2\,|v_n(x)|)\, |v_n(x)| \\ &\to 0 \text{ because } v(x) = 0.\end{aligned}$$

In summary, we have

$$\frac{f(u_n)}{\|u_n\|_{(H^1(\Omega))^3}} v_n \to 0$$

a.e. on Ω. It has been shown that

$$\lim_{n\to+\infty}\int_\Omega \left|\frac{f(u_n)}{\|u_n\|_{(H^1(\Omega))^3}} v_n\right| dx = 0.$$

- Let us now show that the term $\displaystyle\int_{\Gamma_1}\left|\frac{g(x)}{\|u_n\|_{(H^1(\Omega))^3}} v_n\right| \to 0$ as $n\to+\infty$.

  We have

$$\begin{aligned} 0 \le \int_{\Gamma_1} \frac{|g(x)|\,|v_n|}{\|u_n\|_{(H^1(\Omega))^3}} d\Gamma &\le \frac{\|g(x)\|_{(L^2(\Gamma_1))^3}\,\|v_n\|_{(L^2(\Gamma_1))^3}}{\|u_n\|_{(H^1(\Omega))^3}} \\ &\le \frac{C_0 C\,\|v_n\|_{(H^1(\Omega))^3}}{\|u_n\|_{(H^1(\Omega))^3}} \\ &\le \frac{C_0 C}{\|u_n\|_{(L^2(\Omega))^3}} \to 0 \text{ when } n\to+\infty. \end{aligned}$$

Because we have $\|u_n\|_{(L^2(\Omega))^3}\to+\infty$ when $n\to+\infty$. It has been shown that

$$\lim_{n\to+\infty}\int_{\Gamma_1}\frac{|g(x)|\,|v_n|}{\|u_n\|_{(H^1(\Omega))^3}} d\Gamma = 0.$$

So $\lim\limits_{n\to+\infty} X_n = 0$, which is a contradiction with $X_n \ge \alpha k$ for all $n\in\mathbb{N}^*$. Thus, we have showed that there is $R>0$ such as: $(u = h(t,u)) \Longrightarrow \|u\|_{(L^2(\Omega))^3} < R$. This proves (1). Then the existence of solution to (P) is proved.

Uniqueness. We suppose that f does not depend to u. Let u_1 and u_2 be two solutions of this problem:

$$\begin{aligned} &\int_\Omega \sum_{i,j=1}^{3}\sum_{k,h=1}^{3} a_{ijkh}(x)\,\varepsilon_{kh}(u_i)\varepsilon_{ij}(v)dx \\ =& \int_\Omega f(x)\,v(x)\,dx + \int_{\Gamma_1} g(x)\,v(x)\,d\Gamma, \quad i=1,2;\forall v\in V. \end{aligned}$$

Subtracting term to term and substituting v by u_1-u_2, we obtain,

$$\int_\Omega \sum_{i,j=1}^{3}\sum_{k,h=1}^{3} a_{ijkh}(x)\,(\varepsilon_{kh}(u_1-u_2))\varepsilon_{ij}(u_1-u_2)dx = 0,$$

by Korn's inequality, and the hypothesis (iii), we have

$$\alpha\,\|u_1-u_2\|^2_{(H^1(\Omega))^3} \le 0,$$

so, $u_1=u_2$. This completes the proof of Theorem 4.3. □

5. Conclusion

In this work, we have studied the existence and the uniqueness of solutions of the mixed problem for a nonlinear elasticity system in a regular and bounded domain by using Schauder's fixed point theorem and the technique of topological degree. Next future work, we will concentrate on the same problem but with ε is nonlinear, and we will also prove a theorem of existence and uniqueness of solutions in Sobolev spaces with variable exponents.

References

[1] Ciarlet, P.G., *Mathematical elasticity, Vol. I: Three-Dimensional Elasticity*, North-Holland, Amsterdam, 1988.

[2] Dautray, R., Lions, J.L., *Analyse Mathématique et Calcul Numérique pour les Sciences et les Techniques*, Vol. 1, Masson, 1984.

[3] Lions, J.L., *Quelques Méthode de Résolution des Problèmes aux Limites non Linéaires*, Dunod, Paris, 1969.

[4] Merouani, B., *Solutions singulières du système de l'élasticité dans un polygone pour différentes conditions aux limites*, Maghreb Math. Rev., **5**(1996), no. 1 & 2, 95-112.

[5] Merouani, B., *Quelques problèmes aux limites pour le système de Lamé dans un secteur plan*, C.R.A.S., T. 304, Série I, no. 13, 1987.

[6] Merouani, B., Boufenouche, R., *Trigonometric series adapted for the study of Dirichlet boundary-value problems of Lamé systems*, Electron. J. Differential Equations, **2015**(2015), no. 181, 1-6.

[7] Rabinowitz, P.H., *Théorie du degré topologique et application à des problèmes aux limites non linéaires (Lecture note by H. Berestycki)*, Report 75010, Laboratoire d'analyse Numériqe, Université Pierre et Marie Curie, Paris, 1989.

[8] Raviart, P.A., Tomas, J.M., *Introduction à l'Analyse Numériques des Équations aux Dérivées Partielles*, Masson, Paris, 1983.

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Global nonexistence and blow-up results for a quasi-linear evolution equation with variable-exponent nonlinearities

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Abstract. In this paper, we consider a class of quasi-linear parabolic equations with variable exponents,

$$a\left(x,t\right)u_t-\Delta_{m(.)}u=f_{p(.)}\left(u\right)$$

in which $f_{p(.)}\left(u\right)$ the source term, $a(x,t)>0$ is a nonnegative function, and the exponents of nonlinearity $m(x)$, $p(x)$ are given measurable functions. Under suitable conditions on the given data, a finite-time blow-up result of the solution is shown if the initial datum possesses suitable positive energy, and in this case, we precise estimate for the lifespan T^* of the solution. A blow-up of the solution with negative initial energy is also established.

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1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$, $n\geq 1$ with a smooth boundary $\Gamma=\partial\Omega$. We consider the following initial-boundary value problem:

$$\begin{cases} a\left(x,t\right)u_t-\Delta_{m(.)}u=f_{p(.)}\left(u\right), \quad x\in\Omega,\ t>0 \\ u\left(x,t\right)=0 \text{ on } \Gamma,\ t\geq 0 \\ u(x,0)=u_0(x),\ x\in\Omega, \end{cases} \tag{1.1}$$

where

$$\Delta_{m(.)}u=\operatorname{div}\left(\left|\nabla u\right|^{m(x)-2}\nabla u\right)$$

called the $m(.)$–Laplacian operator. This operator can be extended to a monotone operator between the space $W_0^{1,m(.)}(\Omega)$ and its dual as

$$\begin{cases} -\Delta_{m(.)}u : W_0^{1,m(.)}(\Omega) \to W^{-1,m'(.)}(\Omega), \\ < -\Delta_{m(.)}u, \phi(x) >_{m(.)} = \int_\Omega |\nabla u|^{m(x)-2} \nabla u \nabla \phi(x)\, \mathrm{d}x, \\ \text{where } 2 < m_1 \le m(x) \le m_2 < \infty. \end{cases}$$

where $< .,. >_{m(.)}$ denotes the duality pairing between $W_0^{1,m(.)}(\Omega)$ and $W^{-1,m'(.)}(\Omega)$,

$$\frac{1}{m(x)} + \frac{1}{m'(x)} = 1.$$

$f_{p(.)}(u)$ is a general source term depends on $p(.)$, the coefficients $a(x,.)$ is a nonnegative function, the exponents $p(.)$ and $m(.)$ are given measurable functions on $\overline{\Omega}$ such that:

$$2 < m_1 \le m(x) \le m_2 < p_1 \le p(x) \le p_2 \le m_*(x), \tag{1.2}$$

where, for any function ψ, we set

$$\psi_2 = ess \sup_{x\in\Omega} \psi(x), \quad \psi_1 = ess \inf_{x\in\Omega} \psi(x).$$

and

$$m_*(x) = \begin{cases} \frac{nm(x)}{(n-m(x))_2} & \text{if } n > m_2 \\ +\infty & \text{if } n \le m_2. \end{cases}$$

We also assume that $m(.)$ satisfies the following Zhikov–Fan uniform local continuity condition:

$$|m(x) - m(y)| \le \frac{M}{|\log|x-y||}, \text{ for all } x, \ y \text{ in } \Omega \text{ with } |x-y| < \frac{1}{2}, \ M > 0. \tag{1.3}$$

A considerable effort has been devoted to the study of problem (1.1) in the case of constant variable when $p(x) = p$ =constant and $m(x) = m$ =constant. The problem (1.1) with the usual m-Laplacian operator $\Delta_m u = \operatorname{div}\left(|\nabla u|^{m-2} \nabla u\right)$, ($m$ =constant ≥ 2); ($m = 2$, $\Delta_m u = \Delta u$), has been extensively studied concerning existence, nonexistence and long-time dynamics. For results of the nature and in the case when $p(x) = p$ =constant≥ 2 and $m(x) = m$ =constant> 2, we refer the reader to [14, 18, 21] related to the equation

$$a(x) u_t - \operatorname{div}\left(|\nabla u|^{m-2} \nabla u\right) = f_p(u), \ x \in \Omega, \ t > 0.$$

When $m(x) = m = 2$, $a(x,t) = 1$ and $f_{p(.)}(u) = u^{p(x)}$, problem (1.1) becomes the following

$$u_t - \Delta u = u^{p(x)}, \ x \in \Omega, \ t > 0. \tag{1.4}$$

The problem (1.4) arises from many important mathematical models in engineering and physical sciences. For example, nuclear science, chemical reactions, heat transfer, population dynamics, biological sciences, etc., and have interested a great deal of attention in the research, see [1, 2, 4, 7, 12] and the references therein. For problem (1.4), Hua Wang et al. [15] established a blow-up result with positive initial energy under some suitable assumptions on the parameters $p(.)$ and u_0. In [12], the authors proved that there are non-negative solutions with a blow-up in finite time if and only if $p_2 > 1$. The authors in [20] obtained the solution of problem (1.1) blows up in a finite time when the initial energy is positive. In [8], authors based on the idea as in [5] derived the lower bounds for the time of blow-up if the solutions blow up.

This work is extend the results established in bounded domains to general problem as in (1.1) in the case, when the exponents $m(.)$ and $p(.)$ are given measurable functions on $\overline{\Omega}$ and satisfy (1.2) and $f_{p(.)}(u)$ is a more generalized source term. We note that the presence of the variable-exponent nonlinearities and the coefficient $a(x,t)$ in this problem make analysis in the paper somewhat harder than that in the related ones. The goal of the current project is to study the blow-up phenomenon of solutions to the problem (1.1) in the framework of the Lebesgue and Sobolev spaces with variable exponents, we will establish a blow-up result and give a precise estimate for the lifespan T^* of the solution in this case. The method used here is the concavity method. However, because of the presence of the variable-exponent nonlinearities in our problem, our argument is considerably different and it is more abbreviated. The present report is organized as follows. In Sections 2, the Orlicz-Sobolev function spaces are introduced, and a brief description of their main properties are presented. In Sections 3, the blow up for positive initial energy of problem (1.1) is stated. Section 4 provides proof of the blow-up for negative initial energy of problem (1.1).

2. Preliminaries

In this section, some well-known results and facts from the theory of Sobolev spaces with variable exponents are recalled and listed (for details, see [9, 10, 11, 13, 17]). Throughout the rest of this report, Ω is assumed to be a bounded domain of $\mathbb{R}^n$, $n \geq 2$ with a smooth boundary Γ, assuming that $p(.)$ is a measurable function on $\overline{\Omega}$ and satisfy the following Zhikov–Fan uniform local continuity condition:

$$|p(x) - p(y)| \leq \frac{M}{|\log|x-y||}, \text{ for all } x,\ y \text{ in } \Omega \text{ with } |x-y| < \frac{1}{2},\ M > 0.$$

Let $p : \Omega \to [1, \infty]$ be a measurable function. $L^{p(\cdot)}(\Omega)$ denotes the set of measurable functions u on Ω such that

$$\varrho_{p(.)}(u) = \int_{\Omega} |u(x)|^{p(x)}\, \mathrm{d}x.$$

The variable-exponent space $L^{p(\cdot)}$ equipped with the Luxemburg norm

$$\|u\|_{p(.)} = \|u\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{\lambda > 0,\ \ \varrho_{p(.)}\left(\frac{u}{\lambda}\right) \leq 1\right\},$$

is a Banach space. In general, variable-exponent Lebesgue spaces are similar to classical Lebesgue spaces in many aspects; see the first discussion of $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$ spaces by Kovàcik and Rákosnik in [17].

Here are some properties of the space $L^{p(\cdot)}(\Omega)$, which will be used in the study of a problem (1.1).

- It follows directly from the definition of the norm that

$$\min\left(\|u\|_{p(.)}^{p_1}, \|u\|_{p(.)}^{p_2}\right) \leq \varrho_{p(.)}(u) \leq \max\left(\|u\|_{p(.)}^{p_1}, \|u\|_{p(.)}^{p_2}\right).$$

- The following generalized Hölder inequality

$$\int_\Omega |u(x)\, v(x)|\, \mathrm{d}x \leq \left(\frac{1}{p_1} + \frac{1}{(p_1)'}\right) \|u\|_{p(x)} \|v\|_{p'(x)} \leq 2\, \|u\|_{p(x)} \|v\|_{p'(x)}$$

applies for all $u \in L^{p(.)}(\Omega)$, $v \in L^{p'(.)}(\Omega)$ with $p(x) \in (1,\infty)$, $p'(x) = \frac{p(x)}{p(x)-1}$.

- If condition (2.4) is fulfilled, Ω has a finite measure, and p, q are variable exponents such that $p(x) \leq q(x)$ almost everywhere in Ω, then the embedding $L^{q(.)}(\Omega) \hookrightarrow L^{p(.)}(\Omega)$ is continuous.
- The Sobolev space $W_0^{1,p(.)}(\Omega)$ with $p(x) \in [p_1, p_2] \subset (1,\infty)$, and $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$, is defined as

$$\left\{ \begin{array}{c} W_0^{1,p(.)}(\Omega) = \left\{u \in L^{p(.)}(\Omega) \mid \ |\nabla u|^{p(x)} \in L^1(\Omega),\ u = 0 \text{ on } \partial\Omega\right\}, \\ \|u\|_{W_0^{1,p(.)}(\Omega)} = \|u\|_{1,p(.)} = \sum_i \|D_i u\|_{p(.),\Omega} + \|u\|_{p(.),\Omega}, \end{array} \right\}$$

and $W^{-1,p'(.)}(\Omega)$ is defined in the same way as the usual Sobolev spaces (see [9]).

- An equivalent norm of $W_0^{1,p(.)}(\Omega)$ is given by

$$\|u\|_{W_0^{1,p(.)}(\Omega)} = \|\nabla u\|_{p(.),\Omega}.$$

Furthermore, we set $W_0^{1,p(.)}(\Omega)$, to be the closure of $C_0^\infty(\Omega)$ in $W^{1,p(.)}(\Omega)$. Here we note that the space $W_0^{1,p(.)}(\Omega)$ is usually defined in a different way for the variable exponent case. However (see Diening et al [9]), both definitions are equivalent under (1.3). The $\left(W_0^{1,p(.)}(\Omega)\right)'$ is the dual space of $W_0^{1,p(.)}(\Omega)$ with respect to the inner product in $L^2(\Omega)$ and is defined as $W^{-1,p'(.)}(\Omega)$, in the same way as the classical Sobolev spaces, where $\frac{1}{p(.)} + \frac{1}{p(.)'} = 1$.

- If $p \in C\left(\overline{\Omega}\right)$, $q : \Omega \to [1,+\infty)$ is a measurable function and $\underset{x\in\Omega}{ess\inf}\,(p^*(x) - q(x)) > 0$ with $p^*(x) = \frac{np(x)}{(n-p(x))_2}$, then $W_0^{1,p(.)}(\Omega) \hookrightarrow L^{q(.)}(\Omega)$ is continuous and compact.

Lemma 2.1. ([9]) *Let Ω be a bounded domain of $\mathbb{R}^n$, $p(.)$ and $m(.)$ satisfy* (1.2) *and* (1.3)*, then*

$$B_0 \|\nabla u\|_{m(.)} \geq \|u\|_{p(.)}, \text{ for all } u \in W_0^{1,m(.)}(\Omega), \tag{2.1}$$

where the optimal constant of Sobolev embedding B_0 is depend on $p_{1,2}$ and $|\Omega|$.

Lemma 2.2 (Poincaré's Inequality). ([9]) *Let Ω be a bounded domain of $\mathbb{R}^n$ and $m(.)$ satisfies* (1.3)*, then*

$$D_0 \|\nabla u\|_{m(.)} \geq \|u\|_{m(.)}, \text{ for all } u \in W_0^{1,m(.)}(\Omega), \tag{2.2}$$

where the optimal constant of Sobolev embedding D_0 is depend on $m_{1,2}$ and $|\Omega|$.

2.1. Mathematical assumptions

In this section, we establish the blow-up result for solutions with positive energy. Let the function $f_{p(.)} \in C^0(\mathbb{R}, \mathbb{R}^+)$, with the primitive

$$F(u) = \int_0^u f_{p(.)}(\eta)\, \mathrm{d}\eta, \tag{2.3}$$

satisfies

$$\left|f_{p(.)}(s)\right| \le C_0 |s|^{p(.)-1}, \quad p(x) F(s) \le s f_{p(.)}(s), \; s \in \mathbb{R}, \; C_0 > 0. \tag{2.4}$$

A simple typical example of these functions is

$$f_{p(.)}(s) = |s|^{p(x)-2} s.$$

Assume that $a(x,t)$ is a positive function which belongs to the space $W^{1,\infty}(0,\infty; L^\infty(\Omega))$ and that $a_t(x,t) \le 0$ a.e. for $t \ge 0$. Let

$$B_1 = \max\left(1, B_0, \left(\frac{1}{C_0}\right)^{\frac{1}{p_1}}\right), \quad \alpha_1 = \left(\frac{1}{B_1^{p_1} C_0}\right)^{\frac{m_2}{p_1 - m_2}}, \quad \alpha_0 = \|\nabla u_0\|_{m(.)}^{m_2}, \tag{2.5}$$

and

$$E_0 = \left(\frac{1}{B_1^{p_1} C_0}\right)^{\frac{m_2}{p_1 - m_2}} \left(\frac{1}{m_2} - \frac{1}{p_1}\right) = \left(\frac{1}{m_2} - \frac{1}{p_1}\right) \alpha_1. \tag{2.6}$$

3. Main result

In this section, we present our main blow-up result. We start with a local existence result for the problem (1.1), which can be established by combining the arguments of [3, 6], the following theorem, which confirms the existence of a local solution is a direct result.

