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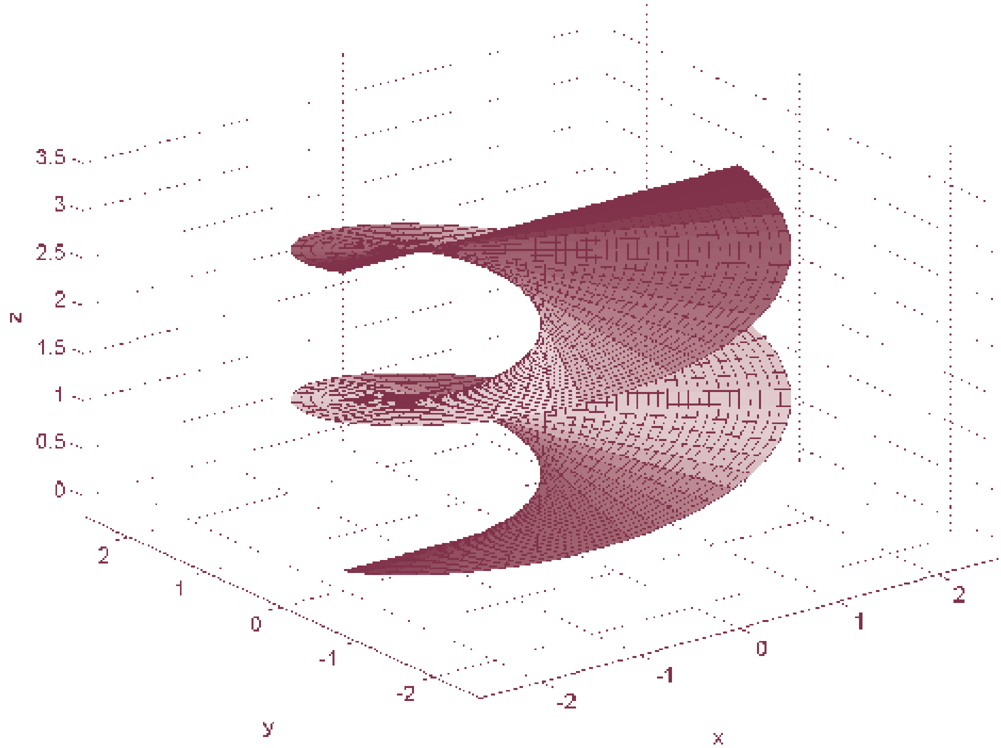

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Graph-directed random fractal interpolation function

Ildikó Somogyi and Anna Soós

Dedicated to Professor Gheorghe Coman on the occasion of his 85th anniversary.

Abstract. Barnsley introduced in [1] the notion of fractal interpolation function (FIF). He said that a fractal function is a (FIF) if it possess some interpolation properties. It has the advantage that it can be also combined with the classical methods or real data interpolation. Hutchinson and Rüschendorf [7] gave the stochastic version of fractal interpolation function. In order to obtain fractal interpolation functions with more flexibility, Wang and Yu [9] used instead of a constant scaling parameter a variable vertical scaling factor. Also the notion of fractal interpolation can be generalized to the graph-directed case introduced by Deniz and Özdemir in [5]. In this paper we study the case of a stochastic fractal interpolation function with graph-directed fractal function.

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1. Introduction

In the construction of a fractal interpolation function Barnsley used the theory of iterated function system [1], [3],[2]. For this we will consider two separable metric spaces (X, d_X) and (Y, d_Y) and a given collection of N bijections $L_i : X \to X_i$ such that

$$\{X_i = L_i(X) | i \in \{1, 2, ..N\}\}$$

$$\cup_{i=1}^{N} X_i = X \quad \text{and} \quad int(X_i) \cap int(X_j) = \emptyset, \quad \text{for} \quad i \neq j.$$

For $g_i : X_i \to Y$, $i \in \{1, 2, ..N\}$, define $\sqcup_i g_i : X \to Y$ by

$$(\sqcup_i g_i)(x) = g_j(x) \quad \text{for} \quad x \in X_j.$$

Assume that mappings $F_i : X \times Y \to Y$, $F_i(x, \cdot) \in Lip^{<1}(Y)$, $x \in X$ are given, $i \in \{1, 2, ...N\}$, where $Lip^{<1}(Y)$ is the set of all Lipschitz functions with Lipschitz

constant less that 1.
Let $\mathbf{F} = \{F_1, F_2, ..., F_N\}$, then $\{X, \mathbf{F}\}$ is a so-called Iterated Function System (IFS).
Denote $\alpha_i = LipF_i$.
For $f : X \to Y$, define the operator $\mathbf{F} : L_\infty(X, Y) \to Y^X$ by

$$\mathbf{F}f = \sqcup_i F_i(L_i^{-1}, f \circ L_i^{-1}).$$

Then f is a selfsimilar fractal function if $\mathbf{F}f = f$.
Let $\Gamma := \{(x_0, y_0), ..., (x_N, y_N) \in (X \times Y)\}$ be the set of interpolation points.
A fractal function f has the interpolation properties with respect to Γ if

$$f(x_j) = y_j \quad \text{for all} \quad j = 0, 1, ..., N.$$

Denote

$$C^*(X, Y) := \{f \in C(X, Y)| \quad f(x_j) = y_j, \quad j \in \{1, 2, ..., N\}\}.$$

Theorem 1.1 (Barnsley, [2]**).** *Let Γ be a set of interpolation points and let $\{X, \mathbf{F}\}$ be the IFS. Suppose*

$$F_i(x_0, y_0) = y_{i-1}, \quad F_i(x_N, y_N) = y_i$$

for all $i \in \{1, 2, ..., N\}$ and $\alpha_\infty := \max \alpha_i < 1$. Then there exists a selfsimilar fractal function $f^* \in C^*(X, Y)$ such that $\mathbf{F}f^* = f^*$.

In order to obtain more various (FIF) in many papers the classical interpolation methods are combined with these fractal interpolation functions, [4],[8].

2. Stochastic fractal interpolation function

Let $(\Omega, \mathcal{K}, P)$ be a probability space and $\Gamma := \{(x_i, y_i), i = 0, 1, ..., N\}$ be a set of interpolation points in $X \times Y$.

Let $L_i : X \to X$ be contractiv Lipschitz maps such that $L_i(x_0) = x_{i-1}$ and $L_{i-1}(x_N) = x_i$ for all $i \in \{1, ..., N\}$.

The IFS $\{X, \mathbb{F}\}$ is defined by $F_i : X \times Y \to Y$ such that $F_i(x, \cdot) \in Lip^{<1}(Y)$ for all $x \in X$ and

$$F_i(x_0, y_0) = y_{i-1} \quad \text{with probability 1 (a.s.)}$$

and

$$F_i(x_N, y_N) = y_i \quad \text{with probability 1 (a.s.)}$$

for all $i \in \{1, ..., N\}$.

$$F_i(x, y) = \alpha_i y + q_i(x), i = 1, 2, .., N,$$

where α_i are random variables defined on Ω satisfying

$$\|\alpha_i\|_\infty = \sup\{|\alpha_i(\omega)| : \omega \in \Omega\} < 1, \quad i = 1, 2, ..., N.$$

The random function $\mathbb{F}$ is defined up to probability distribution by

$$\mathbb{F}f = \sqcup_i F_i(L_i^{-1}, f^{(i)} \circ L_i^{-1}),$$

where $\mathbb{F}, f^{(1)}, ..., f^{(N)}$ are independent and $f^{(i)} \stackrel{d}{=} f$, for $i = 1, 2, ..., N$.
We say f is a random fractal function, if

$$\mathbb{F}f \stackrel{d}{=} f,$$

and it has the interpolation properties with respect to Γ if $f(x_i) = y_i$ a. s. for all $i \in \{0, 1, ..., N\}$.
We will consider

$$C_\omega(X, Y) := \{f : \Omega \times X \to Y, f \quad \text{continuous} \quad a.s.\}$$

and

$$C^*_\omega(X, Y) := \{g \in C_\omega(X, Y) | g(x_i) = y_i \ a.s., \ i \in \{1, ..., N\}\}.$$
$$\mathbb{L}_\infty := \{g : \Omega \times X \to Y | ess \sup_\omega ess \sup_x d_Y(g^\omega(x), a) < \infty\}$$

for some $a \in X$.
For $f, g \in \mathbb{L}_\infty$ we define

$$d^*_\infty(f, g) := ess \sup_\omega d_\infty(f^\omega, g^\omega),$$

where

$$d_\infty(f, g) = ess \sup_x d_Y(f(x), g(x)).$$

Theorem 2.1. *Let Γ be a set of interpolation points in $X \times Y$ and let $\{X, \mathbb{F}\}$ be the IFS defined above. If $\lambda_\infty := ess \sup_\omega \max_i \alpha^\omega_i < 1$ and*

$$ess \sup_\omega \max_i d_Y(F_i(a, f(a)), a) < \infty \tag{2.1}$$

for some $a \in X$, then there exists $f^* \in C^*_\omega(X, Y)$ such that $\mathbb{F}f^* = f^*$. Moreover, f^* is unique up to probability distribution.

Example 2.2. $X = [0, 1]$, $Y = \mathbb{R}$, $N > 0$.

$$\Gamma := \{(x_i, y_i) \in [0, 1] \times \mathbb{R} | 0 = x_0 < x_1 < ... < x_N = 1\}.$$
$$L_i : X \to X_i, \quad L_i(x) := a_i x + d_i, \ a_i, d_i \in \mathbb{R}, \quad i \in \{1, 2, ..., N\}.$$
$$F_i : X \times Y \to Y, i = \{1, 2, .., N\},$$
$$F_i(x, y) := \alpha_i y + q_i(x), q_i(x) = c_i x + e_i,$$

α_i is a random variable, $\lambda_\infty := ess \sup_\omega \max_i \alpha_i < 1$.

We can compute a_i, c_i, d_i, e_i by the conditions $L_i(x_0) = x_{i-1}, \quad L_i(x_N) = x_i$

$$F_i(x_0, y_0) = y_{i-1}, \quad F_i(x_N, y_N) = y_i \quad a.s.$$

for all $i \in \{1, ..., N\}$.
$W_i : X \times Y \to X \times Y$ $W_i(x, y) = (L_i(x), F_i(x, y))$, $i \in \{1, 2, , ..., N\}$.
Using $\mathbb{W} := (W_1, ..., W_N)$, IFS $\{X, \mathbb{W}\}$

$$\mathbb{W}_i : X \times Y \to L \times Y, \quad \mathbb{W}_i(x, y) = (L_i(x), F_i(x, y)) \quad i = 1, , ..., N,$$

for any $K_0 \subset X \times U$

$$K_n = \mathbb{W}K_{n-1} = \cup_{i=0}^N W^\omega_i K_{n-1} = \mathbb{W}^n(K_0).$$

Then

$$ess \sup_\omega d_H(\mathbb{W}^n(K_0), graph f^*) \to 1$$

as $n \to \infty$, d_H denotes the Hausdorff distance.

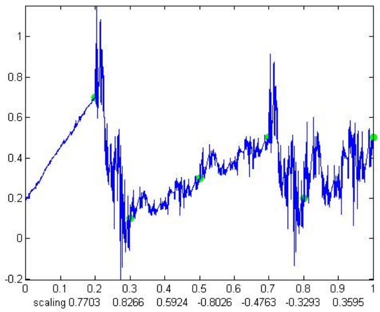

FIGURE 1. Fractal interpolation function with variable parameter, {(0,0.2),(0.2,0.7),(0.3,0.1),(0.5,0.3),(0.7,0.5),(0.8,0.2),(1,0.5)}

3. Graph directed fractal interpolation function

Let $\mathcal{G} = (V, E)$ be a graph, V is the set of vertices and E is the set of edges. For $\alpha, \beta \in V$, let $E^{\alpha,\beta}$ be the set of edges from α to β, and $K^{\alpha,\beta}$ is the number of elements of $E^{\alpha,\beta}$. Also let $\{X^\alpha \mid \alpha \in V\}$ be a set of complete metric spaces and $\phi_i^{\alpha\beta} : X^\beta \to X^\beta$ are contraction mappings, for $i = 1, 2, ..., K^{\alpha\beta}$. Then from [6] it follows that there exists a unique family of nonempty compact sets $A^\alpha \subset X^\alpha$ such that $A^\alpha = \cup_{\beta \in V} \cup_{i=1}^{k} \phi_i^{\alpha\beta}(A^\beta)$. Then $\{X^\alpha, \phi_i^{\alpha\beta}\}$ is a graph-directed iterated function system. Let

$$\Gamma^p = \{(x_0^p, y_0^p), (x_1^p, y_1^p), ..., (x_{N_p}^p, y_{N_p}^p)\} \tag{3.1}$$

be the data sets in $\mathbb{R}^2$, where $N_p \geq 2$, for all $p = 1, 2, ..., n$. These data points satisfy the following condition in order that the maps from the iterated function system to be contractions:

$$\frac{x_i^l - x_{i-1}^l}{x_{N_p}^p - x_0^p} < 1, \tag{3.2}$$

for all $p \neq l, \quad p, l = 1, 2, ..., n, \quad i = 1, 2, ..., N_l$. In [5] we can find the proof regarding the existence of a graph-directed fractal function:

Theorem 3.1. *If we condsider the data set Γ^p in $\mathbb{R}^2$ for $p = 1, 2, ..., n$ satisfying (3.2), then there exists a graph-directed iterated function system, with attractors A_p, $p = 1, 2, ..., n$, such that A_p is the graph of a function which interpolates the data set Γ^p for each p.*

In the case $n = 2$ the construction of these iterated function systems can be done using the method given in [5].

4. Graph directed random fractal interpolation function

Let $(\Omega, \mathcal{K}, P)$ be a probability space and $\{X^\alpha \mid \alpha \in V\}$ a set of complete separable metric spaces and $\Phi_i^{\alpha\beta} : \Omega \times X^\beta \to X^\alpha$ are random variables. Then there exists

$A^\alpha \subseteq \Omega \times X^\alpha$ defined up to probability distribution by

$$A^\alpha \overset{d}{=} \cup_{\beta \in V} \cup_{i=1}^{k} \Phi_i^{\alpha\beta}(A^\beta).$$

The system $\{\Omega \times X^\alpha, \Phi_i^{\alpha\beta}\}$ is the graph directed random iterated function system and A^α is the attractor of the system.

Theorem 4.1. *Let $\Gamma^p = \{(x_0^p, y_0^p), (x_1^p, y_1^p), ..., (x_{N_p}^p, y_{N_p}^p)\}$ be the data sets in $\mathbb{R}^2$ which satisfies (3.2), then there exists a graph directed random iterated function system with attractor A^α such that A^α is the graph of a random function which interpolates Γ^α for each α.*

Proof. We will construct a graph directed random iterated function system for which Theorem 2 holds. Let $n = 2$ and

$$\Gamma^1 = \{(x_0^1, y_0^1), ..., (x_N^1, y_N^1)\},$$
$$\Gamma^2 = \{(x_0^2, y_0^2), ..., (x_M^2, y_M^2)\},$$

where $N, M \geq 2$. Suppose

$$\frac{x_i^1 - x_{i-1}^1}{x_M^2 - x_0^2} < 1 \text{ and } \frac{x_j^2 - x_{j-1}^2}{x_N^1 - x_0^1} < 1$$

$\forall i = 1, ..., N, \quad j = 1, ..., M.$

Let $\mathcal{G} = (V, E)$ such that $V = \{1, 2\}$ and $K^{11} + K^{12} = N$, $K^{21} + K^{22} = M$ and $\Phi_i^{\alpha\beta} : \Omega \times \mathbb{R}^2 \to \mathbb{R}^2$, $i = 1, ..., K^{\alpha\beta}$, $\alpha, \beta \in \{1, 2\}$

$$\Phi_i^{\alpha\beta}(x, y) = \begin{pmatrix} a_i^{\alpha\beta} & 0 \\ c_i^{\alpha\beta} & d_i^{\alpha\beta} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e_i^{\alpha\beta} \\ f_i^{\alpha\beta} \end{pmatrix}.$$

Suppose

$$\begin{cases} \Phi_i^{11}(x_0^1 y_0^1) = & (x_{i-1}^1, y_{i-1}^1) \text{ a.s.} \\ \Phi_i^{11}(x_N^1 y_N^1) = & (x_i^1, y_i^1) \text{ for } i = 1, 2, ..., K^{11} \end{cases}$$

$$\begin{cases} \Phi_{i-k^{11}}^{12}(x_0^2 y_0^2) = & (x_{i-1}^1, y_{i-1}^1) \text{ a.s.} \\ \Phi_{i-K^{11}}^{12}(x_M^2 y_M^2) = & (x_i^1, y_i^1) \text{ for } i = K^{11} + 1, ..., N \end{cases}$$

$$\begin{cases} \Phi_i^{21}(x_0^1 y_0^1) = & (x_{i-1}^2, y_{i-1}^2) \text{ a.s.} \\ \Phi_i^{21}(x_N^1 y_N^1) = & (x_i^2, y_i^2) \text{ for } i = 1, 2, ..., K^{21} \end{cases}$$

$$\begin{cases} \Phi_{i-K^{21}}^{22}(x_0^2 y_0^2) = & (x_{i-1}^2, y_{i-1}^2) \text{ a.s.} \\ \Phi_{i-K^{21}}^{22}(x_M^2 y_M^2) = & (x_i^2, y_i^2) \text{ for } i = K^{21}, ..., M. \end{cases}$$

$\forall i = 1, ..., K^{11}.$

From these conditions we have the following equations:

$$\begin{cases} x_{i-1}^1 = & a_i^{11} x_0^1 + e_i^{11} \\ y_{i-1}^1 = & c_i^{11} x_0^1 + d_i^{11} y_0^1 + f_i^{11} \\ x_i^1 = & a_i^{11} x_N^1 + e_i^{11} \\ y_i^1 = & c_i^{11} x_N^1 + d_i^{11} y_N^1 + f_i^{11} \end{cases}$$

$\forall i = K^{11}+1, ..., N$

$$\begin{cases} x^1_{i-1} = & a^{12}_{i-K^{11}} x^2_0 + e^{12}_{i-K^{11}} \\ y^1_{i-1} = & c^{12}_{i-K^{11}} x^2_0 + d^{12}_{i-K^{11}} y^2_0 + f^{12}_{i-K^{11}} \\ x^1_i = & a^{12}_{i-K^{11}} x^2_M + e^{12}_{i-K^{11}} \\ y^1_i = & c^{12}_{i-K^{11}} x^2_M + d^{12}_{i-K^{11}} y^2_M + f^{12}_{i-K^{11}} \end{cases}$$

$\forall i = 1, ..., K^{21}$.

$$\begin{cases} x^2_{i-1} = & a^{21}_i x^1_0 + e^{21}_i \\ y^2_{i-1} = & c^{21}_i x^1_0 + d^{21}_i y^1_0 + f^{21}_i \\ x^2_i = & a^{21}_i x^1_N + e^{21}_i \\ y^2_i = & c^{21}_i x^1_N + d^{21}_i y^1_N + f^{21}_i \end{cases}$$

$\forall i = K^{21}+1, ..., M$

$$\begin{cases} x^2_{i-1} = & a^{22}_{i-K^{21}} x^2_0 + e^{22}_{i-K21} \\ y^2_{i-1} = & c^{22}_{i-K^{21}} x^2_0 + d^{22}_{i-K^{21}} y^2_0 + f^{22}_{i-K^{21}} \\ x^2_i = & a^{22}_{i-K^{21}} x^2_M + e^{22}_{i-K^{21}} \\ y^2_i = & c^{22}_{i-K^{21}} x^2_M + d^{22}_{i-K^{21}} y^2_M + f^{22}_{i-K^{21}} \end{cases}$$

where $d^{\alpha\beta}_i$ is a random variable.
In this way we obtain $a^{\alpha,\beta}_i, c^{\alpha,\beta}_i, e^{\alpha,\beta}_i, f^{\alpha,\beta}_i,\ \alpha, \beta \in \{1, 2\},\ i = 1, ..., K^{\alpha\beta}$

$$\begin{cases} a^{11}_i = & \frac{x^1_i - x^1_{i-1}}{x^1_N - x^1_0} \\ e^{11}_i = & \frac{x^1_N x^1_{i-1} - x^1_0 x^1_i}{x^1_N - x^1_0} \\ c^{11}_i = & \frac{y^1_i - y^1_{i-1}}{x^1_N - x^1_0} - d^{11}_i \frac{y^1_N - y^1_0}{x^1_N - x^1_0} \\ f^{11}_i = & \frac{x^1_N y^1_{i-1} - x^1_0 y^1_i}{x^1_N - x^1_0} - d^{11}_i \frac{x^1_N y^1_0 - x^1_0 y^1_N}{x^1_N - x^1_0} \end{cases}$$

$$\begin{cases} a^{12}_i = & \frac{x^1_i - x^1_{i-1}}{x^2_M - x^2_0} \\ e^{12}_i = & \frac{x^2_M x^1_{i-1} - x^2_0 x^1_i}{x^2_M - x^2_0} \\ c^{12}_i = & \frac{y^1_i - y^1_{i-1}}{x^2_M - x^2_0} - d^{12}_i \frac{y^2_M - y^2_0}{x^2_M - x^2_0} \\ f^{12}_i = & \frac{x^2_M y^1_{i-1} - x^2_0 y^1_i}{x^2_M - x^2_0} - d^{12}_i \frac{x^2_M y^2_0 - x^2_0 y^2_M}{x^2_M - x^2_0} \end{cases}$$

$$\begin{cases} a^{21}_i = & \frac{x^2_i - x^2_{i-1}}{x^1_N - x^1_0} \\ e^{21}_i = & \frac{x^1_N x^2_{i-1} - x^1_0 x^2_i}{x^1_N - x^1_0} \\ c^{11}_i = & \frac{y^2_i - y^2_{i-1}}{x^1_N - x^1_0} - d^{21}_i \frac{y^1_N - y^1_0}{x^1_N - x^1_0} \\ f^{21}_i = & \frac{x^1_N y^2_{i-1} - x^1_0 y^2_i}{x^1_N - x^1_0} - d^{21}_i \frac{x^1_N y^1_0 - x^1_0 y^1_N}{x^1_N - x^1_0} \end{cases}$$

$$\begin{cases} a_i^{22} = & \frac{x_i^2 - x_{i-1}^2}{x_M^2 - x_0^2} \\ e_i^{22} = & \frac{x_M^2 x_{i-1}^2 - x_0^2 x_i^2}{x_M^2 - x_0^2} \\ c_i^{22} = & \frac{y_i^2 - y_{i-1}^2}{x_M^2 - x_0^2} - d_i^{22} \frac{y_M^2 - y_0^2}{x_M^2 - x_0^2} \\ f_i^{12} = & \frac{x_M^2 y_{i-1}^2 - x_0^2 y_i^2}{x_M^2 - x_0^2} - d_i^{22} \frac{x_M^2 y_0^2 - x_0^2 y_M^2}{x_M^2 - x_0^2} \end{cases}$$

Suppose $ess\sup_{\omega}\max_{i} d_i^{\alpha\beta} < 1$, for all $\alpha, \beta \in \{1,2\}$ and $i = 1, ..., K^{\alpha,\beta}$.

Then $\Phi_i^{\alpha\beta}$ is a contraction and $\{\Omega \times \mathbb{R}^2, \Phi_i^{\alpha\beta}\}$ is a graph directed random iterated function system. We will prove that this graph directed random iterated function system satisfies the theorem.

Let

$$C_1^{\omega} = \{f \mid f : \Omega \times [x_0^1, x_N^1] \to \mathbb{R}, f^{\omega}(x_0^1) = y_0^1, f^{\omega}(x_N^1) = y_N^1, \text{ cont. a.s.}\}$$

$$C_2^{\omega} = \{g \mid g : \omega \times [x_0^2, x_M^2] \to \mathbb{R}, g^{\omega}(x_0^2) = y_0^2, g^{\omega}(x_M^2) = y_M^2, \text{ cont. a.s.}\}$$

For $f_1, f_2 \in C_1^{\omega}$ we define

$$d_{\infty}^*(f_1, f_2) = ess\sup_{\omega} d_{\infty}(f_1^{\omega}, f_2^{\omega})$$

where

$$d_{\infty}(f_1, f_2) = \max_{x}\{|f_1^{\omega}(x) - f_2^{\omega}(x)|, x \in [x_0^1, x_N^1]\}.$$

$(C_1^{\omega}, d_{\omega}^*)$ and $(C_2^{\omega}, d_{\omega}^*)$ are complete metric spaces, hence $C_1^{\omega} \times C_2^{\omega}$ is also a complete metric space with

$$\tilde{f}(\omega, x) = \begin{cases} C_i^{11} I_i^{-1}(x) + d_i^{11} f(\omega, I_i^{-1}(x) + f_i^{11}) & \text{if } x \in [x_{i-1}^1, x_i^1], \\ & i = 1, ..., K^{11} \\ C_{i-K^{11}}^{12} I_i^{-1}(x) + d_{i-K^{11}}^{12} g(\omega, I_i^{-1}(x)) + f_{i-K^{11}}^{12}) & \text{if } x \in [x_{i-1}^1, x_i^1], \\ & i = K^{11} + 1, ..., N, \end{cases}$$

$$\tilde{g}(\omega, y) = \begin{cases} C_j^{21} J_j^{-1}(y) + d_j^{21} f(\omega, J_j^{-1}(y) + f_j^{21}) & \text{if } y \in [x_{j-1}^2, x_j^2], \\ & j = 1, ..., K^{21} \\ C_{j-K^{21}}^{22} J_j^{-1}(y) + d_{j-K^{21}}^{22} g(\omega, J_j^{-1}(y)) + f_{j-K^{21}}^{22}) & \text{if } y \in [x_{ij-1}^2, x_j^2], \\ & j = K^{21} + 1, ..., M, \end{cases}$$

where

$$\begin{array}{lll} I_i : [x_0^1, x_N^1] \to [x_{i-1}^1, x_i^1],\ I_i(x) & = & a_i^{11} x + e_i^{11}, \text{ for } i = 1, ..., K^{11} \\ I_i : [x_0^2, x_M^2] \to [x_{i-1}^1, x_i^1],\ I_i(x) & = & a_{i-K^{11}}^{12} x + e_{i-K^{11}}^{12}, \text{ for } i = K^{11} + 1, ..., N \\ J_i : [x_0^1, x_N^1] \to [x_{i-1}^1, x_i^1],\ J_i(x) & = & a_i^{21} x + e_i^{21}, \text{ for } i = 1, ..., K^{21} \\ J_i : [x_0^2, x_M^2] \to [x_{i-1}^2, x_i^2],\ J_i(x) & = & a_{i-K^{21}}^{22} x + e_{i-K^{21}}^{22}, \text{ for } i = K^{21} + 1, ..., M. \end{array}$$

We have

$$\tilde{f}(\omega, x_0^1) = y_0^1 \text{ a. s., } \tilde{f}(\omega, x_N^1) = y_N^1 \text{ a. s.}$$

$$\tilde{g}(\omega, x_0^2) = y_0^2 \text{ a. s., } \tilde{g}(\omega, x_M^2) = y_M^2 \text{ a. s.}$$

One can show that $\tilde{f}$ and $\tilde{g}$ are continuous functions a.s.. We have to show that T is a contraction.

$$d^*_\infty(f_1, f_2) = ess\sup_\omega \max_x\{|f_1(\omega, x) - f_2(\omega, x)|\}$$

$$\max_{x\in[x^1_0, x^1_{K^{11}}]}\{|f_1(\omega, x) - f_2(\omega, x)|\} = \max_{i=1,\ldots,K^{11}} \{|d^{11}_i||f_1(\omega, I^{-1}_i(x)) -$$
$$- \quad f_2(\omega, I^{-1}_i(x))| \ , x \in [x^1_{i-1}, x^1_i]\}\} \leq ess\sup_\omega\{d^{11}_i, i = 1, ..., K^{11}\} \cdot d_\infty(f_1, f_2)$$

$$\max_{x\in[x^1_{K^{11}}, x^1_M]}\{|f_1(\omega, x) - f_2(\omega, x)|\} = \max_{i=K^{11}+1,\ldots,N} \{|d^{12}_{i-K^{11}}||g_1(\omega, I^{-1}_i(x)) -$$
$$- \quad g_2(\omega, I^{-1}_i(x))| \ , x \in [x^1_{i-1}, x^1_i]\}\} \leq ess\sup_\omega\{d^{12}_i, i = 1, ..., K^{12}\} \cdot d_\infty(f_1, f_2)$$

$$\begin{aligned} d^*_\infty(f_1, f_2) \quad &\leq \quad \max\{ess\sup_\omega\{d^{12}_i, i = 1, ..., K^{12}\}, ess\sup_\omega\{d^{11}_i, i = 1, ..., K^{11}\}\} \cdot \\ &\cdot \quad \max\{d^*_\infty(f_1, f_2), d^*_\infty(g_1, g_2)\} \end{aligned}$$

similarly

$$\begin{aligned} d^*_\infty(g_1, g_2) \quad &\leq \quad \max\{ess\sup_\omega\{d^{21}_i, i = 1, ..., K^{21}\}, ess\sup_\omega\{d^{22}_i, i = 1, ..., K^{22}\}\} \cdot \\ &\cdot \quad \max\{d^*_\infty(f_1, f_2), d^*_\infty(g_1, g_2)\}. \end{aligned}$$

So

$$\begin{aligned} d(T(f_1, g_1), T(f_2, g_2)) \quad &= \quad \max\{d^*_\infty(\tilde{f}_1, \tilde{f}_2), d^*_\infty(\tilde{g}_1, \tilde{g}_2)\} \leq \\ &\leq \quad r \cdot \max\{d^*_\infty(f_1.f_2), d^*_\infty(g_1, g_2)\}, \end{aligned}$$

where

$$\begin{aligned} r \quad = \quad \max\Big\{ & ess\sup_\omega\{d^{21}_i, i = 1, ..., K^{21}\}, ess\sup_\omega\{d^{22}_i, i = 1, ..., K^{22}\} \ , \\ & ess\sup_\omega\{d^{12}_i, i = 1, ..., K^{12}\}, ess\sup_\omega\{d^{11}_i, i = 1, ..., K^{11}\}\Big\} < 1. \end{aligned}$$

Using Banach fixed point theorem, T has a unique fixed point (f_0, g_0):

$$T(f_0, g_0) = (f_0, g_0).$$

Let F and G be the graph of f_0 and g_0:

$$\begin{aligned} f_0(\omega, a^{11}_i x + e^{11}_i) \quad &= \quad c^{11}_i x + d^{11}_i f_0(\omega, x) + f^{11}_i \text{ for } i = 1, ..., K^{11} \\ f_0(\omega, a^{12}_i y + e^{12}_i) \quad &= \quad c^{12}_i y + d^{12}_i g_0(\omega, y) + f^{12}_i \text{ for } i = 1, ..., K^{12}, \end{aligned}$$

which imply:

$$F = \bigcup_{i=1}^{K^{11}} \Phi^{11}_i(F) \cup \bigcup_{i=1}^{K^{12}} \Phi^{12}_i(G)$$

similarly

$$G = \bigcup_{i=1}^{K^{21}} \Phi^{21}_i(F) \cup \bigcup_{i=1}^{K^{22}} \Phi^{22}_i(G).$$

According to the uniqueness of the solution, the graph of f_0 and g_0 are the attractor of the fractal interpolation function. □

In the last few years the method of fractal interpolation was widely used in signal processing, computer geometry, image compression and of course in approximation theory. The stochastic type fractal interpolation method and the graph-directed random fractal interpolation function present more flexibility and therefore it can be applied much better in the case of real data interpolation.

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Shepard operator of least squares thin-plate spline type

Teodora Cătinaş and Andra Malina

Dedicated to Professor Gheorghe Coman on the occasion of his 85th anniversary.

Abstract. We obtain some new Shepard type operators based on the classical, the modified Shepard methods and the least squares thin-plate spline function. Given some sets of points, we compute some representative subsets of knot points following an algorithm described by J. R. McMahon in 1986.

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Keywords: Scattered data, Shepard operator, least squares approximation, thin-plate spline, knot points.

1. Preliminaries

One of the best suited methods for approximating large sets of data is the Shepard method, introduced in 1968 in [16]. It has the advantages of a small storage requirement and an easy generalization to additional independent variables, but it suffers from no good reproduction quality, low accuracy and a high computational cost relative to some alternative methods [14], these being the reasons for finding new methods that improve it (see, e.g.,[1]-[8], [17], [18]). In this paper we obtain some new operators based on the classical, the modified Shepard methods and the least squares thin-plate spline.

Let f be a real-valued function defined on $X \subset \mathbb{R}^2$, and $(x_i, y_i) \in X$, $i = 1, ..., N$ some distinct points. Denote by $r_i(x, y)$ the distances between a given point $(x, y) \in X$ and the points (x_i, y_i), $i = 1, ..., N$. The bivariate Shepard operator is defined by

$$(S_\mu f)(x, y) = \sum_{i=1}^{N} A_{i,\mu}(x, y) f(x_i, y_i), \tag{1.1}$$

where

$$A_{i,\mu}(x,y)=\frac{\prod\limits_{\substack{j=1\\ j\neq i}}^{N} r_j^{\mu}(x,y)}{\sum\limits_{k=1}^{N}\prod\limits_{\substack{j=1\\ j\neq k}}^{N} r_j^{\mu}(x,y)}, \tag{1.2}$$

with the parameter $\mu > 0$.

It is known that the bivariate Shepard operator S_μ reproduces only the constants and that the function $S_\mu f$ has flat spots in the neighborhood of all data points.

Franke and Nielson introduced in [10] a method for improving the accuracy in reproducing a surface with the bivariate Shepard approximation. This method has been further improved in [9], [15], [14], and it is given by:

$$(Sf)(x,y)=\frac{\sum\limits_{i=1}^{N} W_i(x,y)\, f(x_i,y_i)}{\sum\limits_{i=1}^{N} W_i(x,y)}, \tag{1.3}$$

with

$$W_i(x,y)=\left[\frac{(R_w-r_i)_+}{R_w r_i}\right]^2, \tag{1.4}$$

where R_w is a radius of influence about the node (x_i, y_i) and it is varying with i. R_w is taken as the distance from node i to the jth closest node to (x_i, y_i) for $j > N_w$ (N_w is a fixed value) and j as small as possible within the constraint that the jth closest node is significantly more distant than the $(j-1)$st closest node (see, e.g. [14]). As it is mentioned in [11], this modified Shepard method is one of the most powerful software tools for the multivariate approximation of large scattered data sets.

2. The Shepard operators of least squares thin-plate spline type

Consider f a real-valued function defined on $X \subset \mathbb{R}^2$, and $(x_i, y_i) \in X$, $i = 1, ..., N$ some distinct points. We introduce the Shepard operator based on the least squares thin-plate spline in four ways.

Method 1. *We consider*

$$(S_1 f)(x,y)=\sum_{i=1}^{N} A_{i,\mu}(x,y) F_i(x,y), \tag{2.1}$$

where $A_{i,\mu}$, $i = 1, ..., N$, are defined by (1.2), for a given parameter $\mu > 0$ and the least squares thin-plate splines are given by

$$F_i(x,y)=\sum_{j=1}^{i} C_j d_j^2 \log(d_j) + ax + by + c, \quad i=1,...,N, \tag{2.2}$$

with $d_j = \sqrt{(x-x_j)^2 + (y-y_j)^2}$.

For the second way, we consider a smaller set of $k \in \mathbb{N}^*$ knot points $(\hat{x}_j, \hat{y}_j)$, $j = 1, ..., k$ that will be representative for the original set. This set is obtained following the next steps (see, e.g., [12] and [13]):

Algorithm 2.1. 1. Generate k random knot points, with $k < N$;
2. Assign to each point the closest knot point;
3. If there exist knot points for which there is no point assigned, move the knot to the closest point;
4. Compute the next set of knot points as the arithmetic mean of all corresponding points;
5. Repeat steps 2-4 until the knot points do not change for two successive iterations.

Method 2. *For a given $k \in \mathbb{N}^*$, we consider the representative set of knot points $(\hat{x}_j, \hat{y}_j)$, $j = 1, ..., k$. The Shepard operator of least squares thin-plate spline is given by*

$$(S_2 f)(x, y) = \sum_{i=1}^{k} A_{i,\mu}(x, y) F_i(x, y), \tag{2.3}$$

where $A_{i,\mu}$, $i = 1, ..., k$, are defined by

$$A_{i,\mu}(x, y) = \frac{\prod\limits_{\substack{j=1\\ j\neq i}}^{k} r_j^{\mu}(x, y)}{\sum\limits_{p=1}^{k} \prod\limits_{\substack{j=1\\ j\neq p}}^{k} r_j^{\mu}(x, y)},$$

for a given parameter $\mu > 0$.

The least squares thin-plate spline are given by

$$F_i(x, y) = \sum_{j=1}^{i} C_j d_j^2 \log(d_j) + ax + by + c, \quad i = 1, ..., k, \tag{2.4}$$

with $d_j = \sqrt{(x - \hat{x}_j)^2 + (y - \hat{y}_j)^2}$.

For Methods 1 and 2, the coefficients C_j, a, b, c of F_i are found such that to minimize the expressions

$$E = \sum_{i=1}^{N'} [F_i(x_i, y_i) - f(x_i, y_i)]^2,$$

considering $N' = N$ for the first case and $N' = k$ for the second one. There are obtained systems of the following form (see, e.g., [12]):

$$\begin{pmatrix} 0 & d_{12}^2 \log d_{12} & \cdots & d_{1N'}^2 \log d_{1N'} & x_1 & y_1 & 1 \\ d_{21}^2 \log d_{21} & 0 & \cdots & d_{2N'}^2 \log d_{2N'} & x_2 & y_2 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ d_{N'1}^2 \log d_{N'1} & d_{N'2}^2 \log d_{N'2} & \cdots & 0 & x_{N'} & y_{N'} & 1 \\ x_1 & x_2 & \cdots & x_{N'} & 0 & 0 & 0 \\ y_1 & y_2 & \cdots & y_{N'} & 0 & 0 & 0 \\ 1 & 1 & \cdots & 1 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ C_{N'} \\ a \\ b \\ c \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_{N'} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

with $d_{ij}^2 = (x_i - x_j)^2 + (y_i - y_j)^2$, $f_i = f(x_i, y_i)$, $i, j = 1, \ ..., \ N'$.

Next we consider the improved form of the Shepard operator given in (1.3).

Method 3. *We consider Shepard operator of least squares thin-plate spline type of the following form:*

$$(S_3 f)(x,y) = \frac{\sum_{i=1}^{N} W_i\,(x,y)\,F_i(x,y)}{\sum_{i=1}^{N} W_i\,(x,y)}, \tag{2.5}$$

with W_i given by (1.4), F_i given by (2.2), for $i = 1, ..., N$.

The coefficients C_j, a, b, c of F_i, $i = 1, ..., N$ are determined in order to minimize the expression

$$E = \sum_{i=1}^{N} [F_i(x_i, y_i) - f(x_i, y_i)]^2.$$

Method 4. *For a given $k \in \mathbb{N}^*$, we consider the representative set of knot points $(\hat{x}_j, \hat{y}_j)$, $j = 1, ..., k$, obtained applying the Algorithm 2.1. In this case, we introduce the Shepard operator of least squares thin-plate spline type by the following formula:*

$$(S_4 f)(x,y) = \frac{\sum_{i=1}^{k} W_i\,(x,y)\,F_i(x,y)}{\sum_{i=1}^{k} W_i\,(x,y)}, \tag{2.6}$$

with W_i given by (1.4) and F_i given by (2.4), for $i = 1, ..., k$.

The coefficients C_j, a, b, c of F_i, $i = 1, ..., k$ are determined in order to minimize the expression

$$E = \sum_{i=1}^{k} [F_i(x_i, y_i) - f(x_i, y_i)]^2.$$

3. Numerical examples

We consider the following test functions (see, e.g., [9], [15], [14]):

$$\begin{aligned} &\text{Gentle:} \quad f_1(x,y) = \exp[-\tfrac{81}{16}((x-0.5)^2 + (y-0.5)^2)]/3, \\ &\text{Saddle:} \quad f_2(x,y) = \frac{(1.25 + \cos 5.4y)}{6 + 6(3x-1)^2}, \\ &\text{Sphere:} \quad f_3(x,y) = \sqrt{64 - 81((x-0.5)^2 + (y-0.5)^2)}/9 - 0.5. \end{aligned} \tag{3.1}$$

Table 1 contains the maximum errors for approximating the functions (3.1) by the classical and the modified Shepard operators given, respectively, by (1.1) and (1.3), and the errors of approximating by the operators introduced in (2.1), (2.3), (2.5) and (2.6). We consider three sets of $N = 100$ random points for each function in $[0,1] \times [0,1]$, $k = 25$ knots, $\mu = 3$ and $N_w = 19$.

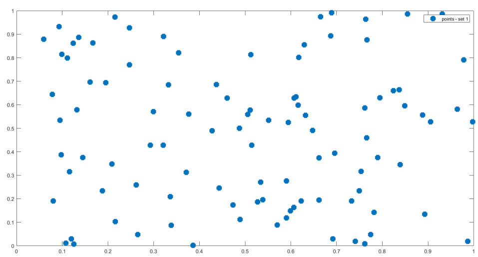
Remark 3.1. The approximants $S_2 f_i$, $S_4 f_i$, $i = 1, 2, 3$ have better approximation properties although the number of knot points is smaller than the number of knot points considered for the approximants $S_1 f_i$, $S_3 f_i$ $i = 1, 2, 3$, so this illustrates the benefits of the algorithm of choosing the representative set of points.

In Figures 2, 4, 6 we plot the graphs of f_1, f_2, f_3 and of the corresponding Shepard operators $S_1 f_i$, $S_2 f_i$, $S_3 f_i$ and $S_4 f_i$, $i = 1, 2, 3$, respectively.

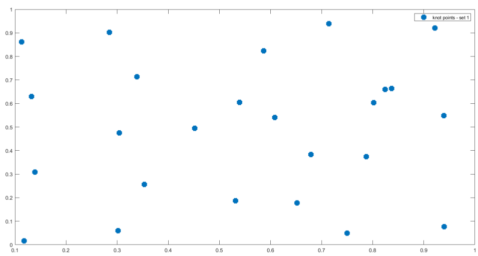
In Figures 1, 3, 5 we plot the sets of the given points and the corresponding sets of the representative knot points.

TABLE 1. Maximum approximation errors.

	f_1	f_2	f_3
$S_\mu f$	0.0864	0.1095	0.1936
Sf	0.0724	0.0970	0.1770
$S_1 f$	0.1644	0.4001	0.6595
$S_2 f$	0.1246	0.2858	0.3410
$S_3 f$	0.1578	0.3783	0.6217
$S_4 f$	0.1212	0.2834	0.3399

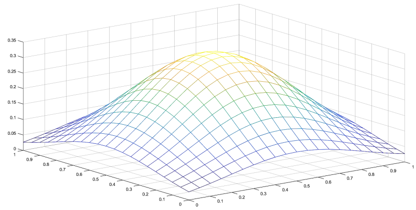


First set of given points.

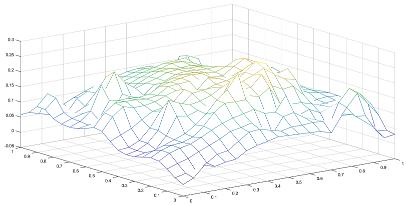


First set of representative knot points.

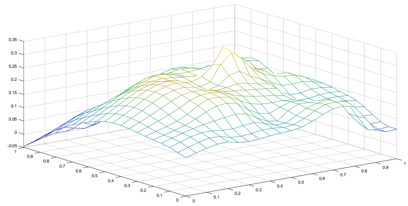
FIGURE 1. First sets of points.



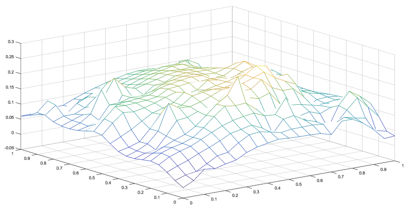
Function f_1.



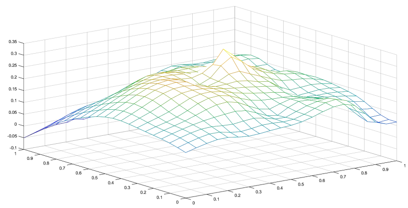
$S_1 f_1$



$S_2 f_1$

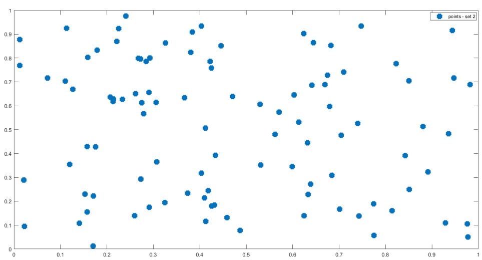


$S_3 f_1$

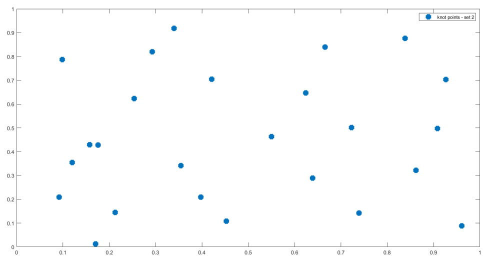


$S_4 f_1$

FIGURE 2. Graphs for f_1.

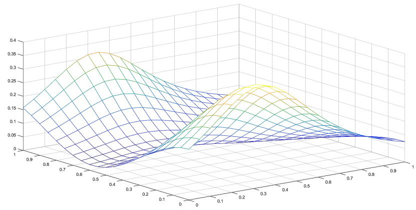


Second set of given points.

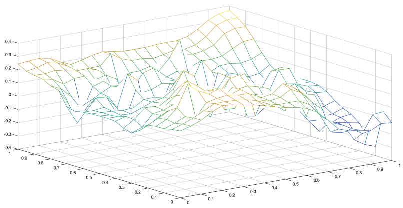


Second set of representative knot points.

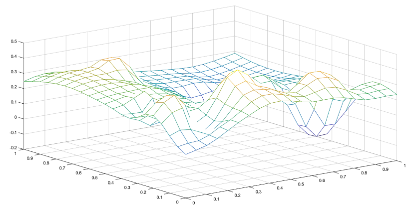
FIGURE 3. Second sets of points.



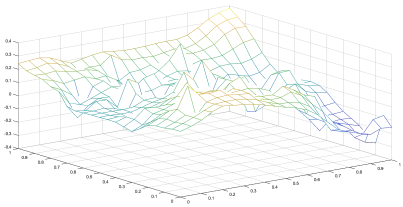
Function f_2.



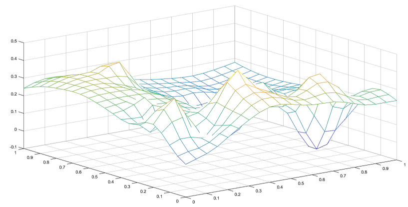
$S_1 f_2$



$S_2 f_2$

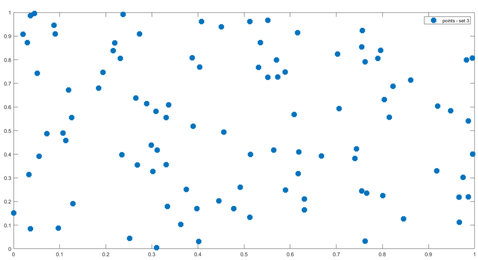


$S_3 f_2$

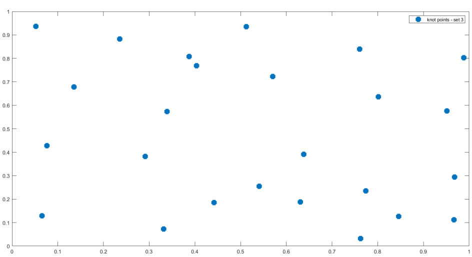


$S_4 f_2$

FIGURE 4. Graphs for f_2.

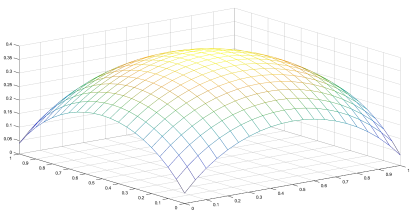


Third set of given points.

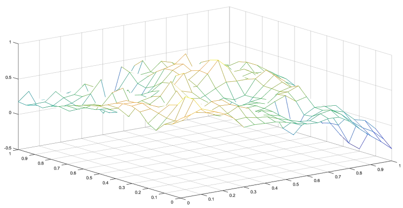


Third set of representative knot points.

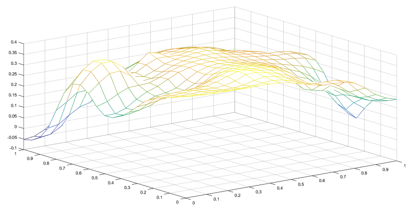
FIGURE 5. Third sets of points.



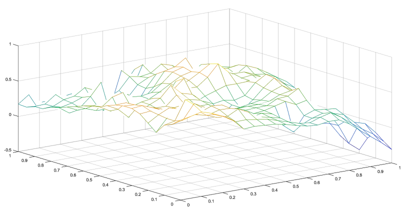
Function f_3.



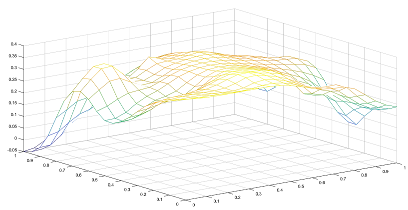
$S_1 f_3$



$S_2 f_3$



$S_3 f_3$



$S_4 f_3$

FIGURE 6. Graphs for f_3.

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A numerical method for two-dimensional Hammerstein integral equations

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Dedicated to Professor Gheorghe Coman on the occasion of his 85th anniversary.

Abstract. In this paper we investigate a collocation method for the approximate solution of Hammerstein integral equations in two dimensions. As in [8], collocation is applied to a reformulation of the equation in a new unknown, thus reducing the computational cost and simplifying the implementation. We start with a special type of piecewise linear interpolation over triangles for a reformulation of the equation. This leads to a numerical integration scheme that can then be extended to any bounded domain in $\mathbb{R}^2$, which is used in collocation. We analyze and prove the convergence of the method and give error estimates. As the quadrature formula has a higher degree of precision than expected with linear interpolation, the resulting collocation method is superconvergent, thus requiring fewer iterations for a desired accuracy. We show the applicability of the proposed scheme on numerical examples and discuss future research ideas in this area.

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1. Introduction

Integral equations are a special topic in Applied Mathematics, as they are an important tool for modeling many applications in fields ranging from engineering, computer graphics to astrophysics, chemistry, quantum mechanics and more (see [13]). They also arise in reformulations of initial and boundary value problems for ordinary and partial differential equations.

Having such a wide variety of applications, they have been studied extensively, especially form the approximation perspective. Numerical solutions have been obtained using moving least squares [5], Adomian decomposition, [4], kernel methods [6], collocation [8, 3, 10, 7], Galerkin and Nyström methods [9]. Also, good results were obtained using wavelets [11, 12] and other iterative methods [1]. For more details on approximating methods for integral equations, see [2].

In this paper, we consider the following integral equation of Hammerstein type

$$u(x) \quad = \quad \int_D k(x,y) g\left(y, u(y)\right) dy + f(x), \;\; x \in D \subset \mathbb{R}^2, \tag{1.1}$$

with a smooth kernel k and $g : D \times \mathbb{R} \to \mathbb{R}$ a continuous nonlinear function. Later, other assumptions will be made on k, g and f.

As in [8], collocation is applied to a reformulation of the equation in a new unknown, thus reducing the computational cost and simplifying the implementation. For the new integral equation, we define a collocation scheme based on linear interpolation and show that at the collocation nodes, it converges faster than over the entire domain, thus requiring fewer iterations for a given accuracy. These two aspects make this method much more efficient from the computational point of view.

In operator form, we write (1.1) as

$$u \quad = \quad Ku + f, \tag{1.2}$$

where

$$(Ku)(x) = \int_D k(x,y) g\left(y, u(y)\right) dy. \tag{1.3}$$

Following the ideas in [8], we reformulate (1.1) for the new unknown

$$v(y) := g(y, u(y)).$$

By (1.1), v must satisfy the equation

$$v(x) \quad = \quad g\left(x, \int_D k(x,y) v(y) dy + f(x)\right), \;\; x \in D \tag{1.4}$$

and u is given by

$$u(x) \quad = \quad \int_D k(x,y) v(y) dy + f(x), \;\; x \in D. \tag{1.5}$$

We define a collocation scheme for v in equation (1.4), which will be then used to find an approximate solution of (1.5).
We are interested in finding a numerical solution of equation (1.1), which approximates the exact solution, assumed to exist. To this end, we work under the following assumptions:

(A1) The equation (1.3) has an isolated solution u^* with non-zero index, which is assumed to be smooth enough;
(A2) The integral operator $K : C(D) \to C(D)$ defined by (1.3) is completely continuous;
(A3) The derivative $g_u(y, u)$ exists and is continuous on $D \times \mathbb{R}$;
(A4) The function $f \in C(D)$.

