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On implicit φ-Hilfer fractional differential equations with the p-Laplacian operator

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Abstract. In this paper, we establish the existence and uniqueness of solutions for a new class of nonlocal boundary implicit φ-Hilfer fractional differential equations involving the p-Laplacian operator. The existence results are derived using the topological degree method for condensing maps and the Banach contraction principle. Moreover, we investigate the Ulam-Hyers and generalized Ulam-Hyers stability of our main problem. To illustrate the applicability of our theoretical results, we provide an example.

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1. Introduction

Fractional differential equations have gained significant importance due to their ability to accurately model a wide range of physical phenomena across various fields, including chemistry, physics, biology, engineering, viscoelasticity, electrical engineering, signal processing, electrochemistry, and controllability (see [21, 22]). The motivation for introducing new fractional derivatives is twofold: first, to capture certain dynamic behaviors of physical systems that are not well represented by existing fractional derivatives, and second, to preserve key properties of the standard derivative. Numerous types of fractional derivatives have been proposed, such as the Riemann-Liouville, Caputo, Hadamard, Hilfer, and Katugampola derivatives [1, 11, 13, 17], each offering unique advantages in capturing specific phenomenological behaviors.

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Recently, Sousa and Oliveira [18] introduced a new class of fractional derivatives known as the φ-Hilfer fractional derivative. This derivative generalizes several well-known fractional derivatives through appropriate choices of the function φ. For instance, it reduces to the Hilfer fractional derivative ${}^{H}\mathfrak{D}^{\alpha,\beta}(\theta)$ when $\varphi(\theta)=\theta$; the Caputo fractional derivative ${}^{C}\mathcal{D}^{\alpha}(\theta)$ and the Riemann-Liouville fractional derivative ${}^{RL}\mathcal{D}^{\alpha}(\theta)$ when $\varphi(\theta)=\theta$ with $\beta\to 1$ and $\beta\to 0$, respectively; the Hadamard fractional derivative ${}^{H}\mathcal{D}^{\alpha}(\theta)$ when $\varphi(\theta)=\ln(\theta)$ and $\beta\to 0$; and the Katugampola fractional derivative ${}^{\rho}\mathcal{D}^{\alpha}(\theta)$ when $\varphi(\theta)=\theta^{\rho}$ and $\beta\to 0$, while taking the φ-Caputo fractional derivative ${}^{C}\mathcal{D}^{\alpha;\varphi}(\theta)$ when $\beta\to 1$ [1, 6, 7, 8, 11, 13, 17]. Thus, this provides a flexible framework for modeling different physical phenomena depending on the choice of φ. Due to this versatility, it has attracted considerable attention and has been extensively studied by various researchers (see [2, 3, 5, 19, 20]).

In [15], Mali et al. investigated the existence and Ulam-Hyers stability of implicit φ-Hilfer fractional differential equations of the form

$$\begin{cases} {}^{H}\mathfrak{D}^{\alpha,\beta,\varphi}_{0^+}z(\theta)=h(\theta,z(\theta),{}^{H}\mathfrak{D}^{\alpha,\beta,\varphi}_{0^+}z(\theta)), & \theta\in(a,b]\\ z(a)=0, \quad z(b)=\displaystyle\sum_{i=1}^{m}a_i\mathfrak{I}^{\nu_i,\varphi}_{0^+}z(\xi_i), \end{cases}$$

where ${}^{H}\mathfrak{D}^{\alpha,\beta,\varphi}_{0^+}$ is the φ-Hilfer fractional derivative of order $\alpha\in(1,2)$ with type $\beta\in[0,1]$, $\xi_i\in(a,b]$, $a_i\in\mathbb{R}$, $\mathfrak{I}^{\nu_i,\varphi}_{0^+}$ is the φ-Riemann-Liouville fractional integral of order ν_i, and $h:(a,b]\times\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$ is a given continuous function.

In [4], Alsaedi et al. established the existence and uniqueness of solutions for the following implicit φ-Hilfer differential equation involving the p-Laplacian operator

$$\begin{cases} \left(\Phi_p^{H}\mathfrak{D}^{\alpha,\beta,\varphi}_{0^+}z(\theta)\right)'+h(\theta,z(\theta))=0, & \theta\in[0,1],\\ z(0)=0, \quad {}^{H}\mathfrak{D}^{\alpha,\beta,\varphi}_{0^+}z(0)=0, \quad z(T)=\displaystyle\sum_{i=1}^{m}a_i\mathfrak{I}^{\nu_i,\varphi}_{0^+}z(\xi_i), \end{cases}$$

where ${}^{H}\mathfrak{D}^{\alpha,\beta,\varphi}_{0^+}$ is the $\varphi-$Hilfer fractional derivative of orders $\alpha\in(1,2]$ with type $\beta\in[0,1]$, $\xi_i\in(0,1)$, $a_i\in\mathbb{R}$, Φ_p denotes the p-Laplacian operator such that $\Phi_p(\zeta)=|\zeta|^{p-2}\zeta$, $p>1$ and $h:[0,1]\times\mathbb{R}\longrightarrow\mathbb{R}$ is a given continuous function.

Motivated by the aforementioned works, we establish the existence and stability of solutions to the following implicit φ-Hilfer fractional differential equation

$$\begin{cases} {}^{H}\mathfrak{D}^{\alpha,\beta,\varphi}_{0^+}\Phi_p^{H}\mathfrak{D}^{\alpha',\beta',\varphi}_{0^+}z(\theta)=h\Big(\theta,z(\theta),{}^{H}\mathfrak{D}^{\alpha,\beta,\varphi}_{0^+}\Phi_p^{H}\mathfrak{D}^{\alpha',\beta',\varphi}_{0^+}z(\theta)\Big), & \theta\in\Delta\\ z(0)=0, \quad {}^{H}\mathfrak{D}^{\alpha',\beta',\varphi}_{0^+}z(0)=0, \quad z(T)=\displaystyle\sum_{i=1}^{m}a_i\mathfrak{I}^{\nu_i,\varphi}_{0^+}z(\xi_i), \end{cases} \tag{1.1}$$

where $\Delta=[0,T]$ and ${}^{H}\mathfrak{D}^{\alpha,\beta,\varphi}_{0^+}$ and ${}^{H}\mathfrak{D}^{\alpha',\beta',\varphi}_{0^+}$ are the φ-Hilfer fractional derivatives of orders $\alpha\in(0,1)$ with types $\beta\in[0,1]$, $\alpha'\in(1,2)$ with types $\beta,\beta'\in[0,1]$ respectively, $\xi_i\in(0,T)$, $a_i\in\mathbb{R}$ and $h:\Delta\times\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$ is a continuous function such that

($\mathcal{A}_1$) There exist constants $\kappa_1>0$ and $0<\kappa_2<1$ such that

$$|h(\theta,z_1,\eta_1)-h(\theta,z_1,\eta_2)|\le\kappa_1\mid z_1-z_2\mid+\kappa_2|\eta_1-\eta_2|,$$

for any z_1, η_1, z_2, $\eta_2 \in \mathbb{R}$ and $\theta \in \Delta$.