Theorem 3.1. *For all $u_0 \in W_0^{1,m(.)}(\Omega)$, there exists a number $T_0 \in (0,T]$ such that the problem (1.1) has a strong solution u on $[0,T_0]$ satisfying*

$$u \in C([0,T_0]; W_0^{1,m(.)}(\Omega)) \cap C([0,T_0]; L^{p(.)}(\Omega)) \cap W^{1,2}([0,T_0]; L^2(\Omega)).$$

4. Blow up for positive initial energy

This section first presents our main blow-up result and its proof for the problem (1.1). For this purpose, we start by the following lemma defining the energy of the solution.

Lemma 4.1. *The corresponding energy to problem (1.1) is given by*

$$E(t) = \int_\Omega \frac{1}{m(x)} |\nabla u(x,t)|^{m(x)}\, \mathrm{d}x - \int_\Omega F(u(x,t))\, \mathrm{d}x, \tag{4.1}$$

furthermore, by the easily verified formula

$$\frac{\mathrm{d}E(t)}{\mathrm{d}t} = -\int_\Omega a(x,t)\, u_t^2(x,t)\, \mathrm{d}x \le 0, \tag{4.2}$$

the inequality $E(t) \le E(0)$ is obtained.

Now, we are in a position to state our main theorem results.

Theorem 4.2. *If the initial data $u_0 \in W^{1,m(.)}(\Omega)$ are such that $u_0 \neq 0$,*

$$E(0) = \int_\Omega \frac{1}{m(x)} |\nabla u_0(x)|^{m(x)} \mathrm{d}x - \int_\Omega F(u_0(x))\, \mathrm{d}x \le E_0, \tag{4.3}$$

then there exists T^* such that $\limsup_{t\to T^*} \|u(.,t)\|_2 = +\infty$. Moreover, if $E(0) < E_0$, then the T^* can be bounded above as:

$$T^* \le \frac{8\left\|\sqrt{a_0}u_0\right\|^2_{L^2(\Omega)}}{(p_1-2)^2(E_0 - E(0))}, \tag{4.4}$$

where $a(x,0) := a_0$ and $u(x,0) := u_0$.

In order to prove the main theorem, we recall the following lemmas.

Lemma 4.3. ([16, Lemma1.1] and [19, Logarithmic convexity methods]) *Assume that $\varphi \in C^2([0,T))$ satisfying:*

$$\varphi''\varphi - (1+\alpha)(\varphi')^2 \ge 0, \quad \alpha > 0,$$

and

$$\varphi(0) > 0, \quad \varphi'(0) > 0,$$

then

$$\varphi \to \infty \text{ as } t \to t_1 \le t_2 = \frac{\varphi(0)}{\alpha\varphi'(0)}.$$

Lemma 4.4. *Suppose $E(0) < E_0$ and $\alpha_1 < \alpha_0 \le B_1^{-m_2}$. Then it exists a constant $\alpha_2 > \alpha_1$ such that:*

$$\|\nabla u\|^{m_2}_{m(.)} \ge \alpha_2 > \alpha_1 \text{ for all } t \ge 0.$$

Proof. Thanks to (2.3) and (2.1), we have for any $t \ge 0$

$$\begin{aligned} E(t) &= \int_\Omega \frac{1}{m(x)} |\nabla u(x,t)|^{m(x)} \mathrm{d}x - \int_\Omega F(u(x,t))\, \mathrm{d}x \\ &\ge \frac{1}{m_2}\min\left(\|\nabla u\|^{m_1}_{m(.)}, \|\nabla u\|^{m_2}_{m(.)}\right) - \int_\Omega \frac{C_0}{p(x)} |u(x,t)|^{p(x)} \mathrm{d}x \\ &\ge \frac{1}{m_2}\min\left(\|\nabla u\|^{m_1}_{m(.)}, \|\nabla u\|^{m_2}_{m(.)}\right) - \frac{C_0}{p_1}\max\left(B_1^{p_1}\|\nabla u\|^{p_1}_{m(.)}, B_1^{p_2}\|\nabla u\|^{p_2}_{m(.)}\right) \quad (4.5) \\ &= \frac{1}{m_2}\min\left(\alpha^{\frac{m_1}{m_2}}, \alpha\right) - \frac{C_0}{p_1}\max\left((\alpha B_1^{m_2})^{\frac{p_1}{m_2}}, (\alpha B_1^{m_2})^{\frac{p_2}{m_2}}\right) : = g(\alpha), \ \forall \alpha \in [0, +\infty[\end{aligned}$$

where $\alpha = \|\nabla u\|^{m_2}_{m(.)}$. Now if we let

$$h(\alpha) = \frac{1}{m_2}\alpha - \frac{C_0}{p_1}(\alpha B_1^{m_2})^{\frac{p_1}{m_2}}$$

Notice that $h(\alpha) = g(\alpha)$, for $0 < \alpha < B_1^{-m_2}$. It is easy to check that the function $h(\alpha)$ is increasing for $0 < \alpha < \alpha_1$ and decreasing for $\alpha_1 < \alpha \le +\infty$.

Because $E(0) < E_0 = h(\alpha_1)$, there exists a positive constant $\alpha_2 \in (\alpha_1, +\infty)$ such that $h(\alpha_2) = E(0)$. Then we have

$$h(\alpha_0) = g(\alpha_0) \leq E(0) = h(\alpha_2).$$

It implies that $\alpha_0 \geq \alpha_2 > \alpha_1$.

To show that $\|\nabla u(x,t)\|_{m(.)}^{m_2} \geq \alpha_2$ we reason by absurd while supposing that

$$\|\nabla u(x,t^*)\|_{m(.)}^{m_2} < \alpha_2$$

for a some t^*. Then by the continuity of $\|\nabla u(.,t)\|_{m(.)}$-norm with respect to time variable, one can choose t^* such that

$$\alpha_2 > \|\nabla u(x,t^*)\|_{m(.)}^{m_2} > \alpha_1.$$

The monotonicity of $h(\alpha)$, gives

$$E(t^*) \geq h(\|\nabla u(x,t)\|_{m(.)}^{m_2}) > h(\alpha_2) = E(0)$$

it is impossible because $E(0) \geq E(t)$ for all $t \geq 0$. Then, for all time $t \geq 0$:

$$\|\nabla u\|_{m(.)}^{m_2} \geq \alpha_2 > \alpha_1. \tag{4.6}$$

□

Proof of Theorem 1. **Case 1:** $E(0) < E_0$. The goal is to construct a suitable function which satisfies the conditions in Lemma (4.3). Following the arguments of [22, 23], for our purpose, we define the following suitable function

$$\begin{aligned}\varphi(t) = &\int_0^t \int_\Omega a(x,s)\, u^2(x,s)\, \mathrm{d}x\mathrm{d}s + \int_0^t \int_\Omega (s-t)\, a_t(x,s)\, u^2(x,s)\, \mathrm{d}x\mathrm{d}s \\ &+ (T_0 - t) \int_\Omega a_0(x)\, u_0^2(x)\, \mathrm{d}x + \beta(t+t_0)^2, \ t < T_0\end{aligned} \tag{4.7}$$

where t_0, T_0 and β are positive constants to be determined later. Then using equation (1.1) and integration by parts, to obtains

$$\begin{aligned}\varphi'(t) = &\int_\Omega a(x,t)\, u^2(x,t)\, \mathrm{d}x - \int_0^t \int_\Omega a_t(x,s)\, u^2(x,s)\, \mathrm{d}x\mathrm{d}s \\ &- \int_\Omega a_0(x)\, u_0^2(x)\, \mathrm{d}x + 2\beta(t+t_0) \\ = &\, 2\int_0^t \int_\Omega a(x,s)\, u(x,s)\, u_t(x,s)\, \mathrm{d}x\mathrm{d}s + 2\beta(t+t_0),\end{aligned} \tag{4.8}$$

and

$$\varphi''(t) = 2\int_\Omega a(x,t)\, u(x,t)\, u_t(x,t)\, \mathrm{d}x + 2\beta. \tag{4.9}$$

Then, due to (2.4) and (4.6), the following is obtained

$$\varphi''(t) \geq -2\int_\Omega |\nabla u(x,t)|^{m(x)}\,\mathrm{d}x + 2\int_\Omega p(x) F(u)\,\mathrm{d}x + 2\beta$$
$$\geq -2\int_\Omega |\nabla u(x,t)|^{m(x)}\,\mathrm{d}x + 2p_1\left(\int_\Omega \frac{1}{m(x)}|\nabla u(x,t)|^{m(x)}\,\mathrm{d}x - E(t)\right) + 2\beta$$
$$\geq \left(\frac{2p_1}{m_2} - 2\right)\int_\Omega |\nabla u(x,t)|^{m(x)}\,\mathrm{d}x - 2p_1 E(t) + 2\beta$$
$$\geq \left(\frac{2p_1}{m_2} - 2\right)\int_\Omega |\nabla u(x,t)|^{m(x)}\,\mathrm{d}x$$
$$+2p_1\int_0^t\int_\Omega a(x,s)\,u_t^2(x,s)\,\mathrm{d}x\mathrm{d}s - 2p_1E(0) + 2\beta$$
$$\geq \left(\frac{2p_1}{m_2} - 2\right)\min\left(\|\nabla u\|_{m(.)}^{m_1}, \|\nabla u\|_{m(.)}^{m_2}\right)$$
$$+2p_1\int_0^t\int_\Omega a(x,s)\,u_t^2(x,s)\,\mathrm{d}x\mathrm{d}s - 2p_1E(0) + 2\beta$$
$$\geq \left(\frac{2p_1}{m_2} - 2\right)\min\left(\alpha_2^{\frac{m_1}{m_2}}, \alpha_2\right)$$
$$+2p_1\int_0^t\int_\Omega a(x,s)\,u_t^2(x,s)\,\mathrm{d}x\mathrm{d}s - 2p_1E(0) + 2\beta$$
$$\geq 2p_1\left(\frac{1}{m_2} - \frac{1}{p_1}\right)\min\left(\alpha_1^{\frac{m_1}{m_2}}, \alpha_1\right)$$
$$-2p_1E(0) + 2\beta + 2p_1\int_0^t\int_\Omega a(x,s)\,u_t^2(x,s)\,\mathrm{d}x\mathrm{d}s$$
$$= 2p_1\left(\frac{1}{m_2} - \frac{1}{p_1}\right)\alpha_1 - 2p_1E(0) \quad \text{(by (2.5))}$$
$$+2\beta + 2p_1\int_0^t\int_\Omega a(x,s)\,u_t^2(x,s)\,\mathrm{d}x\mathrm{d}s$$
$$= 2p_1(E_0 - E(0)) + 2\beta + 2p_1\int_0^t\int_\Omega a(x,s)\,u_t^2(x,s)\,\mathrm{d}x\mathrm{d}s$$

Now, let $\beta = 2(E_0 - E(0)) > 0$, and note that $p_1 > 2$, then

$$\varphi''(t) \geq (p_1+2)\beta + (p_1+2)\int_0^t\int_\Omega a(x,s)\,u_t^2(x,s)\,\mathrm{d}x\mathrm{d}s \tag{4.10}$$

From (4.7), (4.8), (4.9) and (4.10), we have

$$\begin{cases} \varphi(0) = T_0\int_\Omega a_0(x)\,u_0^2(x)\,\mathrm{d}x + \beta t_0^2 > 0; \\ \varphi'(0) = 2\beta t_0 > 0; \\ \varphi''(t) \geq (p_1+2)\beta > 0 \ \forall t \geq 0. \end{cases}$$

Therefore φ and φ' are both positive. Since $a_t(x,t) \leq 0$, for all $x \in \Omega$ and $t \geq 0$, we have

$$\varphi(t) \geq \int_0^t \int_\Omega a(x,s)\, u^2(x,s)\, \mathrm{d}x\mathrm{d}s + \beta(t+t_0)^2, \tag{4.11}$$

Thus, from (4.7)-(4.10) and (4.11), the following inferred for all $(\zeta, \eta) \in \mathbb{R}^2$

$$\begin{aligned}
&\varphi(t)\,\zeta^2 + \varphi'(t)\,\zeta\eta + \frac{\eta^2}{p_1+2}\varphi''(t)\\
&\geq \left(\int_0^t \int_\Omega a(x,s)\, u^2(x,s)\, \mathrm{d}x\mathrm{d}s + \beta(t+t_0)^2\right)\zeta^2\\
&+2\zeta\eta \int_0^t \int_\Omega a(x,s)\, u(x,s)\, u_t(x,s)\, \mathrm{d}x\mathrm{d}s + 2\zeta\eta\beta(t+t_0)\\
&+\beta\eta^2 + \eta^2 \int_0^t \int_\Omega a(x,s)\, u_t^2(x,s)\, \mathrm{d}x\mathrm{d}s \geq 0,
\end{aligned}$$

which implies that

$$\varphi(t)\,\frac{\varphi''(t)}{p_1+2} - \left(\frac{\varphi'(t)}{2}\right)^2 \geq 0,$$

subsequently

$$\varphi(t)\,\varphi''(t) - \frac{p_1+2}{4}\left(\varphi'(t)\right)^2 \geq 0. \tag{4.12}$$

Then using Lemma (4.3), to infer $\varphi(t) \to \infty$ as $t \to T^*$, where,

$$T^* \leq \frac{\varphi(0)}{\left(\frac{p_1-2}{4}\right)\varphi'(0)} = \frac{2\left(T_0 \left\|\sqrt{a_0}u_0\right\|^2_{L^2(\Omega)} + \beta t_0^2\right)}{(p_1-2)\,\beta t_0}.$$

Now we go to choose appropriate t_0 and T_0. Let t_0 be any number which depends only on p_1, $E_0 - E(0)$ and $\|u_0\|_{L^2(\Omega)}$ as

$$t_0 > \frac{\left\|\sqrt{a_0}u_0\right\|^2_{L^2(\Omega)}}{(p_1-2)\left(E_0 - E(0)\right)}.$$

Fix t_0, then T_0 can be picking as

$$T_0 = \frac{2\left(T_0 \left\|\sqrt{a_0}u_0\right\|^2_{L^2(\Omega)} + \beta t_0^2\right)}{(p_1-2)\,\beta t_0},$$

so that

$$T_0 = \frac{2\left(E_0 - E(0)\right) t_0^2}{(p_1-2)\left(E_0 - E(0)\right) t_0 - \left\|\sqrt{a_0}u_0\right\|^2_{L^2(\Omega)}},$$

Therefore the lifespan of the solution $u(x,t)$ is bounded by

$$T^* \leq \inf_{t\geq t_0} \frac{2\left(E_0 - E\left(0\right)\right)t^2}{\left(p_1-2\right)\left(E_0 - E\left(0\right)\right)t - \left\|\sqrt{a_0}u_0\right\|^2_{L^2(\Omega)}},$$
$$= \frac{8\left\|\sqrt{a_0}u_0\right\|^2_{L^2(\Omega)}}{\left(p_1-2\right)^2\left(E_0 - E\left(0\right)\right)}.$$

Case 2: $E(0) = E_0$. For this case, actually we consider the following claim

Claim 4.5. There exists $t^* > 0$ such that $E(t^*) < E_0$.

Suppose Claim is not true which means that $E(t) = E_0$ for all $t \geq 0$. Then by the continuity of $\left\|\nabla u\left(.,t\right)\right\|_{m(.)}$ there exists a t_0 small enough, such that

$$E(t) = E_0 \text{ and } \left\|\nabla u\left(.,t\right)\right\|^{m_2}_{m(.)} \geq \alpha_2 > \alpha_1 \text{ for all } t \in [0,t_0]$$

Then we consider the solution of (1.1) on $[0,t_0]$,

$$0 = E(t) - E_0 = -\int_0^{t_0}\int_\Omega a\left(x,t\right)u_t^2\left(x,t\right)\mathrm{d}x\mathrm{d}t$$

which turns out to be

$$\int_\Omega a\left(x,t\right)u_t\left(x,t\right)u\left(x,t\right)\mathrm{d}x = 0 \text{ a.e. on } [0,t_0]$$

And consequently, due to the equation (1.1),

$$\int_\Omega a\left(x,t\right)u_t\left(x,t\right)u\left(x,t\right)\mathrm{d}x \tag{4.13}$$
$$= -\int_\Omega \left|\nabla u\left(x,t\right)\right|^{m(x)}\mathrm{d}x + \int_\Omega u\left(x,t\right)f_{p(.)}\left(u\left(x,t\right)\right)\mathrm{d}x = 0 \text{ a.e.on } (0,t_0].$$

On the other hand,

$$E_0 = E\left(t\right) = \int_\Omega \frac{1}{m\left(x\right)}\left|\nabla u\left(x,t\right)\right|^{m(x)}\mathrm{d}x - \int_\Omega F\left(u\left(x,t\right)\right)\mathrm{d}x$$
$$\geq \frac{1}{m_2}\int_\Omega \left|\nabla u\left(x,t\right)\right|^{m(x)}\mathrm{d}x - \frac{1}{p_1}\int_\Omega u\left(x,t\right)f_{p(.)}\left(u\left(x,t\right)\right)\mathrm{d}x$$
$$= \left(\frac{1}{m_2} - \frac{1}{p_1}\right)\int_\Omega \left|\nabla u\left(x,t\right)\right|^{m(x)}\mathrm{d}x \text{ (by (4.13))}$$
$$> \left(\frac{1}{m_2} - \frac{1}{p_1}\right)\min\left(\alpha_1^{\frac{m_1}{m_2}},\alpha_1\right) \text{ (by (4.6))}$$
$$= \left(\frac{1}{m_2} - \frac{1}{p_1}\right)\alpha_1 = E_0 \text{ (by (2.5) and (2.6))}$$

which is a contradiction.

The proof of Theorem (4.2) is complete since one can apply the previous case (**Case 1**) after changing the time origin to t^*. □

5. Blow up for negative initial energy

This section is devoted to the main blow-up result and its proof in the case when $E(0) \leq 0$.

Assume that $a(x,t)$ is a positive function which belongs to the space $W^{1,\infty}(0,\infty;L^{\infty}(\Omega))$ and that $a_t(x,t) \geq 0$ a.e. for $t \geq 0$.

The next Lemma gives the desired blow-up result.