The ideas and results described in this paper work for any closed bounded domain $D \subset \mathbb{R}^2$ that can be triangulated in a smooth way. For simplicity, we restrict the discussion to the case of a rectangular region $D = [a, b] \times [c, d]$.

The rest of the paper is organized as follows: in Section 2, we define a collocation scheme for equation (1.4), based on a special type of linear interpolation on a triangular region. We prove the convergence and give error estimates (for both u nd v) in Section 3. Section 4 shows the applicability of the proposed method to numerical examples, where the theoretical error bounds are confirmed by the numerical results. We draw some important conclusions and discuss future research ideas in Section 5.

2. Numerical method

2.1. Preliminaries for collocation

Let us recall the collocation method in the general framework of projection methods. Consider a set of nodes $\{x_1, \dots, x_n\} \subset D$ and let $\{l_1, \dots, l_n\}$ be a set of functions defined on D such that

$$l_j(x_i) = \delta_{ij}, 1 \le i, j \le n.$$

Denote by $D_n = \text{span}\{l_1, \dots, l_n\}$ and define the interpolatory projection operator $P_n : D \to D_n$ by

$$(P_n u)(x) = \sum_{j=1}^{n} u(x_j) l_j(x), \ x \in D. \tag{2.1}$$

Then $P_n : C(D) \to C(D)$ is a linear operator (see e.g. [2]) and its norm is given by

$$||P_n|| = \sup_{x \in D} \sum_{j=1}^{n} |l_j(x)|.$$

We will assume that $||P_n|| < \infty$ and that

$$\lim_{n \to \infty} ||u - P_n u|| = 0, \text{ for all } u \in C(D). \tag{2.2}$$

Let v^* be the solution of (1.4) corresponding to u^*. Using P_n, we define an approximation of v^* by

$$v_n(x) = P_n v(x) = \sum_{j=1}^{n} v_n(x_j) l_j(x). \tag{2.3}$$

The values $\{v_n(x_j)\}_{j=1}^{n}$ are determined by forcing equation (1.4) to be true at the collocation points. This leads to the system

$$v_n(x_i) = g\left(x_i, \sum_{j=1}^{n} v_n(x_j) \int_D k(x_i, y) l_j(y) dy + f(x_i)\right), i = 1, \dots, m, \tag{2.4}$$

or

$$\sum_{j=1}^{n} v_n(x_j) l_j(x_i) = g\left(x_i, \sum_{j=1}^{n} v_n(x_j) \int_D k(x_i, y) l_j(y) dy + f(x_i)\right). \tag{2.5}$$

It is worth mentioning that the integrals on the right hand side only have to be evaluated once, not at every iteration, since they are dependent only on the basis functions, *not* on v_n. This reduces the computational cost of the method and simplifies the implementation.

From (2.5), the approximate solution of (1.5) is found by

$$\begin{aligned} u_n(x) &= \int_D k(x,y)v_n(y)dy + f(x) \\ &= \sum_{j=1}^{n} v_n(x_j) \int_D k(x,y)l_j(y)dy + f(x). \end{aligned} \tag{2.6}$$

For the two approximate solutions, the following result holds:

Theorem 2.1. ([8, Theorem 2]) *Assume conditions* (A1) − (A4) *hold and that the operator* P_n *defined in* (2.1) *satisfies* (2.2). *Then*

$$||v_n - v^*|| \to 0, \quad ||u_n - u^*|| \to 0, \quad \text{as} \quad n \to 0.$$

Moreover, there exists an $n_0 \in \mathbb{N}$ *and a constant* c*, independent of* n*, such that for all* $n \geq n_0$*,*

$$||u_n - u^*|| \leq c \inf_{\phi \in D_n} ||\phi - v^*||.$$

This means that u_n converges to u^* at least as fast as v_n converges to v^*.

2.2. Interpolation-based collocation

To define the projection operator P_n, we start with piecewise linear interpolation of an unknown function on a triangle. First, we consider the unit simplex $\sigma = \{(s,t) \mid 0 \leq s, t, s+t \leq 1\}$. Let h be a continuous function on σ and denote by $w = 1 - s - t$. To approximate h, we use linear interpolation

$$h(s,t) \approx \sum_{i=1}^{3} h(\tau_i)l_i(s,t), \tag{2.7}$$

where the nodes

$$\tau_1 = \left(\frac{1}{2}, \frac{1}{2}\right), \tau_2 = \left(\frac{1}{2}, 0\right), \tau_3 = \left(0, \frac{1}{2}\right) \tag{2.8}$$

are the midpoints of the three sides of σ and

$$l_1(s,t) = 1 - 2w, \quad l_2(s,t) = 1 - 2t, \quad l_3(s,t) = 1 - 2s \tag{2.9}$$

are the corresponding Lagrange interpolation basis functions. Obviously, the approximation formula (2.7) is exact for all polynomials of degree less than or equal to 1.

This formula can be extended to any triangle Δ, using an affine mapping $T : \sigma \underset{onto}{\overset{1-1}{\longrightarrow}} \Delta$ given by

$$x = (\xi, \eta) = T(s,t) = wz_1 + tz_2 + sz_3, \tag{2.10}$$

where z_1, z_2, z_3 are the vertices of Δ. This mapping transforms a polynomial of degree r in (s,t) into a polynomial of the same degree in (ξ, η) (and its inverse acts the same

way on polynomials in (ξ, η)).
Denote by

$$q_i = T(\tau_i), i = 1, 2, 3. \tag{2.11}$$

For a given $h \in C(\Delta)$, define $P_n h$ by

$$\begin{aligned} P_n h(x) &= P_n h(T(s,t)) \\ &= \sum_{i=1}^{3} h(q_i) l_i(s,t), \quad (s,t) \in \sigma. \end{aligned} \tag{2.12}$$

Then the approximation formula

$$h(x) \approx P_n h(x) = \sum_{i=1}^{3} h(q_i) l_i(s,t), \tag{2.13}$$

is still exact for polynomials of degree $r \leq 1$.
From general interpolation theory, we have the following error bound for this approximation (see e.g. [2]).

Lemma 2.2. *Let Δ be a planar triangle and assume $h \in C^2(\Delta)$. Then, the following holds*

$$||h - P_n h||_\infty \leq c\delta^2 ||D^2 h||_\infty, \tag{2.14}$$

where $\delta = diameter(\Delta)$ and $D^2 h = \max_{0 \leq i \leq 2} \left| \frac{\partial^2 h(\xi,\eta)}{\partial \xi^i \partial \eta^{2-i}} \right|$. The constant c is independent of both h and Δ.

Now, to define our collocation method, let $\mathcal{T}_n = \{\Delta_1, ..., \Delta_n\}$ be a triangulation of D with grid size δ_n and $T_k : \sigma \to \Delta_k$ be defined as in (2.10), for every $k = 1, \ldots, n$. At each iteration, the triangulation is refined by splitting each triangle into four triangles, obtained by connecting the midpoints of the three sides. The new triangulation, $\mathcal{T}_{4n}$, has four times as many triangles and grid size $\delta_{4n} = \frac{1}{2}\delta_n$.
For a function $h \in C(D)$, restrict it to some $\Delta \in \mathcal{T}_n$ and use (2.13) to approximate it on Δ.
Integrating (2.7), we obtain the quadrature formula

$$\int_\sigma h(s,t) d\sigma \approx \frac{1}{6}\left[h\left(\frac{1}{2}, \frac{1}{2}\right) + h\left(\frac{1}{2}, 0\right) + h\left(0, \frac{1}{2}\right)\right]. \tag{2.15}$$

Using the same affine mapping (2.10), this leads to a quadrature formula for integrals on Δ

$$\begin{aligned} \int_\Delta h(y) dy &\approx \int_\Delta P_n h(y) dy \\ &= \sum_{i=1}^{3} h(q_i) \int_\sigma l_i(s,t) J_T(s,t) \, d\sigma, \end{aligned} \tag{2.16}$$

where J_T is the Jacobian of the transformation given in (2.10).
Thus, we have the approximation

$$\int_D h(y)dy = \sum_{k=1}^{n} \int_{\Delta_k} h(y)dy \tag{2.17}$$

$$\approx \sum_{k=1}^{n}\sum_{i=1}^{3} h(q_{k,i}) \int_{\sigma} l_i(s,t)J_{T_k}(s,t)\, d\sigma, \tag{2.18}$$

where $q_{k,i} = T_k(\tau_i), i = 1, 2, 3$.
Then the collocation method is given by

$$v_n(q_i) = g\left(q_i, \sum_{j=1}^{3} v_n(q_j) \int_D k(q_i, y)l_j(y)dy + f(q_i)\right), \tag{2.19}$$

which leads to the system

$$v_n(q_i) = g\left(q_i, \sum_{k=1}^{n}\sum_{j=1}^{3} v_n(q_{k,j}) \int_{\sigma} k(q_i, T_k(s,t))l_j(s,t)J_{T_k}(s,t)d\sigma + f(q_i)\right), \tag{2.20}$$

for all $i = 1, \ldots, 3n$. Once all the unknowns $v_n(q_i)$ are found from this system, we have, for each $x = T_k(s,t) \in \Delta_k$,

$$\begin{aligned} v_n(x) &= \sum_{i=1}^{3} v_n(q_i)l_i(s,t), \\ u_n(x) &= \sum_{i=1}^{3} v_n(q_i) \int_{\sigma} k(x, T_k(s,t))l_i(s,t)J_{T_k}(s,t)d\sigma + f(x). \end{aligned} \tag{2.21}$$

3. Convergence and error analysis

We write the system (2.20) in operator form as

$$(I - P_n\mathcal{K})v_n = 0, \tag{3.1}$$

with

$$\mathcal{K}(v)(x) = g\left(x, \int_D k(x,y)v(y)dy + f(x)\right) \tag{3.2}$$

and P_n defined as in (2.12).

Since we are using piecewise linear interpolation in our collocation method, we have the following well known general result (see, e.g. [2, p. 177]).

Theorem 3.1. *Assuming* A1 − A4 *hold, the operators* $I - P_n\mathcal{K}$ *are invertible on* $C(D)$ *and have uniformly bounded inverses, for all sufficiently large* n*, say* $n \geq n_0$*. In addition, the following error bounds hold:*

$$||v^* - v_n||_\infty \leq \left|\left|(I - P_n\mathcal{K})^{-1}\right|\right| \cdot ||v^* - P_n v^*||, \; n \geq n_0$$

and

$$||v^* - v_n||_\infty \le O(\delta^2),\ \ n \ge n_0, \tag{3.3}$$

where $\delta = \delta_n$ *denotes the mesh size of the triangulation* $\mathcal{T}_n$.

Now, this result holds in general, when using linear spline approximation. However, because of our particular choice of collocation (and interpolation) nodes, the approximation has higher order than $O(\delta^2)$. Notice that the quadrature formula (2.15) has degree of precision $d = 2$, *higher* than expected with interpolation of degree 1. Then formula (2.16) also has degree of precision $d = 2$. This will lead to a higher rate of convergence at the collocation nodes than $O(\delta^2)$, i. e., we get *superconvergence*.

Theorem 3.2. *Assume the conditions* A1 − A4 *hold, that* $k \in C^2(D \times D)$ *and that* $v^* \in C^3(D)$. *Then*

$$\max_{1\le i\le n} |v^*(q_i) - v_n(q_i)|,\quad \max_{1\le i\le n} |u^*(q_i) - u_n(q_i)| \le O(\delta^3). \tag{3.4}$$

Proof. The proof is computational and follows the same ideas as the ones given for similar results e.g. in [3, 10].
Since v^* is the exact solution of (1.4), $v^* = \mathcal{K}v^*$. By the interpolation formula (2.13), $v_n = P_n\mathcal{K}v_n$. Then

$$\begin{aligned}(I - P_n\mathcal{K})(v^* - v_n) &= v^* - v_n - P_n\mathcal{K}v^* + P_n\mathcal{K}v_n\\ &= \mathcal{K}v^* - P_n\mathcal{K}v^* \ = \ (I - P_n)\mathcal{K}v^*\end{aligned}$$

and, thus, under our assumptions, for each $i = 1, \dots, n$,

$$\begin{aligned}|(I - P_n\mathcal{K})(v^* - v_n)(q_i)| &= \Bigg| g\left(q_i, \int_D k(q_i, y)v^*(y)dy + f(q_i)\right)\\ &\quad - g\left(q_i, \int_D k(q_i, y)P_n v^*(y)dy + f(q_i)\right)\Bigg|\\ &\le c\left|\sum_{k=1}^{n}\int_{\Delta_k} k(q_i, y)(I - P_n)v^*(y)dy\right|.\end{aligned}$$

Now, on each triangle Δ_k, let p_j denote Taylor polynomial expansions of v^* around a suitable point in Δ_k, for $j = 1, 2$. Then

$$\begin{aligned}||v^* - p_j||_\infty &\le c\delta^{j+1},\\ ||p_2 - p_1||_\infty &\le c\delta^2.\end{aligned} \tag{3.5}$$

Also, since $k \in C^2(D \times D)$, there exists a constant k_0 such that, for all $y \in \Delta_k$,

$$|k(q_i, y) - k_0| \le c\delta. \tag{3.6}$$

Since the interpolation formula (2.13) has degree of precision 1, we have

$$k(q_i, y)(I - P_n)p_1(y) \ = \ 0$$

and because the quadrature formula (2.15) has degree of precision 2, it follows that

$$k_0 \int\limits_{\Delta_k} (I - P_n)p_2(y)dy \quad = \quad 0.$$

Then, we can write

$$\begin{aligned}
\left| \int\limits_{\Delta_k} k(q_i, y)(I - P_n)v^*(y)dy \right| \quad &\leq \quad c \left| \int\limits_{\Delta_k} k(q_i, y)(I - P_n)(v^* - p_2)(y)dy \right. \\
&+ \quad \int\limits_{\Delta_k} (k(q_i, y) - k_0)(I - P_n)(p_2 - p_1)(y)dy \\
&- \quad \left. \int\limits_{\Delta_k} k_0(I - P_n)p_1(y)dy \right| .
\end{aligned}$$

Now, using the bounds (3.5), (3.6), we get

$$\begin{aligned}
\max_{1 \leq i \leq n} |v^*(q_i) - v_n(q_i)| \quad &\leq \quad \max_{1 \leq i \leq n} |(I - P_n)(\mathcal{K}v^*)(q_i)| \\
&\leq \quad c\delta^3 \sum_{k=1}^{n} \int\limits_{\Delta_k} dy \\
&= \quad O(\delta^3) \cdot n \cdot \text{Area}(\Delta_k) \\
&= \quad O(\delta^3) \cdot O(\delta^{-2}) \cdot O(\delta^2) \quad = \quad O(\delta^3).
\end{aligned}$$

Then by Theorem 2.1, we also have

$$\max_{1 \leq i \leq n} |u^*(q_i) - u_n(q_i)| \leq O(\delta^3). \qquad \square$$

4. Numerical experiments

We will use the notation $x = (x_1, x_2), y = (y_1, y_2) \in D = [a, b] \times [c, d]$.

Example 4.1. Let us consider the integral equation

$$u(x_1, x_2) \quad = \quad \frac{1}{2} \int\limits_0^1 \int\limits_0^1 \frac{x_1^2 + 1}{y_1^2 + 2y_2^2} u^2(y_1, y_2)\ dy_1\ dy_2 + 2x_2^2, \tag{4.1}$$

for $(x_1, x_2) \in [0, 1] \times [0, 1]$, with exact solution $u^*(x_1, x_2) = x_1^2 + 2x_2^2 + 1$.

Here, we have

$$\begin{aligned}
k(x, y) \quad &= \quad k(x_1, x_2, y_1, y_2) \quad = \quad \frac{x_1^2 + 1}{y_1^2 + 2y_2^2}, \\
g(y, u(y)) \quad &= \quad \frac{1}{2} u^2(y), \\
f(x) \quad &= \quad f(x_1, x_2) \quad = \quad 2x_2^2
\end{aligned}$$

and $D = [0,1] \times [0,1]$.
We start with $n = 2$ triangles that cover D and then refine the triangulation as described earlier. We compute the errors

$$\begin{aligned} e_n(v) &= \max_{1 \le i \le n} |v^*(q_i) - v_n(q_i)| \text{ and} \\ e_n(u) &= \max_{1 \le i \le n} |u^*(q_i) - u_n(q_i)| \end{aligned}$$

Also, we look at the ratios

$$r_1 = \frac{e_n(v)}{e_{4n}(v)}, \quad r_2 = \frac{e_n(u)}{e_{4n}(u)}$$

from one iteration to the next. If indeed the numerical method has order of convergence $O(\delta^d)$, then these ratios should equal approximately 2^d.
We approximate the integrals needed for the coefficients of the nonlinear system (2.20) using tiled adaptive quadratures (function *integral2* in Matlab).

In Table 1, we give the errors $e_n(v)$ and $e_n(u)$, as well as the values $\log_2 r_1$ and $\log_2 r_2$, for each iteration.

n	$e_n(v)$	$\log_2 r_1$	$e_n(u)$	$\log_2 r_2$
2	$1.121e-1$		$2.382e-2$	
8	$1.874e-2$	2.58	$3.397e-3$	2.81
32	$2.582e-3$	2.86	$4.519e-4$	2.91
128	$3.296e-4$	2.97	$5.653e-5$	2.99

TABLE 1. Errors in Example 4.1

The table shows that both r_1 and r_2 approach the value 2^3, which is consistent with the superconvergence $O(\delta^3)$ proved in Theorem 3.2.

Example 4.2. Next, we consider the equation

$$u(x_1, x_2) = \frac{1}{8} \int_0^2 \int_0^2 e^{2x_1 + x_2} \left(y_1^2 - y_2\right) \ln\left(u(y_1, y_2)\right) dy_1\, dy_2, \tag{4.2}$$

for $(x_1, x_2) \in [0,2] \times [0,2]$, whose exact solution is $u^*(x_1, x_2) = e^{2x_1 + x_2}$.

In this case, $D = [0,2] \times [0,2]$ and we can take

$$\begin{aligned} k(x,y) &= \frac{1}{8} e^{2x_1 + x_2}, \\ g(y, u(y)) &= \ln\left(u(y)\right), \\ f(x) &= 0. \end{aligned}$$

Again, we start with $n = 2$ triangles and proceed as before.
The errors are shown in Table 2. The results show, again, an $O(\delta^3)$ rate of convergence.

n	$e_n(v)$	$\log_2 r_1$	$e_n(u)$	$\log_2 r_2$
2	$2.143e-1$		$3.628e-2$	
8	$3.344e-2$	2.68	$5.356e-3$	2.76
32	$4.735e-3$	2.82	$7.428e-4$	2.85
128	$6.086e-4$	2.96	$9.415e-5$	2.98

Table 2. Errors in Example 4.2

5. Conclusions

We have developed a collocation method for the approximate solution of nonlinear Hammerstein integral equations over bounded domains of $\mathbb{R}^2$. The numerical scheme is based on a special choice of linear spline interpolation over triangles. Having a higher degree of precision of the quadrature formula than expected, the resulting collocation method is superconvergent at the collocation nodes, converging faster than over the entire domain. This is one major advantage of the proposed numerical scheme over other projection-type collocation methods. Another important aspect is the fact that by applying collocation to a newly reformulated integral equation, the integrals needed for the coefficients of the linear system only have to be evaluated once, not at every iteration, which reduces the computational cost of the method and simplifies the implementation. Thus, this is a very efficient numerical method.

These ideas can be taken further, considering more complicated domains, higher dimensions or other types of interpolation.

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Kantorovich-type operators associated with a variant of Jain operators

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Dedicated to Professor Gheorghe Coman on the occasion of his 85th birthday, with high esteem.

Abstract. This note focuses on a sequence of linear positive operators of integral type in the sense of Kantorovich. The construction is based on a class of discrete operators representing a new variant of Jain operators. By our statements, we prove that the integral family turns out to be useful in approximating continuous signals defined on unbounded intervals. The main tools in obtaining these results are moduli of smoothness of first and second order, K-functional and Bohman-Korovkin criterion.

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1. Introduction

The starting point of the paper is a class of operators introduced by G.C. Jain [8]. The construction is based on a Poisson-type distribution with two parameters given by

$$w_\beta(k;\alpha) = \frac{\alpha}{k!}(\alpha + k\beta)^{k-1}e^{-(\alpha+k\beta)},\ k \in \mathbb{N}_0,$$

for $\alpha > 0$ and $|\beta| < 1$. With the help of Lagrange inversion, in [1, *Lemma 1*] was proved

$$\sum_{k=0}^{\infty} w_\beta(k;\alpha) = 1. \tag{1.1}$$

Considering $w_\beta(k;0) = \delta_{k,0}$, Kronecker's delta symbol, Jain defined the operators

$$(P_n^{[\beta]}f)(x) = \sum_{k=0}^{\infty} w_\beta(k;nx) f\left(\frac{k}{n}\right),\ x \geq 0, \tag{1.2}$$

where $\beta \in [0,1)$ and $f \in C(\mathbb{R}_+)$ whenever the above series is convergent. Here $C(\mathbb{R}_+)$ stands for the space of real-valued continuous functions defined on $\mathbb{R}_+ = [0,\infty)$. Clearly, $P_n^{[\beta]}$, $n \in \mathbb{N}$, are linear and positive operators.

In recent years, the investigation of these operators have been invigorated obtaining new properties as well as various generalizations. For a brief synthesis, [2] can be consulted.

Set $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$. We consider the functions $e_0(x) = 1$, $e_m(x) = x^m$, $x \geq 0$. The first three monomials represent the so-called Korovkin test functions, having an essential role in the study of the convergence of any sequence of linear positive operators towards the identity operator. For the operators defined by (1.2), the following identities

$$P_n^{[\beta]} e_0 = e_0,\ P_n^{[\beta]} e_1 = \frac{1}{1-\beta} e_1,\ P_n^{[\beta]} e_2 = \frac{1}{(1-\beta)^2} e_2 + \frac{1}{n(1-\beta)^3} e_1, \tag{1.3}$$

take place, see [8, *Eqs. (2.13)-(2.14)*]. We mention that using the Stirling numbers of the second kind, all $P_n^{[\beta]} e_j$ moments were explicitly calculated in [1, *Proposition 1*].

Many classical linear positive operators preserve e_0 and e_1, which implies that they have affine functions as fixed points. Such operators are also called Markov type. This property becomes useful in the study of the approximation properties which the operators enjoy. Pursuing this goal, in [7] the authors introduced and investigated the following variant of Jain operators

$$(D_n^{[\beta]} f)(x) = \sum_{k=0}^{\infty} w_\beta(k; u_n(x)) f\left(\frac{k}{n}\right),\ f \in C(\mathbb{R}_+),\ x \geq 0, \tag{1.4}$$

where $u_n(x) = n(1-\beta)x$, $x \geq 0$. The following identities

$$D_n^{[\beta]} e_0 = e_0,\ D_n^{[\beta]} e_1 = e_1,\ D_n^{[\beta]} e_2 = e_2 + \frac{1}{n(1-\beta)^2} e_1, \tag{1.5}$$

hold, see [7, *Lemma 2.1*].

We remind that for $\beta = 0$, $P_n^{[0]} = D_n^{[0]}$, $n \geq 1$, turn into well-known Szász-Mirakjan operators, see [14], [11].

The aim of this paper is to define an integral Kantorovich-type generalization of $D_n^{[\beta]}$, $n \geq 1$, operators and to establish some approximation properties. These will be achieved in the next two sections.

We specify that we wished the presentation to be self-contained to be accessible to a wide audience.

2. Integral type construction

Kantorovich-type constructions are based on replacing the values of the function f on the nodes k/n, $k \geq 0$, with average values of the function obtained by integrals on intervals of the form $I_{n,k} = \left[\frac{k}{n}, \frac{k+1}{n}\right]$, $k \geq 0$. The utility of this type of operators is given by the fact that such classes can approximate functions belonging to larger spaces.

The first approach in Kantorovich sense of $P_n^{[\beta]}$, $n \geq 1$, operators was achieved by Umar and Razi [15]. They defined and analyzed the operators given by the formula

$$(\widetilde{P}_n^{[\beta]} f)(x) = n \sum_{k=0}^{\infty} w_\beta(k; nx) \int_{k/n}^{(k+1)/n} f(t)dt, \tag{2.1}$$

where f is locally integrable function and the right hand side of relation (2.1) is finite.

Our proposal for an integral extension of the operators defined by (1.4) has the following form

$$(\widetilde{D}_n^{[\beta]} f)(x) = \frac{1}{\lambda_n} \sum_{k=0}^{\infty} w_\beta(k; u_n(x)) \int_{k\lambda_n}^{(k+1)\lambda_n} f(t)dt, \; x \geq 0, \tag{2.2}$$

where

(i) $(\lambda_n)_{n\geq 1}$ is a sequence of strictly decreasing positive numbers such that $\lim_{n\to\infty} \lambda_n = 0$,

(ii) f belongs to the space of integrable functions defined on $\mathbb{R}_+$ such that the series in (2.2) is absolutely convergent.

In the above construction we used a flexible net on $\mathbb{R}_+$ namely $(k\lambda_n)_{k\geq 0}$. The operators also admit the integral representation

$$(\widetilde{D}_n^{[\beta]} f)(x) = \int_0^{\infty} K_n^*(x,t) f(t)dt, \; x \geq 0,$$

with the kernel

$$K_n^*(x,t) = \frac{1}{\lambda_n} \sum_{k=0}^{\infty} w_\beta(k; u_n(x)) \chi_{n,k}(t),$$

where $\chi_{n,k}$ is the characteristic function of the interval $[k\lambda_n, (k+1)\lambda_n]$ with respect to $\mathbb{R}_+$, $k \geq 0$.

For particular case $\lambda_n = \dfrac{1}{n}$ and $u_n(x) := nx$, we reobtain the operators defined at (2.1). Further, if we choose $\beta = 0$ in (2.1), the operators turn into Szász-Mirakjan-Kantorovich operators introduced by Butzer [4, *Eq. (5)*]. Clearly, for each $n \in \mathbb{N}$, $\widetilde{D}_n^{[\beta]}$ is a linear positive operator. In what follows we establish some computational results.

Lemma 2.1. *Let $\widetilde{D}_n^{[\beta]}$, $n \in \mathbb{N}$, be the operators defined by (2.2). For each $n \in \mathbb{N}$ the following identities*

$$\widetilde{D}_n^{[\beta]} e_0 = e_0, \tag{2.3}$$

$$\widetilde{D}_n^{[\beta]} e_1 = n\lambda_n e_1 + \frac{1}{2}\lambda_n, \tag{2.4}$$

$$\widetilde{D}_n^{[\beta]} e_2 = (n\lambda_n)^2 e_2 + n\lambda_n^2 \left(1 + \frac{1}{(1-\beta)^2}\right) e_1 + \frac{1}{3}\lambda_n^2 \tag{2.5}$$

take place.

Proof. Taking in view the definition of $D_n^{[\beta]}$ operators and identity (1.1), we immediately deduce $\widetilde{D}_n^{[\beta]} e_0 = D_n^{[\beta]} e_0$, as well as the following identities

$$(\widetilde{D}_n^{[\beta]} e_1)(x) = n\lambda_n (D_n^{[\beta]} e_1)(x) + \frac{1}{2}\lambda_n (D_n^{[\beta]} e_0)(x),$$

$$(\widetilde{D}_n^{[\beta]} e_2)(x) = (n\lambda_n)^2 (D_n^{[\beta]} e_2)(x) + n\lambda_n^2 (D_n^{[\beta]} e_1)(x) + \frac{1}{3}\lambda_n^2 (D_n^{[\beta]} e_0)(x).$$

Using (1.5), the proof is ended. □

We indicate the first two central moments of the operators. Set $\varphi_x(t) = t - x$, $(t,x) \in \mathbb{R}_+ \times \mathbb{R}_+$.

$$\begin{aligned}(\widetilde{D}_n^{[\beta]} \varphi_x)(x) &= (n\lambda_n - 1)x + \frac{1}{2}\lambda_n,\\ (\widetilde{D}_n^{[\beta]} \varphi_x^2)(x) &= (n\lambda_n - 1)^2 x^2 + n\lambda_n^2 \left(1 + \frac{1}{(1-\beta)^2} - \frac{1}{n\lambda_n}\right) x + \frac{1}{3}\lambda_n^2.\end{aligned} \tag{2.6}$$

We recall the Bohman-Korovkin criterion. Briefly speaking, this theorem says: if a sequence of linear and positive operators approximates uniformly the test functions e_k, $k = \overline{0,2}$, then it approximates all continuous functions defined on a compact interval.

Remark 2.2. Based on Bohman-Korovkin theorem, studying relations (1.3) it can be observed that the sequence $(P_n^{[\beta]})_{n\geq 1}$ does not tend to the identity operator. To turn it into an approximation process we will proceed as follows. For each $n \in \mathbb{N}$, the constant β will be replaced by a number $\beta_n \in [0,1)$. If $\lim_{n\to\infty} \beta_n = 0$, then

$$\lim_{n\to\infty} (P_n^{[\beta_n]} e_j)(x) = e_j(x),\ j \in \{0,1,2\},$$

uniformly on any compact interval $K \subset \mathbb{R}_+$. Consequently,

$$\lim_{n\to\infty} (P_n^{[\beta_n]} f)(x) = f(x),\ \text{uniformly in } x \in K.$$

What is important to point out is that for the Kantorovich variant defined at (2.2) we no longer have to make this change on the β parameter, as will be seen in the next paragraph.

3. Approximation properties of $\widetilde{D}_n^{[\beta]}$ operators

We establish sufficient conditions for the sequence $(\widetilde{D}_n^{[\beta]})_{n\geq 1}$ to become an approximation process in a certain specified space.

Throughout the paragraph we use standard notations. Set $B(X)$ the Banach space of all real-valued bounded functions defined on X, endowed with the norm of the uniform convergence (briefly, sup-norm) defined by $\|f\| = \sup_{x\in X} |f(x)|$ for every $f \in B(X)$. Also, set $C_B(X) = C(X) \cap B(X)$, endowed with the sup-norm.

We mention that the operators $\widetilde{D}_n^{[\beta]}$, $n \geq 1$, are non expansive in the space $B(\mathbb{R}_+)$, this means

$$\|\widetilde{D}_n^{[\beta]} f\| \leq \|f\|, \tag{3.1}$$

for any local integrable function f belonging to $B(\mathbb{R}_+)$. As a consequence, for each $n \in \mathbb{N}$, $\widetilde{D}_n^{[\beta]}$ maps continuously $C_B(\mathbb{R}_+)$ into itself.

Theorem 3.1. *Let the operators $\widetilde{D}_n^{[\beta]}$, $n \in \mathbb{N}$, be defined by (2.2) such that the following condition*

$$\lim_{n\to\infty} n\lambda_n = 1 \tag{3.2}$$

is fulfilled. For any compact interval $K \subset \mathbb{R}_+$, the following relation

$$\lim_{n\to\infty} \widetilde{D}_n^{[\beta]} f = f \text{ uniformly on } K$$

occurs, provided f is continuous and bounded on $\mathbb{R}_+$.

Proof. A direct way to prove this result is to use a general result established by Altomare [3, *Theorem 4.1*] which says

Let X be a locally compact subset of $\mathbb{R}^d$, $d \geq 1$. Consider a lattice subspace E of $F(X)$ containing the set $\mathcal{T} = \left\{\mathbf{1}, pr_1, \ldots, pr_d, \sum_{i=1}^{d} pr_j^2\right\}$ and let $(L_n)_{n\geq 1}$ be a sequence of positive linear operators from E into $F(X)$ such that for every $g \in \mathcal{T}$,

$$\lim_{n\to\infty} L_n(g) = g \text{ uniformly on compact subsets of } X.$$

Then, for every $f \in E \cap C_B(X)$,

$$\lim_{n\to\infty} L_n(f) = f \text{ uniformly on compact subsets of } X.$$

In the above $F(X)$ stands for the linear space of all real-valued functions defined on X and the function $pr_j : \mathbb{R}^d \to \mathbb{R}$ indicates the j-th coordinate function,

$$pr_j(x) = x_j,\ x = (x_j)_{1\leq j\leq d} \in \mathbb{R}^d.$$

Applying the above result for $d = 1$, $X = \mathbb{R}_+$, $E = C(\mathbb{R}_+)$, the set $\mathcal{T}$ will consist of the test functions e_j, $j \in \{0, 1, 2\}$. The formulas (2.3)-(2.5) correlated with hypothesis (3.2) complete the proof. □

In order to obtain the error of approximation we use the modulus of continuity defined as follows

$$\begin{aligned}\omega_f(\delta) \equiv \omega(f;\delta) &= \sup\{|f(x') - f(x'')| :\ x', x'' \in \mathbb{R}_+,\ |x' - x''| \leq \delta\}\\ &= \sup_{0\leq h\leq\delta}\ \sup_{x\in\mathbb{R}_+} |f(x+h) - f(x)|,\end{aligned}$$

where $\delta \geq 0$ and $f \in B(\mathbb{R}_+)$.

Theorem 3.2. *Let the operators $\widetilde{D}_n^{[\beta]}$, $n \in \mathbb{N}$, be defined by (2.2). For any local integrable function defined on $\mathbb{R}_+$ belonging to $B(\mathbb{R}_+)$, we get*

$$|(\widetilde{D}_n^{[\beta]} f)(x) - f(x)| \leq \left(1 + \sqrt{\tau(x) + \lambda_n^2 \alpha_n^{-1}}\right) \omega\left(f; \sqrt{\alpha_n}\right),\ x \geq 0, \tag{3.3}$$

where

$$\tau(x) = \max_{x\geq 0}\{x, x^2\} \text{ and } \alpha_n = (n\lambda_n - 1)^2 + \left(1 + \frac{1}{(1-\beta)^2}\right) n\lambda_n^2,\ n \geq 1. \tag{3.4}$$

Proof. To achieve the statement, we appeal to an old result established by Shisha and Mond [13]: if T is a linear and positive operator, then one has

$$|(Tf)(x) - f(x)| \le |f(x)|\,|(Te_0)(x) - 1| + \left((Te_0)(x) + \frac{1}{\delta}\sqrt{(Te_0)(x)(T\varphi_x^2)(x)}\right)\omega(f;\delta),\ \ \delta > 0,$$

for every bounded function f. Using this inequality for $\widetilde{D}_n^{[\beta]}$ operators, we take into account relation (2.3). Based on (2.6), we can write

$$(\widetilde{D}_n^{[\beta]}\varphi_x^2)(x) \le \alpha_n\tau(x) + \lambda_n^2,\ \ n \ge 1. \tag{3.5}$$

Choosing $\delta := \sqrt{\alpha_n}$ we arrive at (3.3) and the proof is over. □

Remark 3.3. Let us suppose that f is uniformly continuous on $\mathbb{R}_+$. In this case it is known that $\lim\limits_{\delta\to 0^+}\omega(f;\delta) = 0$, see, e.g., the monograph [5, *page 40*]. If we request that the condition (3.2) to be fulfilled, then relation (3.3) leads to the fact that $((\widetilde{D}_n^{[\beta]}f)(x))_{n\ge 0}$ is pointwise convergent to $f(x)$ for any $x \in \mathbb{R}_+$. Also, (3.2) guarantees that the upper-bound for the error of approximation has the magnitude $\mathcal{O}\,(1/\sqrt{n})$.

As a special case we indicate the rate of convergence of our operators by means of the elements of γ-Hölder continuous class

$$Lip_M(\gamma) = \{f : \mathbb{R}_+ \to \mathbb{R} \mid\ |f(t) - f(x)| \le M|t-x|^\gamma,\ (t,x) \in \mathbb{R}_+ \times \mathbb{R}_+\}, \tag{3.6}$$

where $0 < \gamma \le 1$ and M is a nonnegative constant independent of f.

Theorem 3.4. *Let the operators* $\widetilde{D}_n^{[\beta]}$, $n \in \mathbb{N}$, *be defined by (2.2). For any function* $f \in Lip_M(\gamma)$ *the following inequality*

$$|(\widetilde{D}_n^{[\beta]}f)(x) - f(x)| \le M(\alpha_n\tau(x) + \lambda_n^2)^{\gamma/2},\ \ x \ge 0, \tag{3.7}$$

holds, where τ *and* α_n *are defined at (3.4).*

Proof. Considering relations (2.3) and (3.6) we can write

$$|(\widetilde{D}_n^{[\beta]}f)(x) - f(x)| \le \widetilde{D}_n^{[\beta]}(|f - f(x)|;x) \le M\widetilde{D}_n^{[\beta]}(|\varphi_x|^\gamma;x). \tag{3.8}$$

At this point we apply Hölder inequalities with conjugate numbers

$$p := 2/\gamma,\ \ q := 2/(2-\gamma)$$

inferring

$$\int_{k\lambda_n}^{(k+1)\lambda_n}|\varphi_x(t)|^\gamma dt \le \lambda_n^{\frac{2-\gamma}{2}}\left(\int_{k\lambda_n}^{(k+1)\lambda_n}\varphi_x^2(t)dt\right)^{\gamma/2}.$$

Returning at (3.8) we get

$$\begin{aligned}|(\widetilde{D}_n^{[\beta]}f)(x) - f(x)| &\le M\sum_{k=0}^{\infty} w_\beta(k;u_n(x))\left(\frac{1}{\lambda_n}\int_{k\lambda_n}^{(k+1)\lambda_n}\varphi_x^2(t)dt\right)^{\gamma/2}\\ &\le M(\widetilde{D}_n^{[\beta]}\varphi_x^2)^{\gamma/2}(x).\end{aligned}$$

Using the inequality (3.5) we reach (3.7) and the proof is complete. □

We can also consider functions satisfying another Lipschitz type condition defined by Szász [14, *Eq. (8)*]. Set

$$Lip_M^*(\gamma) = \left\{ f : \mathbb{R}_+ \to \mathbb{R} \mid \ |f(t) - f(x)| \leq M \frac{|t-x|^\gamma}{(t+x)^{\gamma/2}},\ t>0,\ x>0 \right\},$$

where $0 < \gamma \leq 1$ and M is a nonnegative constant independent of f.

Since $(t+x)^{-\gamma/2} < x^{-\gamma/2}$, for $t > 0$ and $x > 0$, following a demonstration path similar to that indicated in Theorem 3.4, we can state

Remark 3.5. For any function $f \in Lip_M^*(\gamma)$ it takes place

$$|(\widetilde{D}_n^{[\beta]} f)(x) - f(x)| \leq \frac{M}{x^{\gamma/2}} (\alpha_n \tau(x) + \lambda_n^2)^{\gamma/2},\ x > 0,$$

where τ and α_n are defined at (3.4).

We focus on establishing the degree of approximation in terms of the second modulus of smoothness $\omega_2(f;\cdot)$ of a function $f \in C_B(\mathbb{R}_+)$. It is given as follows

$$\omega_2(f;\delta) = \sup_{0\leq h\leq\delta} \sup_{x\geq 0} |f(x+2h) - 2f(x+h) + f(x)|,\ \delta \geq 0.$$

Also, a subtle measurement of the error of approximation is provided by K-functional introduced by Peetre [12]. If X_0, X_1 are two Banach spaces with X_1 continuously embedded in X_0, $X_1 \hookrightarrow X_0$, the K-functional is defined for each $f \in X_0$ by the formula

$$K(f,\delta;X_0,X_1) \equiv K(f;\delta) = \inf_{g\in X_1} (\|f-g\|_{X_0} + \delta\|g\|_{X_1}),\ \delta > 0.$$

This quantity describes properties of approximation of $f \in X_0$. More detailed, the inequality $K(f;\delta) < \varepsilon$ for $\delta > 0$ implies that f can be approximated with the error $\|f-g\|_{X_0} < \varepsilon$ in X_0 by an element $g \in X_1$ whose norm is not too large, namely $\|g\|_{X_1} < \varepsilon\delta^{-1}$. For our purpose, we choose

$$X_0 = C_B(\mathbb{R}_+),\ X_1 = C_B^2(\mathbb{R}_+) = \{g \in C_B(\mathbb{R}_+) :\ g', g'' \in C_B(\mathbb{R}_+)\},$$

both spaces being endowed with the sup-norm $\|\cdot\|$. Thus, we will use

$$K(f;\delta) = \inf_{g\in C_B^2(\mathbb{R}_+)} (\|f-g\| + \delta\|g''\|).$$

Based for example on [10, *Proposition 6.1*], between ω_2 and K-functional the following relations hold: the positive constants c_1 and c_2 exist such that

$$c_1\omega_2(f;\delta) \leq K(f;\delta^2) \leq c_2\omega_2(f;\delta),\ \delta > 0. \tag{3.9}$$

Theorem 3.6. *Let the operators* $\widetilde{D}_n^{[\beta]}$, $n \in \mathbb{N}$, *be defined by (2.2).*

For every $f \in C_B(\mathbb{R}_+)$ *the following inequality*

$$|(\widetilde{D}_n^{[\beta]} f)(x) - f(x)| \leq M\omega_2(f;\delta_n(x)) + \omega\left(f;(n\lambda_n - 1)x + \frac{1}{2}\lambda_n\right),\ x \geq 0, \tag{3.10}$$

holds, where M *is a constant independent of* f *and*

$$\delta_n(x) = \frac{1}{2}\left(\alpha_n\tau(x) + \lambda_n^2 + \left((n\lambda_n - 1)x + \frac{1}{2}\lambda_n\right)^2\right)^{1/2}.$$

The quantities α_n and τ are defined by (3.4).

Proof. Our approach follows a route similar to the proof that appears in [6, *Theorem 3.2*] aiming at a Kantorovich modification for Szász-Mirakjan operators based on Jain and Pethe operators [9].

At first we define the operators $E_n^{[\beta]} : C_B(\mathbb{R}_+) \to C(\mathbb{R}_+)$,

$$(E_n^{[\beta]} f)(x) = (\widetilde{D}_n^{[\beta]} f)(x) - f\left(n\lambda_n x + \frac{1}{2}\lambda_n\right) + f(x). \tag{3.11}$$

Using relations (2.3) and (2.4), obviously $E_n^{[\beta]} e_k = e_k$ for $k \in \{0, 1\}$. Therefore, the first central moment $E_n^{[\beta]}\varphi_x$ is null. Since $\widetilde{D}_n^{[\beta]}$ verifies (3.1), we have

$$|(E_n^{[\beta]} h)(x)| \le 3\|f\|, \;\; x \ge 0, \tag{3.12}$$

for any $h \in C_B(\mathbb{R}_+)$. Let $g \in C_B^2(\mathbb{R}_+)$ be arbitrarily chosen. By Taylor's expansion with integral form of the remainder, we get

$$g(t) = g(x) + (t-x)g'(x) + \int_x^t (t-u)g''(u)du,$$

for $t \ge 0$ and $x \ge 0$. Applying $E_n^{[\beta]}$ on both sides, we can write successively

$$\begin{aligned}(E_n^{[\beta]} g)(x) - g(x) &= g'(x)(E_n^{[\beta]}\varphi_x)(x) + E_n^{[\beta]}\left(\int_{xe_0}^{e_1} (e_1 - ue_0)g''(u)du; x\right)\\ &= \widetilde{D}_n^{[\beta]}\left(\int_{xe_0}^{e_1} (e_1 - ue_0)g''(u)du; x\right)\\ &\quad - \int_x^{n\lambda_n x + \frac{1}{2}\lambda_n} \left(n\lambda_n x + \frac{1}{2}\lambda_n - u\right) g''(u)du.\end{aligned}$$

In the above we used (3.11). Considering the increase

$$\left|\int_x^t \varphi_x(u)g''(u)du\right| \le \|g''\| \left|\int_x^t |u-x|du\right| \le \|g''\|\varphi_x^2(t),$$

it allows us to write

$$|(E_n^{[\beta]} g)(x) - g(x)| \le \|g''\| \left((\widetilde{D}_n^{[\beta]}\varphi_x^2)(x) + \left((n\lambda_n - 1)x + \frac{1}{2}\lambda_n\right)^2\right).$$

Returning at (3.11), with the help of (3.12), definition of modulus of continuity $\omega(f;\cdot)$ and (3.5), we get

$$\begin{aligned}&|(\widetilde{D}_n^{[\beta]} f)(x) - f(x)|\\ &\le |E_n^{[\beta]}(f-g;x)| + |(E_n^{[\beta]} g)(x) - g(x)| + |g(x) - f(x)| + \left|f\left(n\lambda_n x + \frac{1}{2}\lambda_n\right) - f(x)\right|\\ &\le 4\|f-g\| + \left((\widetilde{D}_n^{[\beta]}\varphi_x^2)(x) + \left((n\lambda_n - 1)x + \frac{1}{2}\lambda_n\right)^2\right)\|g''\| + \omega\left(f; (n\lambda_n - 1)x + \frac{1}{2}\lambda_n\right)\\ &\le 4\|f-g\| + \left(\alpha_n\tau(x) + \lambda_n^2 + \left((n\lambda_n - 1)x + \frac{1}{2}\lambda_n\right)^2\right)\|g''\| + \omega\left(f; (n\lambda_n - 1)x + \frac{1}{2}\lambda_n\right).\end{aligned}$$

Taking infimum with respect to all $g \in C_B^2(\mathbb{R}_+)$ and using (3.9) we arrive at (3.10) which concludes the proof. □

Remark 3.7. Based on the fact that $\lim_{n\to\infty} \lambda_n = 0$ and relation (3.2) takes place, we deduce $\lim_{n\to\infty} \delta_n(x) = 0$ for any $x \geq 0$.

Conclusion. In this article we propose an integral version in Kantorovich sense of a family of discrete operators recently obtained from genuine Jain operators. The proposed construction involves sub-intervals of the form $[k\lambda_n, (k+1)\lambda_n]$, $k \geq 0$, these being used in several previous studies. A notable aspect in the fact that the proposed integral variant is an approximation process for any fixed parameter β belonging to the interval $[0, 1)$. In order to have the same quality, in the discrete operators β has to be replaced with a sequence of parameters $(\beta_n)_{n\geq 1}$ such that $\lim_{n\to\infty} \beta_n = 0$. For the newly created sequence of linear positive operators, the highlighted approximation properties involve locally integrable functions in different functions spaces. Upper bounds of approximation error have been established using the first and second order modulus of smoothness.

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Sharp inequalities for the rates of convergence of the iterates of some operators which preserve the constants

Marius Mihai Birou

Dedicated to Professor Gheorghe Coman on the occasion of his 85th anniversary.

Abstract. In this paper we give estimates for the rates of convergence for the iterates of some positive linear operators which preserve only the constants. We obtain sharp inequalities when we use both continuous functions and differentiable functions. We present some optimal results for the Cesaro, Stancu and Schurer operators.

Mathematics Subject Classification (2010): 41A36, 41A25.

Keywords: Positive linear operators, iterates, convergence, sharp inequalities.

1. Introduction

Starting with the articles [9] and [8] of R.P. Kelisky, T.J. Rivlin and respectively S. Karlin, Z. Ziegler, the iterates of the positive linear operators were intensively studied.

The convergence of the sequence of the iterates of some positive linear operators which preserve only the constants was proved in [3], [14], [7], [13], [15], [4], [5], [6], [2].

On the other hand, estimations of the rates of convergence for the iterates of some positive operators preserving the constants were given in [10] using moduli of smoothness. In [1] the authors got sharp inequalities for the iterates of the Bernstein operators. In [12] the author obtained an estimate of the convergence rate for the iterations of linear and positive operators that reproduce linear functions in the case of differentiable functions.

In this note we obtain inequalities for the rates of convergence of the iterates of some positive linear operators $L : C[a, b] \to C[a, b]$ which preserve only the constants and have the interpolation point $x = a$ or $x = b$. In Section 2 we get these estimations both for continuous functions (using moduli of smoothness and divided difference) and for differentiable functions. The inequalities (2.1), (2.5), (2.6), (2.8), (2.9) and (2.12) are sharp in sense that we get equality if we take $f = e_1$. In Section 3 we determine the best constants in some inequalities involving the iterates of Cesaro, Stancu and Schurer operators.

Throughout the paper we use the following notations and definitions:

• the the monomial functions: $e_i : [a, b] \to \mathbb{R}$, $e_i(x) = x^i$, $i = 0, 1, \dots$;

• the first and and the second moduli of smoothness of the functiom $f \in C[a, b]$:

$$\omega_1(f, \delta) = \sup \{f(x + h) - f(x) : x, x + h \in [a, b],\ 0 \le h \le \delta\},$$

and respectively

$$\omega_2(f, \delta) = \sup \{f(x + h) - 2f(x) + f(x - h) : x, x \pm h \in [a, b],\ 0 \le h \le \delta\},$$

where $\delta \ge 0$,

• the divided difference of the function $f \in C[a, b]$ on the distinct points $x_1, x_2 \in [a, b]$:

$$[x_1, x_2; f] = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

2. Main results

Theorem 2.1. *Let $L : C[a, b] \to C[a, b]$ be a positive linear operator which preserves only the constants and has interpolation point $x = a$. If*

$$L^k e_1(x) > a,\ x \in (a, b],$$

then we have, for every $f \in C[a, b]$ and $x \in [a, b]$,

$$\left|L^k f(x) - f(a)\right| \le \frac{4}{b - a}\lambda_k(x)\omega_1(f, \lambda_k(x)) + 3\omega_2(f, \lambda_k(x)), \tag{2.1}$$

where

$$\lambda_k(x) = \frac{1}{2}\sqrt{(b - a)(L^k e_1(x) - a)}. \tag{2.2}$$

Proof. Let $f \in C[a, b]$ and $0 < \delta \le (b - a)/2$. If F is a positive linear functional on $C[a, b]$, then from the optimal result of Păltănea [11] we have:

$$|f(x) - F(f)| \le f(x)|F(e_0) - 1| + \frac{1}{\delta}|F(e_1 - xe_0)|\omega_1(f, \delta) \tag{2.3}$$

$$+ \left(F(e_0) + \frac{1}{2\delta^2}F(e_1 - xe_0)^2\right)\omega_2(f, \delta),\ x \in [a, b].$$

Taking $F(f) = f(a)$ we get

$$|f - f(a)| \le \frac{e_1 - ae_0}{\delta}\omega_1(f, \delta) + \left(e_0 + \frac{(e_1 - ae_0)^2}{2\delta^2}\right)\omega_2(f, \delta)$$

$$\le \frac{e_1 - ae_0}{\delta}\omega_1(f, \delta) + \left(e_0 + \frac{(b - a)(e_1 - ae_0)}{2\delta^2}\right)\omega_2(f, \delta).$$

Since L preserves the constant functions, it follows that

$$\left|L^k f - f(a)\right| \le \frac{1}{\delta}\left(L^k e_1 - a e_0\right)\omega_1(f,\delta) + \left(e_0 + \frac{(b-a)(L^k e_1 - a e_0)}{2\delta^2}\right)\omega_2(f,\delta). \tag{2.4}$$

If we take in (2.4)

$$\delta = \lambda_k(x),\ x \in (a,b],$$

where λ_k is given by (2.2) we get that (2.1) holds for $x \in (a,b]$.
For $x = a$, due the interpolation property of L, we have $L^k f(a) = f(a)$. Therefore (2.1) is also true for $x = a$. This completes the proof. □

Theorem 2.2. *Let $L : C[a,b] \to C[a,b]$ be a positive linear operator which preserves only the constants and has interpolation point $x = b$. If*

$$L^k e_1(x) < b,\ x \in [a,b),$$

then we have, for every $f \in C[a,b]$ and $x \in [a,b]$,

$$\left|L^k f(x) - f(b)\right| \le \frac{4}{b-a}\mu_k(x)\omega_1(f,\mu_k(x)) + 3\omega_2(f,\mu_k(x)), \tag{2.5}$$

where

$$\mu_k(x) = \frac{1}{2}\sqrt{(b-a)(b - L^k e_1(x))}.$$

Proof. Taking $F(f) = f(b)$ in (2.3) we get

$$\begin{aligned}|f - f(b)| &\le \frac{be_0 - e_1}{\delta}\omega_1(f,\delta) + \left(e_0 + \frac{(be_0 - e_1)^2}{2\delta^2}\right)\omega_2(f,\delta)\\ &\le \frac{be_0 - e_1}{\delta}\omega_1(f,\delta) + \left(e_0 + \frac{(b-a)(be_0 - e_1)}{2\delta^2}\right)\omega_2(f,\delta).\end{aligned}$$

The conclusion follows analogous as in Theorem 2.1. □

Theorem 2.3. *Let $L : C[a,b] \to C[a,b]$ be a positive linear operator which preserves constants and has the interpolation point $x = a$. Then, for every $f \in C[a,b]$ and $x \in [a,b]$ we have*

$$m_a(L^k(e_1)(x) - a) \le L^k(f)(x) - f(a) \le M_a(L^k(e_1)(x) - a), \tag{2.6}$$

where $m_a, M_a \in \mathbb{R}$ such that $m_a \le [a,t;f] \le M_a$ when $t \in (a,b]$.

