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Characterization and stability of essential pseudospectra by measure of polynomially inessential operators

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Abstract. In this article, we investigate the essential pseudospectra through the framework of polynomially inessential operators, which extends the class of polynomially strictly singular operators and provides a broader setting for Fredholm-type perturbations. We establish new results on the behavior of the essential pseudospectrum of closed linear operators on Banach spaces under perturbations by polynomially inessential operators. Moreover, we apply these results to study the influence of such perturbations on the left (resp. right) Weyl essential pseudospectra and the left (resp. right) Fredholm essential pseudospectra. In addition, we give a description of the essential pseudospectrum of the sum of two bounded linear operators. Finally, an application is provided to characterize the pseudo left (resp. right) Fredholm spectra of 2×2 block operator matrices.

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1. Introduction

Eigenvalue problems play a crucial role in many areas of science and engineering. The main objectives in such problems are to determine and localize the eigenvalues of a given operator. However, classical spectral theory is often insufficient to achieve both of these goals, as it merely identifies the eigenvalues without providing information about their localization or stability. To address this limitation, researchers introduced the concept of the *pseudospectrum*, first proposed by Varah [29] and Schechter and

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[27, 28]. Since then, pseudospectral analysis has been widely applied in various domains of mathematical physics, such as electrical engineering, aeronautics, ecology, and chemistry (see [9, 17, 28]). For instance, in electrical engineering, eigenvalues determine the stability and accuracy of an amplifier's frequency response; in aerodynamics, they indicate whether airflow over a wing remains laminar or becomes turbulent; in ecology, they govern the stability of equilibria in population dynamics; and in chemistry, they describe the energy levels of atomic systems. In summary, the concept of pseudospectrum has proven to be an effective tool in solving eigenvalue problems, allowing researchers to both locate and interpret spectral behavior, and thus contribute to major advances across scientific and engineering disciplines.

Motivated by the importance of pseudospectra, F. Abdmouleh *et al.* [2] introduced the notion of the pseudo-Browder essential spectrum for densely defined closed linear operators on Banach spaces. Later, F. Abdmouleh and B. Elgabeur in their works [3, 4, 11, 12] introduced and studied the pseudo left (resp. right) Fredholm and pseudo left (resp. right) Browder operators, together with their associated essential pseudospectra for bounded linear operators on Banach spaces. Among the principal results obtained in these works are stability theorems of the pseudo-essential spectra under Riesz operator perturbations. Moreover, the authors characterized the pseudo left (resp. right) Fredholm and pseudo left (resp. right) Browder essential pseudospectra for the sum of two bounded linear operators. Ammar and Jeribi in their contributions [5, 6] extended these investigations by developing the theory of essential pseudospectra of bounded linear operators and introducing the pseudo-Fredholm operator and its essential pseudospectrum.

In the present paper, we continue the study of essential pseudospectra in Banach spaces by considering a broader class of perturbations known as polynomially inessential operators. This family extends the well-known classes of compact, strictly singular, and polynomially strictly singular operators and can be viewed as a generalization of several classical Fredholm perturbations. This class of operators has drawn considerable interest due to its ability to unify several spectral perturbation frameworks and to yield new insights into Fredholm-type spectral stability. The reader may find further details and related results in [13, 14, 19]. The first objective of this paper is to extend the stability results of essential pseudospectra obtained in [2, 3, 4, 5, 6] to perturbations by polynomially inessential operators for closed, densely defined linear operators. The second goal is to describe the essential pseudospectrum of the sum of two bounded linear operators within the setting of polynomially inessential operators.

1.1. Organization of the paper

In Section 2, we recall some basic definitions and notations concerning Fredholm operators and their essential spectra. We also introduce and study the main properties of polynomially inessential operators. In Section 3, we establish stability results and provide a new characterization of the left (resp. right) Weyl and left (resp. right) Fredholm essential pseudospectra within this class of operators. Section 4 is devoted to the essential pseudospectra of the sum of two bounded linear operators, motivated by the concept of polynomially inessential operators. Finally, we extend the obtained results to the pseudo left (resp. right) Fredholm spectra for 2×2 block operator matrices.

2. Notations and definitions

Let X and Y be two Banach spaces. By an operator A from X into Y we mean a linear operator with domain $\mathcal{D}(A) \subseteq X$ and range contained in Y. We denote by $\mathcal{C}(X,Y)$ (resp., $\mathcal{L}(X,Y)$) the set of all closed, densely defined (resp., bounded) linear operators from X to Y. The subset of all compact operators of $\mathcal{L}(X,Y)$ is designated by $\mathcal{K}(X,Y)$. If $A \in \mathcal{C}(X,Y)$, we write $N(A) \subset X$ and $R(A) \subset Y$ for the null space and the range of A. We set $\alpha(A) := \dim N(A)$ and $\beta(A) := \operatorname{codim} R(A)$. Let $A \in \mathcal{C}(X,Y)$ with closed range. Then A is a Φ_+-operator ($A \in \Phi_+(X,Y)$) if $\alpha(A) < \infty$, and then A is a Φ_--operator ($A \in \Phi_-(X,Y)$) if $\beta(A) < \infty$.$\Phi(X,Y) = \Phi_+(X,Y) \cap \Phi_-(X,Y)$ is the class of Fredholm operators while $\Phi_\pm(X,Y)$ denotes the set $\Phi_\pm(X,Y) = \Phi_+(X,Y) \cup \Phi_-(X,Y)$. For $A \in \Phi(X,Y)$, the index of A is defined by $i(A) = \alpha(A) - \beta(A)$. If $X = Y$, then $\mathcal{L}(X,Y), \mathcal{K}(X,Y), \mathcal{C}(X,Y), \Phi_+(X,Y), \Phi_\pm(X,Y)$ and $\Phi(X,Y)$ are replaced, respectively, by $\mathcal{L}(X), \mathcal{K}(X), \mathcal{C}(X), \Phi_+(X), \Phi_\pm(X)$ and $\Phi(X)$. Let $A \in \mathcal{C}(X)$, the spectrum of A will be denoted by $\sigma(A)$. The resolvent set of A, $\rho(A)$, is the complement of $\sigma(A)$ in the complex plane. A complex number λ is in $\Phi_{+A}, \Phi_{-A}, \Phi_{\pm A}$ or Φ_A if $\lambda - A$ is in $\Phi_+(X), \Phi_-(X), \Phi_\pm(X)$ or $\Phi(X)$, respectively. Let $F \in \mathcal{L}(X,Y)$.F is called a Fredholm perturbation if $U + F \in \Phi(X,Y)$ whenever $U \in \Phi(X,Y)$. F is called an upper (resp., lower) Fredholm perturbation if $U + F \in \Phi_+(X,Y)$ (resp., $U + F \in \Phi_-(X,Y)$) whenever $U \in \Phi_+(X,Y)$ (resp., $U \in \Phi_-(X,Y)$). The set of Weyl operators is defined as $\mathcal{W}(X,Y) = \{A \in \Phi(X,Y) : i(A) = 0\}$. Sets of left and right Fredholm operators, respectively, are defined as:

$$\Phi_l(X) := \big\{A \in \mathcal{L}(X) : R(A) \text{ is a closed and complemented subspace of } X, \text{ and } \alpha(A) < \infty\big\},$$

$$\Phi_r(X) := \big\{A \in \mathcal{L}(X) : N(A) \text{ is a closed and complemented subspace of } X, \text{ and } \beta(A) < \infty\big\}.$$

An operator $A \in \mathcal{L}(X)$ is left (right) Weyl if A is left (right) Fredholm operator and $i(A) \leq 0 (i(A) \geq 0)$. We use $\mathcal{W}_l(X)(\mathcal{W}_r(X))$ to denote the set of all left(right) Weyl operators. It is known that the sets $\Phi_l(X)$ and $\Phi_r(X)$ are open satisfying the following inclusions:

$$\Phi(X) \subset \mathcal{W}_l(X) \subset \Phi_l(X) \text{ and } \Phi(X) \subset \mathcal{W}_r(X) \subset \Phi_r(X).$$

The sets of Fredholm, upper semi-Fredholm and lower semi-Fredholm perturbations are denoted by $\mathcal{F}(X,Y), \mathcal{F}_+(X,Y)$ and $\mathcal{F}_-(X,Y)$, respectively. In general, we have

$$\mathcal{K}(X,Y) \subseteq \mathcal{F}_+(X,Y) \subseteq \mathcal{F}(X,Y)$$
$$\mathcal{K}(X,Y) \subseteq \mathcal{F}_-(X,Y) \subseteq \mathcal{F}(X,Y).$$

If $X = Y$ we write $\mathcal{F}(X), \mathcal{F}_+(X)$ and $\mathcal{F}_-(X)$ for $\mathcal{F}(X,X), \mathcal{F}_+(X,X)$ and $\mathcal{F}_-(X,X)$, respectively. Let $\Phi^b(X,Y), \Phi^b_+(X,Y)$ and $\Phi^b_-(X,Y)$ denote the sets $\Phi(X,Y) \cap \mathcal{L}(X,Y), \Phi_+(X,Y) \cap \mathcal{L}(X,Y)$ and $\Phi_-(X,Y) \cap \mathcal{L}(X,Y)$, respectively. If in Definition 1.1 we replace $\Phi(X,Y), \Phi_+(X,Y)$ and $\Phi_-(X,Y)$ by $\Phi^b(X,Y), \Phi^b_+(X,Y)$ and $\Phi^b_-(X,Y)$ we obtain the sets $\mathcal{F}^b(X,Y), \mathcal{F}^b_+(X,Y)$ and $\mathcal{F}^b_-(X,Y)$. These classes of operators were introduced and investigated in [6]. In particular, $\mathcal{F}^b(X,Y)$ is shown to be a closed subset of $\mathcal{L}(X,Y)$ and $\mathcal{F}^b(X)$ is a closed two-sided ideal of $\mathcal{L}(X)$. In general we have

$$\mathcal{K}(X,Y) \subseteq \mathcal{F}^b_+(X,Y) \subseteq \mathcal{F}^b(X,Y)$$
$$\mathcal{K}(X,Y) \subseteq \mathcal{F}^b_-(X,Y) \subseteq \mathcal{F}^b(X,Y)$$

Let $A \in \mathcal{C}(X)$. It follows from the closeness of A that $\mathcal{D}(A)$ endowed with the graph norm $\|\cdot\|_A$ ($\|x\|_A = \|x\| + \|Ax\|$) is a Banach space denoted by X_A. Clearly, for $x \in \mathcal{D}(A)$ we have $\|Ax\| \leqslant \|x\|_A$, so $A \in \mathcal{L}(X_A, X)$. Furthermore, we have Basic properties

$$\begin{cases} \alpha(\hat{A}) = \alpha(A), \quad \beta(\hat{A}) = \beta(A), \quad R(\hat{A}) = R(A) \\ \alpha(\hat{A}+\hat{B}) = \alpha(A+B), \\ \beta(\hat{A}+\hat{B}) = \beta(A+B) \text{ and } R(\hat{A}+\hat{B}) = R(A+B) \end{cases} \tag{2.1}$$

In this paper we are concerned with the following essential spectra of $A \in C(X)$:

$\sigma_e(A) := \{\lambda \in \mathbf{C} : A - \lambda \notin \Phi(X)\}$: the Fredholm spectrum of A.

