Stud. Univ. Babeş-Bolyai Math. 70(2025), No. 4, 709–720
DOI: 10.24193/subbmath.2025.4.12

Lefschetz admissible dominated spaces for maps with an inclusion property

Donal O'Regan

Abstract. We consider the notion of a Lefschetz admissible dominated space and we present some fixed point results for compact maps with a selection property.

Mathematics Subject Classification (2010): 47H10, 54H25.

Keywords: Fixed points, set–valued maps, admissible spaces.

1. Introduction

In this paper we consider two general classes of maps, namely the $HYAd$ and $HYAdC$ maps which have a very general selection property (motivated by KLU [10], $HLPY$ [11], Scalzo [17] and Wu [18] maps). Using a result that NES(compact) and $SANES$(compact) spaces are Lefschetz spaces (see [6, 13]) we establish new Lefschetz fixed point theorems for NES, $SANES$ and Lefschetz admissible and admissible dominated spaces. Our results improve and complement those in the literature (see [2, 3, 6, 13, 14, 16] and the references therein).

First we describe the maps considered in this paper. Let H be the Čech homology functor with compact carriers and coefficients in the field of rational numbers K from the category of Hausdorff topological spaces and continuous maps to the category of graded vector spaces and linear maps of degree zero. Thus $H(X) = \{H_q(X)\}$ (here X is a Hausdorff topological space) is a graded vector space, $H_q(X)$ being the q–dimensional Čech homology group with compact carriers of X. For a continuous map $f : X \to X$, $H(f)$ is the induced linear map $f_\star = \{f_{\star\, q}\}$ where $f_{\star\, q} : H_q(X) \to H_q(X)$. A space X is acyclic if X is nonempty, $H_q(X) = 0$ for every $q \geq 1$, and $H_0(X) \approx K$.

Received 23 July 2025; Accepted 15 September 2025.

Let X, Y and Γ be Hausdorff topological spaces. A continuous single valued map $p : \Gamma \to X$ is called a Vietoris map (written $p : \Gamma \Rightarrow X$) if the following two conditions are satisfied:
(i). for each $x \in X$, the set $p^{-1}(x)$ is acyclic
(ii). p is a perfect map i.e. p is closed and for every $x \in X$ the set $p^{-1}(x)$ is nonempty and compact.

Let $\phi : X \to Y$ be a multivalued map (note for each $x \in X$ we assume $\phi(x)$ is a nonempty subset of Y). A pair (p, q) of single valued continuous maps of the form $X \overset{p}{\leftarrow} \Gamma \overset{q}{\rightarrow} Y$ is called a selected pair of ϕ (written $(p, q) \subset \phi$) if the following two conditions hold:
(i). p is a Vietoris map
and
(ii). $q\left(p^{-1}(x)\right) \subset \phi(x)$ for any $x \in X$.

Now we define the admissible maps of Gorniewicz [7]. An upper semicontinuous map $\phi : X \to 2^Y$ (nonempty subsets of Y) with compact values is said to be admissible (and we write $\phi \in Ad(X, Y)$) provided there exists a selected pair (p, q) of ϕ. An example of an admissible map is a Kakutani map. An upper semicontinuous map $\phi : X \to CK(Y)$ is said to be Kakutani (and we write $\phi \in Kak(X, Y)$); here Y is a Hausdorff topological vector space and $CK(Y)$ denotes the family of nonempty, convex, compact subsets of Y. Another example is an acyclic map which we now describe. Let X and Z be subsets of Hausdorff topological spaces and let $F : X \to K(Z)$ i.e. F has nonempty compact values. Recall a nonempty topological space is said to be acyclic if all its reduced Čech homology groups over the rationals are trivial. Now we consider maps $F : X \to Ac(Z)$ i.e. $F : X \to K(Z)$ with acyclic values (i.e. F has nonempty acyclic compact values). We say $F \in AC(X, Z)$ (i.e. F is an acyclic map) if $F : X \to Ac(Z)$ is upper semicontinuous.

Next we consider a general class of maps, namely the PK maps of Park (which include Kak and Ad maps). Let X and Y be Hausdorff topological spaces. Given a class $\mathbf{X}$ of maps, $\mathbf{X}(X, Y)$ denotes the set of maps $F : X \to 2^Y$ (nonempty subsets of Y) belonging to $\mathbf{X}$, and $\mathbf{X}_c$ the set of finite compositions of maps in $\mathbf{X}$. We let

$$\mathbf{F}(\mathbf{X}) = \{Z : \; Fix\, F \neq \emptyset \;\text{ for all }\; F \in \mathbf{X}(Z, Z)\}$$

where $Fix\, F$ denotes the set of fixed points of F.

The class $\mathbf{U}$ of maps is defined by the following properties:
(i). $\mathbf{U}$ contains the class C of single valued continuous functions;
(ii). each $F \in \mathbf{U}_c$ is upper semicontinuous and compact valued; and
(iii). $B^n \in \mathbf{F}(\mathbf{U}_c)$ for all $n \in \{1, 2, ...\}$; here $B^n = \{x \in \mathbf{R}^n : \; \|x\| \leq 1\}$.

We say $F \in PK(X, Y)$ if for any compact subset K of X there is a $G \in \mathbf{U}_c(K, Y)$ with $G(x) \subseteq F(x)$ for each $x \in K$.