Proof. We have

$$f(x) - f(a) = \begin{cases} [a,x;f](x-a), & x \in (a,b] \\ 0, & x = a \end{cases}$$

It follows

$$m_a(e_1 - a) \le f - f(a) \le M_a(e_1 - a). \tag{2.7}$$

Applying k times the operator L on (2.7) we get the conclusion. □

Remark 2.4. From Theorem 2.3 we get the following criterion for the convergence of the iterates (see also [5, Corolar 2]): if $L : C[a,b] \to C[a,b]$ is a positive linear operator which preserves the constants, has the interpolation point $x = a$ and satisfies the condition

$$\lim_{k\to\infty} L^k e_1 = a, \text{ uniformly on } [a,b],$$

then for every $f \in C[a,b]$ we have

$$\lim_{k\to\infty} L^k f = f(a), \text{ uniformly on } [a,b].$$

Theorem 2.5. *Let $L : C[a,b] \to C[a,b]$ be a positive linear operator which preserves constants and has the interpolation point $x = b$. Then, for every $f \in C[a,b]$ and $x \in [a,b]$ we have*

$$m_b(b - L^k(e_1)(x)) \leq f(b) - L^k(f)(x) \leq M_b(b - L^k(e_1)(x)), \tag{2.8}$$

where $m_b, M_b \in \mathbb{R}$ such that $m_b \leq [t,b;f] \leq M_b$ for every $t \in [a,b)$.

The proof follows analogous with that of Theorem 2.3 using the formula

$$f(b) - f(x) = \begin{cases} [x,b;f](b-x), & x \in [a,b) \\ 0, & x = b \end{cases}$$

Remark 2.6. From Theorem 2.5 we get the following criterion for the convergence of the iterates: if $L : C[a,b] \to C[a,b]$ is a positive linear operator which preserves the constants, has the interpolation point $x = b$ and satisfies the condition

$$\lim_{k\to\infty} L^k e_1 = b, \text{ uniformly on } [a,b],$$

then for every $f \in C[a,b]$ we have

$$\lim_{k\to\infty} L^k f = f(b), \quad \text{uniformly on } [a,b].$$

Theorem 2.7. *Let $L : C[a,b] \to C[a,b]$ be a positive linear operator which preserves constants and has the interpolation point $x = a$. Then, for every $f \in C^1[a,b]$ and $x \in [a,b]$ we have*

$$m'(L^k(e_1)(x) - a) \leq L^k(f)(x) - f(a) \leq M'(L^k(e_1)(x) - a), \tag{2.9}$$

where $m', M' \in \mathbb{R}$ such that $m' \leq f'(t) \leq M'$, $t \in [a,b]$ and

$$\left|L^k(f)(x) - f(a)\right| \leq \overline{M'}(L^k(e_1)(x) - a),$$

where $\overline{M'} = \max_{t\in[a,b]} |f'(t)|$.

Proof. If $x \in (a,b]$, then using the mean value theorem it follows that there exists $\xi \in (a,x)$ such that

$$f(x) - f(a) = (x-a)f'(\xi). \tag{2.10}$$

If $x = a$ the formula (2.10) also holds for every $\xi \in [a,b]$.
Therefore

$$m'(e_1 - a) \leq f - f(a) \leq M'(e_1 - a). \tag{2.11}$$

Applying k times the operator L on (2.11) we get (2.9). The proof is ended. ☐

Theorem 2.8. *Let $L : C[a,b] \to C[a,b]$ be a positive linear operator which preserves constants and has the interpolation point $x = b$. Then, for every $f \in C^1[a,b]$ and $x \in [a,b]$ we have*

$$m'(b - L^k(e_1)(x)) \leq f(b) - L^k(f)(x) \leq M'(b - L^k(e_1)(x)), \tag{2.12}$$

where $m', M' \in \mathbb{R}$ such that $m' \leq f'(t) \leq M'$, $t \in [a,b]$ and

$$\left|L^k(f)(x) - f(b)\right| \leq \overline{M'}(b - L^k(e_1)(x)),$$

where $\overline{M'} = \max_{t\in[a,b]} |f'(t)|$.

The proof follows analogous with that of Theorem 2.7 using the mean value theorem:

$$f(b) - f(x) = (b - x)f'(\xi),\ \xi \in (a,b).$$

3. Applications

We consider the following positive linear operators which preserve only the constants:

- Cesaro operator

$$C : C[0,1] \to C[0,1],\ C(f)(x) = \begin{cases} f(0), & x = 0 \\ \dfrac{1}{x}\displaystyle\int_0^x f(t)dt, & x > 0 \end{cases},\ x \in [0,1]$$

- Bernstein-Stancu operators (see [16])

$$S_{n,\alpha} : C[0,1] \to C[0,1],\ S_{n,\alpha}(f)(x) = \sum_{i=0}^{n} \binom{n}{i} x^i (1-x)^{n-i} f\left(\frac{i+\alpha}{n+\alpha}\right),$$

$$x \in [0,1],\ n = 0,1,\ldots,\ \alpha > 0,$$

and

$$S_{n,\beta} : C[0,1] \to C[0,1],\ S_{n,\beta}(f)(x) = \sum_{i=0}^{n} \binom{n}{i} x^i (1-x)^{n-i} f\left(\frac{i}{n+\beta}\right),$$

$$x \in [0,1],\ n = 0,1,\ldots,\ \beta > 0,$$

- Schurer operator

$$S_{n,p} : C[0,1] \to C[0,1],\ S_{n,p}(f)(x) = \sum_{i=0}^{n-p} \binom{n-p}{i} x^i (1-x)^{n-p-i} f\left(\frac{i}{n}\right),$$

$$x \in [0,1],\ n,p \in \mathbb{N},\ n \geq p.$$

The operators C, $S_{n,\beta}$, $S_{n,p}$ have the interpolation point $x = 0$ while the operator $S_{n,\alpha}$ interpolates the continuous functions at $x = 1$. For every $k \geq 0$ we have by induction (see also [5] for the operators C, $S_{n,\beta}$, $S_{n,p}$):

$$C^k e_1 = \frac{1}{2^k} e_1,\ S^k_{n,\alpha} e_1 = \left(\frac{n}{n+\beta}\right)^k e_1, S^k_{n,p} e_1 = \left(\frac{n-p}{n}\right)^k e_1,$$

$$S^k_{n,\alpha}e_1 = e_0 + \left(\frac{n}{n+\alpha}\right)^k (e_1 - e_0).$$

From Theorem 2.1 and Theorem 2.2 we have:

Theorem 3.1. *For every $f \in C[0,1]$ and $x \in [0,1]$ we have:*

1. $$\left|C^k f(x) - f(0)\right| \le 2\sqrt{\frac{x}{2^k}} \cdot \omega_1\left(f, \frac{1}{2}\sqrt{\frac{x}{2^k}}\right) + 3\omega_2\left(f, \frac{1}{2}\sqrt{\frac{x}{2^k}}\right),$$
2. $$\left|S^k_{n,\alpha} f(x) - f(0)\right| \le$$ $$2\sqrt{\left(\frac{n}{n+\beta}\right)^k x} \cdot \omega_1\left(f, \frac{1}{2}\sqrt{\left(\frac{n}{n+\beta}\right)^k x}\right) + 3\omega_2\left(f, \frac{1}{2}\sqrt{\left(\frac{n}{n+\beta}\right)^k x}\right),$$
3. $$\left|S^k_{n,p} f(x) - f(0)\right| \le$$ $$2\sqrt{\left(\frac{n-p}{n}\right)^k x} \cdot \omega_1\left(f, \frac{1}{2}\sqrt{\left(\frac{n-p}{n}\right)^k x}\right) + 3\omega_2\left(f, \frac{1}{2}\sqrt{\left(\frac{n-p}{n}\right)^k x}\right),$$
4. $$\left|S^k_{n,\beta} f(x) - f(1)\right| \le$$ $$2\sqrt{\left(\frac{n}{n+\alpha}\right)^k x} \cdot \omega_1\left(f, \frac{1}{2}\sqrt{\left(\frac{n}{n+\alpha}\right)^k x}\right) + 3\omega_2\left(f, \frac{1}{2}\sqrt{\left(\frac{n}{n+\alpha}\right)^k x}\right).$$

Using Theorem 2.3, Theorem 2.5 and respectively Theorem 2.7, Theorem 2.8 we get the following sharp estimates:

Theorem 3.2. *Let $f \in C[0,1]$. If $m_0, M_0, m_1, M_1 \in \mathbb{R}$ such that $m_0 \le [0,t;f] \le M_0$, $t \in (0,1]$ and $m_1 \le [t,1;f] \le M_1$, $t \in [0,1)$ then for every $k \ge 0$ we have:*

1. *$m_0 c_1(k) e_1 \le C^k(f) - f(0)e_0 \le M_0 c_1(k) e_1$, where $c_1(k) = \frac{1}{2^k}$,*
2. *$m_0 c_2(k,n,\beta) e_1 \le S^k_{n,\beta}(f) - f(0)e_0 \le M_0 c_2(k,n,\beta) e_1$, where $c_2(k,n,\beta) = \left(\frac{n}{n+\beta}\right)^k$,*
3. *$m_0 c_3(k,n,p) e_1 \le S^k_{n,p}(f) - f(0)e_0 \le M_0 c_3(k,n,p) e_1$, where $c_3(k,n,p) = \left(\frac{n-p}{n}\right)^k$,*
4. *$m_1 c_4(k,n,\alpha)(e_0 - e_1) \le f(1)e_0 - S^k_{n,\alpha}(f) \le M_1 c_4(k,n,\alpha)(e_0 - e_1)$, where $c_4(k,n,\alpha) = \left(\frac{n}{n+\alpha}\right)^k$.*

Theorem 3.3. *Let $f \in C^1[0,1]$. If $m', M' \in \mathbb{R}$ such that $m' \le f'(t) \le M'$, $t \in [0,1]$, then for every $k \ge 0$ we have:*

1. *$m' c_1(k) e_1 \le C^k(f) - f(0)e_0 \le M' c_1(k) e_1$,*
2. *$m' c_2(k,n,\beta) e_1 \le S^k_{n,\beta}(f) - f(0)e_0 \le M' c_2(k,n,\beta) e_1$,*
3. *$m' c_3(k,n,p) e_1 \le S^k_{n,p}(f) - f(0)e_0 \le M' c_3(k,n,p) e_1$,*
4. *$m' c_4(k,n,\alpha)(e_0 - e_1) \le f(1)e_0 - S^k_{n,\alpha}(f) \le M' c_4(k,n,\alpha)(e_0 - e_1)$,*

where the constants $c_1(k)$, $c_2(k,n,\beta)$, $c_3(k,n,p)$, $c_4(k,n,\alpha)$ are given in Theorem 3.2.

The constants $c_1(k)$, $c_2(k,n,\beta)$, $c_3(k,n,p)$, $c_4(k,n,\alpha)$ in Theorem 3.2 and Theorem 3.3 are the best possible: for $f = e_1$ we get equality.

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Barbashin conditions for uniform instability of evolution operators

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Dedicated to Professor Gheorghe Coman on the occasion of his 85th anniversary.

Abstract. The aim of the present paper is to give some characterization theorems of Barbashin type for the uniform exponential instability and uniform polynomial instability behavior of evolution operators. Also, some examples which illustrate the connections between the concepts presented are given.

Mathematics Subject Classification (2010): 47B01, 34D05.

Keywords: Evolution operator, uniform instability, Barbashin conditions.

1. Introduction

In the last period significant progress has been made in the study of exponential stability, dichotomy and trichotomy in Banach spaces. A great number of papers that describe the asymptotic behavior of evolution operators in the exponential case was published, see for example [7], [8], [9], [11] and the references therein. In particular, the uniform exponential instability was studied in [5], [4], [12], [13], [15].

Later, the need for a new approach arose from the fact that in some situations, in particular for nonautonomous systems, the exponential stability is too stringent. In this sense a polynomial asymptotic behavior was introduced by L. Barreira and C. Valls ([2]) for the continuous case, respectively by A.J.G. Bento and C.M.Silva ([3]) for discrete-time systems.

Also, another interesting idea in this area can be found in [10] where A.L. Sasu, M. Megan and B. Sasu give some theorems of characterization for the concept of uniform exponential instability in terms of Banach function spaces. Recently, the same authors proposed in [14] an overview in the framework of Banach sequence spaces and their applications in the asymptotic theory of variational equations.

In this paper we focus on the concepts of uniform exponential instability, $*$-uniform exponential instability, uniform polynomial instability and $*$-uniform polynomial instability for evolution operators in Banach spaces.

We obtain some characterization theorems of Barbashin type ([1]) for the concepts mentioned above, assuming that the evolution operator has exponential decay, $*$-exponential decay, respectively polynomial decay, $*$-polynomial decay. Also, we establish the connections between the notions defined in the paper and the decay properties, by giving some illustrative examples in this sense.

2. Preliminaries

Let X be a real or complex Banach space and X^* its dual space. We denote by $\mathcal{B}(X)$ the Banach algebra of all bounded linear operators acting on X. We denote by I the identity operator and the norms on X, X^* and on $\mathcal{B}(X)$ will be denoted by $\|.\|$. By Δ and T we will denote the following sets

$$\Delta = \{(t,s) \in \mathbb{R}_+^2 : t \geq s\}, \qquad T = \{(t,s,t_0) \in \mathbb{R}_+^3 : t \geq s \geq t_0\}.$$

Definition 2.1. An application $U : \Delta \to \mathcal{B}(X)$ is said to be an *evolution operator* on X if the following relations are satisfied:

(e_1) $U(t,t) = I$ for all $t \geq 0$

(e_2) $U(t,s)U(s,t_0) = U(t,t_0)$ for all $(t,s,t_0) \in T$. In addition,

(e_3) if for all $(t,s) \in \Delta$ the linear operator $U(t,s)$ is bijective then we say that the evolution operator U is reversible.

In this case, we denote by $V : \Delta \to \mathcal{B}(X)$ the inverse of the evolution operator U, which means that $V(t,s) = U(t,s)^{-1}$.

Remark 2.2. If $U : \Delta \to \mathcal{B}(X)$ is a reversible evolution operator, then the following properties hold:

(i) $V(t,t) = I$ for all $t \geq 0$

(ii) $V(t,t_0) = V(s,t_0)V(t,s)$ for all $(t,s,t_0) \in T$.

Definition 2.3. An evolution operator $U : \Delta \to \mathcal{B}(X)$ is said to be *strongly measurable* if for all $(s,x) \in \mathbb{R}_+ \times X$, the mapping $t \mapsto \|U(t,s)x\|$ is measurable on $[s,\infty)$.

Definition 2.4. An evolution operator $U : \Delta \to \mathcal{B}(X)$ is said to be $*$-*strongly measurable* if for all $(s,x^*) \in \mathbb{R}_+ \times X^*$, the mapping $t \mapsto \|U(t,s)^*x^*\|$ is measurable on $[0,t)$.

Definition 2.5. The evolution operator $U : \Delta \to \mathcal{B}(X)$ *has uniform exponential decay (u.e.d.)* if there exist two constants $M \geq 1$ and $\omega > 0$ such that:

$$\|x\| \leq Me^{\omega(t-s)}\|U(t,s)x\|, \text{ for all } (t,s,x) \in \Delta \times X.$$

Definition 2.6. The evolution operator $U : \Delta \to \mathcal{B}(X)$ is said to be *uniformly exponentially instable (u.e.is.)* if there exist $N \geq 1$ and $\nu > 0$ such that:

$$\|x\| \leq Ne^{-\nu(t-s)}\|U(t,s)x\|, \text{ for all } (t,s,x) \in \Delta \times X.$$

Definition 2.7. The evolution operator $U : \Delta \to \mathcal{B}(X)$ *has uniform exponential growth (u.e.g.)* if there exist two constants $M \geq 1$ and $\omega > 0$ such that:

$$\|U(t,s)x\| \leq Me^{\omega(t-s)}\|x\|, \text{ for all } (t,s,x) \in \Delta \times X.$$

Definition 2.8. The evolution operator $U : \Delta \to \mathcal{B}(X)$ is said to be *uniformly exponentially stable (u.e.s.)* if there exist $N \geq 1$ and $\nu > 0$ such that:

$$\|U(t,s)x\| \leq Ne^{-\nu(t-s)}\|x\|, \text{ for all } (t,s,x) \in \Delta \times X.$$

Definition 2.9. The evolution operator $U : \Delta \to \mathcal{B}(X)$ has *uniform polynomial decay (u.p.d.)* if there exist two constants $M \geq 1$ and $\omega > 0$ such that:

$$(s+1)^{\omega}\|x\| \leq M(t+1)^{\omega}\|U(t,s)x\|, \text{ for all } (t,s,x) \in \Delta \times X.$$

Definition 2.10. The evolution operator $U : \Delta \to \mathcal{B}(X)$ is *uniformly polynomially instable (u.p.is.)* if there exist $N \geq 1$ and $\nu > 0$ such that:

$$(t+1)^{\nu}\|x\| \leq N(s+1)^{\nu}\|U(t,s)x\|, \text{ for all } (t,s,x) \in \Delta \times X.$$

Definition 2.11. The evolution operator $U : \Delta \to \mathcal{B}(X)$ *has $*$-uniform exponential decay ($*$-u.e.d.)* if there exist two constants $M \geq 1$ and $\omega > 0$ such that:

$$\|x^*\| \leq Me^{\omega(t-s)}\|U(t,s)^*x^*\|, \text{ for all } (t,s,x^*) \in \Delta \times X^*.$$

Definition 2.12. The evolution operator $U : \Delta \to \mathcal{B}(X)$ is said to be *$*$-uniformly exponentially instable ($*$-u.e.is.)* if there exist $N \geq 1$ and $\nu > 0$ such that:

$$\|x^*\| \leq Ne^{-\nu(t-s)}\|U(t,s)^*x^*\|, \text{ for all } (t,s,x^*) \in \Delta \times X^*.$$

Definition 2.13. The evolution operator $U : \Delta \to \mathcal{B}(X)$ has *$*$-uniform polynomial decay ($*$-u.p.d.)* if there exist $M \geq 1$ and $\omega > 0$ such that:

$$(s+1)^{\omega}\|x^*\| \leq M(t+1)^{\omega}\|U(t,s)^*x^*\|, \text{ for all } (t,s,x^*) \in \Delta \times X^*.$$

Definition 2.14. The evolution operator $U : \Delta \to \mathcal{B}(X)$ is *$*$-uniformly polynomially instable ($*$-u.p.is.)* if there exist $N \geq 1$ and $\nu > 0$ such that:

$$(t+1)^{\nu}\|x^*\| \leq N(s+1)^{\nu}\|U(t,s)^*x^*\|, \text{ for all } (t,s,x^*) \in \Delta \times X^*.$$

Remark 2.15. Let $U : \Delta \to \mathcal{B}(X)$ be a reversible evolution operator. Then, U has uniform exponential decay if and only if V has uniform exponential growth.

Remark 2.16. Let $U : \Delta \to \mathcal{B}(X)$ be a reversible evolution operator. Then, U is uniformly exponentially instable if and only if V is uniformly exponentially stable.

Remark 2.17. The following diagram illustrates the connections between the instability concepts and the decay properties mentioned in the paper.

$$\begin{array}{ccc} u.e.is. & \Rightarrow & u.p.is. \\ \Downarrow & & \Downarrow \\ u.e.d. & \Leftarrow & u.p.d. \end{array}$$

In general, the converse implications are not true. Next, we will present some examples which clarify the relations given above.

Example 2.18. (Evolution operator which has u.e.d, but it is not u.e.is.)
Let $X = \mathbb{R}$ and $U : \Delta \to \mathcal{B}(\mathbb{R}),\ U(t,s)x = e^{s-t}x.$
Then, U has uniform exponential decay, but it is not uniformly exponentially instable. Indeed, we have that

$$\|U(t,s)\| = e^{s-t} = e^{-(t-s)} \geq e^{-2(t-s)},$$

which implies that U u.e.d. for $\omega = 2, M = 1$.
If we suppose that U is u.e.is., then there exist $N \geq 1, \nu > 0$ such that $e^{\nu(t-s)} \leq Ne^{s-t}$, for all $(t,s) \in \Delta$.
For $s = 0$ and $t \to \infty$ it results that $\infty \leq N$, contradiction.

Example 2.19. (Evolution operator which is u.p.is., but it is not u.e.is.)
We consider $X = \mathbb{R}$ and the application $u : [1,\infty) \to \mathbb{R}_+^*,\ u(t) = t^2 + 1$. Then

$$U : \Delta \to \mathcal{B}(\mathbb{R}), U(t,s)x = \frac{u(t)}{u(s)}x,$$

is an evolution operator which is uniformly polynomially instable, but it is not uniformly exponentially instable.

Proof. See [13]. □

Example 2.20. (Evolution operator which has u.p.d., but it is not u.p.is.)
We consider $X = \mathbb{R}$ and the evolution operator

$$U : \Delta \to \mathcal{B}(\mathbb{R}),\ U(t,s)x = \frac{\varphi(s)}{\varphi(t)}x, \text{ where}$$

$$\varphi : \mathbb{R}_+ \to [1,\infty)\ \varphi(t) = t + 1.$$

Then, U has uniform polynomial decay, but it is not uniformly polynomially instable. Indeed, if we compute the norm of the operator U we obtain immediately that U has u.p.d. for $M = \omega = 1$.
If we suppose that U is u.p.is., then there exist $N \geq 1$ and $\nu > 0$ such that

$$\left(\frac{t+1}{s+1}\right)^{\nu} \leq N\left(\frac{s+1}{t+1}\right), \text{ for all } t \geq s \geq 0.$$

For $s = 0$ we obtain $(t+1)^{\nu+1} \leq N$, which for $t \to \infty$ yields to a contradiction.

Example 2.21. (Evolution operator that has u.e.d., but not u.p.d.)
We consider the evolution operator given in Example 2.18. We have that

$$\|U(t,s)\| = e^{s-t} \geq e^{-2(t-s)},$$

which implies that U has uniform exponential decay for $M = 1$ and $\omega = 2$.
If we suppose that U has u.p.d., it results that there exist $M \geq 1$ and $\omega > 0$ such that

$$\left(\frac{s+1}{t+1}\right)^{\omega} \leq Me^{s-t}, \text{ for all } (t,s) \in \Delta.$$

For $s = 0$ we obtain $M \geq \dfrac{e^t}{(t+1)^{\omega}}$ which for $t \to \infty$ yields to a contradiction, so U has not uniform polynomial decay.

We define $U_1 : \Delta \to B(X),\ U_1(t,s) = U(e^t - 1, e^s - 1)$ the evolution operator associated to U.

Proposition 2.22. *The evolution operator $U : \Delta \to \mathcal{B}(X)$ has $*$-uniform polynomial decay if and only if the evolution operator $U_1 : \Delta \to \mathcal{B}(X)$ has $*$-uniform exponential decay.*

Proof. It results in a similar manner as Proposition 2.12 from [6]. □

Proposition 2.23. *The evolution operator $U : \Delta \to \mathcal{B}(X)$ is $*$-uniformly polynomially instable if and only if $U_1 : \Delta \to \mathcal{B}(X)$ is $*$-uniformly exponentially instable.*

Proof. It follows using analogous arguments with those used to prove Proposition 2.13 in [6]. □

3. The main results

The results of this section are some characterization theorems of Barbashin type for the uniform exponential instability, $*$-uniform exponential instability, respectively for the uniform polynomial instability and $*$-uniform polynomial instability for evolution operators in Banach spaces.

Theorem 3.1. *Let U be a $*$-strongly measurable evolution operator with $*$-uniform exponential decay. Then U is $*$-uniformly exponentially instable if and only if there exist the constants $B > 1$ and $b > 0$ such that*

$$\int_0^t \frac{e^{-bs}}{\|U(t,s)^* x^*\|} ds \leq \frac{Be^{-bt}}{\|x^*\|},$$

for all $(t, x^*) \in \mathbb{R}_+ \times (X^* \setminus \{0\})$.

Proof. Necessity. Let $b \in (0, \nu)$. We suppose that U is $*$-uniformly exponentially instable. Then, there exist $N \geq 1, \nu > 0$ such that

$$e^{\nu(t-s)} \|x^*\| \leq N \|U(t,s)^* x^*\|, \text{ for all } (t,s,x^*) \in \Delta \times X^*,$$

which is equivalent to

$$\frac{1}{\|U(t,s)^* x^*\|} \leq \frac{Ne^{-\nu(t-s)}}{\|x^*\|}, \text{ for all } (t,s,x^*) \in \Delta \times X^*, \text{ that implies}$$

$$\int_0^t \frac{e^{-bs}}{\|U(t,s)^* x^*\|} ds \leq \frac{N}{\|x^*\|} \int_0^t e^{-bs} \cdot e^{-\nu(t-s)} ds = \frac{Ne^{-\nu t}}{\|x^*\|} \int_0^t e^{(\nu-b)s} ds \leq \frac{Be^{-bt}}{\|x^*\|},$$

where $B = 1 + \dfrac{N}{\nu - b}$.

Sufficiency. For $t \geq s+1$ we have

$$\frac{e^{-bs}}{\|U(t,s)^*x^*\|} = \int_s^{s+1} \frac{e^{-bs}}{\|U(t,s)^*x^*\|} d\tau = \int_s^{s+1} \frac{e^{-bs}}{\|U(\tau,s)^*U(t,\tau)^*x^*\|} d\tau$$

$$\leq \frac{1}{M} \int_s^{s+1} \frac{e^{-bs} \cdot e^{\omega(\tau - s)}}{\|U(t,\tau)^*x^*\|} d\tau = \frac{e^{\omega}}{M} \int_s^{s+1} e^{b(\tau - s)} \cdot \frac{e^{-b\tau}}{\|U(t,\tau)^*x^*\|} d\tau$$

$$\leq \frac{e^{\omega+b}}{M} \int_0^{t} \frac{e^{-b\tau}}{\|U(t,\tau)^*x^*\|} d\tau \leq \frac{Be^{\omega+b}}{M} \cdot \frac{e^{-bt}}{\|x^*\|} \leq N_1 \frac{e^{-bt}}{\|x^*\|},$$

where $N_1 = 1 + \dfrac{Be^{\omega+b}}{M}$. So, we obtained that

$$e^{b(t-s)}\|x^*\| \leq N_1 \|U(t,s)^*x^*\|, \text{ for all } t \geq s+1, \ s \geq 0. \tag{3.1}$$

For $t \in [s, s+1]$ we apply the $*$-decay property and we obtain that

$$\|U(t,s)^*x^*\| \geq Me^{-\omega(t-s)}\|x^*\|, \text{ which implies}$$

$$e^{b(t-s)}\|x^*\| \leq \frac{e^{(\omega+b)(t-s)}}{M}\|U(t,s)^*x^*\| \leq \frac{e^{\omega+b}}{M}\|U(t,s)^*x^*\| \leq N_2\|U(t,s)^*x^*\|,$$

where $N_2 = 1 + \dfrac{e^{\omega+b}}{M}$. So, we have that

$$e^{b(t-s)}\|x^*\| \leq N_2 \|U(t,s)^*x^*\|, \text{ for all } t \in [s, s+1], \ s \geq 0 \tag{3.2}$$

From (3.1) and (3.2) we obtain that

$$e^{b(t-s)}\|x^*\| \leq N \|U(t,s)^*x^*\|, \text{ for all } (t,s,x) \in \Delta \times X,$$

where $N = \max\{N_1, N_2\}$, so the theorem is proved. □

Corollary 3.2. *Let U be a $*$-strongly measurable evolution operator with $*$-uniform polynomial decay. Then U is $*$-uniformly polynomially instable if and only if there exist the constants $B > 1$ and $b > 0$ such that*

$$\int_0^t \frac{(s+1)^{-b-1}}{\|U(t,s)^*x^*\|} ds \leq \frac{B(t+1)^{-b}}{\|x^*\|},$$

for all $(t, x^*) \in \mathbb{R}_+ \times (X^* \setminus \{0\})$.

Proof. If U is $*$-uniformly polynomially instable with $*$-uniform polynomial decay, then, from Proposition 2.22 and Proposition 2.23, this is equivalent to U_1 is $*$-uniformly exponentially instable with $*$-uniform exponential decay, which is equivalent from Theorem 3.1 that there exist $B > 1$ and $b > 0$ such that

$$\int_0^t \frac{e^{-bs}}{\|U(e^t - 1, e^s - 1)^*x^*\|} ds \leq \frac{Be^{-bt}}{\|x^*\|}, \tag{3.3}$$

for all $(t, x^*) \in \mathbb{R}_+ \times (X^* \setminus \{0\})$. Using the change of variables $e^s - 1 = u$ relation (3.3) is equivalent to

$$\int_0^{e^t-1} \frac{e^{-b\ln(1+u)}}{\|U(e^t-1,u)^* x^*\|} \frac{du}{1+u} \leq \frac{Be^{-bt}}{\|x^*\|}. \tag{3.4}$$

Denoting by $v = e^t - 1$ relation (3.4) becomes

$$\int_0^{v} \frac{(1+u)^{-b-1}}{\|U(v,u)^* x^*\|} du \leq \frac{B(v+1)^{-b}}{\|x^*\|},$$

so we conclude that the proof is complete. □

Theorem 3.3. *Let $U : \Delta \to \mathcal{B}(X)$ be a strongly measurable and reversible evolution operator with uniform exponential decay. Then U is uniformly exponentially instable if and only if there exist $B > 1$ and $b \in (0,1)$ such that*

$$\int_0^{t} \frac{\|V(t,s)x\|}{e^{bs}} ds \leq \frac{B\|x\|}{e^{bt}}, \text{ for all } (t,x) \in \mathbb{R}_+ \times X. \tag{3.5}$$

Proof. If $U : \Delta \to \mathcal{B}(X)$ is a reversible evolution operator with uniform exponential decay, then from Remark 2.15 we have that $V : \Delta \to \mathcal{B}(X)$ has uniform exponential growth.

Necessity. Let $b \in (0, \nu)$. We suppose that U is uniformly exponentially instable which implies from Remark 2.16 that V is uniformly exponentially stable. Then, we have

$$\int_0^{t} \frac{\|V(t,s)x\|}{e^{bs}} ds \leq N \int_0^{t} \frac{e^{-\nu(t-s)}}{e^{bs}} \|x\| ds = Ne^{-\nu t}\|x\| \int_0^{t} e^{(\nu-b)s} ds$$
$$= \frac{N}{\nu-b}\|x\| \left(e^{-bt} - e^{-\nu t}\right) \leq \frac{B}{e^{bt}}\|x\|,$$

where $B = 1 + \dfrac{N}{\nu - b}$.

Sufficiency. If $t \geq s+1$ we obtain

$$e^{bt}\|V(t,s)x\| = \int_s^{s+1} e^{bt}\|V(\tau,s)V(t,\tau)x\| d\tau \leq M \int_s^{s+1} e^{bt}e^{\omega(\tau-s)}\|V(t,\tau)x\| d\tau$$
$$= M \int_s^{s+1} e^{(\tau-s)(b+\omega)} e^{b(t+s)} \frac{\|V(t,\tau)x\|}{e^{b\tau}} d\tau$$
$$\leq Me^{b+\omega}e^{b(t+s)} \int_0^{t} \frac{\|V(t,\tau)x\|}{e^{b\tau}} d\tau \leq Ne^{bs}\|x\|,$$

where $N = BMe^{b+\omega}$.
So, we obtained

$$\|V(t,s)x\| \le Ne^{-b(t-s)\|x\|}, \text{ for all } t \ge s+1,\ s \ge 0. \tag{3.6}$$

If $t \in [s, s+1)$ we apply the growth property and we have

$$e^{bt}\|V(t,s)x\| \le Me^{bt}e^{\omega(t-s)}\|x\| = Me^{(b+\omega)(t-s)}e^{bs}\|x\| \le Ne^{bs}\|x\|,$$

which implies

$$\|V(t,s)x\| \le Ne^{-b(t-s)\|x\|}, \text{ forall } t \in [s, s+1),\ s \ge 0. \tag{3.7}$$

Finally, from (3.6) and (3.7) we obtain that V is uniformly exponentially stable which means from Remark 2.16 that U is uniformly exponentially instable. □

Corollary 3.4. *Let $U : \Delta \to \mathcal{B}(X)$ be a strongly measurable and reversible evolution operator with uniform polynomial decay. Then U is uniformly polynomially instable if and only if there exist $B > 1$ and $b \in (0,1)$ such that*

$$\int_0^t \frac{\|V(t,s)x\|}{(s+1)^{b+1}}ds \le \frac{B\|x\|}{(t+1)^b}, \text{ for all } (t,x) \in \mathbb{R}_+ \times X.$$

Proof. It results immediately using the same idea as in the proof of Corollary 3.2. □

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Approximations of the solution of a stochastic Ginzburg-Landau equation

Brigitte E. Breckner and Hannelore Lisei

Dedicated to Professor Gheorghe Coman on the occasion of his 85th anniversary.

Abstract. This paper presents a method to approximate the solution of a stochastic Ginzburg-Landau equation with multiplicative noise term. Error estimates for the approximation of the solution are given.

Mathematics Subject Classification (2010): 60H15, 34K28, 35Q56.

Keywords: Stochastic Ginzburg-Landau equation, power-type nonlinearity, multiplicative noise.

1. Introduction

The complex Ginzburg-Landau equation on a bounded domain G in $\mathbb{R}$ or $\mathbb{R}^2$ with sufficiently regular boundary ∂G is

$$\begin{aligned} dX(t) &= (a_1 + \mathrm{i}a_2)\Delta X(t)dt + (\lambda_1 + \mathrm{i}\lambda_2)|X(t)|^2 X(t)dt + \gamma X(t)dt, \\ X(0) &= X_0, \end{aligned} \tag{1.1}$$

where $X : G \times [0, \infty) \to \mathbb{C}$ and $a_1, a_2, \lambda_1, \lambda_2, \gamma$ are certain real parameters.

Throughout this paper i is the imaginary unit, $\mathrm{Re}z$, $\mathrm{Im}z$ are, respectively, the real part and imaginary part of a complex number z, $\bar{z}$ denotes its complex conjugate and $|z| = \sqrt{(\mathrm{Re}z)^2 + (\mathrm{Im}z)^2}$ its modulus. Further we use the notations: $\mathbb{R}^* := \mathbb{R} \setminus \{0\}$ and $\mathbb{N}^* := \{1, 2, \ldots\}$.

Equation (1.1) is a nonlinear Schrödinger type equation with complex coefficients and power-type nonlinearity. Different forms of this evolution equation have applications in physics, see for example [5, 6, 8, 10, 11, 12, 13].

In this paper we consider a method to approximate the solution of the following stochastic complex Ginzburg-Landau evolution equation in dimension one perturbed by a multiplicative noise term

$$dX(t) = \mathrm{i}a\Delta X(t)\,dt - \lambda|X(t)|^2 X(t)\,dt + \gamma X(t)dt + \mathrm{i}\sum_{k=1}^{\infty} b_k(t)X(t)\,dW_k(t) \qquad (1.2)$$

with initial condition $X(0) = X_0$ and homogeneous Neumann boundary condition. Here, X is a complex-valued stochastic process depending on $t \in [0,T]$ and $x \in G \subset \mathbb{R}$, $a \in \mathbb{R}^*$, $\lambda, \gamma, T > 0$ are fixed, $(W_k)_{k\in\mathbb{N}}$ is a sequence of independent real-valued Wiener processes and $(b_k)_{k\in\mathbb{N}}$ is a sequence of real-valued functions, whose properties will be detailed later. The stochastic equation (1.2) corresponds to the case when in the deterministic equation (1.1) the parameters are $a_1 = 0, a_2 = a \in \mathbb{R}^*$ and $\lambda_1 = -\lambda < 0, \lambda_2 = 0, \gamma > 0$.

The existence of the solution of the stochastic Ginzburg-Landau equations is studied for example in [9, 1, 2, 5] with different noise terms than in our paper. In [9] the Galerkin method for the stochastic equation is used, while in [1] there are studied mild solutions and Strichartz' estimates are applied. In [5] the equation is studied on a three dimensional torus and has an additive noise term. A similar noise term is considered in [2], where the martingale solution is investigated.

In this paper we prove the existence of the solution by using a deterministic Ginzburg-Landau type equation. Moreover, we present a method to approximate the solution of (1.2) and give error estimates for this approximation. In the context of computer simulations error estimates are very important.

The paper has the following structure: Section 2 contains some notations, preliminary results, and the variational formulation of the stochastic, as well as the deterministic Ginzburg-Landau equation. In Section 3 we prove the existence and uniqueness of the considered evolution equation. In the last section we give some approximation results and error estimates for both the solution of the deterministic and stochastic Ginzburg-Landau equation.

2. Preliminaries

For simplicity we take the domain $G = (0,1)$. Consider the complex Hilbert spaces $H := L^2(0,1)$ and $V := H^1(0,1)$, the inner product in H is given by

$$(u,v) := \int_0^1 u(x)\,\bar{v}(x)\,dx, \qquad \text{for all } u,v \in H,$$

while the inner product in V is

$$(u,v)_V := \int_0^1 \left[u(x)\,\bar{v}(x) + \frac{du}{dx}(x)\,\frac{d\bar{v}}{dx}(x)\right] dx, \qquad \text{for all } u,v \in V.$$

The corresponding norms in H and V are $\|\cdot\|$ and $\|\cdot\|_V$, respectively. Furthermore, let V^* be the dual space of V and $\langle \cdot,\cdot \rangle$ the duality pairing of V^* and V.

Let $A : V \to V^*$ be the operator defined by

$$\langle Au, v\rangle := \int_0^1 \frac{du}{dx}(x)\,\frac{d\bar{v}}{dx}(x)\,dx, \qquad \text{for all } u,v \in V. \qquad (2.1)$$

Let $(\mu_k)_{k\in\mathbb{N}}$ be the increasing sequence of real eigenvalues and let $(h_k)_{k\in\mathbb{N}}$ be the corresponding eigenfunctions of A with respect to homogeneous Neumann boundary conditions. The eigenfunctions $(h_k)_{k\in\mathbb{N}}$ form an orthonormal system in H and they are orthogonal in V. Obviously, for all $u \in H$ and all $v \in V$, it holds that

$$u = \sum_{k=1}^{\infty} (u, h_k) h_k, \qquad Av = \sum_{k=1}^{\infty} \mu_k (v, h_k) h_k \,,$$

and

$$\operatorname{Im}\langle Av, v\rangle = 0, \tag{2.2}$$

$$\operatorname{Re}\langle Av, v\rangle = \langle Av, v\rangle = \sum_{k=1}^{\infty} \mu_k \left|(v, h_k)\right|^2 \geq 0.$$

Moreover,

$$\|Av\|_{V^*} \leq \|v\|_V, \quad \text{for all } v \in V. \tag{2.3}$$

Recall that $V \hookrightarrow H$ is a compact embedding, (V, H, V^*) is a triplet of rigged Hilbert spaces (Gelfand triple), and $\langle Au, v\rangle = (Au, v)$, for each $u, v \in V$, such that $Au \in H$.

In [4, Lemma 1.1] it is stated that

$$\sup_{x\in[0,1]} |v(x)|^2 \leq \|v\| \left(\|v\| + 2 \left\| \frac{dv}{dx} \right\| \right) \leq 2\|v\|_V^2, \text{ for all } v \in V. \tag{2.4}$$

Recall that $H^1(0,1) \hookrightarrow C[0,1]$.

For each $n \in \mathbb{N}^*$ set $H_n := \operatorname{sp}\{h_1, h_2, \ldots, h_n\}$ with the norm being induced from H. The norms $\|\cdot\|$ and $\|\cdot\|_V$ are equivalent on H_n.

The map $\Pi_n : H \to H_n$ defined by $\Pi_n h := \sum_{k=1}^{n} (h, h_k) h_k$ is the orthogonal projection of H onto H_n. Then, $\Pi_n h = h$, for each $h \in H_n$, and $\|\Pi_n h\| \leq \|h\|$, for each $h \in H$. Moreover, for each $h \in H_n$ it holds

$$Ah = \sum_{k=1}^{n} \mu_k (h, h_k) h_k \in H_n,$$

and then

$$\langle Ah, h\rangle = (Ah, h) = \sum_{k=1}^{n} \mu_k \left|(h, h_k)\right|^2 = \|h\|_V^2 - \|h\|^2, \tag{2.5}$$

$$(Ah, Ah) = \sum_{k=1}^{n} \mu_k^2 \left|(h, h_k)\right|^2, \text{ for each } h \in H_n. \tag{2.6}$$

Further, we mention some results used throughout the paper.

Lemma 2.1. *Let $u, v \in V$ such that $Av \in H$ and $|u|^2 v \in V$, then*

$$\operatorname{Re}(|u|^2 v, Av) \geq 0. \tag{2.7}$$

Especially, if $|v|^2 v \in V$, it holds

$$\operatorname{Re}(|v|^2 v, Av) \geq 0. \tag{2.8}$$

Proof. We have

$$\begin{aligned}\mathrm{Re}(Av,|u|^2v) &= \mathrm{Re}\int_0^1 \frac{dv}{dx}(x)\frac{d|u|^2\bar{v}}{dx}(x)\,dx\\ &= \mathrm{Re}\int_0^1 \left(\bar{v}(x)\frac{dv}{dx}(x)\frac{d|u|^2}{dx}(x)+|u(x)|^2\left|\frac{dv}{dx}(x)\right|^2\right)dx\\ &= \int_0^1 \left(\left(\frac{1}{2}\frac{d|v|^2}{dx}(x)\right)\frac{d|u|^2}{dx}(x)+|u(x)|^2\left|\frac{dv}{dx}(x)\right|^2\right)dx \geq 0.\end{aligned}$$

Therefore,

$$\mathrm{Re}(|u|^2v,Av)=\mathrm{Re}\overline{(|u|^2v,Av)}=\mathrm{Re}(Av,|u|^2v)\geq 0.$$

□

Lemma 2.2. *Let $z_1, z_2 \in \mathbb{C}$. Then the following inequalities hold*

$$\left||z_1|^2z_1-|z_2|^2z_2\right| \leq 3\left(|z_1|^2+|z_2|^2\right)|z_1-z_2|; \tag{2.9}$$

$$Re\left(\left(|z_1|^2z_1-|z_2|^2z_2\right)(\bar{z}_1-\bar{z}_2)\right)\geq 0. \tag{2.10}$$

Proof. See, e.g., [9, Lemma 7.2, Lemma 7.3]. □

Lemma 2.3. *Let S be a bounded set in $L^2([0,T];V)$, which is equicontinuous in $C([0,T];V^*)$. Then, S is relatively compact in $L^2([0,T];H)$.*

Proof. We use [14, Theorem 4.1] applied for $V \hookrightarrow H \hookrightarrow V^*$, where $V \hookrightarrow H$ is compact. □

In what follows we assume that $(\Omega,\mathcal{F},P)$ is a complete probability space and $(W_k)_{k\in\mathbb{N}}$ a sequence of independent real-valued standard Brownian motions on $[0,T]$ generating an increasing family of σ-algebras $\left(\mathcal{F}_t\right)_{t\in[0,T]}$. For each $k\geq 1$, let $b_k:[0,T]\to\mathbb{R}$ be square integrable functions such that

$$\sum_{k=1}^{\infty}\int_0^T b_k^2(s)\,ds<\infty. \tag{2.11}$$

Throughout the paper let $\lambda,\gamma,T>0$, $a\in\mathbb{R}^*$, $X_0\in V$ be fixed.

Definition 2.4. An $\left(\mathcal{F}_t\right)_{t\in[0,T]}$ adapted process

$$X\in L^2(\Omega;C([0,T];H))\cap L^4(\Omega\times[0,T];V)$$

is called a **variational solution of the stochastic Ginzburg-Landau equation** (1.2) with initial condition $X_0\in V$ if

$$\begin{aligned}(X(t),v) =&(X_0,v)-\mathrm{i}a\int_0^t\langle AX(s),v\rangle ds-\lambda\int_0^t(|X(s)|^2X(s),v)ds\\ &+\gamma\int_0^t(X(s),v)ds+\mathrm{i}\sum_{k=1}^{\infty}\int_0^t b_k(s)(X(s),v)dW_k(s)\end{aligned} \tag{2.12}$$

holds for all $t\in[0,T]$, $v\in V$, and a.e. $\omega\in\Omega$.

Remark 2.5. Let (2.11) be satisfied. Recall that the real-valued stochastic integral with respect to countably many Brownian motions

$$R(t) := \sum_{k=1}^{\infty} \int_0^t b_k(s) dW_k(s), \qquad t \in [0,T],$$

is a continuous square integrable martingale with respect to the filtration $\big(\mathcal{F}_t\big)_{t\in[0,T]}$, see [3, Lemma 2.1], having the quadratic variation equal to

$$[R]_t = \sum_{k=1}^{\infty} \int_0^t b_k^2(s) ds, \ t \in [0,T].$$

Moreover, for each $U \in L^2(\Omega; C([0,T];H))$ the H-valued stochastic integral

$$I(t) := \sum_{k=1}^{\infty} \int_0^t b_k(s) U(s)\, dW_k(s), \qquad t \in [0,T],$$

is a continuous square integrable H-valued martingale with respect to the filtration $\big(\mathcal{F}_t\big)_{t\in[0,T]}$ and

$$E\Big(\|I(t)\|^2\Big) = E \sum_{k=1}^{\infty} \int_0^t b_k^2(s) \|U(s)\|^2 ds, \ t \in [0,T].$$

Similar to the method from the paper [7], we associate to (2.12) a deterministic equation, which has the same initial condition $X_0 \in V$. For this we denote

$$Y(t) := \exp\left(-\gamma t - \frac{1}{2}\sum_{k=1}^{\infty} \int_0^t b_k^2(s)\, ds - \mathrm{i} \sum_{k=1}^{\infty} \int_0^t b_k(s)\, dW_k(s)\right)$$

for all $t \in [0,T]$ and a.e. $\omega \in \Omega$. The $\big(\mathcal{F}_t\big)_{t\in[0,T]}$ adapted real-valued process $(Y(t))_{t\in[0,T]}$ is the solution of the following stochastic linear differential equation

$$Y(t) = 1 - \gamma \int_0^t Y(s)\, ds - \sum_{k=1}^{\infty} \int_0^t b_k^2(s) Y(s)\, ds - \mathrm{i} \sum_{k=1}^{\infty} \int_0^t b_k(s) Y(s)\, dW_k(s)$$

for all $t \in [0,T]$ and a.e. $\omega \in \Omega$, where the stochastic integral in this equation is a real-valued continuous martingale (see Remark 2.5). For all $t \in [0,T]$ let

$$B(t) := \frac{1}{|Y(t)|^2} = \exp\left(2\gamma t + \sum_{k=1}^{\infty} \int_0^t b_k^2(s)\, ds\right), \tag{2.13}$$

and then $0 < B(t) \leq B(T) < \infty$ for all $t \in [0,T]$.

Definition 2.6. If $Z \in C([0,T];H)) \cap L^4([0,T];V)$ satisfies the evolution equation

$$(Z(t), v) = (X_0, v) - \mathrm{i}a \int_0^t \langle AZ(s), v\rangle\, ds - \lambda \int_0^t B(s)(|Z(s)|^2 Z(s), v)\, ds \tag{2.14}$$

for all $t \in [0,T]$ and all $v \in V$, then Z is called a **variational solution of the deterministic Ginzburg-Landau type equation**.

3. Existence results

Theorem 3.1. *There exists a unique solution*

$$Z \in C([0,T];H) \cap L^4([0,T];V)$$

of (2.14)*. Moreover, the following inqualities hold*

$$\sup_{t\in[0,T]} \|Z(t)\|^2 \le \|X_0\|^2,$$

$$\int_0^T \|Z(t)\|_V^2 dt \le T\|X_0\|_V^2, \quad \int_0^T \|Z(t)\|_V^4 dt \le T\|X_0\|_V^4.$$

Proof. Fix $n \in \mathbb{N}^*$. We consider the finite dimensional deterministic equation corresponding to (2.14)

$$(Z_n(t), h_j) = (X_0, h_j) - \mathrm{i}a\int_0^t \langle AZ_n(s), h_j\rangle\, ds - \lambda\int_0^t B(s)(|Z_n(s)|^2 Z_n(s), h_j)\, ds, \quad (3.1)$$

for all $t \in [0,T]$, $j \in \{1, ..., n\}$. In what follows we study the existence and uniqueness of the solution $Z_n \in C([0,T];H_n)$ of (3.1).

From (2.2) and (2.10) it follows by standard arguments that the solution of (3.1) is unique in $C([0,T];H_n)$, and that the solution of (2.14) is unique in $C([0,T];H) \cap L^4([0,T];V)$.

The existence of $Z_n \in C([0,T];H_n)$ follows from the finite dimensional theory for differential equations with locally Lipschitz nonlinearities. Note that (2.9) assures that the nonlinearity in (3.1) is locally Lipschitz. The solution is global on $[0,T]$ by the estimate (3.4) below: By taking the complex conjugate in (3.1) we have

$$\overline{(Z_n(t), h_j)} = \overline{(X_0, h_j)} + \mathrm{i}a\int_0^t \overline{\langle AZ_n(s), h_j\rangle} ds - \lambda\int_0^t B(s)\overline{(|Z_n(s)|^2 Z_n(s), h_j)} ds \quad (3.2)$$

for all $t \in [0,T]$, $j \in \{1, ..., n\}$.

For $z \in \mathbb{C}$ we recall the following identities

$$z - \bar{z} = 2\mathrm{i}\ \mathrm{Im}z \text{ and } z + \bar{z} = 2\ \mathrm{Re}z. \quad (3.3)$$

By using (3.1), (3.2), the chain rule for the product

$$(Z_n(\cdot), h_j) \cdot \overline{(Z_n(\cdot), h_j)},\ j \in \{1, ..., n\},$$

as well as (3.3) and the property

$$\|Z_n(t)\|^2 = \sum_{j=1}^n |(Z_n(t), h_j)|^2 \in \mathbb{R},\ t \in [0,T],$$

we obtain for all $t \in [0,T]$

$$\|Z_n(t)\|^2 = \|\Pi_n X_0\|^2 + 2a\mathrm{Im}\int_0^t \langle AZ_n(s), Z_n(s)\rangle\, ds$$
$$- 2\lambda\mathrm{Re}\int_0^t B(s)(|Z_n(s)|^2 Z_n(s), Z_n(s))\, ds.$$

By (2.2) and (2.8) we conclude

$$\sup_{t\in[0,T]} \|Z_n(t)\|^2 \le \|X_0\|^2. \tag{3.4}$$

Further we obtain estimates for $\sup_{t\in[0,T]} \|Z_n(t)\|_V^2$. Using (3.1) and (3.2) as above, we have for all $t \in [0,T]$ and $j \in \{1,...,n\}$

$$\begin{aligned}\mu_j|(Z_n(t),h_j)|^2 =& \mu_j|(X_0,h_j)|^2 + 2a\mathrm{Im}\int_0^t \mu_j^2|(Z_n(s),h_j)|^2 ds \\ &- 2\lambda\mathrm{Re}\int_0^t B(s)(|Z_n(s)|^2 Z_n(s), \mu_j(Z_n(s),h_j)h_j)\, ds.\end{aligned}$$

Summing up from $j=1$ to n, then, by (2.5) and (2.6), we obtain for all $t\in[0,T]$

$$(AZ_n(t), Z_n(t)) = (A\Pi_n X_0, \Pi_n X_0) - 2\lambda\mathrm{Re}\int_0^t B(s)(|Z_n(s)|^2 Z_n(s), AZ_n(s))ds.$$

By (2.8) and (2.5) it follows for all $t\in[0,T]$

$$\|Z_n(t)\|_V^2 - \|Z_n(t)\|^2 \le (A\Pi_n X_0, \Pi_n X_0) \le \langle AX_0, X_0\rangle = \|X_0\|_V^2 - \|X_0\|^2.$$

Therefore by (3.4) it follows

$$\sup_{t\in[0,T]} \|Z_n(t)\|_V^2 \le \|X_0\|_V^2. \tag{3.5}$$

Then, we conclude $Z_n \in L^4([0,T];V)$.
By (3.1) and (2.3) we have for all $r,t\in[0,T]$ with $r<t$

$$\begin{aligned}\|Z_n(t)-Z_n(r)\|_{V^*}^2 &\le 2\int_r^t \|AZ_n(s)\|_{V^*}^2 ds + 2\lambda^2\int_r^t B^2(s)\||Z_n(s)|^2 Z_n(s)\|_{V^*}^2 ds \\ &\le 2\int_r^t \|Z_n(s)\|_V^2 ds + 2\lambda^2 CB^2(T)\int_r^t \||Z_n(s)|^2 Z_n(s)\|^2 ds,\end{aligned}$$

where C is the embedding constant of $H \hookrightarrow V^*$. Moreover, by (2.4) we write for all $r,t\in[0,T]$ with $r<t$

$$\int_r^t \||Z_n(s)|^2 Z_n(s)\|^2 ds = \int_r^t \left(\int_0^1 |Z_n(s)|^6 dx\right) ds \le 4\int_r^t \|Z_n(s)\|_V^4 \|Z_n(s)\|^2 ds.$$

Using these estimates, as well as (3.4) and (3.5), it follows for all $r,t\in[0,T]$ with $r<t$

$$\|Z_n(t)-Z_n(r)\|_{V^*}^2 \le 2(t-r)\|X_0\|_V^2 \left(1+4\lambda^2 CB^2(T)\|X_0\|_V^2\|X_0\|^2\right).$$

We observe that $S := (Z_n)_{n\ge1}$ is equicontinuous in $C([0,T];V^*)$ and it is bounded in $L^2([0,T];V)$ and also in $L^4([0,T];V)$ (see (3.5)).

