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Integral solution to a parabolic equation involving the fractional p-Laplacian operator

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Abstract. The aim of this work is to study the existence and uniqueness of integral solutions for a class of non-local parabolic equations. There are two main results. First, we use a subdifferential technique to verify the existence and uniqueness of weak solutions when the initial data belong to L^2. Secondly, the existence and uniqueness of an integral solution is demonstrated by extending the study to initial data in L^1 space. To overcome the difficulties caused by non-local terms, the proposed strategy combines new approaches with sophisticated strategies derived from the theory of accretive operators. These results contribute to a better understanding of nonlocal evolution equations and their applications.

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1. Introduction

The main purpose of our work is to study a parabolic equation that includes the fractional p-Laplacian operator, which is modeled as follows:

$$\begin{cases} \frac{\partial u}{\partial t} + (-\Delta)_p^s u = f & \text{in } Q_T :=]0,T[\times\Omega \\ u = 0 & \text{on }]0,T[\times(\mathbb{R}^N \setminus \Omega) \\ u(0,x) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1.1}$$

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with $s \in (0,1)$, $p > 1$, $ps < N$, $T > 0$, Ω is a bounded open subset of $\mathbb{R}^N$ and $(-\Delta)^s_p$ is the fractional p-Laplacian operator defined as follows

$$\begin{aligned}(-\Delta)^s_p u(x) :&= P.V \int_{\mathbb{R}^N} \frac{\Theta(u(x)-u(y))}{\mid x-y \mid^{N+sp}}\, dy \\ &= \lim_{\sigma\to 0} \int_{\mathbb{R}^N \setminus B_\sigma(x)} \frac{\Theta(u(x)-u(y))}{\mid x-y \mid^{N+sp}}\, dy,\end{aligned}$$

where $\Theta(\xi) = \mid \xi \mid^{p-2} \xi$, P.V. often abbreviated to signify "in the principal value sense" and $B_\sigma(x) = \{z \in \mathbb{R}^N : \mid x - z \mid < \sigma\}$. Additionally, we consider that u_0 and f are nonnegative functions, satisfying the conditions

$$f \in L^1(Q_T) \quad \text{and} \quad u_0 \in L^1(\Omega). \tag{1.2}$$

There has been a growing interest in the study of nonlocal operators due to their relevance in both pure mathematics and practical applications. These operators naturally emerge in various fields, such as water waves [14, 15, 31], crystal dislocations [33], elasticity problems [29], game theory [11], phase transition [3, 10, 30], Lévy processes [5], flame propagation [12] and quasi-geostrophic flows [13, 21]. A key example of these nonlocal operators is the fractional p-Laplacian operator, which is crucial in various physical and mathematical contexts, including its application in image processing [6, 20, 17].

The value of $(-\Delta)^s_p u(x)$ for each $x \in \Omega$ is not solely dependent on the values of the Ω, but also on the entire $\mathbb{R}^N$, which is a typical characteristic of this operator. The reason is that $u(x)$ is the expected value for a random variable that is linked to a jump process that can leap far from the point x without any prior knowledge. Brownian motion's continuity properties lead to landing on $\partial\Omega$ upon exiting Ω in the classical case. However, exiting can result in landing anywhere outside of Ω as a result of the jump process's nature. In this regard, assigned values of u in $\mathbb{R}^N \setminus \Omega$ rather than on $\partial\Omega$ is how the non-homogeneous Dirichlet boundary condition works.

Numerous references in the literature address the nonlocal Laplacian operator (when $p = 2$). For example, in Ref. [18], Leonori et al. considered problem (1.1) and established both the existence and uniqueness of the solution using duality and approximation techniques, showing that it resides in an appropriate fractional Sobolev space.

In Ref. [35], Vázquez studied problem (1.1) with $f = 0$. He proved the existence of a solution to problem (1.1), referred to as the friendly giant, which takes the form $U(t,x) = t^{\frac{-1}{p-2}} F(x)$, with $F(x)$ is a positive function in Ω, C^α-Hölder continuous and solves an interesting nonlocal elliptic problem. In [34], Vázquez worked on problem (1.1) with $f = 0$ and with the initial condition $\lim_{t\to 0} u(t,x) = u_0(x)$. He demonstrated that for each mass $K > 0$, there is only one self-similar solution with initial data $K\delta(x)$ taking the form

$$V(t,x,K) = K^{sp\gamma} t^{-N\gamma} G(K^{(2-p)\gamma} x t^{-\gamma}), \tag{1.3}$$

where $\gamma = \dfrac{1}{sp - N(2-p)}$ and $G(r)$ is a positive, continuous, radially symmetric function with $G(r) \underset{+\infty}{\sim} r^{-(N+sp)}$. He also showed in the case $u_0 \in L^1(\mathbb{R}^N)$, the estimate $\lim\limits_{t\to+\infty} \| u(t) - V_K(t) \|_1 = 0$, with V_K defined by (1.3).

