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Some existence results for a class of parabolic equations with nonlinear boundary conditions

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Abstract. In this paper, we are interested in studying the weak solutions for the following nonlinear parabolic problem:

$$\begin{cases} u_t - \Delta_p u + |u|^{p-2}u = 0 & \text{in } \ \Omega, \ t > 0, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = g(u) & \text{on } \ \partial\Omega, \ t > 0, \\ u(x;0) = u_0(x) & \text{in } \ \Omega. \end{cases}$$

Using the Galerkin approximation and a family of potential wells, we establish the existence of a global weak solution under appropriate conditions. Additionally, we provide a result on the blow-up and asymptotic behavior of certain solutions with positive initial energy.

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1. Introduction and main results

Parabolic problems are an important class of partial differential equations (PDEs) that frequently appear in models of physical phenomena, such as heat diffusion (see [2, 4, 6, 8, 5]), diffusion of substances in fluids, and population dynamics (see [13, 12, 11, 3]). The motivation for studying parabolic problems lies in their ability to describe time-evolving processes in which a quantity diffuses or propagates in a medium. For example, in engineering, parabolic PDEs are used to design heat management systems, such as cooling systems in electronic devices, or to study the properties of materials under various temperature conditions (see [23, 19, 21, 1]).

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In this paper, we deal with the following nonlinear parabolic problem:

$$\begin{cases} u_t - \Delta_p u + |u|^{p-2}u = 0 & \text{in } \Omega,\ t > 0, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = g(u) & \text{on } \partial\Omega,\ t > 0, \\ u(x;0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ is a bounded domain with smooth boundary $\partial\Omega$ and $g(u)$ satisfies the following conditions:

$$\text{(C)} \quad \begin{cases} g \in C^1 \text{ and } g(0) = g'(0) = 0; \\ g(u) \text{ is monotone, concave for } u < 0 \text{ and convex for } u > 0; \\ (q+1)G(u) \leqslant ug(u), |ug(u)| \leqslant \mu|G(u)|; \end{cases}$$

where

$$G(u) = \int_0^u g(s)\mathrm{d}s,$$

$$\begin{cases} 2 < q+1 \leqslant \mu < \infty \text{ if } n = 2, \\ 2 < q+1 \leqslant \mu \leqslant \frac{2(n-1)}{n-2} \text{ if } n \geqslant 3, \end{cases} \quad \text{and} \quad \begin{cases} 1 \leq \mu \leq p^\partial \text{ if } p \neq n, \\ 1 \leq \mu < \infty \text{ if } p = n, \end{cases}$$

with

$$p^\partial := \begin{cases} \frac{p(n-1)}{n-p} \text{ if } 1 < p < n, \\ \infty \text{ if } p \geq n. \end{cases}$$

Equations of the form

$$u_t - \Delta_p u = u^q,$$

are also called the non-Newtonian filtration equations, which are known as fast diffusive for $1 < p < 2$, and as slow diffusive for $p > 2$.
From a physical point of view, the nonlinear boundary value condition:

$$|\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = g(u),$$

can be physically interpreted as the nonlinear radial law (see [9, 16]).

In the literature, many works have been devoted to nonlinear parabolic problems (see [17, 10, 15]). For example, Y. Li and C. Xie [17] have considered the following Dirichlet problem:

$$\begin{cases} u_t - \Delta_p u = \lambda|u|^{q-2}u & \text{in } \Omega \times (0,T), \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x;0) = u_0(x) & \text{in } \Omega. \end{cases}$$

The authors have given a complete overview of the explosion criteria for the above problem. More specifically, in the critical case $q = p > 2$, they proved that if $\lambda > \lambda_1$, there are no global non-trivial weak solutions, and that if $\lambda \leq \lambda_1$, all weak solutions are global, where λ_1 is the first eigenvalue of the following eigenvalue problem:

$$\begin{cases} -\Delta_p \varphi = \lambda \mid \varphi \mid^{p-2} \varphi & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega. \end{cases}$$