It follows that there exist $U \in L^2([0,T];H)\cap L^4([0,T];V)$ and a subsequence $(Z_{n_k})_{k\ge1}$ which is:
- strongly convergent in $L^2([0,T];H)$ to U (by Lemma 2.3)

and
- weakly convergent in $L^4([0,T];V)$ and, also in $L^2([0,T];V)$, to U.

Recall that $L^4([0,T];V) \hookrightarrow L^2([0,T];V) \hookrightarrow L^2([0,T];H)$; these are reflexive Banach spaces and we can use [15, Proposition 21.23(i), Proposition 21.35(c)].

Since $(Z_{n_k})_{k\geq 1}$ is strongly convergent to U in $L^2([0,T];H)$, one can prove by using (2.9) that $\big(|Z_{n_k}|^2 Z_{n_k}\big)_{k\geq 1}$ is weakly convergent to $|U|^2U$ in $L^2([0,T];H)$. In (3.1) we take n_k instead of n, then let $k\to\infty$, and using the above convergence results, we get for all $j\in\mathbb{N}^*$ that

$$(U(t),h_j) = (X_0,h_j) - \mathrm{i}a\int_0^t \langle AU(s),h_j\rangle\,ds - \lambda\int_0^t B(s)(|U(s)|^2U(s),h_j)\,ds \tag{3.6}$$

holds for a.e. $t\in[0,T]$.

There exists an H-valued function that is equal to U for a.e. $t\in[0,T]$ and is equal to the right side of (3.6) for all $t\in[0,T]$. This function we denote by Z. By the properties of U we have

$$Z\in C([0,T];H)\cap L^4([0,T];V)$$

and Z is the solution of (2.14).

The estimate

$$\sup_{t\in[0,T]}\|Z(t)\|^2 \leq \|X_0\|^2$$

is obtained similarly to (3.4) by using (2.12). By the weak convergence of $(Z_{n_k})_{k\geq 1}$ to Z in $L^2([0,T];V)$ and also in $L^4([0,T];V)$, we get from (3.5)

$$\int_0^T \|Z(t)\|_V^2 dt \leq \liminf_{k\to\infty}\int_0^T \|Z_{n_k}(t)\|_V^2 dt \leq T\|X_0\|_V^2,$$

$$\int_0^T \|Z(t)\|_V^4 dt \leq \liminf_{k\to\infty}\int_0^T \|Z_{n_k}(t)\|_V^4 dt \leq T\|X_0\|_V^4.$$

□

Remark 3.2. In [13, Chapter IV, Theorem 5.1] and [8, Chap. 10, Théorème 10.1] the reader may find alternative ideas for the proof of the existence of the solution of (2.14). The purpose of our detailed proof, using classical methods from partial differential equations, was to obtain the estimates stated in Theorem 3.1, which will be used in the computation of error bounds in Section 4.

Theorem 3.3. *There exists a unique variational solution of* (2.12)

$$X\in L^2(\Omega;C([0,T];H))\cap L^4(\Omega\times[0,T];V).$$

Moreover, $X\in C([0,T];H)\cap L^4([0,T];V)$ for a.e. $\omega\in\Omega$ and the following inqualities hold for a.e. $\omega\in\Omega$

$$\sup_{t\in[0,T]}\|X(t)\|^2 \leq B(T)\|X_0\|^2,$$

$$\int_0^T \|X(t)\|_V^2 dt \leq TB(T)\|X_0\|_V^2, \quad \int_0^T \|X(t)\|_V^4 dt \leq TB^2(T)\|X_0\|_V^4.$$

Proof. By the Itô formula and the uniqueness of the solution of (2.14) one has that

$$X(t) := Z(t)Y^{-1}(t), \qquad \text{for all } t \in [0,T] \text{ and a.e. } \omega \in \Omega,$$

is the unique solution of (2.12). The estimates for X follow from Theorem 3.1 and we also have

$$E \sup_{t\in[0,T]} \|X(t)\|^2 \le B(T)\|X_0\|^2,$$

$$E\int_0^T \|X(t)\|_V^2 dt \le TB(T)\|X_0\|_V^2, \quad E\int_0^T \|X(t)\|_V^4 dt \le TB^2(T)\|X_0\|_V^4. \qquad \square$$

4. Approximation of the solution

We will approximate the solution of (2.12) by a sequence of stochastic processes $(X_N)_{N\ge 1}$, where, for each $N \in \mathbb{N}^*$, we consider $X_N := Z_N Y^{-1}$, Z_N being the solution of the following *linearized deterministic problem* in variational formulation

$$(Z_N(t), v) = (X_0, v) - \mathrm{i}a\int_0^t \langle AZ_N(s), v\rangle\, ds - \lambda\int_0^t B(s)(|Z_{N-1}(s)|^2 Z_N(s), v)\, ds, \quad (4.1)$$

for all $t \in [0,T]$ and all $v \in V$. We take $Z_0 := X_0$.

Theorem 4.1. *For each $N \in \mathbb{N}^*$ there exists a unique solution*

$$Z_N \in C([0,T]; H) \cap L^4([0,T]; V)$$

of (4.1).

Proof. The result is obtained successively: Let $N \ge 1$. If

$Z_{N-1} \in C([0,T]; H) \cap L^4([0,T]; V)$, then $Z_N \in C([0,T]; H) \cap L^4([0,T]; V)$

is a solution of (4.1).

The existence and uniqueness of the solution of (4.1) is proved analogously to Theorem 3.1. We use (2.7), Lemma 2.3, and the Galerkin method associated to (4.1) in order to obtain the following estimates for each $N \in \mathbb{N}^*$

$$\sup_{t\in[0,T]} \|Z_N(t)\|^2 \le \|X_0\|^2,$$

$$\int_0^T \|Z_N(t)\|_V^2 dt \le T\|X_0\|_V^2, \quad \int_0^T \|Z_N(t)\|_V^4 dt \le T\|X_0\|_V^4. \qquad \square$$

Theorem 4.2. *For each $N \in \mathbb{N}^*$ it holds*

$$\sup_{t\in[0,T]} \|Z_N(t) - Z(t)\|^2 \le \frac{3\lambda TB(T)}{2^{N-4}}\|X_0\|^2\|X_0\|_V^2 \exp\left(20\lambda TB(T)\|X_0\|_V^2\right).$$

Proof. By using (4.1) and (2.14), we have

$$\|Z_N(t) - Z(t)\|^2 = 2a\mathrm{Im}\int_0^t \langle AZ_N(s) - AZ(s), Z_N(s) - Z(s)\rangle\, ds \quad (4.2)$$

$$- 2\lambda\mathrm{Re}\int_0^t B(s)(|Z_{N-1}(s)|^2 Z_N(s) - |Z(s)|^2 Z(s), Z_N(s) - Z(s))\, ds,$$

for all $t \in [0,T]$. We define

$$e(t) = \exp\left(-20\lambda \int_0^t B(s)\|Z(s)\|_V^2 ds\right), \qquad \text{for all } t \in [0,T].$$

Then, by (4.2) and (2.2), we have for all $t \in [0,T]$

$$\begin{aligned} & e(t)\|Z_N(t) - Z(t)\|^2 = \qquad (4.3)\\ & -2\lambda \text{Re} \int_0^t e(s)B(s)(|Z_{N-1}(s)|^2 Z_N(s) - |Z(s)|^2 Z(s), Z_N(s) - Z(s))\, ds \\ & -20\lambda \int_0^t e(s)B(s)\|Z(s)\|_V^2 \|Z_N(s) - Z(s)\|^2 ds. \end{aligned}$$

We compute

$$\begin{aligned} & -\text{Re}(|Z_{N-1}|^2 Z_N - |Z|^2 Z, Z_N - Z) \\ & = -\text{Re}(|Z_{N-1}|^2 (Z_N - Z), Z_N - Z) + \text{Re}((|Z|^2 - |Z_{N-1}|^2)Z, Z_N - Z). \end{aligned}$$

Denote $Q = Z_{N-1}, R = Z_N$ (we omit writing the dependence on s and x). Due to the definition of the scalar product $(\cdot,\cdot)$ in H, we write

$$\begin{aligned} & -\text{Re}(|Z_{N-1}|^2(Z_N - Z), Z_N - Z) = -\int_0^1 |Q|^2|R - Z|^2 dx \\ & = -\int_0^1 \Big(|Q - Z|^2 + |Z|^2 + 2\text{Re}[(Q - Z)\bar{Z}]\Big)|R - Z|^2 dx \\ & \leq \int_0^1 \Big(-|Q - Z|^2|R - Z|^2 - |Z|^2|R - Z|^2 + 2|Q - Z||Z||R - Z|^2\Big) dx \\ & \leq \int_0^1 \Big(-\frac{1}{2}|Q - Z|^2|R - Z|^2 + |Z|^2|R - Z|^2\Big) dx, \end{aligned}$$

as well as

$$\begin{aligned} & \text{Re}((|Z|^2 - |Z_{N-1}|^2)Z, Z_N - Z) = \text{Re} \int_0^1 (|Z|^2 - |Q|^2)Z(\bar{R} - \bar{Z}) dx \\ & = \text{Re} \int_0^1 \Big(-|Q - Z|^2 - 2\text{Re}[(Q - Z)\bar{Z}]\Big)Z(\bar{R} - \bar{Z}) dx \\ & \leq \int_0^1 \Big(\frac{1}{2}|Q - Z|^2|R - Z|^2 + |Z|^2|R - Z|^2 + \frac{3}{2}|Z|^2|Q - Z|^2\Big) dx\,. \end{aligned}$$

Then,

$$\begin{aligned} & -2\lambda \text{Re}(|Z_{N-1}|^2 Z_N - |Z|^2 Z, Z_N - Z) \\ & \leq \lambda \int_0^1 \Big(4|Z|^2|Z_N - Z|^2 + 3|Z|^2|Z_{N-1} - Z|^2\Big) dx \\ & \leq 8\lambda\|Z\|_V^2\|Z_N - Z\|^2 + 6\lambda\|Z\|_V^2\|Z_{N-1} - Z\|^2, \end{aligned}$$

where in the last inequality we apply (2.4). Using the result obtained above, we get, by (4.3), for all $t \in [0,T]$

$$\begin{aligned} e(t)\|Z_N(t) - Z(t)\|^2 + 12\lambda \int_0^t e(s)B(s)\|Z(s)\|_V^2 \|Z_N(s) - Z(s)\|^2 ds \qquad (4.4) \\ \leq 6\lambda \int_0^t e(s)B(s)\|Z(s)\|_V^2 \|Z_{N-1}(s) - Z(s)\|^2 ds. \end{aligned}$$

This implies, for each $N \in \mathbb{N}^*$

$$\begin{aligned} &\int_0^T e(s)B(s)\|Z(s)\|_V^2 \|Z_N(s) - Z(s)\|^2 ds \\ &\leq \frac{1}{2} \int_0^T e(s)B(s)\|Z(s)\|_V^2 \|Z_{N-1}(s) - Z(s)\|^2 ds \\ \ldots &\leq \frac{1}{2^N} \int_0^T e(s)B(s)\|Z(s)\|_V^2 \|X_0 - Z(s)\|^2 ds \\ &\leq \frac{1}{2^{N-1}} B(T)\big(\|X_0\|^2 + \sup_{s\in[0,T]} \|Z(s)\|^2\big) \int_0^T \|Z(s)\|_V^2 ds \\ &\leq \frac{TB(T)}{2^{N-2}} \|X_0\|^2 \|X_0\|_V^2. \end{aligned}$$

Note that in the last inequality we take into consideration the estimates from Theorem 3.1. Using the above result in (4.4), we obtain for each $N \in \mathbb{N}^*$

$$\begin{aligned} \sup_{t\in[0,T]} \|Z_N(t) - Z(t)\|^2 &\leq 6\lambda \frac{TB(T)}{2^{N-3} e(T)} \|X_0\|^2 \|X_0\|_V^2 \\ &\leq \frac{3\lambda TB(T)}{2^{N-4}} \|X_0\|^2 \|X_0\|_V^2 \exp\left(20\lambda TB(T)\|X_0\|_V^2\right). \end{aligned}$$

□

Applying Theorem 3.3 and Theorem 4.2, we obtain the main result of our paper.

Theorem 4.3. *For a.e. $\omega \in \Omega$ and for each $N \in \mathbb{N}^*$ let*

$$X_N(t) := Z_N(t)Y^{-1}(t), \quad \textit{for all } t \in [0,T].$$

The following approximation result holds for a.e. $\omega \in \Omega$ and all $N \in \mathbb{N}^*$

$$\sup_{t\in[0,T]} \|X_N(t) - X(t)\|^2 \leq \frac{3T\lambda B^2(T)}{2^{N-4}} \|X_0\|^2 \|X_0\|_V^2 \exp\left(20\lambda TB(T)\|X_0\|_V^2\right),$$

where

$$B(T) = \exp\left(2\gamma T + \sum_{k=1}^{\infty} \int_0^T b_k^2(s)\, ds\right).$$

In particular, this implies

$$P\Big(\lim_{N\to\infty} \sup_{t\in[0,T]} \|X_N(t) - X(t)\|^2 = 0\Big) = 1$$

and

$$\lim_{N\to\infty} E \sup_{t\in[0,T]} \|X_N(t) - X(t)\|^2 = 0.$$

Remark 4.4. We can obtain similar results as in this paper, if:
1) we consider homogeneous Dirichlet or periodic boundary conditions;
2) instead of the nonlinear term $|X|^2X$ we take $|X|^{2\sigma}X$, where $\sigma \geq 1$, or combined power-type nonlinearities such as $|X|^{2\sigma_1}X + |X|^{2\sigma_2}X$, where $\sigma_1, \sigma_2 \geq 1$;
3) $\gamma \leq 0$;
4) in (2.12) the operator $-\mathrm{i}aA$ is replaced by $-(a_1+\mathrm{i}a_2)A$, where $a_2 \in \mathbb{R}^*$ and $a_1 > 0$;
5) for each $k \geq 1$, we assume $b_k : \Omega \times [0,T] \to \mathbb{R}$ to be $\left(\mathcal{F}_t\right)_{t\in[0,T]}$ adapted processes satisfying

$$E\left(\exp\left(3\sum_{k=1}^{\infty}\int_0^T b_k^2(s)\,ds\right)\right) < \infty.$$

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Solution of nonlinear equations via Padé approximation. A Computer Algebra approach

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Dedicated to Professor Gheorghe Coman on the occasion of his 85th anniversary.

Abstract. We generate automatically several high order numerical methods for the solution of nonlinear equations using Padé approximation and Maple CAS.

Mathematics Subject Classification (2010): 65H05, 65D05, 65D99.

Keywords: Nonlinear equation, Padé approximation, Numerical methods, Computer Algebra.

1. Introduction

Consider the nonlinear scalar equation

$$f(x) = 0, \tag{1.1}$$

where $f : D \subseteq \mathbb{R} \to \mathbb{R}$ is a continuous and differentiable as many times as necessary. Let α be a solution of (1.1). Let $\mathcal{R}_{m,p}$ be the set of rational functions with degree of numerator m and degree of denominator p. Suppose f has a formal Taylor series

$$f(z) = c_0 + c_1 z + c_2 z^2 + \cdots .$$

For any pair $(m,p) \in \mathbb{N} \times \mathbb{N}$, $r_{m,p} \in \mathcal{R}_{m,p}$ is the type (m,p) Padé approximant to f if their Taylor series at $z = 0$ agree as far as possible:

$$(f - r_{m,p})(z) = O(z^{max}) \tag{1.2}$$

We will use three different strategies based on Padé approximation in order to obtain automatically high order method:

- a direct strategy;
- inverse interpolation;
- modified methods.

The features of Maple CAS allow us to generate methods of arbitrary orders. See [4] or [6] for details. The `pade` procedure from the `numapprox` package computes a Padé approximation of degree (m, p) about a given point. The paper [3] and the book [2] contain several interesting examples of using Computer Algebra for the derivation of numerical methods. In the sequel we will consider one-step methods, i.e. methods of the form

$$x_{n+1} = F(x_n), \qquad x_0 \text{ given.}$$

For the sake of brevity we will use the notations $f_n = f(x_n)$ and $f_n^{(k)} = f^{(k)}(x_n)$.

2. The direct approach

The first strategy is to approximate f by its (m, p) Padé approximant $r_{m,p} \in \mathcal{R}_{m,p}$ and to solve the equation $r_{m,p}(x) = 0$. The iteration will have the form

$$x_{n+1} = F(x_n),$$

where $F(x)$ is the root of $r_{m,p}(x) = 0$ as a function of x. In order to avoid the solution of higher order equations, we will choose $m = 1$.

For example, for $m = 1$ and $p = 0$, we obtain the Newton's method.

```
> restart;
> with(numapprox):
> F:=pade(f(t),t=x[n],[1,0]):
> G:=collect(solve(%,t),x[n]);
```

$$G := x_n - \frac{f(x_n)}{\mathrm{D}(f)(x_n)}$$

or,

$$x_{n+1} = x_n - \frac{f_n}{f_n'}.$$

For $m = 1$ and $p = 1$, we obtain Halley's method.

```
> F:=pade(f(t),t=x[n],[1,1]):
> G:=collect(solve(%,t),x[n]);
```

$$G := x_n - 2\,\frac{\mathrm{D}(f)(x_n)\,f(x_n)}{2\,(\mathrm{D}(f)(x_n))^2 - \left(D^{(2)}\right)(f)(x_n)\,f(x_n)}$$

or,

$$x_{n+1} = x_n - \frac{2 f_n' f_n}{2\,(f_n')^2 - f_n'' f_n.}$$

This formula was obtained using direct Padé approximation in [2].

These are in fact particular cases of Householder-type methods. They could be obtained by considering $(1, p)$ Padé approximation and solving the equation $r_{1,p} = 0$.

Their order is $p+2$. If $f \in C^{p+1}(V)$, where V is a neighborhood of α, Househelder showed in [9] that the general form of iteration is

$$x_{n+1} = x_n + (p+1) \left. \frac{\left(\frac{1}{f}\right)^{(p)}}{\left(\frac{1}{f}\right)^{(p+1)}} \right|_{x_n} .$$

The generation of such a method is straightforward with the following one-line Maple code

```
> Phi:=(x,p)->x+(p+1)*(D@@(p))(1/f)(x)/(D@@(p+1))(1/f)(x):
```

We give two examples, for $p = 2$ and $p = 3$. The results were converted to mathematical notation.

```
> F_2:=x+normal(Phi(x,2)-x);
> F_3:=x+normal(Phi(x,3)-x);
```

$$F_2 := x - 3\frac{\left[2f'^2(x) - f''(x)f(x)\right] f(x)}{f'''(x)f^2(x) + 6f'^3(x) - 6f''(x)f'(x)f(x)} \tag{2.1}$$

$$F_3 := x + \frac{4\left[f'''(x)f^2(x) + 6f'^3(x) - 6f''(x)f'(x)f(x)\right] f(x)}{Q(x)}, \tag{2.2}$$

where

$$Q(x) = f^{(4)}(x)f^3(x) - 8f'''(x)f'(x)f^2(x) - 24f'^4(x) + 36f''(x)f'^2(x)f(x) - 6f''^2(x)f^2(x) \tag{2.3}$$

3. Inverse interpolation

Suppose there exists $g = f^{-1}$ on a neighborhood V of α. The inverse interpolation consists of approximating

$$\alpha = g(0),$$

by the value of an interpolant $\widehat{g}$ for g at 0

$$\alpha = \widehat{g}(0).$$

In this section we will use inverse Padé interpolation. The formula we look for will have the form

$$x_{k+1} = r_{m,p}(x_k), \qquad k = 0, 1, ,$$

where $r_{m,p}$ is the (m,p) Padé approximant for $g(0)$. For details on inverse interpolation see [1], [5], [7]. The paper [7] uses rational interpolation to derive methods for the solution of scalar nonlinear equations. The Maple procedure `invpade` generates the iteration function based on (m,p)-inverse Padé interpolation.

```
> invPade:=proc(m::nonnegint,p::nonnegint)
> local f,x;
> x+collect(eval(pade((f@@(-1))(y),y=f(x),[m,p]),y=0)-x,
> x,simplify);
> end proc;
```

We give examples for $(m,p) \in \{(1,1),(2,1),(2,2)\}$. The results were edited, in order to fit on page.

Formula for $(1,1)$ is the Halley's formula.

```
> F11:=invPade(1,1);
```

$$F_{11} := x + 2\frac{f'(x)f(x)}{f''(x)f(x) - 2f'^2(x)}$$

Formula for $(2,1)$ was given and studied in [10].

```
> F21:=invPade(2,1);
> convert(%,diff);
```

$$F_{21} := x - \frac{f(x)\left[f(x)f'(x)f'''(x) - \frac{3}{2}f(x)f''^2(x) + 3f'^2(x)f''(x)\right]}{f'(x)\left[f(x)f'(x)f'''(x) - 3f(x)f''^2(x) + 3f'^2(x)f''(x)\right]} \tag{3.1}$$

Note that the formula for $(1,2)$ is different from (2.1) (that is, the direct approach and inverse interpolation generates different formulas for (1,2) pair of degrees). The $(2,2)$-type formula is

$$F_{22} = x + \frac{U}{V}, \tag{3.2}$$

where

$$U = 6ff'\left[f\left(f'\right)^2 f^{(4)} - 6ff'f''f''' + 6f\left(f''\right)^3 + 4f'''\left(f'\right)^3 - 6\left(f''\right)^2\left(f'\right)^2\right](x)$$

and

$$\begin{aligned} V = f^2 &\left(3\left(f'\right)^2 f^{(4)} f'' - 4\left(f'\right)^2\left(f'''\right)^2 - 6f'\left(f''\right)^2 f''' + 9\left(f''\right)^4\right)(x) \\ &- 6f\left(f'\right)^2\left(\left(f'\right)^2 f^{(4)} - 8f'f''f''' + 9\left(f''\right)^3\right)(x) \\ &- 12\left(f'\right)^4\left(2f'f''' - 3\left(f''\right)^2\right)(x). \end{aligned}$$

4. Modified methods

Following the ideas of Sebah and Gourdon [8], we look for an iteration of the form

$$x_{n+1} = x_n + h_n + a_2\frac{h_n^2}{2!} + a_3\frac{h_n^3}{3!} + \cdots, \tag{4.1}$$

where $h_n = -\frac{f(x_n)}{f'(x_n)}$. Under the assumptions that f is sufficiently differentiable and $h_n + a_2\frac{h_n^2}{2!} + a_3\frac{h_n^3}{3!} + \cdots$ is small, we start from Taylor expansion of $f(x_{n+1})$ about x_n, and using the side-relation $f(x_n) + h_n f'(x_n) = 0$, we try to choose a_n's so that to cancel as many terms as possible in the expansion.

The Maple procedure `modPade` below returns the coefficients (a_k) and the modified method (4.1) truncated to a given number of terms.

```
> modPade:=proc(nmax::nonnegint)
> local k, inc,dT, dT2, sol, a, ec, so, it, n ;
> inc:=h+add(a[k]*h^k/k!,k=2..max(nmax+1,3));
> dT:=convert(taylor(f(x[n]+t),t=0,nmax+1),polynom);
> dT:=simplify(subs(t=inc,dT),[f(x[n])+h*D(f)(x[n])=0]):
> dT2:=collect(dT,h,simplify):
> for k from 2 to nmax+1 do
> ec[k]:=coeff(dT2,h,k);
> end;
> so:=solve([seq(ec[k],k=2..nmax+1)],[seq(a[k],k=2..nmax+1)]);
> assign(so);
> it:=x[n]+eval(subs(h=-f(x[n])/D(f)(x[n]),factor(inc)));
> return a,it;
> end proc:
```

modPade computes for a_k, $k = 2, \ldots, 6$, the following values

$$a_2 = -\frac{f_n''}{f_n'}$$

$$a_3 = \frac{3\left(f_n''\right)^2 - f_n''' f_n'}{\left(f_n'\right)^2}$$

$$a_4 = -\frac{f_n^{(4)}\left(f_n'\right)^2 - 10 f_n''' f_n'' f_n' + 15\left(f_n''\right)^3}{\left(f_n'\right)^3}$$

$$a_5 = \frac{105\left(f_n''\right)^4 - 105 f_n'''\left(f_n''\right)^2 f_n' + 15 f_n^{(4)} f_n''\left(f_n'\right)^2 + 10\left(f_n'\right)^2\left(f_n'''\right)^2 - f_n^{(5)}\left(f_n'\right)^3}{\left(f^{(4)}\right)^4}$$

$$a_6 = -\frac{7}{\left(f_n'\right)^5}\Big(135\left(f_n''\right)^5 - 180 f_n'''\left(f_n''\right)^3 f_n' + 30 f_n^{(4)}\left(f_n''\right)^2\left(f_n'\right)^2 + 40 f_n''\left(f_n'''\right)^2\left(f_n'\right)^2$$
$$-3 f_n^{(5)} f_n''\left(f_n'\right)^3 - 5 f_n''' f_n^{(4)}\left(f_n'\right)^3\Big)$$

For $n_{\max} = 4$, modPade gives the fourth-order formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f''(x_n) f^2(x_n)}{2\left(f'(x_n)\right)^3} + \frac{\left(f'''(x_n) f'(x_n) - 3\left(f''(x_n)\right)^2\right) f^3(x_n)}{6\left(f'(x_n)\right)^5} \tag{4.2}$$

For $n_{\max} = 5$, modPade gives the fifth-order formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f''(x_n) f^2(x_n)}{2\left(f'(x_n)\right)^3} + \frac{\left(f'''(x_n) f'(x_n) - 3\left(f''(x_n)\right)^2\right) f^3(x_n)}{6\left(f'(x_n)\right)^5}$$
$$- \frac{\left(f^{(4)}(x_n)\left(f'(x_n)\right)^2 - 10 f'''(x_n) f''(x_n) f'(x_n) + 15\left(f''(x_n)\right)^3\right) f^4(x_n)}{24\left(f'(x_n)\right)^7} \tag{4.3}$$

Remark 4.1. These methods are the same as Chebyshev methods and could be generated using inverse Taylor interpolation (see [1, 7]).

5. Numerical examples

We wish to compare the different iterations on the solution of the equation

$$xe^x + x^2 - 6 = 0. \tag{5.1}$$

First, we compute the solution using `fsolve` function with `Digits` set to 400.

```
> Digits:=400:
> eq:=x*exp(x)+x^2-6:
> root1:=fsolve(eq,x);
```

root1 :=1.257169468081542443224161713705996802920131265042900761\
14235516200997511308305661557912016056910371859828810\
140558803113433921630435939810988753086636 . . .

Then, for each method we execute a small number of iteration steps and count the number of correct digits and compute the absolute error as the modulus of the difference between `root1` and the computed approximation.

- Padé $(1, 2)$, order 4 (formula (2.1))

$x_1 = 1.26(257\dots)$ 2 digits

$x_2 = 1.2571694681(095\dots)$ 10 digits

$x_3 = 1.2571694680815424432241617137059968029201312(853\dots)$ 43 digits

$x_4 = 1.25716946808154244322416171370599680292013126504(\dots)$ 176 digits

- inverse Padé $(2, 1)$, order 4 (formula (3.1))

$x_1 = 1.2(727\dots)$ 1 digits

$x_2 = 1.2571694(737\dots)$ 8 digits

$x_3 = 1.2571694680815424432241617137059969(004\dots)$ 34 digits

$x_4 = 1.25716946808154244322416171370599680292013126 50(\dots)$ 137 digits

- modified method, order 4 (formula (4.2))

$x_1 = 1.3(106\dots)$ 1 digits

$x_2 = 1.25717(411\dots)$ 5 digits

$x_3 = 1.257169468081542443224(458\dots)$ 21 digits

$x_4 = 1.25716946808154244322416171370599680292013126504(\dots)$ 86 digits

- Padé $(1, 3)$, order 5 (formulas (2.2) and (2.3))

$x_1 = 1.257(703\dots)$ 3 digits

$x_2 = 1.257169468081542443(624\dots)$ 18 digits

$x_3 = 1.2571694680815424432241617137059968029201 31265(\dots)$ 94 digits

$x_4 = 1.25716946808154244322416171370599680292013126504(\dots)$ 472 digits

Note that this method was tested for `Digits` *set to* 500.

- inverse Padé $(2,2)$, order 5 (formula (3.2))

  $x_1 = 1.26(\ldots)$ 2 digits

  $x_2 = 1.2571694680815(682\ldots)$ 13 digits

  $x_3 = 1.257169468081542443224161713705996802920131265(\ldots)$ 69 digits

  $x_4 = 1.2571694680815424432241617137059968029201312650 4(\ldots)$ 348 digits

- modified method, order 5 (formula (4.3))

  $x_1 = 1.(2846\ldots)$ 1 digits

  $x_2 = 1.257169(479\ldots)$ 7 digits

  $x_3 = 1.257169468081542443224161713705996802920(249\ldots)$ 39 digits

  $x_4 = 1.25716946808154244322416171370599680292013126 50(\ldots)$ 199 digits

Tables 1 and 2 give the error after each iteration for 4th order and for 5th order methods, respectively.

Iteration	Padé $(1,2)$	Inverse Padé $(2,1)$	Modified order 4
1	$5.4033e-03$	$1.5528e-02$	$5.3445e-02$
2	$2.7982e-11$	$5.6144e-09$	$4.6404e-06$
3	$2.0247e-44$	$9.7495e-35$	$2.9607e-22$
4	$5.5508e-177$	$8.8659e-138$	$4.9061e-87$

TABLE 1. Errors for each iteration, 4th order methods

Iteration	Padé $(1,3)$	Inverse Padé $(2,2)$	Modified order 5
1	$5.3370e-04$	$3.7722e-03$	$2.7441e-02$
2	$4.0001e-19$	$2.5751e-14$	$1.0904e-08$
3	$9.4690e-95$	$3.8318e-70$	$1.1775e-40$
4	$7.0386e-473$	$2.7954e-349$	$1.7284e-200$

TABLE 2. Errors for each iteration, 5th order methods

6. Conclusions

All methods presented computes a large number of correct digits in a small number of iterations. Direct Padé and inverse Padé methods are superior to modified methods. Direct Padé methods, (in fact, Householder methods) have a better accuracy than methods based on inverse Padé interpolation of the same total degree, at least for equation (5.1). The approach presented in this paper could be useful in the context of symbolic computation, when a large number of digits is required, and to automatically generate numerical methods for the solution of nonlinear equations.

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Complex left Caputo fractional inequalities

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Abstract. Here we present several complex left Caputo type fractional inequalities of well known kinds, such as of Ostrowski, Poincare, Sobolev, Opial and Hilbert-Pachpatte.

Mathematics Subject Classification (2010): 26D10, 26D15, 30A10.

Keywords: Complex inequalities, fractional inequalities, Caputo fractional derivative.

1. Introduction

We are motivated by the following result for functions of complex variable: Complex Ostrowski type inequality

Theorem 1.1. (see [3]) *Let f be holomorphic in G, an open domain and suppose $\gamma \subset G$ is a smooth path from $z(a) = u$ to $z(b) = w$. If $v = z(x)$ with $x \in (a, b)$, then $\gamma_{u,w} = \gamma_{u,v} \cup \gamma_{v,w}$,*

$$\left| f(v)(w-u) - \int_\gamma f(z)\,dz \right| \leq \|f'\|_{\gamma_{u,v};\infty} \int_{\gamma_{u,v}} |z-u|\,|dz| + \|f'\|_{\gamma_{v,w};\infty} \int_{\gamma_{v,w}} |z-w|\,|dz|$$
$$\leq \left[\int_{\gamma_{u,v}} |z-u|\,|dz| + \int_{\gamma_{v,w}} |z-w|\,|dz| \right] \|f'\|_{\gamma_{u,w};\infty},$$

and

$$\left| f(v)(w-u) - \int_\gamma f(z)\,dz \right| \leq \max_{z\in\gamma_{u,v}} |z-u|\,\|f'\|_{\gamma_{u,v};1} + \max_{z\in\gamma_{v,w}} |z-w|\,\|f'\|_{\gamma_{v,w};1}$$
$$\leq \max\left\{ \max_{z\in\gamma_{u,v}} |z-u|\,,\, \max_{z\in\gamma_{v,w}} |z-w| \right\} \|f'\|_{\gamma_{u,w};1}\,.$$

If $p, q > 1$ *with* $\frac{1}{p} + \frac{1}{q} = 1$, *then*

$$\left| f(v)(w-u) - \int_{\gamma} f(z)\, dz \right| \leq \left(\int_{\gamma_{u,v}} |z-u|^q \, |dz| \right)^{\frac{1}{q}} \|f'\|_{\gamma_{u,v};p} + \left(\int_{\gamma_{v,w}} |z-w|^q \, |dz| \right)^{\frac{1}{q}} \|f'\|_{\gamma_{v,w};p}$$

$$\leq \left(\int_{\gamma_{u,v}} |z-u|^q \, |dz| + \int_{\gamma_{v,w}} |z-w|^q \, |dz| \right)^{\frac{1}{q}} \|f'\|_{\gamma_{u,w};p} .$$

Above $|\cdot|$ *is the complex absolute value.*

We are also motivated by the next complex Opial type inequality:

Theorem 1.2. (see [2]) *Let* $f : D \subseteq \mathbb{C} \to \mathbb{C}$ *be an analytic function on the domain* D *and let* $x, y, w \in D$. *Suppose* γ *is a smooth path parametrized by* $z(t)$, $t \in [a, b]$ *with* $z(a) = x$, $z(c) = y$, *and* $z(b) = w$, *where* $c \in [a, b]$ *is floating. Assume that* $f^{(k)}(x) = 0$, $k = 0, 1, ..., n$, $n \in \mathbb{Z}_+$, *and* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. *Then*

1)

$$\left| \int_a^b f(z(t)) \, f^{(n+1)}(z(t)) \, z'(t) \, dt \right|$$

$$\leq \int_a^b |f(z(t))| \left| f^{(n+1)}(z(t)) \right| |z'(t)| \, dt$$

$$\leq \frac{1}{2^{\frac{1}{q}} n!} \left[\int_a^b \left(\int_a^c |z(c) - z(t)|^{pn} \, |z'(t)| \, dt \right) |z'(c)| \, dc \right]^{\frac{1}{p}}$$

$$\cdot \left(\int_a^b \left| f^{(n+1)}(z(t)) \right|^q |z'(t)| \, dt \right)^{\frac{2}{q}} ,$$

equivalently it holds

2)

$$\left| \int_{\gamma_{x,w}} f(z) \, f^{(n+1)}(z) \, dz \right| \leq \int_{\gamma_{x,w}} |f(z)| \left| f^{(n+1)}(z) \right| |dz|$$

$$\leq \frac{1}{2^{\frac{1}{q}} n!} \left[\int_a^b \left(\int_{\gamma_{x,y}} |z(c) - z|^{pn} \, |dz| \right) |z'(c)| \, dc \right]^{\frac{1}{p}} \left(\int_{\gamma_{x,w}} \left| f^{(n+1)}(z) \right|^q |dz| \right)^{\frac{2}{q}} .$$

Here we utilize on $\mathbb{C}$ the results of [1] which are for general Banach space valued functions.

Mainly we give different cases of the left fractional $\mathbb{C}$-Ostrowski type inequality and we continue with the left fractional: $\mathbb{C}$-Poincaré like and Sobolev like inequalities.

We present an Opial type left $\mathbb{C}$-fractional inequality, and we finish with the Hilbert-Pachpatte left $\mathbb{C}$-fractional inequalities.

2. Background

In this section all integrals are of Bochner type.
We need

Definition 2.1. (see [4]) A definition of the Hausdorff measure h_α goes as follows: if (T,d) is a metric space, $A \subseteq T$ and $\delta > 0$, let $\Lambda(A,\delta)$ be the set of all arbitrary collections $(C)_i$ of subsets of T, such that $A \subseteq \cup_i C_i$ and $diam(C_i) \leq \delta$ ($diam$ =diameter) for every i. Now, for every $\alpha > 0$ define

$$h_\alpha^\delta(A) := \inf\left\{\sum (diam C_i)^\alpha \mid (C_i)_i \in \Lambda(A,\delta)\right\}. \tag{2.1}$$

Then there exists $\lim_{\delta\to 0} h_\alpha^\delta(A) = \sup_{\delta>0} h_\alpha^\delta(A)$, and $h_\alpha(A) := \lim_{\delta\to 0} h_\alpha^\delta(A)$ gives an outer measure on the power set $\mathcal{P}(T)$, which is countably additive on the σ-field of all Borel subsets of T. If $T = \mathbb{R}^n$, then the Hausdorff measure h_n, restricted to the σ-field of the Borel subsets of $\mathbb{R}^n$, equals the Lebesgue measure on $\mathbb{R}^n$ up to a constant multiple. In particular, $h_1(C) = \mu(C)$ for every Borel set $C \subseteq \mathbb{R}$, where μ is the Lebesgue measure.

Definition 2.2. ([1]) Let $[a,b] \subset \mathbb{R}$, X be a Banach space, $\nu > 0$; $n := \lceil \nu \rceil \in \mathbb{N}$, $\lceil\cdot\rceil$ is the ceiling of the number, $f : [a,b] \to X$. We assume that $f^{(n)} \in L_1([a,b],X)$. We call the Caputo-Bochner left fractional derivative of order ν:

$$\left(D_{*a}^\nu f\right)(x) := \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} f^{(n)}(t)\,dt, \quad \forall\, x \in [a,b]. \tag{2.2}$$

If $\nu \in \mathbb{N}$, we set $D_{*a}^\nu f := f^{(\nu)}$ the ordinary X-valued derivative, defined similarly to the numerical one, and also set $D_{*a}^0 f := f$.

By [1] $\left(D_{*a}^\nu f\right)(x)$ exists almost everywhere in $x \in [a,b]$ and $D_{*a}^\nu f \in L_1([a,b],X)$.
If $\left\|f^{(n)}\right\|_{L_\infty([a,b],X)} < \infty$, then by [1] $D_{*a}^\nu f \in C([a,b],X)$.
We need the left-fractional Taylor's formula:

Theorem 2.3. ([1]) *Let $n \in \mathbb{N}$ and $f \in C^{n-1}([a,b],X)$, where $[a,b] \subset \mathbb{R}$ and X is a Banach space, and let $\nu \geq 0 : n = \lceil \nu \rceil$. Set*

$$F_x(t) := \sum_{i=0}^{n-1} \frac{(x-t)^i}{i!} f^{(i)}(t), \quad \forall\, t \in [a,x], \tag{2.3}$$

where $x \in [a,b]$.
Assume that $f^{(n)}$ exists outside a μ-null Borel set $B_x \subseteq [a,x]$, such that

$$h_1(F_x(B_x)) = 0, \ \forall\, x \in [a,b]. \tag{2.4}$$

We also assume that $f^{(n)} \in L_1([a,b],X)$. Then

$$f(x) = \sum_{i=0}^{n-1} \frac{(x-a)^i}{i!} f^{(i)}(a) + \frac{1}{\Gamma(\nu)} \int_a^x (x-z)^{\nu-1} \left(D_{*a}^\nu f\right)(z)\,dz, \tag{2.5}$$

$\forall\, x \in [a,b]$.

Next we mention an Ostrowski type inequality at left fractional level for Banach valued functions.

Theorem 2.4. ([1]) *Let $\nu \geq 0$, $n = \lceil \nu \rceil$. Here all as in Theorem 2.3. Assume that $f^{(i)}(a) = 0$, $i = 1, ..., n-1$, and that $D_{*a}^{\nu} f \in L_{\infty}([a,b], X)$. Then*

$$\left\| \frac{1}{b-a} \int_a^b f(x)\, dx - f(a) \right\| \leq \frac{\|D_{*a}^{\nu} f\|_{L_{\infty}([a,b],X)}}{\Gamma(\nu+2)} (b-a)^{\nu}. \tag{2.6}$$

We mention an Ostrowski type L_p fractional inequality:

Theorem 2.5. ([1]) *Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, and $\nu > \frac{1}{q}$, $n = \lceil \nu \rceil$. Here all as in Theorem 2.3. Assume that $f^{(k)}(a) = 0$, $k = 1, ..., n-1$, and $D_{*a}^{\nu} f \in L_q([a,b], X)$, where X is a Banach space. Then*

$$\left\| \frac{1}{b-a} \int_a^b f(x)\, dx - f(a) \right\| \leq \frac{\|D_{*a}^{\nu} f\|_{L_q([a,b],X)}}{\Gamma(\nu)(p(\nu-1)+1)^{\frac{1}{p}} \left(\nu + \frac{1}{p}\right)} (b-a)^{\nu - \frac{1}{q}}. \tag{2.7}$$

It follows

Corollary 2.6. ([1]) *(to Theorem 2.5, case of $p = q = 2$). Let $\nu > \frac{1}{2}$, $n = \lceil \nu \rceil$. Here all as in Theorem 2.3. Assume that $f^{(k)}(a) = 0$, $k = 1, ..., n-1$, and $D_{*a}^{\nu} f \in L_2([a,b], X)$. Then*

$$\left\| \frac{1}{b-a} \int_a^b f(x)\, dx - f(a) \right\| \leq \frac{\|D_{*a}^{\nu} f\|_{L_2([a,b],X)}}{\Gamma(\nu)\left(\sqrt{2\nu-1}\right)\left(\nu + \frac{1}{2}\right)} (b-a)^{\nu - \frac{1}{2}}. \tag{2.8}$$

Next comes the L_1 case of fractional Ostrowski inequality:

Theorem 2.7. ([1]) *Let $\nu \geq 1$, $n = \lceil \nu \rceil$, and all as in Theorem 2.3. Assume that $f^{(k)}(a) = 0$, $k = 1, ..., n-1$, and $D_{*a}^{\nu} f \in L_1([a,b], X)$. Then*

$$\left\| \frac{1}{b-a} \int_a^b f(x)\, dx - f(a) \right\| \leq \frac{\|D_{*a}^{\nu} f\|_{L_1([a,b],X)}}{\Gamma(\nu+1)} (b-a)^{\nu-1}. \tag{2.9}$$

We continue with a Poincaré like fractional inequality:

Theorem 2.8. ([1]) *Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, and $\nu > \frac{1}{q}$, $n = \lceil \nu \rceil$. Here all as in Theorem 2.3. Assume that $f^{(k)}(a) = 0$, $k = 0, 1, ..., n-1$, and $D_{*a}^{\nu} f \in L_q([a,b], X)$, where X is a Banach space. Then*

$$\|f\|_{L_q([a,b],X)} \leq \frac{(b-a)^{\nu}}{\Gamma(\nu)(p(\nu-1)+1)^{\frac{1}{p}} (q\nu)^{\frac{1}{q}}} \|D_{*a}^{\nu} f\|_{L_q([a,b],X)}. \tag{2.10}$$

Next comes a Sobolev like fractional inequality.

Theorem 2.9. ([1]) *All as in the last Theorem 2.8. Let $r > 0$. Then*

$$\|f\|_{L_r([a,b],X)} \leq \frac{(b-a)^{\nu - \frac{1}{q} + \frac{1}{r}}}{\Gamma(\nu)(p(\nu-1)+1)^{\frac{1}{p}} \left(r\left(\nu - \frac{1}{q}\right)+1\right)^{\frac{1}{r}}} \|D_{*a}^{\nu} f\|_{L_q([a,b],X)}. \tag{2.11}$$

We mention the following Opial type fractional inequality:

Theorem 2.10. ([1]) *Let* $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, *and* $\nu>\frac{1}{q}$, $n:=\lceil\nu\rceil$. *Let* $[a,b]\subset\mathbb{R}$, X *a Banach space, and* $f\in C^{n-1}([a,b],X)$. *Set*

$$F_x(t):=\sum_{i=0}^{n-1}\frac{(x-t)^i}{i!}f^{(i)}(t),\quad\forall\ t\in[a,x],\ \text{where } x\in[a,b]. \tag{2.12}$$

Assume that $f^{(n)}$ *exists outside a* μ*-null Borel set* $B_x\subseteq[a,x]$, *such that*

$$h_1(F_x(B_x))=0,\quad\forall\ x\in[a,b]. \tag{2.13}$$

We also assume that $f^{(n)}\in L_\infty([a,b],X)$.
Assume also that $f^{(k)}(a)=0$, $k=0,1,...,n-1$. *Then*

$$\int_a^x\|f(w)\|\,\|(D_{*a}^{\nu}f)(w)\|\,dw$$

$$\leq\frac{(x-a)^{\nu-1+\frac{2}{p}}}{2^{\frac{1}{q}}\Gamma(\nu)\left((p(\nu-1)+1)(p(\nu-1)+2)\right)^{\frac{1}{p}}}\left(\int_a^x\|(D_{*a}^{\nu}f)(z)\|^q\,dz\right)^{\frac{2}{q}}, \tag{2.14}$$

$\forall\ x\in[a,b]$.

We finish this section with a Hilbert-Pachpatte left fractional inequality:

Theorem 2.11. ([1]) *Let* $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, *and* $\nu_1>\frac{1}{q}$, $\nu_2>\frac{1}{p}$, $n_i:=\lceil\nu_i\rceil$, $i=1,2$. *Here* $[a_i,b_i]\subset\mathbb{R}$, $i=1,2$; X *is a Banach space. Let* $f_i\in C^{n_i-1}([a_i,b_i],X)$, $i=1,2$. *Set*

$$F_{x_i}(t_i):=\sum_{j_i=0}^{n_i-1}\frac{(x_i-t_i)^{j_i}}{j_i!}f_i^{(j_i)}(t_i), \tag{2.15}$$

$\forall\ t_i\in[a_i,x_i]$, *where* $x_i\in[a_i,b_i]$; $i=1,2$. *Assume that* $f_i^{(n_i)}$ *exists outside a* μ*-null Borel set* $B_{x_i}\subseteq[a_i,x_i]$, *such that*

$$h_1(F_{x_i}(B_{x_i}))=0,\quad\forall\ x_i\in[a_i,b_i];\ i=1,2. \tag{2.16}$$

We also assume that $f_i^{(n_i)}\in L_1([a_i,b_i],X)$, *and*

$$f_i^{(k_i)}(a_i)=0,\quad k_i=0,1,...,n_i-1;\ i=1,2, \tag{2.17}$$

and

$$\left(D_{*a_1}^{\nu_1}f_1\right)\in L_q([a_1,b_1],X),\quad\left(D_{*a_2}^{\nu_2}f_2\right)\in L_p([a_2,b_2],X).$$

Then

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\frac{\|f_1(x_1)\|\,\|f_2(x_2)\|\,dx_1dx_2}{\left(\frac{(x_1-a_1)^{p(\nu_1-1)+1}}{p(p(\nu_1-1)+1)}+\frac{(x_2-a_2)^{q(\nu_2-1)+1}}{q(q(\nu_2-1)+1)}\right)}$$

$$\leq\frac{(b_1-a_1)(b_2-a_2)}{\Gamma(\nu_1)\Gamma(\nu_2)}\left\|D_{*a_1}^{\nu_1}f_1\right\|_{L_q([a_1,b_1],X)}\left\|D_{*a_2}^{\nu_2}f_2\right\|_{L_p([a_2,b_2],X)}. \tag{2.18}$$

3. Main results

We need a special case of Definition 2.2 over $\mathbb{C}$.

Definition 3.1. Let $[a,b] \subset \mathbb{R}$, $\nu > 0$; $n := \lceil \nu \rceil \in \mathbb{N}$, $\lceil \cdot \rceil$ is the ceiling of the number and $f \in C^n([a,b],\mathbb{C})$. We call Caputo-Complex left fractional derivative of order ν:

$$\left(D_{*a}^{\nu} f\right)(x) := \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} f^{(n)}(t)\,dt, \quad \forall\, x \in [a,b], \tag{3.1}$$

where the derivatives $f', \ldots f^{(n)}$ are defined as the numerical derivative.

If $\nu \in \mathbb{N}$, we set $D_{*a}^{\nu} f := f^{(\nu)}$ the ordinary $\mathbb{C}$-valued derivative and also set $D_{*a}^{0} f := f$.

Notice here (by [1]) that $D_{*a}^{\nu} f \in C([a,b],\mathbb{C})$.

We make

Remark 3.2. Suppose γ is a smooth path parametrized by $z(t)$, $t \in [a,b]$ (i.e. there exists $z'(t)$ and is continuous) and from now on f is a complex function which is continuous on γ.

Put $z(a) = u$ and $z(b) = w$ with $u, w \in \mathbb{C}$. We define the integral of f on $\gamma_{u,w} = \gamma$ as

$$\int_{\gamma} f(z)\,dz = \int_{\gamma_{u,w}} f(z)\,dz := \int_a^b f(z(t))\, z'(t)\,dt = \int_a^b h(t)\,dt, \tag{3.2}$$

where $h(t) := f(z(t))\,z'(t)$, $t \in [a,b]$.

We notice that the actual choice of parametrization of γ does not matter.

This definition immediately extends to paths that are piecewise smooth. Suppose γ is parametrized by $z(t)$, $t \in [a,b]$, which is differentiable on the intervals $[a,c]$ and $[c,b]$, then assuming that f is continuous on γ we define

$$\int_{\gamma_{u,w}} f(z)\,dz := \int_{\gamma_{u,v}} f(z)\,dz + \int_{\gamma_{v,w}} f(z)\,dz,$$

where $v := z(c)$. This can be extended for a finite number of intervals.

We also define the integral with respect to arc-length

$$\int_{\gamma_{u,w}} f(z)\,|dz| := \int_a^b f(z(t))\,|z'(t)|\,dt$$

and the length of the curve γ is then

$$l(\gamma) = \int_{\gamma_{u,w}} |dz| := \int_a^b |z'(t)|\,dt.$$

We mention also the triangle inequality for the complex integral, namely

$$\left| \int_{\gamma} f(z)\,dz \right| \leq \int_{\gamma} |f(z)|\,|dz| \leq \|f\|_{\gamma,\infty}\, l(\gamma), \tag{3.3}$$

where $\|f\|_{\gamma,\infty} := \sup_{z \in \gamma} |f(z)|$.

We give the following left-fractional $\mathbb{C}$-Taylor's formula:

Theorem 3.3. *Let $h \in C^n([a,b],\mathbb{C})$, $n = \lceil \nu \rceil$, $\nu \geq 0$. Then*

$$h(t) = \sum_{i=0}^{n-1} \frac{(t-a)^i}{i!} h^{(i)}(a) + \frac{1}{\Gamma(\nu)} \int_a^t (t-\lambda)^{\nu-1} (D_{*a}^{\nu} h)(\lambda)\, d\lambda, \tag{3.4}$$

$\forall\ t \in [a,b]$, in particular it holds,

$$f(z(t))\, z'(t) = \sum_{i=0}^{n-1} \frac{(t-a)^i}{i!} \left(f(z(a))\, z'(a)\right)^{(i)} + \frac{1}{\Gamma(\nu)} \int_a^t (t-\lambda)^{\nu-1} \left(D_{*a}^{\nu} f(z(\cdot))\, z'(\cdot)\right)(\lambda)\, d\lambda, \tag{3.5}$$

$\forall\ t \in [a,b]$.