$(\mathcal{A}_2)$ There exist constants $\tau_1, \tau_2, \tau_3 > 0$ with $0 < \tau_2 < 1$ such that

$$|h(\theta, z, \eta)| \leq \tau_1 |z| + \tau_2 |\eta| + \tau_3,$$

for any z, $\eta \in \mathbb{R}$ and $\theta \in \Delta$.

This paper is organized as follows: In Section 2, we introduce the necessary definitions and results required for proving the existence theorems. In Section 3 we investgate the existence and uniqueness of solutions for our main problem. In Section 4, we analyze the Ulam-Hyers-type stability results. Finally, we provide an example to validate the obtained results.

2. Preliminaries

Let $C(\Delta, \mathbb{R})$ be the Banach space of all continuous functions $z : \Delta \to \mathbb{R}$ equipped with the norm $\|z\| = \sup_{\theta \in \Delta} |z(\theta)|$, and let $B_r(0)$ denote the closed ball centered at 0 with radius r. We introduce the space

$$\mathcal{A}l^n(\Delta) = \{\varphi \in C^n(\Delta, \mathbb{R}) \text{ such that } \varphi'(\theta) > 0 \text{ for all } \theta \in \Delta\}.$$

Definition 2.1. [18] *Let $\alpha > 0$, $h : \Delta \longrightarrow \mathbb{R}$ an integrable function and $\varphi \in \mathcal{A}l^1(\Delta)$. Then the φ-Riemann-Liouville fractional integral of order α of the function h is given by*

$$\mathfrak{I}_{0^+}^{\alpha,\varphi} h(\theta) = \frac{1}{\Gamma(\alpha)} \int_0^\theta \varphi'(s) \Omega_\varphi^{\alpha-1}(\theta, s) h(s) ds,$$

where $\Omega_\varphi^\alpha(\theta, s) = (\varphi(\theta) - \varphi(s))^\alpha$ and $\Gamma(.)$ is the Gamma function.

Definition 2.2. [18] *Let $\alpha > 0$, $h \in C^n(\Delta, \mathbb{R})$ and $\varphi \in \mathcal{A}l^n(\Delta)$. Then the φ-hilfer fractional derivative of order α and type $\beta \in [0, 1]$ of the function h is given by*

$${}^H\mathfrak{D}_{0^+}^{\alpha,\beta,\varphi} h(\theta) = \mathfrak{I}_{0^+}^{\beta(n-\alpha),\varphi} \left(\frac{1}{\varphi'(\theta)} \frac{d}{d\theta} \right)^n \mathfrak{I}_{0^+}^{(1-\beta)(n-\alpha),\varphi} h(\theta),$$

where $n = [\alpha] + 1$ and $[\alpha]$ is the part integer of α.

Proposition 2.3. [18] *Let $h \in C^n(\Delta, \mathbb{R})$, then the following holds*

1) ${}^H\mathfrak{D}_{0^+}^{\alpha,\beta,\varphi} \mathfrak{I}_{0^+}^{\alpha,\varphi} h(\theta) = h(\theta)$

2) $\mathfrak{I}_{0^+}^{\alpha,\varphi}\, {}^H\mathfrak{D}_{0^+}^{\alpha,\beta,\varphi} h(\theta) = h(\theta) - \sum_{i=1}^{n} \frac{\Omega_\varphi^{\gamma-1}(\theta, 0)}{\Gamma(\gamma - i + 1)} h_\varphi^{[n-i]} \mathfrak{I}_{0^+}^{(1-\beta)(n-\alpha),\varphi} h(0),$

where $h_\varphi^{[n]} h(\theta) = \left(\frac{1}{\varphi'(\theta)} \frac{d}{d\theta} \right)^n h(\theta)$ *and* $\gamma = \alpha + \beta(n - \alpha)$.

Lemma 2.4. [14] *Let Φ_p be the p-Laplacian operator. Then*

1. *If $1 < p < 2$, $z_1 z_2 > 0$ and $|z_1|, |z_2| \geq m > 0$, we have*

$$|\Phi_p(z_1) - \Phi_p(z_2)| \leq (p - 1) m^{p-2} |z_1 - z_2|.$$

2. *If $p > 2$, $z_1 z_2 > 0$ and $|z_1|, |z_2| \leq M$, we have*

$$|\Phi_p(z_1) - \Phi_p(z_2)| \leq (p - 1) M^{p-2} |z_1 - z_2|.$$

3. Φ_p *is invertible with* $\Phi_p^{-1}(z) = \Phi_{p'}(z)$, *such that* $\frac{1}{p} + \frac{1}{p'} = 1$.

Definition 2.5. [9] *Let E be a Banach space and O a bounded subset of E. The Kuratowski measure of noncompactness is the map μ: $O \longrightarrow \mathbb{R}_+$ given by*

$$\mu(O) = \inf\{\rho > 0 : \ O \subseteq \bigcup_{i=1}^{n} O_i \ \text{and} \ diam(O_i) \leq \rho\}.$$

Proposition 2.6. [9] *Let O, O_1, O_2 be bounded subsets of E. Then the Kuratowski measure of noncompactness μ satisfies the following*

1. *O is relatively compact $\Leftrightarrow$ $\mu(O) = 0$*
2. $\mu(\kappa O) = |\kappa|\mu(O), \quad \kappa \in \mathbb{R}$.
3. $\mu(O_1 + O_2) \leq \mu(O_1) + \mu(O_2)$.
4. $O_1 \subset O_2 \Rightarrow \mu(O_1) \leq \mu(O_2)$.
5. $\mu(O_1 \cup O_2) = \max\{\mu(O_1), \ \mu(O_2)\}$.
6. *$\mu(O) = \mu(\overline{O}) = \mu(convO)$ where $\overline{O}$ and $convO$ represent the closure and the convex hull of the set O, respectively.*

Definition 2.7. [9] *Let $\mathcal{F} : \mathcal{W} \subset E \to E$ be a continuous bounded map. Then $\mathcal{F}$ is called*