Also, Y. Jingxue et al. in [10] studied the following nonlinear parabolic problem:

$$\begin{cases} u_t - \Delta_q u = u^{q_1} & \text{in } \Omega,\ t>0, \\ |\nabla u|^{q-2}\frac{\partial u}{\partial \nu} = u^{q_2} & \text{on } \partial\Omega,\ t>0, \\ u(x;0) = u_0(x) & \text{in } \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary, $q>1, q_1, q_2 > 0$ are all constants and $u_0 \in L^\infty(\Omega) \cap W^{1,q}(\Omega)$. The authors studied the critical exponents of the explosion of the evolutionary q-Laplacian with multiple sources, and showed that the critical values of (q_1, q_2) are $q_1^* = 1$ and $q_2^* = \min\{1, 2(q-1)/q\}$.
Recently, in [15], A. Lamaizi et al. have considered the following problem:

$$\begin{cases} u_t - \Delta_p u + |u|^{p-2}u = 0 & \text{in } \Omega,\ t>0, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda |u|^q u & \text{on } \partial\Omega,\ t>0, \\ u(x;0) = u_0(x) & \text{in } \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is an open bounded domain for $n \geq 2$ with smooth boundary $\partial\Omega$, $\lambda > 0$, p and q satisfy

$$(H) \qquad \frac{2n}{n+2} \leq p < +\infty\ ,\ p < 2+q \quad \text{and} \quad \begin{cases} 1 \leq q+2 \leq p^{\partial} \text{ if } p \neq n, \\ 1 \leq q+2 < \infty \text{ if } p = n. \end{cases}$$

By using the Galerkin approximation, they established the existence of global weak solution and finite time blow-up under some suitable conditions. So a natural question arises, can we obtain some qualitative results such as the existence and blow up of solutions if we replace the term $\lambda|u|^q u$ by the function $g(u)$ which satisfies condition (C) ? Then, the goal of this article is to give a positive answer to this question, more precisely, we will establish existence and blow up results by applying Galerkin approximation and similar techniques to those used in [15].

In order to investigate our problem, it is necessary to define some functionals and sets:

$$B(u) = \frac{1}{p}\|u\|_{1,p}^p - \int_{\partial\Omega} G(u)d\rho,$$
$$A(u) = \|u\|_{1,p}^p - \int_{\partial\Omega} ug(u)d\rho,$$
$$S = \left\{u \in W^{1,p}(\Omega) \mid A(u) > 0, B(u) < h\right\} \cup \{0\},$$

where

$$h = \inf_{u \in Y} B(u),$$
$$Y = \left\{u \in W^{1,p}(\Omega) \mid A(u) = 0, \|u\|_{1,p} \neq 0\right\},$$

and

$$U = \left\{u \in W^{1,p}(\Omega) \mid A(u) < 0, B(u) < h\right\}.$$

We are now in a position to state the main results of this paper.

Theorem 1.1. (**Global Existence**)
Let $u_0(x) \in W^{1,p}(\Omega)$ and $g(u)$ satisfy (C). Suppose that $0 < B(u_0) < h$ and $A(u_0) > 0$. Then, problem (1.1) admits a global weak solution $u(t) \in L^\infty\left(0,\infty; W^{1,p}(\Omega)\right) \cap C\left([0,T]; L^2(\Omega) \times L^2(\partial\Omega,\rho)\right)$ with $u_t(t) \in L^2\left(0,\infty; L^2(\Omega)\right)$ and $u(t) \in S$ for $0 \leqslant t < \infty$.

Theorem 1.2. (**Finite Time Blow-up**)
Let $u_0(x) \in W^{1,p}(\Omega)$ and $g(u)$ satisfy (C). Suppose that $B(u_0) < h$ and $A(u_0) < 0$. Then, the weak solution of problem (1.1) must blow up in finite time i.e. there exists a $T > 0$ such that

$$\lim_{t \to T} \int_0^t \|u\|_2^2 \, \mathrm{d}\tau = +\infty. \tag{1.2}$$

Theorem 1.3. (**Asymptotic Behavior**)
Let $u_0(x) \in W^{1,p}(\Omega)$ and $g(u)$ satisfy (C). Suppose also that $B(u_0) < h$ and $A(u_0) > 0$. Then, for the weak global solution $u(t)$ of problem (1.1) there exists a constant $\omega > 0$ such as:

$$\|u(t)\|_2^2 \leq \|u_0\|_2^2 \, \mathrm{e}^{-\omega t}, \quad 0 \leq t < \infty. \tag{1.3}$$