Next we recall the following fixed point result [15] for PK maps. Recall a nonempty subset W of a Hausdorff topological vector space E is said to be admissible if for any nonempty compact subset K of W and every neighborhood V of 0 in E

there exists a continuous map $h : K \to W$ with $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a finite dimensional subspace of E (for example every nonempty convex subset of a locally convex space is admissible).

Theorem 1.1. *Let X be an admissible convex set in a Hausdorff topological vector space and $F \in PK(X, X)$ be a closed compact map. Then F has a fixed point in X.*

For a subset K of a topological space X, we denote by $Cov_X(K)$ the directed set of all coverings of K by open sets of X (usually we write $Cov(K) = Cov_X(K)$). Given a map $F : X \to 2^X$ and $\alpha \in Cov(X)$, a point $x \in X$ is said to be an α–fixed point of F if there exists a member $U \in \alpha$ such that $x \in U$ and $F(x) \cap U \neq \emptyset$.

Given two maps $F, G : X \to 2^Y$ and $\alpha \in Cov(Y)$, F and G are said to be α–close if for any $x \in X$ there exists $U_x \in \alpha$, $y \in F(x) \cap U_x$ and $w \in G(x) \cap U_x$. Of course, given two single valued maps $f, g : X \to Y$ and $\alpha \in Cov(Y)$, then f and g are α–close if for any $x \in X$ there exists $U_x \in \alpha$ containing both $f(x)$ and $g(x)$. We say f and g are α-homotopic if there is a homotopy $h_t : X \to Y$ $(t \in [0, 1])$ joining f and g such that for each $x \in X$ the values $h_t(x)$ belong to a common $U_x \in \alpha$ for all $t \in [0, 1]$. We say f and g are homotopic if there is a homotopy $h_t : X \to Y$ $(t \in [0, 1])$ joining f and g. We recall the following result [2].

Theorem 1.2. *Let X be a regular topological space, $F : X \to 2^X$ an upper semicontinuous map with closed values and suppose there exists a cofinal covering $\theta \subseteq Cov_X(\overline{F(X)})$ such that F has an α–fixed point for every $\alpha \in \theta$. Then F has a fixed point.*

Remark 1.3. From Theorem 1.2 in proving the existence of fixed points in uniform spaces for upper semicontinuous compact maps with closed values it suffices [3, page 298] to prove the existence of approximate fixed points (since open covers of a compact set A admit refinements of the form $\{U[x] : x \in A\}$ where U is a member of the uniformity [9, page 199], so such refinements form a cofinal family of open covers). Note also that uniform spaces are regular (in fact completely regular [5]). Also note in Theorem 1.2 if F is compact valued, then the assumption that X is regular can be removed. We note here that when we apply Theorem 1.2 we will assume the space is uniform. Of course one could consider other appropriate spaces (like regular (Hausdorff) spaces) as well.

Let Q be a class of topological spaces. A space Y is an extension space for Q (written $Y \in ES(Q)$) if for any pair (X, K) in Q with $K \subseteq X$ closed, any continuous function $f_0 : K \to Y$ extends to a continuous function $f : X \to Y$. A space Y is an approximate extension space for Q (written $Y \in AES(Q)$) if for any $\alpha \in Cov(Y)$ and any pair (X, K) in Q with $K \subseteq X$ closed, and any continuous function $f_0 : K \to Y$ there exists a continuous function $f : X \to Y$ such that $f|_K$ is α–close to f_0. A space Y is a neighborhood extension space for Q (written $Y \in NES(Q)$) if $\forall X \in Q$, $\forall K \subseteq X$ closed in X, and any continuous function $f_0 : K \to Y$ there exists a continuous extension $f : U \to Y$ of f_0 over a neighborhood U of K in X. A space Y is an approximate neighborhood extension space for Q (written $Y \in ANES(Q)$) if $\forall \alpha \in Cov(Y)$, $\forall X \in Q$, $\forall K \subseteq X$ closed in X, and any continuous function

$f_0 : K \to Y$ there exists a neighborhood U_α of K in X and a continuous function $f_\alpha : U_\alpha \to Y$ such that $f_\alpha|_K$ and f_0 are α–close. A space Y is a strongly approximate neighborhood extension space for Q (written $Y \in SANES(Q)$) if $\forall \alpha \in Cov(Y)$, $\forall X \in Q$, $\forall K \subseteq X$ closed in X, and any continuous function $f_0 : K \to Y$ there exists a neighborhood U_α of K in X and a continuous function $f_\alpha : U_\alpha \to Y$ such that $f_\alpha|_K$ and f_0 are α–close and homotopic.

Next we describe the maps due to Wu [18]. Let X and Y be subsets lying in Hausdorff topological vector spaces and we say $\Phi \in W(X,Y)$ if $\Phi : X \to 2^Y$ and there exists a lower semicontinuous map $\theta : X \to 2^Y$ with $\overline{co}\,(\theta(x)) \subseteq \Phi(x)$ for $x \in X$. Next we recall a selection theorem [1] (see the proof in Theorem 1.1 there) for Wu maps.

