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On a class of nonlinear discrete problems of Kirchhoff type

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Abstract. In view of variational methods and critical points theory, we study the existence of solutions for a discrete boundary value problem, which is a discrete variant of a continuous $(p_1(x), p_2(x))$-Kirchhoff-type problem, with a real parameter $\lambda > 0$.

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1. Introduction

In this paper, we are concerned with the following anisotropic discrete problem of Kirchhoff-type

$$(P)\quad \begin{cases} -\displaystyle\sum_{i=1}^{2} M_i\left(\phi_i(u(r))\right)\Delta(|\triangle u(r-1)|^{p_i(r-1)-2}) = \lambda f(r,u(r)), \quad r\in[1,N]_{\mathbb{Z}}, \\ u(0)=u(N+1)=0, \end{cases}$$

where $N \geq 2$ is a positive integer, $[1,N]_{\mathbb{Z}}$ is the discrete interval $[1,N]_{\mathbb{Z}} := \{1,2,\ldots,N\}$, Δ denotes the forward difference operator defined by $\Delta u(r) = u(r+1) - u(r)$, and for $i = 1, 2$

$$\phi_i(u) = \sum_{r=1}^{N+1} \frac{|\Delta u(r-1)|^{p_i(r-1)}}{p_i(r-1)}.$$

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For every fixed $r \in [0, N]_{\mathbb{Z}}$, the function $f(r, .) : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous, $M_i : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$, $i = 1, 2$, are continuous functions that satisfy some conditions which will be stated later on and $p_1, p_2 : [0, N+1]_{\mathbb{Z}} \to [2; +\infty)$ are two given functions.

Put

$$p_i^+ := \max_{r\in[0,N+1]_{\mathbb{Z}}} p_i(r), \quad p_i^- := \min_{r\in[0,N+1]_{\mathbb{Z}}} p_i(r),$$
$$P_M^+ := \max\{p_1^+, p_2^+\}, \quad p_m^- := \min\{p_1^-, p_2^-\}.$$

In recent years, a great attention has been focused on studying nonlocal equations and corresponding problems involving fractional Sobolev spaces. To be more precise, the Kirchhoff type equations involving variable exponent growth conditions have been studied in many papers and by many authors, this interest is justified by its various applications in many fields of research. In fact, there are applications concerning image restoration [8], electro-rheological fluids and stationary thermo-rheological viscous flows of non-Newtonian fluids [16, 17]. For some interesting results we refer to [1, 9] and the references therein. Problem (P) is related to the stationary problem of a model introduced by Kirchhoff [14]. More precisely, Kirchhoff presented a model given by

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0, \tag{1.1}$$

which extends the classical D'Alembert's wave equation by taking into consideration the effects of the changes in the length of the strings during the vibrations. On the other hand, stationary counterpart of (1.1) is given as

$$\begin{cases} \left(a + b\int_\Omega |\nabla u|^2 dx\right)\Delta u = f(x, u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

which has received much attention from many authors, where new methods and a new functional analysis framework for the problem were proposed (see, e.g.,) [2, 7, 10], for some interesting results. Later, the study of Kirchhoff type equations has been extended to the case of nonlocal elliptic boundary value problem driving by the $p(x)$-Laplacian like [11],

$$-M\left(\int_\Omega |\nabla u|^{p(x)} dx\right)\Delta_{p(x)} u = f(x, u) \ \text{ in } \Omega. \tag{1.2}$$

Problem (P) may be viewed as a discretization of the nonlocal equation ([4])

$$\begin{aligned} &-M_1\left(\int_\Omega \frac{|\nabla u|^{p_1(x)} dx}{p_1(x)}\right)\operatorname{div}\left(|\nabla u|^{p_1(x)-2}\nabla u\right) \\ &-M_2\left(\int_\Omega \frac{|\nabla u|^{p_2(x)} dx}{p_2(x)}\right)\operatorname{div}\left(|\nabla u|^{p_2(x)-2}\nabla u\right) \\ &\quad = f(x, u) \quad \text{in } \Omega, \ u = 0 \ \text{ on } \partial\Omega. \end{aligned}$$

There are few papers concerning existence results for anisotropic discrete boundary-value problems of Kirchhoff type. To the best of our knowledge, the first study dealing with this class of problems was conducted by Z. Yucedag (see [19]). In this work, we are inspired by the results in [4, 5, 20] and the ideas introduced in the above mentioned papers, we employ a mountain pass lemma in Theorem 4.2 and

Ekeland's variational principle [12] in Theorem 4.5 to get our main results. More precisely, we consider the previous problem ([4]) and investigate its parametric version in the discrete case. We are proving existence of solutions for appropriate value of parameter λ and under suitable assumptions on the nonlinear term and the Kirchhoff functions M_1, M_2.