2. Preliminaries

Throughout this work, we denote the Lebesgue space $L^p(\Omega)$ by :

$$L^p(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ measurable such that } \int_\Omega |u(x)|^p dx < +\infty \right\},$$

endowed with the norm

$$\|u\|_p = \left(\int_\Omega |u(x)|^p dx \right)^{\frac{1}{p}}.$$

For $p = \infty$, we denote

$$L^\infty(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ measurable such that } \operatorname{ess-sup}_\Omega |u| < +\infty \right\},$$

equipped with the norm

$$\operatorname{ess-sup}_\Omega |u| = \inf\{C > 0 \text{ such that } |u(x)| \leq C \text{ a.e. } \Omega\}.$$

Next, for simplicity, we denote

$$\langle u, v \rangle = \int_\Omega uv \, dx \text{ and } \langle u, v \rangle_0 = \oint_{\partial\Omega} uv \, d\rho.$$

Moreover, we denote the usual Sobolev space on Ω

$$W^{1,p}(\Omega) = \{u \in L^p(\Omega) : |\nabla u| \in L^p(\Omega)\},$$

endowed with the norm

$$\|u\|_{1,p} = \|u\|_p + \|\nabla u\|_p,$$

or to the equivalent norm

$$\|u\|_{1,p} = \left(\|u\|_p^p + \|\nabla u\|_p^p\right)^{\frac{1}{p}}, \text{ if } 1 \le p < +\infty.$$

Proposition 2.1. (See [2]). The trace operator $u: W^{1,p}(\Omega) \to L^q(\partial\Omega, \rho)$ is continuous if and only if

$$\begin{cases} 1 \le q \le p^\partial \text{ if } p \ne n, \\ 1 \le q < \infty \text{ if } p = n, \end{cases}$$

Let X be a Banach space and $T > 0$. Denote the following spaces:

$$C([0,T];X) = \{u : [0,T] \longrightarrow X \text{ continue }\},$$

$$L^p(0,T;X) = \left\{ u : [0,T] \longrightarrow X \text{ is a measurable such that } \int_0^T \|u(t)\|_X^p dt < \infty \right\},$$

equipped with the norms

$$\|u\|_{L^p(0,T;X)} = \left(\int_0^T \|u(t)\|_X^p dt\right)^{\frac{1}{p}},$$

and

$$L^\infty(0,T;X) = \{u : [0,T] \longrightarrow X \text{ is a measurable such that } : \exists C > 0; \|u(t)\|_X < C \text{ a.e.t}\},$$

endowed with the norm

$$\|u\|_{L^\infty(0,T;X)} = \inf \{C > 0; \|u(t)\|_X < C \text{ a.e.t}\}.$$

In addition, for $\theta > 0$ we define

$$A_\theta(u) = \theta \|u\|_{1,p}^p - \int_{\partial\Omega} u g(u) \mathrm{d}\rho,$$

$$h(\theta) = \inf_{u \in Y_\theta} B(u),$$

$$Y_\theta = \left\{u \in W^{1,p}(\Omega) \mid A_\theta(u) = 0, \|u\|_{1,p} \ne 0\right\},$$

$$S_\theta = \left\{u \in W^{1,p}(\Omega) \mid A_\theta(u) > 0, B(u) < h(\theta)\right\} \cup \{0\},$$

and

$$U_\theta = \left\{u \in W^{1,p}(\Omega) \mid A_\theta(u) < 0, B(u) < h(\theta)\right\}.$$

3. Proof of main results

3.1. Proof of Theorem 1.1

In this part, we prove the global existence result. Firstly, we give the definition of the weak solution and some lemmas which will be used later.

Definition 3.1. Let $u(x,0) = u_0 \in W^{1,p}(\Omega)$. A **weak solution** of problem (1.1) is a function $u : \Omega \times (0;T) \to \mathbb{R}$ such that:
(i) $u = u(x,t) \in L^\infty\left(0,\infty; W^{1,p}(\Omega)\right) \cap C\left([0,T]; L^p(\Omega) \times L^p(\partial\Omega,\rho)\right);$
(ii) $u_t(t) \in L^2\left(0,\infty; L^2(\Omega)\right);$
(iii) for any $v \in W^{1,p}(\Omega)$, and for almost all $t \in [0,T]$ it holds

$$\langle u_t, v\rangle + \langle |u|^{p-2}u, v\rangle + \langle |\nabla u|^{p-2}\nabla u, \nabla v\rangle = \langle g(u), v\rangle_0; \tag{3.1}$$