Theorem 1.4. *Let X be a paracompact subset of a Hausdorff topological vector space and Y a metrizable complete subset of a Hausdorff locally convex linear topological space. Suppose $\Phi \in W(X,Y)$ and let $\theta : X \to 2^Y$ be a lower semicontinuous map with $\overline{co}\,(\theta(x)) \subseteq \Phi(x)$ for $x \in X$. Then there exists an upper semicontinuous map $\Psi : X \to CK(Y)$ (collection of nonempty convex compact subsets of Y) with $\Psi(x) \subseteq \overline{co}\,(\theta(x)) \subseteq \Phi(x)$ for $x \in X$.*

Remark 1.5. Let X be paracompact and Y a metrizable subset of a complete Hausdorff locally convex linear topological space E and $\Phi \in W(X,Y)$ with $\theta : X \to 2^Y$ a lower semicontinuous map and $\overline{co}\,(\theta(x)) \subseteq \Phi(x)$ for $x \in X$. Note [12] that $\overline{co}\,\theta : X \to 2^Y$ (since $\overline{co}\,(\theta(x)) \subseteq \Phi(x) \subseteq Y$ for $x \in X$) is lower semicontinuous, so from Michael's selection theorem there exists a continuous (single valued) map $f : X \to Y$ with $f(x) \in \overline{co}\,(\theta(x))$ for $x \in X$, so consequently $f(x) \in \overline{co}\,(\theta(x)) \subseteq \Phi(x)$ for $x \in X$.

Let Z be a subset of a Hausdorff topological space Y_1 and W a subset of a Hausdorff topological vector space Y_2 and G a multifunction. We say $F \in HLPY(Z,W)$ [11] if W is convex and there exists a map $S : Z \to W$ (i.e. $S : Z \to P(W)$ (collection of subsets of W)) with $co\,(S(x)) \subseteq F(x)$ for $x \in Z$, $S(x) \neq \emptyset$ for each $x \in Z$ and $Z = \bigcup\{\,int\,S^{-1}(w) : \ w \in W\}$; here $S^{-1}(w) = \{z \in Z : \ w \in S(z)\}$ and note $S(x) \neq \emptyset$ for each $x \in Z$ is redundant since if $z \in Z$ then there exists a $w \in W$ with $z \in int\,S^{-1}(w) \subseteq S^{-1}(w)$ so $w \in S(z)$ i.e. $S(z) \neq \emptyset$. For the selection theorem below see [11].

Theorem 1.6. *Let X be a paracompact subset of a Hausdorff topological space, Y a convex subset of a Hausdorff topological vector space and $F \in HLPY(X,Y)$ (let $S : X \to 2^Y$ with $co\,(S(x)) \subseteq F(x)$ for $x \in X$ and $X = \bigcup\{\,int\,S^{-1}(w) : \ w \in Y\}$). Then there exists a continuous (single–valued) map $f : X \to Y$ with $f(x) \in co\,S(x) \subseteq F(x)$ for all $x \in X$.*

Remark 1.7. These maps are related to the DKT maps in the literature and $F \in DKT(Z,W)$ [4] if W is convex and there exists a map $S : Z \to W$ with $co\,(S(x)) \subseteq F(x)$ for $x \in Z$, $S(x) \neq \emptyset$ for each $x \in Z$ and the fibre $S^{-1}(w)$ is open (in Z) for each $w \in W$. Note these maps were motivated from the Fan maps.

Let X be a subset of a Hausdorff topological space and Y a subset of a Hausdorff topological vector space. We say $T : X \to 2^Y$ has the strong continuous inclusion property (SCIP) [10] at $x \in X$ if there exists an open set $U(x)$ in X containing x and a $F^x : U(x) \to 2^Y$ such that $F^x(w) \subseteq T(w)$ for all $w \in U(x)$ and $co\, F^x : U(x) \to 2^Y$ is compact valued and upper semicontinuous. We write $T \in KLU(X,Y)$ if T has the SCIP at every $x \in X$.

In this paper our map T will be a compact map so T has the SCIP is equivalent to T has the CIP [8].

Remark 1.8. These maps contain as a special case the Scalzo maps [17] in the literature (see [10, p. 12]).

Next we recall a selection theorem [10].

Theorem 1.9. *Let X be a paracompact subset of a Hausdorff topological space, Y a subset of a Hausdorff topological vector space and $T \in KLU(X,Y)$. Then there exists an upper semicontinuous map $G : X \to CK(Y)$ with $G(w) \subseteq co\, T(w)$ for all $w \in X$.*

Finally we present some preliminaries on the Lefschetz set for Ad maps needed in Section 2. Let $D(X,Y)$ be the set of all pairs $X \overset{p}{\Leftarrow} \Gamma \overset{q}{\to} Y$ where p is a Vietoris map and q is continuous. We will denote every such diagram by (p,q). Given two diagrams (p,q) and (p',q'), where $X \overset{p'}{\Leftarrow} \Gamma' \overset{q'}{\to} Y$, we write $(p,q) \sim (p',q')$ if there are maps $f : \Gamma \to \Gamma'$ and $g : \Gamma' \to \Gamma$ such that $q' \circ f = q$, $p' \circ f = p$, $q \circ g = q'$ and $p \circ g = p'$. The equivalence class of a diagram $(p,q) \in D(X,Y)$ with respect to $\sim$ is denoted by

$$\phi = \{X \overset{p}{\Leftarrow} \Gamma \overset{q}{\to} Y\} : X \to Y$$

or $\phi = [(p,q)]$ and is called a morphism from X to Y. We let $M(X,Y)$ be the set of all such morphisms. For any $\phi \in M(X,Y)$ a set $\phi(x) = q\, p^{-1}(x)$ where $\phi = [(p,q)]$ is called an image of x under a morphism ϕ.