In Ref.[22], Mazón et al. considered the following problem

$$\begin{cases} \frac{\partial u}{\partial t} = (-\Delta)_p^s u & \text{in }]0,T[\times\Omega \\ u = 0 & \text{on }]0,T[\times(\mathbb{R}^N \setminus \Omega) \\ u(0,x) = u_0(x) & \text{in } \Omega. \end{cases} \tag{1.4}$$

The authors showed that problem (1.4) has a unique strong solution when $u_0 \in L^2(\Omega)$, using Nonlinear Semigroup Theory.

In [32], Teng et al. establish the existence and uniqueness of nonnegative renormalized and entropy solutions to problem (1.1) with L^1 data. They also demonstrate the equivalence of these renormalized and entropy solutions and provide a comparison result.

In Ref. [19], Liao established the local Hölder regularity for weak solutions. The author's proof employs DeGiorgi's iterative technique while enhancing DiBenedetto's intrinsic scaling method. In Ref. [7], Brasco et al. showed space-time Hölder continuity estimates for weak solutions, specifying the exponents explicitly. Their approach relies on a method of iterated discrete differentiation inspired by Moser's technique.

Drawing from the existing literature, we examine the existence and uniqueness of integral solutions for problem (1.1). As far as we are aware, there have been no prior results concerning this specific type of solution for problem (1.1). For this argument, our work is original.

Demonstrating the existence and uniqueness of an integral solution to problem (1.1) presents several challenges. One significant challenge is the involvement of nonlocal and nonlinear fractional p-Laplacian operator. Another issue arises when considering the condition (1.2). Fortunately, innovative methods introduced by Akagi et al. [1], Alaa et al. [2], [25] and Roubicek [28] offer solutions to this challenge. We will be employing these techniques.

The outline of this paper is: Section 2 provides a review notations and definitions for the pertinent fractional Sobolev-type spaces, along with some subdifferential calculus tools. Section 3 focuses on the solvability of problem (1.1) when the data belongs to L^2, we demonstrate the existence and uniqueness of a weak solution using maximal monotone operator theory. In section 4, we prove that problem (1.1) admits a unique integral solution when the data are only in L^1 performing accretive operator theory.

2. Foundational concepts and intermediate outcomes

The main purpose of this section is to provide a summary of certain definitions and fundamental properties of fractional Sobolev spaces, which are essential for the discussion of Problem (1.1). Additionally, we will revisit some characteristics of the subdifferentials of lower semi-continuous, convex and proper functionals within a Hilbert space.

2.1. Fractional Sobolev spaces

We now recall the primary notations and definitions for the pertinent fractional Sobolev-type spaces that will be used throughout this paper. Let Ω be an open subset in $\mathbb{R}^N$, $p \in [1, \infty)$ and $s \in (0,1)$, we define the fractional Sobolev space $W^{s,p}(\Omega)$ as follows

$$W^{s,p}(\Omega) = \left\{ \phi \in L^p(\Omega) \text{ such that } \frac{\mid \phi(x) - \phi(z) \mid}{\mid x - z \mid^{\frac{N}{p}+s}} \in L^p(\Omega \times \Omega) \right\},$$

which is a Banach space furnished with the following norm

$$\| \phi \|_{W^{s,p}(\Omega)} = \left(\| \phi \|^p_{L^p(\Omega)} + [\phi]^p_{W^{s,p}(\Omega)} \right)^{\frac{1}{p}},$$

where

$$[\phi]_{W^{s,p}(\Omega)} = \left(\int_\Omega \int_\Omega \frac{\mid \phi(x) - \phi(z) \mid^p}{\mid x - z \mid^{N+sp}} \, dx \, dz \right)^{\frac{1}{p}}.$$

$$W^{s,p}_0(\Omega) = \left\{ \phi \in W^{s,p}(\mathbb{R}^N) \text{ such that } \phi = 0 \; in \; \mathbb{R}^N \setminus \Omega \right\},$$

provided with the following norm

$$\| \phi \|_{W^{s,p}_0(\Omega)} = [\phi]_{W^{s,p}(\Omega)}.$$

To learn more about fractional Sobolev spaces, the reader is recommended to read papers [8, 24, 26].

