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Existence and multiplicity of solutions to an abstract Cauchy problem for an evolution equation involving fractional dissipation term of Caputo type

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Abstract. The aim of this work is to investigate the existence and multiplicity of solutions to a nonhomogenious quasi-linear second order evolution equation involving a fractional dissipation term of Caputo type in abstract framework. Some criteria on the existence of at least one or two solutions are obtained by using some well known fixed point theorems for the sum of two operators. An example is presented to validate our analysis.

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1. Introduction

Evolution problems have been an area of wide research for a decade. Their role in modeling real world scenarios is crucial in sciences, physics and engineering. However, the loss of energy often produces friction effects in real world models [14, 19, 20]. Fractional dissipation which extend the classical dissipation mechanisms where fractional derivatives are involved provide an appropriate description of memory effects.
In recent years, special attention has been focused on fractional derivatives, both in their interpretation and as a nonlocal dissipation. More details can be found in

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[4, 13, 22, 23, 24, 25].
The following abstract system has been extensively studied in the literature

$$u'' + \phi(\|\mathcal{A}^{\frac{1}{2}}u\|^2)\mathcal{A}u + V(t) = U(t). \tag{1.1}$$

Many authors have developed several methods and different techniques to study this kind of problems in the last decade. See for instance [2, 7, 11, 12, 16, 17, 18].
It is remarkable that no paper in the aforementioned literature discussed such models by means of topological methods which offer a key tool to overcome many analysis difficulties and guarantee the well-posedness of solutions. The use of topological methods has been extensively researched in both qualitative and quantitative analysis of nonlinear boundary value problems, see for instance [5, 15, 26] and the references therein. Figueiredo et al. in [9] used Leray-Schauder degree and the bifurcation method to study the existence and uniqueness of positive solution for the following non-homogeneous elliptic Kirchhoff problem with nonlinear reaction term

$$\begin{aligned} -\phi\left(x, \|u\|^2\right)\Delta \quad u &= \varrho u^q \quad in\ \Omega \\ u &= 0 \quad on\ \partial\Omega, \end{aligned}$$

where $\Omega \subseteq \mathbb{R}^N, N \geq 1$ is a bounded regular domain, $0 < q \leq 1, \varrho \in \mathbb{R}$ and

$$\phi(x,s) = a(x) + b(x)s, \quad \|u\|^2 = \int_\Omega |\nabla u|^2 dx,$$

with $a, b \in C^\gamma(\overline{\Omega}), \gamma \in (0,1)$ and $a(x) \geq a_0 > 0, b \geq 0$.

The authors in [1] applied sup- and super-solutions methods to investigate the existence of positive solutions for the following Kirchhoff type problem

$$\begin{aligned} -\phi\left(\textstyle\int_\Omega |\nabla u|^p dx\right)\Delta_p u &= \varrho f(u) \quad in\ \Omega \\ u > 0 \quad in\ \Omega, \quad u &= 0 \quad on\ \partial\Omega, \end{aligned}$$

where $1 < p < N, \phi : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous and increasing function, ϱ is a parameter and $f : [0, +\infty) \to \mathbb{R}$ is a C^1 semipositone nondecreasing function.

The authors in [28] used the method of upper and lower solutions to investigate the existence and multiplicity of solutions for the following Kirchhoff type problem

$$\begin{aligned} -\phi\left(\textstyle\int_\Omega |\nabla u|^2 dx\right)\Delta u(x) &= f(x, u(x), \nabla u(x)) - g(x, u(x), \nabla u(x)), \quad in\ \Omega \\ u &= 0 \quad on\ \partial\Omega, \end{aligned}$$

where $\Omega \subseteq \mathbb{R}^N$ is a bounded and smooth domain and $\phi \in C([0, +\infty), (0, +\infty))$ with $\phi(t)$ nondecreasing on $[0, +\infty)$ and $\phi(t) \geq \phi(0) > 0, \forall t \geq 0$.