Consider vector spaces over a field K. Let E be a vector space and $f : E \to E$ an endomorphism. Now let $N(f) = \{x \in E : f^{(n)}(x) = 0 \text{ for some } n\}$ where $f^{(n)}$ is the n^{th} iterate of f, and let $\tilde{E} = E \setminus N(f)$. Since $f(N(f)) \subseteq N(f)$ we have the induced endomorphism $\tilde{f} : \tilde{E} \to \tilde{E}$. We call f admissible if $dim\, \tilde{E} < \infty$; for such f we define the generalized trace $Tr(f)$ of f by putting $Tr(f) = tr(\tilde{f})$ where tr stands for the ordinary trace.

Let $f = \{f_q\} : E \to E$ be an endomorphism of degree zero of a graded vector space $E = \{E_q\}$. We call f a Leray endomorphism if (i). all f_q are admissible and (ii). almost all $\tilde{E}_q$ are trivial. For such f we define the generalized Lefschetz number $\Lambda(f)$ by

$$\Lambda(f) = \sum_q (-1)^q\, Tr\,(f_q).$$

With Čech homology functor extended to a category of morphisms we have the following well known result [7] (note the homology functor H extends over this category i.e. for a morphism

$$\phi = \{X \overset{p}{\Leftarrow} \Gamma \overset{q}{\to} Y\} : X \to Y$$

we define the induced map

$$H(\phi) = \phi_\star : H(X) \to H(Y)$$

by putting $\phi_\star = q_\star \circ p_\star^{-1}$). If $\phi : X \to Y$ and $\psi : Y \to Z$ are two morphisms (here X, Y and Z are Hausdorff topological spaces) then $(\psi \circ \phi)_\star = \psi_\star \circ \phi_\star$. Two morphisms $\phi, \psi \in M(X,Y)$ are homotopic (written $\phi \sim \psi$) provided there is a morphism $\chi \in M(X \times [0,1], Y)$ such that $\chi(x,0) = \phi(x)$, $\chi(x,1) = \psi(x)$ for every $x \in X$ (i.e. $\phi = \chi \circ i_0$ and $\psi = \chi \circ i_1$, where $i_0, i_1 : X \to X \times [0,1]$ are defined by $i_0(x) = (x,0)$, $i_1(x) = (x,1)$). Recall the following result [10]: If $\phi \sim \psi$ then $\phi_\star = \psi_\star$.

A map $\phi \in Ad(X,X)$ is said to be a Lefschetz map if for each selected pair $(p,q) \subset \phi$ the linear map $q_\star p_\star^{-1} : H(X) \to H(X)$ (the existence of $p_\star^{-1}$ follows from the Vietoris Theorem) is a Leray endomorphism. If $\phi : X \to X$ is a Lefschetz map, we define the Lefschetz set $\mathbf{\Lambda}(\phi)$ (or $\mathbf{\Lambda}_X(\phi)$) by

$$\mathbf{\Lambda}(\phi) = \left\{ \Lambda(q_\star p_\star^{-1}) : \ (p,q) \subset \phi \right\}.$$

A Hausdorff topological space X is said to be a Lefschetz space (for the class Ad) provided every compact $\phi \in Ad(X,X)$ is a Lefschetz map and $\mathbf{\Lambda}(\phi) \neq \{0\}$ implies ϕ has a fixed point.

Now we recall the fixed point results in [6, 13] needed in Section 2. Let X be a subset of a Hausdorff topological vector space. We say $F \in HYAd(X,X)$ if $F : X \to 2^X$ and there exists a map $\Phi \in Ad(X,X)$ with $\Phi(x) \subseteq co\,(F(x))$ for $x \in X$.

Let $X \in NES$(compact) (and X a subset of a Hausdorff topological vector space) and $F \in HYAd(X,X)$ with $co\,F$ a compact map. Then there exists a compact map $\Phi \in Ad(X,Y)$ with $\Phi(x) \subseteq co\,(F(x))$ for $x \in X$ (note Φ is a compact map since $co\,F$ is a compact map). In [6, 13] we showed that the Lefschetz set $\mathbf{\Lambda}(\Phi)$ is well defined and if $\mathbf{\Lambda}(\Phi) \neq \{0\}$ then Φ (so consequently $co\,F$) has a fixed point.

Let X be a subset of a Hausdorff topological vector space. We say $F \in HYAdC(X,X)$ if $F : X \to 2^X$ and there exists a map $\Phi \in Ad(X,X)$ with $\Phi(x) \subseteq F(x)$ for $x \in X$.

Let $X \in NES$(compact) and $F \in HYAdC(X,X)$ with F a compact map. Then there exists a compact map $\Phi \in Ad(X,Y)$ with $\Phi(x) \subseteq F(x)$ for $x \in X$. In [6, 13] we showed that the Lefschetz set $\mathbf{\Lambda}(\Phi)$ is well defined and if $\mathbf{\Lambda}(\Phi) \neq \{0\}$ then Φ (so consequently F) has a fixed point.