Proof. By Theorem 2.3. ☐

It follows a left fractional $\mathbb{C}$-Ostroswski type inequality

Theorem 3.4. *Let $n \in \mathbb{N}$ and $h \in C^n([a,b],\mathbb{C})$, where $[a,b] \subset \mathbb{R}$, and let $\nu \geq 0 : n = \lceil \nu \rceil$. Assume that $h^{(i)}(a) = 0$, $i = 1,...,n-1$. Then*

$$\left| \frac{1}{b-a} \int_a^b h(t)\, dt - f(a) \right| \leq \frac{\left\| D_{*a}^{\nu} h \right\|_{\infty,[a,b]}}{\Gamma(\nu+2)} (b-a)^{\nu}, \tag{3.6}$$

in particular when $h(t) := f(z(t))\, z'(t)$ and $\left(f(z(t))\, z'(t)\right)^{(i)} |_{t=a} = 0$, $i = 1,...n-1$, we get

$$\left| \frac{1}{b-a} \int_{\gamma_{u,w}} f(z)\, dz - f(u)\, z'(a) \right| = \left| \frac{1}{b-a} \int_a^b f(z(t))\, z'(t)\, dt - f(z(a))\, z'(a) \right|$$

$$\leq \frac{\left\| D_{*a}^{\nu} f(z(t))\, z'(t) \right\|_{\infty,[a,b]}}{\Gamma(\nu+2)} (b-a)^{\nu}. \tag{3.7}$$

Proof. By Theorem 2.4. ☐

The corresponding $\mathbb{C}$-Ostrowski type L_p inequality follows:

Theorem 3.5. *Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, and $\nu > \frac{1}{q}$, $n = \lceil \nu \rceil$. Here $h \in C^n([a,b],\mathbb{C})$. Assume that $h^{(i)}(a) = 0$, $i = 1,...,n-1$. Then*

$$\left| \frac{1}{b-a} \int_a^b h(t)\, dt - h(a) \right| \leq \frac{\left\| D_{*a}^{\nu} h \right\|_{L_q([a,b],\mathbb{C})}}{\Gamma(\nu) \left(p(\nu-1)+1\right)^{\frac{1}{p}} \left(\nu + \frac{1}{p}\right)} (b-a)^{\nu - \frac{1}{q}}, \tag{3.8}$$

in particular when $h(t) := f(z(t))z'(t)$ *and* $(f(z(t))z'(t))^{(i)}|_{t=a} = 0$, $i = 1, ...n-1$, *we get:*

$$\left|\frac{1}{b-a}\int_{\gamma_{u,w}} f(z)\,dz - f(u)\,z'(a)\right| = \left|\frac{1}{b-a}\int_a^b f(z(t))\,z'(t)\,dt - f(z(a))\,z'(a)\right|$$
$$\leq \frac{\|D_{*a}^{\nu}(f(z(t))\,z'(t))\|_{L_q([a,b],\mathbb{C})}}{\Gamma(\nu)(p(\nu-1)+1)^{\frac{1}{p}}\left(\nu+\frac{1}{p}\right)}(b-a)^{\nu-\frac{1}{q}}. \tag{3.9}$$

Proof. By Theorem 2.5. □

It follows

Corollary 3.6. *(to Theorem* 3.5*, case of* $p = q = 2$*). We have that*

$$\left|\frac{1}{b-a}\int_{\gamma_{u,w}} f(z)\,dz - f(u)\,z'(a)\right| \leq \frac{\|D_{*a}^{\nu}(f(z(t))\,z'(t))\|_{L_2([a,b],\mathbb{C})}}{\Gamma(\nu)\sqrt{2\nu-1}\left(\nu+\frac{1}{2}\right)}(b-a)^{\nu-\frac{1}{2}}. \tag{3.10}$$

We continue with an L_1 fractional $\mathbb{C}$-Ostrowski type inequality:

Theorem 3.7. *Let* $\nu \geq 1$, $n = \lceil \nu \rceil$*. Assume that* $h \in C^n([a,b],\mathbb{C})$*, where*

$$h(t) := f(z(t))\,z'(t),$$

and such that $h^{(i)}(a) = 0$, $i = 1, ..., n-1$*. Then*

$$\left|\frac{1}{b-a}\int_{\gamma_{u,w}} f(z)\,dz - f(u)\,z'(a)\right| \leq \frac{\|D_{*a}^{\nu}(f(z(t))\,z'(t))\|_{L_1([a,b],\mathbb{C})}}{\Gamma(\nu+1)}(b-a)^{\nu-1}. \tag{3.11}$$

Proof. By Theorem 2.7. □

It follows a Poincaré like $\mathbb{C}$-fractional inequality:

Theorem 3.8. *Let* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$*, and* $\nu > \frac{1}{q}$, $n = \lceil \nu \rceil$*. Let* $h \in C^n([a,b],\mathbb{C})$*. Assume that* $h^{(i)}(a) = 0$, $i = 1, ..., n-1$*. Then*

$$\|h\|_{L_q([a,b],\mathbb{C})} \leq \frac{(b-a)^{\nu}\,\|D_{*a}^{\nu}h\|_{L_q([a,b],\mathbb{C})}}{\Gamma(\nu)(p(\nu-1)+1)^{\frac{1}{p}}(q\nu)^{\frac{1}{q}}}, \tag{3.12}$$

in particular when $h(t) := f(z(t))z'(t)$ *and* $(f(z(t))z'(t))^{(i)}|_{t=a} = 0$, $i = 1, ...n-1$, *we get:*

$$\|f(z(t))\,z'(t)\|_{L_q([a,b],\mathbb{C})}$$
$$\leq \frac{(b-a)^{\nu}}{\Gamma(\nu)(p(\nu-1)+1)^{\frac{1}{p}}(q\nu)^{\frac{1}{q}}}\|D_{*a}^{\nu}(f(z(t))\,z'(t))\|_{L_q([a,b],\mathbb{C})}. \tag{3.13}$$

Proof. By Theorem 2.8. □

The corresponding Sobolev like inequality follows:

Theorem 3.9. *All as in Theorem* 3.8. *Let* $r > 0$. *Then*

$$\|f(z(t))\,z'(t)\|_{L_r([a,b],\mathbb{C})}$$
$$\leq \frac{(b-a)^{\nu-\frac{1}{q}+\frac{1}{r}}}{\Gamma(\nu)\,(p(\nu-1)+1)^{\frac{1}{p}}\left(r\left(\nu-\frac{1}{q}\right)+1\right)^{\frac{1}{r}}}\,\|D_{*a}^{\nu}(f(z(t))\,z'(t))\|_{L_q([a,b],\mathbb{C})}. \tag{3.14}$$

Proof. By Theorem 2.9. □

We continue with an Opial type $\mathbb{C}$-fractional inequality

Theorem 3.10. *Let* $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, *and* $\nu>\frac{1}{q}$, $n:=\lceil\nu\rceil$, $h\in C^n([a,b],\mathbb{C})$. *Assume* $h^{(k)}(a)=0$, $k=0,1,...,n-1$. *Then*

$$\int_a^x |h(t)|\,|(D_{*a}^{\nu}h)(t)|\,dt$$
$$\leq \frac{(x-a)^{\nu-1+\frac{2}{p}}}{2^{\frac{1}{q}}\Gamma(\nu)\,((p(\nu-1)+1)(p(\nu-1)+2))^{\frac{1}{p}}}\left(\int_a^x |(D_{*a}^{\nu}h)(t)|^q\,dt\right)^{\frac{2}{q}}, \tag{3.15}$$

$\forall\ x\in[a,b]$, *in particular when* $h(t):=f(z(t))\,z'(t)$ *and* $(f(z(t))\,z'(t))^{(i)}\,|_{t=a}=0$, $i=1,...n-1$, *we get:*

$$\int_a^x |f(z(t))|\,|(D_{*a}^{\nu}(f(z(t))\,z'(t)))|\,|z'(t)|\,dt$$
$$\leq \frac{(x-a)^{\nu-1+\frac{2}{p}}}{2^{\frac{1}{q}}\Gamma(\nu)\,((p(\nu-1)+1)(p(\nu-1)+2))^{\frac{1}{p}}}\left(\int_a^x |D_{*a}^{\nu}(f(z(t))\,z'(t))|^q\,dt\right)^{\frac{2}{q}}, \tag{3.16}$$

$\forall\ x\in[a,b]$.

Proof. By Theorem 2.10. □

We finish with Hilbert-Pachpatte left $\mathbb{C}$-fractional inequalities:

Theorem 3.11. *Let* $p,q>1:\frac{1}{p}+\frac{1}{q}=1$, *and* $\nu_1>\frac{1}{q}$, $\nu_2>\frac{1}{p}$, $n_i:=\lceil\nu_i\rceil$, $i=1,2$. *Let* $h_i\in C^{n_i}([a_i,b_i],\mathbb{C})$, $i=1,2$. *Assume* $h_i^{(k_i)}(a_i)=0$, $k_i=0,1,...,n_i-1$; $i=1,2$. *Then*

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\frac{|h_1(t_1)|\,|h_2(t_2)|\,dt_1dt_2}{\left(\frac{(t_1-a_1)^{p(\nu_1-1)+1}}{p(p(\nu_1-1)+1)}+\frac{(t_2-a_2)^{q(\nu_2-1)+1}}{q(q(\nu_2-1)+1)}\right)}$$
$$\leq \frac{(b_1-a_1)(b_2-a_2)}{\Gamma(\nu_1)\,\Gamma(\nu_2)}\left\|D_{*a_1}^{\nu_1}h_1\right\|_{L_q([a_1,b_1],\mathbb{C})}\left\|D_{*a_2}^{\nu_2}h_2\right\|_{L_p([a_2,b_2],\mathbb{C})}, \tag{3.17}$$

in particular when $h_1(t_1):=f_1(z_1(t_1))\,z_1'(t_1)$ *and* $h_2(t_2):=f_2(z_2(t_2))\,z_2'(t_2)$, *with* $h_i^{(k_i)}(a_i)=0$, $k_i=0,1,...n_i-1$; $i=1,2$, *we get:*

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\frac{|f_1(z_1(t_1))\,z_1'(t_1)|\,|f_2(z_2(t_2))\,z_2'(t_2)|\,dt_1dt_2}{\left(\frac{(t_1-a_1)^{p(\nu_1-1)+1}}{p(p(\nu_1-1)+1)}+\frac{(t_2-a_2)^{q(\nu_2-1)+1}}{q(q(\nu_2-1)+1)}\right)}\leq\frac{(b_1-a_1)(b_2-a_2)}{\Gamma(\nu_1)\,\Gamma(\nu_2)}\cdot$$
$$\left\|D_{*a_1}^{\nu_1}(f_1(z_1(t_1))\,z_1'(t_1))\right\|_{L_q([a_1,b_1],\mathbb{C})}\left\|D_{*a_2}^{\nu_2}(f_2(z_2(t_2))\,z_2'(t_2))\right\|_{L_p([a_2,b_2],\mathbb{C})}. \tag{3.18}$$

Proof. By Theorem 2.11. □

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Classes of an univalent integral operator

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Abstract. In this paper we introduce a new general integral operator for analytic functions in the open unit disk **U** and we obtain sufficient conditions for univalence of this integral operator.

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1. Introduction

Let $\mathcal{A}$ be the class of the functions f which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} :| z |< 1\}$ and $f(0) = f^{'}(0) - 1 = 0$.

We denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions $f \in \mathcal{A}$, which are univalent in $\mathbb{U}$.

We consider the integral operator

$$\mathcal{T}_n(z) = \left\{ \delta \int_0^z t^{\delta-1} \prod_{i=1}^{n} \left[\left(\frac{f_i(t)}{t} \right)^{\alpha_i - 1} \cdot \left(g_i(t)' \right)^{\beta_i} \cdot \left(\frac{h_i(t)}{k_i(t)} \right)^{\gamma_i} \cdot \left(\frac{{h_i}'(t))}{{k_i}'(t)} \right)^{\delta_i} \right] \mathrm{dt} \right\}^{\frac{1}{\delta}} \tag{1.1}$$

for $f_i, g_i, h_i, k_i \in \mathcal{A}$ and the complex numbers $\delta, \alpha_i, \beta_i, \gamma_i, \delta_i$, with $\delta \neq 0$, $i = \overline{1, n}$, $n \in \mathbb{N} \setminus \{0\}$.

Remark 1.1. The integral operator $\mathcal{T}_n$ defined by (1.1), is a general integral operator of Pfaltzgraff, Kim-Merkes and Ovesea types which extends also the other operators as follows:

i) For $n = 1$, $\delta = 1$, $\alpha_1 - 1 = \alpha_1$ and $\beta_1 = \gamma_1 = \delta_1 = 0$ we obtain the integral operator which was studied by Kim-Merkes [7].

$$\mathcal{F}_\alpha(z) = \int_0^z \left(\frac{f(t)}{t} \right)^{\alpha} \mathrm{dt},$$

ii) For $n = 1$, $\delta = 1$ and $\alpha_1 - 1 = \gamma_1 = \delta_1 = 0$ we obtain the integral operator which was studied by Pfaltzgraff [18].

$$\mathcal{G}_\alpha(z) = \int_0^z (f'(t))^\alpha \,\mathrm{dt},$$

iii) For $\alpha_i - 1 = \alpha_i$ and $\beta_i = \gamma_i = \delta_i = 0$ we obtain the integral operator which was defined and studied by D. Breaz and N. Breaz [2].

$$\mathcal{D}_n(z) = \left[\delta \int_0^z t^{\delta-1} \prod_{i=1}^n \left(\frac{f_i(t)}{t}\right)^{\alpha_i} \mathrm{dt}\right]^{\frac{1}{\delta}},$$

this integral operator is a generalization of the integral operator introduced by Pascu and Pescar [12].

iv) For $\alpha_i - 1 = \gamma_i = \delta_i = 0$ we obtain the integral operator which was defined and studied by D. Breaz, Owa and N. Breaz [4]

$$\mathcal{I}_n(z) = \left[\delta \int_0^z t^{\delta-1} \prod_{i=1}^n [f_i'(t)]^{\alpha_i} \,\mathrm{dt}\right]^{\frac{1}{\delta}},$$

this integral operator is a generalization of the integral operator introduced by Pescar and Owa in [17] .

v) For $\alpha_i - 1 = \alpha_i$ and $\gamma_i = \delta_i = 0$ we obtain the integral operator which was defined and studied by Frasin [5]

$$\mathcal{F}_n(z) = \left[\delta \int_0^z t^{\delta-1} \prod_{i=1}^n \left(\frac{f_i(t)}{t}\right)^{\alpha_i} \left(f_i{}'(t)\right)^{\beta_i} \mathrm{dt}\right]^{\frac{1}{\delta}},$$

this integral operator is a generalization of the integral operator introduced by Ovesea in [9].

vi) For $\alpha_i - 1 = \beta_i = 0$ we obtain the integral operator which was defined and studied by Pescar [13].

$$\mathcal{I}_n(z) = \left[\delta \int_0^z t^{\delta-1} \prod_{i=1}^n \left(\frac{f_i(t)}{g_i(t)}\right)^{\gamma_i} \left(\frac{f_i'(t)}{g_i'(t)}\right)^{\delta_i} \mathrm{dt}\right]^{\frac{1}{\delta}},$$

Thus, the integral operator $\mathcal{T}_n$, introduced here by the formula (1.1), can be considered as an extension and a generalization of these operators above mentioned.

We need the following lemmas.

Lemma 1.2. [11] *Let γ, δ be complex numbers, $Re\gamma > 0$ and $f \in \mathcal{A}$. If*

$$\frac{1-|z|^{2Re\gamma}}{Re\gamma}\left|\frac{zf''(z))}{f'(z)}\right| \leq 1,$$

for all $z \in \mathbb{U}$, then for any complex number δ, $Re\delta \geq Re\gamma$, the function F_δ defined by

$$F_\delta(z) = \left(\delta \int_0^z t^{\delta-1} f'(t)dt\right)^{\frac{1}{\delta}},$$

is regular and univalent in $\mathbb{U}$.

Lemma 1.3. [14] *Let* δ *be complex number,* $Re\delta > 0$ *and* c *a complex number,* $|c| \le 1$, $c \ne -1$, *and* $f \in \mathcal{A}$, $f(z) = z + a_2 z^2 + \ldots$. *If*

$$\left| c\left|z\right|^{2\delta} + \left(1 - |z|^{2\delta}\right) \frac{zf''(z))}{\delta f'(z)} \right| \le 1,$$

for all $z \in \mathbb{U}$, *then the function* F_δ *defined by*

$$F_\delta(z) = \left(\delta \int_0^z t^{\delta - 1} f'(t)\, dt \right)^{\frac{1}{\delta}},$$

is regular and univalent in $\mathbb{U}$.

Lemma 1.4. [8] *Let* f *be the function regular in the disk* $\mathbb{U}_R = \{z \in \mathbb{C} : |z| < R\}$ *with* $|f(z)| < M$, M *fixed. If* $f(z)$ *has in* $z = 0$ *one zero with multiply* $\ge m$, *then*

$$|f(z)| \le \frac{M}{R^m} z^m,$$

the equality for $z \ne 0$ *can hold only if*

$$f(z) = e^{i\theta} \frac{M}{R^m} z^m,$$

where θ *is constant.*

2. Main results

Theorem 2.1. *Let* $\gamma, \delta, \alpha_i, \beta_i, \gamma_i, \delta_i$ *be complex numbers,* $c = Re\gamma > 0$, M_i, N_i, P_i, Q_i, R_i, S_i *real positive numbers and* $f_i, g_i, h_i, k_i \in \mathcal{A}$,

$$\begin{aligned} f_i(z) &= z + a_{2i} z^2 + a_{3i} z^3 + \ldots, \\ g_i(z) &= z + b_{2i} z^2 + b_{3i} z^3 + \ldots, \\ h_i(z) &= z + c_{2i} z^2 + c_{3i} z^3 + \ldots, \\ k_i(z) &= z + d_{2i} z^2 + d_{3i} z^3 + \ldots,\ i = \overline{1,n}. \end{aligned}$$

If

$$\left| \frac{zf_i'(z)}{f_i(z)} - 1 \right| \le M_i, \quad \left| \frac{zg_i'(z)}{g_i(z)} - 1 \right| \le N_i, \quad \left| \frac{zh_i'(z)}{h_i(z)} - 1 \right| \le P_i,$$
$$\left| \frac{zk_i'(z)}{k_i(z)} - 1 \right| \le Q_i, \quad \left| \frac{zh_i''(z)}{h_i'(z)} \right| \le R_i, \quad \left| \frac{zk_i''(z)}{k_i'(z)} \right| \le S_i,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$ *and*

$$\sum_{i=1}^{n} \left[|\alpha_i - 1| M_i + |\beta_i| N_i + |\gamma_i| (P_i + Q_i) + |\delta_i| (R_i + S_i) \right] \le \frac{(2c+1)^{\frac{2c+1}{2c}}}{2}, \qquad (2.1)$$

then, for all δ *complex numbers,* $Re\delta \ge Re\gamma$, *the integral operator* $\mathcal{T}_n$, *given by (1.1) is in the class* $\mathcal{S}$.

Proof. Let us define the function

$$H_n(z) = \int_0^z \prod_{i=1}^n \left[\left(\frac{f_i(t)}{t} \right)^{\alpha_i - 1} \cdot \left(g_i(t)' \right)^{\beta_i} \cdot \left(\frac{h_i(t)}{k_i(t)} \right)^{\gamma_i} \cdot \left(\frac{h_i{}'(t)}{k_i{}'(t)} \right)^{\delta_i} \right] \mathrm{dt},$$

for $f_i, g_i, h_i, k_i \in \mathcal{A}$, $i = \overline{1,n}$.

The function H_n is regular in $\mathbb{U}$ and satisfy the following usual normalization conditions $H_n(0) = H_n^{'}(0) - 1 = 0$.

Now

$$H_n{}'(z) = \prod_{i=1}^n \left[\left(\frac{f_i(t)}{t} \right)^{\alpha_i - 1} \cdot \left(g_i(t)' \right)^{\beta_i} \cdot \left(\frac{h_i(z)}{k_i(z)} \right)^{\gamma_i} \cdot \left(\frac{h_i'(z)}{k_i'(z)} \right)^{\delta_i} \right].$$

We have

$$\frac{zH_n''(z)}{H_n'(z)} = \sum_{i=1}^n \left[(\alpha_i - 1) \left(\frac{zf_i'(z)}{f_i(z)} - 1 \right) + \beta_i \frac{zg_i''(z)}{g_i'(z)} \right]$$
$$+ \sum_{i=1}^n \left[\gamma_i \left(\frac{zh_i'(z)}{h_i(z)} - \frac{zk_i'(z)}{k_i(z)} \right) + \delta_i \left(\frac{zh_i''(z)}{h_i'(z)} - \frac{zk_i''(z)}{k_i'(z)} \right) \right],$$

for all $z \in \mathbb{U}$.

Thus, we have

$$\frac{1 - |z|^{2c}}{c} \left| \frac{zH_n''(z)}{H_n'(z)} \right| = \frac{1 - |z|^{2c}}{c} \sum_{i=1}^n \left[(\alpha_i - 1) \left(\frac{zf_i'(z)}{f_i(z)} - 1 \right) + \beta_i \frac{zg_i''(z)}{g_i'(z)} \right]$$
$$+ \frac{1 - |z|^{2c}}{c} \sum_{i=1}^n \left[\gamma_i \left(\frac{zh_i'(z)}{h_i(z)} - \frac{zk_i'(z)}{k_i(z)} \right) + \delta_i \left(\frac{zh_i''(z)}{h_i'(z)} - \frac{zk_i''(z)}{k_i'(z)} \right) \right],$$

for all $z \in \mathbb{U}$.

Therefore

$$\frac{1 - |z|^{2c}}{c} \left| \frac{zH_n''(z)}{H_n'(z)} \right| \le \frac{1 - |z|^{2c}}{c} \sum_{i=1}^n \left[|\alpha_i - 1| \left| \frac{zf_i'(z)}{f_i(z)} - 1 \right| + |\beta_i| \left| \frac{zg_i''(z)}{g_i'(z)} \right| \right]$$
$$+ \frac{1 - |z|^{2c}}{c} \sum_{i=1}^n \left[|\gamma_i| \left(\left| \frac{zh_i'(z)}{h_i(z)} - 1 \right| + \left| \frac{zk_i'(z)}{k_i(z)} - 1 \right| \right) \right]$$
$$+ \frac{1 - |z|^{2c}}{c} \sum_{i=1}^n \left[|\delta_i| \left(\left| \frac{zh_i''(z)}{h_i'(z)} \right| + \left| \frac{zk_i''(z)}{k_i'(z)} \right| \right) \right],$$

for all $z \in \mathbb{U}$.

By applying the General Schwarz Lemma (1.4) we obtain

$$\left| \frac{zf_i'(z)}{f_i(z)} - 1 \right| \le M_i |z|, \quad \left| \frac{zg_i''(z)}{g_i'(z)} \right| \le N_i |z|, \quad \left| \frac{zh_i'(z)}{h_i(z)} - 1 \right| \le P_i |z|,$$
$$\left| \frac{zk_i'(z)}{k_i(z)} - 1 \right| \le Q_i |z|, \quad \left| \frac{zh_i''(z)}{h_i'(z)} \right| \le R_i |z|, \quad \left| \frac{zK_i''(z)}{K_i'(z)} \right| \le S_i |z|,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$.

Using these inequalities we have

$$\frac{1-|z|^{2c}}{c}\left|\frac{zH_n''(z)}{H_n'(z)}\right|$$

$$\leq \frac{1-|z|^{2c}}{c}\,|z|\sum_{i=1}^{n}\left[|\alpha_i-1|\,M_i+|\beta_i|\,N_i+|\gamma_i|\,(P_i+Q_i)+|\delta_i|\,(R_i+S_i)\right], \qquad (2.2)$$

for all $z\in\mathbb{U}$.

Since

$$\max_{|z|\leq 1}\frac{\left(1-|z|^{2c}\right)|z|}{c}=\frac{2}{(2c+1)^{\frac{2c+1}{2c}}},$$

from (2.2), we obtain

$$\frac{1-|z|^{2c}}{c}\left|\frac{zH_n''(z)}{H_n'(z)}\right|$$

$$\leq \frac{2}{(2c+1)^{\frac{2c+1}{2c}}}\sum_{i=1}^{n}\left[|\alpha_i-1|\,M_i+|\beta_i|\,N_i+|\gamma_i|\,(P_i+Q_i)+|\delta_i|\,(R_i+S_i)\right],$$

and hence, by (2.1) we have

$$\frac{1-|z|^{2c}}{c}\left|\frac{zH_n''(z)}{H_n'(z)}\right|\leq\frac{2}{(2c+1)^{\frac{2c+1}{2c}}}\cdot\frac{(2c+1)^{\frac{2c+1}{2c}}}{2}=1,$$

for all $z\in\mathbb{U}$.

So,

$$\frac{1-|z|^{2c}}{c}\left|\frac{zH_n''(z)}{H_n'(z)}\right|\leq 1. \qquad (2.3)$$

and using (2.3), by Lemma 1.2, it results that the integral operator $\mathcal{T}_n$, given by (1.1) is in the class $\mathcal{S}$. □

If we consider $\delta=1$ in Theorem 2.1, obtain the next corollary:

Corollary 2.2. *Let $\gamma,\alpha_i,\beta_i,\gamma_i,\delta_i$ be complex numbers, $0<Re\gamma\leq 1$, $c=Re\gamma$, M_i,N_i,P_i,Q_i,R_i,S_i real positive numbers and $f_i,g_i,h_i,k_i\in\mathcal{A}$. If*

$$\left|\frac{zf_i'(z)}{f_i(z)}-1\right|\leq M_i,\quad \left|\frac{zg_i''(z)}{g_i'(z)}\right|\leq N_i,\quad \left|\frac{zh_i'(z)}{h_i(z)}-1\right|\leq P_i,$$

$$\left|\frac{zk_i'(z)}{k_i(z)}-1\right|\leq Q_i,\quad \left|\frac{zh_i''(z)}{h_i'(z)}\right|\leq R_i,\quad \left|\frac{zk_i''(z)}{k_i'(z)}\right|\leq S_i,$$

$$\sum_{i=1}^{n}\left[|\alpha_i-1|\,M_i+|\beta_i|\,N_i+|\gamma_i|\,(P_i+Q_i)+|\delta_i|\,(R_i+S_i)\right]\leq\frac{(2c+1)^{\frac{2c+1}{2c}}}{2},$$

then the integral operator $\mathcal{F}_n$ defined by

$$\mathcal{F}_n(z)=\int_0^z\prod_{i=1}^{n}\left[\left(\frac{f_i(t)}{t}\right)^{\alpha_i-1}\cdot\left(g_i(t)'\right)^{\beta_i}\cdot\left(\frac{h_i(t)}{k_i(t)}\right)^{\gamma_i}\cdot\left(\frac{h_i{}'(t))}{k_i{}'(t)}\right)^{\delta_i}\right]dt, \qquad (2.4)$$

is in the class $\mathcal{S}$.

If we consider $\delta = 1$ and $\delta_1 = \delta_2 = ... = \delta_n = 0$ in Theorem 2.1, obtain the next corollary:

Corollary 2.3. *Let $\gamma, \alpha_i, \beta_i, \gamma_i$ be complex numbers, $0 < Re\gamma \leq 1$, $c = Re\gamma$, M_i, N_i, P_i, Q_i real positive numbers and $f_i, g_i, h_i, k_i \in \mathcal{A}$. If*

$$\left|\frac{zf_i'(z)}{f_i(z)} - 1\right| \leq M_i, \quad \left|\frac{zg_i''(z)}{g_i'(z)}\right| \leq N_i,$$

$$\left|\frac{zh_i'(z)}{h_i(z)} - 1\right| \leq P_i, \quad \left|\frac{zk_i'(z)}{k_i(z)} - 1\right| \leq Q_i,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$ and

$$\sum_{i=1}^{n} \left[\left|\alpha_i - 1\right| M_i + \left|\beta_i\right| N_i + \left|\gamma_i\right| (P_i + Q_i)\right] \leq \frac{(2c+1)^{\frac{2c+1}{2c}}}{2},$$

then the integral operator $\mathcal{S}_n$ defined by

$$\mathcal{S}_n(z) = \int_0^z \prod_{i=1}^{n} \left[\left(\frac{f_i(t)}{t}\right)^{\alpha_i - 1} \cdot \left(g_i(t)'\right)^{\beta_i} \cdot \left(\frac{h_i(t)}{k_i(t)}\right)^{\gamma_i}\right] dt, \tag{2.5}$$

is in the class $\mathcal{S}$.

If we consider $\delta = 1$ and $\beta_1 = \beta_2 = ... = \beta_n = 0$ in Theorem 2.1, obtain the next corollary:

Corollary 2.4. *Let $\gamma, \alpha_i, \gamma_i, \delta_i$ be complex numbers, $0 < Re\gamma \leq 1$, $c = Re\gamma$, M_i, P_i, Q_i, R_i, S_i real positive numbers and $f_i, h_i, k_i \in \mathcal{A}$. If*

$$\left|\frac{zf_i'(z)}{f_i(z)} - 1\right| \leq M_i, \quad \left|\frac{zh_i'(z)}{h_i(z)} - 1\right| \leq P_i, \quad \left|\frac{zk_i'(z)}{k_i(z)} - 1\right| \leq Q_i,$$

$$\left|\frac{zh_i''(z)}{h_i'(z)}\right| \leq R_i, \quad \left|\frac{zk_i''(z)}{k_i'(z)}\right| \leq S_i,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$ and

$$\sum_{i=1}^{n} \left[\left|\alpha_i - 1\right| M_i + \left|\gamma_i\right| (P_i + Q_i) + \left|\delta_i\right| (R_i + S_i)\right] \leq \frac{(2c+1)^{\frac{2c+1}{2c}}}{2},$$

then the integral operator $\mathcal{X}_n$ defined by

$$\mathcal{X}_n(z) = \int_0^z \prod_{i=1}^{n} \left[\left(\frac{f_i(t)}{t}\right)^{\alpha_i - 1} \cdot \left(\frac{h_i(t)}{k_i(t)}\right)^{\gamma_i} \cdot \left(\frac{h_i{}'(t))}{k_i{}'(t)}\right)^{\delta_i}\right] dt, \tag{2.6}$$

is in the class $\mathcal{S}$.

If we consider $\delta = 1$ and $\alpha_1 = \alpha_2 = ... = \alpha_n = 0$ in Theorem 2.1, obtain the next corollary:

Corollary 2.5. *Let $\gamma, \beta_i, \gamma_i, \delta_i$ be complex numbers, $0 < Re\gamma \leq 1$, $c = Re\gamma$, N_i, P_i, Q_i, R_i, S_i real positive numbers and $g_i, h_i, k_i \in \mathcal{A}$. If*

$$\left|\frac{zg_i''(z)}{g_i'(z)}\right| \leq N_i, \quad \left|\frac{zh_i'(z)}{h_i(z)} - 1\right| \leq P_i, \quad \left|\frac{zk_i'(z)}{k_i(z)} - 1\right| \leq Q_i,$$

$$\left|\frac{zh_i''(z)}{h_i'(z)}\right| \leq R_i, \quad \left|\frac{zk_i''(z)}{k_i'(z)}\right| \leq S_i,$$

for all $z \in \mathbb{U}$, $i = \overline{1, n}$ and

$$\sum_{i=1}^{n} \left[|\beta_i| N_i + |\gamma_i| (P_i + Q_i) + |\delta_i| (R_i + S_i)\right] \leq \frac{(2c+1)^{\frac{2c+1}{2c}}}{2},$$

then the integral operator $\mathcal{D}_n$ defined by

$$\mathcal{D}_n(z) = \int_0^z \prod_{i=1}^{n} \left[\left(g_i(t)'\right)^{\beta_i} \cdot \left(\frac{h_i(t)}{k_i(t)}\right)^{\gamma_i} \cdot \left(\frac{h_i{}'(t))}{k_i{}'(t)}\right)^{\delta_i}\right] dt, \tag{2.7}$$

is in the class $\mathcal{S}$.

If we consider $\delta = 1$ and $\gamma_1 = \gamma_2 = ... = \gamma_n = 0$ in Theorem 2.1, obtain the next corollary:

Corollary 2.6. *Let $\gamma, \alpha_i, \beta_i, \delta_i$ be complex numbers, $0 < Re\gamma \leq 1$, $c = Re\gamma$, M_i, N_i, R_i, S_i real positive numbers and $f_i, g_i, h_i, k_i \in \mathcal{A}$. If*

$$\left|\frac{zf_i'(z)}{f_i(z)} - 1\right| \leq M_i, \quad \left|\frac{zg_i''(z)}{g_i'(z)}\right| \leq N_i,$$

$$\left|\frac{zh_i''(z)}{h_i'(z)}\right| \leq R_i, \quad \left|\frac{zk_i''(z)}{k_i'(z)}\right| \leq S_i,$$

for all $z \in \mathbb{U}$, $i = \overline{1, n}$ and

$$\sum_{i=1}^{n} \left[|\alpha_i - 1| M_i + |\beta_i| N_i + |\delta_i| (R_i + S_i)\right] \leq \frac{(2c+1)^{\frac{2c+1}{2c}}}{2},$$

then the integral operator $\mathcal{Y}_n$ defined by

$$\mathcal{Y}_n(z) = \int_0^z \prod_{i=1}^{n} \left[\left(\frac{f_i(t)}{t}\right)^{\alpha_i - 1} \cdot \left(g_i(t)'\right)^{\beta_i} \cdot \left(\frac{h_i{}'(t))}{k_i{}'(t)}\right)^{\delta_i}\right] dt, \tag{2.8}$$

is in the class $\mathcal{S}$.

If we consider $n = 1$, $\delta = \gamma = \alpha$ and $\alpha_i - 1 = \beta_i = \gamma_i$ in Theorem 2.1, obtain the next corollary:

Corollary 2.7. *Let α be complex number, $Re\alpha > 0$, M, N, P, Q, R, S real positive numbers and $f, g, h, k \in \mathcal{A}$. If*

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq M, \quad \left|\frac{zg''(z)}{g(z)'}\right| \leq N, \quad \left|\frac{zh'(z)}{h(z)} - 1\right| \leq P,$$

$$\left|\frac{zk'(z)}{k(z)} - 1\right| \leq Q, \quad \left|\frac{zh''(z)}{h'(z)}\right| \leq R, \quad \left|\frac{zk''(z)}{k'(z)}\right| \leq S,$$

for all $z \in \mathbb{U}$, *and*

$$|\alpha - 1|\,(M + N + P + Q + R + S) \leq \frac{(2Re\alpha + 1)^{\frac{2Re\alpha+1}{2Re\alpha}}}{2},$$

then the integral operator $\mathcal{T}$ *defined by*

$$\mathcal{T}(z) = \left[\alpha \int_0^z \left(f(t) \cdot g'(t) \cdot \frac{h(t)}{k(t)} \cdot \frac{h'(t))}{k'(t)}\right)^{\alpha-1} dt\right]^{\frac{1}{\alpha}}, \tag{2.9}$$

is in the class $\mathcal{S}$.

Theorem 2.8. *Let* $\gamma, \alpha_i, \beta_i, \gamma_i, \delta_i$ *be complex numbers,* $c = Re\gamma > 0$ *and* $f_i, h_i, k_i \in \mathcal{S}$, ${g_i}', {h_i}', {k_i}' \in \mathcal{P}$. *If*

$$4\sum_{i=1}^{n} |\alpha_i - 1| + 2\sum_{i=1}^{n} |\beta_i| + 8\sum_{i=1}^{n} |\gamma_i| + 4\sum_{i=1}^{n} |\delta_i| \leq \frac{c}{2}, \quad for \quad 0 < c < 1 \tag{2.10}$$

or

$$4\sum_{i=1}^{n} |\alpha_i - 1| + 2\sum_{i=1}^{n} |\beta_i| + 8\sum_{i=1}^{n} |\gamma_i| + 4\sum_{i=1}^{n} |\delta_i| \leq \frac{1}{2}, \quad for \quad c \geq 1 \tag{2.11}$$

then, for any complex numbers δ, $Re\delta \geq c$, *the integral operator* $\mathcal{T}_n$ *defined in (1.1) is in the class* $\mathcal{S}$.

Proof. After the same steps as in the proof of Theorem 2.1., we get

$$\begin{aligned}\frac{1-|z|^{2c}}{c}\left|\frac{zH_n''(z)}{H_n'(z)}\right| \leq{}& \frac{1-|z|^{2c}}{c}\sum_{i=1}^{n}\left[|\alpha_i - 1|\left(\left|\frac{zf_i'(z)}{f_i(z)}\right| + 1\right) + |\beta_i|\left|\frac{zg_i''(z)}{g_i'(z)}\right|\right] \\ &+ \frac{1-|z|^{2c}}{c}\sum_{i=1}^{n}\left[|\gamma_i|\left(\left|\frac{zh_i'(z)}{h_i(z)}\right| + 1 + \left|\frac{zk_i'(z)}{k_i(z)}\right| + 1\right)\right] \\ &+ \frac{1-|z|^{2c}}{c}\sum_{i=1}^{n}\left[|\delta_i|\left(\left|\frac{zh_i''(z)}{h_i'(z)}\right| + \left|\frac{zk_i''(z)}{k_i'(z)}\right|\right)\right],\end{aligned}$$

for all $z \in \mathbb{U}$.

Since $f_i, h_i, k_i \in \mathcal{S}$ we have

$$\left|\frac{zf_i'(z)}{f_i(z)}\right| \leq \frac{1+|z|}{1-|z|}, \quad \left|\frac{zh_i'(z)}{h_i(z)}\right| \leq \frac{1+|z|}{1-|z|}, \quad \left|\frac{zk_i'(z)}{k_i(z)}\right| \leq \frac{1+|z|}{1-|z|},$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$.

For ${g_i}', {h_i}', {k_i}' \in \mathcal{P}$ we have

$$\left|\frac{zg_i''(z)}{g_i'(z)}\right| \leq \frac{2|z|}{1-|z|^2}, \quad \left|\frac{zh_i''(z)}{h_i'(z)}\right| \leq \frac{2|z|}{1-|z|^2}, \quad \left|\frac{zk_i''(z)}{k_i'(z)}\right| \leq \frac{2|z|}{1-|z|^2},$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$.

Using these relations we get

$$\frac{1-|z|^{2c}}{c}\left|\frac{zH_n''(z)}{H_n'(z)}\right| \leq \frac{1-|z|^{2c}}{c}\left(\frac{1+|z|}{1-|z|} + 1\right)\sum_{i=1}^{n}|\alpha_i - 1|$$

$$+\frac{1-|z|^{2c}}{c}\cdot\frac{2|z|}{1-|z|^2}\sum_{i=1}^{n}|\beta_i|+\frac{1-|z|^{2c}}{c}\left(\frac{1+|z|}{1-|z|}+1+\frac{1+|z|}{1-|z|}+1\right)\sum_{i=1}^{n}|\gamma_i|$$

$$+\frac{1-|z|^{2c}}{c}\left(\frac{2|z|}{1-|z|^2}+\frac{2|z|}{1-|z|^2}\right)\sum_{i=1}^{n}|\delta_i|$$

$$\leq\frac{1-|z|^{2c}}{c}\cdot\frac{2}{1-|z|}\sum_{i=1}^{n}|\alpha_i-1|++\frac{1-|z|^{2c}}{c}\cdot\frac{2|z|}{1-|z|^2}\sum_{i=1}^{n}|\beta_i|$$

$$+\frac{1-|z|^{2c}}{c}\cdot\frac{4}{1-|z|}\sum_{i=1}^{n}|\gamma_i|+\frac{1-|z|^{2c}}{c}\cdot\frac{4|z|}{1-|z|^2}\sum_{i=1}^{n}|\delta_i|, \tag{2.12}$$

for all $z\in\mathbb{U}$.

For $0<c<1$, we have $1-|z|^{2c}\leq 1-|z|^2$, $z\in\mathbb{U}$ and by (2.12), we obtain

$$\frac{1-|z|^{2c}}{c}\left|\frac{zH_n''(z)}{H_n'(z)}\right|\leq\frac{4}{c}\sum_{i=1}^{n}|\alpha_i-1|+\frac{2}{c}\sum_{i=1}^{n}|\beta_i|+\frac{8}{c}\sum_{i=1}^{n}|\gamma_i|+\frac{4}{c}\sum_{i=1}^{n}|\delta_i|, \tag{2.13}$$

for all $z\in\mathbb{U}$.

From (2.10) and (2.13) we have

$$\frac{1-|z|^{2c}}{c}\left|\frac{zH_n''(z)}{H_n'(z)}\right|\leq 1. \tag{2.14}$$

for all $z\in\mathbb{U}$ and $0<c<1$.

For $c\geq 1$ we have $\frac{1-|z|^{2c}}{c}\leq 1-|z|^2$, for all $z\in\mathbb{U}$ and by (2.12), we obtain

$$\frac{1-|z|^{2c}}{c}\left|\frac{zH_n''(z)}{H_n'(z)}\right|\leq 4\sum_{i=1}^{n}|\alpha_i-1|+2\sum_{i=1}^{n}|\beta_i|+8\sum_{i=1}^{n}|\gamma_i|+4\sum_{i=1}^{n}|\delta_i|, \tag{2.15}$$

for all $z\in\mathbb{U}$ and $c\geq 1$.

From (2.11) and (2.15) we obtain

$$\frac{1-|z|^{2c}}{c}\left|\frac{zH_n''(z)}{H_n'(z)}\right|\leq 1. \tag{2.16}$$

for all $z\in\mathbb{U}$ and $c\geq 1$.

And by (2.14), (2.16) and Lemma 1.2, it results that the integral operator $\mathcal{T}_n$, defined by (1.1) is in the class $\mathcal{S}$. □

If we consider $\delta=1$ in Theorem 2.8, we obtain the next corollary:

Corollary 2.9. *Let $\gamma,\alpha_i,\beta_i,\gamma_i,\delta_i$ be complex numbers, $0<Re\gamma\leq 1$ and $f_i,h_i,k_i\in\mathcal{S}$, ${g_i}',{h_i}',{k_i}'\in\mathcal{P}$. If*

$$4\sum_{i=1}^{n}|\alpha_i-1|+2\sum_{i=1}^{n}|\beta_i|+8\sum_{i=1}^{n}|\gamma_i|+4\sum_{i=1}^{n}|\delta_i|\leq\frac{Re\gamma}{2},\quad for\ \ 0<c<1$$

then the integral operator $\mathcal{F}_n$ defined by (2.4) belongs to the class $\mathcal{S}$.

If we consider $\delta=1$ and $\beta_1=\beta_2=...=\beta_n=0$ in Theorem 2.8, we obtain the next corollary:

Corollary 2.10. *Let $\gamma, \alpha_i, \gamma_i, \delta_i$ be complex numbers, $0 < Re\gamma \leq 1$ and $f_i, h_i, k_i \in \mathcal{S}$, ${h_i}', {k_i}' \in \mathcal{P}$. If*

$$4\sum_{i=1}^{n} |\alpha_i - 1| + 8\sum_{i=1}^{n} |\gamma_i| + 4\sum_{i=1}^{n} |\delta_i| \leq \frac{Re\gamma}{2}, \quad for \quad 0 < c < 1$$

then the integral operator $\mathcal{X}_n$ defined by (2.6) belongs to the class $\mathcal{S}$.

If we consider $\delta = 1$ and $\gamma_1 = \gamma_2 = ... = \gamma_n = 0$ in Theorem 2.8, we obtain the next corollary:

Corollary 2.11. *Let $\gamma, \alpha_i, \beta_i, \delta_i$ be complex numbers, $0 < Re\gamma \leq 1$ and $f_i \in \mathcal{S}$, ${g_i}', {h_i}', {k_i}' \in \mathcal{P}$. If*

$$4\sum_{i=1}^{n} |\alpha_i - 1| + 2\sum_{i=1}^{n} |\beta_i| + 4\sum_{i=1}^{n} |\delta_i| \leq \frac{Re\gamma}{2}, \quad for \quad 0 < c < 1$$

then the integral operator $\mathcal{Y}_n$ defined by (2.8) belongs to the class $\mathcal{S}$.

If we consider $\delta = 1$ and $\alpha_1 = \alpha_2 = ... = \alpha_n = 0$ in Theorem 2.8, we obtain the next corollary:

Corollary 2.12. *Let $\gamma, \beta_i, \gamma_i, \delta_i$ be complex numbers, $0 < Re\gamma \leq 1$ and $h_i, k_i \in \mathcal{S}$, ${g_i}', {h_i}', {k_i}' \in \mathcal{P}$. If*

$$2\sum_{i=1}^{n} |\beta_i| + 8\sum_{i=1}^{n} |\gamma_i| + 4\sum_{i=1}^{n} |\delta_i| \leq \frac{Re\gamma}{2}, \quad for \quad 0 < c < 1$$

then the integral operator $\mathcal{D}_n$ defined by (2.7) belongs to the class $\mathcal{S}$.

If we consider $\delta = 1$ and $\delta_1 = \delta_2 = ... = \delta_n = 0$ in Theorem 2.8, we obtain the next corollary:

Corollary 2.13. *Let $\gamma, \alpha_i, \beta_i, \gamma_i$ be complex numbers, $0 < Re\gamma \leq 1$ and $f_i, h_i, k_i \in \mathcal{S}$, ${g_i}' \in \mathcal{P}$. If*

$$4\sum_{i=1}^{n} |\alpha_i - 1| + 2\sum_{i=1}^{n} |\beta_i| + 8\sum_{i=1}^{n} |\gamma_i| \leq \frac{Re\gamma}{2}, \quad for \quad 0 < c < 1$$

then the integral operator $\mathcal{S}_n$ defined by (2.5) belongs to the class $\mathcal{S}$.

Theorem 2.14. *Let $\gamma, \delta, \alpha_i, \beta_i, \gamma_i, \delta_i$ be complex numbers, $Re\gamma > 0$, M_i, N_i, P_i real positive numbers and $f_i, g_i, h_i, k_i \in \mathcal{A}$. If*

$$\left|\frac{zf_i'(z)}{f_i(z)} - 1\right| \leq M_i, \quad \left|\frac{zg_i''(z)}{g_i'(z)}\right| \leq 1, \quad \left|\frac{zh_i'(z)}{h_i(z)} - 1\right| \leq N_i,$$

$$\left|\frac{zk_i'(z)}{k_i(z)} - 1\right| \leq P_i, \quad \left|\frac{zh_i''(z)}{h_i'(z)}\right| \leq 1, \quad \left|\frac{zk_i''(z)}{k_i'(z)}\right| \leq 1,$$

for all $z \in \mathbb{U}$, $i = \overline{1,n}$ *and*

$$|c| \leq 1 - \frac{1}{|\delta|}\left[(2+M_i)\sum_{i-1}^{n}|\alpha_i - 1| + \sum_{i=1}^{n}|\beta_i|\right]$$
$$- \frac{1}{|\delta|}\left[(N_i + P_i + 4)\sum_{i=}^{n}|\gamma_i| + 2\sum_{i=1}^{n}|\delta_i|\right], \tag{2.17}$$

where $c \in \mathbb{C}$, $c \neq -1$, *then the integral operator* $\mathcal{T}_n$, *defined by (1.1) is in the class* $\mathcal{S}$.

Proof. Also, a simple computation yields

$$\left| |c\,|z|^{2\delta} + \left(1 - |z|^{2\delta}\right)\frac{zH_n''(z)}{\delta H_n'(z)}\right|$$
$$\leq |c| + \frac{1}{|\delta|}\sum_{i-1}^{n}|\alpha_i - 1|\left(\left|\frac{zf_i'(z)}{f_i(z)}\right| + 1\right) + \frac{1}{|\delta|}\sum_{i-1}^{n}|\beta_i|\left|\frac{zg_i''(z)}{g_i'(z)}\right|$$
$$+ \frac{1}{|\delta|}\sum_{i=1}^{n}|\gamma_i|\left[\left(\left|\frac{zh_i'(z)}{h_i(z)}\right| + 1\right) + \left(\left|\frac{zk_i'(z)}{k_i(z)}\right| + 1\right)\right]$$
$$+ \frac{1}{|\delta|}\sum_{i=1}^{n}|\delta_i|\left(\left|\frac{zh_i''(z)}{h_i'(z)}\right| + \left|\frac{zk_i''(z)}{k_i'(z)}\right|\right), \tag{2.18}$$

for all $z \in \mathbb{U}$.

Using these inequalities from hypothesis we have

$$\left| |c\,|z|^{2\delta} + \left(1 - |z|^{2\delta}\right)\frac{zH_n''(z)}{\delta H_n'(z)}\right| \leq |c| + \frac{1}{|\delta|}\left[(2+M_i)\sum_{i-1}^{n}|\alpha_i - 1| + \sum_{i=1}^{n}|\beta_i|\right]$$
$$+ \frac{1}{|\delta|}\left[(N_i + P_i + 4)\sum_{i=}^{n}|\gamma_i| + 2\sum_{i=1}^{n}|\delta_i|\right],$$

for all $z \in \mathbb{U}$. and hence, by inequality (2.17) we have

$$\left| |c\,|z|^{2\delta} + \left(1 - |z|^{2\delta}\right)\frac{zH_n''(z)}{\delta H_n'(z)}\right| \leq 1, \tag{2.19}$$

for all $z \in \mathbb{U}$.

Applying Lemma 1.3, we conclude that the integral operator $\mathcal{T}_n$, given by (1.1) is in the class $\mathcal{S}$. □

If we consider $\delta = \gamma = \alpha$ and $\alpha_i - 1 = \beta_i = \gamma_i$ and $n = 1$ in Theorem 2.14, we obtain the next corollary:

Corollary 2.15. *Let* α *be complex number,* $Re\alpha > 0$ M, N, P *real positive numbers, and* $f, g, h, k \in \mathcal{A}$. *If*

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq M, \quad \left|\frac{zg''(z)}{g'(z)}\right| \leq 1, \quad \left|\frac{zh'(z)}{h(z)} - 1\right| \leq N,$$
$$\left|\frac{zk'(z)}{k(z)} - 1\right| \leq P, \quad \left|\frac{zh''(z)}{h'(z)}\right| \leq 1, \quad \left|\frac{zk''(z)}{k'(z)}\right| \leq 1,$$

for all $z \in \mathbb{U}$ *and*

$$|c| \leq 1 - \left|\frac{\alpha - 1}{\alpha}\right| (M_i + N_i + P_i + 8), \qquad c \in \mathbb{C}, \qquad c \neq -1,$$

then the integral operator $\mathcal{T}$*, given by (2.9) is in the class* $\mathcal{S}$*.*

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Sufficient conditions for analytic functions defined by Frasin differential operator

Tariq Al-Hawary

Abstract. Very recently, Frasin [7] introduced the differential operator $\mathcal{I}_{m,\lambda}^{\zeta} f(z)$ defined as

$$\mathcal{I}_{m,\lambda}^{\zeta} f(z) = z + \sum_{n=2}^{\infty} \left(1 + (n-1) \sum_{j=1}^{m} \binom{m}{j} (-1)^{j+1} \lambda^j \right)^{\zeta} a_n z^n.$$

The current work contributes to give an application of the differential operator $\mathcal{I}_{m,\lambda}^{\zeta} f(z)$ to the differential inequalities in the complex plane.

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1. Introduction and preliminaries

Let $\mathcal{A}$ be the class of all normalized analytic functions in $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ that has a Taylor-Maclaurin series expansion of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

For a function f in $\mathcal{A}$, and using the binomial series

$$(1-\lambda)^m = \sum_{j=0}^{m} \binom{m}{j} (-1)^j \lambda^j \qquad (m \in \mathbb{N},\ j \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mathbb{N} = \{1, 2, \cdots\}),$$

let $\mathcal{I}_{m,\lambda}^{\zeta} f(z)$ be the differential operator defined as follows:

$$\begin{aligned} \mathcal{I}^0 f(z) &= f(z), \\ \mathcal{I}_{m,\lambda}^1 f(z) &= (1-\lambda)^m f(z) + (1-(1-\lambda)^m) z f'(z) = \mathcal{I}_{m,\lambda} f(z),\ \lambda > 0; m \in \mathbb{N}, \\ \mathcal{I}_{m,\lambda}^{\zeta} f(z) &= \mathcal{I}_{m,\lambda}(\mathcal{I}^{\zeta-1} f(z)) \qquad (\zeta \in \mathbb{N}). \end{aligned} \tag{1.2}$$

For $f \in \mathcal{A}$, we see that

$$\mathcal{I}_{m,\lambda}^{\zeta} f(z) = z + \sum_{n=2}^{\infty} \left(1 + (n-1) \sum_{j=1}^{m} \binom{m}{j} (-1)^{j+1} \lambda^j \right)^{\zeta} a_n z^n, \ \zeta \in \mathbb{N}_0. \tag{1.3}$$

Using (1.3), it is easily verified that

$$C_j^m(\lambda) z (\mathcal{I}_{m,\lambda}^{\zeta} f(z))' = \mathcal{I}_{m,\lambda}^{\zeta+1} f(z) - (1 - C_j^m(\lambda)) \mathcal{I}_{m,\lambda}^{\zeta} f(z), \ \ \zeta \in \mathbb{N}_0, \tag{1.4}$$

where $C_j^m(\lambda) := \sum\limits_{j=1}^{m} \binom{m}{j} (-1)^{j+1} \lambda^j$.