1. *μ-Lipschitz if there exists $\kappa \geq 0$ such that*
$$\mu(\mathcal{F}(O)) \leq \kappa\mu(O) \quad \text{for all } O \subset \mathcal{W} \text{ bounded.}$$
Furthermore, if $\kappa < 1$ it is called a strict μ-contraction.
2. *μ-condensing if*
$$\mu(\mathcal{F}(O)) < \mu(O) \quad \text{for all } \ O \subset \mathcal{W} \ \text{bounded with } \mu(O) > 0.$$

Definition 2.8. [9] *Let $\mathcal{Q} : \mathcal{W} \subset E \to E$. The map $\mathcal{F}$ is called Lipschitz if there exists a constant $\kappa \geq 0$ such that*

$$\|\mathcal{F}w_1 - \mathcal{F}w_2\| \leq \kappa\|w_1 - w_2\|, \qquad \text{for all } w_1, w_2 \in \mathcal{W}.$$

Furthermore, $\mathcal{F}$ is called a strict contraction if $\kappa < 1$.

Lemma 2.9. [9] *If $\mathcal{F} : \mathcal{W} \subset E \to E$ is Lipschitz having constant κ, then $\mathcal{F}$ is μ-Lipschitz having the constant κ.*

Lemma 2.10. [9] *If $\mathcal{F}_1, \mathcal{F}_2 : \mathcal{W} \subset E \to E$ are μ-Lipschitz maps having constants κ_1, κ_2 respectively, then $\mathcal{F}_1 + \mathcal{F}_2 : \mathcal{W} \to E$ is μ-Lipschitz map having constant $\kappa_1 + \kappa_2$.*

Lemma 2.11. [9] *If $\mathcal{F} : \mathcal{W} \subset E \to E$ is compact, then $\mathcal{F}$ is μ- Lipschitz having constant $\kappa = 0$.*

Theorem 2.12. [12] *Let $\mathcal{F} : E \to E$ be μ-condensing, consider the set*

$$\mathbb{S}_\epsilon = \Big\{z \in E : \ \text{there exist } \ 0 \leq \epsilon \leq 1 \ \text{ such that } \ z = \epsilon\mathcal{F}(z)\Big\}.$$

If $\mathbb{S}_\epsilon$ is a bounded set in E, then there exists $r > 0$ such that $\mathbb{S}_\epsilon \subset B_r(0)$, then

$$deg(I_d - \epsilon\mathcal{F}, B_r(0), 0) = 1, \qquad \text{for all } \ \epsilon \in [0, 1].$$

Consequently, the operator $\mathcal{F}$ has at least one fixed point and the set of the fixed points of $\mathcal{F}$ lies in $B_r(0)$.

3. Existence results

In this section under certain hypotheses we investigate the existence and uniqueness of solutions for the problem (1.1). To simplify, we introduce the following notation.

$$\begin{aligned} h_z(\theta) &= h(\theta, z(\theta), \mathfrak{I}_{0^+}^{\alpha,\beta,\varphi}\Phi_p\mathfrak{I}_{0^+}^{\alpha',\beta',\varphi}z(\theta)), \quad \gamma' = \alpha' + \beta'(2-\alpha'),\\ \mathbb{A}_{h_z} &= \frac{\mathfrak{I}_{0^+}^{\alpha',\beta',\varphi}\Phi_q\mathfrak{I}_{0^+}^{\alpha,\beta,\varphi}h_z(T) - \mathfrak{I}_{0^+}^{\alpha'+\nu_i,\beta',\varphi}\Phi_q\mathfrak{I}_{0^+}^{\alpha,\beta,\varphi}h_z(\xi_i)}{\varpi\Gamma(\gamma')},\\ \Lambda_1 &= \frac{(p'-1)M^{p'-2}\Omega_\varphi^{\gamma'-1}(T,0)}{|\varpi|\Gamma(\gamma')}\Big(\frac{\Omega_\varphi^{\alpha'+\alpha}(T,0)}{\Gamma(\alpha'+\alpha+1)} + \sum_{i=1}^m |a_i|\frac{\Omega_\varphi^{\nu_i+\alpha'+\alpha}(\xi_i,0)}{\Gamma(\nu_i+\alpha'+\alpha+1)}\Big),\\ \Lambda_2 &= (p'-1)M^{p'-2}\frac{\Omega_\varphi^{\alpha'+\alpha-1}(T,0)}{\Gamma(\alpha'+\alpha)}, \ \varpi = \sum_{i=1}^m \frac{a_i\Omega_\varphi^{\gamma'+\nu_i-1}(\xi_i,0)}{\Gamma(\gamma'+\nu_i)} - \frac{\Omega_\varphi^{\gamma'-1}(T,0)}{\Gamma(\gamma')}. \end{aligned}$$

Lemma 3.1. *A function $z \in C^2(\Delta,\mathbb{R})$ is a solution of problem* (1.1) *if and only if it satisfies the following fractional integral equation*

$$z(\theta) = \mathfrak{I}_{0^+}^{\alpha',\beta',\varphi}\Phi_q\mathfrak{I}_{0^+}^{\alpha,\beta,\varphi}h_z(\theta) + \mathbb{A}_{h_z}\Omega_\varphi^{\gamma'-1}(\theta,0). \tag{3.1}$$

Proof. Let $z \in C^2(\Delta,\mathbb{R})$ be a solution of the problem (1.1). By applying fractional integral $\mathfrak{I}_{0^+}^{\alpha,\varphi}$ to both sides of the first equation in (1.1) and Proposition 2.3, we obtain

$$\Phi_p\big({}^H\mathfrak{D}_{0^+}^{\alpha',\beta',\varphi}z(\theta)\big) = d_0\Omega_\varphi^{\gamma-1}(\theta,0) + \mathfrak{I}_{0^+}^{\alpha,\varphi}h_z(\theta),$$

where

$$d_0 \in \mathbb{R}.$$

Since

$${}^H\mathfrak{D}_{0^+}^{\alpha',\beta',\varphi}z(0) = 0$$

then

$$d_0 = 0,$$

it follows that

$$\Phi_p\big({}^H\mathfrak{D}_{0^+}^{\alpha',\beta',\varphi}z(\theta)\big) = \mathfrak{I}_{0^+}^{\alpha,\beta,\varphi}h_z(\theta). \tag{3.2}$$