A special case of the above (when $\Phi = F$) is the following.

Theorem 1.10. *Let $X \in NES(compact)$ and $F \in Ad(X,X)$ a compact map. Then $\mathbf{\Lambda}(F)$ is well defined and if $\mathbf{\Lambda}(F) \neq \{0\}$ then F has a fixed point i.e. X is a Lefschetz space.*

Let $X \in SANES$(compact) (and X a subset of a Hausdorff topological vector space) and $F \in HYAd(X,X)$ with $co\,F$ a compact map. Then there exists a compact map $\Phi \in Ad(X,Y)$ with $\Phi(x) \subseteq co\,(F(x))$ for $x \in X$. In [13] we showed that the Lefschetz set $\mathbf{\Lambda}(\Phi)$ is well defined and if $\mathbf{\Lambda}(\Phi) \neq \{0\}$ then Φ (so consequently $co\,F$) has a fixed point.

Let $X \in SANES(\text{compact})$ and X a uniform space and $F \in HYAdC(X,X)$ with F a compact map. Then there exists a compact map $\Phi \in Ad(X,Y)$ with $\Phi(x) \subseteq F(x)$ for $x \in X$. In [13] we showed that the Lefschetz set $\mathbf{\Lambda}(\Phi)$ is well defined and if $\mathbf{\Lambda}(\Phi) \neq \{0\}$ then Φ (so consequently F) has a fixed point.

A special case of the above (when $\Phi = F$) is the following.

Theorem 1.11. *Let $X \in SANES(\text{compact})$, X a uniform space and $F \in Ad(X,X)$ a compact map. Then $\mathbf{\Lambda}(F)$ is well defined and if $\mathbf{\Lambda}(F) \neq \{0\}$ then F has a fixed point i.e. X is a Lefschetz space.*

2. Lefschetz Fixed Point Theory

The results in this section are motivated by admissibility in Section 1, Schauder projections and dominating space [6, 7]. Let W be a subset of a Hausdorff topological space.

Definition 2.1. *We say W is NES admissible if for all compact subsets K of W, all $\alpha \in Cov_W(K)$, there exists a continuous function $\pi_\alpha : K \to W$ such that*

(i). π_α and $i : K \hookrightarrow W$ are α–close;

(ii). $\pi_\alpha(K)$ is contained in a subset $C_\alpha \subseteq W$ and $C_\alpha \in NES(\text{compact})$;

(iii). π_α and $i : K \hookrightarrow W$ are homotopic.

Definition 2.2. *We say W is $SANES$ admissible if for all compact subsets K of W, all $\alpha \in Cov_W(K)$, there exists a continuous function $\pi_\alpha : K \to W$ such that*

(i). π_α and $i : K \hookrightarrow W$ are α–close;

(ii). $\pi_\alpha(K)$ is contained in a subset $C_\alpha \subseteq W$, $C_\alpha \in SANES(\text{compact})$ and C_α is a uniform space;

(iii). π_α and $i : K \hookrightarrow W$ are homotopic.

Indeed one could combine the above two definitions and consider the more general situation.

Definition 2.3. *We say W is Lefschetz admissible if for all compact subsets K of W, all $\alpha \in Cov_W(K)$, there exists a continuous function $\pi_\alpha : K \to W$ such that*

(i). π_α and $i : K \hookrightarrow W$ are α–close;

(ii). $\pi_\alpha(K)$ is contained in a subset $C_\alpha \subseteq W$ and C_α is a Lefschetz space;

(iii). π_α and $i : K \hookrightarrow W$ are homotopic.

Remark 2.4. In Definition 2.2 if W is a subset of a Hausdorff topological vector space then W is a uniform space and so automatically C_α is a uniform space (recall a subset of a uniform space is a uniform space). Thus C_α is a uniform space is redundant in Definition 2.2 if W is a subset of a Hausdorff topological vector space or more generally if W is a uniform space.

Theorem 2.5. *Let X be NES admissible (and X a subset of a Hausdorff topological vector space) and $F \in HY Ad(X,X)$ with $co\,F$ a compact map (so in particular there exists a compact map $\Phi \in Ad(X,X)$ with $\Phi(x) \subseteq co\,(F(x))$ for $x \in X$). Then $\mathbf{\Lambda}\,(\Phi)$ is well defined and if $\mathbf{\Lambda}\,(\Phi) \neq \{0\}$ then Φ (so consequently $co\,F$) has a fixed point.*

Proof. Let Φ be as in the statement above. Let $p,\, q : \Gamma \to X$ be a selected pair of Φ and let $K = \overline{\Phi(X)}$. Also let $\alpha \in Cov_X(K)$. Then there exists a continuous function $\pi_\alpha : K \to X$ and a subset C of X with $\pi_\alpha(K) \subseteq C$, $C \in NES$(compact) and π_α and $i : K \hookrightarrow X$ are α–close. Let $q_\alpha = \pi_\alpha\, q$. Note q_α is a compact map and $q \sim q_\alpha$; to see this note there exists a continuous map $h : K \times [0,1] \to X$ with $h(x,0) = \pi_\alpha(x)$ and $h(x,1) = i(x)$ and with $\eta(x,t) = h(q(x),t)$ we have $h : \Gamma \times [0,1] \to X$ (note p is surjective so $p^{-1}(X) = \Gamma$ and so $q : \Gamma \to K$) with $\eta(x,0) = \pi_\alpha q(x) = q_\alpha(x)$ and $\eta(x,1) = i(g(x)) = g(x)$.