From the identity (1.4), we readily have

$$C_j^m(\lambda) z (\mathcal{I}_{m,\lambda}^{\zeta-1} f(z))' = \mathcal{I}_{m,\lambda}^{\zeta} f(z) - (1 - C_j^m(\lambda)) \mathcal{I}_{m,\lambda}^{\zeta-1} f(z), \ \ \zeta \in \mathbb{N}_0 \tag{1.5}$$

and

$$C_j^m(\lambda) z (\mathcal{I}_{m,\lambda}^{\zeta+1} f(z))' = \mathcal{I}_{m,\lambda}^{\zeta+2} f(z) - (1 - C_j^m(\lambda)) \mathcal{I}_{m,\lambda}^{\zeta+1} f(z), \ \ \zeta \in \mathbb{N}_0. \tag{1.6}$$

The above differential operator $\mathcal{I}_{m,\lambda}^{\zeta} f(z)$ was introduced and studied by Frasin [7]. Note that for $m = 1$, we obtain the differential operator $\mathcal{I}_{1,\lambda}^{\zeta}$ defined by Al-Oboudi [1] and for $m = \lambda = 1$, we get Sălăgean differential operator $\mathcal{I}^{\zeta}$ [9] (see also Aouf [2, 3]). Our aim in this work is to provide an application of the differential operator $\mathcal{I}_{m,\lambda}^{\zeta} f(z)$, (see for example, [4, 5, 6, 8, 10]).

For our purpose, using the operator $\mathcal{I}_{m,\lambda}^{\zeta} f(z)$, we define the classes Q and G respectively.

Definition 1.1. Let Q be the set of continuous complex functions $q(r, s, t) : \mathbb{C}^3 \to \mathbb{C}$ in $\mathbb{D} \subset \mathbb{C}^3$ such that $(0, 0, 0) \in \mathbb{D}$, $\ |q(0, 0, 0)| < 1$ and

$$\begin{aligned} & \big| q(e^{i\theta}, [C_j^m(\lambda)\delta + (1 - C_j^m(\lambda))] e^{i\theta}, \\ & [C_j^m(\lambda)]^2 \beta + [C_j^m(\lambda)(2 - C_j^m(\lambda))\delta + (1 - C_j^m(\lambda))^2] e^{i\theta} \big| \\ \geq \ & 1 \end{aligned}$$

whenever

$$\begin{aligned} & \big(e^{i\theta}, [C_j^m(\lambda)\delta + (1 - C_j^m(\lambda))] e^{i\theta}, \\ & [C_j^m(\lambda)]^2 \beta + [C_j^m(\lambda)(2 - C_j^m(\lambda))\delta + (1 - C_j^m(\lambda))^2] e^{i\theta} \big) \\ \in \ & \mathbb{D} \end{aligned}$$

with $\text{Re}\{\beta e^{-i\theta}\} \geq \delta(\delta - 1)$ for real $\theta, \delta \geq 1$.

Definition 1.2. Let G be the set of continuous complex functions $g(r, s, t) : \mathbb{C}^3 \to \mathbb{C}$ in $\mathbb{D} \subset \mathbb{C}^3$ such that $(1, 1, 1) \in \mathbb{D}$, $|g(1, 1, 1)| < L$ $\ (L > 1)$ and

$$\left| g \left(Le^{i\theta}, Le^{i\theta} + C_j^m(\lambda)\delta, \frac{[C_j^m(\lambda)]^2(\delta + \mu) + 3LC_j^m(\lambda)\delta e^{i\theta} + L^2 e^{2i\theta}}{C_j^m(\lambda)(Le^{i\theta} + C_j^m(\lambda)\delta)} \right) \right| \geq L$$

whenever

$$\left(Le^{i\theta}, Le^{i\theta} + C_j^m(\lambda)\delta, \frac{[C_j^m(\lambda)]^2(\delta + \mu) + 3LC_j^m(\lambda)\delta e^{i\theta} + L^2 e^{2i\theta}}{C_j^m(\lambda)(Le^{i\theta} + C_j^m(\lambda)\delta)} \right) \in \mathbb{D}$$

with $\mathrm{Re}\{\mu\} \geq \delta(\delta-1)$ for real $\theta, \delta \geq \frac{L-1}{L+1}$.

2. Main results

To prove our theorems in this section, we recall two lemmas for Miller and Mocanu.

Lemma 2.1. [8] *Let a function $w(z) \in \mathcal{A}$ with $w(z) \neq 0$ in $\mathbb{U}$. If $z_0 = r_0 e^{i\theta}$ $(0 < r_0 < 1)$ and $|w(z_0)| = \max_{|z| \leq r_0} |w(z)|$. Then*

$$z_0 w'(z_0) = \delta w(z_0) \tag{2.1}$$

and

$$Re\left\{1 + \frac{z_0 w''(z_0)}{w'(z_0)}\right\} \geq \delta, \quad \delta \geq 1. \tag{2.2}$$

Lemma 2.2. [8] *Let $w(z) = a + w_k z^k + \cdots$ be analytic in $\mathbb{U}$ with $w(z) \neq a$ and $k \geq 1$. If $z_0 = r_0 e^{i\theta}$ $(0 < r_0 < 1)$ and $|w(z_0)| = \max_{|z| \leq r_0} |w(z)|$. Then*

$$z_0 w'(z_0) = \delta w(z_0) \tag{2.3}$$

and

$$Re\left\{1 + \frac{z_0 w''(z_0)}{w'(z_0)}\right\} \geq \delta, \quad (\delta \in \mathbb{R}) \tag{2.4}$$

where

$$\delta \geq k \frac{|w(z_0) - a|^2}{|w(z_0)|^2 - |a|^2} \geq k \frac{|w(z_0)| - |a|}{|w(z_0)| + |a|}.$$

Applying Lemma 2.1, we prove Theorem 2.3.

Theorem 2.3. *Let $q(r, s, t) \in Q$ and $f(z) \in \mathcal{A}$ such that*

$$\left(\mathcal{I}^{\zeta}_{m,\lambda} f(z), \mathcal{I}^{\zeta+1}_{m,\lambda} f(z), \mathcal{I}^{\zeta+2}_{m,\lambda} f(z)\right) \in \mathbb{D} \subset \mathbb{C}^3 \tag{2.5}$$

and

$$\left|q\left(\mathcal{I}^{\zeta}_{m,\lambda} f(z), \mathcal{I}^{\zeta+1}_{m,\lambda} f(z), \mathcal{I}^{\zeta+2}_{m,\lambda} f(z)\right)\right| < 1 \tag{2.6}$$

for $\zeta \in \mathbb{N}_0$, $m \in \mathbb{N}$, $\lambda > 0$ and $z \in \mathbb{U}$. Then

$$\left|\mathcal{I}^{\zeta}_{m,\lambda} f(z)\right| < 1 \qquad (z \in \mathbb{U}). \tag{2.7}$$

Proof. Let

$$\mathcal{I}^{\zeta}_{m,\lambda} f(z) = w(z),$$

then $w(z) \in \mathcal{A}$ and $w(z) \neq 0$ $(z \in \mathbb{U})$. Using the identity (1.4), we have

$$\mathcal{I}^{\zeta+1}_{m,\lambda} f(z) = C^m_j(\lambda) z w'(z) + (1 - C^m_j(\lambda)) w(z)$$

and

$$\mathcal{I}^{\zeta+2}_{m,\lambda} f(z) = [C^m_j(\lambda)]^2 (z^2 w''(z)) + C^m_j(\lambda)(2 - C^m_j(\lambda)) z w'(z) + (1 - C^m_j(\lambda))^2 w(z).$$

Letting $z_0 = r_0 e^{i\theta}$ $(0 < r_0 < 1)$, $|w(z_0)| = \max_{|z|\le r_0} |w(z)| = 1$, $w(z_0) = e^{i\theta}$ and using (2.1), we have

$$\begin{aligned}\mathcal{I}^{\zeta}_{m,\lambda} f(z_0) &= w(z_0) = e^{i\theta},\\ \mathcal{I}^{\zeta+1}_{m,\lambda} f(z_0) &= C^m_j(\lambda)\delta w(z_0) + (1 - C^m_j(\lambda))w(z_0)\\ &= [C^m_j(\lambda)\delta + (1 - C^m_j(\lambda))]e^{i\theta},\end{aligned}$$

and

$$\begin{aligned}\mathcal{I}^{\zeta+2}_{m,\lambda} f(z_0) &= [C^m_j(\lambda)]^2(z_0^2 w''(z_0)) + C^m_j(\lambda)(2 - C^m_j(\lambda))\delta w(z_0) + (1 - C^m_j(\lambda))^2 w(z_0)\\ &= [C^m_j(\lambda)]^2\beta + [C^m_j(\lambda)(2 - C^m_j(\lambda))\delta + (1 - C^m_j(\lambda))^2]e^{i\theta}.\end{aligned}$$

where

$$\beta = z_0^2 w''(z_0) \quad \text{and} \quad \delta \ge 1\,.$$

Moreover, an application of (2.2) gives

$$\operatorname{Re}\left\{\frac{z_0 w''(z_0)}{w'(z_0)}\right\} = \operatorname{Re}\left\{\frac{z_0^2 w''(z_0)}{\delta e^{i\theta}}\right\} \ge \delta - 1,$$

or

$$\operatorname{Re}\{\beta e^{-i\theta}\} \ge \delta(\delta - 1).$$

Since $q(r, s, t) \in Q$, we have

$$\begin{aligned}&\left|q\left(\mathcal{I}^{\zeta}_{m,\lambda} f(z), \mathcal{I}^{\zeta+1}_{m,\lambda} f(z), \mathcal{I}^{\zeta+2}_{m,\lambda} f(z)\right)\right|\\ =\ &\big|q(e^{i\theta}, [C^m_j(\lambda)\delta + (1 - C^m_j(\lambda))]e^{i\theta},\\ &[C^m_j(\lambda)]^2\beta + [C^m_j(\lambda)(2 - C^m_j(\lambda))\delta + (1 - C^m_j(\lambda))^2]e^{i\theta})\big|\\ >\ &1\end{aligned}$$

which opposes the condition (2.6) of Theorem 2.3. So we have

$$\left|\mathcal{I}^{\zeta}_{m,\lambda} f(z)\right| < 1 \qquad (z \in \mathbb{U}).$$ □

In Theorem 2.3, if $\zeta = 0$, $\lambda = 1$ and $m = 1$ we get

Corollary 2.4. *Let $q(r, s, t) \in Q$ and $f(z) \in \mathcal{A}$ such that*

$$\big(f(z), zf'(z), z^2 f''(z) + zf'(z)\big) \in \mathbb{D} \subset \mathbb{C}^3$$

and

$$\left|q\left(f(z), zf'(z), z^2 f''(z) + zf'(z)\right)\right| < 1, \qquad z \in \mathbb{U}.$$

Then

$$|f(z)| < 1 \qquad (z \in \mathbb{U}).$$

Now, using Lemma 2.2 we will prove the following theorem.

Theorem 2.5. *Let $g(r,s,t) \in G$ and $f(z) \in \mathcal{A}$ satisfy*

$$\left(\frac{\mathcal{I}^{\zeta}_{m,\lambda}f(z)}{\mathcal{I}^{\zeta-1}_{m,\lambda}f(z)}, \frac{\mathcal{I}^{\zeta+1}_{m,\lambda}f(z)}{\mathcal{I}^{\zeta}_{m,\lambda}f(z)}, \frac{\mathcal{I}^{\zeta+2}_{m,\lambda}f(z)}{\mathcal{I}^{\zeta+1}_{m,\lambda}f(z)}\right) \in \mathbb{D} \subset \mathbb{C}^3 \tag{2.8}$$

and

$$\left|g\left(\frac{\mathcal{I}^{\zeta}_{m,\lambda}f(z)}{\mathcal{I}^{\zeta-1}_{m,\lambda}f(z)}, \frac{\mathcal{I}^{\zeta+1}_{m,\lambda}f(z)}{\mathcal{I}^{\zeta}_{m,\lambda}f(z)}, \frac{\mathcal{I}^{\zeta+2}_{m,\lambda}f(z)}{\mathcal{I}^{\zeta+1}_{m,\lambda}f(z)}\right)\right| < L \tag{2.9}$$

for $m \in \mathbb{N}$, $\zeta \geq 1$, $\lambda > 0, L > 1$ and all $z \in \mathbb{U}$. Then

$$\left|\frac{\mathcal{I}^{\zeta}_{m,\lambda}f(z)}{\mathcal{I}^{\zeta-1}_{m,\lambda}f(z)}\right| < L \qquad (z \in \mathbb{U}).$$

Proof. Let

$$\frac{\mathcal{I}^{\zeta}_{m,\lambda}f(z)}{\mathcal{I}^{\zeta-1}_{m,\lambda}f(z)} = w(z) \qquad (\zeta \geq 1). \tag{2.10}$$

Then $w(z)$ is analytic function in $\mathbb{U}$, $w(0) = 1$ and $w(z) \neq 1$. Differentiating (2.10) logarithmically and multiplying by z, we get

$$\frac{z(\mathcal{I}^{\zeta}_{m,\lambda}f(z))'}{\mathcal{I}^{\zeta}_{m,\lambda}f(z)} - \frac{z(\mathcal{I}^{\zeta-1}_{m,\lambda}f(z))'}{\mathcal{I}^{\zeta-1}_{m,\lambda}f(z)} = \frac{zw'(z)}{w(z)}.$$

Using the identities (1.4) and (1.5), we have

$$\frac{\mathcal{I}^{\zeta+1}_{m,\lambda}f(z)}{\mathcal{I}^{\zeta}_{m,\lambda}f(z)} = w(z) + C^m_j(\lambda)\frac{zw'(z)}{w(z)}. \tag{2.11}$$

Differentiating (2.11) logarithmically and multiply by z, we have

$$\begin{aligned}
&\frac{z(\mathcal{I}^{\zeta+1}_{m,\lambda}f(z))'}{\mathcal{I}^{\zeta+1}_{m,\lambda}f(z)} - \frac{z(\mathcal{I}^{\zeta}_{m,\lambda}f(z))'}{\mathcal{I}^{\zeta}_{m,\lambda}f(z)} \\
= \; & \frac{z\left[w(z) + C^m_j(\lambda)\frac{zw'(z)}{w(z)}\right]'}{w(z) + C^m_j(\lambda)\frac{zw'(z)}{w(z)}} \\
= \; & \frac{zw'(z) + C^m_j(\lambda)\left[\frac{zw'(z)}{w(z)} + \frac{z^2w''(z)}{w(z)} - \left(\frac{zw'(z)}{w(z)}\right)^2\right]}{w(z) + C^m_j(\lambda)\frac{zw'(z)}{w(z)}}.
\end{aligned} \tag{2.12}$$

Using the identities (1.4) and (1.6), it follows from (2.12) that

$$\begin{aligned}
&\frac{1}{C_j^m(\lambda)}\frac{\mathcal{I}_{m,\lambda}^{\zeta+2}f(z)}{\mathcal{I}_{m,\lambda}^{\zeta+1}f(z)}\\
=\ &\frac{1}{C_j^m(\lambda)}\frac{\mathcal{I}_{m,\lambda}^{\zeta+1}f(z)}{\mathcal{I}_{m,\lambda}^{\zeta}f(z)}+\frac{zw'(z)+C_j^m(\lambda)\left[\frac{zw'(z)}{w(z)}+\frac{z^2w''(z)}{w(z)}-\left(\frac{zw'(z)}{w(z)}\right)^2\right]}{w(z)+C_j^m(\lambda)\frac{zw'(z)}{w(z)}}\\
=\ &\frac{1}{C_j^m(\lambda)}w(z)+\frac{zw'(z)}{w(z)}+\frac{zw'(z)+C_j^m(\lambda)\left[\frac{zw'(z)}{w(z)}+\frac{z^2w''(z)}{w(z)}-\left(\frac{zw'(z)}{w(z)}\right)^2\right]}{w(z)+C_j^m(\lambda)\frac{zw'(z)}{w(z)}}
\end{aligned}$$

Letting $z_0=r_0e^{i\theta}$ $(0<r_0<1)$, $\max\limits_{|z|\le r_0}|w(z)|=|w(z_0)|=L$, $w(z_0)=Le^{i\theta}$ and using Lemma 2.2 with $a=1$ and $k=1$, we have

$$\begin{aligned}
\frac{\mathcal{I}_{m,\lambda}^{\zeta}f(z_0)}{\mathcal{I}_{m,\lambda}^{\zeta-1}f(z_0)} &= Le^{i\theta},\\
\frac{\mathcal{I}_{m,\lambda}^{\zeta+1}f(z_0)}{\mathcal{I}_{m,\lambda}^{\zeta}f(z_0)} &= Le^{i\theta}+C_j^m(\lambda)\delta,\\
\frac{\mathcal{I}_{m,\lambda}^{\zeta+2}f(z_0)}{\mathcal{I}_{m,\lambda}^{\zeta+1}f(z_0)} &= \frac{[C_j^m(\lambda)]^2(\delta+\mu)+3LC_j^m(\lambda)\delta e^{i\theta}+L^2e^{2i\theta}}{C_j^m(\lambda)(Le^{i\theta}+C_j^m(\lambda)\delta)},
\end{aligned}$$

where

$$\mu=\frac{z_0^2w''(z_0)}{w(z_0)}\quad\text{and}\quad\delta\ge\frac{L-1}{L+1}.$$

Moreover, an application of (2.2) gives $\mathrm{Re}\{\mu\}\ge\delta(\delta-1)$.
Since $g(r,s,t)\in G$, we have

$$\begin{aligned}
&\left|g\left(\frac{\mathcal{I}_{m,\lambda}^{\zeta}f(z_0)}{\mathcal{I}_{m,\lambda}^{\zeta-1}f(z_0)},\frac{\mathcal{I}_{m,\lambda}^{\zeta+1}f(z_0)}{\mathcal{I}_{m,\lambda}^{\zeta}f(z_0)},\frac{\mathcal{I}_{m,\lambda}^{\zeta+2}f(z_0)}{\mathcal{I}_{m,\lambda}^{\zeta+1}f(z_0)}\right)\right|\\
=\ &\left|g\left(Le^{i\theta},Le^{i\theta}+C_j^m(\lambda)\delta,\frac{[C_j^m(\lambda)]^2(\delta+\mu)+3LC_j^m(\lambda)\delta e^{i\theta}+L^2e^{2i\theta}}{C_j^m(\lambda)(Le^{i\theta}+C_j^m(\lambda)\delta)}\right)\right|\\
\ge\ &L
\end{aligned}$$

which contradicts the condition (2.9) of Theorem 2.5. Thus

$$|w(z)|=\left|\frac{\mathcal{I}_{m,\lambda}^{\zeta}f(z)}{\mathcal{I}_{m,\lambda}^{\zeta-1}f(z)}\right|<L.$$

for $m\in\mathbb{N}$, $\zeta\ge1$, $\lambda>0$ and all $z\in\mathbb{U}$. The proof is complete. □

In Theorem 2.5, if $\zeta=1$, $\lambda=1$ and $m=1$ we get

Corollary 2.6. *Let* $g(r,s,t) \in G$ *and* $f(z) \in \mathcal{A}$ *satisfy*

$$\left(\frac{zf'(z)}{f(z)}, \frac{zf''(z)+f'(z)}{f'(z)}, \frac{z^2f^{(3)}(z)+3zf''(z)+f'(z)}{zf''(z)+f'(z)}\right) \in \mathbb{D} \subset \mathbb{C}^3 \tag{2.13}$$

and

$$\left|g\left(\frac{zf'(z)}{f(z)}, \frac{zf''(z)+f'(z)}{f'(z)}, \frac{z^2f^{(3)}(z)+3zf''(z)+f'(z)}{zf''(z)+f'(z)}\right)\right| < L \tag{2.14}$$

for $L > 1$ *and all* $z \in \mathbb{U}$. *Then*

$$\left|\frac{zf'(z)}{f(z)}\right| < L \qquad (z \in \mathbb{U}).$$

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A study of existence and multiplicity of positive solutions for nonlinear fractional differential equations with nonlocal boundary conditions

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Abstract. This paper deals with the existence, uniqueness and the multiplicity of solutions for a class of fractional differential equations boundary value problems involving three-point nonlocal Riemann-Liouville fractional derivative and integral boundary conditions. Our results are based on some well-known tools of fixed point theory such as Banach contraction principle, fixed point index theory and the Leggett-Williams fixed point theorem. As applications, some examples are presented at the end to illustrate the main results.

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1. Introduction

In this paper, we are interested in the existence of solutions for the nonlinear fractional differential equation

$$D_{0+}^{\alpha} u(t) + a(t) f(t, u(t)) = 0, \quad t \in (0,1), \tag{1.1}$$

subject to the boundary condition

$$u^{(i)}(0) = 0,\ i \in \{0,1,2\},\ D_{0+}^{\beta} u(1) = \lambda I_{0+}^{\beta} u(\eta), \tag{1.2}$$

where $D_{0+}^{\alpha},\ D_{0+}^{\beta}$ are the standard Riemann-Liouville fractional derivative of order $\alpha \in (3,4],\ \beta \in [2,3]$, I_{0+}^{β} is the stantard Riemann-Liouville fractional integral of order $\beta \in [2,3]$.

Due to the fact that the tools of fractional calculus has numerous applications in various disciplines of science and engineering such as physics, mechanics, chemistry, biology, aerodynamics, electrodynamics of complex medium, polymer rheology, Bode's analysis of feedback amplifiers, capacitor theory, electrical circuits, electroanalytical chemistry, control theory, fitting of experimental data, involves derivatives of fractional order. In consequence, the subject of fractional differential equations is gaining much importance and attention. Therefore, there have been many papers and books dealing with the theoretical development of fractional calculus and the solutions or positive solutions of boundary value problems for nonlinear fractional differential equations. For more details we refer the reader to [10, 19, 21] and the references cited therein.

Many mathematicians show strong interest in fractional differential equations and many wonderful results have been obtained. The techniques of nonlinear analysis, as the main method to deal with the problems of nonlinear fractional differential equations, plays an essential role in the research of this field, such as establishing the existence and the uniqueness or the multiplicity of solutions to nonlinear fractional differential equations boundary value problems, see [2, 5, 7, 9, 11, 14, 16, 18] and the references therein.

In [17], the authors studied the existence of positive solutions to the following fractional boundary value problem

$$\begin{cases} D_{0+}^{\alpha}u(t)+h(t)f(t,u(t))=0, & t\in(0,1), \\ u(0)=u'(0)=u''(0)=0, & u(1)=\lambda\int_0^{\eta}u(\eta)\,ds, \end{cases}$$

where D_{0+}^{α} are the standard Riemann-Liouville fractional derivative of order $\alpha\in(3,4]$, $\eta\in(0,1]$, and $0\le\frac{\lambda\eta^{\alpha}}{\alpha}<1$.

In [22], the authors studied the boundary value problems of the fractional order differential equation:

$$\begin{cases} D_{0+}^{\alpha}u(t)=f(t,u(t))=0, & t\in(0,1), \\ u(0)=0, & D_{0+}^{\beta}u(1)=aD_{0+}^{\beta}u(\eta), \end{cases}$$

where $1<\alpha\le 2,\ 0<\eta<1,\ 0<a,\ 0<\beta\le 1,\ f\in C([0,1]\times[0,\infty),[0,\infty))$ and $D_{0+}^{\alpha},\ D_{0+}^{\beta}$ are the standard Riemann-Liouville fractional derivative of order $\alpha,\ \beta$. They obtained the multiple positive solutions by the Leray-Schauder nonlinear alternative and the fixed point theorem on cones.

In 2017, Benaicha and Bouteraa [3] studied the existence and uniqueness of solutions for nonlinear fractional differential equation

$$^{c}D^{\alpha}u(t)=f(t,u(t),u'(t)),\quad t\in J=[0,1]$$

subject to three-point boundary conditions

$$\begin{cases} \beta u(0)+\gamma u(1)=u(\eta), \\ u(0)=\int_0^{\eta}u(s)\,ds, \\ \beta^{c}D^{p}u(0)+\gamma^{c}D^{p}u(1)={}^{c}D^{p}u(\eta), \end{cases}$$

where $2 < \alpha \leq 3,\ 1 < p \leq 2\ 0 < \eta < 1,\ \beta, \gamma \in \mathbb{R}^+,\ f\ :\ [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function and ${}^cD^\alpha$ denotes the Caputo fractional derivative of order α.
In 2018, Bouteraa and Benaicha [6] interested in the existence of solutions for the nonlinear fractional differential equation

$$D^\alpha_{0+}u\,(t) + f\,(t, u\,(t)) = 0, \quad t \in (0,1)\,,$$

subject to the boundary conditions

$$u^{(i)}\,(0) = 0,\ i \in \{0, 1, \ldots, n-2\}\,,\ D^\beta_{0+}u\,(1) = \sum_{j=1}^{p} a_j D^\beta_{0+}u\,(\eta_j)\,,$$

where $D^\alpha_{0+},\ D^\beta_{0+}$ are the standard Riemann-Liouville fractional derivative of order $\alpha\,(n-1, n]\,,\ \beta \in [1, n-2]\ for\,n \in \mathbb{N}^*\ and\,n \geq 3$ and $f \in C\,((0,1) \times \mathbb{R}, \mathbb{R})$ is allowed to be singular at $t = 0$ and/or $t = 1$ and $a_j \in \mathbb{R}^+$, $j = 1, 2, \ldots, p$, $0 < \eta_1 < \eta_2 < \ldots < \eta_p < 1,\ for\,p \in \mathbb{N}^+$. The existence and uniqueness of positive solutions for the above nonlocal boundary value problem obtained by applying the iterative method.

Inspired and motivated by the works mentioned above, we focus on the existence of positive solutions for the nonlocal boundary value problem $(1.1)-(1.2)$. The paper is organized as follows. In Section 2, we recall some preliminary facts that will be need in the sequel. In Section 3, we establish the existence, uniqueness and multiplicity of the positive solutions for boundary value problem $(1.1)-(1.2)$ by applying some well-known tools of fixed point theory such as Banach contraction principle, fixed point index theory and the Leggett-Williams fixed point theorem and we give two examples to illustrate our results.

2. Preliminaries

In this section, we recall some definitions and facts which will be used in the later analysis.

Definition 2.1. ([20]) Let E be a real Banach space. A nonempty closed set $K \subset E$ is said to be a cone provided that
(i) $c_1 u + c_2 v \in K$ for all $c_1 \geq 0$, $c_2 \geq 0$, and
(ii) $u \in K$, $-u \in K$ implies $u = 0$.
Every cone K induces an ordering in E given by $u \leq v$ if and only if $v - u \in K$.

Definition 2.2. ([10, 15]) The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $u\,:\,(0,\infty) \to \mathbb{R}$ is given by

$$I^\alpha_{0+}u\,(t) = \frac{1}{\Gamma\,(\alpha)} \int_0^t (t-s)^{\alpha-1}\,u\,(s)\,ds, \quad t > 0,$$

where $\Gamma\,(\cdot)$ is the Euler gamma function, provided that the right side is pointwise defined on $(0,\infty)$.

Definition 2.3. ([10, 15]) The Riemann-Liouville fractional derivative order $\alpha > 0$ of a continuous function u is defined by

$$D_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} u(s)\, ds, \quad t > 0,$$

where $\Gamma(\cdot)$ is the Euler gamma function and $n = \lceil \alpha \rceil + 1$, $\lceil \alpha \rceil$ denotes the integer part of number α, provided that the right side is pointwise defined on $(0, \infty)$.

Lemma 2.4. ([10]) *(i) If $u \in L^p(0,1)$, $1 \le p \le +\infty$, $\beta > \alpha > 0$, then*

$$D_{0+}^{\alpha} I_{0+}^{\beta} u(t) = I_{0+}^{\beta-\alpha} u(t), \; D_{0+}^{\alpha} I_{0+}^{\alpha} u(t) = u(t), \; I_{0+}^{\alpha} I_{0+}^{\beta} u(t) = I_{0+}^{\alpha+\beta} u(t).$$

(ii) If $\beta > \alpha > 0$, then $D^{\alpha} t^{\beta-1} = \frac{\Gamma(\beta) t^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)}$.
(iii) If $\alpha > 0$ and $\gamma \in (-1, +\infty)$, then $I_{0+}^{\alpha} t^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\alpha+\gamma}$.

Lemma 2.5. *([10]) Let $\alpha > 0$ and for any $y(\cdot) \in L^1(0,1)$. Then, the general solution of the fractional differential equation $D_{0+}^{\alpha} u(t) + y(t) = 0$, $0 < t < 1$ is given by*

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\, ds + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n},$$

where $c_0, c_1, ..., c_{n-1}$ are real constants and $n = \lceil \alpha \rceil + 1$.

Now, let $0 < d < l < r$ be given and let β be a nonnegative continuous concave functional on the cone K i.e.,

$$\beta(\lambda u + (1-\lambda) v) \ge \lambda \beta(u) + (1-\lambda) \beta(v),$$

for all $u, v \in K$ and $\lambda \in [0,1]$.
Define the convex sets K_l and $K(\beta, l, r)$ by

$$K_l = \{u \in K \; : \; \|u\| < l\},$$

and

$$K(\beta, l, r) = \{u \in K \; : \; l \le \beta(u), \; \|u\| \le r\}.$$

The key tools in our approaches are the following fixed point theorem and lemmas

Theorem 2.6. (Leggett-Wiliams fixed point (*See* [20])) *Let E be a Banach space and $K \subset E$ be a cone in E. $T : \bar{K}_c \to \bar{K}_c$ be a completely continuous and β be a nonnegative continuous concave functional on K with $\beta(u) \le \|u\|$ for all $u \in K_c$. Suppose there exist $0 < d < l < r \le c$ such that*
(i) $u \in \{K(\beta, l, r) : \beta(u) > l\} \ne \emptyset$ and $\beta(Tu) > l$ for $u \in K(\beta, l, r)$,
(ii) $\|Tu\| < d$ for $\|u\| \le d$,
(iii) $\beta(Tu) > l$ for $u \in K(\beta, l, c)$ with $\|Tu\| > r$.
Then T has at least three positive solutions u_1, u_2, u_3 satisfying

$$\|u_1\| < d, \quad l < \beta(u_2), \quad \|u_3\| > d \text{ and } \beta(u_3) < l.$$

Lemma 2.7. (Krein-Rutman [20]) *Let K be a reproducing cone in a real Banach space E, and $L : E \to E$ be a compact linear operator with $L(K) \subseteq K$ and spectral radius $r(L)$. If $r(L) > 0$, then there exists $\varphi \in K \setminus \{0\}$ such that $L\varphi = r(L)\varphi$.*

Lemma 2.8. (Fixed point index theory [20]) *Let E be a Banach space and K is a cone in E and $\Omega(K)$ is a bounded open subset in K. Furthermore, assume that $T : \overline{\Omega(K)} \to K$ is a completely continuous operator Then the following conclusion hold:*
(i) there exists $u_0 \in K \setminus \{0\}$ such that $Tu + \lambda u_0 \neq u$ for all $u \in \partial\Omega(K)$ and $\lambda \geq 0$, then the fixed point index $i(T, \Omega(K), K) = 0$,
(ii) if $0 \in \Omega(K)$ and $Tu \neq \lambda u$ for all $u \in \partial\Omega(K)$ and $\lambda \geq 1$, then the fixed point index $i(T, \Omega(K), K) = 1$.

Lemma 2.9. *Let $y(\cdot) \in C[0,1]$. Then the solution of the fractional boundary value problem*

$$\begin{cases} D_{0+}^{\alpha} u(t) + y(t) = 0, \\ u^{(i)}(0) = 0, \ i \in \{0,1,2\}, \\ D_{0+}^{\beta} u(1) = \lambda I_{0+}^{\beta} u(\eta), \end{cases} \tag{2.1}$$

is given by

$$u(t) = \int_0^1 G(t,s)\, y(s)\, ds, \tag{2.2}$$

where

$$G(t,s) = \begin{cases} \frac{-P\Gamma(\alpha-\beta)\Gamma(\alpha+\beta)(t-s)^{\alpha-1}+\Delta}{P\Gamma(\alpha)\Gamma(\alpha-\beta)\Gamma(\alpha+\beta)}, & 0 \leq s \leq t \leq 1, \ s \leq \eta, \\ \frac{\Delta}{P\Gamma(\alpha)\Gamma(\alpha-\beta)\Gamma(\alpha+\beta)}, & 0 \leq t \leq s \leq \eta \leq 1, \\ \frac{-P\Gamma(\alpha-\beta)\Gamma(\alpha+\beta)(t-s)^{\alpha-1}+\Lambda}{P\Gamma(\alpha)\Gamma(\alpha-\beta)\Gamma(\alpha+\beta)}, & 0 \leq \eta \leq s \leq t \leq 1, \\ \frac{\Gamma(\alpha)\Gamma(\alpha+\beta)(1-s)^{\alpha-\beta-1}t^{\alpha-1}}{P\Gamma(\alpha)\Gamma(\alpha-\beta)\Gamma(\alpha+\beta)}, & 0 \leq t \leq s \leq 1, \ s \geq \eta, \end{cases} \tag{2.3}$$

where

$$\Delta = t^{\alpha-1}\left[\Gamma(\alpha)\Gamma(\alpha+\beta)(1-s)^{\alpha-\beta-1} - \lambda\Gamma(\alpha)\Gamma(\alpha-\beta)(\eta-s)^{\alpha+\beta-1}\right],$$

and

$$\Lambda = \Gamma(\alpha+\beta)\Gamma(\alpha)(1-s)^{\alpha-\beta-1} t^{\alpha-1},$$

where

$$P = \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)} - \frac{\lambda\Gamma(\alpha)}{\Gamma(\alpha+\beta)}\eta^{\alpha+\beta-1}.$$

Proof. In view of Lemma 2.5, the general solution for the above equation in (2.1) is

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\, ds + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3} + C_4 t^{\alpha-4},$$

where $c_1, c_2, c_3, c_4 \in \mathbb{R}$.
The boundary condition $u(0) = u'(0) = u''(0) = 0$, implies that $c_2 = c_3 = c_4 = 0$. Thus

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\, ds + c_1 t^{\alpha-1}. \tag{2.4}$$

By (2.4) and Lemma 2.4, we get

$$D_{0^+}^{\beta} u(t) = \frac{1}{\Gamma(\alpha-\beta)} \left[c_1 \Gamma(\alpha) t^{\alpha-\beta-1} - \int_0^t (t-s)^{\alpha-\beta-1} y(s)\, ds \right].$$

In view of boundary condition $D_{0^+}^{\beta} u(1) = \lambda I_{0^+}^{\beta} u(\eta)$, we conclude that

$$c_1 = \frac{1}{P} \left[\frac{1}{\Gamma(\alpha-\beta)} \int_0^1 (1-s)^{\alpha-\beta-1} y(s)\, ds - \frac{\lambda}{\Gamma(\alpha+\beta)} \int_0^{\eta} (\eta-s)^{\alpha+\beta-1} y(s)\, ds \right].$$

Therefore, the unique solution of the problem (2.1) is given by

$$u(t) = \frac{t^{\alpha-1}}{P\Gamma(\alpha-\beta)} \int_0^1 (1-s)^{\alpha-\beta-1} y(s)\, ds - \frac{\lambda t^{\alpha-1}}{P\Gamma(\alpha+\beta)} \int_0^{\eta} (\eta-s)^{\alpha+\beta-1} y(s)\, ds$$
$$- \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\, ds.$$

For $t \le \eta$, one has

$$\begin{aligned} u(t) &= \frac{t^{\alpha-1}}{P\Gamma(\alpha-\beta)} \left[\int_0^t (1-s)^{\alpha-\beta-1} y(s)\, ds + \int_t^{\eta} (1-s)^{\alpha-\beta-1} y(s)\, ds \right. \\ &\quad \left. + \int_{\eta}^1 (1-s)^{\alpha-\beta-1} y(s)\, ds \right] - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\, ds \\ &\quad - \frac{\lambda t^{\alpha-1}}{P\Gamma(\alpha+\beta)} \left[\int_0^t (\eta-s)^{\alpha+\beta-1} y(s)\, ds + \int_t^{\eta} (\eta-s)^{\alpha+\beta-1} y(s)\, ds \right] \\ &= \int_0^t \frac{-P\Gamma(\alpha-\beta)\Gamma(\alpha+\beta)(t-s)^{\alpha-1} + \Delta}{P\Gamma(\alpha)\Gamma(\alpha+\beta)\Gamma(\alpha-\beta)} y(s)\, ds \\ &\quad + \int_t^{\eta} \frac{\Delta}{P\Gamma(\alpha)\Gamma(\alpha+\beta)\Gamma(\alpha-\beta)} y(s)\, ds \\ &\quad + \int_{\eta}^1 \frac{\Gamma(\alpha)\Gamma(\alpha+\beta)(1-s)^{\alpha-\beta-1} t^{\alpha-1}}{P\Gamma(\alpha)\Gamma(\alpha+\beta)\Gamma(\alpha-\beta)} y(s)\, ds \\ &= \int_0^1 G(t,s)\, y(s)\, ds. \end{aligned}$$

For $t \geq \eta$, one has

$$u(t) = \int_0^\eta \frac{-P\Gamma(\alpha-\beta)\Gamma(\alpha+\beta)(t-s)^{\alpha-1}+\Delta}{P\Gamma(\alpha)\Gamma(\alpha+\beta)\Gamma(\alpha-\beta)} y(s)\,ds$$

$$+\int_\eta^t \frac{-P\Gamma(\alpha-\beta)\Gamma(\alpha+\beta)(t-s)^{\alpha-1}+\Gamma(\alpha)\Gamma(\alpha+\beta)(1-s)^{\alpha-\beta-1}t^{\alpha-1}}{P\Gamma(\alpha)\Gamma(\alpha+\beta)\Gamma(\alpha-\beta)} y(s)\,ds$$

$$+\int_t^1 \frac{\Gamma(\alpha)\Gamma(\alpha+\beta)(1-s)^{\alpha-\beta-1}t^{\alpha-1}}{P\Gamma(\alpha)\Gamma(\alpha+\beta)\Gamma(\alpha-\beta)} y(s)\,ds$$

$$= \int_0^1 G(t,s)\,y(s)\,ds.$$

The proof is complete. □

We need some properties of function $G(t,s)$ to establish the existence of positive solutions.

Lemma 2.10. *The Green's function $G(t,s)$ has the following properties:*

(i) The function $G(t,s)$ is continuous on $[0,1]\times[0,1]$.

(ii) $G(t,s) > 0$ for all $s \in (0,1)$,

(iii) for all $t,\ s \in (0,1)$, we have $G(t,s) \leq G(1,s)$,

(iv) there exists a positive function $\gamma(s) \in C(0,1)$ such that

$$\min_{\eta\leq t\leq 1} G(t,s) \geq \gamma(s) \max_{0\leq t\leq 1} G(t,s) = \eta^{\alpha-1} G(1,s)\,,\ 0 < s < 1. \tag{2.5}$$

Proof. It is easy to prove (i). Now, we prove $(ii)-(iv)$. Let

$$g_1(t,s) = \frac{\Delta - P\Gamma(\alpha-\beta)\Gamma(\alpha+\beta)(t-s)^{\alpha-1}}{P\Gamma(\alpha)\Gamma(\alpha+\beta)\Gamma(\alpha-\beta)},$$

where Δ defined above.

$$g_2(t,s) = \frac{t^{\alpha-1}\Gamma(\alpha)\Gamma(\alpha+\beta)(1-s)^{\alpha-\beta-1} - P\Gamma(\alpha-\beta)\Gamma(\alpha+\beta)(t-s)^{\alpha-1}}{P\Gamma(\alpha)\Gamma(\alpha+\beta)\Gamma(\alpha-\beta)}$$

$$g_3(t,s) = \frac{t^{\alpha-1}\Gamma(\alpha)\left(\Gamma(\alpha+\beta)(1-s)^{\alpha-\beta-1} - \lambda\Gamma(\alpha-\beta)(\eta-s)^{\alpha+\beta-1}\right)}{P\Gamma(\alpha)\Gamma(\alpha+\beta)\Gamma(\alpha-\beta)}$$

$$g_4(t,s) = \frac{t^{\alpha-1}\Gamma(\alpha)\Gamma(\alpha+\beta)(1-s)^{\alpha-\beta-1}}{P\Gamma(\alpha)\Gamma(\alpha+\beta)\Gamma(\alpha-\beta)}.$$

We will first show that

$$g_1(t,s) > 0,\ 0 \leq \min\{t,\eta\} < 1.$$

To simplify we introduce the abbreviation

$$\triangle_1 = t^{\alpha-1}\Gamma(\alpha)\Gamma(\alpha+\beta)(1-s)^{\alpha-\beta-1}.$$

We can rewrite $\triangle_1$ as

$$\begin{aligned}\triangle_1 &= t^{\alpha-1}\Gamma(\alpha)\Gamma(\alpha+\beta)\left(\frac{\Gamma(\alpha-\beta)}{\Gamma(\alpha-\beta)}-\frac{\lambda\Gamma(\alpha-\beta)}{\Gamma(\alpha+\beta)}\eta^{\alpha+\beta-1}\right.\\ &\quad\left.+\frac{\lambda\Gamma(\alpha-\beta)}{\Gamma(\alpha+\beta)}\eta^{\alpha+\beta-1}\right)(1-s)^{\alpha-\beta-1}\\ &= t^{\alpha-1}\Gamma(\alpha-\beta)\Gamma(\alpha+\beta)\left(\frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}-\frac{\lambda\Gamma(\alpha)}{\Gamma(\alpha+\beta)}\eta^{\alpha+\beta-1}\right.\\ &\quad\left.+\frac{\lambda\Gamma(\alpha)}{\Gamma(\alpha+\beta)}\eta^{\alpha+\beta-1}\right)(1-s)^{\alpha-\beta-1}\\ &= t^{\alpha-1}\Gamma(\alpha-\beta)\Gamma(\alpha+\beta)\left(P+\frac{\lambda\Gamma(\alpha)\eta^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\right)(1-s)^{\alpha-\beta-1},\end{aligned}$$

$$\lambda t^{\alpha-1}\Gamma(\alpha)\Gamma(\alpha-\beta)(\eta-s)^{\alpha+\beta-1}=\lambda t^{\alpha-1}\Gamma(\alpha)\Gamma(\alpha-\beta)\eta^{\alpha+\beta-1}\left(1-\frac{s}{\eta}\right)^{\alpha+\beta-1},$$

and

$$P\Gamma(\alpha-\beta)\Gamma(\alpha+\beta)(t-s)^{\alpha-1}=Pt^{\alpha-1}\Gamma(\alpha-\beta)\Gamma(\alpha+\beta)\left(1-\frac{s}{t}\right)^{\alpha-1}.$$

Thus

$$\begin{aligned}g_1(t,s) &= Q\left\{Pt^{\alpha-1}\Gamma(\alpha-\beta)\Gamma(\alpha+\beta)\left[(1-s)^{\alpha-\beta-1}-\left(1-\frac{s}{t}\right)^{\alpha-1}\right]\right.\\ &\quad\left.+\lambda t^{\alpha-1}\Gamma(\alpha)\Gamma(\alpha-\beta)\left[\eta^{\alpha-\beta-1}(1-s)^{\alpha-\beta-1}-\eta^{\alpha+\beta-1}\left(1-\frac{s}{\eta}\right)^{\alpha+\beta-1}\right]\right\}\\ &> Q\left\{Pt^{\alpha-1}\Gamma(\alpha-\beta)\Gamma(\alpha+\beta)\left[(1-s)^{\alpha-1}-\left(1-\frac{s}{t}\right)^{\alpha-1}\right]\right.\\ &\quad\left.+\lambda t^{\alpha-1}\Gamma(\alpha)\Gamma(\alpha-\beta)\left[\eta^{\alpha+\beta-1}(1-s)^{\alpha+\beta-1}-\eta^{\alpha+\beta-1}\left(1-\frac{s}{\eta}\right)^{\alpha+\beta-1}\right]\right\}\\ &> Q\left\{Pt^{\alpha-1}\Gamma(\alpha-\beta)\Gamma(\alpha+\beta)\left[(1-s)^{\alpha-1}-(1-s)^{\alpha-1}\right]\right.\\ &\quad\left.+\lambda t^{\alpha-1}\Gamma(\alpha)\Gamma(\alpha-\beta)\eta^{\alpha+\beta-1}\left[(1-s)^{\alpha+\beta-1}-(1-s)^{\alpha+\beta-1}\right]\right\}=0,\end{aligned}$$

where $Q=\frac{1}{P\Gamma(\alpha)\Gamma(\alpha-\beta)\Gamma(\alpha+\beta)}$.
We deduce that $g_1(t,s)>0,\ 0\le\min\{t,\eta\}<1$.
By the similar argument we can conclude that

$$g_2(t,s)>0,\ 0<\eta\le s\le t\le 1,\quad g_3(t,s)>0,\ 0\le t\le s\le\eta\le 1,$$

and

$$g_4(t,s)>0,\ 0\le\max\{s,\eta\}\le s\le 1.$$

Therefore $G(t,s)>0$ for any $t,\ s\in(0,1)$.

Now, we show that $G(t,s) \le G(1,s)$ for any $t,\, s \in (0,1)$.
Let $h_1(t,s) = g_1(t,s)\,\Gamma(\alpha)\,\Gamma(\alpha-\beta)\,\Gamma(\alpha+\beta)$. Then, as the above argument but for the derivative of $h_1(t,s)$ with respect to t on $[s,1]$, we have

$$\begin{aligned}
\frac{\partial h_1(t,s)}{\partial t} &= \frac{(\alpha-1)\,t^{\alpha-2}}{P}\left\{P\Gamma(\alpha-\beta)\,\Gamma(\alpha+\beta)\left[(1-s)^{\alpha-\beta-1} - \left(1-\frac{s}{t}\right)^{\alpha-2}\right]\right.\\
&\quad \left. + \lambda\Gamma(\alpha)\,\Gamma(\alpha-\beta)\left[\eta^{\alpha-\beta-1}(1-s)^{\alpha-\beta-1} - \eta^{\alpha+\beta-1}\left(1-\frac{s}{\eta}\right)^{\alpha+\beta-1}\right]\right\}\\
&> \frac{(\alpha-1)\,t^{\alpha-2}}{P}\left\{P\Gamma(\alpha-\beta)\,\Gamma(\alpha+\beta)\left[(1-s)^{\alpha-2} - (1-s)^{\alpha-2}\right]\right.\\
&\quad \left. + \lambda t^{\alpha-1}\Gamma(\alpha)\,\Gamma(\alpha-\beta)\,\eta^{\alpha+\beta-1}\left[(1-s)^{\alpha+\beta-1} - (1-s)^{\alpha+\beta-1}\right]\right\} = 0,
\end{aligned}$$

so, we have $\frac{h_1(t,s)}{\partial t} > 0$, then $g_1(t,s)$ is increasing with respect to t on $[s,1]$.
Next, we show that $g_2(t,s)$ is increasing with respect to t on $[s,1]$.
Let $h_2(t,s) = g_2(t,s)\,\Gamma(\alpha)\,\Gamma(\alpha-\beta)\,\Gamma(\alpha+\beta)$. Then, we have

$$\begin{aligned}
\frac{\partial h_2(t,s)}{\partial t} &= \frac{(\alpha-1)\,t^{\alpha-2}}{P}\left\{\Gamma(\alpha+\beta)\left[\Gamma(\alpha)(1-s)^{\alpha-\beta-1} - P\Gamma(\alpha-\beta)\left(1-\frac{s}{t}\right)^{\alpha-2}\right]\right\}\\
&\ge \frac{(\alpha-1)\,t^{\alpha-2}}{P}\left\{\Gamma(\alpha+\beta)\left[\Gamma(\alpha)(1-s)^{\alpha-\beta-1} - P\Gamma(\alpha-\beta)(1-s)^{\alpha-2}\right]\right\}\\
&\ge \frac{(\alpha-1)(1-s)^{\alpha-2}\,t^{\alpha-2}}{P}\left\{\Gamma(\alpha+\beta)\left[\Gamma(\alpha)(1-s)^{1-\beta} - P\Gamma(\alpha-\beta)\right]\right\}\\
&= \frac{(\alpha-1)\,(t(1-s))^{\alpha-2}}{P}\left\{\Gamma(\alpha+\beta)\left[\Gamma(\alpha)(1-s)^{1-\beta} - P\Gamma(\alpha-\beta)\right]\right\}\\
&= \frac{(\alpha-1)\,(t(1-s))^{\alpha-2}}{P}\left\{\Gamma(\alpha+\beta)\,\Gamma(\alpha)(1-s)^{1-\beta} + \lambda\Gamma(\alpha)\,\Gamma(\alpha-\beta)\,\eta^{\alpha+\beta-1}\right.\\
&\qquad \left. -\Gamma(\alpha)\,\Gamma(\alpha+\beta)\right\}\\
&\ge \frac{(\alpha-1)\,(t(1-s))^{\alpha-2}}{P}\left\{\Gamma(\alpha)\,\Gamma(\alpha+\beta)\left[(1-s)^{1-\beta} - 1\right]\right\} \ge 0,
\end{aligned}$$

so, we have $\frac{h_2(t,s)}{\partial t} > 0$, then $g_2(t,s)$ is increasing with respect to t on $[s,1]$.
Then, we conclude that $G(t,s)$ is increasing with respect to t on $[s,1]$. Hence, $G(t,s) \le G(1,s)$ for $s,\, t \in [0,1]$.
On the hand, we know that

$$\begin{aligned}
\min_{\eta\le t\le 1} G(t,s) &= \begin{cases} \min\limits_{\eta\le t\le 1}\{g_1(t,s), g_3(t,s)\}, & 0 \le s \le \eta,\\ \min\limits_{\eta\le t\le 1}\{g_2(t,s), g_4(t,s)\}, & \eta \le s \le 1,\end{cases}\\
&= \begin{cases} g_1(\eta,s), & 0 \le s \le \eta,\\ g_2(\eta,s), & \eta \le s \le 1.\end{cases}
\end{aligned}$$

Let

$$\gamma(s) \le \begin{cases} \frac{g_1(\eta,s)}{G(1,s)}, & 0 < s \le \eta,\\ \frac{g_2(\eta,s)}{G(1,s)}, & \eta < s \le 1,\end{cases}$$

where

$$G(1,s)=\begin{cases} g_1(1,s), & 0\le s\le\eta,\\ g_2(1,s), & \eta\le s\le 1.\end{cases}$$

$$=\begin{cases} \frac{\Gamma(\alpha)\left(\Gamma(\alpha+\beta)(1-s)^{\alpha-\beta-1}-\lambda\Gamma(\alpha-\beta)(\eta-s)^{\alpha+\beta-1}\right)-P\Gamma(\alpha-\beta)\Gamma(\alpha+\beta)(1-s)^{\alpha-1}}{P\Gamma(\alpha)\Gamma(\alpha+\beta)\Gamma(\alpha-\beta)}, & 0\le s\le\eta\\ \frac{\Gamma(\alpha)\Gamma(\alpha+\beta)(1-s)^{\alpha-\beta-1}-P\Gamma(\alpha-\beta)\Gamma(\alpha+\beta)(1-s)^{\alpha-1}}{P\Gamma(\alpha)\Gamma(\alpha-\beta)\Gamma(\alpha+\beta)}, & \eta\le s\le 1.\end{cases}$$

Therefore, we have

$$\gamma(s)=\eta^{\alpha-1}\in(0,1).$$

Then

$$\min_{\eta\le t\le 1}G(t,s)\ge\gamma(s)\max_{0\le t\le 1}G(t,s)=\eta^{\alpha-1}G(1,s),\ 0<s<1.$$

The proof is complete. □

3. Existence results

We shall consider the Banach space $E=C[0,1]$ equipped with the norm

$$\|u\|=\max_{0\le t\le 1}|u(t)|$$

and let a closed cone $K\subset E$ by

$$K=\{u\in E:\ u(t)\ge 0,\ t\in[0,1]\},$$

where 0 is the the zero function. Obviously, K is a reproducing cone of E.
Define the operator $T:K\to K$ and the linear operator $L:K\to K$ as follows

$$T(u)(t)=\int_0^1 G(t,s)\,a(s)\,f(s,u(s))\,ds,\quad t\in[0,1],\tag{3.1}$$

and

$$L(u)(t)=\int_0^1 G(t,s)\,a(s)\,u(s)\,ds,\quad t\in[0,1],\tag{3.2}$$

where $G(t,s)$ is given by (2.3). It is not hard to see that fixed points of operator T coincide with the solutions to the problem (1.1) – (1.2).
First, for the existence results of problem (1.1) – (1.2), we need the following assumptions.
(H_1) $f:[0,1]\times[0,\infty)\to[0,\infty)$ is continuous function,
(H_2) $a(\cdot)\in L^1(0,1)$ is a nonnegative function, $a(t)$ does not vanish identically on any subinterval of $[0,1]$ and $0<\int_0^1 a(s)(1-s)^{\alpha-\beta-1}s^{\alpha-1}ds<\infty$.