Very recently Precup and Stan [27] discussed by means of fixed point theorems of Banach and Schaefer, the existence of solutions to the following stationary Kirchhoff problem with reaction terms

$$\begin{aligned} -\left(a + b\textstyle\int_\Omega |\nabla u|^2 dx\right)\Delta u &= f + g(x, u, \nabla u), \quad in\ \Omega \\ u|_{\partial\Omega} &= 0. \end{aligned}$$

In this paper, we investigate the following second order evolution problem

$$u''(t) + \phi\left(\|\mathcal{G}(t)u(t)\|^2\right)\mathcal{A}(t)u(t) + \gamma\partial_t^{\alpha,\eta}u(t) = f(u(t)), \quad t \in [0, \varpi], \tag{1.2}$$

$$u(0) = u_0, \quad u'(0) = u_1,$$

where $0 \leq u_0, u_1 \leq \chi$ where χ is a nonnegative constant, $\gamma > 0$, ϕ is is a function that will be specialized in the sequel and $\mathcal{A}(t), \mathcal{G}(t) : \mathcal{H} \to \mathcal{H}$are two families of operators. When $\mathcal{A}(t) = \mathcal{A}$ is a positive self-adjoint operator and $\mathcal{G}(t) = \mathcal{A}^{\frac{1}{2}}$ the equation above coincides with a class of Kirchhoff equations. We begin by introducing some assumptions.

(A1). $\mathcal{H}$ is a Hilbert space with inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ and norm $\|\cdot\|_{\mathcal{H}}$, f is a nonlinear operator with domain

$$\mathcal{D}(f) = \{u \in \mathcal{H} : f(u) \in \mathcal{H}\}$$

and

$$\|f(u)\|_{\mathcal{H}} \leq \theta_0(t) + \sum_{k=1}^{r} \theta_k(t) \|u\|_{\mathcal{H}}^{p_k},$$

where $r \in N^*$, $p_k \geq 0$, $k \in \{1, \ldots, r\}$, $\theta_k \in \mathcal{C}([0, \varpi])$, $0 \leq \theta_k \leq \chi$ on $[0, \varpi]$ for some nonnegative constant χ,

(A2). $\mathcal{A}(t)$ and $\mathcal{G}(t)$ verify

$$\|\mathcal{A}(t)u\|_{\mathcal{H}} \leq \zeta_0(t) + \sum_{k=1}^{n} \zeta_k(t) \|u\|_{\mathcal{H}}^{s_k}, \quad t \in [0, \varpi],$$

where $n \in N^*$, $s_k \geq 0$, $k \in \{1, \ldots, n\}$, $\zeta_k \in \mathcal{C}([0, \varpi])$, $0 \leq \zeta_k \leq \chi$ on $[0, \varpi]$, $k \in \{1, \ldots, n\}$, and

$$\|\mathcal{G}(t)u\|_{\mathcal{H}} \leq \varsigma_0(t) + \sum_{k=1}^{l} \varsigma_k(t) \|u\|_{\mathcal{H}}^{r_k}, \quad t \in [0, \varpi],$$

where $l \in N^*$, $r_k \geq 0$, $k \in \{1, \ldots, l\}$, $\varsigma_k \in \mathcal{C}([0, \varpi])$, $0 \leq \varsigma_k \leq \chi$ on $[0, \varpi]$, $k \in \{1, \ldots, l\}$,

(A3). $\phi \in \mathcal{C}([0, \infty), [0, \infty))$ verifies

$$|\phi(z)| \leq \vartheta_0(t) + \sum_{k=1}^{m} \vartheta_k(t) |z|^{q_k}, \quad t \in [0, \varpi],$$

where $m \in N^*$, $q_k \geq 0$, $k \in \{1, \ldots, m\}$, $\vartheta_k \in \mathcal{C}([0, \varpi])$, $0 \leq \vartheta_k \leq \chi$ on $[0, \varpi]$ for some nonnegative constant χ,

(A4). $\partial_t^{\alpha,\eta}$ is the generalized Caputo fractional differential operator of order $\alpha \in (0, 1)$ (see [6] and [3]). It is given by

$$\partial_t^{\alpha,\eta} u(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{-\eta(t-\tau)} \frac{d}{d\tau} u(\tau) d\tau, \quad \eta \geq 0.$$

In this paper under hypotheses $(A1) - (A4)$ we prove that the problem (1.2) has at least one or two bounded solutions.

Let

$$\mathcal{E} = \mathcal{C}^1([0, \varpi], \mathcal{H})$$

be endowed with the norm

$$\|u\| = \max\left\{\sup_{t\in[0,\varpi]} \|u\|_{\mathcal{H}},\ \sup_{t\in[0,\varpi]} \|u'\|_{\mathcal{H}}\right\}$$

provided it exists. For $u \in \mathcal{H}$, define $\mathcal{J}$ as the identity operator in $\mathcal{H}$.

The paper is organized as follows. In the next section, we give some preliminary definitions, lemmas and theorems. Section 3 is devoted to our main findings. In section 4, we illustrate the main results with an example. Finally, the last section provides a conclusion summarizing the study.