The rest of this article is structured as follows, in section 2, we introduce some basic properties of the investigated space of solutions and provide several inequalities useful in our approach. After we give the variational framework in section 3, and we state and prove the main results in the fourth section.

2. Preliminaries

Solutions to (P) will be considered in a space E defined as

$$E = \{u : [0, N+1]_{\mathbb{Z}} \to \mathbb{R};\ \text{such that } u(0) = u(N+1) = 0\}$$

which is a N-dimensional Hilbert space (see [1]) equipped with the inner product

$$\langle u, v\rangle = \sum_{i=1}^{N+1} \Delta u(i-1)\Delta v(i-1),\ \forall u, v \in E$$

and the corresponding norm is defined by

$$\|u\| = \left(\sum_{i=1}^{N+1} |\Delta u(i-1)|^2\right)^{1/2}. \tag{2.1}$$

We also consider other normes on E, denoted by $|u|_m$ and is namely,

$$|u|_m = \left(\sum_{i=1}^{N} |u(i)|^m\right)^{1/m},\ \forall u \in E \text{ and } m \geq 2. \tag{2.2}$$

It is easy to verify that (see [6])

$$N^{(2-m)/2m}|u|_2 \leq |u|_m \leq N^{1/m}|u|_2,\ \forall u \in E \text{ and } m \geq 2 \tag{2.3}$$

Lemma 2.1. [13] *For any function $p : [0, N+1]_{\mathbb{Z}} \to [2; +\infty)$ and $u \in E$*

$$p^+ := \max_{r\in[0,N+1]_{\mathbb{Z}}} p(r)\ p^- := \min_{r\in[0,N+1]_{\mathbb{Z}}} p(r),$$

we have the following assertions:

(A.1) *If $\|u\| > 1$, we have*

$$\sum_{i=1}^{N+1} \frac{|\Delta u(i-1)|^{p(i-1)}}{p(i-1)} \geq \frac{(\sqrt{N})^{2-p^-}}{p^+}\|u\|^{p^-} - N.$$

(A.2) *If $\|u\| < 1$, we have*

$$\sum_{i=1}^{N+1} \frac{|\Delta u(i-1)|^{p(i-1)}}{p(i-1)} \geq \frac{(\sqrt{N})^{p^+-2}}{p^+}\|u\|^{p^+}.$$

(A.3) *For any $m \geq 2$, there exist a positive constant c_m such that*

$$\sum_{i=1}^{N} |u(i)|^m \leq c_m \sum_{i=1}^{N+1} |\Delta u(i-1)|^m. \tag{2.4}$$

Moreover, from (2.3) *and* (*A*.3)*, we have*

$$|u|_m^m \leq N|u|_2^m \leq c_m N \left(\sum_{i=1}^{N+1} |\Delta u(i-1)|^2 \right)^{\frac{m}{2}} = c_m N \|u\|^m. \tag{2.5}$$

Definition 2.2. *Let $(X, \|.\|)$ be a Banach space and $J \in C^1(X, \mathbb{R})$, we say that J satisfies the Palais-Smale condition (we denote (PS) condition), if any sequence $(u_n) \subset X$ such that $\{J(u_n)\}$ bounded and $J'(u_n) \longrightarrow 0$, there exists a subsequence of (u_n) which is convergent in X.*

Proposition 2.3. *(Mountain Pass Lemma* [18]*). Let $(X, \|.\|)$ be a Banach space and $J \in C^1(X, \mathbb{R})$ satisfies (PS) condition with*

(*i*) *There exist $\varrho, \gamma > 0$ such that $J(u) \geq \gamma$, $\forall u \in X$ with $\|u\| = \varrho$.*
(*ii*) *There exists $e \in X$ with $\|e\| > \varrho$ such that $J(e) < 0$.*