(iv)

$$\int_0^t \|u_t\|_2^2 \, d\tau + B(u) \leqslant B(u_0), \quad \forall t \in [0,T). \tag{3.2}$$

Lemma 3.2. ([20]). Let $g(u)$ satisfy (C). Then,

1. $|G(u)| \leqslant M|u|^\mu$ for some $M > 0$ and all $u \in \mathbb{R}$.
2. $G(u) \geqslant N|u|^{p+1}$ for some $N > 0$ and $|u| \geqslant 1$.
3. The equality $u\left(ug'(u) - g(u)\right) \geqslant 0$ holds only for $u = 0$.

Corollary 3.3. ([20]). Let $g(u)$ satisfy (C). Then,

1. $|ug(u)| \leqslant \mu M|u|^\mu, |g(u)| \leqslant \mu M|u|^{\mu-1}$ for all $u \in \mathbb{R}$.
2. $ug(u) \geqslant (p+1)N|u|^{p+1}$ for $|u| \geqslant 1$.

Lemma 3.4. Let $\theta_1 < \theta_2$ be the two roots of equation $h(\theta) = B(u)$. Then, the sign of $A_\theta(u)$ does not change for $\theta \in (\theta_1, \theta_2)$, provided $0 < B(u) < h$ for some $u \in W^{1,p}(\Omega)$.

Proof. If this were false, there would exist a $\theta_0 \in (\theta_1, \theta_2)$ such as $A_{\theta_0}(u) = 0$. By $B(u) > 0$, we have $\|u\|_{1,p} \neq 0$, consequently $u \in Y_{\theta_0}$. Then, $B(u) \geq h(\theta_0)$, which contradicts

$$B(u) = h(\theta_1) = h(\theta_2) < h(\theta_0).$$

□

Lemma 3.5. Let $g(u)$ satisfy (C), $u_0(x) \in W^{1,p}(\Omega), 0 < e < h$ and $\theta_1 < \theta_2$ be the two roots of equation $h(\theta) = e$. Assume that $A(u_0) > 0$, then all weak solutions $u(t)$ of problem (1.1) with $B(u_0) = e$ belong to S_θ for $\theta \in (\theta_1, \theta_2)$ and $0 \leq t < T$.

Proof. By $B(u_0) = e, A(u_0) > 0$ and Lemma 3.4 we obtain $A_\theta(u_0) > 0$ and $B(u_0) < h(\theta)$ i.e. $u_0(x) \in S_\theta$ for $\theta \in (\theta_1, \theta_2)$. Let $u(t)$ be any weak solution of problem (1.1) with $A(u_0) > 0$ and $B(u_0) = e$, and T be the maximal existence time of $u(t)$. Next, we show that $u(t) \in S_\theta$ for $\theta_1 < \theta < \theta_2$ and $0 < t < T$. If it is false, so it must exists a $\theta_0 \in (\theta_1, \theta_2)$ and $t_0 \in (0,T)$ such as

$$A_{\theta_0}(u(t_0)) = 0, \|u(t_0)\|_{1,p} \neq 0 \text{ or } B(u(t_0)) = h(\theta_0).$$

From (3.2), it follows that

$$\int_0^t \|u_\tau\|_2^2 \, d\tau + B(u) \leq B(u_0) < h(\theta), \quad \theta_1 < \theta < \theta_2, \ 0 \leq t < T. \tag{3.3}$$

As a result $B(u(t_0)) \neq h(\theta_0)$. If $A_{\theta_0}(u(t_0)) = 0, \|u(t_0)\|_{1,p} \neq 0$, thus the definition of $h(\theta)$ means that $B(u(t_0)) \geq h(\theta_0)$, which contradicts (3.3). □

Proof of Theorem 1.1. The idea of the proof is classical, for more details see [7, 14, 22]. Let $w_j(x)$ be a system of base functions in $W^{1,p}(\Omega)$. Define the approximate solutions $u_m(x,t)$ of problem (1.1)

$$u_m(x,t) = \sum_{j=1}^{m} \Phi_{jm}(t) w_j(x), \quad m = 1, 2, \ldots$$

verifying

$$\langle u_{mt}, w_s\rangle + \langle |u_m|^{p-2} u_m, w_s\rangle + \langle |\nabla u_m|^{p-2} \nabla u_m, \nabla w_s\rangle = \langle g(u_m), w_s\rangle_0, \quad s = 1, 2, \ldots, m \tag{3.4}$$

and

$$u_m(x,0) = \sum_{j=1}^{m} a_{jm} w_j(x) \to u_0(x) \quad \text{in } W^{1,p}(\Omega). \tag{3.5}$$