2. Preliminaries

First, we recall the following definitions.

Definition 2.1. *Let $\mathcal{E}$ and $\mathcal{F}$ be real Banach spaces. A map $\mathcal{T} : \mathcal{E} \to \mathcal{F}$ is called expansive if there exists a constant $h > 1$ for which the following inequality holds*

$$\|\mathcal{T}x - \mathcal{T}y\|_{\mathcal{F}} \geq h\|x - y\|_{\mathcal{E}},$$

for any $x, y \in \mathcal{E}$.

Definition 2.2. *A closed, convex set $\mathcal{Q}$ in a Banach space $\mathcal{E}$ is said to be a cone if*

1. *$\sigma x \in \mathcal{Q}$ for any $\sigma \geq 0$ and for any $x \in \mathcal{Q}$,*
2. *$x, -x \in \mathcal{Q}$ implies $x = 0$.*

The following fixed point theorems are the main tools to prove our results. To prove our first existence result we will use the following fixed point theorem.

Theorem 2.3. *([10], [21]) Let $\mathcal{E}$ be a Banach space, $\mathcal{F}$ a closed, convex subset of $\mathcal{E}$, $\mathcal{U}$ be any open subset of $\mathcal{F}$ with $0 \in \mathcal{U}$. Consider two operators $\mathcal{T}$ and $\mathcal{S}$, where*

$$\mathcal{T}x = \varepsilon\, x,\ x \in \overline{\mathcal{U}},$$

for some $\varepsilon > 1$ and $\mathcal{S} : \overline{\mathcal{U}} \to \mathcal{E}$ be such that

(i). *$\mathcal{I} - \mathcal{S} : \overline{\mathcal{U}} \to \mathcal{F}$ continuous, compact and*
(ii). *$\{x \in \partial\mathcal{U} : x = \lambda(\mathcal{I} - \mathcal{S})x\} = \emptyset$, for any $\lambda \in \left(0, \frac{1}{\varepsilon}\right)$.*

Then there exists $x^* \in \overline{\mathcal{U}}$ such that

$$\mathcal{T}x^* + \mathcal{S}x^* = x^*.$$

The next fixed point theorem will allow us to prove existence of at least two nonnegative global classical solutions of the IVP (1.2).

Theorem 2.4. *([8], [29]) Let $\mathcal{Q}$ be a cone of a Banach space $\mathcal{E}$; Ω a subset of $\mathcal{Q}$ and $\mathcal{U}_1$,$\mathcal{U}_2$ and $\mathcal{U}_3$ three open bounded subsets of $\mathcal{Q}$ such that $\overline{\mathcal{U}}_1 \subset \overline{\mathcal{U}}_2 \subset \mathcal{U}_3$ and $0 \in \mathcal{U}_1$. Assume that $\mathcal{T} : \Omega \to \mathcal{E}$ is an expansive mapping, $\mathcal{S} : \overline{\mathcal{U}}_3 \to \mathcal{E}$ is a completely continuous map and $\mathcal{S}(\overline{\mathcal{U}}_3) \subset (\mathcal{I}-\mathcal{T})(\Omega)$. Suppose that $(\mathcal{U}_2\backslash\overline{\mathcal{U}}_1)\cap\Omega \neq \emptyset$, $(\mathcal{U}_3\backslash\overline{\mathcal{U}}_2)\cap\Omega \neq \emptyset$, and there exist $u_0 \in \mathcal{Q}^* = \mathcal{Q}\backslash\{0\}$ and $\epsilon > 0$ such that the following conditions hold:*

(i). *$\mathcal{S}x \neq (\mathcal{I} - \mathcal{T})(x - \lambda u_0)$, for all $\lambda > 0$ and $x \in \partial\mathcal{U}_1 \cap (\Omega + \lambda u_0)$,*
(ii). *$\mathcal{S}x \neq (\mathcal{I} - \mathcal{T})(\lambda x)$, for all $\lambda \geq 1 + \epsilon$, $x \in \partial\mathcal{U}_2$ and $\lambda x \in \Omega$,*

(iii). $\mathcal{S}x \neq (\mathcal{I}-\mathcal{T})(x-\lambda u_0)$, *for all* $\lambda > 0$ *and* $x \in \partial\mathcal{U}_3 \cap (\Omega + \lambda u_0)$.