Then J possesses a critical value $c \geq \gamma$ with

$$c = \inf_{g \in \Gamma} \max_{t \in [0,1]} J(g(t)),$$

where

$$\Gamma := \Big\{ g \in C\left([0,1], X\right) \mid \ g(0) = 0, \ g(1) = e \Big\}.$$

3. Variational framework

Related to problem (P), let us define the following functional $J_\lambda : E \to \mathbb{R}$ by the formula

$$J_\lambda(u) = \Phi(u) - \lambda \Psi(u), \text{ for all } u \in E,$$

where

$$\Phi(u) = \sum_{i=1}^{2} \widehat{M_i}\left(\phi_i(u)\right) \text{ and } \Psi(u) = \sum_{r=1}^{N} F(r, u(r)),$$

such that $\widehat{M_i}(t) = \displaystyle\int_0^t M_i(s) ds$ for $i = 1, 2$ and $F(r, s) = \displaystyle\int_0^s f(r, t) dt.$

The functional J_λ is well-defined on E and is of class $C^1(E,\mathbb{R})$ with derivative given by

$$\begin{aligned}\langle J_\lambda'(u),v\rangle &= \sum_{i=1}^{2} M_i\left(\phi_i(u)\right)\langle \phi_i'(u),v\rangle - \lambda\sum_{r=1}^{N} f(r,u(r))v(r)\\ &= \sum_{i=1}^{2}\left(M_i\left(\phi_i(u)\right)\sum_{r=1}^{N+1}|\Delta u(r-1)|^{p_i(r-1)-2}\Delta u(r-1)\Delta v(r-1)\right)\\ &\quad - \lambda\sum_{r=1}^{N} f(r,u(r))v(r)\end{aligned}$$

for all $u,v\in E$.

Definition 3.1. *A solution of (P) is a function $u\in E$ such that*

$$\sum_{i=1}^{2}\left(M_i\left(\sum_{r=1}^{N+1}\frac{|\Delta u(r-1)|^{p_i(r-1)}}{p_i(r-1)}\right)\sum_{r=1}^{N+1}|\Delta u(r-1)|^{p_i(r-1)-2}\Delta u(r-1)\Delta\varphi(r-1)\right)$$
$$=\lambda\sum_{r=1}^{N} f(r,u(r))\varphi(r),$$

for any $\varphi\in E$.

It is clear that a function $u\in E$ is a solution of the problem (P) if and only if u is a critical point to the functional J_λ.

We impose the following assumptions on the functions M_1, M_2 and the nonlinear term:

(f_0) There exists a function $q:[1,N]_{\mathbb{Z}}\to[2,\infty)$ such that

$$|f(r,t)|\le C_0\left(1+|t|^{q(r)-1}\right),\quad \text{for all } (r,t)\in[1,N]_{\mathbb{Z}}\times\mathbb{R},$$

where C_0 is a positive constant.

(M_0) $M_1,M_2:\mathbb{R}^+\to\mathbb{R}^+$ are continuous functions and satisfy the conditions

$$\begin{aligned}a_1t^{\alpha-1}&\le M_1(t),\\ a_2t^{\alpha-1}&\le M_2(t),\end{aligned}$$

for all $t>0$, where a_1 and a_2 are positive constants and $\alpha>1$.

(M_0') $M_i:\mathbb{R}^+\longrightarrow\mathbb{R}^+$, $i=1,2$, are continuous functions such that

$$m_1t^{\alpha-1}\le M_i(t)\le m_2t^{\alpha-1},$$

for all $t>0$, where m_1,m_2 and α are real numbers such that $0<m_1\le m_2$ and $\alpha>1$.

(f_1) $\displaystyle\lim_{t\to0}\frac{f(r,t)}{|t|^{\alpha P_M^+-1}}=0$, uniformly for all $r\in[0,N]_{\mathbb{Z}}$.

(AR) $\exists s_*>0,\ \kappa>\dfrac{m_2}{m_1}\alpha P_M^+$ such that

$$0\le\kappa F(r,s)\le f(r,s)s,\ |s|\ge s_*,\quad \forall r\in[1,N]_{\mathbb{Z}}.$$

Throughout the sequel, the letters $C,\ \tilde{c},\ c_i,\ i=1,2,...$ stand for positive constants which may differ from line to line.