Lemma 2.1. *([16], Lemma 2.3) for all $u, v \in W^{s,p}_0(\Omega)$, we have*

$$\langle (-\Delta)^s_p u, v \rangle = \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\Theta(u(x) - u(y))(v(x) - v(y))}{\mid x - y \mid^{N+sp}} \, dx \, dy, \tag{2.1}$$

where $\langle .,. \rangle$ denote the duality pairing betwen $W^{s,p}_0(\Omega)$ and $W^{-s,p'}(\Omega)$.

Proposition 2.2. *([26], Proposition 2.1)*
The energy functional $\mathcal{K}^s_p : W^{s,p}_0(\Omega) \to \mathbb{R}$ defined by

$$\mathcal{K}^s_p(u) = \frac{1}{p} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\mid u(x) - u(y) \mid^p}{\mid x - y \mid^{N+sp}} \, dx \, dy,$$

is well defined, bounded and possesses the Gâteaux derivative

$$\langle (\mathcal{K}^s_p)'(u), v \rangle = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\Theta(u(x) - u(y))(v(x) - v(y)}{\mid x - y \mid^{N+sp}} \, dx \, dy, \tag{2.2}$$

for any $u, v \in W^{s,p}_0(\Omega)$.

2.2. Subdifferentials

Consider the Hilbert space $\mathbf{H}$ endowed with the inner product denoted by $(.,.)_{\mathbf{H}}$ and let $\Psi : \mathbf{H} \to [0,+\infty]$ a lower semi-continuous, proper and convex functional defined on the domain

$$\mathcal{D}(\Psi) := \{\vartheta \in \mathbf{H} \ : \ \Psi(\vartheta) < +\infty\}.$$

The functional Ψ has a subdifferential $\partial\Psi$ that is defined through

$$\begin{cases} \mathcal{D}(\partial\Psi) := \{\vartheta \in \mathcal{D}(\Psi) : \exists \xi \in \mathbf{H}, \forall \nu \in \mathbf{H} : \Psi(\nu) - \Psi(\vartheta) \geq (\xi, \nu - \vartheta)_{\mathbf{H}}\}, \\ \partial\Psi(\vartheta) := \{\xi \in \mathbf{H} : \forall \nu \in \mathbf{H} : \Psi(\nu) - \Psi(\vartheta) \geq (\xi, \nu - \vartheta)_{\mathbf{H}}\}. \end{cases}$$

For additional information on the concept of subdifferentials, readers are encouraged to consult [9, 27].

Definition 2.3. *Consider $\mathfrak{B}$ as a Banach space and $x, \tilde{x} \in \mathfrak{B}$. The directional derivative of $||\cdot||$ at x along the vector $\tilde{x}$ is*

$$[x, \tilde{x}] := \lim_{\varepsilon \to 0^+} \frac{\| x + \varepsilon\tilde{x} \| - \| x \|}{\varepsilon}.$$

Proposition 2.4. *(see Proposition 57.2 page 582 in [36])*

i) *For any $x, \varpi_1, \varpi_2 \in L^1(\Omega)$, we have*

$$-[x, \varpi_1 + \varpi_2] \leq [x, -\varpi_1] - [x, \varpi_2] \tag{2.3}$$

$$[x, \varpi_1] \leq \| \varpi_1 \|_{L^1(\Omega)} . \tag{2.4}$$

ii) *For all $\tilde{v} \in \mathcal{Z}$, it follows that*

$$\frac{d}{dt} \|\tilde{v}\|_{L^1(\Omega)} = -\left[\tilde{v}, -\frac{\partial \tilde{v}}{\partial t}\right], \tag{2.5}$$

where $\mathcal{Z} = \{\tilde{v} : Q_T \to \mathbb{R}, \quad \tilde{v}(t) \in L^1(\Omega),$ and $\dfrac{\partial \tilde{v}(t)}{\partial t} \in L^1(\Omega)\}$

3. Existence and Uniqueness of a Weak Solution with L^2 Data

In this section, we show the existence and uniqueness of weak solutions of problem (1.1) with $f \in L^2(Q_T)$ and $u_0 \in L^2(\Omega)$ by using the theory of evolution equations guided by subdifferential operators. The following definition presents the concept of a weak solution to our problem.