Lemma 5.1. *Let $u_0 \in W_0^{1,m(.)}(\Omega)$ such that $\int_\Omega u_0^2 \mathrm{d}x > 0$, $f_{p(.)}$ satisfies* (2.4) *and $E(0) \leq 0$. Then there exists a finite time $T_{\max} < \infty$ such that*

$$\int_\Omega |u(t)|^2 \mathrm{d}x \to \infty \text{ if } t \to T_{\max}.$$

Proof of Lemma (5.1). We then define

$$\phi(t) = \frac{1}{2}\int_\Omega a(x,t)|u(t)|^2 \mathrm{d}x$$

Differentiating ϕ with respect to t, gets

$$\begin{aligned}
\phi'(t) &= \int_\Omega a(x,t)uu_t \mathrm{d}x + \frac{1}{2}\int_\Omega a_t(x,t)|u(t)|^2 \mathrm{d}x \\
&\geq -\int_\Omega \left(|\nabla u|^{m(x)} - uf_{p(.)}(u)\right)\mathrm{d}x \quad (\text{by } (1.1)) \\
&\geq -\int_\Omega \left(|\nabla u|^{m(x)} - p(x)F(u)\right)\mathrm{d}x \quad (\text{by } (2.4)) \\
&\geq -\int_\Omega |\nabla u|^{m(x)} \mathrm{d}x + p_1\int_\Omega F(u)\mathrm{d}x \\
&= -\int_\Omega |\nabla u|^{m(x)} \mathrm{d}x + p_1\int_\Omega \frac{1}{m(x)}|\nabla u(x,t)|^{m(x)} \mathrm{d}x - p_1E(t) \quad (\text{by } (4.1)) \\
&\geq \left(\frac{p_1}{m_2} - 1\right)\int_\Omega |\nabla u|^{m(x)} \mathrm{d}x - p_1E(0) \quad (\text{by } (4.2)) \\
&\geq \left(\frac{p_1}{m_2} - 1\right)\int_\Omega |\nabla u|^{m(x)} \mathrm{d}x = c_0\int_\Omega |\nabla u|^{m(x)} \mathrm{d}x, \quad (c_0 > 0)
\end{aligned}$$

We define the sets

$$\Omega_2 = \{x \in \Omega \mid |\nabla u| \geq 1\} \quad \text{and} \quad \Omega_1 = \{x \in \Omega \mid |\nabla u| < 1\}.$$

So

$$\begin{aligned}
\phi'(t) &\geq c_0\int_{\Omega_2} |\nabla u|^{m_1} \mathrm{d}x + c_0\int_{\Omega_1} |\nabla u|^{m_2} \mathrm{d}x \\
&\geq C_1\left(\left(\int_{\Omega_2} |\nabla u|^2 \mathrm{d}x\right)^{\frac{m_1}{2}} + \left(\int_{\Omega_1} |\nabla u|^2 \mathrm{d}x\right)^{\frac{m_2}{2}}\right),
\end{aligned}$$

Using the fact that $\|\nabla u\|_2 \leq C \|\nabla u\|_q$, for all $q \geq 2$, to obtain

$$\begin{cases} (\phi'(t))^{\frac{2}{m_2}} \geq C_2 \int_{\Omega_1} |\nabla u|^2 \, \mathrm{d}x; \\ (\phi'(t))^{\frac{2}{m_1}} \geq C_3 \int_{\Omega_2} |\nabla u|^2 \, \mathrm{d}x. \end{cases}$$

By addition, leads to

$$(\phi'(t))^{\frac{2}{m_2}} + (\phi'(t))^{\frac{2}{m_1}} \geq C_4 \int_\Omega |\nabla u|^2 \, \mathrm{d}x \tag{5.1}$$

$$\geq C_5 \int_\Omega |u|^2 \, \mathrm{d}x \geq \frac{C_5}{\sup a(x,t)} \phi(t), \ \forall t \geq 0.$$

or

$$(\phi'(t))^{\frac{2}{m_1}} \left(1 + (\phi'(t))^{\frac{2}{m_2} - \frac{2}{m_1}}\right) \geq C_6 \phi(t), \ \forall t \geq 0. \tag{5.2}$$

By (5.1) and the fact that $\phi(t) \geq \phi(0) > 0$ ($\phi'(t) \geq 0$), we have, for each $t > 0$, either

$$\begin{cases} (\phi'(t))^{\frac{2}{m_1}} \geq \frac{C_6}{2} \phi(t) \geq \frac{C_6}{2} \phi(0); \\ \text{or } (\phi'(t))^{\frac{2}{m_2}} \geq \frac{C_6}{2} \phi(t) \geq \frac{C_6}{2} \phi(0) \end{cases}$$

which gives, in turn

$$\begin{cases} \phi'(t) \geq C_7 (\phi(0))^{\frac{m_2}{2}}; \\ \text{or } \phi'(t) \geq C_8 (\phi(0))^{\frac{m_1}{2}}, \end{cases}$$

hence

$$\phi'(t) \geq \alpha = \min\left(C_7 (\phi(0))^{\frac{m_2}{2}}, C_8 (\phi(0))^{\frac{m_1}{2}}\right),$$

since $\frac{1}{p_2} - \frac{1}{p_1} \leq 0$, (5.2) yields

$$(\phi'(t))^{\frac{2}{m_1}} (1 + \alpha)^{\frac{2}{m_2} - \frac{2}{m_1}} \geq C_4 \phi(t), \ \forall t \geq 0.$$

therefore

$$\phi'(t) \geq \beta \phi^{\frac{m_1}{2}}(t), \ \forall t \geq 0.$$

simple integrating then leads to

$$(\phi(t))^{1 - \frac{m_1}{2}} \leq (\phi(0))^{1 - \frac{m_1}{2}} - \frac{m_1 - 2}{2} \beta t, \ \forall t \geq 0.$$

which implies that

$$\phi(t) \geq \frac{1}{\left((\phi(0))^{1 - \frac{m_1}{2}} - \frac{m_1 - 2}{2} \beta t\right)^{\frac{2}{m_1 - 2}}}$$

This show that ϕ blows up in finite time $T_{\max}$ given by the estimate

$$T_{\max} \leq \frac{2 (\phi(0))^{1 - \frac{m_1}{2}}}{(m_1 - 2) \beta}.$$

□

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References

[1] Abita, R., *Blow-up phenomenon for a semilinear pseudo-parabolic equation involving variable source*, Applicable Analysis, 2021.

[2] Abita, R., *Bounds for below-up time in a nonlinear generalized heat equation*, Applicable Analysis, 2020.

[3] Abita, R., Benyattou, B., *Quasilinear parabolic equations with $p(x)$-laplacian diusion terms and nonlocal boundary conditions*, Stud. Univ. Babeş-Bolyai Math., **64**(2019), 101-116.

[4] Acerbi, E., Mingione, G., *Regularity results for stationary eletrorheological fluids*, Arch. Ration. Mech. Anal, **164**(2002), 213-259.

[5] Aiguo, B., Xianfa, S., *Bounds for the blowup time of the solutions to quasi-linear parabolic problems*, Zeitschrift für Angewandte Mathematik und Physik (ZAMP), **65**(2014).

[6] Akagi, G., Ôtani, M., *Evolutions inclusions governed by subdifferentials in reflexive Banach spaces*, J. Evol. Equ., **4**(2004), 519-541.

[7] Antonsev, S.N., *Blow up of solutions to parabolic equations with nonstandard growth conditions*, J. Comput. Appl. Math., **234**(2010), 2633-2645.

[8] Baghaei, K., Ghaemi, M.B., Hesaaraki, M., *Lower bounds for the blow-up time in a semilinear parabolic problem involving a variable source*, Applied Mathematics Letters, **27**(2014), 49-52.

[9] Diening, L., Hästo, P., Harjulehto, P., Ruzicka, M., *Lebesgue and Sobolev Spaces with Variable Exponents*, in: Springer Lecture Notes, Springer-Verlag, Berlin, 2011 and 2017.

[10] Diening, L., Ruzicka, M., *Calderon Zygmund operators on generalized Lebesgue spaces $L^{p(x)}(\Omega)$ and problems related to fluid dynamics*, Preprint Mathematische Fakultät, Albert-Ludwigs-Universität Freiburg, Freiburg, 120((21/2002,04.07.2002)), 197-220.

[11] Fan, X., Shen, J., Zhao, D., *Sobolev embedding theorems for spaces $W^{k,p(x)}(\Omega)$*, J. Math. Anal. Appl., **262**(2001), 749-760.

[12] Ferreira, R., de Pablo, A., Pérez-Llanos, M., Rossi, J.D., *Critical exponents for a semilinear parabolic equation with variable reaction*, Proceedings of the Royal Society of Edinburgh Section A Mathematics, **142A**(2012), 1027-1042.

[13] Fu, Y., *The existence of solutions for elliptic systems with nonuniform growth*, Studia Math., **151**(2002), 227-246.

[14] Fujita, H., *On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$*, J. Fac. Sci. Univ. Tokyo Sect., **13**(1966), no. I, 109-124.

[15] Hua, W., Yijun, H., *On blow-up of solutions for a semilinear parabolic equation involving variable source and positive initial energy*, Applied Mathematics Letters, **26**(2013), no. 10, 1008-1012.

[16] Kalantarov, V., Ladyzhenskaya, O.A., *The occurence of collapse for quasilinear equation of paprabolic and hyperbolic types*, J. Sov. Math., **10**(1978), 53-70.

[17] Kovàcik, O., Rákosnik, J., *On spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$*, Czechoslovak Math. J., **41**(1991).

[18] Ni, W.M., Sacks, P.E., Tavantzis, J., *On the asymptotic behavior of solutions of certain quasilinear parabolic equations*, J. Differential Equations, **54**(1984), 97-120.

[19] Payne, L.E., *Improperly Posed Problems in Partial Differential Equations*, Regional Conference Series in Applied Mathematics, 1975, 1-61.

[20] Xiulan, W., Guo, B., Wenjie, G., *Blow-up of solutions for a semilinear parabolic equation involving variable source and positive initial energy*, Applied Mathematics Letters, **26**(2013), 539-543.

[21] Zhong, T., *The reaction-diffusion equation with lewis function and critical sobolev exponent*, Journal of Mathematical Analysis and Applications, **272**(2002), no. 2, 480-495.

[22] Zhou, Y., *Global nonexistence for a quasilinear evolution equation with critical lower energy*, Arch. Inequal. Appl., **2**(2004), 41-47.

[23] Zhou, Y., *Global nonexistence for a quasilinear evolution equation with a generalized lewis function*, Journal for Analysis and its Applications, **24**(2005), 179-187.

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On a Fredholm-Volterra integral equation

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Abstract. In this paper we give conditions in which the integral equation

$$x(t) = \int_a^c K(t,s,x(s))ds + \int_a^t H(t,s,x(s))ds + g(t),\ t \in [a,b],$$

where $a < c < b$, $K \in C([a,b] \times [a,c] \times \mathbb{B}, \mathbb{B})$, $H \in C([a,b] \times [a,b] \times \mathbb{B}, \mathbb{B})$, $g \in C([a,b], \mathbb{B})$, with $\mathbb{B}$ a (real or complex) Banach space, has a unique solution in $C([a,b], \mathbb{B})$. An iterative algorithm for this equation is also given.

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Keywords: Fredholm-Volterra integral equation, existence, uniqueness, contraction, fiber contraction, Maia theorem, successive approximation, fixed point, Picard operator.

1. Introduction

The following type of integral equation was studied by several authors (see [11], [2], [3], [6], [1], [5], [10], [7], ...),

$$x(t) = \int_a^c K(t,s,x(s))ds + \int_a^t H(t,s,x(s))ds + g(t),\ t \in [a,b], \tag{1.1}$$

where $a < c < b$, $K \in C([a,b] \times [a,c] \times \mathbb{B}, \mathbb{B})$, $H \in C([a,b] \times [a,b] \times \mathbb{B}, \mathbb{B})$, $g \in C([a,b], \mathbb{B})$, with $(\mathbb{B}, |\cdot|)$ a (real or complex) Banach space.

The aim of this paper is to give some conditions on K and H in which the equation (1.1) has a unique solution in $C([a,b], \mathbb{B})$. To do this, we shall use the contraction principle, the fiber contraction principle ([9], [13], [10], [11]) and a variant of Maia fixed point theorem given in [8] (see also [4]).

2. Preliminaries

Let us recall some notions, notations and fixed point results which will be used in this paper.

2.1. Picard operators and weakly Picard operators

Let $(X, \to)$ be an L-space $((X,d), \overset{d}{\to}; (X,\tau), \overset{\tau}{\to}; (X, \|\cdot\|), \overset{\|\cdot\|}{\to}, \rightharpoonup; \ldots)$. An operator $A : (X, \to) \to (X, \to)$ is called weakly Picard operator (WPO) if the sequence $(A^n(x))_{n\in\mathbb{N}}$ converges for all $x \in X$ and the limit (which generally depends on x) is a fixed point of A.

If an operator A is WPO and the fixed point set of A is a singleton, i.e.,

$$F_A = \{x^*\},$$

then, by definition, A is called Picard operator (PO).

For a WPO, $A : (X, \to) \to (X, \to)$, we define the limit operator $A^\infty : (X, \to) \to (X, \to)$, by $A^\infty(x) := \lim_{n\to\infty} A^n(x)$. We remark that, $A^\infty(X) = F_A$, i.e., A^∞ is a set retraction of X on F_A.

2.2. Fiber contraction principle

Regarding this principle, some important results were given in [12] and [13].

Fiber Contraction Theorem. *Let $(X, \to)$ be an L-space, (Y, d) be a metric space, $B : X \to X$, $C : X \times Y \to Y$ and $A : X \times Y \to X \times Y$, $A(x, y) := (B(x), C(x, y))$. We suppose that:*

(i) *(Y, d) is a complete metric space;*
(ii) *B is a WPO;*
(iii) *$C(x, \cdot) : Y \to Y$ is an l-contraction, for all $x \in X$;*
(iv) *$C : X \times Y \to Y$ is continuous.*

Then A is a WPO. Moreover, if B is a PO, then A is a PO.

Generalized Fiber Contraction Theorem. *Let $(X, \to)$ be an L-space and (X_i, d_i), $i = \overline{1,m}$, $m \geq 1$ be metric spaces. Let $A_i : X_0 \times \ldots \times X_i \to X_i$, $i = \overline{0,m}$, be some operators. We suppose that:*

(i) *(X_i, d_i), $i = \overline{1,m}$, are complete metric spaces;*
(ii) *A_0 is a WPO;*
(iii) *$A_i(x_0, \ldots, x_{i-1}, \cdot) : X_i \to X_i$, $i = \overline{1,m}$, are l_i-contractions;*
(iv) *A_i, $i = \overline{1,m}$, are continuous.*

Then the operator $A : X_0 \times \ldots \times X_m \to X_0 \times \ldots \times X_m$, defined by

$$A(x_0, \ldots, x_m) := (A_0(x_0), A_1(x_0, x_1), \ldots, A_m(x_0, \ldots, x_m))$$

is a WPO. Moreover, if A_0 is a PO, then A is a PO.

2.3. A variant of Maia fixed point theorem

We recall here the following variant of Maia fixed point theorem, given by I.A. Rus in [8]:

Theorem 2.1. *Let X be a nonempty set, d and ρ be two metrics on X and $A : X \to X$ be an operator. We suppose that:*

(1) *there exists $c > 0$ such that $d(A(x), A(y)) \leq c\rho(x, y)$, for all $x, y \in X$;*
(2) *(X, d) is a complete metric space;*
(3) *$A : (X, d) \to (X, d)$ is continuous;*

(4) $A : (X, \rho) \to (X, \rho)$ *is an* l*-contraction.*

Then:

(*i*) $F_A = \{x^*\}$;

(*ii*) $A : (X, d) \to (X, d)$ *is PO.*

3. Operatorial point of view on equation (1.1)

Let $X := C([a, b], \mathbb{B})$ and $T : X \to X$ be defined by

$$T(x)(t) := \int_a^c K(t, s, x(s))ds + \int_a^t H(t, s, x(s))ds + g(t),\ t \in [a, b].$$

For $x \in X$, we denote by $u := x\big|_{[a,c]}$ and $v := x\big|_{[c,b]}$. If x is a solution of the equation (1.1) (i.e. a fixed point of T), then

$$u(t) = \int_a^c K(t, s, u(s))ds + \int_a^t H(t, s, u(s))ds + g(t),\ t \in [a, c] \tag{3.1}$$

and

$$v(t) = \int_a^c K(t, s, u(s))ds + \int_a^c H(t, s, u(s))ds + \int_c^t H(t, s, v(s))ds + g(t),\ t \in [c, b]. \tag{3.2}$$

Let $X_1 := C([a, c], \mathbb{B})$, $X_2 := C([c, b], \mathbb{B})$ and

$T_1 : X_1 \to X_1$, $T_1(u)(t) :=$ *the second part of* (3.1),

$T_2 : X_1 \times X_2 \to X_2$, $T_2(u, v)(t) :=$ *the second part of* (3.2).

The mappings T_1 and T_2 allow us to construct the triangular operator

$$\tilde{T} : X_1 \times X_2 \to X_1 \times X_2,\ \tilde{T}(u, v) := (T_1(u), T_2(u, v)),\ \text{for all } (u, v) \in X_1 \times X_2.$$

Remark 3.1. If $(u^*, v^*) \in F_{\tilde{T}}$, then $u^*(c) = v^*(c)$. So the function $x^* \in X$, defined by

$$x^*(t) := \begin{cases} u^*(t),\ t \in [a, c] \\ v^*(t),\ t \in [c, b] \end{cases}$$

is a fixed point of T, i.e., a solution of (1.1).

Remark 3.2. For $(u_0, v_0) \in X_1 \times X_2$ we consider the successive approximations corresponding to the operator $\tilde{T}$, $(u_{n+1}, v_{n+1}) = \tilde{T}(u_n, v_n)$, $n \in \mathbb{N}$. We observe that, for $n \in \mathbb{N}^*$, $u_n(c) = v_n(c)$. So, the function x_n, defined by

$$x_n(t) := \begin{cases} u_n(t),\ t \in [a, c] \\ v_n(t),\ t \in [c, b] \end{cases}$$

is in X.