By applying the operator $\Phi_{p'}$ the inverse operator of Φ_p on both sides of (3.2), we get

$${}^H\mathfrak{D}_{0^+}^{\alpha',\beta',\varphi}z(\theta) = \Phi_{p'}\left(\mathfrak{I}_{0^+}^{\alpha,\varphi}h_z(\theta)\right). \tag{3.3}$$

Applying the operator $\mathfrak{I}_{0^+}^{\alpha',\varphi}$ on both sides of (3.3), we obtain

$$z(\theta) = d_1\Omega_\varphi^{\gamma'-1}(\theta,0) + d_2\Omega_\varphi^{\gamma'-2}(\theta,0) + \mathfrak{I}_{0^+}^{\alpha',\varphi}\Phi_{p'}\mathfrak{I}_{0^+}^{\alpha,\beta,\varphi}h_z(\theta), \text{ where } d_1,\ d_2 \in \mathbb{R}.$$

Using the condition $z(0) = 0$, we conclude that $d_2 = 0$. It follows that

$$z(\theta) = d_1\Omega_\varphi^{\gamma'-1}(\theta,0) + \mathfrak{I}_{0^+}^{\alpha',\varphi}\ \Phi_{p'}\mathfrak{I}_{0^+}^{\alpha,\varphi}h_z(\theta). \tag{3.4}$$

By the condition $z(T)=\sum_{i=1}^{m} a_i \mathfrak{I}_{0^+}^{\nu_i,\varphi} z(\xi_i)$, we get

$$d_1=\frac{\mathfrak{I}_{0^+}^{\alpha',\varphi}\Phi_{p'}\mathfrak{I}_{0^+}^{\alpha,\varphi}h_z(T)-\sum_{i=1}^{m}a_i\mathfrak{I}_{0^+}^{\nu_i+\alpha',\varphi}\Phi_{p'}\mathfrak{I}_{0^+}^{\alpha,\varphi}h_z(\xi_i)}{\Gamma(\gamma')\sum_{i=1}^{m}\frac{a_i\Omega_{\varphi}^{\gamma'+\nu_i-1}(\xi_i,0)}{\Gamma(\gamma'+\nu_i)}-\Omega_{\varphi}^{\gamma'-1}(T,0)}.$$

By substituting d_1 in (3.4), we get the integral equation (3.1).

Conversely, a direct computation shows that if z satisfies the integral equation (3.1), the φ-Hilfer problem (1.1) holds, completing the proof. $\square$

We define the operators $\mathcal{T}_1$, $\mathcal{T}_2$, and $\mathcal{T}: C^2(\Delta,\mathbb{R})\to C^2(\Delta,\mathbb{R})$ as follows

$$\mathcal{T}_1 z(\theta)=\mathbb{A}_{h_z}\Omega_{\varphi}^{\gamma'-1}(\theta,0),$$
$$\mathcal{T}_2 z(\theta)=\mathfrak{I}_{0^+}^{\alpha',\beta',\varphi}\Phi_q\mathfrak{I}_{0^+}^{\alpha,\beta,\varphi}h_z(\theta),$$

and

$$\mathcal{T}z(\theta)=\mathcal{T}_1 z(\theta)+\mathcal{T}_2 z(\theta). \tag{3.5}$$

From (3.5), we can deduce that the problem (1.1) is equivalent to the following operator equation

$$\mathcal{T}z(\theta)=z(\theta),\quad \theta\in\Delta. \tag{3.6}$$

Hence, the problem (1.1) has a solution if only if $\mathcal{T}$ has a fixed point.
For this purpose we shall demonstrate that $\mathcal{T}$ satisfies all the conditions given in Theorem 2.12.

Lemma 3.2. *The operator $\mathcal{T}_1$ is Lipschitz with constant $\frac{\kappa_1\Lambda_1}{1-\kappa_2}$. Furthermore, satisfies the following growth condition*

$$\|\mathcal{T}_1 z\|\leq\Lambda_1\Big(\frac{\tau_1}{1-\tau_2}\|z\|+\frac{\tau_3}{1-\tau_2}\Big),\quad \textit{for all}\ \ z\in C^2(\Delta,\ \mathbb{R}). \tag{3.7}$$

Proof. Let $z_1,z_2\in C^2(\Delta,\mathbb{R})$, then

$$|\mathcal{T}_1 z_1(\theta)-\mathcal{T}_1 z_2(\theta)|=\left|\Omega_{\varphi}^{\gamma'-1}(\theta,0)(\mathbb{A}_{h_{z_1}}-\mathbb{A}_{h_{z_2}})\right|\leq\Omega_{\varphi}^{\gamma'-1}(T,0)|\mathbb{A}_{h_{z_1}}-\mathbb{A}_{h_{z_2}}|.$$

By using Lemma 3.5, we have

$$\begin{aligned}|\mathbb{A}_{h_{z_1}}-\mathbb{A}_{h_{z_2}}|&\leq\frac{1}{|\varpi|\Gamma(\gamma')}\mathfrak{I}_{0^+}^{\alpha',\beta',\varphi}\left|\Phi_q\mathfrak{I}_{0^+}^{\alpha,\beta,\varphi}h_{z_1}(T)-\Phi_q\mathfrak{I}_{0^+}^{\alpha,\beta,\varphi}h_{z_2}(T)\right|\\&\quad+\frac{1}{|\varpi|\Gamma(\gamma')}\mathfrak{I}_{0^+}^{\alpha'+\nu_i,\beta',\varphi}\left|\Phi_q\mathfrak{I}_{0^+}^{\alpha,\beta,\varphi}h_{z_1}(\xi_i)-\Phi_q\mathfrak{I}_{0^+}^{\alpha,\beta,\varphi}h_{z_2}(\xi_i)\right|\\&\leq\frac{(p'-1)M^{p'-2}}{|\varpi|\Gamma(\gamma')}\mathfrak{I}_{0^+}^{\alpha'+\alpha,\beta',\varphi}|h_{z_1}(T)-h_{z_2}(T)|\\&\quad+\frac{(p'-1)M^{p'-2}}{|\varpi|\Gamma(\gamma')}\mathfrak{I}_{0^+}^{\alpha'+\alpha+\nu_i,\beta',\varphi}|h_{z_1}(\xi_i)-h_{z_2}(\xi_i)|.\end{aligned}$$

By using the assumption $(\mathcal{A}_1)$, we obtain

$$|h_{z_1}(\theta) - h_{z_2}(\theta)| \leq \frac{\kappa_1}{1-\kappa_2}|z_1 - z_2|, \quad \text{for all } \theta \in \Delta.$$