Definition 3.1. *A function $u \in C\left([0,T]; L^2(\Omega)\right)$ is said to be a weak solution of problem* (1.1)*, if the following conditions are all satisfied:*

- *$u \in L^2(Q_T)$, $\frac{\partial u}{\partial t} \in L^2(Q_T)$ and $(-\Delta)^s_p u \in L^2(Q_T)$,*
- *$u(0) = u_0$,*
- *for all $w \in W^{s,p}_0(Q_T) \cap L^2(Q_T)$, it holds that*

$$\int_0^T \langle \frac{\partial u}{\partial t}, w\rangle dt + \int_0^T \langle(-\Delta)^s_p u, w\rangle dt = \int_0^T \langle f, w\rangle dt. \tag{3.1}$$

Theorem 3.2. *Assume that $f \in L^2(Q_T)$ and $u_0 \in L^2(\Omega)$. Then there exists a unique weak solution $u = u(t,x)$ of the problem* (1.1)*that fulfills the conditions specified in the Definition* (3.1)*.*

Proof. To prove Theorem 3.2, we integrate the theory of evolution equations (see Chapter III in [9]) with the subdifferential approach. To this end, let us consider the energy functional $\Phi_p^s : L^2(\Omega) \to [0,\infty]$ defined by

$$\Phi_p^s(u) = \begin{cases} \dfrac{1}{2p} \displaystyle\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\, u(x) - u(y) \,|^p}{|\, x - y \,|^{N+sp}} \, dx\, dy \\ \text{if } u \in W_0^{s,p}(\Omega) \cap L^2(\Omega) + \infty \text{ otherwise.} \end{cases} \tag{3.2}$$

Using Fatou's Lemma, we can conclude that Φ_p^s is lower semicontinuous in the space $L^2(\Omega)$. Furthermore, due to the fact that Φ_p^s is proper and convex, we can deduce that the subdifferential $\partial\Phi_p^s$ is identified as a maximal monotone operator in $L^2(\Omega)$. Next, we will demonstrate that

$$\partial\Phi_p^s(u) = (-\Delta)_p^s u.$$

Let $u \in W_0^{s,p}(\Omega) \cap L^2(\Omega)$ and set $h := \partial\Phi_p^s(u)$. Then for all $v \in W_0^{s,p}(\Omega) \cap L^2(\Omega)$, we have

$$\Phi_p^s(v) - \Phi_p^s(u) \geq (h, v - u).$$

Then

$$\frac{1}{2p} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\, v(x) - v(y) \,|^p - |\, u(x) - u(y) \,|^p}{|\, x - y \,|^{N+sp}} \, dx\, dy \geq \int_\Omega h(v - u) dx. \tag{3.3}$$

Taking $v = \varepsilon w + (1-\varepsilon)u$ in (3.3) with $w \in W_0^{s,p}(\Omega) \cap L^2(\Omega)$ and $\varepsilon \in [0,1]$, we obtain

$$\begin{aligned} &\frac{1}{2p} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\, (\varepsilon w + (1-\varepsilon)u)(x) - (\varepsilon w + (1-\varepsilon)u)(y) \,|^p - |\, u(x) - u(y) \,|^p}{|\, x - y \,|^{N+sp}} \, dx\, dy \\ &\qquad \geq \varepsilon \int_\Omega h(w - u) dx. \end{aligned}$$

Consequently

$$\lim_{\varepsilon \to 0} \frac{\Phi_p^s(u + \varepsilon(w - u)) - \Phi_p^s(u)}{2\varepsilon} \geq \int_\Omega h(w - u) dx.$$

Hence

$$\begin{aligned} &\frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\, u(x) - u(y) \,|^{p-2} \, (u(x) - u(y))((w - u)(x) - (w - u)(y))}{|\, x - y \,|^{N+sp}} \, dx\, dy \\ &\qquad \geq \int_\Omega h(w - u) dx. \end{aligned} \tag{3.4}$$

Replacing w by $w + u$ in (3.4), we obtain

$$\begin{aligned} &\frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\, u(x) - u(y) \,|^{p-2} \, (u(x) - u(y))(w(x) - w(y))}{|\, x - y \,|^{N+sp}} \, dx\, dy \\ &\qquad \geq \int_\Omega hw dx. \end{aligned} \tag{3.5}$$

As (3.5) is valid when w is replaced by $-w$, we deduce

$$\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\mid u(x)-u(y)\mid^{p-2}(u(x)-u(y))(w(x)-w(y))}{\mid x-y\mid^{N+sp}}\,dx\,dy=\int_\Omega hwdx, \quad (3.6)$$

for all $w \in W_0^{s,p}(\Omega)\cap L^2(\Omega)$. As a result

$$h=(-\Delta)_p^s u.$$

Consequently, our problem (1.1) is transformed into the following Cauchy problem

$$\begin{cases} \frac{\partial u(t)}{\partial t}+\partial\Phi_p^s u(t)=f(t) & \text{in } L^2(\Omega) \text{ for all } t\in]0,T[,\\ u(0)=u_0. \end{cases}$$

This type of abstract evolution equation was extensively studied by H. Brézis (refer to Chapter III of [9]). Hence, we deduce that for any $f\in L^2(Q_T)$ and $u_0\in L^2(\Omega)$, problem (1.1) admits a unique weak solution as described in Definition (3.1). □

4. Existence and uniquness of integral solutions

In this section, we prove the existence and uniqueness of integral solutions to problem (1.1) when $(f,u_0)\in L^1(Q_T)\times L^1(\Omega)$. To begin, we outline the concept of accretive operators in the context of Banach spaces.