Multiplying (3.4) by $\Phi'_{sm}(t)$ and summing for s yields

$$\int_0^t \|u_{mt}\|_2^2 \, \mathrm{d}\tau + B(u_m) \leqslant B(u_0) < h, \quad \forall t \in [0, T), \tag{3.6}$$

and $u_m \in S$ for $0 \leqslant t < \infty$ (see the proof of Lemma 3.5).
Combining (3.6) and

$$\begin{aligned} B(u_m) = \frac{1}{p}\|u_m\|_{1,p}^p - \int_{\partial\Omega} G(u_m)\,\mathrm{d}\rho &\geqslant \frac{1}{p}\|u_m\|_{1,p}^p - \frac{1}{q+1}\int_{\partial\Omega} u_m g(u_m)\,\mathrm{d}\rho \\ &= \left(\frac{1}{p} - \frac{1}{q+1}\right)\|u_m\|_{1,p}^p + \frac{1}{q+1} A(u_m) \\ &\geqslant \frac{q-p+1}{p(q+1)}\|u_m\|_{1,p}^p, \end{aligned}$$

we obtain

$$\int_0^t \|u_{mt}\|_2^2 \, \mathrm{d}\tau + \frac{q-p+1}{p(q+1)}\|u_m\|_{1,p}^p < h, \quad 0 \leqslant t < \infty. \tag{3.7}$$

From (3.7), we get

$$\|u_m\|_{1,p}^p < \frac{p(q+1)}{q-p+1} h, \quad 0 \leqslant t < \infty, \tag{3.8}$$

$$\||u_m|^{p-2} u_m\|_s^s = \|u_m\|_p^p < \frac{p(q+1)}{q-p+1} h, \quad s = \frac{p}{p-1}, \ 0 \leqslant t < \infty,$$

$$\|u_m\|_{\mu,\partial\Omega} \leqslant C_* \|u_m\|_{1,p} < C_* \left(\frac{p(q+1)}{q-p+1} h\right)^{\frac{1}{p}}, \quad 0 \leqslant t < \infty, \tag{3.9}$$

$$\begin{aligned} \|g(u_m)\|_{r,\partial\Omega}^r &\leqslant \int_{\partial\Omega} \left(\mu M |u_m|^{\mu-1}\right)^r \, \mathrm{d}\rho \\ &= (\mu M)^r \|u_m\|_{\mu,\partial\Omega}^{\mu} \\ &\leqslant (\mu M)^r C_*^{\mu} \left(\frac{p(q+1)}{q-p+1} h\right)^{\frac{\mu}{p}}, \quad r = \frac{\mu}{\mu-1}, \quad 0 \leqslant t < \infty, \end{aligned} \tag{3.10}$$

where C_* is the embedding constant from $W^{1,p}(\Omega)$ into $L^\mu(\partial\Omega)$.
Furthermore

$$\int_0^t \|u_{mt}\|_2^2 \, \mathrm{d}\tau < h, \quad 0 \leqslant t < \infty. \tag{3.11}$$

Therefore, there exist u, ϕ and a subsequence $\{u_v\}$ of $\{u_m\}$ such as

$$\begin{aligned}
&u_v \to u \text{ in } L^\infty\left(0,\infty; W^{1,p}(\Omega)\right) \text{ weakly star},\\
&u_{vt} \to u_t \text{ in } L^2\left(0,\infty; L^2(\Omega)\right) \text{ weakly },\\
&|u_v|^{p-2}u_v \to |u|^{p-2}u \text{ in } L^\infty\left(0,\infty; L^s(\Omega)\right) \text{ weakly star},\\
&g\left(u_v\right) \to \phi \text{ in } L^\infty\left(0,\infty; L^r(\partial\Omega)\right) \text{ weakly star, and a.e. in } \partial\Omega \times [0,\infty).
\end{aligned}$$