4. Main results and proofs

Our main results are the following.

Theorem 4.1. *Assume that the assumptions (f_0) and (M_0) are fulfilled and*

$$q^+ < \alpha p_m^-.$$

Then, for all $\lambda > 0$ the problem (P) has a solution u_λ in E. Moreover, if $f(r,t) > 0$ for all $r \in [1,N]_{\mathbb{Z}}$ and $t > 0$, there exists $\lambda^* > 0$ such that for all $\lambda > \lambda^*$, the solution u_λ is nontrivial.

Proof. Let $u \in E$ such that $\|u\| > 1$. We point out that

$$|u(r)|^{q(r)} \leq |u(r)|^{q^-} + |u(r)|^{q^+},\ \forall r \in [1,N]_{\mathbb{Z}},\ u \in E.$$

Relations (2.2) and (2.3) give

$$\begin{aligned} |u(r)|^{q^-} + |u(r)|^{q^+} &\leq c_{q^+} \sum_{r=1}^{N+1} |\Delta u(r-1)|^{q^+} + c_{q^-} \sum_{r=1}^{N+1} |\Delta u(r-1)|^{q^-} \\ &\leq c_{q^+} N \|u\|^{q^+} + c_{q^-} N \|u\|^{q^-} \\ &\leq c_1 N \|u\|^{q^+}, \end{aligned}$$

which implies that

$$\begin{aligned} \Psi(u) = \sum_{r=1}^{N} F(r,u(r)) &\leq \sum_{r=1}^{N} C_0 \frac{|u(r)|^{q(r)}}{q(r)} + c_2 \\ &\leq \frac{c_3 N}{q^-} \|u\|^{q^+} + c_2. \end{aligned} \tag{4.1}$$

On the other hand, in view of condition (M_0), we have

$$\begin{aligned} \Phi(u) &= \sum_{i=1}^{2} \widehat{M_i}\left(\phi_i(u)\right) \\ &\geqslant \sum_{i=1}^{2} \frac{a_i}{\alpha} \left(\sum_{r=1}^{N+1} \frac{|\Delta u(r-1)|^{p_i(r-1)}}{p(r-1)} \right)^{\alpha} \\ &\geqslant \sum_{i=1}^{2} \frac{a_i}{\alpha \left(P_M^+\right)^{\alpha}} \left(\sum_{r=1}^{N+1} |\Delta u(r-1)|^{p_i(r-1)} \right)^{\alpha} \\ &\geqslant \frac{2A}{\alpha \left(P_M^+ (\sqrt{N})^{p_m^- - 2} \right)^{\alpha}} \|u\|^{\alpha p_m^-} - c_4,\ \text{with } A = max\{a_1;a_2\}, \end{aligned} \tag{4.2}$$

for $u \in E$ with $\|u\| > 1$.
We combine the inequalities (4.1) and (4.2) with each other, this fact gives

$$J_\lambda(u) \geqslant \frac{2A}{\alpha\left(P_M^+(\sqrt{N})^{p_m^- -2}\right)^{\alpha}}\|u\|^{\alpha p_m^-} - \lambda\frac{c_3 N}{q^-}\|u\|^{q^+} - c_5. \tag{4.3}$$

Since $\alpha p_m^- > q^+ \geq q^-$, we infer that $J_\lambda(u) \to +\infty$ as $\|u\| \to +\infty$.

J_λ is coercive, weakly lower semicontinuous on E. Therefore, it has a minimum point $u_\lambda \in E$ and thus, a solution of (P).

To complete the proof, it is enough to show that u_λ is not trivial.
For this reason, letting $s > 1$ be a fixed real and $r_0 \in [1, N]_{\mathbb{Z}}$. Defining the function $u_0 : [0, N+1]_{\mathbb{Z}} \to \mathbb{R}$ by

$$\begin{cases} u_0\left(r_0\right) = s, \\ u_0(r) = 0 \quad \text{for } r \in [0, N+1]_{\mathbb{Z}} \backslash \left\{r_0\right\}. \end{cases}$$