Definition 4.1. *An operator* $\mathbf{A}$ *defined on a subset* $\mathcal{D}(\mathbf{A})$ *of a Banach space* $\mathfrak{B}$ *with values in* $\mathfrak{B}$ *is said to be accretive if*

$$\text{for all}\, w_1,w_2\in\mathcal{D}(\mathbf{A}):\ [w_1-w_2,\mathbf{A}(w_1)-\mathbf{A}(w_2)]\geq 0. \quad (4.1)$$

By ([36], Section 57.2), we know that (4.1) is equivalent to

$$\text{for all}\,\lambda>0; w_1,w_2\in\mathcal{D}(\mathbf{A}):\ \| w_1-w_2\|\leq\| w_1+\lambda\mathbf{A}(w_1)-w_2-\lambda\mathbf{A}(w_2)\|. \quad (4.2)$$

In this section, we will use $\mathfrak{B}=L^1(\Omega)$ and we introduce the operator $\mathbf{A}$, which is defined as follows

$$\mathcal{D}(\mathbf{A}):=\{w\in W_0^{s,p}(\Omega)\cap L^2(\Omega);\ (-\Delta)_p^s w\in L^2(\Omega)\} \quad (4.3)$$

$$\mathbf{A}(w)=(-\Delta)_p^s w. \quad (4.4)$$

Lemma 4.2. *The operator* $\mathbf{A}$ *specified by* (4.3)-(4.4) *is accretive with respect to the* $L^1(\Omega)$ *norm.*

Proof. To establish Lemma (4.2), we consider a strictly positive value λ and verify (4.2). for this purpose, let us consider for $i=1,2, f_i\in L^2(\Omega)$ and we consider the respective weak solution $u_i\in W_0^{s,p}(\Omega)\cap L^2(\Omega)$ of the following boundary-value problem

$$\begin{cases} u_i+\lambda(-\Delta)_p^s u_i=f_i & \text{in } \Omega\\ u_i=0 & \text{on } \mathbb{R}^N\setminus\Omega. \end{cases} \quad (4.5)$$

Remark 4.3. By results obtained in Section 3, we have Φ_p^s is proper, convex and lower semicontinuous. In addition, we know that $\mathbf{A}=\Phi_p^s$. Hence, $\mathbf{A}$ is a maximal monotone operator. Thus, the operator $I+\lambda\mathbf{A}$ is bijective from $\mathcal{D}(\mathbf{A})$ into $L^2(\Omega)$. Consequently, for any $f_i\in L^2(\Omega)$, there exist a unique weak solution $u_i\in W_0^{s,p}(\Omega)\cap L^2(\Omega)$ to problem (4.5).

To verify (4.2), we show that

$$\|u_1 - u_2\|_{L^1(\Omega)} \leq \|f_1 - f_2\|_{L^1(\Omega)}. \tag{4.6}$$

The weak solution to (4.5) satisfies for any $v \in W^{s,p}(\Omega) \cap L^2(\Omega)$, the equality

$$\int_\Omega u_i(x)v(x) + \lambda \int_\Omega \langle (-\Delta)_p^s u_i, v\rangle = \int_\Omega f_i(x)v(x)\, \mathrm{d}x. \tag{4.7}$$

To demonstrate (4.6), we subtract the equation (4.7) for $i = 2$ from the equation (4.7) for $i = 1$, and choose
$v = \operatorname{sgn}_\varepsilon(u_1 - u_2) \in W_0^{s,p}(\Omega) \cap L^2(\Omega)$ where

$$\operatorname{sgn}_\varepsilon(\zeta) = \begin{cases} -1 & \text{for } \zeta \leq -\varepsilon, \\ \zeta/\varepsilon & \text{for } -\varepsilon \leq \zeta \leq \varepsilon, \\ 1 & \text{for } \zeta \geq \varepsilon. \end{cases}$$