Consequently, from Lemma (1.3) in [18], we deduce $\phi = g(u)$. In (3.4) for fixed s letting $m = v \to \infty$, we have

$$\langle u_t, w_s\rangle + \langle |u|^{p-2}u, w_s\rangle + \langle |\nabla u|^{p-2}\nabla u, \nabla w_s\rangle = \langle g\left(u\right), w_s\rangle_0, \quad \forall s,$$

and

$$\langle u_t, v\rangle + \langle |u|^{p-2}u, v\rangle + \langle |\nabla u|^{p-2}\nabla u, \nabla v\rangle = \langle g\left(u\right), v\rangle_0, \ \forall v \in W^{1,p}(\Omega).$$

By (3.5), we obtain $u(x,0) = u_0(x)$ in $W^{1,p}(\Omega)$. Then $u(t)$ is a global weak solution of problem (1.1). Finally, by applying Lemma 3.5 we deduce that the solution $u(t) \in S$.

3.2. Proof of Theorem 1.2

To prove Theorem 1.2, we need the following auxiliary lemmas.

Lemma 3.6. Let $g(u)$ satisfy (C), and $A_\theta(u) < 0$. Then, $\|u\|_{1,p} > z(\theta)$. In particular, if $A(u) < 0$, then $\|u\|_{1,p} > z(1)$.

Where

$$z(\theta) = \left(\frac{\theta}{aC_*^\mu}\right)^{1/(\mu-2)},$$

with C_* is the embedding constant form $W^{1,p}(\Omega)$ into $L^\mu(\partial\Omega)$, and

$$a = \sup \frac{ug(u)}{|u|^\mu}.$$

Proof. $A_\delta(u) < 0$ gives

$$\theta\|u\|_{1,p}^2 < \int_{\partial\Omega} ug(u)\mathrm{d}\rho \leqslant a\|u\|_{\mu,\partial\Omega}^\mu \leqslant aC_*^\mu \|u\|_{1,p}^{\mu-2}\|u\|_{1,p}^2, \tag{3.12}$$

then $\|u\|_{1,p} > z(\theta)$. □

Lemma 3.7. Let $g(u)$ satisfy (C), and $u_0(x) \in W^{1,p}(\Omega)$. Suppose that $0 < e < h$, $\theta_1 < \theta_2$ are the two roots of equation $h(\theta) = e$. Then, all weak solutions of problem (1.1) with $B\left(u_0\right) = e$ belong to $\theta \in (\theta_1, \theta_2)$, provided $A\left(u_0\right) < 0$.

Proof. Let $u(t)$ be any solution of problem (1.1) with $B(u_0) = e$, $A(u_0) < 0$ and T be the existence time of $u(t)$. First from $B(u_0) = e$, $A(u_0) < 0$ and Lemma 3.4 we can deduce $A_\theta(u_0) < 0$ and $B(u_0) < h(\theta)$, i.e. $u_0(x) \in U_\theta$ for $\theta \in (\theta_1, \theta_2)$.
Next, we prove $u(t) \in U_\theta$ for $\theta_1 < \theta < \theta_2$ and $0 < t < T$. If it is false, let $t_0 \in (0, T)$ be the first time such that $u(t) \in U_\theta$ for $0 \leqslant t < t_0$ and $u(t_0) \in \partial U_\theta$, i.e. $A_\theta(u(t_0)) = 0$ or $B(u(t_0)) = h(\theta)$ for some $\theta \in (\theta_1, \theta_2)$. So (3.3) implies $B(u(t_0)) = h(\theta)$ is impossible. If $A_\theta(u(t_0)) = 0$, thus $A_\theta(u(t)) < 0$ for $t \in (0, t_0)$ and Lemma 3.6 yield $\|u(t)\|_{1,p} > z(\theta)$ and $\|u(t_0)\|_{1,p} \geqslant z(\theta)$. Therefore by the definition of $h(\theta)$ we have $B(u(t_0)) \geqslant h(\theta)$ which contradicts (3.3). □

Proof of Theorem 1.2. Let $u(t)$ be any weak solution of problem (1.1) with $B(u_0) < h$ and $A(u_0) < 0$. We consider the auxiliary function

$$\varphi_1(t) = \int_0^t \|u\|_2^2 \, \mathrm{d}\tau.$$

A direct calculation gives

$$\dot{\varphi}_1(t) = \|u\|_2^2,$$

and

$$\ddot{\varphi}_1(t) = 2\langle u_t, u\rangle = 2\left(\langle g(u), u\rangle - \|u\|_{1,p}^p\right) = -2A(u). \tag{3.13}$$