Also note $q_\alpha(\Gamma) = \pi_\alpha q(\Gamma) \subseteq \pi_\alpha(K) \subseteq C$. Let

$$p_\alpha : p^{-1}(C) \to C,\ \overline{q_\alpha} : p^{-1}(C) \to C,\ q'_\alpha : \Gamma \to C$$

denote contractions of the appropriate maps and note p_α is a Vietoris map (see [6, pp 214]). Also let

$$i_C : C \hookrightarrow X \quad \text{and} \quad j : p^{-1}(C) \hookrightarrow \Gamma$$

be inclusions. Note $(p_\alpha, \overline{q_\alpha})$ is a selected pair of $\pi_\alpha \Phi$ (note $p_\alpha : p^{-1}(C) \to C$ and $\overline{q_\alpha} : p^{-1}(C) \to C$) and $\pi_\alpha\Phi \in Ad(C,C)$ is a compact map (note the composition of Ad maps is an Ad map [6] and note $\Phi \in Ad(C,K)$ [7, 13]). Now since $C \in NES$(compact) then Theorem 1.10 guarantees that $(\overline{q_\alpha})_\star (p_\alpha^{-1})_\star$ is a Leray endomorphism. Next we note (recall $i_C q'_\alpha = q_\alpha$)

$$(i_C)_\star (q'_\alpha)_\star p_\star^{-1} = (i_C q'_\alpha)_\star p_\star^{-1} = (q_\alpha)_\star p_\star^{-1} = q_\star p_\star^{-1}$$

since $q \sim q_\alpha$ (so [7] $q_\star = (q_\alpha)_\star$). Also we have (note [16] $p_\star p_\star^{-1} = i_\star$ and $i_C p_\alpha = p\, j$ and $q'_\alpha j = \overline{q_\alpha}$)

$$(q'_\alpha)_\star p_\star^{-1} (i_C)_\star = (q'_\alpha)_\star j_\star (p_\alpha^{-1})_\star = (\overline{q_\alpha})_\star (p_\alpha^{-1})_\star.$$

Now [6, pp 214 (see (1.3))] (here $E' = H(C)$, $E'' = H(X)$, $u = (i_C)_\star$, $v = (q'_\alpha)_\star p_\star^{-1}$, $f' = (\overline{q_\alpha})_\star (p_\alpha^{-1})_\star$, $f'' = q_\star p_\star^{-1}$ and note $u\,f' = (i_C)_\star (\overline{q_\alpha})_\star (p_\alpha^{-1})_\star = (i_C)_\star (q'_\alpha)_\star p_\star^{-1} (i_C)_\star = q_\star p_\star^{-1} (i_C)_\star = f''\,u$) guarantees that $q_\star p_\star^{-1}$ is a Leray endomorphism and $\mathbf{\Lambda}\,(q_\star p_\star^{-1}) = \mathbf{\Lambda}\,((\overline{q_\alpha})_\star (p_\alpha^{-1})_\star)$ i.e. $\mathbf{\Lambda}\,(\Phi)$ is well defined and $\mathbf{\Lambda}\,(\Phi) \subseteq \mathbf{\Lambda}\,(\pi_\alpha \Phi)$.

Now assume $\mathbf{\Lambda}\,(\Phi) \neq \{0\}$. Then $\mathbf{\Lambda}\,(\pi_\alpha\Phi) \neq \{0\}$. Since $C \in NES$(compact) then Theorem 1.10 guarantees that there exists a $x_\alpha \in C$ with $x_\alpha \in \pi_\alpha \Phi(x_\alpha)$. Since π_α and i are α–close then Φ has an α–fixed point of Φ; note $x_\alpha = \pi_\alpha(y_\alpha)$ where $y_\alpha \in \Phi(x_\alpha)$ so there exists a $U_\alpha \in \alpha$ with $\pi_\alpha(y_\alpha) \in U_\alpha$ and $y_\alpha \in U_\alpha$ i.e. $x_\alpha \in U_\alpha$ and $y_\alpha \in U_\alpha$ i.e. $x_\alpha \in U_\alpha$ and $\Phi(x_\alpha) \cap U_\alpha \neq \emptyset$ (note $y_\alpha \in U_\alpha$ and $y_\alpha \in \Phi(x_\alpha)$). Thus Φ has an α–fixed point (for each $\alpha \in Cov_X(K)$). Now Theorem 1.2 and Remark 1.3 (note Hausdorff topological vector spaces are uniform spaces) guarantee that Φ (so consequently $co\,(F)$) has a fixed point. □

The same argument as in Theorem 2.5 establishes the following result (here X is a subset of a Hausdorff topological space).

Theorem 2.6. *Let X be NES admissible with X a uniform space and $F \in HYAdC(X,X)$ with F a compact map (so in particular there exists a compact map $\Phi \in Ad(X,X)$ with $\Phi(x) \subseteq F(x)$ for $x \in X$). Then $\mathbf{\Lambda}(\Phi)$ is well defined and if $\mathbf{\Lambda}(\Phi) \neq \{0\}$ then Φ (so consequently F) has a fixed point.*

The same argument as in Theorem 2.5 (with Theorem 1.10 replaced by Theorem 1.11) yields the following results.