Thus we obtain

$$\int_\Omega (u_1(x) - u_2(x)) \operatorname{sgn}_\varepsilon(u_1 - u_2)\, \mathrm{d}x + \lambda \int_\Omega \langle (-\Delta)_p^s u_1 - (-\Delta)_p^s u_2, \operatorname{sgn}_\varepsilon(u_1 - u_2)\rangle$$
$$= \int_\Omega (f_1(x) - f_2(x)) \operatorname{sgn}_\varepsilon(u_1 - u_2)\, \mathrm{d}x \leq \int_\Omega |f_1 - f_2|\, \mathrm{d}x = \|f_1 - f_2\|_{L^1(\Omega)}.$$

The term multiplied by λ is positive, moreover

$$\lim_{\varepsilon \to 0} \int_\Omega (u_1 - u_2) \operatorname{sgn}_\varepsilon(u_1 - u_2)\, \mathrm{d}x = \int_\Omega |u_1 - u_2|\, \mathrm{d}x = \|u_1 - u_2\|_{L^1(\Omega)}.$$

Thus, (4.6) is proved. □

We will also need the subsequent result. For simplicity, we use the notation $u(t) = u(t, x)$, $f(t) = f(t, x)$, and so forth. The notation $[.,.]$ will be used to denote the derivative in a given direction of the norm in $L^1(\Omega)$.

Lemma 4.4. *Let $f, g \in L^2(Q_T)$, $u_0, w_0 \in L^2(\Omega)$ and let u, w be two weak solutions to the problem* (1.1)*, hence for all $0 \leq \sigma \leq t \leq T$ we get*

$$\| u(t) - w(t) \|_{L^1(\Omega)} \leq \| u(\sigma) - w(\sigma) \|_{L^1(\Omega)} + \int_\sigma^t [u(\varsigma) - w(\varsigma), f(\varsigma) - g(\varsigma)] d\varsigma. \tag{4.8}$$

Proof. By Lemma 4.2, we have the operator defined by (4.3)-(4.4) is accretive, hence (4.1) holds. Then for $u_1, u_2 \in \mathcal{D}(\mathbf{A})$ we obtain

$$[u_1 - u_2, (-\Delta)_p^s u_1 - (-\Delta)_p^s u_2] \geq 0.$$

We have $(-\Delta)_p^s u \in L^2(Q_T)$. Hence, we can take $u_1 := u(t) \in \mathcal{D}(\mathbf{A})$ for a.e $t \in [0, T]$. similarly, we choose $u_2 := w(t) \in \mathcal{D}(\mathbf{A})$ for a.e $t \in [0, T]$. Utilizing these substitutions and considering (1.1), $(-\Delta)_p^s u(t)$ can be substituted by $f(t) - \frac{\partial u(t)}{\partial t}$. This leads to the result that, for almost every $t \in [0, T]$,

$$-[u(t) - w(t), -\frac{\partial u(t)}{\partial t} + f(t) - (-\Delta)_p^s w(t)] \leq 0. \tag{4.9}$$

Using successively (2.3), (2.5) and the inequality (4.9), we deduce the following estimate,

$$
\begin{aligned}
&\| u(t) - w(t) \|_{L^1(\Omega)} - \| u(\sigma) - w(\sigma) \|_{L^1(\Omega)} \\
&= \int_\sigma^t \frac{d}{dt} \| u(\varsigma) - w(\varsigma) \|_{L^1(\Omega)} \, d\varsigma \\
&= \int_\sigma^t -[u(\varsigma) - w(\varsigma), \frac{\partial w(\varsigma)}{\partial \varsigma} - \frac{\partial u(\varsigma)}{\partial \varsigma}] d\varsigma \\
&\leq \int_\sigma^t [u(\varsigma) - w(\varsigma), -\frac{\partial w(\varsigma)}{\partial \varsigma} + f(\varsigma) - (-\Delta)_p^s w(\varsigma)] \\
&\quad - [u(\varsigma) - w(\varsigma), -\frac{\partial u(\varsigma)}{\partial \varsigma} + f(\varsigma) - (-\Delta)_p^s w(\varsigma)] \\
&\leq \int_\sigma^t [u(\varsigma) - w(\varsigma), -\frac{\partial w(\varsigma)}{\partial \varsigma} + f(\varsigma) - (-\Delta)_p^s w(\varsigma)].
\end{aligned}
$$

We have

$$
-\frac{\partial w(\varsigma)}{\partial \varsigma} - (-\Delta)_p^s w(\varsigma) = -g(\varsigma),
$$

consequently

$$
\| u(t) - w(t) \|_{L^1(\Omega)} \leq \| u(\sigma) - w(\sigma) \|_{L^1(\Omega)} + \int_\sigma^t [u(\varsigma) - w(\varsigma), f(\varsigma) - g(\varsigma)].
$$