Thus, we get

$$\begin{aligned}|\mathcal{T}_1 z_1(\theta) - \mathcal{T}_1 z_2(\theta)| &\leq \frac{\kappa_1 (p'-1) M^{p'-2} \Omega_\varphi^{\gamma'-1}(T,0)}{(1-\kappa_2)|\varpi|\Gamma(\gamma')} \\ &\quad \times \Big(\frac{\Omega_\varphi^{\alpha'+\alpha}(T,0)}{\Gamma(\alpha'+\alpha+1)} + \sum_{i=1}^{m} |a_i| \frac{\Omega_\varphi^{\nu_i+\alpha'+\alpha}(\xi_i,0)}{\Gamma(\nu_i+\alpha'+\alpha+1)} \Big) \|z_1 - z_2\|.\end{aligned}$$

Taking supremum over θ we get

$$\|\mathcal{T}_1 z_1 - \mathcal{T}_1 z_2\| \leq \frac{\kappa_1 \Lambda_1}{1-\kappa_2} \|z_1 - z_2\|.$$

Thus, $\mathcal{T}_1$ is Lipschitz having constant $\dfrac{\kappa_1 \Lambda_1}{1-\kappa_2}$.
To demonstrate the growth condition (3.7), we proceed as follows:

$$\begin{aligned}|\mathcal{T}_1 z(\theta)| &\leq \Omega_\varphi^{\gamma'-1}(T,0)|\mathbb{A}_{h_z}| \\ &\leq \frac{\Omega_\varphi^{\gamma'-1}(T,0)}{|\varpi|\Gamma(\gamma')} \big| \mathfrak{I}_{0^+}^{\alpha',\beta',\varphi} \Phi_q \mathfrak{I}_{0^+}^{\alpha,\beta,\varphi} h_z(T) - \mathfrak{I}_{0^+}^{\alpha'+\nu_i,\beta',\varphi} \Phi_q \mathfrak{I}_{0^+}^{\alpha,\beta,\varphi} h_z(\xi_i) \big|.\end{aligned}$$

By Lemma (3.5), we get

$$\begin{aligned}|\mathcal{T}_1 z(\theta)| &\leq \frac{(p'-1) M^{p'-2} \Omega_\varphi^{\gamma'-1}(T,0)}{|\varpi|\Gamma(\gamma')} \\ &\quad \times \Big(\mathfrak{I}_{0^+}^{\alpha'+\alpha,\beta',\varphi} |h_z(T)| + \mathfrak{I}_{0^+}^{\alpha'+\alpha+\nu_i,\beta',\varphi} |h_z(\xi_i)| \Big).\end{aligned}$$

By using $(\mathcal{A}_2)$, we have

$$|h_z(\theta)| \leq \frac{1}{1-\tau_2}(\tau_1|z| + \tau_3), \quad \theta \in \Delta.$$

Thus, we obtain

$$\begin{aligned}|\mathcal{T}_1 z(\theta)| &\leq \frac{(p'-1) M^{p'-2} \Omega_\varphi^{\gamma'-1}(T,0)}{|\varpi|\Gamma(\gamma')} \\ &\quad \times \Big(\mathfrak{I}_{0^+}^{\alpha'+\alpha,\beta',\varphi} \Big(\frac{\tau_1\|z\| + \tau_3}{1-\tau_2} \Big) + \mathfrak{I}_{0^+}^{\alpha'+\alpha+\nu_i,\beta',\varphi} \Big(\frac{\tau_1\|z\| + \tau_3}{1-\tau_2} \Big) \Big).\end{aligned}$$

Taking supremum over θ we get

$$\|\mathcal{T}_1 z\| \leq \Lambda_1 \Big(\frac{\tau_1}{1-\tau_2}\|z\| + \frac{\tau_3}{1-\tau_2} \Big).$$

□

Lemma 3.3. *The operator $\mathcal{T}_2$ is continuous. Furthermore, $\mathcal{T}_2$ satisfies the following growth condition*

$$\|\mathcal{T}_2 z\| \leq \Lambda_2(\tau_1\|z\| + \tau_3), \quad \textit{for all} \;\; z \in C^2(\Delta, \mathbb{R}). \tag{3.8}$$

Proof. Let $z_n \in C^2(\Delta,\mathbb{R})$ a sequence that converges to $z \in C^2(\Delta,\mathbb{R})$ for each $t \in \Delta$. Then we have

$$\begin{aligned}|\mathcal{T}_2 z(\theta) - \mathcal{T}_2 z_n(\theta)| &= \left|\mathfrak{I}_{0^+}^{\alpha',\beta',\varphi}\Phi_q \mathfrak{I}_{0^+}^{\alpha,\beta,\varphi} h_z(\theta) - \mathfrak{I}_{0^+}^{\alpha',\beta',\varphi}\Phi_q \mathfrak{I}_{0^+}^{\alpha,\beta,\varphi} h_{z_n}(\theta)\right| \\ &\le \mathfrak{I}_{0^+}^{\alpha',\beta',\varphi}\left|\Phi_q \mathfrak{I}_{0^+}^{\alpha,\beta,\varphi} h_z(\theta) - \Phi_q \mathfrak{I}_{0^+}^{\alpha,\beta,\varphi} h_{z_n}(\theta)\right|.\end{aligned}$$

By Lemma 3.5 and $(\mathcal{A}_1)$, we get

$$\begin{aligned}|\mathcal{T}_2 z(\theta) - \mathcal{T}_2 z_n(\theta)| &\le (p'-1)M^{p'-2}\mathfrak{I}_{0^+}^{\alpha'+\alpha,\beta',\varphi}|h_z(\theta) - h_{z_n}(\theta)| \\ &\le (p'-1)M^{p'-2}\frac{\kappa_1}{1-\kappa_2}\Omega_\varphi^{\alpha'+\alpha-1}(T,0)\|z - z_n\|.\end{aligned}$$

By taking supremum over θ, we obtain

$$\|\mathcal{T}_2 z - \mathcal{T}_2 z_n\| \le \frac{\kappa_1}{1-\kappa_2}\Lambda_2\|z - z_n\|,$$

it follows that

$$\|\mathcal{T}_2 z - \mathcal{T}_2 z_n\| \to 0 \quad \text{as} \quad n \to \infty.$$