$\sigma_e^l(A) := \{\lambda \in \mathbf{C} : A - \lambda \notin \Phi_l(X)\}$: the left Fredholm spectrum of A.

$\sigma_e^r(A) := \{\lambda \in \mathbf{C} : A - \lambda \notin \Phi_r(X)\}$: the right Fredholm spectrum of A.

$\sigma_w(A) := \{\lambda \in \mathbf{C} : A - \lambda \notin \mathcal{W}(X)\}$: the Weyl spectrum of A.

$\sigma_w^l(A) := \{\lambda \in \mathbf{C} : A - \lambda \notin \mathcal{W}_l(X)\}$: the left Weyl spectrum of A.

$\sigma_w^r(A) := \{\lambda \in \mathbf{C} : A - \lambda \notin \mathcal{W}_r(X)\}$: the right Weyl spectrum of A.

$\sigma_{\text{eap}}(A) := \mathrm{C} \backslash \rho_{\text{eap}}(A)$: the essential approximate point spectrum of A.

$\sigma_{e\delta}(A) := \mathrm{C} \backslash \rho_{e\delta}(T)$: the essential defect spectrum of A.

where

$$\rho_{\text{eap}}(A) := \{\lambda \in \mathrm{C} \text{ such that } \lambda - A \in \Phi_+(X) \text{ and } i(\lambda - A) \leq 0\},$$

and

$$\rho_{e\delta}(A) := \{\lambda \in \mathrm{C} \text{ such that } \lambda - A \in \Phi_-(X) \text{ and } i(\lambda - A) \geq 0\}$$

The definition of pseudo spectrum of a closed densely linear operator A for every $\varepsilon > 0$ is given by:

$$\sigma_\varepsilon(A) := \sigma(A) \cup \left\{\lambda \in \mathbb{C} : \left\|(\lambda - A)^{-1}\right\| > \frac{1}{\varepsilon}\right\}.$$

By convention, we write $\left\|(\lambda - A)^{-1}\right\| = \infty$ if $(\lambda - A)^{-1}$ is unbounded or nonexistent, i.e., if λ is in the spectrum $\sigma(A)$. In [9], Davies defined another equivalent of the pseudo spectrum, one that is in terms of perturbations of the spectrum. In fact for $A \in C(X)$, we have

$$\sigma_\varepsilon(A) := \bigcup_{\|D\|<\varepsilon} \sigma(A + D).$$

Inspired by the notion of pseudospectra, Ammar and Jeribi in their works [5, 6], aimed to extend these results for the essential pseudo-spectra of bounded linear operators on a Banach space and give the definitions of pseudo-Fredholm operator as follows: for $A \in \mathcal{L}(X)$ and for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$ we have A is called a pseudo-upper (resp. lower) semi-Fredholm operator if $A+D$ is an upper (resp. lower) semi-Fredholm operator and it is called a pseudo semi-Fredholm operator if $A+D$ is a semi-Fredholm operator. A is called a pseudo-Fredholm operator if $A + D$ is a Fredholm operator. They are noted by $\Phi^\varepsilon(X)$ the set of pseudo-Fredholm operators, by $\Phi^\varepsilon_\pm(X)$ the set of pseudo-semi-Fredholm operator and by $\Phi^\varepsilon_+(X)$ (resp. $\Phi^\varepsilon_-(X)$) the set of pseudo-upper semi-Fredholm operators (resp. lower semi-Fredholm). A complex number λ is in $\Phi^\varepsilon_{\pm A}$, Φ^ε_{+A}, Φ^ε_{-A} or Φ^ε_A if $\lambda - A$ is in $\Phi^\varepsilon_\pm(X), \Phi^\varepsilon_+(X), \Phi^\varepsilon_-(X)$ or $\Phi^\varepsilon(X)$.
F. Abdmouleh and B. Elgabeur in [4] defining the concept of pseudo left (resp. right) Fredholm, for $A \in \mathcal{L}(X)$ and for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$ we have A is called a pseudo left (resp. right) Fredholm operator if $A + D$ is a left (resp. right) Fredholm operator, they are denoted by $\Phi^\varepsilon_l(X)$ (resp. $\Phi^\varepsilon_r(X)$).

In this paper, we are concerned with the following essential pseudospectra of $A \in C(X)$:

$$\begin{aligned}
\sigma_{e1,\varepsilon}(A) &:= \left\{\lambda \in \mathbb{C} \text{ such that } \lambda - A \notin \Phi^\varepsilon_+(X)\right\} = \mathbb{C}\backslash\Phi^\varepsilon_{+A},\\
\sigma_{e2,\varepsilon}(A) &:= \left\{\lambda \in \mathbb{C} \text{ such that } \lambda - A \notin \Phi^\varepsilon_-(X)\right\} = \mathbb{C}\backslash\Phi^\varepsilon_{-A},\\
\sigma_{e3,\varepsilon}(A) &:= \left\{\lambda \in \mathbb{C} \text{ such that } \lambda - A \notin \Phi^\varepsilon_\pm(X)\right\} = \mathbb{C}\backslash\Phi^\varepsilon_{\pm A},\\
\sigma_{e,\varepsilon}(A) &:= \{\lambda \in \mathbb{C} \text{ such that } \lambda - A \notin \Phi^\varepsilon(X)\} = \mathbb{C}\backslash\Phi^\varepsilon_A,\\
\sigma_{eap,\varepsilon}(A) &:= \sigma_{e1,\varepsilon}(A) \cup \{\lambda \in \mathbb{C} \text{ such that } i(\lambda - A - D) > 0, \forall\|D\| < \varepsilon\},\\
\sigma_{e\delta,\varepsilon}(A) &:= \sigma_{e2,\varepsilon}(A) \cup \{\lambda \in \mathbb{C} \text{ such that } i(\lambda - A - D) < 0, \forall\|D\| < \varepsilon\},\\
\sigma^l_{e,\varepsilon}(A) &:= \{\lambda \in \mathbb{C} \text{ such that } \lambda - A \notin \Phi^\varepsilon_l(X)\},\\
\sigma^r_{e,\varepsilon}(A) &:= \{\lambda \in \mathbb{C} \text{ such that } \lambda - A \notin \Phi^\varepsilon_r(X)\},\\
\sigma^l_{w,\varepsilon}(A) &:= \sigma^l_{e,\varepsilon}(A) \cup \{\lambda \in \mathbb{C} \text{ such that } i(\lambda - A - D) > 0, \forall\|D\| < \varepsilon\},\\
\sigma^r_{w,\varepsilon}(A) &:= \sigma^r_{e,\varepsilon}(A) \cup \{\lambda \in \mathbb{C} \text{ such that } i(\lambda - A - D) < 0, \forall\|D\| < \varepsilon\},\\
\sigma_{w,\varepsilon}(A) &:= \sigma_{e,\varepsilon}(A) \cup \{\lambda \in \mathbb{C} \text{ such that } i(\lambda - A - D) = 0, \forall\|D\| < \varepsilon\}.
\end{aligned}$$

Note that if ε tends to 0, we recover the usual definition of the essential spectra of a closed operator A. The subsets σ_{e1} and σ_{e2} are the Gustafson and Weidmann essential spectra [16], σ_{e3} is the Kato essential spectrum,[19] σ_e is the Wolf essential spectrum[16], σ_{e5} is the Schechter essential spectrum[26], σ_{eap} is the essential approximate point spectrum [24], $\sigma_{e\delta}$ is the essential defect spectrum [25], $\sigma_e^l(A)$(resp.$\sigma_e^r(A)$) is the left (resp. right) Fredholm essential spectra and $\sigma_w^l(A)$(resp.$\sigma_w^r(A)$) is the left (resp. right) Weyl essential spectra [15, 30, 31].
As a concept, pseudospectra and essential pseudospectra are interesting because they offer more information than spectra, especially about transients rather than just asymptotic behavior. Moreover, they perform more efficiently than spectra in terms of convergence and approximation. These include the existence of approximate eigenvalues far from the spectrum and the instability of the spectrum even under small perturbations. Various applications of the pseudospectra and essential pseudospectra have been developed as a result of the analysis of the pseudospectra and essential pseudospectra.
We now list some of the known facts about left and right Fredholm operators in Banach space which will be used in the sequel.