Then $\mathcal{T}+\mathcal{S}$ *has at least two non-zero fixed points* $x_1, x_2 \in \mathcal{Q}$ *such that*

$$x_1 \in \partial\mathcal{U}_2 \cap \Omega \text{ and } x_2 \in (\overline{\mathcal{U}}_3 \setminus \overline{\mathcal{U}}_2) \cap \Omega$$

or

$$x_1 \in (\mathcal{U}_2 \setminus \mathcal{U}_1) \cap \Omega \text{ and } x_2 \in (\overline{\mathcal{U}}_3 \setminus \overline{\mathcal{U}}_2) \cap \Omega.$$

We give also some auxiliary results which will be used in the sequel. To this end, for any $u \in \mathcal{E}$, let us define the operators

$$\begin{aligned}\mathcal{N}u(t) &= u(t) - u_0 - u_1 t \\ &\quad - \int_0^t (t-s)\left(\phi\left(\|\mathcal{G}(t)u(s)\|^2\right)\mathcal{A}(t)u(s) + \gamma\partial_s^{\alpha,\eta}u(s) - f(u(s))\right) ds,\end{aligned}$$

for all $t \in [0,\varpi]$. Set

$$\begin{aligned}\vartheta_1 &= 2\chi + \chi\varpi \\ &\quad + \left(\chi^2\left(1+\sum_{k=1}^{m}\left(\chi\left(1+\sum_{k=1}^{l}\chi^{r_k}\right)\right)\right)\right)^{q_k} \\ &\quad \times \left(1+\sum_{k=1}^{n}\chi^{s_k}\right) + \chi\left(1+\sum_{k=1}^{r}\chi^{p_k}\right)\varpi^2 + \frac{\gamma\chi}{\Gamma(2-\alpha)}\varpi^{3-\alpha}.\end{aligned}$$

The following lemma is going to be needed in what follows.

Lemma 2.5. *Suppose that* $(A1)-(A4)$ *hold. If* $u \in \mathcal{E}$ *and* $\|u\| \leq \chi$, *then*

$$\|\mathcal{N}u(t)\|_{\mathcal{H}} \leq \vartheta_1, \quad t \in [0,\varpi].$$

Proof. We have

$$\begin{aligned}\|\mathcal{G}(t)u(t)\|_{\mathcal{H}} &\leq \varsigma_0(t) + \sum_{k=1}^{l}\varsigma_k(t)\|u(t)\|_{\mathcal{H}}^{r_k} \\ &\leq \chi\left(1+\sum_{k=1}^{l}\chi^{r_k}\right), \quad t \in [0,\varpi],\end{aligned}$$

$$\begin{aligned}\left|\phi\left(\|\mathcal{G}(t)u(t)\|^2\right)\right| &\leq \vartheta_0(t) + \sum_{k=1}^{m}\vartheta_k(t)\|\mathcal{G}(t)u(t)\|_{\mathcal{H}}^{q_k} \\ &\leq \chi\left(1+\sum_{k=1}^{m}\left(\chi\left(1+\sum_{k=1}^{l}\chi^{r_k}\right)\right)^{q_k}\right), \quad t \in [0,\varpi],\end{aligned}$$

$$
\begin{aligned}
\|\partial_t^{\alpha,\eta} u(t)\|_{\mathcal{H}} &\leq \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} e^{-\eta(t-\tau)} \|u'(\tau)\|_{\mathcal{H}} d\tau \\
&\leq \frac{\chi}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} d\tau \\
&= \frac{\chi}{\Gamma(2-\alpha)} t^{1-\alpha}, \quad t \in [0, \varpi],
\end{aligned}
$$

$$
\begin{aligned}
\|f(u(t))\|_{\mathcal{H}} &\leq \theta_0(t) + \sum_{k=1}^{r} \theta_k(t) \|u(t)\|_{\mathcal{H}}^{p_k} \\
&\leq \chi \left(1 + \sum_{k=1}^{r} \chi^{p_k}\right), \quad t \in [0, \varpi],
\end{aligned}
$$