Thus, $\mathcal{T}_2$ is a continuous.
Now let us demonstrate (3.8). By using Lemma 3.5, we get

$$\begin{aligned}|\mathcal{T}_2 z(\theta)| &= \left|\mathfrak{I}_{0^+}^{\alpha',\beta',\varphi}\Phi_q \mathfrak{I}_{0^+}^{\alpha,\beta,\varphi} h_z(\theta)\right| \le \mathfrak{I}_{0^+}^{\alpha',\beta',\varphi}\Phi_q \mathfrak{I}_{0^+}^{\alpha,\beta,\varphi}|h_z(\theta)| \\ &\le (p'-1)M^{p'-2}\mathfrak{I}_{0^+}^{\alpha',\beta',\varphi}\mathfrak{I}_{0^+}^{\alpha,\beta,\varphi}|h_z(\theta)|.\end{aligned}$$

By using $(\mathcal{A}_2)$, we get

$$|\mathcal{T}_2 z(\theta)| \le (p'-1)M^{p'-2}\frac{\Omega_\varphi^{\alpha'+\alpha-1}(T,0)}{\Gamma(\alpha'+\alpha)}\Big(\frac{\tau_1}{1-\tau_2}\|z\| + \frac{\tau_3}{1-\tau_2}\Big).$$

Hence

$$\|\mathcal{T}_2 z\| \le \Lambda_2\Big(\frac{\tau_1}{1-\tau_2}\|z\| + \frac{\tau_3}{1-\tau_2}\Big).$$

□

Lemma 3.4. *The operator $\mathcal{T}_2$ is a compact. Consequently $\mathcal{T}_2$ is μ-Lipschitz with zero constant.*

Proof. Let $z \in B_r$, then by (3.8) we get

$$\|\mathcal{T}_2 z\| \le \frac{\Lambda_2}{1-\tau_2}(\tau_1 r + \tau_3).$$

It follows that $\mathcal{T}_2(B_r)$ is uniformly bounded.
Let $\theta_1, \theta_2 \in \Delta$ such that $0 < \theta_1 < \theta_2 < T$. Then by Lemma 3.5 and $(\mathcal{A}_2)$, we get

$$\begin{aligned}|\mathcal{T}_2 z(\theta_2) - \mathcal{T}_2 z(\theta_1)| \leq & \Big|\int_{\theta_1}^{\theta_2} \frac{\varphi'(s)\Omega_{\varphi}^{\alpha'-1}(\theta_2, s)}{\Gamma(\alpha')} \Phi_q \mathfrak{I}_{0^+}^{\alpha,\beta,\varphi} \frac{(\tau_1 r + \tau_3)}{(1-\tau_2)} ds\Big| \\ & + \Big|\int_0^{\theta_1} \frac{\varphi'(s)(\Omega_{\varphi}^{\alpha'-1}(\theta_2, s) - \Omega_{\varphi}^{\alpha'-1}(\theta_1, s))}{\Gamma(\alpha')} \\ & \times \Phi_q \mathfrak{I}_{0^+}^{\alpha,\beta,\varphi} \frac{(\tau_1 r + \tau_3)}{(1-\tau_2)} ds\Big| \\ \leq & \frac{(p'-1)M^{p'-2}(\tau_1 r + \tau_3)}{(1-\tau_2)\Gamma(\alpha'+\alpha+1)} \\ & \times \Big(\Omega_{\varphi}^{\alpha'+\alpha}(\theta_2, 0) - \Omega_{\varphi}^{\alpha'+\alpha}(\theta_2, \theta_1) + \Omega_{\varphi}^{\alpha'+\alpha}(\theta_1, 0)\Big).\end{aligned}$$

Thus

$$|\mathcal{T}_2 z(\theta_1) - \mathcal{T}_2 z(\theta_2)| \to 0 \quad \text{as} \quad \theta_1 \to \theta_2,$$

which implies that the set $\mathcal{T}_2(B_r)$ is equicontinuous. By applying the Arzelà–Ascoli Theorem [10], $\mathcal{T}_2$ is compact. Consequently, by Lemma 2.11, the operator $\mathcal{T}_2$ is μ-Lipschitz having constant zero. □

Theorem 3.5. *Assume that $(\mathcal{A}_1) - (\mathcal{A}_2)$ hold, if*

$$\frac{\kappa_1 \Lambda_1}{1-\kappa_2} < 1. \tag{3.9}$$

Then the problem (1.1) has at least one solution in $C^2(\Delta, \mathbb{R})$. Moreover, the set of solutions is bounded in $C^2(\Delta, \mathbb{R})$.

Proof. The operators $\mathcal{T}_1$, $\mathcal{T}_2$, and $\mathcal{T}$ are bounded and continuous. Moreover, $\mathcal{T}_1$ is μ-Lipschitz having constant $\frac{\kappa_1 \Lambda_1}{1-\kappa_2}$ and $\mathcal{T}_2$ is μ-Lipschitz having zero constant. By the condition 3.9 and both Lemma 2.10 and Lemma 2.9, we deduce that $\mathcal{T}$ is μ-condensing.
Now, we consider the following set

$$\mathbb{S}_{\varepsilon} = \big\{z \in C^2(\Delta, \mathbb{R}) : \text{ there exists } \varepsilon \in [0,1] \text{ such that } z = \varepsilon \mathcal{T} z\big\}.$$

Let $z \in \mathbb{S}_{\epsilon}$, then $z = \varepsilon \mathcal{T} z = \varepsilon(\mathcal{T}_1 z + \mathcal{T}_2 z)$, it follows

$$\begin{aligned}\|z\| = & \leq \varepsilon(\|\mathcal{T}_1 z\| + \|\mathcal{T}_2 z\|) \\ & \leq \frac{\Lambda_1 + \Lambda_2}{1-\tau_2}(\tau_1\|z\| + \tau_3).\end{aligned}$$

Hence, the set $\mathbb{S}_{\varepsilon}$ is bounded. Since all conditions of Theorem 2.12 hold. Then the operator $\mathcal{T}$ has at least one fixed point, which represents a solution to the problem (1.1). □

Theorem 3.6. *Assume that $(\mathcal{A}_1) - (\mathcal{A}_2)$ hold, if*

$$\frac{\kappa_1(\Lambda_1 + \Lambda_2)}{1-\kappa_2} < 1. \tag{3.10}$$

Then the problem (1.1) has a unique solution $z \in C^2(\Delta, \mathbb{R})$.