□

Lemma 4.5. *Let $f, g \in L^2(Q_T)$, $u_0, v_0 \in L^2(\Omega)$ and let u, w be two weak solutions to Problem* (1.1)*, hence*

$$
\| u - w \|_{C([0,T];L^1(\Omega))} \quad \leq \| f - g \|_{L^1(Q_T)} + \| u_0 - w_0 \|_{L^1(\Omega)} . \tag{4.10}
$$

Proof. Using (4.8) with $\sigma = 0$, moreover taking into account (2.4), we obtain

$$
\begin{aligned}
&\| u(t) - w(t) \|_{L^1(\Omega)} - \| u_0 - w_0 \|_{L^1(\Omega)} \\
&\leq \int_0^t [u - w, f - g] \\
&\leq \int_0^t \| f - g \|_{L^1(\Omega)} \\
&\leq \| f - g \|_{L^1(Q_T)} .
\end{aligned} \tag{4.11}
$$

□

Remark 4.6. For any $(f, u_0) \in L^1(Q_T) \times L^1(\Omega)$, it is possible to construct a sequence of smooth data $(f_n, u_{0n}) \in L^2(Q_T) \times L^2(\Omega)$ such that

$$
f_n \to f \textit{ in } L^1(Q_T) \text{ and } u_{0n} \to u_0 \textit{ in } L^1(\Omega). \tag{4.12}
$$

Let (u_n, u_m) be the weak solutions to problem (1.1) corresponding to the data sets (f_n, u_{0n}) and (f_m, u_{0m}), by Lemma 4.5, it can be inferred that

$$
\| u_n - u_m \|_{C([0,T];L^1(\Omega))} \quad \leq \| f_n - f_m \|_{L^1(Q_T)} + \| u_{0n} - u_{0m} \|_{L^1(\Omega)} . \tag{4.13}
$$

Therefore, every weak solution u_n with data that satisfies (4.12), constitutes a Cauchy sequence within $C([0,T];L^1(\Omega))$.

The remark 4.6 prompts the introduction of a new type of solution for problem (1.1), which we will term a generalized solution to (1.1).

Definition 4.7. *We call u a generalized solution to* (1.1) *when it is the limit in $C([0,T];L^1(\Omega))$ of a sequence of weak solutions whose corresponding data satisfies the requirements of* (4.12)*.*

Utilizing Remark 4.6 along with Definition 4.7, it is possible to rigorously demonstrate the subsequent proposition.

Proposition 4.8. *Suppose that $f \in L^1(Q_T)$ and $u_0 \in L^1(\Omega)$, hence*

(i) (1.1) *possesses only one generalized solution as described in Definition 4.7. Moreover, this generalized solution is independent of the specific sequences chosen to satisfy condition* (4.12)*.*

(ii) *every weak solution to problem* (1.1) *is the generalized solution.*

Definition 4.9. *Let $f \in L^1(Q_T)$ and $u_0 \in L^1(\Omega)$, we call $u \in C([0,T];L^1(\Omega))$ an integral solution of* (1.1) *if*

$$\| u(t) - w(t) \|_{L^1(\Omega)} \leq \| u_0 - w_0 \|_{L^1(\Omega)} + \int_0^t [u(\varsigma) - w(\varsigma), f(\varsigma) - g(\varsigma)] d\varsigma \tag{4.14}$$

holds for any $t \in [0,T]$, in which w is a generalized solution to

$$\begin{cases} \frac{\partial w}{\partial t} + (-\Delta)^s_p w = g & \text{in }]0,T[\times\Omega \\ w = 0 & \text{on }]0,T[\times(\mathbb{R}^N \setminus \Omega) \\ w(0,x) = w_0(x) & \text{in } \Omega, \end{cases} \tag{4.15}$$

for $g \in L^1(Q_T)$ and $w_0 \in L^1(\Omega)$.

In the following theorem, we present the second main result of our paper.

Theorem 4.10. *Suppose that $f \in L^1(Q_T)$ and $u_0 \in L^1(\Omega)$. Then problem* (1.1) *admits a unique integral solution.*

Proof. We propose dividing the demonstration of Theorem 4.10 into two phases: the first phase will concentrate on establishing the existence of an integral solution to (1.1), and the second phase will focus on proving the uniqueness of the integral solution derived.