Remark 3.3. Let $Y \subset X_1 \times X_2$ be defined by

$$Y := \{(u,v) \in X_1 \times X_2 \mid u(c) = v(c)\}.$$

The operator $R : X \to Y$, defined by $R(x) := (x\big|_{[a,c]}, x\big|_{[c,b]})$ is a bijection. From the above definitions, it is clear that $T(x) = (R^{-1}\tilde{T}R)(x)$ and the n^{th} iterate of T is $T^n = R^{-1}\tilde{T}^n R$.

In conclusion, to study the equation (1.1) (which is equivalent with $x = T(x)$) it is sufficient to study the fixed point of the operator $\tilde{T}$. If $(u^*, v^*) \in F_{\tilde{T}}$ then $R^{-1}(u^*, v^*) \in F_T$.

4. Existence and uniqueness of solution of equation (1.1)

In what follows, in addition to the continuity of H, K and g, we suppose on K and H that:

(i) There exists $L_1 \in C([a,b] \times [a,c], \mathbb{B})$ such that:

$$|K(t,s,\xi) - K(t,s,\eta)| \leq L_1(t,s)|\xi - \eta|, \text{ for all } t \in [a,b], \ s \in [a,c], \ \xi, \eta \in \mathbb{B}.$$

(ii) There exists $L_2 \in C([a,b] \times [a,b], \mathbb{B})$ such that:

$$|H(t,s,\xi) - H(t,s,\eta)| \leq L_2(t,s)|\xi - \eta|, \text{ for all } t, s \in [a,b], \ \xi, \eta \in \mathbb{B}.$$

(iii) $\left(\displaystyle\int_{[a,c]\times[a,c]} \big(L_1(t,s) + L_2(t,s)\big)^2 dtds \right)^{\frac{1}{2}} < 1.$

The basic result of our paper is the following.

Theorem 4.1. *In the above conditions we have that:*

(1) *The equation* (1.1) *has in* $C([a,b], \mathbb{B})$ *a unique solution* x^*.

(2) *The operator* $\tilde{T}$ *is a Picard operator with respect to* $\overset{unif.}{\to}$. *Let* $F_{\tilde{T}} = \{(u^*, v^*)\}$.

(3) *The operator* T *is a Picard operator with respect to* $\overset{unif.}{\to}$ *and* $F_T = \{x^*\}$. *Moreover,* $x^* = R^{-1}(u^*, v^*)$.

Proof. From the remarks which were given in §3, it is sufficient to prove that the operator $\tilde{T}$ is a Picard operator with respect to the uniform convergence on $X_1 \times X_2$.

In order to apply the Fiber contraction principle, we shall prove that:

(j) $T_1 : (X_1, \overset{unif.}{\to}) \to (X_1, \overset{unif.}{\to})$ is a Picard operator;

(jj) $T_2(u, \cdot) : (X_2, \|\cdot\|_\tau) \to (X_2, \|\cdot\|_\tau)$ is a contraction.

Let us prove (j).

We consider on X_1, the norms $\|\cdot\|_\infty$ and $\|\cdot\|_{L^2}$. By using the assumptions (i) and (ii), we have the following estimations:

$$|T_1(u_1)(t) - T_1(u_2)(t)| \leq \int_a^c |K(t,s,u_1(s)) - K(t,s,u_2(s))|ds$$
$$+ \int_a^t |H(t,s,u_1(s)) - H(t,s,u_2(s))|ds$$

$$\leq \int_a^c L_1(t,s)|u_1(s)-u_2(s)|ds + \int_a^c L_2(t,s)|u_1(s)-u_2(s)|ds$$

$$\overset{\substack{\text{Hölder's}\\ \text{inequality}}}{\leq} \left(\int_a^c L_1(t,s)^2 ds\right)^{\frac{1}{2}} \left(\int_a^c |u_1(s)-u_2(s)|^2 ds\right)^{\frac{1}{2}}$$

$$+ \left(\int_a^c L_2(t,s)^2 ds\right)^{\frac{1}{2}} \left(\int_a^c |u_1(s)-u_2(s)|^2 ds\right)^{\frac{1}{2}}.$$

By taking the $\max\limits_{t\in[a,c]}$ in the above inequalities, there exists a real positive constant

$$c := \max_{t\in[a,c]} \left(\int_a^c L_1(t,s)^2 ds\right)^{\frac{1}{2}} + \max_{t\in[a,c]} \left(\int_a^c L_2(t,s)^2 ds\right)^{\frac{1}{2}}$$

such that

$$\|T_1(u_1)-T_1(u_2)\|_\infty \leq c\|u_1-u_2\|_{L^2}, \text{ for all } u_1,u_2\in X_1.$$

On the other hand, we have that

$$\begin{aligned}\|T_1(u_1)-T_1(u_2)\|_{L^2} &= \left(\int_a^c |T_1(u_1)(t)-T_1(u_2)(t)|^2 dt\right)^{\frac{1}{2}} \\ &\leq \left(\int_a^c \left(\int_a^c (L_1(t,s)ds + L_2(t,s))^2 ds\right) \|u_1-u_2\|_{L^2}^2 dt\right)^{\frac{1}{2}} \\ &= \left(\int_a^c \int_a^c (L_1(t,s)+L_2(t,s))^2\, dsdt\right)^{\frac{1}{2}} \|u_1-u_2\|_{L^2}, \\ &\text{for all } u_1,u_2\in X_1.\end{aligned}$$

By using the assumption (iii), it follows that the operator T_1 is a contraction with respect to $\|\cdot\|_{L^2}$ on X_1.

The conclusion follows from the variant of Maia theorem.

Let us prove (jj).

For $t\in[c,b]$ and $M_{L_2} := \max\limits_{t,s\in[c,b]} L_2(t,s)$, we have that

$$\begin{aligned}|T_2(u,v_1)(t)-T_2(u,v_2)(t)| &\leq \int_c^t |H(t,s,v_1(s))-H(t,s,v_2(s))|ds \\ &\leq \int_c^t L_2(t,s)|v_1(s)-v_2(s)|ds \\ &\leq M_{L_2}\int_c^t |v_1(s)-v_2(s)|e^{-\tau(s-c)}e^{\tau(s-c)}ds \\ &\leq M_{L_2}\|v_1-v_2\|_\tau \int_c^t e^{\tau(s-c)}ds \leq M_{L_2}\|v_1-v_2\|_\tau \frac{e^{\tau(t-c)}}{\tau}.\end{aligned}$$

It follows that

$$|T_2(u,v_1)(t)-T_2(u,v_2)(t)|e^{-\tau(t-c)} \leq \frac{M_{L_2}}{\tau}\|v_1-v_2\|_\tau.$$

By taking $\max_{t\in[c,b]}$ and by choosing $\tau > M_{L_2}$, there exists a real positive constant

$$l := \frac{M_{L_2}}{\tau} < 1$$

such that

$$\|T_2(u,v_1) - T_2(u,v_2)\|_\tau \le l\|v_1 - v_2\|_\tau,\ \text{for all } v_1, v_2 \in X_2. \qquad \square$$

Remark 4.2. Let $\mathbb{K} := \mathbb{R}$ or $\mathbb{C}$, $|\cdot|$ be a norm on $\mathbb{B} := \mathbb{K}^m$ ($|\cdot|_1$, $|\cdot|_2$, $|\cdot|_\infty$, $\ldots$), $a < c < b$, $K = (K_1,\ldots,K_m) \in C([a,b],\mathbb{K}^m)$ and $H = (H_1,\ldots,H_m) \in C([a,b],\mathbb{R}^m)$. In this case, the equation (1.1) takes the following form

$$\begin{cases} x_1(t) &= \int_a^c K_1(t,s,x_1(s),\ldots,x_m(s))ds \\ & \quad + \int_a^t H_1(t,s,x_1(s),\ldots,x_m(s))ds,\ t\in[a,b] \\ & \vdots \\ x_m(t) &= \int_a^c K_m(t,s,x_1(s),\ldots,x_m(s))ds \\ & \quad \int_a^t H_m(t,s,x_1(s),\ldots,x_m(s))ds,\ t\in[a,b]. \end{cases} \tag{4.1}$$

From Theorem 4.1 we have an existence and uniqueness result for the system (4.1).

In the case when $\mathbb{B}$ is a Banach space of infinite sequences with elements in $\mathbb{K}$ ($c(\mathbb{K})$, $C_p(\mathbb{K})$, $m(\mathbb{K})$, $l^p(\mathbb{K})$, $\ldots$) we have from Theorem 4.1 an existence and uniqueness result for an infinite system of Fredholm-Volterra integral equations.

References

[1] Bolojan, O.-M., *Fixed Point Methods for Nonlinear Differential Systems with Nonlocal Conditions*, Casa Cărţii de Ştiinţă, Cluj-Napoca, 2013.

[2] Boucherif, A., *Differential equations with nonlocal boundary conditions*, Nonlinear Anal., **47**(2001), 2419-2430.

[3] Boucherif, A., Precup, R., *On the nonlocal initial value problem for first order differential equations*, Fixed Point Theory, **4**(2003), 205-212.

[4] Filip, A.-D., *Fixed Point Theory in Kasahara Spaces*, Casa Cărţii de Ştiinţă, Cluj-Napoca, 2015.

[5] Nica, O., *Nonlocal initial value problems for first order differential systems*, Fixed Point Theory, **13**(2012), 603-612.

[6] Petruşel, A., Rus, I.A., *A class of functional integral equations with applications to a bilocal problem*, 609-631. In: Topics in Mathematical Analysis and Applications (Rassias, Th.M. and Tóth, L., Eds.), Springer, 2014.

[7] Precup, R., *Methods in Nonlinear Integral Equations*, Kluwer, Dordrecht-Boston-London, 2002.

[8] Rus, I.A., *On a fixed point theorem of Maia*, Stud. Univ. Babeş-Bolyai Math., **22**(1977), no.. 1, 40-42.

[9] Rus, I.A., *Picard operators and applications*, Sci. Math. Jpn., **58**(2003), 191-219.

[10] Rus, I.A., *Abstract models of step method which imply the convergence of successive approximations*, Fixed Point Theory, **9**(2008), no. 1, 293-307.

[11] Rus, I.A., *Some variants of contraction principle in the case of operators with Volterra property: step by step contraction principle*, Adv. Theory Nonlinear Anal. Appl., **3**(2019), no. 3, 111-120.

[12] Rus, I.A., Şerban, M.-A., *Basic problems of the metric fixed point theory and the relevance of a metric fixed point theorem*, Carpathian J. Math., **29**(2013), no. 2, 239-258.

[13] Şerban, M.-A., *Teoria Punctului Fix pentru Operatori Definiţi pe Produs Cartezian*, Presa Univ. Clujeană, Cluj-Napoca, 2002.

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Multiplicative perturbations of local C-cosine functions

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Abstract. We establish some left and right multiplicative perturbations of a local C-cosine function $C(\cdot)$ on a complex Banach space X with non-densely defined generator, which can be applied to obtain some new additive perturbation results concerning $C(\cdot)$.

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1. Introduction

Let X be a Banach space over $\mathbb{F}$ (=$\mathbb{R}$ or $\mathbb{C}$) with norm $\|\cdot\|$, and let $L(X)$ denote the set of all bounded linear operators on X. For each $0 < T_0 \leq \infty$ and each injection $C \in L(X)$, a family $C(\cdot)(= \{C(t) \,|\, 0 \leq t < T_0\})$ in $L(X)$ is called a local C-cosine function on X if it is strongly continuous, $C(0) = C$ on X and satisfies

(1.1) $2C(t)C(s) = C(t+s)C + C(|t-s|)C$ on X for all $0 \leq t, s, t+s < T_0$

(see [7], [10], [14], [20], [22], [24], [26]). In this case, the generator of $C(\cdot)$ is a linear operator A in X defined by

$$D(A) = \{x \in X \mid \lim_{h\to 0^+} 2(C(h)x - Cx)/h^2 \in R(C)\}$$

and $Ax = C^{-1} \lim_{h\to 0^+} 2(C(h)x - Cx)/h^2$ for $x \in D(A)$. Moreover, we say that $C(\cdot)$ is

(1.2) locally Lipschitz continuous, if for each $0 < t_0 < T_0$ there exists a $K_{t_0} > 0$ such that $\|C(t+h) - C(t)\| \leq K_{t_0} h$ for all $0 \leq t, h, t+h \leq t_0$;

(1.3) exponentially bounded, if $T_0 = \infty$ and there exist $K, \omega \geq 0$ such that $\|C(t)\| \leq Ke^{\omega t}$ for all $t \geq 0$;

(1.4) exponentially Lipschitz continuous, if $T_0 = \infty$ and there exist $K, \omega \geq 0$ such that $\|C(t+h) - C(t)\| \leq Khe^{\omega(t+h)}$ for all $t, h \geq 0$.

In general, a local C-cosine function is also called a C-cosine function if $T_0 = \infty$ (see [17], [6], [4], [13]), a C-cosine function may not be exponentially bounded (see [13]), and the generator of a local C-cosine function may not be densely defined (see [17], [6]). Moreover, a local C-cosine function is not necessarily extendable to the half line $[0, \infty)$ (see [22]) except for $C = I$ (identity operator on X). Perturbations of local C-cosine functions with or without the exponential boundedness have been extensively studied by many authors appearing in [2,6,8-17,19,23,25]. Some interesting applications of this topic are also illustrated there. In particular, Li has obtained some right-multiplicative perturbation theorems for local C-cosine functions in which the operator C may not commute with the bounded perturbation operator B on X, which satisfies an estimation that is similar to the condition (2.6) below. In this case, $C^{-1}A(I+B)C$ generates a local C-cosine function on X when $CA(I+B) \subset A(I+B)C$ (see [18]). Along this line, Li and Liu also establish some left-multiplicative perturbation theorems for local C-cosine functions on X with densely defined generators. In this case, $(I+B)A$ generates a local C-cosine function on X when $C^{-1}(I+B)AC = (I+B)A$ (see [20]). Just as continuous work of this topic, Kuo shows that $A+B$ generates a local C-cosine function on X when either B is a bounded linear operator from $[D(A)]$ into $R(C)$ such that $R(C^{-1}B) \subset D(A)$ (see [14]) or B is a bounded linear operator on X which commutes with $C(\cdot)$ on X (see [15] or Theorem 2.13 below). The purpose of this paper is to establish some left and right multiplicative perturbation theorems for local C-cosine functions just as results in [18,20] when the generator A of a perturbed local C-cosine function $C(\cdot)$ may not be densely defined, the perturbation operator B is only a bounded linear operator from $\overline{D(A)}$ into $R(C)$, and the assumption of $C^{-1}(I+B)AC = (I+B)A$ is not necessary, which together with Theorem 2.13 can be applied to obtain some new Miyadera type additive perturbation theorems just as results in [15] for local C-cosine functions (see Theorems 2.14 and 2.16 below). An illustrative example concerning these results is also presented in the final part of this paper.

2. Perturbation theorems

In this section, we first note some basic properties of a local C-cosine function and known results about connections between the generator of a local C-cosine function and strong solutions of the following abstract Cauchy problem:

$$\mathrm{ACP}(A, f, x, y) \begin{cases} u''(t) = Au(t) + f(t) & \text{for } t \in (0, T_0) \\ u(0) = x, u'(0) = y, \end{cases}$$

where $x, y \in X$ and f is an X-valued function defined on a subset of $[0, T_0)$.

Proposition 2.1. *(see* [4], [11], [13], [22]*). Let A be the generator of a local C-cosine function $C(\cdot)$ on X. Then*

(2.1) *A is closed and $C^{-1}AC = A$;*
(2.2) *$C(t)x \in D(A)$ and $C(t)Ax = AC(t)x$ for all $x \in D(A)$ and $0 \le t < T_0$;*

(2.3) $\int_0^t\int_0^s C(r)xdrds \in D(A)$ *and* $A\int_0^t\int_0^s C(r)xdrds = C(t)x - Cx$ *for all* $x \in D(A)$ *and* $0 \leq t < T_0$;

(2.4) $D(A) = \{x \in X | C(t)x - Cx = \int_0^t\int_0^s C(r)y_x drds$ *for all* $0 \leq t < T_0$ *and for some* $y_x \in X\}$ *and* $Ax = y_x$ *for each* $x \in D(A)$;

(2.5) $R(C(t)) \subset \overline{D(A)}$ *for* $0 \leq t < T_0$.

Definition 2.2. Let $A : D(A) \subset X \to X$ be a closed linear operator in a Banach space X with domain $D(A)$ and range $R(A)$. A function $u : [0, T_0) \to X$ is called a (strong) solution of ACP(A, f, x, y), if $u \in C^2((0, T_0), X) \cap C^1([0, T_0), X) \cap C((0, T_0), [D(A)])$ and satisfies ACP(A, f, x, y). Here $[D(A)]$ denotes the Banach space $D(A)$ with norm $|\cdot|$ defined by $|x| = \|x\| + \|Ax\|$ for $x \in D(A)$.

Theorem 2.3. (see [11], [13]) *A generates a local C-cosine function $C(\cdot)$ on X if and only if $C^{-1}AC = A$ and for each $x \in X$, ACP$(A, Cx, 0, 0)$ has a unique (strong) solution $u(\cdot, x)$ in $C^2([0, T_0), X)$. In this case, we have*

$$u(t, x) = j_1 * C(t)x \left(= \int_0^t j_1(t - s)C(s)xds \right)$$

for all $x \in X$ and $0 \leq t < T_0$. Here $j_k(t) = t^k/k!$ for all $t \in \mathbb{R}$ and $k \in \mathbb{N} \cup \{0\}$.

Proposition 2.4. *(see [11], [13]) Let A be the generator of a local C-cosine function $C(\cdot)$ on X, $x, y \in X$ and $f \in L^1_{loc}([0, T_0), X) \cap C((0, T_0), X)$. Then ACP$(A, Cf, Cx, Cy)$ has a (strong) solution u in $C^2([0, T_0), X)$ if and only if*

$$v(\cdot) = C(\cdot)x + S(\cdot)y + S * f(\cdot) \in C^2([0, T_0), X).$$

In this case, $u = v$ on $[0, T_0)$. Here $S(\cdot) = j_0 * C(\cdot)$ and $S * f(\cdot) = \int_0^{\cdot} S(\cdot - s)f(s)ds$.

We next establish a new right-multiplicative perturbation theorem for locally Lipschitz continuous and exponentially Lipschitz continuous local C-cosine functions in which B is only a bounded linear operator from $\overline{D(A)}$ into $R(C)$.