Proof. Let $z_1, z_2 \in C^2(\Delta, \mathbb{R})$, then by $(\mathcal{A}_1)$ we have

$$\begin{aligned}
|\mathcal{T}_2 z_1(\theta) - \mathcal{T}_2 z_2(\theta)| &= \Big| \int_0^\theta \frac{\varphi'(s)\Omega_\varphi^{\alpha'-1}(\theta,s)}{\Gamma(\alpha')} \Phi_q \mathfrak{I}_{0^+}^{\alpha,\beta,\varphi}(h_{z_1}(s) - h_{z_2}(s))ds \Big| \\
&\leq (p'-1)M^{p'-2} \int_0^\theta \frac{\varphi'(s)\Omega_\varphi^{\alpha'-1}(\theta,s)}{\Gamma(\alpha')} \mathfrak{I}_{0^+}^{\alpha,\beta,\varphi} |h_{z_1}(s) - h_{z_2}(s)| ds \\
&\leq (p'-1)M^{p'-2} \int_0^T \frac{\varphi'(s)\Omega_\varphi^{\alpha'-1}(T,s)}{\Gamma(\alpha')} \mathfrak{I}_{0^+}^{\alpha,\beta,\varphi} \frac{\kappa_1 \|z_1 - z_2\|}{1-\kappa_2} ds \\
&\leq \frac{\kappa_1}{1-\kappa_2} \Lambda_2 \|z_1 - z_2\|.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
|\mathcal{T} z_1 - \mathcal{T} z_2| &\leq |\mathcal{T}_1 z_1 - \mathcal{T}_1 z_2| + |\mathcal{T}_2 z_1 - \mathcal{T}_2 z_2| \\
&\leq \frac{\kappa_1}{1-\kappa_2} \Lambda_1 \|z_1 - z_2\| + \frac{\kappa_1}{1-\kappa_2} \Lambda_2 \|z_1 - z_2\|.
\end{aligned}$$

It follows

$$\|\mathcal{T} z_1 - \mathcal{T} z_2\| \leq \frac{\kappa_1(\Lambda_1 + \Lambda_2)}{1-\kappa_2} \|z_1 - z_2\|.$$

Thus, $\mathcal{T}$ is a contraction. By Banach's contraction principle, we deduce that $\mathcal{T}$ has a unique fixed point which corresponds to the unique solution of the problem (1.1). □

4. Ulam Stability Results

In this section, we investigate the Ulam-Hyers ($\mathbb{UH}$) and generalized Ulam-Hyers ($\mathbb{GUH}$) stability of our problem (1.1) under certain conditions.

Definition 4.1. [16] *The problem* (1.1) *is said to be Ulam-Hyers stable if there is a constant $c_h > 0$ such that for each $\delta > 0$ and each solution $\bar{z} \in C^2(\Delta, \mathbb{R})$ of the inequality*

$$\left| {}^H\mathfrak{D}_{0^+}^{\alpha,\beta,\varphi} \Phi_p \left({}^H\mathfrak{D}_{0^+}^{\alpha',\beta',\varphi} \bar{z}(\theta) \right) - h_{\bar{z}}(\theta) \right| \leq \delta, \quad \theta \in \Delta, \tag{4.1}$$

there exists a solution $z \in C^2(\Delta, \mathbb{R})$ of problem (1.1) *with*

$$|\bar{z}(\theta) - z(\theta)| \leq c_h \delta, \quad \theta \in \Delta.$$

Definition 4.2. [16] *The problem* (1.1) *is said to be generalized Ulam-Hyers stable if there exists a function $\Psi \in C(\mathbb{R}^+, \mathbb{R}^+)$ with $\Psi(0) = 0$ such that for each solution $\bar{z} \in C^2(\Delta, \mathbb{R})$ of the inequality (4.1), there exists a solution $z \in C^2(\Delta, \mathbb{R})$ of problem* (1.1) *with*

$$|\bar{z}(\theta) - z(\theta)| \leq \Psi(\delta), \quad \theta \in \Delta.$$

Remark 4.3. Definition 4.1 $\Rightarrow$ Definition 4.2.

Remark 4.4. A function $\bar{z} \in C^2(\Delta, \mathbb{R})$ is a solution of the inequality (4.1) if and only if there exists a function $\Psi_1 \in C(\Delta, \mathbb{R})$ such that

1. $|\Psi_1(\theta)| \leq \delta$, for all $\theta \in \Delta$

2. ${}^{H}\mathfrak{D}_{0^+}^{\alpha,\beta,\varphi}\Phi_p\big({}^{H}\mathfrak{D}_{0^+}^{\alpha',\beta',\varphi}\bar{z}(\theta)\big) = h_{\bar{z}}(\theta) + \Psi_1(\theta), \quad \theta \in \Delta.$

Lemma 4.5. *If $\bar{z} \in C^2(\Delta, \mathbb{R})$ is a solution to the inequality* (4.1) *then it satisfies*

$$|\bar{z} - \mathcal{T}\bar{z}| \leq (\Lambda_1 + \Lambda_2)\delta.$$

Proof. By Remark 4.4 and Lemma 3.1, we have

$$\bar{z}(\theta) = \mathfrak{I}_{0^+}^{\alpha',\beta',\varphi}\Phi_q\mathfrak{I}_{0^+}^{\alpha,\beta,\varphi}\big(h_{\bar{z}}(\theta) + \Psi(\theta)\big) + \Omega_{\varphi}^{\gamma'-1}(\theta,0)\mathbb{A}_{h_{\bar{z}}+\Psi}.$$

Then, we obtain

$$\begin{aligned}
|\bar{z}(\theta) - \mathcal{T}\bar{z}(\theta)| &= \Big|\mathfrak{I}_{0^+}^{\alpha',\beta',\varphi}\Phi_q\mathfrak{I}_{0^+}^{\alpha,\beta,\varphi}\big(h_{\bar{z}}(\theta) + \Psi(\theta)\big) + \Omega_{\varphi}^{\gamma'-1}(\theta,0)\mathbb{A}_{h_{\bar{z}+\Psi}} \\
&\qquad - \mathfrak{I}_{0^+}^{\alpha',\beta',\varphi}\Phi_q\mathfrak{I}_{0^+}^{\alpha,\beta,\varphi}h_{\bar{z}}(\theta) - \Omega_{\varphi}^{\gamma'-1}(\theta,0)\mathbb{A}_{h_{\bar{z}}}\Big| \\
&\leq \big|\mathfrak{I}_{0^+}^{\alpha',\beta',\varphi}\Phi_q\mathfrak{I}_{0^+}^{\alpha,\beta,\varphi}\Psi(\theta) + \Omega_{\varphi}^{\gamma'-1}(\theta,0)\mathbb{A}_{\Psi}\big| \\
&\leq \mathfrak{I}_{0^+}^{\alpha',\beta',\varphi}\Phi_q\mathfrak{I}_{0^+}^{\alpha,\beta,\varphi}\delta + \frac{\mathfrak{I}_{0^+}^{\alpha',\beta',\varphi}\Phi_q\mathfrak{I}_{0^+}^{\alpha,\beta,\varphi}\delta + \mathfrak{I}_{0^+}^{\alpha'+\nu_i,\beta',\varphi}\Phi_q\mathfrak{I}_{0^+}^{\alpha,\beta,\varphi}\delta}{|\varpi|\Gamma(\gamma')} \\
&\leq (\Lambda_1 + \Lambda_2)\delta.
\end{aligned}$$