We have

$$\begin{aligned} J_\lambda\left(u_0\right) &= \sum_{i=1}^{2} \widehat{M}i\left(\frac{s^{p_i(r_0-1)}}{p_i\left(r_0-1\right)} + \frac{s^{p_i(r_0)}}{p_i\left(r_0\right)}\right) - \lambda F(r_0, s) \\ &\leq \widehat{M}_1\left(2\frac{s^{P_M^+}}{p_m^-}\right) + \widehat{M}_2\left(2\frac{s^{P_M^+}}{p_m^-}\right) - \lambda F(r_0, s). \end{aligned}$$

Since $F(r_0, s) > 0$, so there exists $\lambda^* > 0$ large enough such that $J_\lambda\left(u_0\right) < 0$ for any $\lambda \in [\lambda^*, +\infty)$. It follows that for any $\lambda \geq \lambda^*$, $J_\lambda(u_\lambda) \leq J_\lambda\left(u_0\right) < 0$. Then, u_λ is nontrivial because $J_\lambda(0) = 0$.

□

Theorem 4.2. *Assume that the assumptions (f_0), (f_1), (M_0') and (AR) hold. Suppose additionally that the function q satisfies*

$$\alpha P_M^+ < q^-. \tag{4.4}$$

Then, for any $\lambda \in (0, +\infty)$ the problem (P) admits at least a nontrivial solution.

In order to prove existence of solution, we shall use the mountain pass theorem, so we start by proving that J_λ satisfies (PS) condition.

Lemma 4.3. *Under assumptions (M_0') and (AR), for any $\lambda > 0$, the functional J_λ satisfies the (PS) condition*

Proof. Let $(u_n) \subset E$ be a sequence such that $|J_\lambda\left(u_n\right)| \leq \tilde{c}$ and $J_\lambda'\left(u_n\right) \to 0$. As E is a finite dimensional space, it is enough to show that (u_n) is bounded. If not, we can find a subsequence, still denoted (u_n) such that $\|u_n\| \to \infty$ as $n \to \infty$. Thus, we may consider that $\|u_n\| > 1$ for any integer n. Then, we get

$$\begin{aligned}\tilde{c}+\|u_n\| \geq & J_\lambda(u_n)-\frac{1}{\kappa}\left\langle J_\lambda'(u_n), u_n\right\rangle \\ = & \sum_{i=1}^{2} \widehat{M_i}(\phi_i(u_n))-\lambda \sum_{r=1}^{N} F(r, u_n(r)) \\ & -\frac{1}{\kappa} \sum_{i=1}^{2} M_i(\phi_i(u_n)) \sum_{r=1}^{N+1}|\Delta u_n(r-1)|^{p_i(r-1)}+\frac{\lambda}{\kappa} \sum_{r=1}^{N} f(r, u_n(k)) u_n(r).\end{aligned}$$

By (AR), (A_1) and (M_0'), we infer that

$$\begin{aligned}\tilde{c}+\|u_n\| \geq & \sum_{i=1}^{2} \widehat{M_i}(\phi_i(u_n))-\frac{P_M^+}{\kappa} \sum_{i=1}^{2} M_i(\phi_i(u_n))(\phi_i(u_n)) \\ & +\frac{\lambda}{\kappa} \sum_{r=1}^{N}(f(r, u_n(r)) u_n(r)-\kappa F(r, u_n(r))) \\ \geq & \left(\frac{m_1}{\alpha}-\frac{m_2 P_M^+}{\kappa}\right) \sum_{i=1}^{2}\left(\sum_{r=1}^{N+1} \frac{|\Delta u_n(r-1)|^{p_i(r-1)}}{p_i(r-1)}\right)^\alpha-c_6 \\ \geq & \left(\frac{m_1}{\alpha}-\frac{m_2 P_M^+}{\kappa}\right) \frac{2}{\left(P_M^+(\sqrt{N})^{p_m^- -2}\right)^\alpha}\|u\|^{\alpha p_m^-}-c_7.\end{aligned}$$

Dividing the above inequality by $\|u_n\|^{\alpha p_m^-}$, and using the fact that $\kappa>\frac{m_2}{m_1}\alpha P_M^+$ we pass to the limit as $n\to\infty$, to obtain a contradiction. So, (u_n) is bounded in E and thus, J_λ satisfies the Palais-Smale condition. □