Theorem 2.7. *Let X be $SANES$ admissible (and X a subset of a Hausdorff topological vector space) and $F \in HYAd(X,X)$ with $co\,F$ a compact map (so in particular there exists a compact map $\Phi \in Ad(X,X)$ with $\Phi(x) \subseteq co\,(F(x))$ for $x \in X$). Then $\mathbf{\Lambda}(\Phi)$ is well defined and if $\mathbf{\Lambda}(\Phi) \neq \{0\}$ then Φ (so consequently $co\,F$) has a fixed point.*

Theorem 2.8. *Let X be $SANES$ admissible with X a uniform space and $F \in HYAdC(X,X)$ with F a compact map (so in particular there exists a compact map $\Phi \in Ad(X,X)$ with $\Phi(x) \subseteq F(x)$ for $x \in X$). Then $\mathbf{\Lambda}(\Phi)$ is well defined and if $\mathbf{\Lambda}(\Phi) \neq \{0\}$ then Φ (so consequently F) has a fixed point.*

Also the same argument as in Theorem 2.5 establishes the following results.

Theorem 2.9. *Let X be Lefschetz admissible (and X a subset of a Hausdorff topological vector space) and $F \in HYAd(X,X)$ with $co\,F$ a compact map (so in particular there exists a compact map $\Phi \in Ad(X,X)$ with $\Phi(x) \subseteq co\,(F(x))$ for $x \in X$). Then $\mathbf{\Lambda}(\Phi)$ is well defined and if $\mathbf{\Lambda}(\Phi) \neq \{0\}$ then Φ (so consequently $co\,F$) has a fixed point.*

Theorem 2.10. *Let X be Lefschetz admissible with X a uniform space and $F \in HYAdC(X,X)$ with F a compact map (so in particular there exists a compact map $\Phi \in Ad(X,X)$ with $\Phi(x) \subseteq F(x)$ for $x \in X$). Then $\mathbf{\Lambda}(\Phi)$ is well defined and if $\mathbf{\Lambda}(\Phi) \neq \{0\}$ then Φ (so consequently F) has a fixed point.*

A special case of Theorem 2.10 (with $\Phi = F$) is the following.

Theorem 2.11. *Let X be Lefschetz admissible with X a uniform space and let $F \in Ad(X,X)$ be a compact map. Then $\mathbf{\Lambda}(F)$ is well defined and if $\mathbf{\Lambda}(F) \neq \{0\}$ then F has a fixed point i.e. X is a Lefschetz space.*

We will now define admissible dominating spaces. Again we could consider NES admissible dominated, $SANES$ admissible dominated and Lefschetz admissible dominated. The argument presented below is exactly the same for the three situations so we will just consider Lefschetz admissible dominated (which of course contains NES admissible dominated and $SANES$ admissible dominated).

Let X be a Hausdorff topological space and let $\alpha \in Cov\,(X)$. X is said to be Lefschetz admissible α–dominated (respectively, NES admissible α–dominated; respectively $SANES$ admissible α–dominated) if there exists a Lefschetz admissible space X_α (respectively, a NES admissible space X_α; respectively a $SANES$ admissible space X_α) with X_α a uniform space, and two continuous mappings $r_\alpha : X_\alpha \to X$, $s_\alpha : X \to X_\alpha$ such that $r_\alpha\, s_\alpha : X \to X$ and the identity $i_X : X \to X$ are α–close and

$r_\alpha\, s_\alpha \sim i_X$ (i.e. there exists a homotopy h joining $r_\alpha s_\alpha$ and i_X). X is said to be Lefschetz admissible dominated (respectively, NES admissible dominated; respectively, $SANES$ admissible dominated) if X is Lefschetz admissible α–dominated (respectively, NES admissible α–dominated; respectively $SANES$ admissible α–dominated) for every $\alpha \in Cov\,(X)$.

Theorem 2.12. *Let X be Lefschetz admissible dominated (and X a subset of a Hausdorff topological vector space) and $F \in HY\!Ad(X,X)$ with $co\,F$ a compact map (so in particular there exists a compact map $\Phi \in Ad(X,X)$ with $\Phi(x) \subseteq co\,(F(x))$ for $x \in X$). Then $\mathbf{\Lambda}\,(\Phi)$ is well defined and if $\mathbf{\Lambda}\,(\Phi) \neq \{0\}$ then Φ (so consequently $co\,F$) has a fixed point.*