and

$$
\begin{aligned}
\|\mathcal{A}(t)u(t)\|_{\mathcal{H}} &\leq \zeta_0(t) + \sum_{k=1}^{n} \zeta_k(t) \|u(t)\|_{\mathcal{H}}^{s_k} \\
&\leq \chi \left(1 + \sum_{k=1}^{n} \chi^{s_k}\right), \quad t \in [0, \varpi].
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\|\mathcal{N}u(t)\|_{\mathcal{H}} \\
&\leq \|u(t)\|_{\mathcal{H}} + u_0 + u_1 t \\
&\quad + \int_0^t (t-s) \left(\left|\phi\left(\|\mathcal{G}(t)u(s)\|^2\right)\right| \|\mathcal{A}(t)u(s)\|_{\mathcal{H}} + \gamma \|\partial_s^{\alpha,\eta} u(s)\|_{\mathcal{H}} + \|f(u(s))\|_{\mathcal{H}} \right) ds \\
&\leq 2\chi + \chi t + \int_0^t (t-s) \Bigg(\chi^2 \left(1 + \sum_{k=1}^{m} \left(\chi \left(1 + \sum_{k=1}^{l} \chi^{r_k}\right)\right)^{q_k}\right) \left(1 + \sum_{k=1}^{n} \chi^{s_k}\right) \\
&\quad + \gamma \frac{\chi}{\Gamma(2-\alpha)} s^{1-\alpha} + \chi \left(1 + \sum_{k=1}^{r} \chi^{p_k}\right) \Bigg) ds \\
&\leq 2\chi + \chi t \\
&\quad + \left(\chi^2 \left(1 + \sum_{k=1}^{m} \left(\chi \left(1 + \sum_{k=1}^{l} \chi^{r_k}\right)\right)\right)^{q_k} \left(1 + \sum_{k=1}^{n} \chi^{s_k}\right) + \chi \left(1 + \sum_{k=1}^{r} \chi^{p_k}\right) \right) t^2 \\
&\quad + \frac{\gamma\chi}{\Gamma(2-\alpha)} t^{3-\alpha}
\end{aligned}
$$

$$\begin{aligned}
&\leq 2\chi + \chi\varpi \\
&\quad + \left(\chi^2\left(1+\sum_{k=1}^{m}\left(\chi\left(1+\sum_{k=1}^{l}\chi^{r_k}\right)\right)\right)^{q_k}\left(1+\sum_{k=1}^{n}\chi^{s_k}\right)+\chi\left(1+\sum_{k=1}^{r}\chi^{p_k}\right)\right)\varpi^2 \\
&\quad + \frac{\gamma\chi}{\Gamma(2-\alpha)}\varpi^{3-\alpha} \\
&= \vartheta_1, \quad t\in[0,\varpi].
\end{aligned}$$

□

Let

$$\varpi_1 = \max\{\varpi, \varpi^2\}.$$

For $u \in \mathcal{E}$, define the operator

$$\mathcal{S}_2 u(t) = \frac{A}{\varpi_1}\int_0^t (t-s)\mathcal{N}u(s)ds, \quad t\in[0,\varpi],$$

where A is a positive constant.

Lemma 2.6. *For $u \in \mathcal{E}$ and $\|u\| \leq \chi$, one has $\|\mathcal{S}_2 u\| \leq A\vartheta_1$.*

Proof. We have

$$\begin{aligned}
\|\mathcal{S}_2 u\|_{\mathcal{H}} &\leq \frac{A}{\varpi_1}\int_0^t (t-s)\|\mathcal{N}u(s)\|_{\mathcal{H}}ds \\
&\leq \frac{A}{\varpi_1}\vartheta_1\int_0^t (t-s)ds \\
&\leq \frac{A}{\varpi_1}\vartheta_1\varpi^2 \\
&\leq A\vartheta_1, \quad t\in[0,\varpi],
\end{aligned}$$

and

$$\begin{aligned}
\left\|\frac{d}{dt}\mathcal{S}_2 u\right\|_{\mathcal{H}} &\leq \frac{A}{\varpi_1}\int_0^t \|\mathcal{N}u(s)\|_{\mathcal{H}}ds \\
&\leq \frac{A}{\varpi_1}\vartheta_1\varpi \\
&\leq A\vartheta_1, \quad t\in[0,\varpi].
\end{aligned}$$

□

Lemma 2.7. *If $u \in \mathcal{E}$ verifies the equation*

$$\mathcal{S}_2 u(t) = D, \quad t\in[0,\varpi],$$

for some constant D, then u is a solution to the problem (1.2).