Theorem 2.5. *Let $C(\cdot)$ be a locally Lipschitz continuous local C-cosine function on X with generator A. Assume that B is a bounded linear operator from $\overline{D(A)}$ into $R(C)$ such that $CB = BC$ on $\overline{D(A)}$, and for each $0 < t_0 < T_0$ there exists an $M_{t_0} > 0$ such that $(S * C^{-1}Bf)(t) \in D(A)$ and*

$$\|A(S * C^{-1}B)[f(t) - f(s)]\| \leq M_{t_0} \int_s^t \|f(r)\|dr \tag{2.6}$$

for all $f \in C([0, t_0], \overline{D(A)})$ and $0 \leq s < t \leq t_0$. Then $A(I + C^{-1}BC)$ generates a locally Lipschitz continuous local C-cosine function $T(\cdot)$ on X satisfying

$$T(\cdot)x = C(\cdot)x + A(S * C^{-1}BT)(\cdot)x \quad \text{on } [0, T_0) \tag{2.7}$$

for all $x \in X$.

Proof. Let $x \in X$ and $0 < t_0 < T_0$ be fixed.
We define $U : C([0,t_0],\overline{D(A)}) \to C([0,t_0],\overline{D(A)})$ by

$$U(f)(\cdot) = C(\cdot)x + A(S * C^{-1}Bf)(\cdot)$$

on $[0,t_0]$ for all $f \in C([0,t_0],\overline{D(A)})$. Then U is well-defined. By induction, we obtain from (2.6) that

$$\begin{aligned}\|U^n f(t) - U^n g(t)\| &= \|U(U^{n-1}f)(t) - U(U^{n-1}g)(t)\| \\ &= \|AS * C^{-1}B(U^{n-1}f - U^{n-1}g)(t)\| \\ &\leq M_{t_0}^n \int_0^t j_{n-1}(t-s)\|f(s)-g(s)\|ds \\ &\leq M_{t_0}^n j_n(t_0)\|f-g\|\end{aligned}$$

for all $f, g \in C([0,t_0],\overline{D(A)})$, $0 \leq t \leq t_0$ and $n \in \mathbb{N}$. Here

$$\|f-g\| = \max_{0\leq s\leq t_0} \|f(s)-g(s)\|.$$

It follows from the contraction mapping theorem that there exists a unique function w_{x,t_0} in $C([0,t_0],\overline{D(A)})$ such that

$$w_{x,t_0}(\cdot) = C(\cdot)x + AS * C^{-1}Bw_{x,t_0}(\cdot)$$

on $[0,t_0]$. In this case, we set $w_x(t) = w_{x,t_0}(t)$ for all $0 \leq t \leq t_0 < T_0$, then $w_x(\cdot)$ is a unique function in $C([0,T_0),\overline{D(A)})$ such that

$$w_x(\cdot) = C(\cdot)x + AS * C^{-1}Bw_x(\cdot)$$

on $[0,T_0)$. Since

$$\begin{aligned}j_1 * w_x(\cdot) &= j_1 * C(\cdot)x + Aj_1 * S * C^{-1}Bw_x(\cdot) \\ &= j_0 * S(\cdot)x + S * C^{-1}Bw_x(\cdot) - Bj_1 * w_x(\cdot)\end{aligned}$$

on $[0,T_0)$, we have

$$(I+B)j_1 * w_x(t) = j_0 * S(t)x + S * C^{-1}Bw_x(t) \in D(A)$$

for all $0 \leq t < T_0$. Clearly, $j_1 * w_x$ is the unique function u_x in $C^2([0,T_0),X)$ such that

$$u_x(\cdot) = j_0 * S(\cdot)x + AS * C^{-1}Bu_x(\cdot)$$

on $[0,T_0)$. Since $j_0 * S(\cdot)x + S * C^{-1}Bw_x(\cdot) \in C^2([0,T_0),X)$, we obtain from Proposition 2.4 that

$$j_0 * S(\cdot)x + S * C^{-1}Bw_x(\cdot) = (I+B)j_1 * w_x(\cdot)$$

is the unique solution of ACP$(A, Cx + Bw_x, 0, 0)$ in $C^2([0,T_0),X)$. This implies that

$$A(I+B)j_1 * w_x + Cx + Bw_x = (I+B)w_x$$

on $[0,T_0)$, and so $A(I+B)j_1 * w_x + Cx = w_x$ on $[0,T_0)$. Hence, $j_1 * w_x$ is a solution of ACP$(A(I+B), Cx, 0, 0)$ in $C^2([0,T_0),X)$. To prove the uniqueness of solutions of

ACP$(A(I+B), Cx, 0, 0)$.
Suppose that $u \in C([0,T_0),X)$ and satisfies $A(I+B)j_1 * u + Cx = u$ on $[0,T_0)$. Then

$$\begin{aligned} j_1 * (S * u - S * j_0 Cx) =& j_1 * S * A(I+B)j_1 * u \\ =& Aj_1 * S * (I+B)j_1 * u \\ =& S * (I+B)j_1 * u - Cj_1 * (I+B)j_1 * u \\ =& S * j_1 * u + S * Bj_1 * u - Cj_1 * (I+B)j_1 * u \end{aligned}$$

on $[0,T_0)$, and so $-S * j_2(\cdot)Cx = S * Bj_1 * u(\cdot) - Cj_1 * (I+B)j_1 * u(\cdot)$ on $[0,T_0)$. Hence,

$$\begin{aligned} -S * j_0(\cdot)x =& (S * C^{-1}Bj_1 * u)''(\cdot) - (I+B)j_1 * u(\cdot) \\ =& AS * C^{-1}Bj_1 * u(\cdot) + Bj_1 * u(\cdot) - (I+B)j_1 * u(\cdot) \\ =& AS * C^{-1}Bj_1 * u(\cdot) - j_1 * u(\cdot) \end{aligned}$$

on $[0,T_0)$, which implies that $j_1 * u(\cdot) = S * j_0(\cdot)x + AS * C^{-1}Bj_1 * u(\cdot)$ on $[0,T_0)$. Consequently, $j_1 * u = j_1 * w_x$ on $[0,T_0)$ or equivalently, $u = w_x$ on $[0,T_0)$. Clearly, $A(I+B)$ is closed and $A(I+B)C = CA(I+B)$ on $D(A(I+B))$. It follows from Proposition 2.4 that $C^{-1}A(I+B)C$ generates a local C-cosine function $T(\cdot)$ on X satisfying (2.7) for all $x \in X$. Just as in the proof of [27, Theorem 2.5], we have $C^{-1}A(I+B)C = A(I+C^{-1}BC)$. By (2.6), $T(\cdot)$ is also locally Lipschitz continuous. □

Since the condition (2.6) in the proof of Theorem 2.5 is only used to show that $T(\cdot)$ is locally Lipschitz continuous. By slightly modifying the proof of Theorem 2.5, we can obtain the next right-multiplicative perturbation theorem for local C-cosine functions without the local Lipschitz continuity.

Theorem 2.6. *Let $C(\cdot)$ be a local C-cosine function on X with generator A. Assume that B is a bounded linear operator from $\overline{D(A)}$ into $R(C)$ such that $CB = BC$ on $\overline{D(A)}$, and for each $0 < t_0 < T_0$ there exists an $M_{t_0} > 0$ such that $(S * C^{-1}Bf)(t) \in D(A)$ and*

$$\|A(S * C^{-1}Bf)(t)\| \leq M_{t_0} \int_0^t \|f(s)\| ds \tag{2.8}$$

for all $f \in C([0,t_0], \overline{D(A)})$ and $0 \leq t \leq t_0$. Then $A(I + C^{-1}BC)$ generates a local C-cosine function $T(\cdot)$ on X satisfying (2.7)

Corollary 2.7. *Let $C(\cdot)$ be a locally Lipschitz continuous local C-cosine function on X with generator A. Assume that B is a bounded linear operator from $\overline{D(A)}$ into $R(C)$ such that $CB = BC$ on $\overline{D(A)}$ and $C^{-1}Bx \in \overline{D(A)}$ for all $x \in \overline{D(A)}$. Then $A(I+C^{-1}BC)$ generates a locally Lipschitz continuous local C-cosine function $T(\cdot)$ on X satisfying (2.7) for all $x \in X$. Moreover, $T(\cdot)$ is exponentially Lipschitz continuous if $C(\cdot)$ is.*

Proof. Clearly, it suffices to show that for each $0 < t_0 < T_0$ there exists an $M_{t_0} > 0$ such that (2.6) holds for all $f \in C([0,t_0], \overline{D(A)})$ and $0 \leq s < t \leq t_0$. Suppose that

$C_1(t)$ denotes the restriction of $C(t)$ to $\overline{D(A)}$, $C_1'(t)$ the strong derivative of $C_1(t)$ on $\overline{D(A)}$ for all $0 \le t < T_0$, and D^2 the second order derivative of a function. Then $C_1(t)x = Cx + Aj_0 * S(t)x$ and $C_1'(t)x = AS(t)x$ for all $x \in \overline{D(A)}$ and $0 \le t < T_0$. In particular, $AS(\cdot)$ is a strongly continuous family of bounded linear operators on $\overline{D(A)}$, which is also exponentially bounded if $C(\cdot)$ is exponentially Lipschitz continuous. Let $0 < t_0 < T_0$ be given, then $S * C^{-1}Bf(\cdot)$ is twice continuously differentiable on $[0, t_0]$,

$$D^2(S * C^{-1}Bf)(\cdot) = AS * C^{-1}Bf(\cdot) + Bf(\cdot) = C_1' * C^{-1}Bf(\cdot) + Bf(\cdot)$$

on $[0, t_0]$ and

$$\begin{aligned}\|A(S * C^{-1}B[f(t) - f(s)])\| &= \|C_1' * C^{-1}B[f(t) - f(s)]\| \\ &\le \sup_{0 \le r \le t_0} \|AS(r)\| \|C^{-1}B\| \int_s^t \|f(r)\| dr\end{aligned}$$

for all $f \in C([0, t_0], \overline{D(A)})$ and $0 \le s < t \le t_0$. It follows from Theorem 2.3 that $A(I + C^{-1}BC)$ generates a locally Lipschitz continuous local C-cosine function $T(\cdot)$ on X satisfying (2.7) for all $x \in X$. Combining the local Lipschitz continuity of $C^{-1}BT(\cdot)$ with the exponential boundedness of $AS(\cdot)$, we get that $AS * C^{-1}BT)(\cdot)$ is exponentially Lipschitz continuous if $C(\cdot)$ is. Consequently, $T(\cdot)$ is exponentially Lipschitz continuous if $C(\cdot)$ is. □

Corollary 2.8. *Let $C(\cdot)$ be a local C-cosine function on X with generator A. Assume that B is a bounded linear operator from $\overline{D(A)}$ into $R(C)$ such that $CB = BC$ on $\overline{D(A)}$ and $C^{-1}Bx \in D(A)$ for all $x \in \overline{D(A)}$. Then $A(I + C^{-1}BC)$ generates a local C-cosine function $T(\cdot)$ on X satisfying*

$$T(\cdot)x = C(\cdot)x + S * AC^{-1}BT(\cdot)x \quad \text{on } [0, T_0) \tag{2.9}$$

for all $x \in X$. Moreover, $T(\cdot)$ is also exponentially bounded (resp., norm continuous) if $C(\cdot)$ is.

Proof. By the assumption of $C^{-1}Bx \in D(A)$ for all $x \in \overline{D(A)}$, we can apply the following estimation to replace the condition (2.8):

$$\|(S * AC^{-1}Bf(t))\| \le \sup_{0 \le r \le t_0} \|S(r)\| \|AC^{-1}B\| \int_0^t \|f(r)\| dr$$

for all $f \in C([0, t_0], \overline{D(A)})$ and $0 \le t \le t_0$. Clearly, $S(\cdot)AC^{-1}B$ is also exponentially bounded (resp., norm continuous) if $C(\cdot)$ is. By (2.9) and the boundedness of $AC^{-1}B$, we have

$$T(\cdot)x = C(\cdot)x + SAC^{-1}B * T(\cdot)x \quad \text{on } [0, T_0) \tag{2.10}$$

for all $x \in X$, which together with Gronwall's inequality implies that $T(\cdot)$ is exponentially bounded (resp., norm continuous) if $C(\cdot)$ is. □

When $\rho((I + C^{-1}BC)A)$ (resolvent set of $(I + C^{-1}BC)A$) is nonempty, we can apply Theorem 2.5 to obtain the next left-multiplicative perturbation theorem concerning locally Lipschitz continuous local C-cosine functions on X in which the generator A of a perturbed local C-cosine function may not be densely defined, B is only a bounded linear operator from $\overline{D(A)}$ into $R(C)$, and $C^{-1}(I+B)AC$ and $(I+B)A$ both may not be equal.

Theorem 2.9. *Under the assumptions of Theorem 2.5. Assume that $\rho((I+C^{-1}BC)A)$ is nonempty. Then $(I+C^{-1}BC)A$ generates a locally Lipschitz continuous local C-cosine function $U(\cdot)$ on X satisfying*

$$\begin{aligned} &U(\cdot)x \\ =&Cx+[\lambda-(I+C^{-1}BC)A](I+C^{-1}BC)j_1*T(\cdot)A[\lambda-(I+C^{-1}BC)A]^{-1}x \end{aligned} \tag{2.11}$$

on $[0,T_0)$ for all $x\in X$. Here $\lambda\in\rho((I+C^{-1}BC)A)$ is fixed and $T(\cdot)$ is given as in (2.7).

Proof. Just as in the proof of [27, Theorem 2.9], we have

$$(I+C^{-1}BC)ACx=C(I+C^{-1}BC)Ax$$

for all $x\in D((I+C^{-1}BC)A)$. We set $P=I+C^{-1}BC$ and

$$u_x(\cdot)=Cx+(\lambda-PA)Pj_1*T(\cdot)A(\lambda-PA)^{-1}x$$

on $[0,T_0)$ for all $x\in X$, then $u_x\in C([0,T_0),X)$ and

$$\begin{aligned} &A(\lambda-PA)^{-1}u_x(\cdot) \\ =&A(\lambda-PA)^{-1}Cx+A(Pj_1*T(\cdot))A(\lambda-PA)^{-1}x \\ =&A(\lambda-PA)^{-1}Cx+T(\cdot)A(\lambda-PA)^{-1}x-CA(\lambda-PA)^{-1}x \\ =&A(\lambda-PA)^{-1}Cx+T(\cdot)A(\lambda-PA)^{-1}x-A(\lambda-PA)^{-1}Cx \\ =&T(\cdot)A(\lambda-PA)^{-1}x \end{aligned}$$

on $[0,T_0)$, and so

$$PA(\lambda-PA)^{-1}j_1*u_x(\cdot)=Pj_1*T(\cdot)A(\lambda-PA)^{-1}x$$

on $[0,T_0)$. Hence,

$$\begin{aligned} -j_1*u_x(\cdot)+\lambda(\lambda-PA)^{-1}j_1*u_x(\cdot)=&PA(\lambda-PA)^{-1}j_1*u_x(\cdot) \\ =&Pj_1*T(\cdot)A(\lambda-PA)^{-1}x \\ =&(\lambda-PA)^{-1}u_x(\cdot)-(\lambda-PA)^{-1}Cx \end{aligned}$$

on $[0,T_0)$, which implies that $j_1*u_x(t)\in D(PA)$ for all $0\le t<T_0$. Consequently,

$$PA(\lambda-PA)^{-1}j_1*u_x(t)\in D(PA)$$

for all $0\le t<T_0$ and $PAj_1*u_x=u_x-Cx$ on $[0,T_0)$. This shows that j_1*u_x is a solution of ACP$(PA,Cx,0,0)$ in $C^2([0,T_0),X)$. In order to show the uniqueness. Suppose that $v\in C([0,T_0),X)$ and $v=PAj_1*v$ on $[0,T_0)$. We set $u=A(\lambda-PA)^{-1}v$ on $[0,T_0)$, then

$$\begin{aligned} Pj_1*u=&PA(\lambda-PA)^{-1}j_1*v \\ =&(\lambda-PA)^{-1}PAj_1*v \\ =&(\lambda-PA)^{-1}v \end{aligned}$$

on $[0, T_0)$, and so $APj_1 * u = A(\lambda - PA)^{-1}v = u$ on $[0, T_0)$. Hence, $u = 0$ on $[0, T_0)$, which implies that $(\lambda - PA)^{-1}v = 0$ on $[0, T_0)$ or equivalently, $v = 0$ on $[0, T_0)$. We conclude from Theorem 2.3 that $(I + C^{-1}BC)A$ generates a local C-cosine function $U(\cdot)$ on X satisfying (2.11) for all $x \in X$. Clearly, for each $y \in X$,

$$(PA)Pj_1 * T(\cdot)y = P(AP)j_1 * T(\cdot)y = PT(\cdot)y - PCy$$

on $[0, T_0)$. It follows from the right-hand side of (2.11) that $U(\cdot)$ is also locally Lipschitz continuous. □

By slightly modifying the proof of Theorem 2.9, we can obtain the next left-multiplicative perturbation theorem for local C-cosine functions in which the generator A of a perturbed local C-cosine function may not be densely defined, B is only a bounded linear operator from $\overline{D(A)}$ into $R(C)$, and $C^{-1}(I + B)AC$ and $(I + B)A$ both may not be equal.

Theorem 2.10. *Under the assumptions of Theorem 2.6. Assume that $\rho((I+C^{-1}BC)A)$ is nonempty. Then $(I + C^{-1}BC)A$ generates a local C-cosine function $U(\cdot)$ on X satisfying (2.11) for all $x \in X$. Moreover, $U(\cdot)$ is exponentially bounded (resp., norm continuous, locally Lipschitz continuous, or exponentially Lipschitz continuous) if $T(\cdot)$ is. Here $T(\cdot)$ is given as in (2.7).*

Corollary 2.11. *Under the assumptions of Corollary 2.7.*
Assume that $\rho((I+C^{-1}BC)A)$ is nonempty. Then $(I+C^{-1}BC)A$ generates a locally Lipschitz continuous local C-cosine function $U(\cdot)$ on X satisfying (2.11) for all $x \in X$. Moreover, $U(\cdot)$ is exponentially Lipschitz continuous if $C(\cdot)$ is.

Corollary 2.12. *Under the assumptions of Corollary 2.8.*
Assume that $\rho((I + C^{-1}BC)A)$ is nonempty. Then $(I + C^{-1}BC)A$ generates a local C-cosine function $U(\cdot)$ on X satisfying (2.11) for all $x \in X$. Moreover, $U(\cdot)$ is also exponentially bounded (resp., norm continuous) if $C(\cdot)$ is.