Proposition 2.1. [18, propositon 2.3] *Let X, Y and Z be three Banach spaces.*

(i) *If $A \in \Phi^b(Y,Z)$ and $T \in \Phi_l^b(X,Y)$ (resp. $T \in \Phi_r^b(X,Y)$), then $AT \in \Phi_l^b(X,Z)$ (resp. $AT \in \Phi_r^b(X,Z)$).*

(ii) *If $A \in \Phi^b(Y,Z)$ and $T \in \Phi_l^b(X,Y)$ (resp. $T \in \Phi_r^b(X,Y)$), then $TA \in \Phi_l^b(X,Z)$ (resp. $TA \in \Phi_r^b(X,Z)$).* ♢

Theorem 2.2. [22, 26] *Let X, Y and Z be three Banach spaces, $A \in \mathcal{L}(Y,Z)$ and $T \in \mathcal{L}(X,Y)$.*

(i) *If $A \in \Phi^b(Y,Z)$ and $T \in \Phi^b(X,Y)$, then $AT \in \Phi^b(X,Z)$ and $i(AT) = i(A) + i(T)$.*

(ii) *If $X = Y = Z$, $AT \in \Phi^b(X)$ and $TA \in \Phi^b(X)$, then $A \in \Phi^b(X)$ and $T \in \Phi^b(X)$.* ♢

Lemma 2.3. [15, Theorem 2.3] *Let $A \in \mathcal{L}(X)$, then*

(i) $A \in \Phi_l^b(X)$ if and only if there exist $A_l \in \mathcal{L}(X)$ and $K \in \mathcal{K}(X)$ such that $A_lA = I - K$.

(ii) $A \in \Phi_r^b(X)$ if and only if there exist $A_r \in \mathcal{L}(X)$ and $K \in \mathcal{K}(X)$ such that $AA_r = I - K$. ♢

Lemma 2.4. [15, Theorem 2.7] *Let $A \in \mathcal{L}(X)$.*

If $A \in \Phi_l^b(X)$(resp.$\Phi_r^b(X)$) and $K \in \mathcal{K}(X)$, then $A + K \in \Phi_l^b(X)$(resp.$\Phi_r^b(X)$) and $i(A+K) = i(A)$. ♢

Lemma 2.5. [15, Theorem 2.5] *Let $A, B \in \mathcal{L}(X)$, If $A \in \Phi_l^b(X)$(resp.$\Phi_r^b(X)$) and $B \in \Phi_l^b(X)$(resp.$\Phi_r^b(X)$) then $AB \in \Phi_l^b(X)$(resp.$\Phi_r^b(X)$) and,*

$$i(A+B) = i(A) + i(B). \qquad ♢$$

We close with the following Lemma.

Lemma 2.6. [8, Lemma 3.4] *Let $A \in \mathcal{L}(X)$.*

(i) If $AB \in \Phi_l^b(X)$ then $B \in \Phi_l^b(X)$.

(ii) If $AB \in \Phi_r^b(X)$ then $A \in \Phi_r^b(X)$.

Definition 2.7. *Let X be a Banach space.*

(i) *An operator $A \in \mathcal{L}(X)$ is said to have a left Fredholm inverse if there exists $A_l \in \mathcal{L}(X)$ such that $I - A_l A \in \mathcal{K}(X)$.*

(ii) *An operator $A \in \mathcal{L}(X)$ is said to have a right Fredholm inverse if there exists $A_r \in \mathcal{L}(X)$ such that $I - AA_r \in \mathcal{K}(X)$.* ♢

We know by the classical theory of Fredholm operators, see for example [19], that A belong to $\Phi(X)$ if it possesses a left, right or two-sided Fredholm inverse, respectively.

We define these sets $\mathcal{I}nvF^l_A(X)$ and $\mathcal{I}nvF^r_A(X)$ by:

$$\mathcal{I}nv^F_{A,l}(X) := \{A_l \in \mathcal{L}(X) : A_l \text{ is a left Fredholm inverse of } A\},$$

$$\mathcal{I}nv^F_{A,r}(X) := \{A_r \in \mathcal{L}(X) : A_r \text{ is a right Fredholm inverse of } A\}.$$

Definition 2.8. *An operator $S \in \mathcal{L}(X, Y)$ is to be strictly singular if for every infinite dimensional subspace M of X, the restriction of S to M is not a homeomorphism.*

Let $\mathcal{S}(X, Y)$ denote the set of strictly singular operators from X into Y. Note that $\mathcal{S}(X, Y)$ is a closed subspace of $\mathcal{L}(X, Y)$. In general, strictly singular operators are not compact (see [13, 14]) and if $X = Y, \mathcal{S}(X)$ is a closed two-sided ideal of $\mathcal{L}(X)$ containing $\mathcal{K}(X)$. If X is a Hilbert space, then $\mathcal{K}(X) = \mathcal{S}(X)$. For basic properties of strictly singular operators, we refer to [14, 19].

Definition 2.9. *An minimal polynomial P is the unitary polynomial of smaller degree that cancels an endomorphism, that is to say a linear application of a vector space in itself.*

In the following, we define the set of polynomially strict singular operators will denote by $\mathcal{P}_{\mathcal{S}}$, as follows:

$\mathcal{P}_{\mathcal{S}} = \{A \in \mathcal{L}(X)$, such that there exists a nonzero complex polynomial $P(z) := \sum_{k=0}^{p} a_k z^k$, satisfying $P\left(\frac{1}{n}\right) \neq 0, \forall n \in \mathbb{Z}^*$ and $P(A) \in \mathcal{S}(X)\}$.

Polynomially Inessential Operators

We now introduce and study the class of *polynomially inessential operators*, which provides a natural polynomial extension of the ideal of inessential operators. This class contains, as particular cases, both compact and polynomially strictly singular operators.

Definition 2.10 (Polynomially Inessential Operators). *Let X be a complex Banach space. We denote by $\mathcal{IIO}(X)$ the ideal of inessential operators on X, that is,*

$$\mathcal{IIO}(X) = \{T \in \mathcal{L}(X) : I - ST \text{ is Fredholm for every } S \in \mathcal{L}(X)\}.$$

An operator $A \in \mathcal{L}(X)$ is said to be polynomially inessential *if there exists a nonzero complex polynomial $P(z) = a_0 + a_1 z + \cdots + a_n z^n$ such that*

$$P(A) \in \mathcal{IIO}(X).$$

The set of all polynomially inessential operators on X is denoted by

$$\mathcal{PIIO}(X) := \{A \in \mathcal{L}(X) : \exists P \in \mathbb{C}(X) \setminus \{0\},\ P(A) \in \mathcal{IIO}(X)\}.$$

Remark 2.11. By definition, we clearly have

$$\mathcal{P}_{\mathcal{S}}(X) \subset \mathcal{PIIO}(X),$$

where $PS(X)$ denotes the set of polynomially strictly singular operators. Consequently, $\mathcal{PIIO}(X)$ also contains $S(X)$ and $K(X)$. Thus, $\mathcal{PIIO}(X)$ is a strictly larger ideal than these classes in general Banach spaces.

Theorem 2.12 (Basic Properties of $\mathcal{PIIO}(X)$). *Let X be a complex Banach space. Then the following properties hold:*

1. *$\mathcal{PIIO}(X)$ is a closed two-sided ideal of $\mathcal{L}(X)$.*
2. *The following inclusions hold:*
$$K(X) \subset S(X) \subset \mathcal{P}_{\mathcal{S}}(X) \subset \mathcal{PIIO}(X) \subset \mathcal{IIO}(X) \subset \mathcal{L}(X).$$
3. *If X is a Hilbert space, then*
$$K(X) = S(X) = \mathcal{P}_{\mathcal{S}}(X) = \mathcal{PIIO}(X) = \mathcal{IIO}(X).$$
4. *$\mathcal{PIIO}(X)$ is norm-closed: if $A_n \in \mathcal{PIIO}(X)$ and $A_n \to A$ in operator norm, then $A \in \mathcal{PIIO}(X)$.*
5. *$\mathcal{PIIO}(X)$ is stable under compact perturbations: if $A \in \mathcal{PIIO}(X)$ and $K \in K(X)$, then $A + K \in \mathcal{PIIO}(X)$.*

Proof. (1) Let $A \in \mathcal{PIIO}(X)$ and $B, C \in \mathcal{L}(X)$. By definition, there exists $P \in \mathbb{C}[z] \setminus \{0\}$ such that $P(A) \in \mathcal{IIO}(X)$. Since $\mathcal{IIO}(X)$ is a two-sided ideal of $\mathcal{L}(X)$, we have $BP(A)C \in \mathcal{IIO}(X)$. As $BP(A)C = Q(BAC)$ for some polynomial Q, it follows that $BAC \in \mathcal{PIIO}(X)$.

(2) The inclusion chain follows directly from the definitions and the fact that $\mathcal{IIO}(X)$ contains both $S(X)$ and $K(X)$ (see [23, 21]).

(3) If X is a Hilbert space, it is known that $K(X) = S(X) = \mathcal{IIO}(X)$ (see [21]). Hence, all intermediate polynomial extensions coincide.

(4) The norm-closedness follows from the closedness of $\mathcal{IIO}(X)$ in $\mathcal{L}(X)$ and the continuity of the polynomial functional calculus.

(5) Compact perturbations preserve inessentiality, i.e., if $T \in \mathcal{IIO}(X)$ and $K \in K(X)$, then $T + K \in \mathcal{IIO}(X)$ (see [1]). Applying this to $P(A)$ yields $P(A + K) \in \mathcal{IIO}(X)$, so $A + K \in \mathcal{IIO}(X)$. ☐

Remark 2.13. The class $\mathcal{PIIO}(X)$ can be seen as a *polynomial extension of the inessential ideal*, and therefore as a natural setting for the study of *polynomial perturbations of Fredholm operators*. In particular, if $A \in \mathcal{PIIO}(X)$ and $P(A) \in \mathcal{IIO}(X)$, then $0 \in \sigma_e(P(A))$, which implies that the essential and Browder essential spectra of $P(A)$ coincide at 0.

Proposition 2.14. *Let $A \in \mathcal{PIIO}(X)$, that is, there exists a nonzero complex polynomial*

$$P(z) = a_0 + a_1 z + \cdots + a_p z^p$$

such that $P(A) \in \mathcal{IIO}(X)$, where $\mathcal{IIO}(X)$ denotes the ideal of inessential operators on the Banach space X. Let $\lambda \in \mathbb{C}$ be such that $P(\lambda) \neq 0$. Then the following hold:

(i) *The element $\pi_{\mathcal{IIO}}(A) - \lambda I$ is invertible in the quotient algebra $\mathcal{L}(X)/\mathcal{IIO}(X)$, where $\pi_{\mathcal{IIO}} : \mathcal{L}(X) \to \mathcal{L}(X)/\mathcal{IIO}(X)$ is the canonical projection.*

(ii) *Consequently, $A - \lambda I$ is* invertible modulo *$\mathcal{E}(X)$, that is, $A - \lambda I$ is a Fredholm element relative to the ideal $\mathcal{IIO}(X)$, and its relative index is zero.*

(iii) *If, in addition, $\mathcal{IIO}(X) \subset K(X)$ (for instance, if X is a Hilbert space), then $A - \lambda I$ is a Fredholm operator in the classical sense and its Fredholm index is zero.*

Proof. Let $\lambda \in \mathbb{C}$ with $P(\lambda) \neq 0$. Using the standard polynomial division, we can write

$$P(z) = P(\lambda) + (z - \lambda)Q(z),$$

where Q is a polynomial of degree $p - 1$.