Proof. We have

$$\int_0^t (t-s)\mathcal{N}u(s)ds = D, \quad t \in [0, \varpi],$$

so two times differentiation with respect to t leads to

$$\mathcal{N}u(t) = 0, \quad t \in [0, \varpi].$$

□

3. Main results

Theorem 3.1. *Assume that assumptions $(A1)-(A4)$ are satisfied, then the problem* (1.2) *has at least one bounded solution in $\mathcal{E}$.*

Proof. In the sequel, $\widetilde{\mathcal{F}}$ denotes the set of all equi-continuous and relatively compact in $\mathcal{H}$ families of functions in $\mathcal{E}$ with respect to the norm $\|\cdot\|$. Let, $\mathcal{F} = \overline{\widetilde{\mathcal{F}}}$ and

$$\mathcal{U} = \left\{ u \in \mathcal{F} : \|u\| < \chi \quad \text{and if} \quad \|u\| \geq \frac{\chi}{2}, \quad \text{then} \quad \|u(0)\| > \frac{\chi}{2} \right\}.$$

For $u \in \overline{\mathcal{U}}$, $\epsilon > 1$ and $t \in [0, \varpi]$, define the operators $\mathcal{T}$ and $\mathcal{S}$ as follows

$$\mathcal{T}u(t) = \epsilon u(t),$$

$$\mathcal{S}u(t) = u(t) - \epsilon u(t) - \epsilon \mathcal{S}_2 u(t).$$

For $u \in \overline{\mathcal{U}}$, we have

$$\begin{aligned} \|(\mathcal{I}-\mathcal{S})u\| &= \|\epsilon u + \epsilon \mathcal{S}_2 u\| \\ &\leq \epsilon\|u\| + \epsilon\|\mathcal{S}_2 u\| \\ &\leq \epsilon\vartheta_1 + \epsilon A \vartheta_1. \end{aligned}$$

Thus, $\mathcal{S} : \overline{\mathcal{U}} \to \mathcal{E}$ is continuous and $\overline{\mathcal{U}}$ is compact, then $(\mathcal{I}-\mathcal{S})(\overline{\mathcal{U}})$ resides in a compact subset of $\mathcal{F}$. Now, suppose that there is a $u \in \partial\mathcal{U}$ such that

$$u = \lambda(\mathcal{I}-\mathcal{S})u$$

or

$$u = \lambda\epsilon\left(u + \mathcal{S}_2 u\right),$$

for some $\lambda \in \left(0, \frac{1}{\epsilon}\right)$. Then, using that $\mathcal{S}_2 u(0) = 0$ and $\|u\| = \chi > \frac{\chi}{2}$, we have $\|u(0)\| > \frac{\chi}{2}$ and

$$u(0) = \lambda\epsilon u(0),$$

and

$$\|u(0)\| = \lambda\epsilon\|u(0)\|$$

whereupon $\lambda\epsilon = 1$, which is a contradiction. Consequently

$$\{u \in \partial\mathcal{U} : u = \lambda_1(\mathcal{I}-\mathcal{S})u\} = \emptyset$$

for any $\lambda_1 \in \left(0, \frac{1}{\epsilon}\right)$. Then, from Theorem 2.3, it follows that the operator $\mathcal{T}+\mathcal{S}$ has a fixed point $u^* \in \mathcal{F}$. Therefore

$$\begin{aligned} u^*(t) &= \mathcal{T}u^*(t) + \mathcal{S}u^*(t) \\ &= \epsilon u^*(t) - u^*(t) + \epsilon u^*(t) - \epsilon \mathcal{S}_2 u^*(t), \quad t \in [0, \varpi], \end{aligned}$$

whereupon

$$0 = \mathcal{S}_2 u^*(t), \quad t \in [0, \varpi].$$

Hence, we conclude that u^* is a solution to the problem (1.2).

□

Theorem 3.2. *Assume that hypotheses* (A1) – (A4) *hold, then the problem* (1.2) *has at least two nonnegative bounded solutions in* $\mathcal{E}$.

Proof. Let r, L, R_1 be positive constants that satisfy the following conditions

$$r < L < R_1 \leq \chi, \ A\vartheta_1 < \frac{L}{5}.$$

Set

$$\widetilde{\mathcal{Q}} = \{u \in \mathcal{E} : \langle uw, w\rangle_{\mathcal{H}} \geq 0, \quad w \in \mathcal{H}, \quad \text{on} \quad [0, \varpi]\}.$$

With $\mathcal{Q}$ we will denote the set of all equi-continuous families in $\widetilde{\mathcal{Q}}$. For $v \in \mathcal{E}$ and $t \in [0, \varpi]$, define the operators $\mathcal{T}_1$ and $\mathcal{S}_3$ as follows

$$\mathcal{T}_1 v(t) = (1 + m\epsilon)v(t) - \epsilon \frac{L}{10}\mathcal{J},$$

$$\mathcal{S}_3 v(t) = -\epsilon \mathcal{S}_2 v(t) - m\epsilon v(t) - \epsilon \frac{L}{10}\mathcal{J}.$$