Proof. Let Φ be as in the statement above. Let $\alpha \in Cov\,(X)$ and let $p,\, q : \Gamma \to X$ be a selected pair of Φ. Now since X is Lefschetz admissible dominated then there exists a Lefschetz space X_α with X_α a uniform space and two continuous mappings $r_\alpha : X_\alpha \to X$, $s_\alpha : X \to X_\alpha$ such that $r_\alpha\, s_\alpha : X \to X$ and the identity $i_X : X \to X$ are α–close and $r_\alpha\, s_\alpha \sim i_X$ (note in particular [7] that $(r_\alpha)_\star (s_\alpha)_\star = (i_X)_\star$). Let $\Phi_\alpha = s_\alpha \Phi\, r_\alpha$ and note $\Phi \in Ad(X_\alpha, X_\alpha)$. Also from [7, Section 40] there exists a selected pair (p_α, q_α) of Φ_α with $(q_\alpha)_\star (p_\alpha^{-1})_\star = (s_\alpha)_\star q_\star p_\star^{-1} (r_\alpha)_\star$. Since X_α is Lefschetz admissible then (see Theorem 2.11) $(q_\alpha)_\star (p_\alpha^{-1})_\star$ is a Leray endomorphism. Next note since $(r_\alpha)_\star (s_\alpha)_\star = (i_X)_\star$ that

$$(r_\alpha)_\star (s_\alpha)_\star q_\star p_\star^{-1} = (i_X)_\star q_\star p_\star^{-1} = q_\star p_\star^{-1}$$

and from above

$$(s_\alpha)_\star q_\star p_\star^{-1} (r_\alpha)_\star = (q_\alpha)_\star (p_\alpha^{-1})_\star.$$

Now [6, pp 214 (see (1.3))] (here $E' = H(X_\alpha)$, $E'' = H(X)$, $u = (r_\alpha)_\star$, $v = (s_\alpha)_\star q_\star p_\star^{-1}$, $f' = (q_\alpha)_\star (p_\alpha^{-1})_\star$, $f'' = q_\star p_\star^{-1}$ and note $u\,f' = (r_\alpha)_\star (q_\alpha)_\star (p_\alpha^{-1})_\star = (r_\alpha)_\star (s_\alpha)_\star q_\star p_\star^{-1} (r_\alpha)_\star = q_\star p_\star^{-1} (r_\alpha)_\star = f''\,u$) guarantees that $q_\star p_\star^{-1}$ is a Leray endomorphism and $\mathbf{\Lambda}\,(q_\star p_\star^{-1}) = \mathbf{\Lambda}\,((q_\alpha)_\star (p_\alpha^{-1})_\star)$ i.e. $\mathbf{\Lambda}\,(\Phi)$ is well defined and $\mathbf{\Lambda}\,(\Phi) \subseteq \mathbf{\Lambda}\,(\Phi_\alpha)$.

Now assume $\mathbf{\Lambda}\,(\Phi) \neq \{0\}$. Then $\mathbf{\Lambda}\,(\Phi_\alpha) \neq \{0\}$. Now since X_α is Lefschetz admissible then Theorem 2.11 guarantees that there exists a $x_\alpha \in X_\alpha$ with $x_\alpha \in \Phi_\alpha(x_\alpha)$. Since $r_\alpha s_\alpha$ and i_X are α–close then Φ has an α–fixed point of Φ; note $x_\alpha \in s_\alpha \Phi r_\alpha(x_\alpha)$ so $x_\alpha = s_\alpha(y)$ for some $y \in \Phi r_\alpha(x_\alpha)$, so with $z = r_\alpha(x_\alpha)$ then $z = r_\alpha s_\alpha(y)$ for some $y \in \Phi(z)$ and now since $r_\alpha\, s_\alpha$ and i_X are α–close then there exists a $U \in \alpha$ with $r_\alpha s_\alpha(y) \in U$ and $i_X(y) \in U$ i.e. $z \in U$ and $y \in U$ i.e. $z \in U$ and $\Phi(z) \cap U \neq \emptyset$ (note $y \in U$ and $y \in \Phi(z)$). Thus Φ has an α–fixed point (for each $\alpha \in Cov_X(X)$). Now Theorem 1.2 and Remark 1.3 guarantee that Φ (so consequently $co\,(F)$) has a fixed point. □

Remark 2.13. (i). From the above analysis note that one could replace the condition $r_\alpha\, s_\alpha \sim i_X$ (in the definition of Lefschetz admissible α–dominated) with any condition that guarantees $(r_\alpha)_\star (s_\alpha)_\star = (i_X)_\star$.

(ii). Also note from the above analysis we only need X to be Lefschetz admissible α–dominated for every $\alpha \in Cov_X\,(\overline{\Phi(X)})$ (here Φ is as in the proof of Theorem 2.12) to deduce Theorem 2.12.

The same argument as in Theorem 2.12 establishes the following result.

Theorem 2.14. *Let X be Lefschetz admissible dominated with X a uniform space and $F \in HYAdC(X, X)$ with F a compact map (so in particular there exists a compact map $\Phi \in Ad(X, X)$ with $\Phi(x) \subseteq F(x)$ for $x \in X$). Then $\mathbf{\Lambda}(\Phi)$ is well defined and if $\mathbf{\Lambda}(\Phi) \neq \{0\}$ then Φ (so consequently F) has a fixed point.*

A special case of Theorem 2.14 (with $\Phi = F$) is the following.

Theorem 2.15. *Let X be Lefschetz admissible dominated with X a uniform space and let $F \in Ad(X, X)$ be a compact map. Then $\mathbf{\Lambda}(F)$ is well defined and if $\mathbf{\Lambda}(F) \neq \{0\}$ then F has a fixed point i.e. X is a Lefschetz space.*

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Donal O'Regan
School of Mathematical and Statistical Sciences
University of Galway
Ireland
e-mail: `donal.oregan@nuigalway.ie`