By (3.13), (3.2) and

$$\int_{\partial\Omega} ug(u)\mathrm{d}\rho \geqslant (q+1)\int_{\partial\Omega} G(u)\mathrm{d}\rho$$

we can deduce

$$\begin{aligned}\ddot{\varphi}_1(t) &\geqslant 2(q+1)\int_0^t \|u_t\|_2^2 \, \mathrm{d}\tau + (q-1)\|u\|_{1,p}^2 - 2(q+1)B(u_0)\\ &\geqslant 2(q+1)\int_0^t \|u_t\|_2^2 \, \mathrm{d}\tau + (q-1)\dot{\varphi}_1(t) - 2(q+1)B(u_0),\end{aligned}$$

and

$$\begin{aligned}\varphi_1\ddot{\varphi}_1 - \frac{q+1}{2}(\dot{\varphi}_1)^2 \geqslant& 2(q+1)\left[\int_0^t \|u\|_2^2 \, \mathrm{d}\tau \int_0^t \|u_t\|_2^2 \, \mathrm{d}\tau - \left(\int_0^t \langle u, u_t\rangle \mathrm{d}\tau\right)^2\right]\\ &+ (q-1)\varphi_1\dot{\varphi}_1 - (q+1)\|u_0\|_2^2\dot{\varphi}_1\\ &- 2(q+1)B(u_0)\varphi_1 + \frac{q+1}{2}\|u_0\|_2^2.\end{aligned}$$

Making use of the Hölder inequality, we get

$$\begin{aligned}\varphi_1\ddot{\varphi}_1 - \frac{q+1}{2}(\dot{\varphi}_1)^2 \geqslant& (q-1)\varphi_1\dot{\varphi}_1 - (q+1)\|u_0\|_2^2\dot{\varphi}_1\\ &- 2(q+1)B(u_0)\varphi_1 + \frac{q+1}{2}\|u_0\|_2^2.\end{aligned} \tag{3.14}$$

1. If $B(u_0) \leqslant 0$, then

$$\varphi_1\ddot{\varphi}_1 - \frac{q+1}{2}(\dot{\varphi}_1)^2 \geqslant (q-1)\varphi_1\dot{\varphi}_1 - (q+1)\|u_0\|_2^2\dot{\varphi}_1.$$

The next task is to prove that $A(u) < 0$ for $t > 0$. Otherwise, we assume the existence of a $t_0 > 0$ so that $A(u(t_0)) = 0$.
Next, let $t_0 > 0$ be the first time such as $A(u(t)) = 0$, thus $A(u(t)) < 0$ for $t \in [0, t_0)$. From Lemma 3.6 we obtain $\|u\|_{1,p} > z(1)$ for $t \in (0, t_0)$. Consequently, we obtain $\|u(t_0)\|_{1,p} \geqslant z(1)$ and $B(u(t_0)) \geqslant h$ which contradicts (3.2). Then, from (3.13) we have $\ddot{\varphi}_1(t) > 0$ for $t > 0$. By this and $\dot{\varphi}_1(0) = \|u_0\|_2^2 \geqslant 0$, then there exists a $t_0 \geqslant 0$ such as $\dot{\varphi}_1(t_0) > 0$ and

$$\varphi_1(t) \geqslant \dot{\varphi}_1(t_0)(t-t_0) + \varphi_1(t_0) \geqslant \dot{\varphi}_1(t_0)(t-t_0), \quad t \geqslant t_0.$$

Therefore for sufficiently large t we can deduce

$$(q-1)\varphi_1 > (q+1)\|u_0\|_2^2,$$

and

$$\varphi_1(t)\ddot{\varphi}_1(t) - \frac{q+1}{2}(\dot{\varphi}_1(t))^2 > 0. \tag{3.15}$$