Applying the polynomial functional calculus to A, we get

$$P(A) = P(\lambda)I + (A - \lambda I)Q(A).$$

Since $P(A) \in \mathcal{IIO}(X)$ by hypothesis, its class in the quotient algebra

$$B := \mathcal{L}(X)/\mathcal{IIO}(X)$$

vanishes:

$$\pi_{\mathcal{IIO}}(P(A)) = 0.$$

Hence, in B, we have the relation

$$0 = P(\lambda)I_B + (\pi_{\mathcal{IIO}}(A) - \lambda I_B)Q(\pi_{\mathcal{IIO}}(A)).$$

Since $P(\lambda) \neq 0$ is an invertible scalar, we can rearrange this equation to obtain

$$(\pi_{\mathcal{IIO}}(A) - \lambda I_B)\left(-\frac{1}{P(\lambda)}Q(\pi_{\mathcal{IIO}}(A))\right) = I_B.$$

Thus, $\pi_{\mathcal{IIO}}(A) - \lambda I_B$ admits a right inverse in B. Repeating the argument with the polynomial identity

$$P(z) = P(\lambda) + Q_1(z)(z - \lambda)$$

provides a left inverse as well. Therefore, $\pi_{\mathcal{IIO}}(A) - \lambda I_B$ is invertible in the quotient algebra B. This proves (i).

Statement (ii) follows immediately, since invertibility in the quotient algebra $IIO(X)/\mathcal{E}(X)$ means that there exists $B \in \mathcal{L}(X)$ such that

$$(A - \lambda I)B - I \in \mathcal{IIO}(X) \quad \text{and} \quad B(A - \lambda I) - I \in \mathcal{IIO}(X).$$

By definition, this means that $A - \lambda I$ is invertible modulo $\mathcal{IIO}(X)$, or equivalently, that it is a Fredholm element relative to $\mathcal{IIO}(X)$. The index of a Fredholm element in a quotient algebra is always zero.

Finally, for (iii), if $\mathcal{IIO}(X) \subset K(X)$, then invertibility modulo $\mathcal{IIO}(X)$ implies invertibility modulo $K(X)$, since any inverse modulo $\mathcal{IIO}(X)$ also works modulo the smaller ideal $K(X)$. Hence, $A - \lambda I$ is Fredholm in the classical sense and $\text{ind}(A - \lambda I) = 0$. In particular, this holds whenever X is a Hilbert space, where $\mathcal{IIO}(X) = K(X)$. □

Remark 2.15. The above result generalizes the classical statement for polynomially strictly singular operators (see [7, Corollary 2.1]), where the ideal $S(X)$ is replaced by $\mathcal{IIO}(X)$. In this more general setting, the conclusion concerns invertibility modulo $\mathcal{IIO}(X)$, which coincides with the usual Fredholm property only when $\mathcal{IIO}(X) = K(X)$, for instance on Hilbert spaces.

3. Stability of essential pseudospectra by means of polynomially inessential perturbations of operators

The following theorem provides a practical criterion for the stability of some essential pseudospectra for perturbed linear operators.

Theorem 3.1. *Let $\varepsilon > 0$ and consider $A, B \in \mathcal{C}(X)$. Assume that there are $A_0,\ B_0 \in \mathcal{L}(X)$ and $S_1,\ S_2 \in \mathcal{PIIO}(X)$ such that*

$$AA_0 = I - S_1, \tag{3.1}$$

$$BB_0 = I - S_2. \tag{3.2}$$

(i) *If $0 \in \Phi_A \cap \Phi_B, A_0 - B_0 \in \mathcal{F}_+(X)$ and $i(A) = i(B)$ then*

$$\sigma_{\text{eap},\varepsilon}(A) = \sigma_{\text{eap},\varepsilon}(B). \tag{3.3}$$

(ii) *If $0 \in \Phi_A \cap \Phi_B, A_0 - B_0 \in \mathcal{F}_-(X)$ and $i(A) = i(B)$ then*

$$\sigma_{e\delta,\varepsilon}(A) = \sigma_{e\delta,\varepsilon}(B). \tag{3.4}$$

(iii) *If $A_0 - B_0 \in \mathcal{F}(X)$, then*

$$\sigma_{e,\varepsilon}(A) = \sigma_{e,\varepsilon}(B).$$

If, further, $0 \in \Phi_A \cap \Phi_B$ such that $i(A) = i(B)$, then

$$\sigma_{w,\varepsilon}(A) = \sigma_{w,\varepsilon}(B). \tag{3.5}$$

Proof. Let λ be a complex number, Equations (3.1) and (3.2) imply

$$(\lambda - A - D)A_0 - (\lambda - B - D)B_0 = S_1 - S_2 + (\lambda - D)(A_0 - B_0). \tag{3.6}$$

(i) Let $\lambda \notin \sigma_{\text{eap},\varepsilon}(B)$, then $\lambda \in \Phi^{\varepsilon}_{+B}$ such that $i(\lambda - B - D) \le 0$, for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$. Since $B + D$ is closed and $\mathcal{D}(B + D) = \mathcal{D}(B)$ endowed with the graph norm is a Banach space denoted by X_{B+D}. We can regard $B + D$ as an operator from X_{B+D} into X. This will be denoted by $\widehat{B+D}$. Using Equation (2.1) we can show that

$$\lambda - \widehat{B+D} \in \Phi^b_+(X_B, X) \text{ and } i(\lambda - \widehat{B+D}) \leq 0.$$

Moreover, since $S_2 \in \mathcal{PIIO}(X)$, applying Proposition 2.14, we obtain $I - S_2 \in \Phi(X)$. Applying [26], Theorem 2.7, p.171 and Equation (3.2), we get $B_0 \in \Phi^b(X, X_B)$. That is $(\lambda - \widehat{B+D})B_0 \in \Phi^b_+(X)$. This asserts that $A_0 - B_0 \in \mathcal{F}_+(X)$ and taking into account Equation (3.6), $(\lambda - \widehat{A+D})A_0 \in \Phi^b_+(X)$ and

$$i((\lambda - \widehat{A+D})A_0) = i((\lambda - \widehat{B+D})B_0). \tag{3.7}$$

A similar reasoning as before combining Equations (2.1) and (3.1), Proposition 2.14 and [26], Corollary 1.6, p. 166, [26], Theorem 2.6, p. 170 shows that $A_0 \in \Phi^b(X, X_A)$ where $X_A := (\mathcal{D}(A), \|.\|_A)$. By [26], Theorem 1.4, p. 108 one sees that

$$A_0 S = I - F \text{ on } X_A,$$

where $S \in \mathcal{L}(X_A, X)$ and $F \in \mathcal{K}(X_A)$, by Equation (3.2) we have

$$(\lambda - \widehat{B+D})A_0 S = (\lambda - \widehat{A+D}) - (\lambda - \widehat{A+D})F.$$

Combining the fact that $S \in \Phi^b(X_A, X)$ with [[26], Theorem 6.6, p. 129], we show that
$(\lambda - \widehat{A+D})A_0 S \in \Phi^b_+(X_A, X)$. Following [[26], Theorem 6.3, p. 128], we derive $(\lambda - \widehat{A+D}) \in \Phi^b_+(X_A, X)$. Thus, Equation (2.1) asserts that

$$(\lambda - A - D) \in \Phi_+(X). \tag{3.8}$$

On the other hand, the assumptions $S_1, S_2 \in \mathcal{PIIO}(X)$, Equations (3.1), (3.2) and Proposition 2.1, [[26], Theorem 2.3 , p. 111] reveals that

$$i(A) + i(A_0) = i(I - S_1) = 0 \text{ and } i(B) + i(B_0) = i(I - S_2) = 0,$$

since $i(A) = i(B)$. That is $i(A_0) = i(B_0)$.

Using Equation (3.7) and [[22], Theorem 2.3, p. 111], we can write

$$i(\lambda - A - D) + i(A_0) = i(\lambda - B - D) + i(B_0).$$

Therefore

$$i(\lambda - A - D) \leq 0, \forall D \in \mathcal{L}(X), \|D\| < \varepsilon. \tag{3.9}$$

Using Equations (3.8) and (3.9), we conclude that

$$\lambda \notin \sigma_{eap,\varepsilon}(A).$$

Therefore we prove the inclusion

$$\sigma_{eap,\varepsilon}(A) \subset \sigma_{eap,\varepsilon}(B).$$

The opposite inclusion follows from symmetry and we obtain Equation (3.3).

(ii) The proof of Equation (3.4) may be checked in a similar way to that in *(i)*. It suffices to replace $\sigma_{eap,\varepsilon}(.)$, $\Phi_+(.)$, $i(.) \leq 0$, [26], Theorem 6.6, p. 129, [26], Theorem 6.3, p. 128 by $\sigma_{e\delta,\varepsilon}(.)$, $\Phi_-(.)$, $i(.) \geq 0$, [22], Theorem 5 (i), p. 150, [26], Theorem 6.7, p. 129 respectively. The details are therefore omitted.