Theorem 2.13. *(see* [15]*) Let A be the generator of a local C-cosine function $C(\cdot)$ on X. Assume that B is a bounded linear operator on X which commutes with $C(\cdot)$ on X. Then $A + B$ is the generator of a local C-cosine function $T_B(\cdot)$ on X satisfying*

$$T_B(t)x = \sum_{n=0}^{\infty} \int_0^t j_{n-1}(s)j_n(t-s)C(|t-2s|)B^n x ds$$

for all $x \in X$ and $0 \le t < T_0$.

Combining Theorem 2.10 with Theorem 2.13, the next new result concerning the additive perturbations of a local C-cosine function on X is also attained in which the generator of a perturbed local C-cosine function may not be densely defined.

Theorem 2.14. *Let $C(\cdot)$ be a local C-cosine function on X with generator A, and let B be a bounded linear operator from $[D(A)]$ into $R(C^2)$ such that $CB = BC$ on $D(A)$. Assume that $\rho_C(A)$ and $\rho(A+B)$ both are nonempty, and for each $0 < t_0 < T_0$ there exists an $M_{t_0} > 0$ such that*

$$|S * C^{-2}Bf(t)| \le M_{t_0} \int_0^t |f(s)|ds \tag{2.12}$$

for all $f \in C([0,t_0],[D(A)])$ *and* $0 \le t \le t_0$. *Then* $A+B$ *generates a local* C*-cosine function* $V(\cdot)$ *on* X.

Proof. Let $\lambda \in \rho_C(A)$ be fixed. We set $\widetilde{B} = C^{-1}B(A-\lambda)^{-1}C$ and $C(-t) = C(t)$ for all $0 \le t < T_0$. Then $\widetilde{B}$ is a bounded linear operator from X into $R(C)$ such that $C\widetilde{B} = \widetilde{B}C$, $A-\lambda$ is the generator of the local C-cosine function $T_{-\lambda}(\cdot)$ on X satisfying

$$j_0 * T_{-\lambda}(t)x = \sum_{n=0}^{\infty} \int_0^t j_{n-1}(s)j_n(t-s)S(t-2s)(-\lambda)^n x ds$$

for all $x \in X$ and $0 \le t < T_0$, and $(A-\lambda)^{-1}C^2 = C(A-\lambda)^{-1}C$. Here

$$\int_0^t j_{-1}(s)j_0(t-s)S(t-2s)xds = S(t)x.$$

Since the norm $|\cdot|_{A-\lambda}$ on $D(A)$ defined by $|x|_{A-\lambda} = \|x\| + \|(A-\lambda)x\|$ for all $x \in D(A)$, is equivalent to $|\cdot|$, we may assume that (2.12) holds under $|\cdot|_{A-\lambda}$. Since

$$(I + C^{-1}\widetilde{B}C)(A-\lambda) = A - \lambda + B$$

and $\rho(A+B)$ is nonempty we have $\rho((I+C^{-1}\widetilde{B}C)(A-\lambda))$ is also nonempty. It is not difficult to see that

$$\begin{aligned}
&\int_0^t j_{n-1}(s)j_n(t-s)S(t-2s)xds \\
&= \sum_{k=0}^{n} \frac{(n-1+k)!}{(n-1)!k!}(-1)^k \frac{1}{2^{n+k}}[j_{n-k}(j_{n-1+k} * S)](t)x \\
&+ \sum_{k=0}^{n-1} \frac{(n+k)!}{n!k!}(-1)^k \frac{1}{2^{n+k+1}}[j_{n-1-k}(j_{n+k} * S)](t)x
\end{aligned} \tag{2.13}$$

for each $n \in \mathbb{N}$, $x \in X$ and $0 \le t < T_0$. Let $0 < t_0 < T_0$ and $f \in C([0,t_0],X)$ be fixed. Then

$$\begin{aligned}
&[j_{n-k}(j_{n-1+k} * S)] * C^{-1}\widetilde{B}f(t) \\
&= \int_0^t j_{n-k}(t-s)(j_{n-1+k} * S)(t-s)C^{-1}\widetilde{B}f(s)ds \\
&= \frac{1}{(n-k)!} \sum_{m=0}^{n-k} \binom{n-k}{m} (-1)^m t^{n-k-m} \int_0^t j_{n-1+k} * S(t-s)C^{-1}\widetilde{B}s^m f(s)ds \\
&= \sum_{m=0}^{n-k} (-1)^m j_{n-k-m}(t) j_{n-1+k} * S * C^{-1}\widetilde{B}j_m f)(t) \\
&= \sum_{m=0}^{n-k} (-1)^m j_{n-k-m}(t) S * C^{-1}\widetilde{B}[j_{n-1+k} * (j_m f)](t)
\end{aligned} \tag{2.14}$$

and

$$
\begin{aligned}
&[j_{n-1-k}(j_{n+k} * S)] * C^{-1}\widetilde{B}f(t) \\
=&\int_0^t j_{n-1-k}(t-s)(j_{n+k} * S)(t-s)C^{-1}\widetilde{B}f(s)ds \\
=&\frac{1}{(n-1-k)!}\sum_{m=0}^{n-1-k} \binom{n-1-k}{m}(-1)^m t^{n-1-k-m}\int_0^t j_{n+k} * S(t-s)C^{-1}\widetilde{B}s^m f(s)ds \\
=&\sum_{m=0}^{n-1-k}(-1)^m j_{n-1-k-m}(t)j_{n+k} * S * (C^{-1}\widetilde{B}j_m f)(t) \\
=&\sum_{m=0}^{n-1-k}(-1)^m j_{n-1-k-m}(t)S * C^{-1}\widetilde{B}[j_{n+k} * (j_m f)](t)
\end{aligned} \tag{2.15}
$$

for all $0 \leq t \leq t_0$. By (2.12), we have

$$
\begin{aligned}
&\|(A-\lambda)j_{n-k-m}(t)S * C^{-1}\widetilde{B}[j_{n-1+k} * (j_m f)](t)\| \\
\leq& j_{n-k-m}(t_0)\|(A-\lambda)S * C^{-1}\widetilde{B}[j_{n-1+k} * (j_m f)](t)\| \\
=& j_{n-k-m}(t_0)\|(A-\lambda)S * C^{-2}B(A-\lambda)^{-1}C[j_{n-1+k} * (j_m f)](t)\| \\
\leq& j_{n-k-m}(t_0)M_{t_0}\int_0^t |(A-\lambda)^{-1}C[j_{n-1+k} * (j_m f)](s)|_{A-\lambda}ds \\
\leq& j_{n-k-m}(t_0)M_{t_0}(\|(A-\lambda)^{-1}C\| + \|C\|)\int_0^t \|[j_{n-1+k} * (j_m f)](s)\|ds
\end{aligned} \tag{2.16}
$$

for all $0 \leq t \leq t_0$. Since

$$
\begin{aligned}
&\int_0^t \|[j_{n-1+k} * (j_m f)](s)\|ds \\
\leq& \int_0^t j_{n-1+k}(s)j_m(s)\int_0^s \|f(s)\|ds \\
=&\frac{(n+k-1+m)!}{(n-1+k)!m!}[j_{n+k+m}(t)\int_0^t \|f(r)\|dr - \int_0^t \|f(s)\|ds] \\
\leq&\frac{(n+k-1+m)!}{(n-1+k)!m!}j_{n+k+m}(t)\int_0^t \|f(r)\|dr
\end{aligned} \tag{2.17}
$$

for all $0 \leq t \leq t_0$, we have

$$
\|(A-\lambda)j_{n-k-m}(t)S * (C^{-1}\widetilde{B}[j_{n-1+k} * (j_m f)](t)\| \tag{2.18}
$$

$$
\leq j_{n-k-m}(t_0)M_{t_0}(\|(A-\lambda)^{-1}C\| + \|C\|)\frac{(n+k-1+m)!}{(n-1+k)!m!}j_{n+k+m}(t)\int_0^t \|f(r)\|dr
$$

for all $0 \le t \le t_0$. Similarly, we can apply (2.12) and (2.15) to obtain

$$\begin{aligned}
&\|(A-\lambda)j_{n-1-k-m}(t)S*(C^{-1}\widetilde{B}[j_{n+k}*(j_m f)](t)\| \qquad (2.19)\\
\le& j_{n-1-k-m}(t_0)M_{t_0}(\|(A-\lambda)^{-1}C\|+\|C\|)\int_0^t \|[j_{n+k}*(j_m f)](s)\|ds\\
\le& j_{n-1-k-m}(t_0)M_{t_0}(\|(A-\lambda)^{-1}C\|+\|C\|)\frac{(n+k+m)!}{(n+k)!m!}j_{n+k+m-1}(t)\int_0^t \|f(r)\|dr
\end{aligned}$$

for all $0 \le t \le t_0$. By (2.13), we have

$$\begin{aligned}
&j_0 * T_{-\lambda} * C^{-1}\widetilde{B}f(t) = S * C^{-1}\widetilde{B}f(t) +\\
&\sum_{n=1}^{\infty}(-\lambda)^n \sum_{k=0}^{n} \frac{(n-1+k)!}{(n-1)!k!}(-1)^k \frac{1}{2^{n+k}}[j_{n-k}(j_{n-1+k}*S)] * C^{-1}\widetilde{B}f(t)\\
&+\sum_{n=1}^{\infty}(-\lambda)^n \sum_{k=0}^{n-1} \frac{(n+k)!}{n!k!}(-1)^k \frac{1}{2^{n+k+1}}[j_{n-1-k}(j_{n+k}*S)] * C^{-1}\widetilde{B}f(t) \qquad (2.20)
\end{aligned}$$

for all $0 \le t \le t_0$. By (2.14) and (2.18), we have

$$\begin{aligned}
&\|(A-\lambda)\sum_{k=0}^{n} \frac{(n-1+k)!}{(n-1)!k!}(-1)^k \frac{1}{2^{n+k}}[j_{n-k}(j_{n-1+k}*S)] * C^{-1}\widetilde{B}f(t)\|\\
=&\|(A-\lambda)\sum_{k=0}^{n}\sum_{m=0}^{n-k} \frac{(n-1+k)!}{(n-1)!k!}(-1)^{k+m} \frac{1}{2^{n+k}} j_{n-k-m}(t)S\\
&* C^{-1}\widetilde{B}[j_{n-1+k}*(j_m f)](t)\|\\
\le&\sum_{k=0}^{n}\sum_{m=0}^{n-k} \frac{(n-1+k)!}{(n-1)!k!}\frac{1}{2^{n+k}}\frac{(n-1+k+m)!}{(n-1+k)!m!} j_{n-k-m}(t_0)M_{t_0}(\|(A-\lambda)^{-1}C\|\\
&+\|C\|)j_{n+k+m}(t)\int_0^t \|f(r)\|dr\\
\le&\sum_{k=0}^{n}\frac{t_0^{2n}}{n!k!2^{n+k}}\sum_{m=0}^{n-k}\frac{1}{m!}M_{t_0}(\|(A-\lambda)^{-1}C\|+\|C\|)\int_0^t \|f(r)\|dr\\
\le&\frac{t_0^{2n}}{n!2^n}e^{1/2}eM_{t_0}(\|(A-\lambda)^{-1}C\|+\|C\|)\int_0^t \|f(r)\|dr. \qquad (2.21)
\end{aligned}$$

Similarly, we can apply (2.15) and (2.19) to show that

$$\begin{aligned}
&\|(A-\lambda)\sum_{k=0}^{n-1}\frac{(n-1+k)!}{n!k!}(-1)^k\frac{1}{2^{n+k+1}}[j_{n-1-k}(j_{n+k}*S)]*C^{-1}\widetilde{B}f(t)\| \\
=&\|(A-\lambda)\sum_{k=0}^{n-1}\sum_{m=0}^{n-1-k}\frac{(n+k)!}{n!k!}(-1)^k\frac{1}{2^{n+k+1}}(-1)^m j_{n-1-k-m}(t)S \\
&*C^{-1}\widetilde{B}[j_{n+k}*(j_m f)](t)\| \\
\le&\sum_{k=0}^{n-1}\sum_{m=0}^{n-1-k}\frac{(n+k)!}{n!k!}\frac{1}{2^{n+k+1}}\frac{(n+k+m)!}{(n+k)!m!}j_{n-1-k-m}(t_0)M_{t_0}(\|(A-\lambda)^{-1}C\| \\
&+\|C\|)j_{n+k+m-1}(t)\int_0^t\|f(r)\|dr \\
\le&\sum_{k=0}^{n-1}\frac{t_0^{2n}}{(n-1)!k!2^{n+k}}\sum_{m=0}^{n-1-k}\frac{1}{m!}M_{t_0}(\|(A-\lambda)^{-1}C\|+\|C\|)\int_0^t\|f(r)\|dr \\
\le&\frac{t_0^{2n}}{(n-1)!2^n}e^{1/2}eM_{t_0}(\|(A-\lambda)^{-1}C\|+\|C\|)\int_0^t\|f(r)\|dr. \qquad (2.22)
\end{aligned}$$

Combining (2.20)-(2.22), we get that there exists an $\widetilde{M_{t_0}}>0$ such that

$$\|(A-\lambda)j_0*T_{-\lambda}*C^{-1}\widetilde{B}f(t)\|\le\widetilde{M_{t_0}}\int_0^t\|f(s)\|ds$$

for all $f\in C([0,t_0],X)$ and $0\le t\le t_0$. It follows from Theorem 2.5 that $A+B-\lambda$ generates a local C-cosine function $U(\cdot)$ on X, which implies that $A+B$ generates a local C-cosine function $V(\cdot)$ on X. □

Just as in the proof of Corollary 2.8, we can apply Theorems 2.13 and 2.14 to obtain the next corollary.

Corollary 2.15. *Let $C(\cdot)$ be a local C-cosine function on X with generator A, and let B be a bounded linear operator from $[D(A)]$ into $R(C^2)$ such that $CB=BC$ on $D(A)$ and $C^{-2}Bx\in D(A)$ for all $x\in D(A)$. Assume that $\rho_C(A)$ and $\rho(A+B)$ both are nonempty. Then $A+B$ generates a local C-cosine function $V(\cdot)$ on X given as in the proof of Theorem 2.14. Moreover, $V(\cdot)$ is exponentially bounded (resp., norm continuous) if $C(\cdot)$ is.*

By slightly modifying the proof of Theorem 2.14, the following additive perturbation results are also attained when $\widetilde{B}$ denotes the restriction of $B(A-\lambda)^{-1}$ to $\overline{D(A)}$, and the assumptions that B is a bounded linear operator from $[D(A)]$ into $R(C^2)$ and $\rho_C(A)$ is nonempty are replaced by assuming that B is a bounded linear operator from $[D(A)]$ into $R(C)$ and $\rho(A)$ is nonempty.

Theorem 2.16. *Let $C(\cdot)$ be a local C-cosine function on X with generator A, and let B be a bounded linear operator from $[D(A)]$ into $R(C)$ such that $CB=BC$ on $D(A)$.*

Assume that $\rho(A)$ *and* $\rho(A+B)$ *both are nonempty, and for each* $0 < t_0 < T_0$ *there exists an* $M_{t_0} > 0$ *such that*

$$|S * C^{-1}Bf(t)| \leq M_{t_0} \int_0^t |f(s)|ds \tag{2.23}$$

for all $f \in C([0,t_0],[D(A)])$ *and* $0 \leq t \leq t_0$. *Then* $A+B$ *generates a local* C*-cosine function on* X.

Corollary 2.17. *Let* $C(\cdot)$ *be a local* C*-cosine function on* X *with generator* A*, and let* B *be a bounded linear operator from* $[D(A)]$ *into* $R(C)$ *such that* $CB = BC$ *on* $D(A)$ *and* $C^{-1}Bx \in D(A)$ *for all* $x \in D(A)$. *Assume that* $\rho(A)$ *and* $\rho(A+B)$ *both are nonempty. Then* $A+B$ *generates a local* C*-cosine function on* X*, which is also exponentially bounded (resp., norm continuous) if* $C(\cdot)$ *is.*

Remark 2.18. The conclusions of Corollaries 2.7 and 2.11 are still true when the assumption that $R(C^{-1}B) \subset \overline{D(A)}$ is replaced by assuming that

$$R(C^{-1}B) \subset \{x \in X \,|\, C(\cdot)x \in C^1([0,T_0),X)\}.$$

We end this paper with a simple illustrative example.

Example 2.19. Let $X = L^\infty(\mathbb{R})$, and $A_0 : D(A_0) \subset X \to X$ be defined by

$$D(A_0) = W^{1,\infty}(\mathbb{R})$$

and $A_0 f = -f'$ for all $f \in D(A_0)$, then $A = A_0^2$ generates a locally Lipschitz continuous local C-cosine function $C(\cdot)(= \{C(t)|0 \leq t < T_0\})$ on X and

$$\overline{D(A)} = \overline{W^{2,\infty}(\mathbb{R})} = C_0(\mathbb{R})$$

(see [1, Example 3.15.5] and [17, Theorem 18.3]). Here $C = (\lambda - A_0)^{-1}$ with $\lambda \in \rho(A_0)$ and $0 < T_0 \leq \infty$ are fixed. Applying Corollary 2.7, we get that $A(I + C^{-1}BC)$ generates a locally Lipschitz continuous local C-cosine function $T(\cdot)$ on $L^\infty(\mathbb{R})$ satisfying (2.7) when B is a bounded linear operator from $C_0(\mathbb{R})$ into $W^{1,\infty}(\mathbb{R})$ such that $(\lambda - A_0)^{-1}B = B(\lambda - A_0)^{-1}$ on $C_0(\mathbb{R})$ and $R((\lambda - A_0)B) \subset C_0(\mathbb{R})$.

References

[1] Arendt, W., Batty, C.J.K., Hieber, H., Neubrander, F., *Vector-Valued Laplace Transforms and Cauchy Problems*, 96, Birkhäuser Verlag, Basel-Boston-Berlin, 2001.

[2] Engel, K.-J., *On singular perturbations of second order Cauchy problems*, Pacific J. Math., **152**(1992), 79-91.

[3] Fattorini, H.O., *Second Order Linear Differential Equations in Banach Spaces*, 108, North-Holland Math. Stud., North-Holland, Amsterdam, 1985.

[4] Gao, M.C., *Local C-semigroups and C-cosine functions*, Acta Math. Sci., **19**(1999), 201-213.

[5] Goldstein, J.A., *Semigroups of Linear Operators and Applications*, Oxford Univ. Press, Oxford, 1985.