Since, for $t > 0$

$$\left(\varphi_1^{-\beta}(t)\right)'' = -\frac{\beta}{\varphi_1^{\beta+2}(t)}\left(\varphi_1(t)\ddot{\varphi}_1(t) - (\beta+1)\dot{\varphi}_1(t)^2\right),$$

we see that for $\beta = \frac{q-1}{2}$ we have $\left(\varphi_1^{-\beta}(t)\right)'' < 0$. Therefore $\varphi_1^{-\beta}(t)$ is concave for sufficiently large t, and there exists a finite time T for which $\varphi_1^{-\beta}(t) \to 0$. In other words,

$$\lim_{t\to T^-}\varphi_1(t) = +\infty.$$

2. If $0 < B(u_0) < h$, thus by Lemma 3.7, we have $u(t) \in U_\theta$ for $1 < \theta < \theta_2$ and $t > 0$, where θ_2 is the larger root of equation $h(\theta) = B(u_0)$. Therefore $A_\theta(u) < 0$ and from Lemma 3.6 we deduce $\|u\|_{1,p} > z(\theta)$ for $1 < \theta < \theta_2$ and $t > 0$. Then, we have $A_{\theta_2}(u) \leqslant 0$ and $\|u\|_{1,p} \geqslant z(\theta_2)$ for $t > 0$. Thus (3.13) gives

$$\ddot{\varphi}_1(t) = -2A(u) = 2(\theta_2-1)\|u\|_{1,p}^p - 2A_{\theta_2}(u) \geqslant 2(\theta_2-1)z^p(\theta_2) > 0, \quad t \geqslant 0,$$
$$\dot{\varphi}_1(t) \geqslant 2(\theta_2-1)z^p(\theta_2)t + \dot{\varphi}_1(0) \geqslant 2(\theta_p-1)z^p(\theta_2)t, \quad t \geqslant 0,$$
$$\varphi_1(t) \geqslant (\theta_2-1)z^p(\theta_2)t^2 + \varphi_1(0) = (\theta_2-1)z^p(\theta_2)t^2, \quad t \geqslant 0.$$

Therefore for sufficiently large t we get

$$\frac{1}{2}(q-1)\varphi_1(t) > (q+1)\|u_0\|_2^2,$$
$$\frac{1}{2}(q-1)\dot{\varphi}_1(t) > 2(q+1)B(u_0).$$

Hence from (3.14) we again obtain (3.15) for sufficiently large t. The remainder of the proof is similar to that in the proof of (i).

3.3. Proof of Theorem 1.3

Under the conditions of Theorem 1.3, and according to Theorem 1.1, the problem (1.1) has a global weak solution. Next, multiplying (3.1) by any $h(t) \in C[0,\infty)$, we have

$$\langle u_t, h(t)v\rangle + \langle |u|^{p-2}u, h(t)v\rangle + \langle |\nabla u|^{p-2}\nabla u, \nabla(h(t)v)\rangle = \langle g(u), h(t)v\rangle_0,\ \forall v \in W^{1,p}(\Omega),$$

and $t \in (0,T)$, consequently

$$\langle u_t, \varphi\rangle + \langle |u|^{p-2}u, \varphi\rangle + \langle |\nabla u|^{p-2}\nabla u, \nabla\varphi\rangle = \langle g(u), \varphi\rangle_0, \quad \forall \varphi \in L^\infty\left(0,\infty; W^{1,p}(\Omega)\right), \tag{3.16}$$

and $t \in (0,T)$.

Setting $\varphi = u$, (3.16) implies

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_2^2 + A(u) = 0, \quad 0 \le t < \infty. \tag{3.17}$$

By $0 < B(u_0) < h, A(u_0) > 0$ and Lemma 3.5, we obtain $u(t) \in S_\theta$ for $\theta_1 < \theta < \theta_2$ and $0 \le t < \infty$, where $\theta_1 < \theta_2$ are the two roots of equation $h(\theta) = B(u_0)$. Consequently, we get $A_\theta(u) \ge 0$ for $\theta_1 < \theta < \theta_2$ and $A_{\theta_1}(u) \ge 0$ for $0 \le t < \infty$. Then, (3.17) leads to

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_2^2 + (1-\theta_1)\|u\|_{1,p}^p + A_{\theta_1}(u) = 0, \quad 0 \le t < \infty,$$

accordingly

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_2^2 + (1-\theta_1)\|u\|_2^2 \le 0, \quad 0 \le t < \infty.$$

Finally, Gronwall's inequality, leads to

$$\|u\|_2^2 \le \|u_0\|_2^2\,\mathrm{e}^{-2(1-\theta_1)t}, \quad 0 \le t < \infty.$$

This completes the proof of the Theorem 1.3.

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