(iii) If $\lambda \notin \sigma_{e,\varepsilon}(B)$, then $\lambda - B - D \in \Phi(X)$. Since B is closed, its domain $\mathcal{D}(B)$ becomes a Banach space X_B for the graph norm $\|.\|_B$. The use of Equation (2.1) leads to $\lambda - \widehat{B+D} \in \Phi^b(X_B, X)$. Moreover, Equation (3.2), Proposition 2.1 and [26], Theorem 5.13 reveals that $B_0 \in \Phi^b(X, X_B)$ and consequently $(\lambda - \widehat{B+D})B_0 \in \Phi^b(X)$. Following with the assumption, Equation (3.6) and [26], Theorem 5.13, leads to estimate $(\lambda - \widehat{A+D})A_0 \in \Phi^b(X)$ with

$$i[(\lambda - \widehat{A+D})A_0] = i[(\lambda - \widehat{B+D})B_0]. \tag{3.10}$$

Since $A \in \mathcal{C}(X)$, proceeding as above, Equation (3.1) implies that $A_0 \in \Phi^b(X, X_A)$. By [26], Theorem 5.4 we can write

$$A_0 S = I - F \text{ on } X_A, \tag{3.11}$$

where $S \in \mathcal{L}(X_A, X)$ and $F \in \mathcal{F}(X_A)$. Taking into account Equation (3.11) we infer that

$$(\lambda - \widehat{A+D})A_0 S = (\lambda - \widehat{A+D}) - (\lambda - \widehat{A+D})F.$$

Therefore, since $S \in \Phi^b(X_A, X)$, the use of [26], Theorem 6.6 amounts to

$$(\lambda - \widehat{A+D})A_0 S \in \Phi^b(X_A, X).$$

Applying [26],Theorem 6.3, we prove that $(\lambda - \widehat{A+D}) \in \Phi^b(X_A, X)$ and consequently

$$(\lambda - A - D) \in \Phi(X).$$

Thus $\lambda \notin \sigma_{e,\varepsilon}(A)$. This implies that $\sigma_{e,\varepsilon}(A) \subset \sigma_{e,\varepsilon}(B)$. Conversely, if $\lambda \notin \sigma_{e,\varepsilon}(A)$, we can easily derive the opposite inclusion.

Now, we prove Equation (3.5). If $\lambda \notin \sigma_{w,\varepsilon}(B)$, then, $\lambda \in \Phi_B^\varepsilon$ and $i(\lambda - B - D) = 0$, for all $D \in \mathcal{L}(X)$ with $\|D\| < \varepsilon$. On the other hand, since $S_1, S_2 \in \mathcal{PIIO}(X)$ and $i(A) = i(B) = 0$, using the Atkinson theorem, we obtain $i(A_0) = i(B_0) = 0$. This together with Equation (3.10) gives $i(\lambda - \widehat{A+D}) = i(\lambda - \widehat{B+D})$. Consequently $i(\lambda - A - D) = 0$, for all $D \in \mathcal{L}(X)$ with $\|D\| < \varepsilon$.. Hence $\lambda \notin \sigma_{w,\varepsilon}(A)$, which proves the inclusion $\sigma_{w,\varepsilon}(A) \subset \sigma_{w,\varepsilon}(B)$. The opposite inclusion follows by symmetry.

□

In the following theorems, we give some perturbation results of the pseudo left, pseudo right Fredholm and pseudo left, pseudo right Weyl spectra for a bounded linear operator in a Banach space.

Theorem 3.2. *Let A and B be two operators in $\mathcal{L}(X)$ and $\lambda \in \mathbb{C}$. The following statements hold:*

(i) *Assume that $\lambda - A \in \Phi_l(X)$ and for all $D \in \mathcal{L}(X)$ with $\|D\| < \varepsilon$, there exists $A_l \in \mathcal{I}nv^F_{\lambda - A - D, l}(X)$ such that $BA_l \in \mathcal{PIIO}(X)$, then*

$$\sigma^l_{e,\varepsilon}(A + B) \subseteq \sigma^l_{e,\varepsilon}(A).$$

(ii) *Assume that* $\lambda - A \in \Phi_r(X)$ *and for all* $D \in \mathcal{L}(X)$ *with* $\|D\| < \varepsilon$, *there exists* $A_r \in \mathcal{I}nv^F_{\lambda-A-D,r}(X)$ *such that* $A_rB \in \mathcal{PIIO}(X)$, *then*

$$\sigma^r_{e,\varepsilon}(A+B) \subseteq \sigma^r_{e,\varepsilon}(A).$$

Proof. (i) Let $\lambda \notin \sigma^{\text{left}}_{\text{e},\varepsilon}(A)$, $\lambda - A - D \in \Phi^\varepsilon_l(X)$. As A_l is a left Fredholm inverse of $\lambda - A - D$, for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$. then by Lemma 2.3 there exists a compact operator $K \in \mathcal{K}(X)$ such that

$$A_l(\lambda - A - D) + K = I.$$

Then, we can write

$$\lambda - A - B - D = (I - BA_l)(\lambda - A - D) - BK. \tag{3.12}$$

Using the fact that $BA_l \in \mathcal{PIIO}(X)$ and according to Proposition 2.14, we have $I - BA_l \in \Phi(X)$. Consequently, by Lemma 2.5 we get

$$(I - BA_l)(\lambda - A - D) \in \Phi_l(X),\ \forall D \in \mathcal{L}(X),\ \|D\| < \varepsilon.$$

Thus, combining the fact that $BK \in \mathcal{K}(X)$ with the use of Equation (3.12) and Lemma 2.4, we have $\lambda-A-B-D \in \Phi_l(X)$, for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$. Therefore, $\lambda \notin \sigma^l_{\text{e},\varepsilon}(A+B)$ as required.

(ii) Let $\lambda \notin \sigma^r_{\text{e},\varepsilon}(A)$, then $\lambda - A - D \in \Phi_r(X)$, for all $D \in \mathcal{L}(X)$ such that $\|D\| < \varepsilon$. Since A_r is a right Fredholm inverse of $\lambda - A - D$. From Lemma 2.3 we infer there exists a compact operator $K \in \mathcal{K}(X)$ such that

$$(\lambda - A - D)A_\text{r} = I - K\ \forall D \in \mathcal{L}(X),\ \|D\| < \varepsilon.$$

Then, we can write $\lambda - A - B - D$ with the following form

$$\lambda - A - B - D = (\lambda - A - D)(I - A_\text{r}B) - KB,\ \forall D \in \mathcal{L}(X),\ \|D\| < \varepsilon. \tag{3.13}$$

Since $A_\text{r}B \in \mathcal{PIIO}(X)$ then, according to Proposition 2.14, we have $I - A_\text{r}B \in \Phi(X)$. Consequently, by Lemma 2.5, we get

$$(\lambda - A - D)(I - A_\text{r}B) \in \Phi_\text{r}(X),\ \forall D \in \mathcal{L}(X),\ \|D\| < \varepsilon.$$

On the other hand, from Equation (3.13) and Lemma 2.4 and the fact $BK \in \mathcal{K}(X)$ we show that $\lambda - A - B - D \in \Phi_\text{r}(X)$, for all $D \in \mathcal{L}(X)$ and $\|D\| < \varepsilon$. We deduce that, $\lambda \notin \sigma^r_{\text{e},\varepsilon}(A+B)$.

□

Theorem 3.3. *Let* A *and* B *be two operators in* $\mathcal{L}(X)$ *and* $\lambda \in \mathbb{C}$. *The following statements hold:*

(i) *Assume that* $\lambda - A \in \Phi_l(X)$ *and for all* $D \in \mathcal{L}(X)$ *with* $\|D\| < \varepsilon$, *there exists* $A_l \in \mathcal{I}nv^F_{\lambda-A-D,l}(X)$ *such that* $BA_l \in \mathcal{PIIO}(X)$, *then*

$$\sigma^l_{e,\varepsilon}(A+B) \subseteq \sigma^l_{e,\varepsilon}(A).$$

(ii) *Assume that* $\lambda - A \in \Phi_r(X)$ *and for all* $D \in \mathcal{L}(X)$ *with* $\|D\| < \varepsilon$, *there exists* $A_r \in \mathcal{I}nv^F_{\lambda-A-D,r}(X)$ *such that* $A_rB \in \mathcal{PIIO}(X)$, *then*

$$\sigma^r_{e,\varepsilon}(A+B) \subseteq \sigma^r_{e,\varepsilon}(A).$$

Proof. (i) Assume that $\lambda \notin \sigma^{l}_{\mathrm{w},\varepsilon}(A)$, then we have $\lambda - A - D \in \Phi_l(X)$ and $i(\lambda - A - D) \leq 0$. A similar reasoning as above gives $\lambda - A - B - D \in \Phi_l(X)$ and it suffices to prove that $i(\lambda - A - B - D) \leq 0$. Since $BK \in \mathcal{K}(X)$ then, Using Equation 3.12 together with Lemmas 2.4 and 2.5, we obtain that

$$i(\lambda - A - B - D) = i\left(I - BA_l\right) + i(\lambda - A - D).$$

Now, Since $BA_l \in \mathcal{PIIO}(X)$, we get by Proposition 2.14, that $i\left(I - BA_l\right) = 0$. We deduce that

$$i(\lambda - A - B - D) = i\left(\lambda - A - D\right) \leq 0.$$

Finally, we conclude that $\lambda - A - B - D \in \mathcal{W}_l(X)$, which entails that $\lambda \notin \sigma^{\mathrm{l}}_{\mathrm{w},\varepsilon}(A + B)$.

(ii) with the same reasoning of (i). Let $\lambda \notin \sigma^{\mathrm{r}}_{\mathrm{w},\varepsilon}(A)$, then we have $\lambda - A - D \in \Phi_r(X)$ and $i(\lambda - A - D) \geq 0$. Proceeding as the proof above, we establish that $\lambda - A - B - D \in \Phi_r(X)$ and $i(\lambda - A - B - D) \geq 0$. Therefore, $\lambda - A - B - D \in \mathcal{W}_r(X)$ and we deduce that $\lambda \notin \sigma^{\mathrm{r}}_{\mathrm{w},\varepsilon}(A + B)$.

□

Remark 3.4. The results of Theorems 3.1, 3.2 and 3.3 are extensions and an improvements of the results of in [2, 3, 4, 5, 6] to a large class of polynomially strict singular operators. ♢

4. Characterization essential spectrum of two linear bounded operators

This section aims to carry out a new criterion allowing us to investigate some spectral analysis of the sum of two linear bounded operators. We begin by giving the following lemma when we need it in the sequel.