Note that any fixed point $v \in \mathcal{E}$ of the operator $\mathcal{T}_1 + \mathcal{S}_3$ is a solution to the IVP (1.2). Define

$$\begin{aligned} \mathcal{U}_1 &= \mathcal{Q}_r = \{v \in \mathcal{Q} : \|v\| < r\}, \\ \mathcal{U}_2 &= \mathcal{Q}_L = \{v \in \mathcal{Q} : \|v\| < L\}, \\ \mathcal{U}_3 &= \mathcal{Q}_{R_1} = \{v \in \mathcal{Q} : \|v\| < R_1\}, \\ \Omega &= \mathcal{Q}. \end{aligned}$$

1. For $v_1, v_2 \in \Omega$, we have

$$\|\mathcal{T}_1 v_1 - \mathcal{T}_1 v_2\| = (1 + m\epsilon)\|v_1 - v_2\|,$$

   whereupon $\mathcal{T}_1 : \Omega \to \mathcal{E}$ is an expansive operator with a constant $h = 1 + m\epsilon > 1$.
2. For $v \in \overline{\mathcal{Q}_{R_1}}$, we get

$$\begin{aligned} \|\mathcal{S}_3 v\| &\leq \epsilon\|\mathcal{S}_2 v\| + m\epsilon\|v\| + \epsilon\frac{L}{10} \\ &\leq \epsilon\left(A\vartheta_1 + mR_1 + \frac{L}{10}\right). \end{aligned}$$

   Therefore $\mathcal{S}_3(\overline{\mathcal{Q}_{R_1}})$ is uniformly bounded. Since $\mathcal{S}_3 : \overline{\mathcal{Q}_{R_1}} \to \mathcal{E}$ is continuous, we have that $\mathcal{S}_3(\overline{\mathcal{Q}_{R_1}})$ is equi-continuous. Consequently $\mathcal{S}_3 : \overline{\mathcal{Q}_{R_1}} \to \mathcal{E}$ is completely continuous.

3. Let $v_1 \in \overline{\mathcal{Q}_{R_1}}$. Set
$$v_2 = v_1 + \frac{1}{m}\mathcal{S}_2 v_1 + \frac{L}{5m}\mathcal{J}.$$
Take $w \in \mathcal{H}$ arbitrarily. Then
$$\langle v_1 w, w\rangle_{\mathcal{H}} \geq 0,$$
and
$$\begin{aligned}
\langle v_2 w, w\rangle_{\mathcal{H}} &= \langle\left(v_1 + \frac{1}{m}\mathcal{S}_2 v_1 + \frac{L}{5m}\mathcal{J}\right) w, w\rangle_{\mathcal{H}} \\
&= \langle v_1 w, w\rangle_{\mathcal{H}} + \frac{1}{m}\langle \mathcal{S}_2 v_1 w, w\rangle_{\mathcal{H}} \\
&\quad + \frac{L}{5m}\langle w, w\rangle_{\mathcal{H}} \\
&\geq \langle v_1 w, w\rangle_{\mathcal{H}} + \frac{L}{5m}\langle w, w\rangle_{\mathcal{H}} \\
&\quad - \frac{1}{m}\|\mathcal{S}_2 v_1 w\|_{\mathcal{H}}\|w\|_{\mathcal{H}} \\
&\geq \langle v_1 w, w\rangle_{\mathcal{H}} + \frac{L}{5m}\|w\|_{\mathcal{H}}^2 - \frac{1}{m}\|\mathcal{S}_2 v_1\|\|w\|_{\mathcal{H}}^2 \\
&\geq \langle v_1 w, w\rangle_{\mathcal{H}} + \frac{L}{5m}\|w\|_{\mathcal{H}}^2 - \frac{A\vartheta_1}{m}\|w\|_{\mathcal{H}}^2 \\
&= \langle v_1 w, w\rangle_{\mathcal{H}} + \frac{1}{m}\left(\frac{L}{5} - A\vartheta_1\right)\|w\|_{\mathcal{H}}^2 \\
&\geq 0.
\end{aligned}$$
Therefore $v_2 \in \Omega$ and
$$-\epsilon m v_2 = -\epsilon m v_1 - \epsilon \mathcal{S}_2 v_1 - \epsilon\frac{L}{10}\mathcal{J} - \epsilon\frac{L}{10}\mathcal{J}$$
or
$$\begin{aligned}
(\mathcal{I} - \mathcal{T}_1)v_2 &= -\epsilon m v_2 + \epsilon\frac{L}{10}\mathcal{J} \\
&= \mathcal{S}_3 v_1.
\end{aligned}$$
Consequently $\mathcal{S}_3(\overline{\mathcal{Q}_{R_1}}) \subset (\mathcal{I} - \mathcal{T}_1)(\Omega)$.
4. Assume that for any $v_0 \in \mathcal{Q}^* = \mathcal{Q}\backslash\{0\}$, there exist $\lambda > 0$ and $v \in \partial\mathcal{Q}_r \cap (\Omega + \lambda v_0)$ or $v \in \partial\mathcal{Q}_{R_1} \cap (\Omega + \lambda v_0)$ such that
$$\mathcal{S}_3 v = (\mathcal{I} - \mathcal{T}_1)(v - \lambda v_0).$$
Then
$$-\epsilon\mathcal{S}_2 v - m\epsilon v - \epsilon\frac{L}{10}\mathcal{J} = -m\epsilon(v - \lambda v_0) + \epsilon\frac{L}{10}\mathcal{J}$$
or
$$-\mathcal{S}_2 v = \lambda m v_0 + \frac{L}{5}\mathcal{J}.$$
Hence,
$$\|\mathcal{S}_2 v\| = \left\|\lambda m v_0 + \frac{L}{5}\mathcal{J}\right\| > \frac{L}{5}.$$