Lemma 3.1. *Assume (H_1) and (H_2) hold. Then the operators $T:K\to K$ and $L:K\to K$ are completely continuous.*

Proof. For any $u \in K$, it follows from (H_1), (H_2) and Lemma 2.10, $T(u)(t) \geq 0$, $t \in [0,1]$. So, $T : K \to K$ and $L : K \to K$ are continuous.
Let $\Phi \subset K$ be bounded .i.e., there exists a positive constant M such that $f(t,u) \leq M$ for all $t \in [0,1]$, $u \in \Phi$. Then, It follows from (3.1) that

$$\begin{aligned} |Tu(t)| &\leq \frac{Mt^{\alpha-1}}{P\Gamma(\alpha-\beta)} \int_0^1 (1-s)^{\alpha-\beta-1} a(s)\,ds + \frac{M}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} a(s)\,ds \\ &\quad + \frac{\lambda M t^{\alpha-1}}{P\Gamma(\alpha+\beta)} \int_0^\eta (\eta-s)^{\alpha+\beta-1} a(s)\,ds \\ &\leq \frac{M}{P\Gamma(\alpha-\beta)} \int_0^1 a(s)\,ds + \frac{M}{\Gamma(\alpha)} \int_0^1 a(s)\,ds \\ &\quad + \frac{\lambda M}{P\Gamma(\alpha+\beta)} \int_0^1 a(s)\,ds \\ &\leq \frac{M\left(\Gamma(\alpha)\Gamma(\alpha+\beta) + P\Gamma(\alpha+\beta)\Gamma(\alpha-\beta) + \lambda\Gamma(\alpha)\Gamma(\alpha-\beta)\right)}{P\Gamma(\alpha)\Gamma(\alpha+\beta)\Gamma(\alpha-\beta)} \int_0^1 a(s)\,ds. \end{aligned}$$

Thus $\|Tu\| < \infty$ for all $u \in \Phi$. Hence, $\{Tu,\ u \in \Phi\}$ is bounded.
Now, we show that T maps bounded sets into equicontinuous sets of K.
Let $t_1,\ t_2 \in [0,1]$ with $t_1 < t_2$ and $u \in \Phi$ is a bounded set of K. Then

$$\begin{aligned} |Tu(t_2) - Tu(t_1)| &\leq \left| \frac{t_2^{\alpha-1}}{P\Gamma(\alpha-\beta)} \int_0^1 (1-s)^{\alpha-\beta-1} a(s) f(s,u(s))\,ds \right. \\ &\quad \left. - \frac{t_1^{\alpha-1}}{P\Gamma(\alpha-\beta)} \int_0^1 (1-s)^{\alpha-\beta-1} a(s) f(s,u(s))\,ds \right| \\ &\quad + \left| \frac{t_1^{\alpha-1}}{\Gamma(\alpha)} \int_0^{t_1} (t_1-s)^{\alpha-1} a(s) f(s,u(s))\,ds - \frac{t_2^{\alpha-1}}{\Gamma(\alpha)} \int_0^{t_2} (t_2-s)^{\alpha-1} a(s) f(s,u(s))\,ds \right| \\ &\quad + \left| \frac{\lambda t_1^{\alpha-1}}{P\Gamma(\alpha+\beta)} \int_0^\eta (\eta-s)^{\alpha-1} a(s) f(s,u(s))\,ds \right. \\ &\quad \left. - \frac{\lambda t_2^{\alpha-1}}{P\Gamma(\alpha+\beta)} \int_0^\eta (\eta-s)^{\alpha-1} a(s) f(s,u(s))\,ds \right| \\ &\leq \frac{M\left(t_2^{\alpha-1} - t_1^{\alpha-1}\right)}{P\Gamma(\alpha-\beta)} \int_0^1 a(s)\,ds + \frac{\lambda M\left(t_2^{\alpha-1} - t_1^{\alpha-1}\right)}{P\Gamma(\alpha-\beta)} \int_0^1 a(s)\,ds \end{aligned}$$

$$+\frac{M\left(t_1^{\alpha-1}-t_2^{\alpha-1}\right)}{\Gamma(\alpha)}\left|\int_0^{t_1}(t_1-s)^{\alpha-1}a(s)\,ds\right|+\frac{Mt_2^{\alpha-1}}{\Gamma(\alpha)}\left|\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}a(s)\,ds\right|.$$

Obviously, the right hand side of the above inequality tends to zero as $t_2 \to t_1$. Thus $\|(Tu)(t_2)-(Tu)(t_1)\| \to 0$, as $t_2 \to t_1$. This shows that the operator T is completely continuous, by the Arzela-Ascoli theorem.
By the same method we can get that $L : K \to K$ is a completely continuous operator. The proof is complete. □

Now, we present the existence result for the boundary value problem (1.1) – (1.2) via Banach contraction principle.

Theorem 3.2. *Assume* (H_1) *and* (H_2) *hold. Suppose that* $f : [0,1]\times[0,\infty) \to [0,\infty)$ *be a continuous function satisfying the condition*
(H_3) $|f(t,u)-f(t,v)| \le l\,|u-v|$, *for* $t \in [0,1]$, $l > 0$ *and* $u, v \in [0,+\infty)$.
If $0 < \int_0^1 G(1,s)\,a(s)\,ds < 1$, *then the boundary value problem* (1.1) – (1.2) *has a unique positive solution on* $[0,1]$.

Proof. As the first step, by Lemma 2.9 we know that $T : K \to K$.
Now, let $u, v \in K$ and for each $t \in [0,1]$, it follows from assumption (H_3) that

$$\|Tu(t)-Tv(t)\| = \max_{t\in[0,1]}|Tu(t)-Tv(t)|$$

$$\le \int_0^1 G(t,s)\,a(s)\,|f(s,u(s))-f(s,v(s))|\,ds$$

$$\le l\int_0^1 G(1,s)\,a(s)\,|u(s)-v(s)|\,ds$$

$$\le l\int_0^1 G(1,s)\,a(s)\,ds\,\|u-v\|.$$

Thus,

$$\|(Tu)-(Tv)\| \le l\int_0^1 G(1,s)\,a(s)\,ds\,\|u-v\|.$$

Since $l\int_0^1 G(1,s)\,a(s)\,ds < 1$, so T s a contraction. Hence it follows by Banach's contraction principle that the boundary value problem (1.1) – (1.2) has a unique positive solution on $[0,1]$. The proof is complete. □

Now, we are in a position to study the existence of solutions for the boundary value problem (1.1) – (1.2) by applying the fixed point index theory.

Lemma 3.3. *Assume* (H_1) *and* (H_2) *hold. Then the spectral radius of the operator* L *is positive that is* $r(L) > 0$.

Proof. Take $u(t) = t^{\alpha-1} \in E$. Then $\|u\| = 1$. We have

$$Lu(t) = \frac{t^{\alpha-1}}{P\Gamma(\alpha-\beta)} \int_0^1 (1-s)^{\alpha-\beta-1} a(s) u(s) ds$$

$$-\frac{\lambda t^{\alpha-1}}{P\Gamma(\alpha+\beta)} \int_0^\eta (\eta-s)^{\alpha+\beta-1} a(s) u(s) ds - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} a(s) u(s) ds$$

$$= \frac{t^{\alpha-1}}{P\Gamma(\alpha-\beta)} \int_0^1 (1-s)^{\alpha-\beta-1} a(s) s^{\alpha-1} ds$$

$$-\frac{\lambda t^{\alpha-1}\eta^{\alpha+\beta-1}}{P\Gamma(\alpha+\beta)} \int_0^\eta \left(1-\frac{s}{\eta}\right)^{\alpha+\beta-1} a(s) s^{\alpha-1} ds - \frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_0^t \left(1-\frac{s}{t}\right)^{\alpha-1} a(s) s^{\alpha-1} ds$$

$$= t^{\alpha-1} \left\{ -\frac{\lambda \eta^{\alpha+\beta-1}}{P\Gamma(\alpha+\beta)} \int_0^\eta \left(1-\frac{s}{\eta}\right)^{\alpha+\beta-1} a(s) s^{\alpha-1} ds \right.$$

$$\left. + \frac{1}{P\Gamma(\alpha-\beta)} \int_0^1 (1-s)^{\alpha-\beta-1} a(s) s^{\alpha-1} ds - \frac{1}{\Gamma(\alpha)} \int_0^t \left(1-\frac{s}{t}\right)^{\alpha-1} a(s) s^{\alpha-1} ds \right\}$$

$$> t^{\alpha-1} \left\{ -\frac{\lambda \eta^{\alpha+\beta-1}}{P\Gamma(\alpha+\beta)} \int_0^1 \left(1-\frac{s}{\eta}\right)^{\alpha+\beta-1} a(s) s^{\alpha-1} ds \right.$$

$$\left. + \frac{1}{P\Gamma(\alpha-\beta)} \int_0^1 (1-s)^{\alpha-\beta-1} a(s) s^{\alpha-1} - \frac{1}{\Gamma(\alpha)} \int_0^1 \left(1-\frac{s}{t}\right)^{\alpha-1} a(s) s^{\alpha-1} ds \right\}$$

$$> t^{\alpha-1} \left\{ -\frac{\lambda \eta^{\alpha+\beta-1}}{P\Gamma(\alpha+\beta)} \int_0^1 (1-s)^{\alpha+\beta-1} a(s) s^{\alpha-1} ds \right.$$

$$\left. + \frac{1}{P\Gamma(\alpha-\beta)} \int_0^1 (1-s)^{\alpha-\beta-1} a(s) s^{\alpha-1} ds - \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} a(s) s^{\alpha-1} ds \right\}$$

$$= t^{\alpha-1} \left\{ -\frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} a(s) s^{\alpha-1} ds \right.$$

$$\left. + \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-\beta-1} a(s) s^{\alpha-1} ds \right\} = \nu t^{\alpha-1} > 0,$$

where

$$\nu = -\frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} a(s) s^{\alpha-1} ds + \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-\beta-1} a(s) s^{\alpha-1} ds.$$

Since $L : K \to K$, according the monotonicity of L and (H_2), we deduce

$$L^2 u(t) = L(Lu(t)) > L\left(\nu t^{\alpha-1}\right) > \nu L\left(t^{\alpha-1}\right) > \nu^2 t^{\alpha-1}.$$

Repeating the process gives $L^n u(t) > \nu^n t^{\alpha-1}$. So, we get $\|L^n\| > \nu^n$. Hence

$$\|L^n\|^{\frac{1}{n}} > \nu, \quad r(L) = \lim_{n\to\infty} \|L^n\|^{\frac{1}{n}} > \nu > 0.$$

The proof is complete. □

For convenience, we introduce the following notation:

$$f^\infty = \lim_{u\to\infty} \sup \max_{t\in[0,1]} \frac{f(t,u)}{u},$$

$$f_0 = \lim_{u\to 0^+} \inf \min_{t\in[0,1]} \frac{f(t,u)}{u},$$

$$K_c = \{u \in K : \|u\| < c\},$$

$$r(L) = \frac{1}{\mu},\ \mu \in \mathbb{R}^+.$$

Lemma 3.4. *Assume (H_1), (H_2) hold and $\mu < f_0 \le \infty$. Then there exists $\rho_0 > 0$ such that for $\rho \in (0, \rho_0]$, if $u \ne Tu$, $u \in \partial K_\rho$, then $i(T, K_\rho, K) = 0$.*

Proof. It follows from $\mu < f_0$ that there exist $\varepsilon > 0$ and $\rho_0 > 0$ such that for $t \in [0,1]$ and $0 \le u \le \rho_0$ we have

$$f(t,u) \ge (\mu+\varepsilon) u. \tag{3.3}$$

For $0 < \rho < \rho_0$ assume that $u \ne Tu$, $u \in \partial K_\rho$. By Lemma 2.7 and Lemma 2.8 (i), we need only to prove that

$$u \ne Tu + \lambda\varphi,\ \lambda > 0,$$

where $\varphi \in K \setminus \{0\}$ with $L\varphi = r(L)\varphi$.
Otherwise, there exist $u_0 \in \partial K_\rho$ and $\lambda_0 > 0$ such that

$$u_0 \ne Tu_0 + \lambda_0\varphi. \tag{3.4}$$

Then $u_0 \ge Tu_0$ and $u_0 \ge \lambda_0\varphi$.
From (2.1), we get

$$Tu_0(t) = \int_0^1 G(t,s) a(s) f(s, u_0(s)) ds \ge (\mu+\varepsilon) Lu_0(t). \tag{3.5}$$

Considering $u_0 \ge \lambda_0\varphi$, we have

$$lu_0 \ge \lambda_0 L\varphi.$$

For $L\varphi = r(L)\varphi$, $(\mu+\varepsilon) r(L) > 1$, so that $(\mu+\varepsilon) r(L)\varphi > \varphi$.
Thus, we can conclude $Tu_0 \ge (\mu+\varepsilon)\lambda_0 L\varphi > \lambda_0\varphi$.

Together with the boundary conditions in (2.1), we have $u_0 \geq 2\lambda_0\varphi$. By (3.3), we obtain $Tu_0 \geq 2\lambda_0\varphi$. Thus, $u_0 \geq 3\lambda_0\varphi$.
Repeating this process, we get that $u_0 \geq n\lambda_0\varphi$. Hence, we have $\|u_0\| \geq n\lambda_0 \|\varphi\| \to \infty$ as $n \to \infty$. This is a contradiction.
It follows from Lemma 2.8 (ii) that $i(T, K_\rho, K) = 0$ for $\rho \in (0, \rho_0]$. The proof is complete. □

Lemma 3.5. *Assume* (H_1), (H_2) *hold and* $0 \leq f^\infty < \mu$. *Then there exists* $\tau_0 > 0$ *such that for* $\tau > \tau_0$, *if* $\lambda u \neq Tu$, $u \in \partial K_\tau$, *then* $i(T, K_\rho, K) = 1$.

Proof. let $\varepsilon > 0$ satisfy $f^\infty < \mu - \varepsilon$. Then there exist $\tau_1 > 0$ and such that for $u > \tau_1$ and $t \in [0, 1]$, we have

$$f(t, u) \leq (\mu - \varepsilon) u. \tag{3.6}$$

Set $\Psi(t) = \max_{u \in [0, \tau_1]} f(t, u)$. Then, for all $u \in \mathbb{R}^+$ and $t \in [0, 1]$, we have

$$f(t, u) \leq (\mu - \varepsilon) u + \Psi(t). \tag{3.7}$$

Let

$$F = \left\| \int_0^1 G(t, s) a(s) \Psi(s) ds \right\|, \ \tau_0 = \left\| \frac{F}{\mu - \varepsilon} \left(\frac{I}{\mu - \varepsilon} - L \right)^{-1} \right\|.$$

Take $\tau > \tau_0$. We will show that $\lambda u \neq Tu$, *for all* $u \in \partial K_\tau$ and $\lambda \geq 1$.
Otherwise, there exist $u_0 \in \partial K_\tau$ and $\lambda_0 \geq 1$ such that

$$Tu_0 = \lambda_0 u_0. \tag{3.8}$$

Together with (3.7), we have

$$u_0 \leq \lambda u_0 = Tu_0 \leq (\mu - \varepsilon) Lu_0 + F.$$

Then $\frac{F}{\mu-\varepsilon} \geq \left(\frac{I}{\mu-\varepsilon} - L\right) u_0(t)$ for $t \in [0, 1]$. So, $\frac{F}{\mu-\varepsilon} - \left(\frac{I}{\mu-\varepsilon} - L\right) u_0(t) \in K$.
It follows from $L(K) \subset K$ that $u_0(t) \leq \frac{F}{\mu-\varepsilon} \left(\frac{I}{\mu-\varepsilon} - L\right)^{-1}$ $t \in [0, 1]$. Therefore, we have $\|u_0\| \leq \tau_0 < \tau$. This is a contradiction. Thus, we conclude that for all $u \in \partial K_\tau$ and $\lambda \geq 1$

$$Tu \neq \lambda u.$$

It follows from Lemma 2.8 (ii) that $i(T, K_\tau, K) = 1$ for $\tau_0 < \tau$.
The proof is complete. □

Theorem 3.6. *Assume* (H_1), (H_2) *hold,* $\mu < f_0 \leq \infty$ *and* $0 \leq f^\infty \leq \mu$. *Then, the boundary value problem* $(1.1) - (1.2)$ *has at least one positive solution on* $[0, 1]$.

Proof. It follows from $\mu < f_0 \leq \infty$ and Lemma 3.4 that there exist $0 < \rho < \tau$ such that either there exists $u \in \partial K_\rho$ with $u = Tu$ or $i(T, K_\rho, K) = 0$. From $0 \leq f^\infty \leq \mu$ and Lemma 3.5 there exists $\tau > 0$ such that $i(T, K_\tau, K) = 1$. Thus, we can conclude that T has fixed point $u \in K$ with $\rho < \|u\| < \tau$ by the properties of index. Hence, the boundary value problem $(1.1) - (1.2)$ has at least one positive solution on $[0, 1]$.
The proof is complete. □

Now, we are in the position to present the third main results of this paper. The existence and the multiplicity result is based on the Leggett-Williams fixed point theorem.

Theorem 3.7. *Assume* (H_1)*and* (H_2) *hold. Furthermore, suppose that there exist constants* $0 < d < l < c$ *such that*
(H_4) $f(t,u) < Md,\ (t,u) \in [0,1] \times [0,d]$,
(H_5) $f(t,u) \leq Mc,\ for\ (t,u) \in [0,1] \times [0,c]$,
(H_6) $f(t,u) \geq Nl,\ for\ (t,u) \in [\eta,1] \times [l,c]$,
where

$$M = \left(\int_0^1 a(s)\, G(1,s)\, ds \right)^{-1},$$

and

$$N = \left(\int_\eta^1 a(s)\, \gamma(s)\, G(1,s)\, ds \right)^{-1},\ and\ \gamma(s) \in (0,1).$$

Then the boundary value problem $(1.1)-(1.2)$ *has at least three positive solutions* u_1, u_2 *and* u_3 *such that*

$$\|u_1\| < d, \quad l < \beta(u_2), \quad u_3 > d\ with\ \beta(u_3) < l.$$

Proof. Let $\beta(u) = \min_{t\in[\eta,1]} |u(t)|$. Then $\beta(u)$ is nonnegative continuous concave functional on the cone K satisfying $\beta(u) \leq \|u\|$ for all $u \in K$.
Let $u \in \bar{K}_c$, then $\|u\| \leq c$. It follows from (H_5) and Lemma 2.10 (iii) that

$$|Tu(t)| = \left| \int_0^1 G(t,s)\, a(s)\, f(s,u(s))\, ds \right|$$
$$\leq Mc \int_0^1 G(1,s)\, a(s)\, ds = c,$$

which implies that $\|Tu\| \leq c$, which shows that $Tu \in \overline{K}_c$. Hence, we have shown that if (H_5) holds, then T maps $\overline{K}_c$ into $\overline{K}_c$ and by Lemma 3.1, T is completely continuous.
If $u \in \overline{K}_d$, then it follows from (H_4) and Lemma 2.10 (iii) that

$$(Tu)(t) = \int_0^1 G(t,s)\, a(s)\, f(s,u(s))\, ds$$
$$< Md \int_0^1 G(1,s)\, a(s)\, ds = d.$$

We verify that $\{u/K(\beta,l,r) : \beta(u) > l\} \neq \phi$ and $\beta(Tu) > l$ for all $u \in K(\beta,l,r)$. Take $\varphi_0(t) = \frac{l+r}{2}$, for $t \in [0,1]$. Then

$$\varphi_0 \in \{u/u \in K(\beta,l,r),\, \beta(u) > l\}.$$

This shows that

$$\{u/u \in K(\beta,l,r) : \beta(u) > l\} \neq \phi.$$

Finally, we assert that if $u \in K(\beta,l,c)$ and $\|Tu\| > c$, then $\beta(Tu) > l$.
Suppose $u \in K(\beta,l,c)$ and $\|u(t)\| > r$, $t \in [\eta,1]$, then $\|u\| < c$. It follows from (H_6) that

$$\begin{aligned}\beta(Tu) &= \min_{t\in[\eta,1]} (Tu)(t)\\ &\geq \min_{t\in[\eta,1]} \int_0^1 G(t,s)\,a(s)\,f(s,u(s))\,ds\\ &> Nl \int_\eta^1 G(1,s)\,a(s)\,\gamma(s)\,ds = l,\end{aligned}$$

which implies that $\beta(Tu) > l$ for $u \in K(\beta,l,c)$.
To sum up, the hypotheses of Theorem 2.6 hold. Therefore, boundary value problem $(1.1)-(1.2)$ has at least three positive solutions u_1, u_2 *and* u_3 such that

$$\|u_1\| < d, \quad l < \beta(u_2), \quad u_3 > d \text{ with } \beta(u_3) < l.$$

The proof is complete. □

We present two examples to illustrate the applicability of the results shown before.

Example 3.8. Consider the following boundary value problem

$$D_{0^+}^{\frac{7}{2}} u(t) + \frac{1}{(t+\cos t+3)^2}\left(\sin^2 t + \arctan(u) + \frac{|u|}{1+|u|}\right) = 0,\ t \in (0,1), \tag{3.9}$$

$$u(0) = u'(0) = u''(0) = 0,\ D^{\frac{5}{2}} u(1) = \frac{1}{2} I_{0^+}^{\frac{5}{2}} u\left(\frac{1}{2}\right), \tag{3.10}$$

where $\alpha = \frac{7}{2}$, $\beta = \frac{5}{2}$, $\lambda = \frac{1}{2}$, $\eta = \frac{1}{2}$ and

$$f(t,u) = \frac{1}{(t+\cos t+3)^2}\left(\sin^2 t + \arctan(u) + \frac{|u|}{1+|u|}\right).$$

Clearly $l = \frac{2}{9}$ as $|f(t,u) - f(t,v)| \leq \frac{2}{9}|u-v|$.
We take $a(t) = 1$. A simple calculation leads to $P \cong 1,32620$.

Furthermore, by simple computation, we have

$$\frac{1}{M}=\int_0^1 a(s)\,G(1,s)\,ds$$

$$=\frac{\Gamma(\alpha)\Gamma(\alpha+\beta)\int_0^1 ds-P\Gamma(\alpha-\beta)\Gamma(\alpha+\beta)\int_0^1(1-s)^{\frac{5}{2}}ds}{P\Gamma(\alpha)\Gamma(\alpha-\beta)\Gamma(\alpha+\beta)}$$

$$+\frac{\Gamma(\alpha)\Gamma(\alpha+\beta)\int_\eta^1(1-s)\,ds}{P\Gamma(\alpha)\Gamma(\alpha-\beta)\Gamma(\alpha+\beta)}\cong 0,27303,$$

so,

$$0<l\int_0^1 a(s)\,G(1,s)\,ds\leq\frac{2}{9}(0,27303)\cong 0,060673<1.$$

Thus all assumptions of Theorem 3.2 are satisfied. So, by the conclusion of Theorem 3.2, problem (3.9) – (3.10) has a unique solution on $[0,1]$.

Example 3.9. Consider the following boundary value problem

$$D_{0^+}^{\frac{7}{2}}u(t)+f(t,u(t))=0,\qquad t\in(0,1),\tag{3.11}$$

$$u(0)=u'(0)=u''(0)=0,\; D^{\frac{5}{2}}u(1)=\frac{1}{2}I_{0^+}^{\frac{5}{2}}u\left(\frac{1}{2}\right),\tag{3.12}$$

where $\alpha=\frac{7}{2}$, $\beta=\frac{5}{2}$, $\lambda=\frac{1}{2}$, $\eta=\frac{1}{2}$, and here

$$f(t,u)=\begin{cases}10u+t, & (t,u)\in[0,1]\times[0,1],\\ 10, & (t,u)\in[0,1]\times(1,+\infty).\end{cases}$$

We take $a(t)=1$. We see that $f\in C([0,1]\times[0,\infty),[0,\infty))$, so, assumption (H_1) satisfied. And

$$0<\int_0^1 a(s)(1-s)^{\alpha-\beta-1}s^{\alpha-1}ds=\int_0^1(1-s)\,s^{\frac{5}{2}}ds=\frac{4}{63}<\infty,$$

so, assumption (H_2) satisfied.
By simple calculation, we obtain $P\cong 1,32620$, $M\cong 3,66264$ and $N\cong 7218,14758$. Choosing, $d=\frac{1}{4}$, $l=1$ and $c=3$, we have

$$f(t,u)=10u+t\leq 3.5<Md\cong 14,65056,\;(t,u)\in[0,1]\times\left[0,\frac{1}{4}\right],$$

$$f(t,u)=10\leq Ml\cong 10,98792,\;(t,u)\in[0,1]\times(1,3],$$

and

$$f(t,u)=10\geq Nr\cong 9,00765,\;(t,u)\in\left[\frac{1}{2},1\right]\times(1,3].$$

Thus, all assumptions and conditions of Theorem 3.7 are satisfied. Hence Theorem 3.7, implies that the problem (3.11) – (3.12) has at least three solutions u_1,u_2 *and* u_3 such that

$$\|u_1\|<d,\quad l<\beta(u_2),\quad u_3>d\text{ with }\beta(u_3)<l.$$

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Some Bessel type additive inequalities in inner product spaces

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Abstract. In this paper we obtain some additive inequalities related to the celebrated Bessel's inequality in inner product spaces. They complement the results obtained by Boas-Bellman, Bombieri, Selberg and Heilbronn, which have been applied for almost orthogonal series and in Number Theory.

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1. Introduction

Let $(H; \langle\cdot,\cdot\rangle)$ be an inner product space over the real or complex number field $\mathbb{K}$. If $(e_i)_{1\le i\le n}$ are orthonormal vectors in the inner product space H, i.e., $\langle e_i, e_j\rangle = \delta_{ij}$ for all $i, j \in \{1,\dots,n\}$ where δ_{ij} is the Kronecker delta, then the following inequality is well known in the literature as *Bessel's inequality*:

$$\sum_{i=1}^{n} |\langle x, e_i\rangle|^2 \le \|x\|^2 \text{ for any } x \in H. \tag{1.1}$$

For other results related to Bessel's inequality, see [8] – [11] and Chapter XV in the book [14].

In 1941, R. P. Boas [2] and in 1944, independently, R. Bellman [1] proved the following generalization of Bessel's inequality (see also [14, p. 392]):

Theorem 1.1. *If $x, y_1, \dots, y_n$ are elements of an inner product space $(H; \langle\cdot,\cdot\rangle)$, then the following inequality holds*

$$\sum_{i=1}^{n} |\langle x, y_i\rangle|^2 \le \|x\|^2 \left[\max_{1\le i\le n} \|y_i\|^2 + \left(\sum_{1\le i\ne j\le n} |\langle y_i, y_j\rangle|^2\right)^{\frac{1}{2}}\right]. \tag{1.2}$$

It is obvious that (1.2) will give for orthonormal families the well known Bessel inequality.

In [7] we pointed out the following Boas-Bellman type inequalities:

$$\sum_{i=1}^{n} |\langle x, y_i \rangle|^2 \leq \|x\| \max_{1 \leq i \leq n} |\langle x, y_i \rangle| \left\{ \sum_{i=1}^{n} \|y_i\|^2 + \sum_{1 \leq i \neq j \leq n} |\langle y_i, y_j \rangle| \right\}^{\frac{1}{2}}, \tag{1.3}$$

for any x, $y_1, \ldots, y_n$ vectors in the inner product space $(H; \langle \cdot, \cdot \rangle)$.

We also have, see [7]

$$\sum_{i=1}^{n} |\langle x, y_i \rangle|^2 \leq \|x\| \left(\sum_{i=1}^{n} |\langle x, y_i \rangle|^{2p} \right)^{\frac{1}{2p}} \tag{1.4}$$

$$\times \left\{ \left(\sum_{i=1}^{n} \|y_i\|^{2q} \right)^{\frac{1}{q}} + (n-1)^{\frac{1}{p}} \left(\sum_{1 \leq i \neq j \leq n} |\langle y_i, y_j \rangle|^q \right)^{\frac{1}{q}} \right\}^{\frac{1}{2}},$$

for any x, $y_1, \ldots, y_n \in H$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$.

Further, we recall [7] that

$$\sum_{i=1}^{n} |\langle x, y_i \rangle|^2 \leq \|x\|^2 \left\{ \max_{1 \leq i \leq n} \|y_i\|^2 + (n-1) \max_{1 \leq i \neq j \leq n} |\langle y_i, y_j \rangle| \right\}, \tag{1.5}$$

for any x, $y_1, \ldots, y_n \in H$. It is obvious that (1.5) will give for orthonormal families the well known Bessel inequality.

In 1971, E. Bombieri [3] gave the following generalization of Bessel's inequality.

Theorem 1.2. *If x, $y_1, \ldots, y_n$ are vectors in the inner product space $(H; (\cdot, \cdot))$, then the following inequality holds:*

$$\sum_{i=1}^{n} |\langle x, y_i \rangle|^2 \leq \|x\|^2 \max_{1 \leq i \leq n} \left\{ \sum_{j=1}^{n} |\langle y_i, y_j \rangle| \right\}. \tag{1.6}$$

It is obvious that if $(y_i)_{1 \leq i \leq n}$ are orthonormal, then from (1.6) one can deduce Bessel's inequality.

It is not widely known, but it appears in a number of places that, the importance of extensions of the Bombieri and Bessel inequality were first shown by J. Sándor (at a Symposium on Mathematical Inequalities, Sibiu, December, 1984), who proved some generalizations of these inequalities, and who was deeply interested in applications in Number Theory. Also, Bessel's inequality and Gram's inequality have been studied by the author and J. Sándor in [12] as well.

Another generalization of Bessel's inequality was obtained by A. Selberg (see for example [14, p. 394]):

Theorem 1.3. *Let x, $y_1, \dots, y_n$ be vectors in H with $y_i \neq 0$ $(i = 1, \dots, n)$. Then one has the inequality:*

$$\sum_{i=1}^{n} \frac{|\langle x, y_i\rangle|^2}{\sum_{j=1}^{n} |\langle y_i, y_j\rangle|} \leq \|x\|^2 . \tag{1.7}$$

Another type of inequality related to Bessel's result, was discovered in 1958 by H. Heilbronn [13] (see also [14, p. 395]).

Theorem 1.4. *With the assumptions in Theorem* 1.2*, one has*

$$\sum_{i=1}^{n} |\langle x, y_i\rangle| \leq \|x\| \left(\sum_{i,j=1}^{n} |\langle y_i, y_j\rangle| \right)^{\frac{1}{2}} . \tag{1.8}$$

In [8] we obtained the following Bombieri type inequalities

$$\sum_{i=1}^{n} |\langle x, y_i\rangle|^2 \leq \|x\| \max_{1\leq i\leq n} |\langle x, y_i\rangle| \left(\sum_{i,j=1}^{n} |\langle y_i, y_j\rangle| \right)^{\frac{1}{2}} , \tag{1.9}$$

$$\sum_{i=1}^{n} |\langle x, y_i\rangle|^2 \tag{1.10}$$

$$\leq \|x\| \max_{1\leq i\leq n} |\langle x, y_i\rangle|^{\frac{1}{2}} \left(\sum_{i=1}^{n} |\langle x, y_i\rangle|^r \right)^{\frac{1}{2r}} \left[\sum_{i=1}^{n} \left(\sum_{j=1}^{n} |\langle y_i, y_j\rangle| \right)^s \right]^{\frac{1}{2s}} ,$$

where $\frac{1}{r} + \frac{1}{s} = 1$, $s > 1$,

$$\sum_{i=1}^{n} |\langle x, y_i\rangle|^2 \tag{1.11}$$

$$\leq \|x\| \max_{1\leq i\leq n} |\langle x, y_i\rangle|^{\frac{1}{2}} \left(\sum_{i=1}^{n} |\langle x, y_i\rangle| \right)^{\frac{1}{2}} \left[\max_{1\leq i\leq n} \left(\sum_{j=1}^{n} |\langle y_i, y_j\rangle| \right) \right] ,$$

$$\sum_{i=1}^{n} |\langle x, y_i\rangle|^2 \tag{1.12}$$

$$\leq \|x\| \max_{1\leq i\leq n} |\langle x, y_i\rangle|^{\frac{1}{2}} \left(\sum_{i=1}^{n} |\langle x, y_i\rangle|^p \right)^{\frac{1}{2p}} \left[\sum_{i=1}^{n} \left(\sum_{j=1}^{n} |\langle y_i, y_j\rangle|^q \right)^{\frac{1}{q}} \right]^{\frac{1}{2}} ,$$

where $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and

$$\sum_{i=1}^{n} |\langle x, y_i\rangle|^2 \leq \|x\|^2 \left\{ \sum_{i,j=1}^{n} |\langle y_i, y_j\rangle|^2 \right\}^{\frac{1}{2}} \tag{1.13}$$

for any $x \in H$.

It has been shown that for different selection of vectors the upper bound provided by the inequality (1.13) is some time better other times worse than the one obtained by Bombieri above in (1.6).

In this paper we obtain some inequalities related to the celebrated Bessel's inequality in inner product spaces. They complement the results obtained by Boas-Bellman, Bombieri, Selberg and Heilbronn above, which have been applied for almost orthogonal series and in Number Theory.

2. Some results via CBS inequality

We have:

Theorem 2.1. *Let x, $y_1, ..., y_n \in H$ and $\alpha_1, ..., \alpha_n \in \mathbb{C}$. Then*

$$\operatorname{Re}\left(\sum_{j=1}^{n} \alpha_j \langle y_j, x\rangle\right) \le \frac{1}{2}\left[\|x\|^2 + \sum_{k=1}^{n} |\alpha_k|^2 \left(\sum_{j,k=1}^{n} |\langle y_j, y_k\rangle|^2\right)^{1/2}\right]. \tag{2.1}$$

Proof. We have for any x, $y_1, ..., y_n \in H$ and $\alpha_1, ..., \alpha_n \in \mathbb{C}$ that

$$\begin{aligned} 0 &\le \left\|\sum_{j=1}^{n} \alpha_j y_j - x\right\|^2 = \left\|\sum_{j=1}^{n} \alpha_j y_j\right\|^2 - 2\operatorname{Re}\left\langle \sum_{j=1}^{n} \alpha_j y_j, x\right\rangle + \|x\|^2 \\ &= \left\langle \sum_{j=1}^{n} \alpha_j y_j, \sum_{k=1}^{n} \alpha_k y_k\right\rangle - 2\operatorname{Re}\left(\sum_{j=1}^{n} \alpha_j \langle y_j, x\rangle\right) + \|x\|^2 \\ &= \sum_{j,k=1}^{n} \alpha_j \overline{\alpha_k} \langle y_j, y_k\rangle - 2\operatorname{Re}\left(\sum_{j=1}^{n} \alpha_j \langle y_j, x\rangle\right) + \|x\|^2, \end{aligned}$$

which implies the inequality

$$\operatorname{Re}\left(\sum_{j=1}^{n} \alpha_j \langle y_j, x\rangle\right) \le \frac{1}{2}\left[\|x\|^2 + \sum_{j,k=1}^{n} \alpha_j \overline{\alpha_k} \langle y_j, y_k\rangle\right] \tag{2.2}$$

for which the term $\sum_{j,k=1}^{n} \alpha_j \overline{\alpha_k} \langle y_j, y_k\rangle$ is obviously nonnegative for any $y_1, ..., y_n \in H$ and $\alpha_1, ..., \alpha_n \in \mathbb{C}$.
By using the Cauchy-Buniakowski-Schwarz's inequality for double sums,

$$\sum_{j,k=1}^{n} |a_{jk} b_{jk}| \le \left(\sum_{j,k=1}^{n} |a_{jk}|^2\right)^{1/2} \left(\sum_{j,k=1}^{n} |b_{jk}|^2\right)^{1/2}$$

for complex numbers a_{jk}, b_{jk} where j, $k \in \{1, ..., n\}$, then we have

$$\begin{aligned}\sum_{j,k=1}^{n} \alpha_j \overline{\alpha_k} \langle y_j, y_k \rangle &= \left| \sum_{j,k=1}^{n} \alpha_j \overline{\alpha_k} \langle y_j, y_k \rangle \right| \le \sum_{j,k=1}^{n} |\alpha_j \overline{\alpha_k}| \, |\langle y_j, y_k \rangle| \\ &\le \left(\sum_{j,k=1}^{n} |\alpha_j \overline{\alpha_k}|^2 \right)^{1/2} \left(\sum_{j,k=1}^{n} |\langle y_j, y_k \rangle|^2 \right)^{1/2} \\ &= \left(\sum_{j,k=1}^{n} |\alpha_j|^2 |\overline{\alpha_k}|^2 \right)^{1/2} \left(\sum_{j,k=1}^{n} |\langle y_j, y_k \rangle|^2 \right)^{1/2} \\ &= \left(\sum_{j=1}^{n} |\alpha_j|^2 \sum_{k=1}^{n} |\alpha_k|^2 \right)^{1/2} \left(\sum_{j,k=1}^{n} |\langle y_j, y_k \rangle|^2 \right)^{1/2} \\ &= \sum_{k=1}^{n} |\alpha_k|^2 \left(\sum_{j,k=1}^{n} |\langle y_j, y_k \rangle|^2 \right)^{1/2}\end{aligned} \tag{2.3}$$

for any $y_1, ..., y_n \in H$ and $\alpha_1, ..., \alpha_n \in \mathbb{C}$.
By making use of (2.2) and (2.3) we get the desired result (2.1). □

Corollary 2.2. *With the assumptions of Theorem* 2.1 *and for* $p \ge 1$ *we have*

$$\sum_{j=1}^{n} |\langle x, y_j \rangle|^p \le \frac{1}{2} \left[\|x\|^2 + \sum_{k=1}^{n} |\langle x, y_k \rangle|^{2(p-1)} \left(\sum_{j,k=1}^{n} |\langle y_j, y_k \rangle|^2 \right)^{1/2} \right]. \tag{2.4}$$

Proof. If we take in (2.1) $\alpha_j = \langle x, y_j \rangle |\langle x, y_j \rangle|^{p-2}$ then we get

$$\begin{aligned}&\operatorname{Re} \left(\sum_{j=1}^{n} \langle x, y_j \rangle |\langle x, y_j \rangle|^{p-2} \langle y_j, x \rangle \right) \\ &\le \frac{1}{2} \left[\|x\|^2 + \sum_{k=1}^{n} \left| \langle x, y_j \rangle |\langle x, y_j \rangle|^{p-2} \right|^2 \left(\sum_{j,k=1}^{n} |\langle y_j, y_k \rangle|^2 \right)^{1/2} \right],\end{aligned}$$

which is equivalent to (2.4). □

Remark 2.3. If we take in (2.4) $p = 1$, then we get the following Heilbronn type inequality

$$\sum_{j=1}^{n} |\langle x, y_j \rangle| \le \frac{1}{2} \left[\|x\|^2 + n \left(\sum_{j,k=1}^{n} |\langle y_j, y_k \rangle|^2 \right)^{1/2} \right] \tag{2.5}$$

for any x, $y_1, ..., y_n \in H$.

If we take in (2.4) $p=2$, then we get

$$\sum_{j=1}^{n}\left|\langle x,y_j\rangle\right|^2 \le \frac{1}{2}\left[\left\|x\right\|^2+\sum_{k=1}^{n}\left|\langle x,y_k\rangle\right|^2\left(\sum_{j,k=1}^{n}\left|\langle y_j,y_k\rangle\right|^2\right)^{1/2}\right], \tag{2.6}$$

that is equivalent to (see also [10])

$$\left[2-\left(\sum_{j,k=1}^{n}\left|\langle y_j,y_k\rangle\right|^2\right)^{1/2}\right]\sum_{j=1}^{n}\left|\langle x,y_j\rangle\right|^2 \le \left\|x\right\|^2 \tag{2.7}$$

for any $x,\ y_1,...,y_n \in H$.
The inequality (2.7) is meaningful if

$$2 \ge \left(\sum_{j,k=1}^{n}\left|\langle y_j,y_k\rangle\right|^2\right)^{1/2}.$$

Also if

$$1 \ge \left(\sum_{j,k=1}^{n}\left|\langle y_j,y_k\rangle\right|^2\right)^{1/2},$$

then

$$\sum_{j=1}^{n}\left|\langle x,y_j\rangle\right|^2 \le \left[2-\left(\sum_{j,k=1}^{n}\left|\langle y_j,y_k\rangle\right|^2\right)^{1/2}\right]\sum_{j=1}^{n}\left|\langle x,y_j\rangle\right|^2 \le \left\|x\right\|^2, \tag{2.8}$$

for any $x \in H$, which improves Bessel's inequality.
We observe that if the family of vectors $\{y_1,...,y_n\}$ is orthogonal, then

$$\sum_{j,k=1}^{n}\left|\langle y_j,y_k\rangle\right|^2 = \sum_{k=1}^{n}\left\|y\right\|_k^4,$$

so, if we assume that

$$\sum_{k=1}^{n}\left\|y\right\|_k^4 \le 1$$

then by (2.8) we get the refinement of Bessel's inequality

$$\sum_{j=1}^{n}\left|\langle x,y_j\rangle\right|^2 \le \left[2-\left(\sum_{k=1}^{n}\left\|y\right\|_k^4\right)^{1/2}\right]\sum_{j=1}^{n}\left|\langle x,y_j\rangle\right|^2 \le \left\|x\right\|^2. \tag{2.9}$$

Corollary 2.4. *With the assumptions of Theorem* 2.1 *we have*

$$\sum_{j=1}^{n} \frac{|\langle x, y_j\rangle|^2}{\sum_{k=1}^{n} |\langle y_k, y_j\rangle|} \leq \frac{1}{2}\left[\|x\|^2 + \sum_{j=1}^{n} \frac{|\langle x, y_j\rangle|^2}{\left(\sum_{k=1}^{n} |\langle y_k, y_j\rangle|\right)^2}\left(\sum_{j,k=1}^{n} |\langle y_j, y_k\rangle|^2\right)^{1/2}\right], \tag{2.10}$$

for any $x \in H$.

Proof. We take in (2.1)

$$\alpha_j = \frac{\langle x, y_j\rangle}{\sum_{k=1}^{n} |\langle y_k, y_j\rangle|},\ j = 1, ..., n$$

to get (2.10). □

Using the Schwarz's inequality we get from (2.4) that

$$\sum_{j=1}^{n} |\langle x, y_j\rangle|^p \leq \frac{1}{2}\|x\|^2\left[1 + \|x\|^{2(p-2)} \sum_{k=1}^{n} \|y_k\|^{2(p-1)}\left(\sum_{j,k=1}^{n} |\langle y_j, y_k\rangle|^2\right)^{1/2}\right], \tag{2.11}$$

for any x, $y_1, ..., y_n \in H$ and $p \geq 1$.
For $p = 2$ we get

$$\sum_{j=1}^{n} |\langle x, y_j\rangle|^2 \leq \frac{1}{2}\|x\|^2\left[1 + \sum_{k=1}^{n} \|y_k\|^2\left(\sum_{j,k=1}^{n} |\langle y_j, y_k\rangle|^2\right)^{1/2}\right], \tag{2.12}$$

for any x, $y_1, ..., y_n \in H$.
From (2.10) we also get Selberg's type inequality

$$\sum_{j=1}^{n} \frac{|\langle x, y_j\rangle|^2}{\sum_{k=1}^{n} |\langle y_k, y_j\rangle|} \leq \frac{1}{2}\|x\|^2\left[1 + \sum_{j=1}^{n} \frac{\|y_j\|^2}{\left(\sum_{k=1}^{n} |\langle y_k, y_j\rangle|\right)^2}\left(\sum_{j,k=1}^{n} |\langle y_j, y_k\rangle|^2\right)^{1/2}\right], \tag{2.13}$$

for any x, $y_1, ..., y_n \in H$.

Theorem 2.5. *Let x, $y_1, ..., y_n \in H$ and $\alpha_1, ..., \alpha_n \in \mathbb{C}$. Then*

$$\operatorname{Re}\left(\sum_{j=1}^{n} \alpha_j \langle y_j, x\rangle\right) \leq \frac{1}{2}\left[\|x\|^2 + \max_{k\in\{1,...,n\}}\left\{\sum_{j=1}^{n} |\langle y_j, y_k\rangle|\right\} \sum_{k=1}^{n} |a_k|^2\right]. \tag{2.14}$$

Proof. By using the Cauchy-Buniakowski-Schwarz's weighted inequality for double sums,

$$\sum_{j,k=1}^{n} m_{jk} \left|a_{jk} b_{jk}\right| \leq \left(\sum_{j,k=1}^{n} m_{jk} \left|a_{jk}\right|^2\right)^{1/2} \left(\sum_{j,k=1}^{n} m_{jk} \left|b_{jk}\right|^2\right)^{1/2}$$

for complex numbers a_{jk}, b_{jk} and nonnegative numbers m_{jk} where j, $k \in \{1, ..., n\}$, then we have

$$\begin{aligned}
&\sum_{j,k=1}^{n} \alpha_j \overline{\alpha_k} \langle y_j, y_k \rangle \qquad (2.15)\\
&= \left| \sum_{j,k=1}^{n} \alpha_j \overline{\alpha_k} \langle y_j, y_k \rangle \right| \leq \sum_{j,k=1}^{n} \left|\alpha_j \overline{\alpha_k}\right| \left|\langle y_j, y_k \rangle\right| = \sum_{j,k=1}^{n} \left|\alpha_j\right| \left|a_k\right| \left|\langle y_j, y_k \rangle\right| \\
&\leq \left(\sum_{j,k=1}^{n} \left|\langle y_j, y_k \rangle\right| \left|a_j\right|^2\right)^{1/2} \left(\sum_{j,k=1}^{n} \left|\langle y_j, y_k \rangle\right| \left|a_k\right|^2\right)^{1/2} \\
&= \sum_{j,k=1}^{n} \left|a_k\right|^2 \left|\langle y_j, y_k \rangle\right| .
\end{aligned}$$

Now, observe that

$$\begin{aligned}
\sum_{j,k=1}^{n} \left|a_k\right|^2 \left|\langle y_j, y_k \rangle\right| &= \sum_{k=1}^{n} \left|a_k\right|^2 \left(\sum_{j=1}^{n} \left|\langle y_j, y_k \rangle\right|\right) \\
&\leq \max_{k \in \{1,\dots,n\}} \left\{\sum_{j=1}^{n} \left|\langle y_j, y_k \rangle\right|\right\} \sum_{k=1}^{n} \left|a_k\right|^2 ,
\end{aligned}$$

which proves the desired inequality (2.14). □

Corollary 2.6. *With the assumptions of Theorem* 2.5 *and for* $p \geq 1$ *we have*

$$\sum_{j=1}^{n} \left|\langle x, y_j \rangle\right|^p \leq \frac{1}{2} \left[\|x\|^2 + \max_{k \in \{1,\dots,n\}} \left\{\sum_{j=1}^{n} \left|\langle y_j, y_k \rangle\right|\right\} \sum_{k=1}^{n} \left|\langle x, y_k \rangle\right|^{2(p-1)}\right] . \qquad (2.16)$$

Proof. If we take in (2.14) $\alpha_j = \langle x, y_j \rangle \left|\langle x, y_j \rangle\right|^{p-2}$ then we get

$$\begin{aligned}
&\operatorname{Re} \left(\sum_{j=1}^{n} \langle x, y_j \rangle \left|\langle x, y_j \rangle\right|^{p-2} \langle y_j, x \rangle\right) \\
&\leq \frac{1}{2} \left[\|x\|^2 + \max_{k \in \{1,\dots,n\}} \left\{\sum_{j=1}^{n} \left|\langle y_j, y_k \rangle\right|\right\} \sum_{k=1}^{n} \left|\langle x, y_j \rangle \left|\langle x, y_j \rangle\right|^{p-2}\right|^2\right] ,
\end{aligned}$$

which is equivalent to (2.16). □

Remark 2.7. If we take in (2.16) $p = 1$, then we get the following Heilbronn type inequality

$$\sum_{j=1}^{n} |\langle x, y_j \rangle| \leq \frac{1}{2} \left[\|x\|^2 + \max_{k \in \{1,\ldots,n\}} \left\{ \sum_{j=1}^{n} |\langle y_j, y_k \rangle| \right\} \right]. \tag{2.17}$$

for any x, $y_1, ..., y_n \in H$.
If we take in (2.16) $p = 2$, then we get

$$\sum_{j=1}^{n} |\langle x, y_j \rangle|^2 \leq \frac{1}{2} \left[\|x\|^2 + \max_{k \in \{1,\ldots,n\}} \left\{ \sum_{j=1}^{n} |\langle y_j, y_k \rangle| \right\} \sum_{k=1}^{n} |\langle x, y_j \rangle|^2 \right], \tag{2.18}$$

which is equivalent to

$$\left(2 - \max_{k \in \{1,\ldots,n\}} \left\{ \sum_{j=1}^{n} |\langle y_j, y_k \rangle| \right\} \right) \sum_{j=1}^{n} |\langle x, y_j \rangle|^2 \leq \|x\|^2 \tag{2.19}$$

for any x, $y_1, ..., y_n \in H$.
The inequality (2.19) is meaningful if

$$2 \geq \max_{k \in \{1,\ldots,n\}} \left\{ \sum_{j=1}^{n} |\langle y_j, y_k \rangle| \right\}.$$

Also if

$$1 \geq \max_{k \in \{1,\ldots,n\}} \left\{ \sum_{j=1}^{n} |\langle y_j, y_k \rangle| \right\},$$

then

$$\sum_{j=1}^{n} |\langle x, y_j \rangle|^2 \leq \left[2 - \max_{k \in \{1,\ldots,n\}} \left\{ \sum_{j=1}^{n} |\langle y_j, y_k \rangle| \right\} \right] \sum_{j=1}^{n} |\langle x, y_j \rangle|^2 \leq \|x\|^2, \tag{2.20}$$

for any $x \in H$, which improves Bessel's inequality.
We observe that if the family of vectors $\{y_1, ..., y_n\}$ is orthogonal, then

$$\max_{k \in \{1,\ldots,n\}} \left\{ \sum_{j=1}^{n} |\langle y_j, y_k \rangle| \right\} = \max_{k \in \{1,\ldots,n\}} \|y\|_k^2,$$

so, if we assume that $\max_{k \in \{1,\ldots,n\}} \|y\|_k^2 \leq 1$ then by (2.20) we get

$$\sum_{j=1}^{n} |\langle x, y_j \rangle|^2 \leq \left[2 - \max_{k \in \{1,\ldots,n\}} \|y\|_k^2 \right] \sum_{j=1}^{n} |\langle x, y_j \rangle|^2 \leq \|x\|^2, \tag{2.21}$$

for any $x \in H$.

Corollary 2.8. *With the assumptions of Theorem* 2.5 *we have*

$$\sum_{j=1}^{n} \frac{|\langle x, y_j\rangle|^2}{\sum_{k=1}^{n} |\langle y_k, y_j\rangle|} \tag{2.22}$$
$$\leq \frac{1}{2}\left[\|x\|^2 + \max_{j\in\{1,\dots,n\}}\left\{\sum_{k=1}^{n} |\langle y_j, y_k\rangle|\right\}\sum_{k=1}^{n} \frac{|\langle x, y_k\rangle|^2}{\left(\sum_{j=1}^{n} |\langle y_k, y_j\rangle|\right)^2}\right],$$

for any $x \in H$.