Lemma 4.1. [8, Lemma 4.1] *Let $A \in \mathcal{L}(X)$.*

(i) If $C\sigma^l_e(A)$ is connected, then

$$\sigma^l_e(A) = \sigma^l_w(A).$$

(ii) If $C\sigma^r_e(A)$ is connected, then

$$\sigma^r_e(A) = \sigma^r_w(A).$$

Theorem 4.2. *Let A, $B \in \mathcal{L}(X)$ and $\lambda \in \mathbb{C}^*$. For all $D \in \mathcal{L}(X)$ with $\|D\| < \varepsilon$, the following statements hold:*

(i) *Assume that the subsets $C\sigma^l_e(A)$ and $C\sigma^l_e(B)$ are connected, $-\lambda^{-1}ABQ_l \in \mathcal{PIIO}(X)$ and $-\lambda^{-1}BAQ_l \in \mathcal{PIIO}(X)$, for every $Q_l \in \mathcal{I}nv^F_{\lambda-A-B-D,l}(X)$, then we have:*

$$\left[\sigma^l_w(A) \cup \sigma^l_{w,\varepsilon}(B)\right] \setminus \{0\} \subseteq \sigma^l_{w,\varepsilon}(A + B) \setminus \{0\}.$$

(ii) *Assume that the subsets $C\sigma^r_e(A)$ and $C\sigma^r_e(B)$ are connected, $-\lambda^{-1}Q_rAB \in \mathcal{PIIO}(X)$ and $-\lambda^{-1}Q_rBA \in \mathcal{PIIO}(X)$, for every $Q_r \in \mathcal{I}nv^F_{\lambda-A-B-D,r}(X)$, then we have:*

$$\left[\sigma_w^r(A) \cup \sigma_{w,\varepsilon}^r(B)\right] \setminus \{0\} \subseteq \sigma_{w,\varepsilon}^r(A+B) \setminus \{0\}.$$

(iii) *Assume that the subsets* $C\sigma_e^l(A)$, $C\sigma_e^l(B)$, $C\sigma_e^r(A)$ *and* $C\sigma_e^r(B)$ *are connected,* $-\lambda^{-1}ABQ_l \in \mathcal{PIIO}(X)$, $-\lambda^{-1}BAQ_l \in \mathcal{PIIO}(X)$, $-\lambda^{-1}Q_rAB \in \mathcal{PIIO}(X)$ *and* $-\lambda^{-1}Q_rBA \in \mathcal{PIIO}(X)$, *for* $Q_l \in \mathcal{I}nv^F_{\lambda-A-B-D,l}(X)$ *and* $Q_r \in \mathcal{I}nv^F_{\lambda-A-B-D,r}(X)$, *then we have:*

$$\left[\sigma_w(A) \cup \sigma_{w,\varepsilon}(B)\right] \setminus \{0\} \subseteq \sigma_{w,\varepsilon}(A+B) \setminus \{0\}. \qquad \diamondsuit$$

Proof.

Firstly we note two equality which is used repeatedly

$$(\lambda - A)(\lambda - B - D) = A(B+D) + \lambda(\lambda - A - B - D). \tag{4.1}$$

$$(\lambda - B - D)(\lambda - A) = (B+D)A + \lambda(\lambda - A - B - D). \tag{4.2}$$

(i) Let $\lambda \notin \sigma_{w,\varepsilon}^l(A+B) \cup \{0\}$ so we have $\lambda - A - B - D \in \Phi_l(X)$ and $i(\lambda - A - B - D) \leq 0$. Then following to the Lemma 2.3 there exist $Q_l \in \mathcal{L}(X)$ and $K \in \mathcal{K}(X)$ such that $Q_l(\lambda - A - B - D) = I - K$.

So when we use Equation (4.1) we obtain

$$\begin{array}{rcl} (\lambda - A)(\lambda - B - D) & = & A(B+D) + \lambda(\lambda - A - B - D). \\ & = & AB[Q_l(\lambda - A - B - D) + K] + \lambda(\lambda - A - B - D). \\ & = & [ABQ_l + \lambda I](\lambda - A - B - D) + ABK. \\ & = & \lambda[\lambda^{-1}ABQ_l + I](\lambda - A - B - D) + ABK, \end{array}$$

Since $\lambda[\lambda^{-1}ABQ_l + I] \in \Phi(X)$ and $(\lambda - A - B - D) \in \Phi_l(X)$ it follows from Proposition 2.1 that $\lambda[\lambda^{-1}ABQ_l + I](\lambda - A - B - D) \in \Phi_l(X)$. Since $ABK \in \mathcal{K}(X)$, this implies by the use of Lemma 2.4 that

$$\lambda[\lambda^{-1}ABQ_l + I](\lambda - A - B - D) + ABQ_lK \in \Phi_l(X).$$

So $(\lambda - A)(\lambda - B - D) \in \Phi_l(X)$ and as a direct consequence of Lemma 2.6 we obtain

$$\lambda - B - D \in \Phi_l(X), \forall D \in \mathcal{L}(X), \|D\| < \varepsilon. \tag{4.3}$$

In the other hand, when we use the Equation (4.2) we have

$$\begin{array}{rcl} (\lambda - B - D)(\lambda - A) & = & BA + \lambda(\lambda - A - B - D), \\ & = & BA[Q_l(\lambda - A - B - D) + K] + \lambda(\lambda - A - B - D), \\ & = & [BAQ_l + \lambda I](\lambda - A - B - D) + BAK, \\ & = & \lambda[\lambda^{-1}BAQ_l + I](\lambda - A - B - D) + BAK. \end{array}$$

Since $\lambda[\lambda^{-1}BAQ_l + I] \in \Phi(X)$ and $(\lambda - A - B - D) \in \Phi_l(X)$ it follows from Proposition 2.1 that

$$\lambda[\lambda^{-1}BAQ_l + I](\lambda - A - B - D) \in \Phi_l(X).$$

Obviously, since $BAK \in \mathcal{K}(X)$ and applying Lemma 2.4, we find that

$$\lambda[\lambda^{-1}BAQ_l + I](\lambda - A - B - D) + BAK \in \Phi_l(X).$$

So $(\lambda - B - D)(\lambda - A) \in \Phi_l(X)$. Therefore using Lemma 2.6 we obtain

$$\lambda - A \in \Phi_l(X). \tag{4.4}$$

Now, to check the index we must have a discussion according to the sign, thus using the above we have

$$i(\lambda - A) + i(\lambda - B - D) = i(\lambda - A - B - D) \leq 0.$$

Case1: If $i(\lambda - A) \leq 0$

Using Lemma 4.1 the index $i(\lambda - B - D)$ must be negative.Therefore adding this condition to Equations (4.3) and (4.4) we obtain

$$\lambda \notin \left[\sigma_w^l(A) \cup \sigma_{w,\varepsilon}^l(B)\right] \cup \{0\}.$$

Case2: If $i(\lambda - B - D) \leq 0$

Following to Lemma 4.1 the index $i(\lambda - A)$ must be negative.

Then adding this condition to Equations (4.3) and (4.4) we assert

$$\lambda \notin \left[\sigma_w^l(A) \cup \sigma_{w,\varepsilon}^l(B)\right] \cup \{0\}.$$

Case3: If $i(\lambda - A) > 0$.

Following to Lemma 4.1 the index $i(\lambda - B - D)$ should be positif which contradicts the fact that $i(\lambda - A - B - D) \leq 0$.

Case4: If $i(\lambda - B - D) > 0$

Following to Lemma 4.1 the index $i(\lambda - A)$ must be positif which contradicts the fact that $i(\lambda - A - B - D) \leq 0$.

(ii) Let $\lambda \notin \sigma_{w,\varepsilon}^r(A+B) \cup \{0\}$ then $\lambda - A - B - D \in \Phi_r(X)$ and $i(\lambda - A - B - D) \leq 0$. So by Lemma 2.3 there exist $Q_r \in \mathcal{L}(X)$ and $K \in \mathcal{K}(X)$ such that $(\lambda - A - B - D)Q_r = I - K$

So following to the Equation (4.1) we have

$$\begin{aligned}(\lambda - A)(\lambda - B - D) &= AB + \lambda(\lambda - A - B - D),\\ &= [(\lambda - A - B - D)Q_r + K]AB + \lambda(\lambda - A - B - D),\\ &= (\lambda - A - B - D)[Q_r AB + \lambda I] + ABK,\\ &= \lambda(\lambda - A - B - D)[\lambda^{-1} Q_r AB + I] + ABK.\end{aligned}$$

Since $\lambda[\lambda^{-1} Q_r AB + I] \in \Phi(X)$ and $(\lambda - A - B - D) \in \Phi_r(X)$ it follows by Proposition 2.1 that

$$\lambda[\lambda^{-1} Q_r AB + I](\lambda - A - B - D) \in \Phi_r(X).$$

Since $ABK \in \mathcal{K}(X)$ then

$$\lambda[\lambda^{-1} Q_r AB + I](\lambda - A - B - D) + ABK \in \Phi_r(X).$$

So $(\lambda - A)(\lambda - B - D) \in \Phi_r(X)$, following to Lemma 2.6 we infer that

$$\lambda - A \in \Phi_r(X). \tag{4.5}$$

In the other hand, the use of Equation (4.2) assert

$$\begin{aligned}(\lambda - B - D)(\lambda - A) &= BA + \lambda(\lambda - A - B - D),\\ &= BA[(\lambda - A - B - D)Q_r + K]BA + \lambda(\lambda - A - B - D),\\ &= (\lambda - A - B - D)[Q_rBA + \lambda I] + KBA,\\ &= \lambda(\lambda - A - B - D)[\lambda^{-1}Q_rBA + I] + KBA.\end{aligned}$$

Since by hypothesis $[\lambda^{-1}Q_rBA + I] \in \Phi(X)$ and $(\lambda - A - B - D) \in \Phi_r(X)$ we have by Proposition 2.1

$$\lambda(\lambda - A - B - D)[\lambda^{-1}Q_rBA + I] \in \Phi_r(X).$$

Since $KBA \in \mathcal{K}(X)$ we obtain

$$\lambda(\lambda - A - B - D)[\lambda^{-1}Q_rBA + I] + KBA \in \Phi_r(X).$$

So $(\lambda - B - D)(\lambda - A) \in \Phi_r(X)$ then the use of Lemma 2.6 infer that

$$\lambda - B - D \in \Phi_r(X),\ \forall D \in \mathcal{L}(X) \text{ with } \|D\| < \varepsilon. \tag{4.6}$$

Now, to check the index we must have a discussion according to the sign, thus using the above we have

$$i(\lambda - A) + i(\lambda - B - D) = i(\lambda - A - B - D) \geq 0.$$

Case 1: If $i(\lambda - A) \geq 0$

Using Lemma 4.1 the index $i(\lambda - B - D)$ must be positif. Therefore adding this condition to Equations (4.5) and (4.6) we get

$$\lambda \notin \left[\sigma^r_w(A) \cup \sigma^r_{w,\varepsilon}(B)\right] \cup \{0\}.$$

Case 2: If $i(\lambda - B - D) \geq 0$.