[6] Hieber, M., *Integrated Semigroups and Differential Operators on* L^p, Ph.D. Dissertation, Tubingen, 1989.

[7] Huang, F., Huang, T., *Local C-cosine family theory and application*, Chin. Ann. Math., **16**(1995), 213-232.

[8] Kellerman, H., Hiebe, M., *Integrated semigroups*, J. Funct. Anal., **84**(1989), 160-180.

[9] Kostic, M., *Perturbation theorems for convoluted C-semigroups and cosine functions*, Bull. Ci. Sci. Math., **35**(2010), 25-47.

[10] Kostic, M., *Generalized Semigroups and Cosine Functions*, Mathematical Institute SANU, Belgrade, 2011.

[11] Kostic, M., *Abstract Volterra Integro-Differential Equations*, CRC Press, Boca Raton, FI., 2015.

[12] Kostic, M., *Abstract Degenerate Volterra Integro-Differential Equations: Linear Theory and Applications*, Book Manuscript, 2016.

[13] Kuo, C.-C., Shaw, S.-Y., *C-cosine functions and the abstract Cauchy problem I, II*, J. Math. Anal. Appl., **210**(1997), 632-646, 647-666.

[14] Kuo, C.-C., *On perturbation of local integrated cosine functions*, Taiwanese J. Math., **11**(2012), 1613-1628.

[15] Kuo, C.-C., *Perturbations of local C-cosine functions*, Stud. Univ. Babeş-Bolyai Math., **65**(2020), 585-597.

[16] Kuo, C.-C., *Local C-semigroups and complete second order abstract Cauchy problems*, Stud. Univ. Babeş-Bolyai Math., **61**(2016), 211-237.

[17] deLaubenfels, R., *Existence Families, Functional Calculi and Evolution Equations*, 1570, Lecture Notes in Math., Springer-Verlag, Berlin, 1994.

[18] Li, F., *Multiplicative perturbations of incomplete second order abstract differential equations*, Kybernetes, **39**(2008), 1431-1437.

[19] Li, F., Liang, J., *Multiplicative perturbation theorems for regularized cosine functions*, Acta Math. Sinica, **46**(2003), 119-130.

[20] Li, F., Liu, J., *A perturbation theorem for local C-regularized cosine functions*, J. Physics: Conference Series, **96**(2008), 1-5.

[21] Oka, H., *Linear Volterra equation and integrated solution families*, Semigroup Forum, **53**(1996), 278-297.

[22] Shaw, S.-Y., Li, Y.-C., *Characterization and generator of local C-cosine and C-sine functions*, Inter. J. Evolution Equations, **1**(2005), 373-401.

[23] Takenaka, T., Okazwa, N., *A Phillips-Miyadera type perturbation theorem for cosine functions of operators*, Tohoku. Math., **69**(1990), 257-288.

[24] Takenaka, T., Piskarev, S., *Local C-cosine families and N-times integrated local cosine families*, Taiwanese J. Math., **8**(2004), 515-546.

[25] Travis, C.C., Webb, G.F., *Perturbation of strongly continuous cosine family generators*, Colleq. Math., **45**(1981), 277-285.

[26] Wang, S.-W., Gao, M.-C., *Automatic extensions of local regularized semigroups and local regularized cosine functions*, Proc. Amer. Math. Soc., **127**(1999), 1651-1663.

[27] Yeh, N.-S., Kuo, C.-C., *Multiplicative perturbations of local α-times integrated C-semigroups*, Acta Math. Sci., **37B**(2017), 877-888.

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Convexity-preserving properties of set-valued ratios of affine functions

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Abstract. The aim of this paper is to introduce some classes of set-valued functions that preserve the convexity of sets by direct and inverse images. In particular, we show that the so-called set-valued ratios of affine functions represent such a class. To this aim, we characterize them in terms of vector-valued selections that are ratios of affine functions in the classical sense of Rothblum.

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1. Introduction

Various classes of fractional type real-valued or vector-valued functions have been introduced in the literature, being nowadays well recognized for their important applications in scalar and vector optimization (see, e.g., Avriel *et al.* [2], Cambini and Martein [4], Göpfert *et al.* [6], Stancu-Minasian [13], and the references therein).

According to Rothblum [11], a vector-valued function $f : D \to \mathbb{R}^m$, defined on a nonempty convex set $D \subseteq \mathbb{R}^n$, is said to be a ratio of affine functions if there exist a vector-valued affine function $g : \mathbb{R}^n \to \mathbb{R}^m$ and a real-valued affine function $h : \mathbb{R}^n \to \mathbb{R}$, such that

$$D \subseteq \{x \in \mathbb{R}^n \mid h(x) > 0\}$$

and

$$f(x) = \frac{g(x)}{h(x)}, \ \forall\, x \in D. \tag{1.1}$$

These functions are known to preserve the convexity of sets by direct and inverse images. Moreover, they transform the line segments in line segments (possibly degenerated into singletons). This concept along with the above mentioned convexity-preserving properties can be naturally extended for vector-valued functions acting between general real linear spaces.

Recently, Orzan [10] investigated a class of set-valued ratios of affine functions, defined similarly to (1.1), by replacing g with a set-valued function that is affine in the sense of Tan [14]. As shown in [10], these functions preserve the convexity of sets by direct images. A natural question is whether they preserve the convexity of sets by inverse images too. The aim of this paper is not only to give a positive answer to this question, but also to identify some broader classes of set-valued functions that preserve the convexity of sets by direct and inverse images.

We start by recalling some basic definitions of set-valued and convex analysis in Section 2. The concept of set-valued affine function, as defined by Tan [14], is investigated in Section 3. In particular, we show that the inverse of such a function is affine as well (in contrast to other concepts of affine set-valued functions, cf. Kuroiwa *et al.* [7, Ex. 2]). Section 4 is devoted to the ratios of affine functions. In Section 4.1 we briefly present how the classical results of Rothblum [11] can be extended from finite-dimensional Euclidean spaces to general linear spaces. Subsection 4.2 is devoted to our main results. First we introduce the class of set-valued ratios of affine functions, by refining the definition proposed in [10]. Theorem 4.8 gives an explicit representation of these functions, which plays a key role in establishing the convexity-preserving properties of the set-valued ratios of affine functions. Moreover, Theorems 4.10 and 4.11 show that these properties are still valid in some broader classes of set-valued functions. We conclude the paper by rising an interesting open question in Section 5.

2. Preliminaries

Throughout this paper we assume that X and Y are two real linear spaces. As usual in set-valued analysis (see, e.g., Aubin and Frankowska [1]), for any set-valued function $F : X \to \mathcal{P}(Y)$ we denote by

$$\operatorname{dom} F = \{x \in X \mid F(x) \neq \emptyset\}$$

the domain of F. We say that F is proper if $\operatorname{dom} F \neq \emptyset$. The (direct) image of a set $A \subseteq X$ by F is defined as

$$F(A) = \bigcup_{x \in A} F(x).$$

There are different manners to define the inverse image of a set $B \subseteq Y$ by a set-valued map F, two of them being currently used in set-valued analysis [1], namely:

$$F^{-1}(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}, \tag{2.1}$$

$$F^{+1}(B) = \{x \in X \mid F(x) \subseteq B\}. \tag{2.2}$$

The set $F^{-1}(B)$ is called the inverse image of B by F and $F^{+1}(B)$ is called the core of B by F (also known as the lower inverse image and the upper inverse image of B by F, respectively). They are related by

$$F^{+1}(B) = X \setminus F^{-1}(Y \setminus B). \tag{2.3}$$

Remark 2.1. Let $F : X \to \mathcal{P}(Y)$ be a set-valued function. Given a set $B \subseteq Y$ we have

$$F^{-1}(B) \subseteq \operatorname{dom} F, \tag{2.4}$$

$$F^{+1}(B) = \{x \in \operatorname{dom} F \mid F(x) \subseteq B\} \cup (X \setminus \operatorname{dom} F). \tag{2.5}$$

Notice that, instead of (2.2), Berge [3] defined a slightly different concept of upper inverse image of B, namely $\{x \in \operatorname{dom} F \mid F(x) \subseteq B\}$, which actually means $F^{+1}(B) \cap \operatorname{dom} F$.

Remark 2.2. Let $F : X \to \mathcal{P}(Y)$ be a set-valued function. Then, for all sets $A \subseteq X$ and $B \subseteq Y$, the following equivalence holds:

$$A \subseteq F^{+1}(B) \iff F(A) \subseteq B.$$

Remark 2.3. According to [1], the inverse of F is the set-valued function $F^{-1} : Y \to \mathcal{P}(X)$ defined for any $y \in Y$ by

$$F^{-1}(y) := \{x \in X \mid y \in F(x)\}.$$

Notice that $\operatorname{dom} F^{-1} = F(X)$ and $F^{-1}(Y) = \operatorname{dom} F$. Also, for every set $B \subseteq Y$, we have

$$F^{-1}(B) = \bigcup_{y \in B} F^{-1}(y) = \{x \in X \mid F(x) \cap B \neq \emptyset\}. \tag{2.6}$$

It is important to notice that, according to (2.1) and (2.6), one can use without any confusion the same notation, $F^{-1}(B)$, for both the lower inverse image of B by F and the direct image of B by F^{-1}.

Remark 2.4. Every vector-valued function $f : D \to Y$ defined on a nonempty set $D \subseteq X$ can be identified with a set-valued function $F : X \to \mathcal{P}(Y)$,

$$F(x) = \begin{cases} \{f(x)\} & \text{if } x \in D \\ \emptyset & \text{if } x \in X \setminus D. \end{cases}$$

Obviously, $\operatorname{dom} F = D$ and, for all $A \subseteq X$ and $B \subseteq Y$, we have

$$F(A) = f(A \cap D) := \{f(x) \mid x \in A \cap D\},$$
$$F^{-1}(B) = f^{-1}(B) := \{x \in D \mid f(x) \in B\},$$
$$F^{+1}(B) = f^{-1}(B) \cup (X \setminus D), \text{ i.e., } F^{+1}(B) \cap D = f^{-1}(B).$$

In particular, the second relation shows that one can define the set-valued function $f^{-1} : Y \to \mathcal{P}(X)$ as $f^{-1}(y) := F^{-1}(y)$ for all $y \in Y$.

The aim of this paper is to study some classes of set-valued functions that preserve the convexity of sets by direct and inverse images. Therefore we will adopt the following conventional notations in a real linear space V (in particular, X or Y). Given $S, S' \subseteq V$, $\lambda \in \mathbb{R}$ and $v_0 \in V$, we set:

$$S + S' := \{v + v' \mid (v, v') \in S \times S'\}, \quad v_0 + S := \{v_0\} + S,$$

$$\lambda S := \{\lambda v \mid v \in S\}, \text{ and } \frac{S}{\lambda} := \frac{1}{\lambda} S \text{ whenever } \lambda \neq 0.$$

Recall that a set $S \subseteq V$ is called convex if $(1-t)S + tS \subseteq S$ for all $t \in [0,1]$. The convex hull of a set S, i.e., the smallest convex subset of V that contains S, will be denoted by $\operatorname{conv} S$. For convenience, when the convex hull applies only to the first term of a sum of two sets we will simply write $\operatorname{conv} S + S'$ instead of $(\operatorname{conv} S) + S'$. A set $S \subseteq X$ is called affine if $(1-t)S + tS \subseteq S$ for all $t \in \mathbb{R}$. If S is nonempty, then S is affine if and only if there exists a (unique) linear subspace L of V such that $S = v + L$ for all $v \in S$.

3. Affine set-valued functions

Recall that a vector-valued function, $a : E \to Y$, defined on a nonempty affine set $E \subseteq X$, is said to be affine if for any $x^1, x^2 \in E$ and $t \in \mathbb{R}$ we have

$$a((1-t)x^1 + tx^2) = (1-t)a(x^1) + ta(x^2).$$

This concept has been generalized for set-valued affine functions in different ways (see, e.g. Deutsch and Singer [5], Nikodem and Popa [9], Tan [14], and the references therein). The following definition, inspired from [14], is suitable for the purposes of our paper.

Definition 3.1. A set-valued function $G : X \to \mathcal{P}(Y)$ is said to be affine if

$$G((1-t)x^1 + tx^2) = (1-t)G(x^1) + tG(x^2) \tag{3.1}$$

for all $x_1, x_2 \in \operatorname{dom} G$ and $t \in \mathbb{R}$.

Remark 3.2. It can be easily seen that if $G : X \to \mathcal{P}(Y)$ is a set-valued affine function, then $\operatorname{dom} G$ is affine, since for any $x_1, x_2 \in \operatorname{dom} G$ and $t \in [0,1]$, the equality (3.1) implies $G((1-t)x^1 + tx^2) \neq \emptyset$, i.e., $(1-t)x^1 + tx^2 \in \operatorname{dom} G$. Moreover, for every $x \in X$, the set $G(x)$ is affine. Indeed, letting $x_1 = x_2 = x$ in (3.1) we get

$$G(x) = G((1-t)x + tx) = (1-t)G(x) + tG(x)$$

for all $t \in \mathbb{R}$, hence $G(x)$ is affine.

Example 3.3. Let $g : E \to Y$ be a vector-valued affine function, defined on a nonempty affine set $E \subseteq X$. In view of Remark 2.4, we can identify g with the set-valued function $G : X \to \mathcal{P}(Y)$, given by

$$G(x) = \begin{cases} \{g(x)\} & \text{if } x \in E \\ \emptyset & \text{if } x \in X \setminus E. \end{cases}$$

It is easy to check that G is affine.

Proposition 3.4. *For any set-valued function $G : X \to \mathcal{P}(Y)$ the following assertions are equivalent:*

1° *G is affine.*

2° *For all $x^1, x^2 \in \operatorname{dom} G$ and $t \in \mathbb{R}$ we have*

$$(1-t)G(x^1) + tG(x^2) \subseteq G((1-t)x^1 + tx^2).$$

Proof. The implication $1^\circ \Longrightarrow 2^\circ$ is obvious, while $2^\circ \Longrightarrow 1^\circ$ can be seen as a particular instance of a known result by Nikodem and Popa [9, Prop. 2.11], applied to the restriction of G to $E := \operatorname{dom} G$, which is an affine set in view of Remark 3.2. □

Remark 3.5. In the paper by Tan [14] the set-valued affine functions are defined on a nonempty affine set $E \subseteq X$ taking values in $\mathcal{P}_0(Y) := \mathcal{P}(Y) \setminus \{\emptyset\}$. Actually, if $G : X \to \mathcal{P}(Y)$ is a proper set-valued affine function, then the set $E := \operatorname{dom} G$ is nonempty and affine, therefore the restriction of G to E, i.e., $G|_{\operatorname{dom} G} : E \to \mathcal{P}_0(Y)$, is affine in the sense of Tan.

Theorem 3.6. *Let $G : X \to \mathcal{P}(Y)$ be a set-valued affine function. Then the inverse of G, i.e., the set-valued function $G^{-1} : Y \to \mathcal{P}(X)$, is affine.*

Proof. According to Proposition 3.4, we just need to show that for any $y_1, y_2 \in \operatorname{dom} G^{-1} = G(X)$ and $t \in \mathbb{R}$ the following inclusion holds:

$$(1-t)G^{-1}(y_1) + tG^{-1}(y_2) \subseteq G^{-1}((1-t)y_1 + ty_2). \tag{3.2}$$

To this aim, let $x \in (1-t)G^{-1}(y_1) + tG^{-1}(y_2)$. Then there exist $x^1 \in G^{-1}(y_1)$ and $x^2 \in G^{-1}(y_1)$ such that $x = (1-t)x^1 + tx^2$. In view of (2.4), we have $x^1, x^2 \in \operatorname{dom} G$. Taking into account that function G is affine, we deduce that

$$(1-t)y^1 + ty^2 \in (1-t)G(x_1) + tG(x_2) = G((1-t)x^1 + tx^2) = G(x),$$

which entails $x \in G^{-1}((1-t)y^1 + ty^2)$. Thus (3.2) holds. □

Corollary 3.7. *If $g : E \to Y$ is a vector-valued affine function, defined on a nonempty affine set $E \subseteq X$, then the set-valued function $g^{-1} : Y \to \mathcal{P}(X)$ is affine.*

Proof. Follows by Theorem 3.6, in view of Remark 2.4. □

The following two results are based on Tan [14, Props. 4 and 5].

Proposition 3.8. *If $G : X \to \mathcal{P}(Y)$ is a proper set-valued affine function, then there is a unique linear subspace $M \subseteq Y$ such that*

$$G(x) = y + M \tag{3.3}$$

for all $x \in \operatorname{dom} G$ and $y \in G(x)$.

Proposition 3.9. *If $G : X \to \mathcal{P}(Y)$ is a proper set-valued affine function, then G possesses an affine selection, i.e., there exists a vector-valued affine function $g : \operatorname{dom} G \to Y$ such that, for all $x \in \operatorname{dom} G$,*

$$g(x) \in G(x).$$

Corollary 3.10. *If $G : X \to \mathcal{P}(Y)$ is a proper set-valued affine function, then there exist a vector-valued affine function $g : \operatorname{dom} G \to Y$ and a linear subspace $M \subseteq Y$ such that, for all $x \in \operatorname{dom} G$,*

$$G(x) = g(x) + M.$$

Proof. It is a straightforward consequence of Propositions 3.8 and 3.9. □

Remark 3.11. If $Y = \mathbb{R}$, then there are only two types of proper set-valued affine functions $G : \mathbb{R} \to \mathcal{P}(\mathbb{R})$, namely: (i) $G(x) = \{g(x)\}$ for all $x \in \operatorname{dom} G$, where $g : \operatorname{dom} G \to \mathbb{R}$ is an affine function, or (ii) $G(x) = \mathbb{R}$ for all $x \in \operatorname{dom} G$. In particular, when $X = \mathbb{R}$ the domain $\operatorname{dom} G$ is either a singleton or the entire $\mathbb{R}$, in view of Remark 3.2.

4. Ratios of affine functions

4.1. Vector-valued ratios of affine functions

We begin this section by extending the notion of vector-valued ratios of affine functions, originally introduced by Rothblum [11] within finite-dimensional Euclidean spaces, to the framework of general real linear spaces.

Definition 4.1. A vector-valued function $f : D \to Y$, defined on a nonempty convex set $D \subseteq X$, is said to be a ratio of affine functions if there exist a vector-valued affine function $g : X \to Y$ and a real-valued affine function $h : X \to \mathbb{R}$, such that

$$D \subseteq \{x \in X \mid h(x) > 0\}$$

and

$$f(x) = \frac{g(x)}{h(x)}, \ \forall\, x \in D. \tag{4.1}$$

Remark 4.2. Since D is assumed to be nonempty in Definition 4.1, it is understood that the set $\{x \in X \mid h(x) > 0\}$ is nonempty.