This is a contradiction.

5. Let $\epsilon_1 = \frac{2}{5m}$. Suppose that there exist a $v_1 \in \partial \mathcal{Q}_L$ and $\lambda_1 \geq 1 + \epsilon_1$ such that
$$\mathcal{S}_3 v_1 = (\mathcal{I} - \mathcal{T}_1)(\lambda_1 v_1). \tag{3.1}$$
Moreover,
$$-\epsilon \mathcal{S}_2 v_1 - m\epsilon v_1 - \epsilon \frac{L}{10} \mathcal{J} = -\lambda_1 m \epsilon v_1 + \epsilon \frac{L}{10} \mathcal{J},$$
or
$$\mathcal{S}_2 v_1 + \frac{L}{5} \mathcal{J} = (\lambda_1 - 1) m v_1.$$
From here,
$$2\frac{L}{5} > \left\| \mathcal{S}_2 v_1 + \frac{L}{5} \mathcal{J} \right\| = (\lambda_1 - 1) m \|v_1\| = (\lambda_1 - 1) m L$$
and
$$\frac{2}{5m} + 1 > \lambda_1,$$
which is a contradiction.

Therefore all conditions of Theorem 2.4 hold. Hence, the problem (1.2) has at least two solutions u_1 and u_2 so that
$$\|u_1\| = L < \|u_2\| \leq R_1$$
or
$$r \leq \|u_1\| < L < \|u_2\| \leq R_1.$$
□

4. Example

As an illustration of our main results, we consider the following equation
$$u''(t) + \|u'(t)\|^2 u''(t) + \frac{2}{\Gamma\left(\frac{1}{2}\right)} \int_0^t (t-s)^{-\frac{1}{2}} e^{-3(t-\tau)} u'(\tau) d\tau = \|u(t)\|^2 w,$$
for $t \in [0,1]$, and a fixed non zero $w \in \mathcal{H}$, subject to the initial conditions
$$u(0) = u'(0) = 1.$$
Here
$$\begin{aligned} r = m = n = l = s_1 = \varpi = \chi = 1, \\ p_1 = q_1 = r_1 = \gamma = 2, \\ \theta_0(t) = \vartheta_0(t) = \zeta_0(t) = \varsigma_0(t) = 0, \\ \theta_1(t) = \vartheta_1(t) = \zeta_1(t) = \varsigma_1(t) = 1, \quad t \in [0,1], \end{aligned}$$
and $\vartheta_1 = 23 + \frac{4}{\sqrt{\pi}}$. Take
$$R_1 = \frac{9}{10}, \quad L = \frac{3}{5}, \quad r = \frac{2}{5}, \quad A = \frac{L}{10\vartheta_1}.$$
One can check that all conditions of Theorem 3.1 and Theorem 3.2 are fulfilled.

5. Conclusion

We discussed the existence of at least one or two global solutions to an abstract quasi-linear nonhomogeneous evolution equation involving fractional dissipation term subject to initial boundary conditions. We used well known fixed point theorems for the sum of two operators to achieve existence and multiplicity criteria.

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