□

Theorem 4.6. *Assume that the assumptions* $(\mathcal{A}_1) - (\mathcal{A}_2)$ *and the condition* (3.10) *are satisfied. Then the problem (1.1) is Ulam–Hyers stable as well as generalized Ulam–Hyers stable.*

Proof. Let $\bar{z} \in C^2(\Delta, \mathbb{R})$ be a solution of the inequality (4.1) and $z \in C^2(\Delta, \mathbb{R})$ be a solution of (1.1). Then, we have

$$\begin{aligned}
|\bar{z}(\theta) - z(\theta)| &= |\bar{z}(\theta) - \mathcal{T}z(\theta)| \\
&\leq |\bar{z}(\theta) - \mathcal{T}\bar{z}(\theta)| + |\mathcal{T}\bar{z}(\theta) - \mathcal{T}z(\theta)| \\
&\leq (\Lambda_1 + \Lambda_2)\delta + \frac{\kappa_1}{1-\kappa_2}(\Lambda_1 + \Lambda_2)|\bar{z}(\theta) - z(\theta)|.
\end{aligned}$$

Thus

$$|\bar{z} - z| \leq c_h\delta, \qquad \text{where} \;\; c_h = \frac{(\Lambda_2 + \Lambda_1)}{1 - \frac{\kappa_1}{1-\kappa_2}(\Lambda_2 + \Lambda_1)}.$$

Hence, the problem (1.1) is Ulam-Hyers stable. On the other hand, by choosing $\Psi(\delta) = c_h\delta, \;\; \Psi(0) = 0$, we obtain that the problem (1.1) is generalized Ulam-Hyers stable. □

5. An illustrative example

In this section, we consider the following φ-Hilfer fractional differential equation with p-Laplacian operator

$$\begin{cases} {}^H\mathfrak{D}_{0^+}^{\frac{1}{2},\frac{1}{2},\theta+1}\Phi_2^H\mathfrak{D}_{0^+}^{\frac{3}{2},\frac{1}{2},\theta+1}z(\theta) = \dfrac{\sin(z(\theta))+1}{19+e^\theta} + \dfrac{|{}^H\mathfrak{D}_{0^+}^{\frac{1}{2},\frac{1}{2},\theta+1}\Phi_2^H\mathfrak{D}_{0^+}^{\frac{3}{2},\frac{1}{2},\theta+1}z(\theta)|}{(\theta+4)^2}, \\ z(0)=0, \quad {}^H\mathfrak{D}_{0^+}^{\frac{3}{2},\frac{1}{2},\theta+1}z(0)=0, \quad z(1) = \mathfrak{I}_{0^+}^{\frac{1}{4},\theta+1}z\left(\frac{1}{4}\right) + 3\mathfrak{I}_{0^+}^{\frac{5}{4},\theta+1}z\left(\frac{3}{4}\right). \end{cases} \tag{5.1}$$

We observe that problem (5.1) is a special case of problem (1.1) when

$$\alpha=\frac{1}{2},\ \beta=\frac{1}{2},\ \alpha'=\frac{3}{2},\ \beta'=\frac{1}{2},\ \gamma=\frac{3}{4},\ \gamma'=\frac{7}{4},\ T=1,\ \varphi(\theta)=\theta+1,$$

$$p=2,\ p'=2,\ M=1,\ m=2,\ a_1=1,\ a_2=3,\ \nu_1=\frac{1}{4},\ \nu_2=\frac{5}{4},$$

$\xi_1=\frac{1}{4}$, $\xi_2=\frac{3}{4}$ and

$$h_z(\theta) = \frac{\sin(z(\theta))+1}{19+e^\theta} + \frac{|{}^H\mathfrak{D}_{0^+}^{\frac{1}{2},\frac{1}{2},\theta+1}\Phi_2^H\mathfrak{D}_{0^+}^{\frac{3}{2},\frac{1}{2},\theta+1}z(\theta)|}{(\theta+4)^2}.$$

For all $z_1, z_2, \bar{z}_1, \bar{z}_2 \in \mathbb{R}$, and $\theta\in[0,1]$, we have

$$\begin{aligned} &|h_{z_1}(\theta) - h_{z_2}(\theta)| \\ &\le \frac{1}{20}|z_1-z_2| + \frac{1}{16}\left|{}^H\mathfrak{D}_{0^+}^{\frac{1}{2},\frac{1}{2},\theta+1}\Phi_2^H\mathfrak{D}_{0^+}^{\frac{3}{2},\frac{1}{2},\theta+1}z_1 - {}^H\mathfrak{D}_{0^+}^{\frac{1}{2},\frac{5}{2},\theta+1}\Phi_2^H\mathfrak{D}_{0^+}^{\frac{3}{2},\frac{5}{2},\theta+1}z_2\right|. \end{aligned}$$

Thus, the assumption $(\mathcal{A}_1)$ holds, with $\kappa_1=\dfrac{1}{20}$ and $\kappa_2=\dfrac{1}{16}$.
On other hand,

$$|h_z(\theta)| \le \frac{1}{20} + \frac{1}{20}|z| + \frac{1}{16}\left|{}^H\mathfrak{D}_{0^+}^{\frac{1}{2},\frac{5}{2},\theta+1}\Phi_2^H\mathfrak{D}_{0^+}^{\frac{3}{2},\frac{5}{2},\theta+1}z\right|.$$

Hence, the assumption $(\mathcal{A}_2)$ satisfied, with $\tau_1=\tau_2=\dfrac{1}{20}$ and $\tau_3=\dfrac{1}{16}$.
With simple calculations, we get

$$\frac{\kappa_1(\Lambda_1+\Lambda_2)}{1-\kappa_2} = 0.0666666666666667 < 1.$$

Then by virtue of Theorem 3.6, the problem (5.1) has a unique solution in $C^2([0,1],\mathbb{R})$. Moreover, by Theorem 4.6, the problem (5.1) is $\mathbb{UH}$ and $\mathbb{GUH}$ stable.

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