The following propositions extend to the framework of general real linear spaces some results obtained within $\mathbb{R}^n$ by Rothblum (see [11, Props. 1, 2 and 3] along with subsequent remarks). Their proofs are omitted, since they follow the main lines in [11].

Proposition 4.3. *Given a vector-valued function $f : D \to Y$ defined on a nonempty convex set $D \subseteq X$, the following assertions are equivalent:*

1° $\operatorname{conv} f(S) \subseteq f(\operatorname{conv} S)$ *for every set $S \subseteq D$.*
2° *$f(A)$ is convex for every convex set $A \subseteq D$, i.e., f preserves the convexity of sets by direct images.*

Proposition 4.4. *Given a vector-valued function $f : D \to Y$ defined on a nonempty convex set $D \subseteq X$, the following assertions are equivalent:*

1° $f(\operatorname{conv} S) \subseteq \operatorname{conv} f(S)$ *for every set $S \subseteq D$.*
2° *$f^{-1}(B)$ is convex for every convex set $B \subseteq Y$, i.e., function f preserves the convexity of sets by inverse images.*

Proposition 4.5. *Let $D \subseteq X$ be a nonempty convex set. If $f : D \to Y$ is a vector-valued ratio of affine functions, then*

$$\operatorname{conv} f(S) = f(\operatorname{conv} S) \ \textit{ for every set } \ S \subseteq D.$$

Therefore f preserves the convexity of sets by direct and inverse images.

4.2. Set-valued ratios of affine functions

In this section we introduce a class of set-valued ratios of affine functions, by slightly modifying the one proposed by us in [10, Def. 3.3].

Definition 4.6. Let $F : X \to \mathcal{P}(Y)$ be a set-valued function, whose domain $\operatorname{dom} F =: D \subseteq X$ is a nonempty convex set. We say that F is a ratio of affine functions if there exist a proper set-valued affine function $G : X \to \mathcal{P}(Y)$ and a real-valued affine function $h : X \to \mathbb{R}$, such that

$$D \subseteq \{x \in X \mid h(x) > 0\} \cap \operatorname{dom} G$$

and

$$F(x) = \begin{cases} \dfrac{G(x)}{h(x)} & \text{if } x \in D \\ \emptyset & \text{if } x \in X \setminus D. \end{cases} \tag{4.2}$$

Remark 4.7. As we have pointed out in Remark 4.2, since D is nonempty, the set $\{x \in X \mid h(x) > 0\}$ is nonempty as well. In particular, we deduce that any proper set-valued affine function $F : X \to \mathcal{P}(Y)$ is a ratio of affine functions of type (4.2). Indeed, in this case, $D := \operatorname{dom} F$ is a nonempty affine hence convex set, $G(x) = F(x)$ and $h(x) = 1$ for all $x \in X$.

Theorem 4.8. *Let $F : X \to \mathcal{P}(Y)$ be a set-valued ratio of affine functions defined by (4.2). Then there exist a vector-valued ratio of affine functions $f : D \to Y$ and a linear subspace $M \subseteq Y$, such that*

$$F(x) = \begin{cases} f(x) + M & \text{if } x \in D \\ \emptyset & \text{if } x \in X \setminus D. \end{cases} \tag{4.3}$$

Proof. In view of Corollary 3.10, we can find a vector-valued affine function $g : X \to Y$ and a linear subspace $M \subseteq Y$, such that $G(x) = g(x) + M$ for any $x \in D$. Consequently, we get

$$F(x) = \frac{G(x)}{h(x)} = \frac{g(x) + M}{h(x)} = \frac{g(x)}{h(x)} + \frac{M}{h(x)} = \frac{g(x)}{h(x)} + M,\ \forall\, x \in D.$$

Thus, we can define a vector-valued ratio of affine functions, $f : D \to Y$, by

$$f(x) := \frac{g(x)}{h(x)},\ \forall\, x \in D,$$

which satisfies (4.3). □

The following result is a set-valued counterpart of Proposition 4.5 and recovers a similar result obtained in [10, Th. 3.1].

Corollary 4.9. *Let $F : X \to \mathcal{P}(Y)$ be a set-valued ratio of affine functions defined by (4.2). Then, for any set $S \subseteq D$ we have*

$$\operatorname{conv} F(S) = F(\operatorname{conv} S). \tag{4.4}$$

Proof. According to Theorem 4.8, there exist a vector-valued ratio of affine functions $f : D \to Y$ and a linear subspace $M \subseteq Y$ such that F has the form (4.3). Consider an arbitrary set $S \subseteq D$. On the one hand, we have $F(S) = f(S) + M$, hence

$$\operatorname{conv} F(S) = \operatorname{conv}\,(f(S) + M). \tag{4.5}$$

On the other hand, we have $F(\operatorname{conv} S) = f(\operatorname{conv} S) + M$, which in view of Proposition 4.5 means

$$F(\operatorname{conv} S) = \operatorname{conv} f(S) + M. \tag{4.6}$$

Taking into account that M is convex, it is a simple exercise to check that

$$\operatorname{conv} f(S) + M = \operatorname{conv}\,(f(S) + M). \tag{4.7}$$

Thus, (4.4) follows by (4.5), (4.6) and (4.7). □

In what follows we will show that, similarly to vector-valued ratios of affine functions, the set-valued ratios of affine functions preserve the convexity of sets by direct and inverse images. To this aim we establish two preliminary results for more general classes of set-valued functions.

Theorem 4.10. *Consider a set-valued function $F : X \to \mathcal{P}(Y)$ of type (4.3), where $D \subseteq X$ and $M \subseteq Y$ are nonempty convex sets, while $f : D \to Y$ is a vector-valued function that preserves the convexity of sets by direct images. Then, for every convex set $A \subseteq X$, the set $F(A)$ is convex, i.e., F preserves the convexity of sets by direct images.*

Proof. Let $A \subseteq X$ be a convex set. Since $\operatorname{dom} F = D$, we have

$$F(A) = F(A \cap D) \cup F(A \setminus D) = F(A \cap D) = f(A \cap D) + M.$$

By hypothesis, $f(A \cap D)$ is convex as being the image of the convex set $A \cap D$ by f. Since M is convex too, we conclude that $F(A)$ is convex. □

Theorem 4.11. *Consider a set-valued function $F : X \to \mathcal{P}(Y)$ of type (4.3), where $D \subseteq X$ and $M \subseteq Y$ are nonempty convex sets, while $f : D \to Y$ is a vector-valued function that preserves the convexity of sets by inverse images. Then, for every convex set $B \subseteq Y$, the sets $F^{-1}(B)$ and $F^{+1}(B) \cap \operatorname{dom} F$ are convex, i.e., F preserves the convexity of sets by lower inverse images as well as by upper inverse images in the sense of Berge.*

Proof. First notice that $\operatorname{dom} F = D$. For every convex set $B \subseteq Y$ we have

$$\begin{aligned} F^{-1}(B) &= \{x \in X \mid F(x) \cap B \neq \emptyset\} \\ &= \{x \in D \mid (f(x) + M) \cap B \neq \emptyset\} \\ &= \bigcup_{m \in M} \{x \in D \mid f(x) + m \in B\} \\ &= \bigcup_{m \in M} f^{-1}(B - m) \\ &= f^{-1}(B - M). \end{aligned}$$

Since $B - M$ is convex (by convexity of B and M), its inverse image by f, i.e., $f^{-1}(B - M)$, is convex as well. Consequently, the set $F^{-1}(B)$ is convex.
In order to prove that $F^{+1}(B) \cap \operatorname{dom} F$, i.e., $F^{+1}(B) \cap D$, is convex, let

$$x^1, x^2 \in F^{+1}(B) \cap D.$$

We have to show that

$$\operatorname{conv}\{x^1, x^2\} \subseteq F^{+1}(B) \cap D.$$

Since D is convex set, we just have to check that $\operatorname{conv}\{x^1, x^2\} \subseteq F^{+1}(B)$. In view of Remark 2.2, this actually means that $F(\operatorname{conv}\{x^1, x^2\}) \subseteq B$, which by (4.3) reduces to

$$f(\operatorname{conv}\{x^1, x^2\}) + M \subseteq B. \tag{4.8}$$

Observe that, by applying Proposition 4.4 for $S := \{x^1, x^2\}$, we have

$$\begin{aligned} f(\operatorname{conv}\{x^1, x^2\}) + M &\subseteq \operatorname{conv} f(\{x^1, x^2\}) + M \\ &= \operatorname{conv}\{f(x^1), f(x^2)\} + M. \end{aligned}$$

Taking into account that M is convex, we can deduce that

$$\begin{aligned} \operatorname{conv}\{f(x^1), f(x^2)\} + M &= \operatorname{conv}\big((f(x^1) + M) \cup (f(x^2) + M)\big) \\ &= \operatorname{conv}\big(F(x^1) \cup F(x^2)\big). \end{aligned}$$

Finally, recalling that $x^1, x^2 \in F^{+1}(B)$, we have $F(x^1) \cup F(x^2) \subseteq B$, which by convexity of B yields

$$\operatorname{conv}\big(F(x^1) \cup F(x^2)\big) \subseteq B.$$

Hence, the desired inclusion (4.8) holds. □

As a direct consequence of Theorems 4.8, 4.10 and 4.11 we obtain the following result.

Corollary 4.12. *If $F : X \to \mathcal{P}(Y)$ is a set-valued ratio of affine functions defined by* (4.2)*, then the following assertions hold:*

1° *F preserves the convexity of sets by direct images.*
2° *F preserves the convexity of sets by lower inverse images.*
3° *F preserves the convexity of sets by upper inverse images in the sense of Berge.*

Remark 4.13. Corollary 4.12 (1°) extends a result obtained in [10, Cor. 3.1].

Remark 4.14. Assertions 2° and 3° of Corollary 4.12, can be interpreted in terms of generalized convexity. They actually show that the restriction $F|_{\operatorname{dom} F} : D \to \mathcal{P}_0(Y)$ of any set-valued ratio of affine functions defined by (4.2) is quasiconvex and quasiconcave in the sense of Nikodem [8, Th. 7.2]. Assertion 3° also means that F is $(u1)$-type $\{0_Y\}$-quasiconvex in the sense of Seto, Kuroiwa and Popovici [12, Def. 3.3], where 0_Y stands for the origin of Y. Actually, the more general set-valued functions involved in Theorem 4.11 also satisfy these generalized convexity properties as well.

Remark 4.15. In contrast to Corollary 4.12 (3°), if $F : X \to \mathcal{P}(Y)$ is a set-valued (even single-valued) ratio of affine functions defined by (4.2), the upper inverse image $F^{+1}(B)$ of a convex set $B \subseteq Y$ in the sense of Aubin-Frankowska is not necessarily convex, as the following example shows.

Example 4.16. Let $X = Y = \mathbb{R}$ and $D = [0,1]$. Let $F : \mathbb{R} \to \mathcal{P}(\mathbb{R})$ be the set-valued function defined as

$$F(x) = \begin{cases} \{x\} & \text{if } x \in [0,1] \\ \emptyset & \text{if } x \notin [0,1]. \end{cases}$$

In view of Remarks 3.11 and 4.7, F is a set-valued ratio of affine functions of type (4.2), where $G(x) = \{x\}$ and $h(x) = 1$ for all $x \in \mathbb{R}$. Consider the convex set $B = \{0\}$. Although $F^{+1}(B) \cap \operatorname{dom} F = \{0\}$ is convex, the set $F^{+1}(B) = \mathbb{R} \setminus]0,1]$ is not convex.

Remark 4.17. Let $F : X \to \mathcal{P}(Y)$ be a proper set-valued affine function, whose domain is the affine set $D := \operatorname{dom} F$. Notice that F is a set-valued ratio of affine functions, in view of Remark 4.7. Then, for every convex set $B \subseteq Y$, the sets

$$F^{-1}(B) \text{ and } F^{+1}(B) \cap D$$

are convex, according to Corollary 4.12 (2° and 3°).

On the other hand, by Theorem 3.6, the inverse of F, i.e., the set-valued function $F^{-1} : Y \to \mathcal{P}(X)$, is also affine with $\operatorname{dom} F^{-1} = F(X) = F(D)$. Thus, by applying the above arguments to F^{-1} in the role of F, we deduce that for any convex set $A \subseteq X$ the sets

$$(F^{-1})^{-1}(A) \text{ and } (F^{-1})^{+1}(A) \cap F(D)$$

are convex. Of course, the convexity of $(F^{-1})^{-1}(A)$ is simply recovered by Corollary 4.12 (1°) applied to F^{-1} in the role of F, since $(F^{-1})^{-1}(A) = F(A)$. However, the convexity of $(F^{-1})^{+1}(A) \cap F(D)$ is not a simple consequence of the convexity-preserving properties of F. Indeed, by applying (2.3) for F^{-1} in the role of F, we get

$$(F^{-1})^{+1}(A) = Y \setminus F(X \setminus A) = Y \setminus F(D \setminus A),$$

hence

$$\begin{aligned} (F^{-1})^{+1}(A) \cap F(D) &= (Y \setminus F(D \setminus A)) \cap F(D) \\ &= F(D) \setminus F(D \setminus A). \end{aligned}$$

Notice that $F(D) \setminus F(D \setminus A) \subseteq F(D \cap A)$ and $F(D \cap A)$ is convex, according to Corollary 4.12 (1°), since $D \cap A$ is convex. However, $F(D) \setminus F(D \setminus A) \neq F(D \cap A)$ in general, as for instance when $F : \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is the constant ratio of affine functions defined by $F(x) = \{1\}$ for all $x \in \mathbb{R}$ and $A = \{0\} \subseteq D = \mathbb{R}$.

5. Conclusions

Theorem 4.8 gives a useful characterization of the set-valued ratios of affine functions, as reflected in its Corollaries 4.9 and 4.9. Actually, the special structure of these functions, given by Theorem 4.8, suggested us to consider in Theorems 4.10 and 4.11 some broader classes of set-valued functions that enjoy the convexity-preserving properties of the set-valued ratios of affine functions pointed out in Corollary 4.9. An interesting question arises, namely whether it would be possible to extend these classes by replacing the convex set M in Theorems 4.10 and 4.11 (which can be seen as a constant affine set-valued function) by appropriate set-valued functions. In general, this is not true even if M is replaced by a single-valued function that preserves the convexity of sets by direct and inverse images. For instance, consider $D = [0,1] \subseteq X = \mathbb{R}$, $Y = \mathbb{R}^2$ and let $F : \mathbb{R} \to \mathcal{P}(\mathbb{R}^2)$ be the set-valued function defined as

$$F(x) = \begin{cases} f(x) + M(x) & \text{if } x \in [0,1] \\ \emptyset & \text{if } x \notin [0,1], \end{cases}$$

where $f : [0,1] \to \mathbb{R}^2$ is a vector-valued ratio of affine functions given by

$$f(x) = \frac{g(x)}{h(x)} = \frac{(1,1)}{x+1} \quad \text{for all } x \in [0,1]$$

and $M : \mathbb{R} \to \mathcal{P}(\mathbb{R}^2)$ is a single-valued affine function, given by

$$M(x) = \{(x, -x)\} \quad \text{for all } x \in \mathbb{R}.$$

Consider the convex sets $A = [0,1]$ and $B = \operatorname{conv} F(\{0,1\})$. It is easy to check that B is a line segment with end points $(1,1)$ and $(3/2, -1/2)$, while $F(A)$ is an arc of hyperbola joining these points. On the other hand, we have $F^{-1}(B) = \{0,1\}$ and $F^{+1}(B) = \mathbb{R}\setminus]0,1[$. Obviously, the sets $F(A)$, $F^{-1}(B)$ and $F^{+1}(B)$ are not convex. It is worth to mention that similar examples, where F preserves the convexity of sets by direct and inverse images, can be given by considering ratios af affine functions $f = \dfrac{g}{h}$ and $M = \dfrac{G}{h}$ with the same denominator. However, such a configuration always leads to the trivial case where F itself is a ratio of affine functions satisfying the properties in demand by virtue of Corollary 4.9.

References

[1] Aubin, J.-P., Frankowska, H., *Set-Valued Analysis*, Birhäuser, Boston, 1990.

[2] Avriel, M., Diewert, W.E., Schaible S., Zang, I., *Generalized Concavity*, Plenum Press, New York, 1988.

[3] Berge, C., *Topological Spaces Including a Treatment of Multi-Valued Functions, Vector Spaces and Convexity*, Oliver & Boyd, Edinburgh-London, 1963.

[4] Cambini, A., Martein, L., *Generalized Convexity and Optimization: Theory and Applications*, Springer-Verlag, Berlin, 2009.

[5] Deutsch, F., Singer, I., *On single-valuedness of convex set-valued maps*, Set-Valued Anal., **1**(1993), 97-103.

[6] Göpfert, A., Riahi, H., Tammer, C., Zălinescu, C., *Variational Methods in Partially Ordered Spaces*, Springer-Verlag, New York, 2003.

[7] Kuroiwa, D., Popovici, N., Rocca, M., *A characterization of cone-convexity for set-valued functions by cone-quasiconvexity*, Set-Valued Var. Anal., **23**(2015), 295-304.

[8] Nikodem, K., *K-Convex and K-Concave Set-Valued Functions*, Habilitation Dissertation, Scientific Bulletin of Łódź Technical University, nr. 559, 1989.

[9] Nikodem, K., Popa, D., *On single-valuedness of set-valued maps satisfying linear inclusions*, Banach J. Math. Anal., **3**(2009), 44-51.

[10] Orzan, A., *A new class of fractional type set-valued functions*, Carpathian J. Math., **35**(2019), 79-84.

[11] Rothblum, U.G., *Ratios of affine functions*, Math. Program., **32**(1985), 357-365.

[12] Seto, K., Kuroiwa, D., Popovici, N., *A systematization of convexity and quasiconvexity concepts for set-valued maps, defined by l-type and u-type preorder relations*, Optimization, **67**(2018), 1077-1094.

[13] Stancu-Minasian, I.M., *Fractional Programming. Theory, Methods and Applications*, Mathematics and its Applications, Kluwer-Dordrecht 409, 1997.

[14] Tan, D.H., *A note on multivalued affine mappings*, Stud. Univ. Babeş-Bolyai Math., **33**(1988), 55-59.

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