Lemma 4.4. *Suppose that the hypotheses of Theorem 4.2 hold. Then,*

(i) *There exist $\gamma, \varrho>0$ such that $J_\lambda(u)\geq\gamma>0,\ u\in E$ with $\|u\|=\varrho$.*
(ii) *There exists $e\in E$ such that $\|e\|>\varrho$ and $J_\lambda(e)<0$.*

Proof. (i) Taking $\|u\|<1$ and by condition (M_0'), $(A.2)$ and $(A.3)$, it follows that

$$\begin{aligned}J_\lambda(u) &= \sum_{i=1}^{2} \widehat{M_i}\left(\sum_{r=1}^{N+1} \frac{|\Delta u(r-1)|^{p_i(r-1)}}{p_i(r-1)}\right)-\lambda \sum_{r=1}^{N} F(r, u(r)) \\ &\geq \sum_{i=1}^{2} \frac{m_1}{\alpha\left(p_i^+\right)^\alpha} N^{(\frac{P_i^+-2}{2})\alpha}\|u\|^{\alpha p_i^+}-\lambda \sum_{r=1}^{N} F(r, u(r)) \\ &\geq \frac{2 m_1}{\alpha\left(P_M^+\right)^\alpha} N^{(\frac{P_M^+-2}{2})\alpha}\|u\|^{\alpha P_M^+}-\lambda \sum_{r=1}^{N} F(r, u(r)).\end{aligned}$$

From (f_0) and (f_1), it follows that

$$F(r, s) \leq \varepsilon|s|^{\alpha P_M^+}+c_8|s|^{q(r)}, \text{ for all } (r, s) \in[1, N]_{\mathbb{Z}} \times \mathbb{R}, \tag{4.5}$$

where $\varepsilon > 0$ and thus

$$\begin{aligned}\Psi(u) &\leq \varepsilon \sum_{r=1}^{N} |u(r)|^{\alpha P_M^+} + c_8 \sum_{r=1}^{N} |u(r)|^{q(r)} \\ &\leq \varepsilon c_{\alpha P_M^+} N \|u\|^{\alpha P_M^+} + c_9 \left(c_{q^+} N \|u\|^{q^+} + c_{q^-} N \|u\|^{q^-} \right) \\ &\leq \varepsilon c_{\alpha P_M^+} N \|u\|^{\alpha P_M^+} + c_{10} N \|u\|^{q^-}.\end{aligned}$$

Let taking $\varepsilon > 0$ be sufficiently small such that $\lambda \varepsilon c_{\alpha P_M^+} N \leq \dfrac{m_1}{\alpha \left(P_M^+\right)^{\alpha}} N^{(\frac{P_M^+ - 2}{2})\alpha}$. Also, we get

$$\begin{aligned}J_\lambda(u) &\geq 2 \frac{m_1}{\alpha \left(P_M^+\right)^{\alpha}} N^{(\frac{P_M^+ - 2}{2})\alpha} \|u\|^{\alpha P_M^+} - \lambda \varepsilon c_{\alpha p^+} N \|u\|^{\alpha P_M^+} - \lambda c_3 N \|u\|^{q^-} \\ &\geq \frac{m_1}{\alpha \left(P_M^+\right)^{\alpha}} N^{(\frac{P_M^+ - 2}{2})\alpha} \|u\|^{\alpha P_M^+} - \lambda c_3 N \|u\|^{q^-}.\end{aligned}$$

From (4.4) the function $g : [0, 1] \to \mathbb{R}$ defined by

$$g(s) = \frac{m_1}{\alpha \left(P_M^+\right)^{\alpha}} N^{(\frac{P_M^+ - 2}{2})\alpha} - \lambda c_3 N s^{q^- - \alpha P_M^+},$$

is positive in a neighborhood of the origin, the proof of (i) is completed.