Following to Lemma 4.1 the index $i(\lambda - A)$ must be positif.

Then adding this condition to Equations (4.3) and (4.4) we obtain

$$\lambda \notin \left[\sigma^r_w(A) \cup \sigma^r_{w,\varepsilon}(B)\right] \cup \{0\}.$$

Case 3: If $i(\lambda - A) < 0$

Following to Lemma 4.1 the index $i(\lambda - B - D)$ should be negative which contradicts the fact that $i(\lambda - A - B - D) \geq 0$.

Case 4: If $i(\lambda - B - D) < 0$

Following to Lemma 4.1 the index $i(\lambda - A)$ should be negative which contradicts the fact that $i(\lambda - A - B - D) \geq 0$.

(iii) Let $\lambda \notin \sigma_{w,\varepsilon}(A+B) \cup \{0\}$ therefore $\lambda - A - B - D \in \Phi(X)$ and $i(\lambda - A - B - D) = 0$ then there exist $Q_l, Q_r \in \mathcal{L}(X)$ and $K \in \mathcal{K}(X)$ such that $Q_l(\lambda - A - B - D) = I - K$ and $(\lambda - A - B - D)Q_r = I - K$.

Now, according to items (i) and (ii) we get

$$\left[\sigma_w(A) \cup \sigma_{w,\varepsilon}(B)\right] \setminus \{0\} \subseteq \sigma_{w,\varepsilon}(A + B) \setminus \{0\}. \qquad \square$$

Theorem 4.3. *Let A, $B \in \mathcal{L}(X)$ such that $AB = BA$ and $\lambda \in \mathbb{C}^*$. For all $D \in \mathcal{L}(X)$ with $\|D\| < \varepsilon$, the following statements hold:*

(i) *If there exists $Q_l \in \mathcal{I}nv^F_{\lambda-A-B-D,l}(X)$, such that $-\lambda^{-1}ABQ_l \in \mathcal{PIIO}(X)$ then*

$$\sigma^l_{e,\varepsilon}(A+B) \setminus \{0\} = \left[\sigma^l_e(A) \cup \sigma^l_{e,\varepsilon}(B)\right] \setminus \{0\}.$$

(ii) *If there exists $Q_r \in \mathcal{I}nv^F_{\lambda-A-B-D,r}(X)$, such that $-\lambda^{-1}Q_rAB \in \mathcal{PIIO}(X)$ then*

$$\sigma^r_{e,\varepsilon}(A+B) \setminus \{0\} = \left[\sigma^r_e(A) \cup \sigma^r_{e,\varepsilon}(B)\right] \setminus \{0\}.$$

(iii) *If there exists $Q \in \mathcal{I}nv^F_{\lambda-A-B-D,l}(X) \cap \mathcal{I}nv^F_{\lambda-A-B-D,r}(X)$, such that $-\lambda^{-1}QAB \in \mathcal{PIIO}(X)$ and $-\lambda^{-1}ABQ \in \mathcal{PIIO}(X)$ then*

$$\sigma_{e,\varepsilon}(A+B) \setminus \{0\} = \left[\sigma_e(A) \cup \sigma_{e,\varepsilon}(B)\right] \setminus \{0\}. \qquad \diamondsuit$$

Proof.

(i) Let $\lambda \notin \sigma^l_{e,\varepsilon}(A+B) \cup \{0\}$ then $\lambda - A - B - D \in \Phi_l(X)$.

We assume there exists $Q_l \in \mathcal{I}nv^F_{\lambda-A-B-D,l}(X)$, thus, using Equation (4.1) we have

$$\begin{aligned}(\lambda-A)(\lambda-B-D) &= A(B+D) + \lambda(\lambda-A-B-D),\\ &= AB[Q_l(\lambda-A-B-D)+K] + \lambda(\lambda-A-B-D),\\ &= [ABQ_l+\lambda I](\lambda-A-B-D) + ABK,\\ &= \lambda[\lambda^{-1}ABQ_l+I](\lambda-A-B-D) + ABK.\end{aligned}$$

Obviously, $-\lambda^{-1}ABQ_l \in \mathcal{PIIO}(X)$ then by Proposition 2.14 we infer that $\lambda^{-1}ABQ_l + I \in \Phi(X)$. Therefore, by Lemma 2.5 we obtain $[\lambda^{-1}ABQ_l + \lambda I](\lambda - A - B - D) \in \Phi_l(X)$.

Since $ABK \in \mathcal{K}(X)$ and by applying Lemma 2.4 we obtain

$$\lambda[\lambda^{-1}ABQ_l + I](\lambda - A - B - D) + ABK \in \Phi_l(X).$$

We conclude that

$$(\lambda-A)(\lambda-B-D) \in \Phi_l(X),\ \forall D \in \mathcal{L}(X) \text{ with } \|D\| < \varepsilon.$$

Hence, by Lemma 2.6 we deduce that

$$(\lambda - B - D) \in \Phi_l(X),\ \forall D \in \mathcal{L}(X) \text{ with } \|D\| < \varepsilon. \tag{4.7}$$

On the other hand, using the fact that $AB = BA$ and according to the Equation (4.2) we observe that

$$\begin{aligned}(\lambda-B-D)(\lambda-A) &= BA + \lambda(\lambda-A-B-D),\\ &= AB + \lambda(\lambda-A-B-D),\\ &= AB[Q_l(\lambda-A-B-D)+K] + \lambda(\lambda-A-B-D),\\ &= [ABQ_l+\lambda I](\lambda-A-B-D) + ABK,\\ &= \lambda[\lambda^{-1}ABQ_l+I](\lambda-A-B-D) + ABK.\end{aligned}$$

Using the same reasoning we conclude that $(\lambda - B - D)(\lambda - A) \in \Phi_l(X)$. Therefore, by Lemma 2.6 we deduce that

$$(\lambda - A) \in \Phi_l(X). \tag{4.8}$$

Finally, the two Equations (4.7) and (4.8) imply that $\lambda \notin \left[\sigma_e^l(A) \cup \sigma_{e,\varepsilon}^l(B)\right] \cup \{0\}$. So, we obtain

$$\left[\sigma_e^l(A) \cup \sigma_{e,\varepsilon}^l(B)\right] \setminus \{0\} \subset \sigma_{e,\varepsilon}^l(A+B) \setminus \{0\}.$$

The other inclusion is allows us to achieve equality is in [8, Theorem 4.3].

(ii) Let $\lambda \notin \sigma_{e,\varepsilon}^r(A+B) \cup \{0\}$ then $\lambda - A - B - D \in \Phi_r(X)$, for all $D \in \mathcal{L}(X)$ and $\|D\| < \varepsilon$.

We assume there exists $Q_r \in \mathcal{I}nv^F_{\lambda-A-B-D,r}(X)$ thus,

$$\begin{aligned}(\lambda - A)(\lambda - B - D) &= AB + \lambda(\lambda - A - B - D),\\ &= [(\lambda - A - B - D)Q_r + K]AB + \lambda(\lambda - A - B - D),\\ &= (\lambda - A - B - D)\lambda[\lambda^{-1}Q_r AB + I] + KAB.\end{aligned}$$

Evidently, $-\lambda^{-1}Q_r AB \in \mathcal{PIIO}(X)$ and by applying Proposition 2.14 we deduce that $\lambda^{-1}Q_r AB + I \in \Phi(X)$. Since, KAB is compact, then by Lemma 2.4 we obtain

$$(\lambda - A - B - D)\lambda[\lambda^{-1}Q_r AB + I] + KAB \in \Phi_l(X).$$

Consequently, we have $(\lambda - A)(\lambda - B - D) \in \Phi_r(X)$ and by Lemma 2.6 we infer that

$$(\lambda - A) \in \Phi_r(X). \tag{4.9}$$

Further, we have $AB = BA$ so,

$$\begin{aligned}(\lambda - B - D)(\lambda - A) &= BA + \lambda(\lambda - A - B - D),\\ &= AB + \lambda(\lambda - A - B - D),\\ &= [(\lambda - A - B - D)Q_r + K]AB + \lambda(\lambda - A - B - D),\\ &= (\lambda - A - B - D)\lambda[\lambda^{-1}Q_r AB + I] + KAB.\end{aligned}$$

Using the same reasoning we conclude that $(\lambda - B - D)(\lambda - A) \in \Phi_r(X)$. Then, by Lemma 2.6 we deduce that

$$(\lambda - B - D) \in \Phi_r(X),\ \forall D \in \mathcal{L}(X) \text{ with } \|D\| < \varepsilon. \tag{4.10}$$

Finally, the two Equations (4.9) and (4.10) imply that

$$\lambda \notin \left[\sigma_e^r(A) \cup \sigma_{e,\varepsilon}^r(B)\right] \cup \{0\}.$$

So, we obtain

$$\left[\sigma_e^r(A) \cup \sigma_{e,\varepsilon}^r(B)\right] \setminus \{0\} \subset \sigma_{e,\varepsilon}^r(A+B) \setminus \{0\}.$$

The other inclusion is allows us to achieve equality is in [8, Theorem 4.3].

(iii) Let $\lambda \notin \sigma_{e,\varepsilon}(A+B) \cup \{0\}$. Then $\lambda - A - B - D \in \Phi(X)$ means that $\lambda - A - B - D \in \Phi_l(X) \cap \Phi_r(X)$.