Proof. We take in (2.1)

$$\alpha_k = \frac{\langle x, y_k\rangle}{\sum_{j=1}^{n} |\langle y_k, y_j\rangle|},\ k = 1, ..., n$$

to get (2.10). □

Using the Schwarz's inequality we get from (2.16) that

$$\sum_{j=1}^{n} |\langle x, y_j\rangle|^p \tag{2.23}$$
$$\leq \frac{1}{2}\|x\|^2\left[1 + \|x\|^{2(p-2)} \max_{k\in\{1,\dots,n\}}\left\{\sum_{j=1}^{n} |\langle y_j, y_k\rangle|\right\}\sum_{k=1}^{n} \|y_k\|^{2(p-1)}\right],$$

for any x, $y_1, ..., y_n \in H$.
If in this inequality we take $p = 2$, then we get

$$\sum_{j=1}^{n} |\langle x, y_j\rangle|^2 \leq \frac{1}{2}\|x\|^2\left[1 + \max_{k\in\{1,\dots,n\}}\left\{\sum_{j=1}^{n} |\langle y_j, y_k\rangle|\right\}\sum_{k=1}^{n} \|y_k\|^2\right], \tag{2.24}$$

for any x, $y_1, ..., y_n \in H$.
From (2.22) we also get the Selberg type inequality

$$\sum_{j=1}^{n} \frac{|\langle x, y_j\rangle|^2}{\sum_{k=1}^{n} |\langle y_k, y_j\rangle|} \tag{2.25}$$
$$\leq \frac{1}{2}\|x\|^2\left[1 + \max_{j\in\{1,\dots,n\}}\left\{\sum_{k=1}^{n} |\langle y_j, y_k\rangle|\right\}\sum_{k=1}^{n} \frac{\|y_k\|^2}{\left(\sum_{j=1}^{n} |\langle y_k, y_j\rangle|\right)^2}\right],$$

for any x, $y_1, ..., y_n \in H$.

3. Related inequalities

We have:

Theorem 3.1. *Let* $x,\ y_1,...,y_n \in H$ *and* $\alpha_1,...,\alpha_n \in \mathbb{C}$. *Then*

$$\operatorname{Re}\left(\sum_{j=1}^{n}\alpha_j\left\langle y_j,x\right\rangle\right)\le\frac{1}{2}\left[\|x\|^2+\max_{j,k\in\{1,\dots,n\}}\{|\langle y_j,y_k\rangle|\}\left(\sum_{j=1}^{n}|\alpha_j|\right)^2\right] \tag{3.1}$$

and

$$\operatorname{Re}\left(\sum_{j=1}^{n}\alpha_j\left\langle y_j,x\right\rangle\right)\le\frac{1}{2}\left[\|x\|^2+\max_{k\in\{1,\dots,n\}}\left\{|\alpha_k|^2\right\}\sum_{j,k=1}^{n}|\langle y_j,y_k\rangle|\right]. \tag{3.2}$$

Proof. From (2.3) we have

$$\begin{aligned}\sum_{j,k=1}^{n}\alpha_j\overline{\alpha_k}\left\langle y_j,y_k\right\rangle&=\left|\sum_{j,k=1}^{n}\alpha_j\overline{\alpha_k}\left\langle y_j,y_k\right\rangle\right|\le\sum_{j,k=1}^{n}|\alpha_j\overline{\alpha_k}|\,|\langle y_j,y_k\rangle|\\&\le\max_{j,k\in\{1,\dots,n\}}\{|\langle y_j,y_k\rangle|\}\sum_{j,k=1}^{n}|\alpha_j\overline{\alpha_k}|\\&=\max_{j,k\in\{1,\dots,n\}}\{|\langle y_j,y_k\rangle|\}\sum_{j,k=1}^{n}|\alpha_j|\,|\overline{\alpha_k}|\\&=\max_{j,k\in\{1,\dots,n\}}\{|\langle y_j,y_k\rangle|\}\left(\sum_{j=1}^{n}|\alpha_j|\right)^2,\end{aligned}$$

for any $x,\ y_1,...,y_n \in H$ and $\alpha_1,...,\alpha_n \in \mathbb{C}$, which proves (3.1).
Similarly, we have

$$\begin{aligned}\sum_{j,k=1}^{n}\alpha_j\overline{\alpha_k}\left\langle y_j,y_k\right\rangle&=\left|\sum_{j,k=1}^{n}\alpha_j\overline{\alpha_k}\left\langle y_j,y_k\right\rangle\right|\le\sum_{j,k=1}^{n}|\alpha_j\overline{\alpha_k}|\,|\langle y_j,y_k\rangle|\\&\le\max_{j,k\in\{1,\dots,n\}}\{|\alpha_j\overline{\alpha_k}|\}\sum_{j,k=1}^{n}|\langle y_j,y_k\rangle|\\&=\max_{k\in\{1,\dots,n\}}\left\{|\alpha_k|^2\right\}\sum_{j,k=1}^{n}|\langle y_j,y_k\rangle|\end{aligned}$$

for any $x,\ y_1,...,y_n \in H$ and $\alpha_1,...,\alpha_n \in \mathbb{C}$, which proves (3.2). □

Corollary 3.2. *With the assumptions of Theorem* 3.1 *and for* $p\ge 1$ *we have*

$$\sum_{j=1}^{n}|\langle x,y_j\rangle|^p\le\frac{1}{2}\left[\|x\|^2+\max_{j,k\in\{1,\dots,n\}}\{|\langle y_j,y_k\rangle|\}\left(\sum_{j=1}^{n}|\langle x,y_j\rangle|^{p-1}\right)^2\right] \tag{3.3}$$

and

$$\sum_{j=1}^{n} |\langle x, y_j \rangle|^p \leq \frac{1}{2} \left[\|x\|^2 + \max_{k\in\{1,\dots,n\}} \left\{ |\langle x, y_j \rangle|^{2(p-1)} \right\} \sum_{j,k=1}^{n} |\langle y_j, y_k \rangle| \right] \tag{3.4}$$

for any $x,\ y_1, ..., y_n \in H$.

Proof. If we take in (3.1) and (3.2) $\alpha_j = \langle x, y_j \rangle |\langle x, y_j \rangle|^{p-2}$ then we get (3.3) and (3.4). □

Remark 3.3. If we take in (3.3) and (3.4) $p = 1$, then we get

$$\sum_{j=1}^{n} |\langle x, y_j \rangle| \leq \frac{1}{2} \left[\|x\|^2 + n^2 \max_{j,k\in\{1,\dots,n\}} \{ |\langle y_j, y_k \rangle| \} \right] \tag{3.5}$$

and

$$\sum_{j=1}^{n} |\langle x, y_j \rangle| \leq \frac{1}{2} \left[\|x\|^2 + \sum_{j,k=1}^{n} |\langle y_j, y_k \rangle| \right] \tag{3.6}$$

for any $x,\ y_1, ..., y_n \in H$.
If we take in (3.3) and (3.4) $p = 2$, then we get

$$\sum_{j=1}^{n} |\langle x, y_j \rangle|^2 \leq \frac{1}{2} \left[\|x\|^2 + \max_{j,k\in\{1,\dots,n\}} \{ |\langle y_j, y_k \rangle| \} \left(\sum_{j=1}^{n} |\langle x, y_j \rangle| \right)^2 \right] \tag{3.7}$$

and

$$\sum_{j=1}^{n} |\langle x, y_j \rangle|^2 \leq \frac{1}{2} \left[\|x\|^2 + \max_{k\in\{1,\dots,n\}} \left\{ |\langle x, y_k \rangle|^2 \right\} \sum_{j,k=1}^{n} |\langle y_j, y_k \rangle| \right] \tag{3.8}$$

for any $x,\ y_1, ..., y_n \in H$.

Using Schwarz's inequality we have from (3.3) and (3.4) that

$$\sum_{j=1}^{n} |\langle x, y_j \rangle|^p \tag{3.9}$$

$$\leq \frac{1}{2} \|x\|^2 \left[1 + \|x\|^{2(p-2)} \max_{j,k\in\{1,\dots,n\}} \{ |\langle y_j, y_k \rangle| \} \left(\sum_{j=1}^{n} \|y_j\|^{p-1} \right)^2 \right]$$

and

$$\sum_{j=1}^{n} |\langle x, y_j \rangle|^p \tag{3.10}$$

$$\leq \frac{1}{2} \|x\|^2 \left[1 + \|x\|^{2(p-2)} \max_{k\in\{1,\dots,n\}} \left\{ \|y_k\|^{2(p-1)} \right\} \sum_{j,k=1}^{n} |\langle y_j, y_k \rangle| \right]$$

for any $x,\ y_1, ..., y_n \in H$.

For $p = 2$ we get

$$\sum_{j=1}^{n} |\langle x, y_j \rangle|^2 \le \frac{1}{2} \|x\|^2 \left[1 + \max_{j,k \in \{1,\dots,n\}} \{|\langle y_j, y_k \rangle|\} \left(\sum_{j=1}^{n} \|y_j\| \right)^2 \right] \tag{3.11}$$

and

$$\sum_{j=1}^{n} |\langle x, y_j \rangle|^2 \le \frac{1}{2} \|x\|^2 \left[1 + \max_{k \in \{1,\dots,n\}} \left\{ \|y_k\|^2 \right\} \sum_{j,k=1}^{n} |\langle y_j, y_k \rangle| \right] \tag{3.12}$$

for any x, $y_1, \dots, y_n \in H$.
We observe that if $y_1, \dots, y_n \in H$ are such that

$$\max_{j,k \in \{1,\dots,n\}} \{|\langle y_j, y_k \rangle|\} \left(\sum_{j=1}^{n} \|y_j\| \right)^2 \le 1,$$

then (3.1) provides a refinement of Bessel's inequality. Also, if

$$\max_{k \in \{1,\dots,n\}} \left\{ \|y_k\|^2 \right\} \sum_{j,k=1}^{n} |\langle y_j, y_k \rangle| \le 1,$$

then (3.12) also provides a refinement of Bessel's inequality.
By using Hölder's inequality we can provide other inequalities as follows:

Theorem 3.4. *Let x, $y_1, \dots, y_n \in H$ and $\alpha_1, \dots, \alpha_n \in \mathbb{C}$. Then for $r, q > 1$ with $\frac{1}{r} + \frac{1}{q} = 1$*

$$\operatorname{Re} \left(\sum_{j=1}^{n} \alpha_j \langle y_j, x \rangle \right) \le \frac{1}{2} \left[\|x\|^2 + \left(\sum_{j,k=1}^{n} |\langle y_j, y_k \rangle|^r \right)^{1/r} \left(\sum_{j=1}^{n} |\alpha_j|^q \right)^{2/q} \right] \tag{3.13}$$

Proof. From (2.3) and Hölder's inequality we have

$$\begin{aligned}
\sum_{j,k=1}^{n} \alpha_j \overline{\alpha_k} \langle y_j, y_k \rangle &= \left| \sum_{j,k=1}^{n} \alpha_j \overline{\alpha_k} \langle y_j, y_k \rangle \right| \le \sum_{j,k=1}^{n} |\alpha_j \overline{\alpha_k}| \, |\langle y_j, y_k \rangle| \\
&\le \left(\sum_{j,k=1}^{n} |\langle y_j, y_k \rangle|^r \right)^{1/r} \left(\sum_{j,k=1}^{n} |\alpha_j \overline{\alpha_k}|^q \right)^{1/q} \\
&= \left(\sum_{j,k=1}^{n} |\langle y_j, y_k \rangle|^r \right)^{1/r} \left(\sum_{j,k=1}^{n} |\alpha_j|^q \, |\alpha_k|^q \right)^{1/q} \\
&= \left(\sum_{j,k=1}^{n} |\langle y_j, y_k \rangle|^r \right)^{1/r} \left(\sum_{j=1}^{n} |\alpha_j|^q \right)^{2/q},
\end{aligned}$$

for any x, $y_1, \dots, y_n \in H$ and $\alpha_1, \dots, \alpha_n \in \mathbb{C}$, which proves (3.13). □

Corollary 3.5. *With the assumptions of Theorem* 3.4 *and for* $p \geq 1$ *we have*

$$\sum_{j=1}^{n} |\langle x, y_j \rangle|^p \leq \frac{1}{2} \left[\|x\|^2 + \left(\sum_{j,k=1}^{n} |\langle y_j, y_k \rangle|^r \right)^{1/r} \left(\sum_{j=1}^{n} |\langle x, y_j \rangle|^{q(p-1)} \right)^{2/q} \right] \tag{3.14}$$

for any x, $y_1, ..., y_n \in H$. *In particular, we have*

$$\sum_{j=1}^{n} |\langle x, y_j \rangle| \leq \frac{1}{2} \left[\|x\|^2 + n^{2/q} \left(\sum_{j,k=1}^{n} |\langle y_j, y_k \rangle|^r \right)^{1/r} \right] \tag{3.15}$$

and

$$\sum_{j=1}^{n} |\langle x, y_j \rangle|^2 \leq \frac{1}{2} \left[\|x\|^2 + \left(\sum_{j,k=1}^{n} |\langle y_j, y_k \rangle|^r \right)^{1/r} \left(\sum_{j=1}^{n} |\langle x, y_j \rangle|^{2q} \right)^{2/q} \right]. \tag{3.16}$$

We observe that, by Schwarz's inequality we get for $p \geq 1$

$$\sum_{j=1}^{n} |\langle x, y_j \rangle|^p \tag{3.17}$$
$$\leq \frac{1}{2} \|x\|^2 \left[1 + \|x\|^{2(p-2)} \left(\sum_{j,k=1}^{n} |\langle y_j, y_k \rangle|^r \right)^{1/r} \left(\sum_{j=1}^{n} \|y_j\|^{q(p-1)} \right)^{2/q} \right],$$

for any x, $y_1, ..., y_n \in H$, where r, $q > 1$ with $\frac{1}{r} + \frac{1}{q} = 1$.
For $p = 2$, we get

$$\sum_{j=1}^{n} |\langle x, y_j \rangle|^2 \leq \frac{1}{2} \|x\|^2 \left[1 + \left(\sum_{j,k=1}^{n} |\langle y_j, y_k \rangle|^r \right)^{1/r} \left(\sum_{j=1}^{n} \|y_j\|^q \right)^{2/q} \right], \tag{3.18}$$

for any x, $y_1, ..., y_n \in H$, where r, $q > 1$ with $\frac{1}{r} + \frac{1}{q} = 1$.
We observe that if $y_1, ..., y_n \in H$ are such that

$$\left(\sum_{j,k=1}^{n} |\langle y_j, y_k \rangle|^r \right)^{1/r} \left(\sum_{j=1}^{n} \|y_j\|^q \right)^{2/q} \leq 1,$$

where r, $q > 1$ with $\frac{1}{r} + \frac{1}{q} = 1$, then (3.18) provides a refinement of Bessel's inequality.
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Nonlinear systems with a partial Nash type equilibrium

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Abstract. In this paper fixed point arguments and a critical point technique are combined leading to hybrid existence results for a system of three operator equations where only two of the equations have a variational structure. The components of the solution which are associated to the equations having a variational form represent a Nash-type equilibrium of the corresponding energy functionals. The result is achieved by an iterative scheme based on Ekeland's variational principle.

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1. Introduction

Many nonlinear equations can be seen as a problem of fixed point $N(u) = u$, where N is a certain operator. One says that the equation has a variational form if it is equivalent with a critical point equation $E'(u) = 0$. In the paper [7], R. Precup studied systems of the form

$$\begin{cases} N_1(u,v) = u \\ N_2(u,v) = v \end{cases} \tag{1.1}$$

in a Hilbert space, where each of the equations has a variational form, i.e. there are two C^1 functionals E_1 and E_2 such that

$$E_{11}(u,v) = u - N_1(u,v) \text{ and}$$
$$E_{22}(u,v) = v - N_2(u,v),$$

where E_{11} and E_{22} are the partial Fréchet derivatives of E_1 and E_2 with respect to u and v, respectively. Sufficient conditions have been established for that the system admits a solution which is a Nash type equilibrium for the functionals E_1 and E_2, that is

$$\begin{aligned} E_1(u,v) &= \inf_u E(\cdot,v), \\ E_2(u,v) &= \inf_v E(u,\cdot). \end{aligned}$$

Related results are obtained in [1].

The concept of a Nash equilibrium goes back to 1838 when Antoine Augustin Cournot [3] used it in his economics studies about the best output of a firm depending on the outputs of the other firms. The existence of such an equilibrium in the framework of the game theory was proved later in 1951 by John Forbes Nash Jr [5] by using Brouwer's fixed point theorem. Now the concept is also used outside economics to systems of variational equations. From a physical point of view, a Nash-type equilibrium (u,v) for two interconnected mechanisms whose energies are E_1, E_2 is such that the motion of each mechanism is conformed to the minimum energy principle by taking into account the motion of the other.

Also, in the paper [2], a system of type (1.1) is studied under the assumption that only one of the equations, say the second one, has a variational form, and the authors prove the existence of a solution (u,v) such that v minimizes $E(u,\cdot)$, where E is the energy functional associated with the second equation. For the proof, they use a hybrid fixed point - critical point method based on Banach's contraction theorem and Ekeland's variational principle.

The aim of this paper is to combine the techniques used in [7] and [2], for the study of a system of three equations

$$\begin{cases} N_1(u,v,w) = u \\ N_2(u,v,w) = v \\ N_3(u,v,w) = w, \end{cases}$$

where only the last two equations have a variational form. Our goal is to obtain a solution (u,v,w) such that the pair (v,w) is a Nash type equilibrium for the two functionals associated to the last two equations.

2. Main result

Let (X_1,d) be a complete metric space and $(X_2,|\cdot|_2)$, $(X_3,|\cdot|_3)$ be two real Hilbert spaces which are identified with their duals. Denote $X := X_1 \times X_2 \times X_3$. Let $N_i\colon X \to X_i$ $(i=1,2,3)$ be continuous and assume that N_2, N_3 have a variational structure, i.e. there exist functionals $E_2, E_3 : X \to \mathbb{R}$ such that $E_2(u,\cdot,w)$ is Fréchet differentiable for every $(u,w) \in X_1 \times X_3$, $E_3(u,v,\cdot)$ is Fréchet differentiable for every $(u,v) \in X_1 \times X_2$ and

$$\begin{aligned} E_{22}(u,v,w) &= v - N_2(u,v,w), \\ E_{33}(u,v,w) &= w - N_3(u,v,w). \end{aligned}$$

Here E_{22}, E_{33} are the Fréchet derivatives of $E_2(u,\cdot,w)$ and $E_3(u,v,\cdot)$, respectively.

We also assume that the operator $N : X \to X$,

$$N(u, v, w) = (N_1(u, v, w),\ N_2(u, v, w),\ N_3(u, v, w))$$

is a *Perov contraction, i.e.* there is a square matrix $A = [a_{ij}]_{1\le i,j\le 3} \in \mathbf{M}_3(\mathbb{R}_+)$ such that A^k tends to the zero matrix 0_3 as $k \to \infty$ and the following vector Lipschitz condition is satisfied

$$\begin{bmatrix} d\left(N_1(u,v,w),\ N_1(\overline{u},\overline{v},\overline{w})\right) \\ \left|N_2(u,v,w) - N_2(\overline{u},\overline{v},\overline{w})\right|_2 \\ \left|N_3(u,v,w) - N_3(\overline{u},\overline{v},\overline{w})\right|_3 \end{bmatrix} \le A \begin{bmatrix} d(u,\overline{u}) \\ |v-\overline{v}|_2 \\ |w-\overline{w}|_3 \end{bmatrix} \tag{2.1}$$

for every $(u, v, w) \in X$.

Note that the for a square matrix $A \in \mathbf{M}_n(\mathbb{R}_+)$, condition A^k tends to the zero matrix 0_n as $k \to \infty$ is equivalent (see [6]) to each one of the following properties:

(i) The spectral radius of A is less than one;

(ii) $I_n - A$ is invertible and $(I_n - A)^{-1} \in \mathbf{M}_n(\mathbb{R}_+)$;

(iii) $I_n - A$ is invertible and $I_n + M + M^2 + \ ...\ = (I_n - A)^{-1}$.

Here I_n stands for the unit matrix in $\mathbf{M}_n(\mathbb{R})$.

The main result is the following theorem.

Theorem 2.1. *Assume that the above conditions are satisfied. Moreover assume that* $E_2(u,\cdot,w),\ E_3(u,v,\cdot)$ *are bounded from below for every* $(u,v,w) \in X$ *and that are constants* $R_2, R_3, a > 0$ *such that*

$$E_2(u,v,w) \ge \inf_{X_2} E_2(u,\cdot,w) + a \quad \text{for all } (u,w) \in X_1 \times X_3 \text{ and } |v|_2 \ge R_2, \tag{2.2}$$

$$E_3(u,v,w) \ge \inf_{X_3} E_3(u,v,\cdot) + a \quad \text{for all } (u,v) \in X_1 \times X_2 \text{ and } |w|_3 \ge R_3. \tag{2.3}$$

Then the unique fixed point (u^*, v^*, w^*) *ensured by the Perov contraction theorem has the property that* (v^*, w^*) *is a Nash type equilibrium for the pair of functionals* (E_2, E_3), *i.e.*

$$\begin{aligned} E_2(u^*,v^*,w^*) &= \inf_{X_2} E_2(u^*,\cdot,w^*), \\ E_3(u^*,v^*,w^*) &= \inf_{X_3} E_3(u^*,v^*,\cdot). \end{aligned}$$

For the proof we need alternatively one of the following two auxiliary results.

Lemma 2.2. *Let* $(A_{k,p})_{k\ge 1}$, $(B_{k,p})_{k\ge 1}$ *be two sequences of vectors in* $\mathbb{R}^n_+$ *(column vectors) depending on a parameter* p, *such that*

$$A_{k,p} \le M A_{k-1,p} + B_{k,p}$$

for all k *and* p, *where* $M \in \mathbf{M}_n(\mathbb{R}_+)$ *is a matrix with spectral radius less than one. If the sequence* $(A_{k,p})_{k\ge 1}$ *is bounded uniformly with respect to* p *and* $B_{k,p} \to 0_n$ *as* $k \to \infty$ *uniformly with respect to* p, *then* $A_{k,p} \to 0_n$ *as* $k \to \infty$ *uniformly with respect to* p.

Proof. Since $B_{k,p} \to 0_n$ as $k \to \infty$ uniformly with respect to p, for any fixed column vector $\epsilon \in (0,\infty)^n$, we can find k_1 independent of p such that $B_{k,p} \leq \epsilon$ for all $k \geq k_1$ and all p. Then, for $k > k_1$ we have

$$\begin{aligned} A_{k,p} &\leq MA_{k-1,p} + \epsilon \leq M^2 A_{k-2,p} + \epsilon + M\epsilon \\ &\leq \ldots \\ &\leq M^{k-k_1} A_{k_1,p} + \epsilon(I_n + M + \ldots M^{k-k_1}) \\ &\leq M^{k-k_1} A_{k_1,p} + \epsilon\left(I_n - M\right)^{-1}. \end{aligned}$$

The conclusion now follows since $(A_{k,p})_{k\geq 1}$ is bounded uniformly with respect to p and $M^{k-k_1} \to 0_n$ as $k \to \infty$. □

Lemma 2.3. *Let $(x_{k,p})_{k\geq 1}$, $(y_{k,p})_{k\geq 1}$ be two sequences of nonnegative real numbers depending on a parameter p which are bounded uniformly with respect to p. Assume that for all k and p,*

$$ax_{k,p} + by_{k,p} \leq a'x_{k-1,p} + b'y_{k-1,p} + q_{k,p}$$

where $0 < a' < a$, $0 \leq b' < b$, $\frac{b'}{a'} < \frac{b}{a}$, and $(q_{k,p})_{k\geq 1}$ is a sequence of positive real numbers converging to zero uniformly with respect to p. Then $x_{k,p} \to 0$ and $y_{k,p} \to 0$ as $k \to \infty$ uniformly with respect to p.

Proof. By the uniform convergence to zero of $q_{k,p}$, taking $\epsilon > 0$, we can find k_1 independent of p, such that $\frac{q_{k,p}}{a} < \epsilon$ for all $k > k_1$. Consider $k > k_1$ and assume $a > b$. Then

$$\begin{aligned} x_{k,p} + \frac{b}{a}y_{k,p} &\leq \frac{a'}{a}\left(x_{k-1,p} + \frac{b'}{a'}y_{k-1,p}\right) + \epsilon \leq \frac{a'}{a}\left(x_{k-1,p} + \frac{b}{a}y_{k-1,p}\right) + \epsilon \\ &\leq \left(\frac{a'}{a}\right)^2\left(x_{k-2,p} + \frac{b'}{a'}y_{k-2,p}\right) + \epsilon\left(\frac{a'}{a} + 1\right) \\ &\ldots \\ &\leq \left(\frac{a'}{a}\right)^{k-k_1}\left(x_{k_1,p} + \frac{b'}{a'}y_{k_1,p}\right) + \epsilon\left(\left(\frac{a'}{a}\right)^{k-k_1-1} + \ldots + 1\right) \\ &\leq \left(\frac{a'}{a}\right)^{k-k_1}\left(x_{k_1,p} + \frac{b'}{a'}y_{k_1,p}\right) + \epsilon\frac{1}{1-\frac{a'}{a}}. \end{aligned}$$

Taking into account that $\frac{a'}{a} < 1$ and the boundedness of $x_{k,p}$ and $y_{k,p}$, it is clear that $x_{k,p} + \frac{b}{a}y_{k,p} \to 0$ as $k \to \infty$ uniformly with respect to p. This clearly gives the conclusion. The case $a < b$ can be treated analogously. □

Proof of the theorem. First note that since the spectral radius of matrix A is less than one, the elements a_{ii} of the main diagonal are less than one. Consequently, for every $(v,w) \in X_2 \times X_3$, the operator $N_1(\cdot, v, w) : X_1 \to X_1$ is a contraction. We now use an iterative procedure to construct an approximating sequence (u_k, v_k, w_k). We start with some fixed element $(v_0, w_0) \in X_2 \times X_3$. Then, by Banach contraction principle, there exists $u_1 \in X_1$ such that $N_1(u_1, v_0, w_0) = u_1$. Next, for fixed (u_1, w_0), according to Ekeland variational principle, there is $v_1 \in X_2$ such that

$$E_2(u_1, v_1, w_0) \le \inf_{X_2} E_2(u_1, \cdot, w_0) + 1, \qquad |E_{22}(u_1, v_1, w_0)|_2 \le 1.$$

Using again Ekeland variational principle for fixed (u_1, v_1), there is $w_1 \in X_3$ with

$$E_3(u_1, v_1, w_1) \le \inf_{X_3} E_2(u_1, v_1, \cdot) + 1, \qquad |E_{33}(u_1, v_1, w_1)|_3 \le 1.$$

At step k, we find a triple (u_k, v_k, w_k) having the following proprieties:

$$\begin{aligned} &N_1(u_k, v_{k-1}, w_{k-1}) = u_k, \\ &E_2(u_k, v_k, w_{k-1}) \le \inf_{X_2} E_2(u_k, \cdot, w_{k-1}) + \frac{1}{k},\ |E_{22}(u_k, v_k, w_{k-1})|_2 \le \frac{1}{k}, \\ &E_3(u_k, v_k, w_k) \le \inf_{X_3} E_3(u_k, v_k, \cdot) + \frac{1}{k},\ |E_{33}(u_k, v_k, w_{k]}|_3 \le \frac{1}{k}. \end{aligned} \tag{2.4}$$

Our next task is to prove that the sequences u_k, v_k, w_k are Cauchy, which will ensure their convergence. Since $N_1(u_k, v_{k-1}, w_{k-1}) = u_k$, we have

$$\begin{aligned} d(u_{k+p}, u_k) &= d(N_1(u_{k+p}, v_{k+p-1}, w_{k+p-1}),\ N_1(u_k, v_{k-1}, w_{k-1})) \\ &\le a_{11} d(u_{k+p}, u_k) + a_{12}|v_{k+p-1} - v_{k-1}|_2 + a_{13}|w_{k+p-1} - w_{k-1}|_3, \end{aligned}$$

whence

$$d(u_{k+p}, u_k) \le \frac{a_{12}}{1 - a_{11}}|v_{k+p-1} - v_{k-1}|_2 + \frac{a_{13}}{1 - a_{11}}|w_{k+p-1} - w_{k-1}|_3.$$

For the sequence (v_k) and (w_k) we have

$$\begin{aligned} |v_{k+p} - v_k|_2 \le\ &| - N_2(u_{k+p}, v_{k+p}, w_{k+p-1}) + v_{k+p} - v_k + N_2(u_k, v_k, w_{k-1})|_2 \\ &+ |N_2(u_{k+p}, v_{k+p}, w_{k+p-1}) - N_2(u_k, v_k, w_{k-1})|_2, \end{aligned} \tag{2.5}$$

$$\begin{aligned} |w_{k+p} - w_k|_3 \le\ &| - N_3(u_{k+p}, v_{k+p}, w_{k+p}) + w_{k+p} - w_k + N_3(u_k, v_k, w_k)|_3 \\ &+ |N_3(u_{k+p}, v_{k+p}, w_{k+p}) - N_3(u_k, v_k, w_k)|_3. \end{aligned} \tag{2.6}$$

Denote

$$\begin{aligned} \beta_{k,p} &:= | - N_2(u_{k+p}, v_{k+p}, w_{k+p-1}) + v_{k+p} - v_k + N_2(u_k, v_k, w_{k-1})|_2 \\ &= |E_{22}(u_{k+p}, v_{k+p}, w_{k+p-1}) - E_{22}(u_k, v_k, w_{k-1}))|_2, \end{aligned}$$

$$\begin{aligned} \gamma_{k,p} &:= | - N_3(u_{k+p}, v_{k+p}, w_{k+p}) + w_{k+p} - w_k + N_3(u_k, v_k, w_k)|_3 \\ &= |E_{33}(u_{k+p}, v_{k+p}, w_{k+p}) - E_{33}(u_k, v_k, w_{k-}))|_3, \end{aligned}$$

$$x_{k,p} := d(u_{u+p}, u_k), \quad y_{k,p} := |v_{k+p} - v_k|_2, \quad z_{k,p} := |w_{k+p} - w_k|_3.$$

With these notations, using (2.5), (2.6) and the Perov contraction condition, we obtain

$$x_{k,p} \le a_{11}x_{k,p} + a_{12}y_{k-1,p} + a_{13}z_{k-1,p}, \tag{2.7}$$

$$y_{k,p} \le a_{21}x_{k,p} + a_{22}y_{k,p} + a_{23}z_{k-1,p} + \beta_{k,p}, \tag{2.8}$$

$$z_{k,p} \le a_{31}x_{k,p} + a_{32}y_{k,p} + a_{33}z_{k,p} + \gamma_{k,p}. \tag{2.9}$$

For the continuation of the proof we may use either Lemma 2.2 or Lemma 2.3.

1) **Use of Lemma** 2.2. Letting

$$A' = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \text{and} \quad A'' = A - A',$$

the following inequality holds

$$\begin{bmatrix} x_{k,p} \\ y_{k,p} \\ z_{k,p} \end{bmatrix} \leq A' \begin{bmatrix} x_{k,p} \\ y_{k,p} \\ z_{k,p} \end{bmatrix} + A'' \begin{bmatrix} x_{k-1,p} \\ y_{k-1,p} \\ z_{k-1,p} \end{bmatrix} + \begin{bmatrix} 0 \\ \beta_{k,p} \\ \gamma_{k,p} \end{bmatrix}. \tag{2.10}$$

Note that if $\rho(A) < 1$, than also $\rho(A') < 1$. Indeed, one clearly has $A'^k < A^k$, and so if $A^k \to 0$ as $k \to \infty$, then $A'^k \to 0$ too.
Rewriting (2.10) as

$$(I - A') \begin{bmatrix} x_{k,p} \\ y_{k,p} \\ z_{k,p} \end{bmatrix} \leq A'' \begin{bmatrix} x_{k-1,p} \\ y_{k-1,p} \\ z_{k-1,p} \end{bmatrix} + \begin{bmatrix} 0 \\ \beta_{k,p} \\ \gamma_{k,p} \end{bmatrix}$$

and using the fact that $I - A'$ is invertible and its inverse has positive entries, we can multiply by $(I - A')^{-1}$ to obtain

$$\begin{bmatrix} x_{k,p} \\ y_{k,p} \\ z_{k,p} \end{bmatrix} \leq (I - A')^{-1} A'' \begin{bmatrix} x_{k-1,p} \\ y_{k-1,p} \\ z_{k-1,p} \end{bmatrix} + (I - A')^{-1} \begin{bmatrix} 0 \\ \beta_{k,p} \\ \gamma_{k,p} \end{bmatrix}.$$

Observe that $M := (I - A')^{-1} A''$ has the spectral radius less than one. To prove this, it is enough to show that $I - M$ is invertible with the inverse has nonegative entries. Is clear that

$$\begin{aligned} M &= (I - A')^{-1} A'' = (I - A')^{-1}(A - A') = (I - A')^{-1}(I - A' + A - I) \\ &= I - (I - A')^{-1}(I - A), \end{aligned}$$

hence $I - M = (I - A')^{-1}(I - A)$. Because $(I - A')^{-1}$ and $I - A$ are invertible, by taking $Q := (I - A)^{-1}(I - A')$, we have $Q(I - M) = (I - M)Q = I$, hence $I - M$ is invertible and its inverse is Q. One has

$$Q = (I - A)^{-1}(I - A') = (I - A)^{-1}(I - A + A'') = I + (I - A)^{-1} A''$$

and since $(I - A)^{-1} A''$ and I are positive matrices, it follows that Q is also positive. Therefore, the spectral radius of M is less than one.
From (2.2) and (2.3) we have that $y_{k,p}$ and $z_{k,p}$ are bounded uniformly with respect to p. Because of this, is immediate that $x_{k,p}$ is also bounded uniformly with respect to p. Moreover, it is clear that

$$\begin{bmatrix} 0 \\ \beta_{k,p} \\ \gamma_{k,p} \end{bmatrix}$$

converges to zero uniformly with respect to p. Applying Lemma 2.2 we obtain that $x_{k,p}$, $y_{k,p}$, $z_{k,p}$ are convergent to zero uniformly with respect to p. Hence the sequences u_k, v_k and w_k are Cauchy as desired.

2) **Use of Lemma** 2.3. The relations (2.7), (2.8), (2.9) can be rewritten under the form

$$\begin{aligned}
x_{k,p} &\leq a_{11}x_{k,p} + a_{12}y_{k,p} + a_{13}z_{k,p} + a_{12}(y_{k-1,p} - y_{k,p}) + a_{13}(z_{k-1,p} - z_{k,p}),\\
y_{k,p} &\leq a_{21}x_{k,p} + a_{22}y_{k,p} + a_{23}z_{k,p} + \beta_{k,p} + a_{23}(z_{k-1,p} - z_{k,p}),\\
z_{k,p} &\leq a_{31}x_{k,p} + a_{32}y_{k,p} + a_{33}z_{k,p} + \gamma_{k,p},
\end{aligned}$$

which can be put under the vector form

$$\begin{bmatrix} x_{k,p} \\ y_{k,p} \\ z_{k,p} \end{bmatrix} \leq A \begin{bmatrix} x_{k,p} \\ y_{k,p} \\ z_{k,p} \end{bmatrix} + \begin{bmatrix} a_{12}(y_{k-1,p} - y_{k,p}) + a_{13}(z_{k-1,p} - z_{k,p}) \\ \beta_{k,p} + a_{23}(z_{k-1,p} - z_{k,p}) \\ \gamma_{k,p} \end{bmatrix}.$$

Denoting $(I - A)^{-1} = C = [c_{ij}]_{1\leq i,j\leq 3}$ we have

$$\begin{bmatrix} x_{k,p} \\ y_{k,p} \\ z_{k,p} \end{bmatrix} \leq C \begin{bmatrix} a_{12}(y_{k-1,p} - y_{k,p}) + a_{13}(z_{k-1,p} - z_{k,p}) \\ \beta_{k,p} + a_{23}(z_{k-1,p} - z_{k,p}) \\ \gamma_{k,p} \end{bmatrix},$$

whence

$$\begin{aligned}
y_{k,p} \leq{}& c_{21}a_{12}(y_{k-1,p} - y_{k,p}) + c_{21}a_{13}(z_{k-1,p} - z_{k,p}) \\
&+ c_{22}a_{23}(z_{k-1,p} - z_{k,p}) + c_{22}\beta_{k,p} + c_{33}\gamma_{k,p},
\end{aligned} \tag{2.11}$$

$$\begin{aligned}
z_{k,p} \leq{}& c_{31}a_{12}(y_{k-1,p} - y_{k,p}) + c_{31}a_{13}(z_{k-1,p} - w_{k,p}) \\
&+ c_{32}a_{23}(z_{k-1,p} - z_{k,p}) + c_{32}\beta_{k,p} + c_{33}\gamma_{k,p}.
\end{aligned} \tag{2.12}$$

We make the following notations

$$\begin{aligned}
a' &= c_{12}a_{13} + c_{22}a_{2,3} + c_{31}a_{13} + c_{32}a_{23},\\
b' &= c_{21}a_{12} + c_{31}a_{12}.
\end{aligned}$$

Adding (2.11) and (2.12) we obtain

$$y_{k,p} + z_{k,p} \leq a'y_{k-1,p} - a'y_{k,p} + b'z_{k-1,p} - b'z_{k,p} + c_{22}\beta_{k,p} + c_{33}\gamma_{k,p} + c_{32}\beta_{k,p} + c_{33}\gamma_{k,p},$$

whence, with the notations $a = 1 + a'$, $b = 1 + b'$ and

$$q_{k,p} := c_{22}\beta_{k,p} + c_{33}\gamma_{k,p} + c_{32}\beta_{k,p} + c_{33}\gamma_{k,p},$$

one has

$$ay_{k,p} + bz_{k,p} \leq a'y_{k-1,p} + b'z_{k-1,p} + q_{k,p}.$$

Note that the sequence $q_{k,p} := c_{22}\beta_{k,p} + c_{33}\gamma_{k,p} + c_{32}\beta_{k,p} + c_{33}\gamma_{k,p}$ converges to zero as $k \to \infty$ uniformly with respect to p, and that from (2.2) and (2.3), the sequences $(y_{k,p})_{k\geq 1}$, $(z_{k,p})_{k\geq 1}$ are bounded uniformly with respect to p. Also note that if $b' < a'$, then $\frac{b'}{a'} < \frac{b}{a}$ and from Lemma 2.3 we obtain that $y_{k,p}$ and $z_{k,p}$ converge to zero as $k \to \infty$ uniformly with respect to p. Similarly, if $a' < b'$, then we obtain the same conclusion if we apply Lemma 2.3 by interchanging a with b and $y_{k,p}$ with $z_{k,p}$. Next, from (2.7) we deduce that $x_{k,p} \to 0$ as $k \to \infty$ uniformly with respect to p, and as above, that the sequences u_k, v_k and w_k are Cauchy as desired.
Finally the limits u^*, v^*, w^* of the sequences u_k, v_k and w_k give the desired solution of the system after passing to the limit in (2.4). □

3. Application

Consider the system

$$\begin{cases} -u'' + a_1^2 u = f_1(t, u(t), v(t), w(t), u'(t)) \\ -v'' + a_2^2 v = \nabla_y f_2(t, u(t), v(t), w(t)) \\ -w'' + a_3^2 w = \nabla_z f_3(t, u(t), v(t), w(t)) \end{cases} \tag{3.1}$$

with the periodic conditions

$$\begin{aligned} u(0) - u(T) &= u'(0) - u'(T) = 0, \\ v(0) - v(T) &= v'(0) - v'(T) = 0, \\ w(0) - w(T) &= w'(0) - w'(T) = 0, \end{aligned}$$

where $f_2, f_3 : (0,T)\times\mathbb{R}^{k_1}\times\mathbb{R}^{k_2}\times\mathbb{R}^{k_3} \to \mathbb{R}$ and $f_1 : (0,T)\times\mathbb{R}^{k_1}\times\mathbb{R}^{k_2}\times\mathbb{R}^{k_3}\times\mathbb{R}^{k_1} \to \mathbb{R}^{k_1}$. We will assume that $f_1, f_2, f_3, \nabla_y f_2$ and $\nabla_z f_3$ are L^1- Carathéodory functions.

For $i = 1,2,3$, let $H_p^1\left(0,T;\mathbb{R}^{k_i}\right)$ be the closure in $H^1\left(0,T;\mathbb{R}^{k_i}\right)$ of the space $\{u \in C^1\left([0,T];\mathbb{R}^{k_i}\right) : u(0) = u(T), u'(0) = u'(T)\}$. We shall endow this space with the inner product

$$(u,v)_i := (u',v')_{L^2\left(0,T;\mathbb{R}^{k_i}\right)} + a_i^2 (u,v)_{L^2\left(0,T;\mathbb{R}^{k_i}\right)}$$

and the corresponding norm

$$|u|_i = \left(|u'|^2_{L^2\left(0,T;\mathbb{R}^{k_i}\right)} + a_i^2\, |u|^2_{L^2\left(0,T;\mathbb{R}^{k_i}\right)}\right)^{\frac{1}{2}}.$$

Also we consider the operator $J_i\colon (H_p^1\left(0,T;\mathbb{R}^{k_i}\right))' \to H_p^1\left(0,T;\mathbb{R}^{k_i}\right)$ given by $J_i h = u_h \ \left(h \in (H_p^1\left(0,T;\mathbb{R}^{k_i}\right))'\right)$, where $u_h \in H_p^1\left(0,T;\mathbb{R}^{k_i}\right)$ is the weak solution of the problem

$$\begin{cases} -u'' + a_i^2 u = h \quad \text{on } (0,T) \\ u(0) - u(T) = u'(0) - u'(T) = 0 \end{cases} \tag{3.2}$$

For every $h \in L^2\left([0,T];\mathbb{R}^{k_i}\right)$ we have

$$|J_i h|_i^2 = (J_i h, J_i h)_i = (h, J_i h)_{L^2} \le |h|_{L^2} |J_i h|_{L^2} \le \frac{1}{a_i} |h|_{L^2} |J_i h|_i, \tag{3.3}$$

hence

$$|J_1 h_1|_i \le \frac{1}{a_i} |h|_{L^2}.$$

Associate to the second and the third equation from (3.1) the functionals

$$E_2, E_3\colon\ H_p^1(0,T;\mathbb{R}^{k_1}) \times H_p^1(0,T;\mathbb{R}^{k_2}) \times H_p^1(0,T;\mathbb{R}^{k_3}) \to \mathbb{R}$$

defined by

$$E_2(u,v,w) = \frac{1}{2}|v|_2^2 - \int_0^T f_2(t, u(t), v(t), w(t))dt$$

and

$$E_3(u,v,w) = \frac{1}{2}|w|_3^2 - \int_0^T f_3(t, u(t), v(t), w(t))dt.$$

According to [4, Theorem 1.4] we have

$$E_{22}(u,v,w) = L_2 v - \nabla_y f_2(\cdot,u,v,w),$$

or equivalently, for any $\varphi \in H_p^1(0,T;\mathbb{R}^{k_2})$,

$$\begin{aligned}(E_{22}(u,v,w),\varphi) &= (L_2 v,\varphi) - (\nabla_y f_2(u,v,w,u',w'),\varphi) \\ &= (v - J_2\nabla_y f_2,\varphi)_2.\end{aligned}$$

Hence $E_{22}(u,v,w) = v - J_2\nabla_y f_2$. Similarly,

$$E_{33}(u,v,w) = w - J_3\nabla_z f_3.$$

On the other hand, system (3.1) is equivalent to the following fixed point equation

$$\begin{cases} N_1(u,v,w) = u \\ N_2(u,v,w) = v \\ N_3(u,v,w) = w \end{cases}$$

where

$$\begin{aligned} N_1(u,v,w) &= J_1 f_1(\cdot,u,v,w,u'), \\ N_2(u,v,w) &= J_2\nabla_y f_2(\cdot,u,v,w), \\ N_3(u,v,w) &= J_3\nabla_z f_3(\cdot,u,v,w). \end{aligned}$$

Related to f_1, f_2, f_3 we assume that the following Lipschitz conditions hold for some constants a_{ij} :

$$|f_1(t,x_1,...,x_4) - f_1(t,\overline{x_1},...,\overline{x_4})| \le \sum_{j=1}^{4} a_{1j}|x_j - \overline{x_j}|, \tag{3.4}$$

$$|\nabla_y f_2(t,x_1,x_2,x_3) - \nabla_y f_2(t,\overline{x_1},\overline{x_2},\overline{x_3})| \le \sum_{j=1}^{3} a_{2j}|x_j - \overline{x_j}|, \tag{3.5}$$

$$|\nabla_z f_3(t,x_1,x_2,x_3) - \nabla_z f_3(\overline{x_1},\overline{x_2},\overline{x_3})| \le \sum_{j=1}^{3} a_{3j}|x_j - \overline{x_j}|. \tag{3.6}$$

Then

$$\begin{aligned} &|N_1(u,v,w) - N_1(\overline{u},\overline{v},\overline{w})|_1 = |J_1\left(f_1(\cdot,u,v,w,u') - f_1(\cdot,\overline{u},\overline{v},\overline{w},\overline{u}')\right)|_1 \\ &\le \frac{1}{a_1}|f_1(\cdot,u,v,w,u') - f_1(\cdot,\overline{u},\overline{v},\overline{w},\overline{u}')|_{L^2} \\ &\le \frac{1}{a_1}\left(\int_0^T (a_{11}|u(t)-\overline{u}(t)| + a_{14}|u'(t)-\overline{u}'(t)|)^2 dt\right)^{\frac{1}{2}} \\ &+ \frac{a_{12}}{a_1}|v-\overline{v}|_{L^2} + \frac{a_{13}}{a_1}|w-\overline{w}|_{L^2} \\ &\le \frac{1}{a_1}\left(\left(\frac{a_{11}}{a_1}\right)^2 + a_{14}^2\right)^{\frac{1}{2}} |u-\overline{u}|_1 + \frac{a_{12}}{a_1}|v-\overline{v}|_{L^2} + \frac{a_{13}}{a_1}|w-\overline{w}|_{L^2}. \end{aligned}$$

Is clear that $|v-\overline{v}|_{L^2} \leq \frac{1}{a_2}|v-\overline{v}|_2$ and $|w-\overline{w}|_{L^2} \leq \frac{1}{a_3}|w-\overline{w}|_3$. Hence, the above inequality becomes

$$\begin{aligned} & |N_1(u,v,w)-N_1(\overline{u},\overline{v},\overline{w})|_1 \\ \leq \; & \frac{1}{a_1}\left(\left(\frac{a_{11}}{a_1}\right)^2+a_{14}^2\right)^{\frac{1}{2}}|u-\overline{u}|_1+\frac{a_{12}}{a_1a_2}|v-\overline{v}|_2+\frac{a_{13}}{a_1a_3}|w-\overline{w}|_3. \end{aligned}$$

For $N_2(u,v,w)$ we obtain the following estimate

$$\begin{aligned} |N_2(u,v,w)-N_2(\overline{u},\overline{v},\overline{w})|_2 &\leq |J_2\nabla_y f_2(\cdot,u,v,w)-\nabla_y f_2(\cdot,\overline{u},\overline{v},\overline{w})|_2 \\ &\leq \frac{1}{a_2}|\nabla_y f_2(\cdot,u,v,w)-\nabla_y f_2(\cdot,\overline{u},\overline{v},\overline{w})|_{L^2} \\ &\leq \frac{a_{21}}{a_2}|u-\overline{u}|_{L^2}+\frac{a_{22}}{a_2}|v-\overline{v}|_{L^2}+\frac{a_{23}}{a_2}|w-\overline{w}|_{L^2} \\ &\leq \frac{a_{21}}{a_2a_1}|u-\overline{u}|_1+\frac{a_{22}}{a_2^2}|v-\overline{v}|_2+\frac{a_{23}}{a_2a_3}|w-\overline{w}|_3. \end{aligned}$$

Similarly

$$|N_3(u,v,w)-N_3(\overline{u},\overline{v},\overline{w})|_3 \leq \frac{a_{31}}{a_3a_1}|u-\overline{u}|_1+\frac{a_{32}}{a_2a_3}|v-\overline{v}|_2+\frac{a_{33}}{a_3^2}|w-\overline{w}|_3.$$

Therefore, the condition related to (2.1) holds provided that the spectral radius of the matrix

$$A=\begin{bmatrix} \frac{1}{a_1}\left(\left(\frac{a_{11}}{a_1}\right)^2+a_{14}^2\right)^{\frac{1}{2}} & \frac{a_{12}}{a_1a_2} & \frac{a_{13}}{a_1a_3} \\ \frac{a_{21}}{a_2a_1} & \frac{a_{22}}{a_2^2} & \frac{a_{23}}{a_2a_3} \\ \frac{a_{31}}{a_3a_1} & \frac{a_{32}}{a_2a_3} & \frac{a_{33}}{a_3^2} \end{bmatrix} \tag{3.7}$$

is less than one.

In what follows we are trying to establish conditions for $E_2(u,\cdot,w)$ and $E_3(u,v,\cdot)$ to be bounded from below. To this aim, assume that for $i\in\{2,3\}$ and $j\in\{1,2,3,4\}$, there are $\sigma_{ij}\in L^1(0,T;\mathbb{R}_+)$ and $\gamma_i\in\mathbb{R}$ with $\gamma_i^2<\frac{a_i^2}{2}$ such that

$$f_2(t,x,y,z)\leq \gamma_2^2|y|^2+\sigma_{21}(t)|x|+\sigma_{22}(t)|y|+\sigma_{23}(t)|z|+\sigma_{24}(t) \tag{3.8}$$

and

$$f_3(t,x,y,z)\leq \gamma_3^2|z|^2+\sigma_{31}(t)|x|+\sigma_{32}(t)|y|+\sigma_{33}(t)|z|+\sigma_{34}(t). \tag{3.9}$$

Then taking into account the continuous embedding of $H_p^1\left(0,T;\mathbb{R}^{k_i}\right)$ into $C\left([0,T];\mathbb{R}^{k_i}\right)$, we obtain

$$\begin{aligned} E_2(u,v,w) &= \int_0^T\left(\frac{1}{2}|v'(t)|^2+\frac{a_2^2}{2}|v^2(t)|-f_2(t,u(t),v(t),w(t))\right)dt \\ &\geq \int_0^T\left(\frac{1}{2}|v'(t)|^2+\frac{1}{2}(a_2^2-2\gamma_2^2)v^2(t)-\sigma_{21}(t)|u(t)|-\sigma_{22}(t)|v(t)|-\sigma_{23}(t)|w(t)|-\sigma_{24}(t)\right)dt \\ &\geq \left(1-\frac{2\gamma_2^2}{a_2^2}\right)|v|_2^2-C_{21}|u|_1-C_{22}|v|_2-C_{23}\,|w|_3-C_{24} \end{aligned}$$

for some constants C_{2j}, $j-1,2,3,4$.
This shows us that $E_2(u,v,w) \to \infty$ as $|v|_2 \to \infty$. Similarly, $E_3(u,v,w) \to \infty$ as $|w|_3 \to \infty$. Thus the functionals $E_2(u,\cdot,w)$ and $E_3(u,v,\cdot)$ are coercive. Then, as in [7, Lemma 4.1], these functionals are bounded from bellow.
Finally, assume that for $i \in \{2,3\}$, there are L^1-Carathéodory functions $g_{i1}, g_{i2}\colon (0,T)\times\mathbb{R}^{k_i} \to \mathbb{R}$ of coercive type such that

$$g_{21}(t,y) \leq f_2(t,x,y,z) \leq g_{22}(t,y) \tag{3.10}$$

and

$$g_{31}(t,z) \leq f_3(t,x,y,z) \leq g_{32}(t,z) \tag{3.11}$$

for all for all $(x,y,z) \in \mathbb{R}^{k_1}\times\mathbb{R}^{k_2}\times\mathbb{R}^{k_3}$ and $t \in (0,T)$. Here, for example, by the coercivity of $g_{21}(t,y)$ we mean that

$$\frac{1}{2}|v|_2^2 - \int_0^T g_{21}(t,v)\,dt \to \infty \quad \text{as} \quad |v|_2 \to \infty.$$

Fix $a>0$. Using the above assumption one has

$$\inf_{v\in H_p^1} E_2(u,\cdot,w) + a \leq \inf_{v\in H_p^1}\left(\frac{1}{2}|v|_2^2 - \int_0^T g_{21}(t,v)\,dt\right) + a.$$

By the coercivity of g_{22}, there exists $R_2>0$ such that

$$\inf_{v\in H_p^1}\left(\frac{1}{2}|v|_2^2 - \int_0^T g_{21}(t,v)\,dt\right) + a \leq \frac{1}{2}|v|_2^2 - \int_0^T g_{22}(t,v)\,dt,$$

for all $|v|_2 \geq R_2$. Now, for $|v|_2 \geq R_2$ and all $(u,w) \in H_p^1(0,T;\mathbb{R}^{k_1}) \times H_p^1(0,T;\mathbb{R}^{k_3})$, using again (3.10) we obtain

$$E_2(u,v,w) \geq \frac{1}{2}|v|_2^2 - \int_0^T g_{22}(t,v)\,dt \geq \inf_{v\in H_p^1} E_2(u,\cdot,w) + a,$$

as desired. The similar inequality for E_3 can be established analogously.
Under the assumptions (3.4), (3.5),(3.6), (3.8), (3.9), (3.10), (3.11) and if the spectral radius of matrix (3.7) is less than one, then all the hypotheses of Theorem 2.1 are fulfilled.

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