(ii) From (AR), one can deduce

$$F(r, su) \geq s^{\kappa} F(r, u), \forall r \in [1, N]_{\mathbb{Z}} \text{ and } s \geq 1.$$

Therefore, for any $v \in E, v \neq 0$ and $t > 1$, we have

$$\begin{aligned}J_\lambda(tv) &= \sum_{i=1}^{2} \widehat{M}_i \left(\sum_{r=1}^{N+1} \frac{|\Delta tv(r-1)|^{p_i(r-1)}}{p_i(r-1)} \right) - \lambda \sum_{r=1}^{N} F(r, tv(r)) \\ &\leq \frac{m_2}{\alpha \left(p_m^-\right)^{\frac{m_2}{m_1}\alpha}} t^{\frac{m_2}{m_1}\alpha P_M^+} \sum_{i=1}^{2} \sum_{r=1}^{N+1} |\Delta v(r-1)|^{p_i(r-1)} - \lambda t^{\kappa} \sum_{r=1}^{N} F(r, v(r)).\end{aligned}$$

Then, $\lim_{t\to\infty} J_\lambda(tv) = -\infty$, because $\kappa > \dfrac{m_2}{m_1} \alpha P_M^+$. So, we can take $e = tv$ such that $\|e\| > \varrho$ and $J_\lambda(e) < 0$ for some t large enough.

□

Proof. Consequently, Since $J_\lambda(0) = 0$ and from the above lemmas, applying proposition 2.3, we conclude that the problem (P) admits at least a nontrivial solution.

□

Theorem 4.5. *Under assumptions (M_0') and (f_0), there exists $\lambda_* > 0$ such that for any $\lambda \in (0, \lambda_*)$ the problem (P) has a nontrivial solution provided*

$$q^- < \alpha p_m^-.$$

To give the proof of this result, To prove this result, we apply Ekeland's variational principle, which requires the following two auxiliary lemmas.

Lemma 4.6. *There exist $\lambda_* > 0$ and ρ, $a > 0$ such that for all $\lambda \in (0, \lambda_*)$, we have $J_\lambda(u) \geq a > 0$, $\forall u \in E$ with $\|u\| = \rho$.*

Proof. Let us fix $\rho \in (0,1)$, so by conditions (M_0'), (f_0) and relations $(A.2)$, (2.3) we state that for any $u \in E$ with $\|u\| = \rho$, the following holds

$$\begin{aligned} J_\lambda(u) &= \sum_{i=1}^{2} \widehat{M}_i \left(\sum_{r=1}^{N+1} \frac{|\Delta u(r-1)|^{p_i(r-1)}}{p_i(r-1)} \right) - \lambda \sum_{r=1}^{N} F(r, u(r)) \\ &\geq \sum_{i=1}^{2} \frac{m_1}{\alpha \left(p_i^+\right)^\alpha} N^{(\frac{P_i^+-2}{2})\alpha} \|u\|^{\alpha p_i^+} - \lambda \frac{c_{11} c_{q^-}}{q^-} N \|u\|^{q^-} \\ &\geq \frac{2m_1}{\alpha \left(P_M^+\right)^\alpha} N^{(\frac{P_M^+-2}{2})\alpha} \|u\|^{\alpha P_M^+} - \lambda \frac{c_{11} c_{q^-}}{q^-} N \|u\|^{q^-} \\ &= \rho^{q^-} \left(\frac{2m_1}{\alpha \left(P_M^+\right)^\alpha} N^{(\frac{P_M^+-2}{2})\alpha} \rho^{\alpha P_M^+ - q^-} - \lambda c_{12} \right). \end{aligned}$$

Choosing λ_* as

$$\lambda_* = \frac{m_1 \rho^{\alpha P_M^+ - q^-}}{\alpha \left(P_M^+\right)^\alpha c_{12}} N^{(\frac{P_M^+-2}{2})\alpha}, \tag{4.6}$$

then for any $\lambda \in (0, \lambda_*)$ and $u \in E$ with $\|u\| = \rho$, there exists $a > 0$ such that $J_\lambda(u) \geq a$. $\square$

Lemma 4.7. *For any $\lambda \in (0, \lambda_*)$ with λ_* given by* (4.6)*, there exists $v \in E$ such that $v \neq 0$ and $J_\lambda(tv) < 0$, for $t > 0$ sufficiently small.*