Now, by the hypothesis there exists $Q \in \mathcal{I}nv^F_{\lambda-A-B-D,l}(X) \cap \mathcal{I}nv^F_{\lambda-A-B-D,r}(X)$, and by applying the results in statements (i) and (ii) we infer that $(\lambda - A - B - D) \in \Phi_r(X)$ and $(\lambda - A - B - D) \in \Phi_l(X)$, therefore $(\lambda - A - B - D) \in \Phi(X)$.

Also, using the hypothesis that $-\lambda^{-1}QAB \in \mathcal{PIIO}(X))$, $-\lambda^{-1}ABQ \in \mathcal{PIIO}(X)$ and $AB = BA$ we give us this two condition:

$$(\lambda - A)(\lambda - B - D) \in \Phi(X) \text{ and } (\lambda - B - D)(\lambda - A) \in \Phi(X).$$

Therefore, following Theorem 2.2 we obtain $(\lambda - A) \in \Phi(X)$ and $(\lambda - B - D) \in \Phi(X)$ means that $\lambda \notin \left[\sigma_e(A) \cup \sigma_{e,\varepsilon}(B)\right] \cup \{0\}$. Then we get the following inclusion

$$\left[\sigma_e(A) \cup \sigma_{e,\varepsilon}(B)\right] \setminus \{0\} \subseteq \sigma_{e,\varepsilon}(A + B) \setminus \{0\}.$$

The other inclusion is allows us to achieve equality is in [8, Theorem 4.3]. $\square$

The same reasoning of the above theorem, we allow to obtain the result of the following result.

Theorem 4.4. *Let A, $B \in \mathcal{L}(X)$ such that $AB = BA$ and $\lambda \in \mathbb{C}^*$. For all $D \in \mathcal{L}(X)$ with $\|D\| < \varepsilon$, the following statements hold:*

(i) *If there exists $Q_l \in \mathcal{I}nv^F_{\lambda-A-B-D,l}(X)$, such that $-\lambda^{-1}ABQ_l \in \mathcal{PIIO}(X)$ then*

$$\sigma^l_{w,\varepsilon}(A + B) \setminus \{0\} = \left[\sigma^l_{w,\varepsilon}(A) \cup \sigma^l_{w,\varepsilon}(B)\right] \setminus \{0\}.$$

(ii) *If there exists $Q_r \in \mathcal{I}nv^F_{\lambda-A-B-D,r}(X)$, such that $-\lambda^{-1}Q_rAB \in \mathcal{PIIO}(X)$ then*

$$\sigma^r_{w,\varepsilon}(A + B) \setminus \{0\} = \left[\sigma^r_w(A) \cup \sigma^r_{w,\varepsilon}(B)\right] \setminus \{0\}.$$

(iii) *If there exists $Q \in \mathcal{I}nv^F_{\lambda-A-B-D,l}(X) \cap \mathcal{I}nv^F_{\lambda-A-B-D,r}(X)$, such that $-\lambda^{-1}QAB \in \mathcal{PIIO}(X)$ and $-\lambda^{-1}ABQ \in \mathcal{PIIO}(X)$ then*

$$\sigma_{w,\varepsilon}(A + B) \setminus \{0\} = \left[\sigma_w(A) \cup \sigma_{w,\varepsilon}(B)\right] \setminus \{0\}. \qquad \diamondsuit$$

5. Application to bounded 2×2 block operator matrices forms

The objective of this section is to utilize Theorem 4.3 from Section 4 in order to analyze the pseudo left (right)-Fredholm essential spectra of the given operator matrix.

Let X_1 and X_2 be two Banach spaces and consider the 2×2 block operator matrices defined on $X_1 \times X_2$ by:

$$\mathcal{M} := \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}.$$

where $A \in \mathcal{L}(X_1)$, $B \in \mathcal{L}(X_2)$, $C \in \mathcal{L}(X_2, X_1)$ and $D \in \mathcal{L}(X_1, X_2)$.

Next, we define the following matrix:

$$\mathfrak{D} = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix},$$

where $D_1 \in \mathcal{L}(X_1)$, $D_2 \in \mathcal{L}(X_2)$ and $\|\mathfrak{D}\| = \max\{\|D_1\|, \|D_2\|\}$.

In the following theorem, we seek the pseudo left (right)-Fredholm essential spectra of Matrix $\mathcal{M}_C$.

Theorem 5.1. *Let the* 2×2 *block operator matrix* $\mathcal{M}_C$ *and* $\varepsilon > 0$. *In all that follows we will make the following assumptions:*

$$\mathcal{H} : \begin{cases} \|\mathfrak{D}\| < \varepsilon, \\ AC = CB, \\ A \in \Phi(X), B \in \Phi(X), \\ CB \in \mathcal{IIO}(X_1 \times X_2). \end{cases}$$

Then, we have that

(*i*) $\sigma_{e,\varepsilon}^{left}(\mathcal{M}_C)\backslash\{0\} \subseteq \left[\sigma_{e,\varepsilon}^{left}(A) \cup \sigma_{e,\varepsilon}^{left}(B)\right]\backslash\{0\}$.

(*ii*) $\sigma_{e,\varepsilon}^{right}(\mathcal{M}_C)\backslash\{0\} \subseteq \left[\sigma_{e,\varepsilon}^{right}(A) \cup \sigma_{e,\varepsilon}^{right}(B)\right]\backslash\{0\}$.

Proof. We begin by presenting the polynomial P in the specified format:

$$\begin{array}{cccc} P & : & \mathbb{R}^2 & \to & \mathbb{R} \\ & & (x,y) & \mapsto & P(x,y) = x.y \end{array}$$

We can write

$$\mathcal{M} := \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = \begin{pmatrix} 0 & C \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

$= \ \mathcal{M}_{\mathcal{C}} \ + \ \mathcal{M}_{A,B}$.

We have:

$$P(\mathcal{M}_{\mathcal{C}}, \mathcal{M}_{A,B}) = \mathcal{M}_{\mathcal{C}}.\mathcal{M}_{A,B} = \begin{pmatrix} 0 & CB \\ 0 & 0 \end{pmatrix}.$$

it follows from the hypothesis (H) that:

$$P(\mathcal{M}_{\mathcal{C}}, \mathcal{M}_{A,B}) \in \mathcal{IIO}(X_1 \times X_2), \text{and } \mathcal{M}_{\mathcal{C}}.\mathcal{M}_{A,B} \in \mathcal{PIIO}_T(X).$$

Moreover we have $A + B \in \Phi(X)$ then there exist $A_0 \in \mathcal{L}(X)$ and $K \in \mathcal{K}(X)$ such that $A_0(A+B) = I - K$. Then

$$A_0(A+B+D) = I - K', \text{with } K' \in \mathcal{K}(X).$$

Using Theorem 4.3, we obtain that

(*i*)

$$\begin{aligned} \sigma_{e,\varepsilon}^{left}(\mathcal{M}) \setminus \{0\} &= \sigma_{e,\varepsilon}^{left}(\mathcal{M}_C + \mathcal{M}_{A,B}) \setminus \{0\} \\ &= \left[\sigma_{e}^{left}(\mathcal{M}_C) \cup \sigma_{e,\varepsilon}^{left}(\mathcal{M}_{A,B})\right] \setminus \{0\}. \end{aligned}$$

(*ii*)

$$\begin{aligned} \sigma_{e,\varepsilon}^{right}(\mathcal{M}) \setminus \{0\} &= \sigma_{e,\varepsilon}^{right}(\mathcal{M}_C + \mathcal{M}_{A,B}) \setminus \{0\} \\ &= \left[\sigma_{e}^{right}(\mathcal{M}_C) \cup \sigma_{e,\varepsilon}^{right}(\mathcal{M}_{A,B})\right] \setminus \{0\}. \end{aligned}$$

Furthermore, we can readily demonstrate $\sigma_e^{left}(\mathcal{M}_\mathcal{C}) = \sigma_e^{right}(\mathcal{M}_\mathcal{C}) = \{0\}$. Consequently, applying [[3], Theorem 4 (i)], we show that

$$\begin{aligned}\sigma_{e,\varepsilon}^{left}(\mathcal{M}) \setminus \{0\} &= \left[\sigma_e^{left}(\mathcal{M}_C) \cup \sigma_{e,\varepsilon}^{left}(\mathcal{M}_{A,B})\right] \setminus \{0\} \\ &= \left[\{0\} \cup \sigma_{e,\varepsilon}^{left}(\mathcal{M}_{A,B})\right] \setminus \{0\} \\ &= \sigma_{e,\varepsilon}^{left}(\mathcal{M}_{A,B}) \\ &\subseteq \left[\sigma_{e,\varepsilon}^{left}(A) \cup \sigma_{e,\varepsilon}^{left}(B)\right] \setminus \{0\}.\end{aligned}$$

Similarly, we have:

$$\begin{aligned}\sigma_{e,\varepsilon}^{right}(\mathcal{M}) \setminus \{0\} &= \left[\sigma_e^{right}(\mathcal{M}_C) \cup \sigma_{e,\varepsilon}^{right}(\mathcal{M}_{A,B})\right] \setminus \{0\} \\ &= \left[\{0\} \cup \sigma_{e,\varepsilon}^{right}(\mathcal{M}_{A,B})\right] \setminus \{0\} \\ &= \sigma_{e,\varepsilon}^{right}(\mathcal{M}_{A,B}) \\ &\subseteq \left[\sigma_{e,\varepsilon}^{right}(A) \cup \sigma_{e,\varepsilon}^{right}(B)\right] \setminus \{0\}.\end{aligned}$$

□

Conclusion

In this article, we have explored the behavior of essential pseudospectra within the framework of polynomially inessential operators. Building on foundational concepts from Fredholm theory, we have developed new stability results and characterizations for the left and right Weyl and Fredholm essential pseudospectra. These insights not only deepen the understanding of spectral properties for such operators but also provide tools for analyzing operator sums. Furthermore, the extension of these results to 2×2 block operator matrices highlights the robustness of the theory and its applicability in more structured operator settings. These findings open new avenues for future research in the spectral analysis of operator matrices and perturbation theory.

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