Existence of an integral solution:

Consider w as a generalized solution to Problem (4.15), meaning there exists a sequence $(w_n) \in L^2(Q_T)$, such that

$$w_n \longrightarrow w \text{ in } C([0,T];L^1(\Omega))$$

with w_n being the weak solution that fulfills the equation

$$\int_0^T \langle \frac{\partial w_n}{\partial t}, v \rangle dt + \int_0^T \langle (-\Delta)^s_p w_n, v \rangle dt = \int_0^T \langle g_n, v \rangle dt, \tag{4.16}$$

for all $v \in W_0^{s,p}(\Omega) \cap L^2(\Omega)$ with $(g_n, w_{0n}) \in L^2(Q_T) \times L^2(\Omega)$ converging to $(g, w_0) \in L^1(Q_T) \times L^1(\Omega)$.
Furthermore, let u a generalized solution to (1.1).
According to definition 4.9, this implies the existence of a sequence $(u_n) \in L^2(Q_T)$ in such a manner that

$$u_n \longrightarrow u \; in \; C([0,T]; L^1(\Omega))$$

and u_n is a weak solution satisfies

$$\int_0^T \langle \frac{\partial u_n}{\partial t}, v \rangle dt + \int_0^T \langle (-\Delta)_p^s u_n, v \rangle dt = \int_0^T \langle f_n, v \rangle dt, \tag{4.17}$$

for all $v \in W_0^{s,p}(\Omega) \cap L^2(\Omega)$, and

$$f_n \in L^2(Q_T);\; f_n \longrightarrow f \; in \; L^1(Q_T)$$
$$u_{0n} \in L^2(\Omega);\; u_{0n} \longrightarrow u_0 \; in \; L^1(\Omega)$$

holds, then (4.8) with $\sigma = 0$ yields that

$$\| u_n(t) - w_n(t) \|_{L^1(\Omega)} \leq \| u_{0n} - w_{0n} \|_{L^1(\Omega)} + \int_0^t [u_n(\varsigma) - w_n(\varsigma), f_n(\varsigma) - g_n(\varsigma)] \tag{4.18}$$

pass to limit with $n \longrightarrow +\infty$, we have obviously

$$\| u_n(t) - w_n(t) \|_{L^1(\Omega)} \longrightarrow \| u(t) - w(t) \|_{L^1(\Omega)}$$

and

$$\| u_{0n} - w_{0n} \|_{L^1(\Omega)} \longrightarrow \| u_0 - w_0 \|_{L^1(\Omega)}$$

and

$$\limsup_{n \longrightarrow +\infty} \int_0^t [u_n(\varsigma) - w_n(\varsigma), f_n(\varsigma) - g_n(\varsigma)]_\Omega d\varsigma \leq \int_0^t [u(\varsigma) - w(\varsigma), f(\varsigma) - g(\varsigma)] d\varsigma.$$

Altogether, we get

$$\| u(t) - w(t) \|_{L^1(\Omega)} \leq \| u_0 - w_0 \|_{L^1(\Omega)} + \int_0^t [u(\varsigma) - w(\varsigma), f(\varsigma) - g(\varsigma)] d\varsigma \tag{4.19}$$

according to the definition (4.9), We have demonstrated that the generalized solution is, in fact, also an integral solution. This confirms that an integral solution exists.
Uniqueness of integral solution:
Let u and $\tilde{w}$ be two integral solutions to problem (1.1). Then, it follows that

$$\| \tilde{w}(t) - w(t) \|_{L^1(\Omega)} \leq \| u_0 - w_0 \|_{L^1(\Omega)} + \int_0^t [\tilde{w} - w, f - g] d\varsigma. \tag{4.20}$$

Since u satisfies (4.19), we can substitute w with u, then

$$\int_0^t [\tilde{w} - w, f - g] d\varsigma = 0.$$

Consequently, we obtain

$$\| \tilde{w}(t) - u(t) \|_{L^1(\Omega)} \leq \| u_0 - u_0 \|_{L^1(\Omega)} = 0.$$

The result is that $\tilde{w}(t) = u(t) \; for \; all \; t \in [0,T]$. □

Conclusion

In this paper, we have investigated a class of non-local parabolic equations involving a fractional p-Laplacian operator. By employing subdifferential techniques and the theory of accretive operators, we established the existence and uniqueness of weak solutions for initial data in L^2, and extended the analysis to prove the existence and uniqueness of integral solutions when the initial data belong to L^1.

The results obtained contribute to a deeper understanding of regularity and solution structure in non-local evolution equations. They also highlight the effectiveness of combining classical nonlinear analysis tools with methods adapted to non-local operators.

As a direction for future research, it would be interesting to consider the case where the source term f depends on the unknown function u. This would involve studying nonlinearities of the type $f = f(u)$, which may require the use of fixed-point techniques or variational methods. Such an extension could provide further insight into nonlinear dynamics driven by non-local operators and broaden the applicability of the current theory.

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