Proof. Let $t \in (0,1)$ and take $v \in E$ such that $v(\tilde{r}) = 1$ and $v(r) = 0$ for $r \in [1, N]_{\mathbb{Z}} \setminus \{\tilde{r}\}$, with $r_0 \in [1, N]_{\mathbb{Z}}$ such that $q(\tilde{r}) = q^-$.
Using hypothesis (M_0'), (f_0) and relation $(A.2)$, we can write

$$\begin{aligned} J_\lambda(tv) &= \sum_{i=1}^{2} \widehat{M}_i \left(\phi_i(tv)\right) - \lambda \sum_{r=1}^{N} F(r, tv(r)) \\ &\leq \sum_{i=1}^{2} \frac{m_2}{\alpha} t^{\alpha p_m^-} \left(\phi_i(v)\right)^\alpha - \lambda F(\tilde{r}, t) \\ &\leq \frac{4m_2}{\alpha p_m^-} t^{\alpha p_m^-} - \lambda \left(c_{13} + c_{14} \frac{t^{q(\tilde{r})}}{q(\tilde{r})} \right) \\ &\leq \left(\frac{4m_2}{\alpha p_m^-} t^{\alpha p_m^- - q^-} - \frac{\lambda c_{14}}{q^-} \right) t^{q^-} - \lambda c_{13} \end{aligned}$$

So, for all $t < \delta^{\frac{1}{\alpha p_m^- - q^-}}$ such that

$$0 < \delta < \min\left\{1, \frac{\lambda \alpha p_m^- c_{14}}{4m_2 q^-}\right\},$$

we conclude that $J_\lambda(tv) < 0$. □

Proof. From Lemma 4.6 it follows that on the boundary of the ball centered at the origin and of radius ρ in E

$$\inf_{\partial B_\rho(0)} J_\lambda(u) > 0.$$

On the other hand, by Lemma 4.7, there exists $v \in E$ such that $J_\lambda(tv) < 0$ for $t > 0$ small enough. As a result, we have

$$-\infty < \bar{c} = \inf_{B_\rho(0)} J_\lambda(u) < 0.$$

Let $0 < \epsilon < \inf_{\partial B_\rho(0)} J_\lambda - \inf_{B_\rho(0)} J_\lambda$, let's use the Ekeland variational principle [12] to the functional $J_\lambda : \overline{B_\rho(0)} \to \mathbb{R}$, there exists $u_\epsilon \in \overline{B_\rho(0)}$, such that

$$J_\lambda(u_\epsilon) < \inf_{\overline{B_\rho(0)}} J_\lambda + \epsilon$$

$$J_\lambda(u) > J_\lambda(u_\epsilon) - \epsilon \|u - u_\epsilon\|, \quad u \neq u_\epsilon.$$

Since

$$J_\lambda(u_\epsilon) \leq \inf_{B_\rho(0)} J_\lambda + \epsilon \leq \inf_{B_\rho(0)} J_\lambda + \epsilon < \inf_{\partial B_\rho(0)} J_\lambda,$$

we deduce that $u_\epsilon \in B_\rho(0)$.

Let us now define $L_\lambda : \overline{B_\rho(0)} \to \mathbb{R}$ by $L_\lambda(u) = J_\lambda(u) + \epsilon \|u - u_\epsilon\|$. It is obvious that u_ϵ is a minimum point of L_λ and thus

$$\frac{L_\lambda(u_\epsilon + t\varphi) - L_\lambda(u_\epsilon)}{t} \geq 0, \tag{4.7}$$

for $t > 0$ small enough and $\varphi \in B_\rho(0)$. From (4.7) , we have

$$\frac{J_\lambda(u_\epsilon + t\varphi) - J_\lambda(u_\epsilon)}{t} + \epsilon\|\varphi\| \geq 0.$$

Taking the limit as $t \to 0$, we infer that $\langle J'_\lambda(u_\epsilon), \varphi\rangle + \epsilon\|\varphi\| > 0$. Then, $\|J'_\lambda(u_\epsilon)\| \leq \epsilon$. So, there exists a sequence $(\varphi_n) \subset B_\rho(0)$ such that

$$J_\lambda(\varphi_n) \to \bar{c}, \quad J'_\lambda(\varphi_n) \to 0$$

It is clear that (φ_n) is bounded in E. Thus, there exists $\varphi_0 \in E$ and a subsequence, still denoted (φ_n) converges to φ_0 in E.

$$J_\lambda(\varphi_0) = \bar{c} < 0, \quad J'_\lambda(\varphi_0) = 0$$

Consequently, the problem (P) possesses a nontrivial solution. □

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