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On some univalence criteria for certain integral operators

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Abstract. For analytic functions in the open unit disk, we define new general integral operators. The aim of this paper is to study these new operators and related univalence criteria. First of all, we recall some classes of functions defined on the unit disk. We will use functions from these classes to construct our integral operators. Secondly, we recall the univalence criteria that we use in the proofs of our results. Finally, we use the univalence criteria to establish univalence conditions related to our general integral operators.

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1. Introduction and Preliminaries

Let $\mathcal{A}$ be the class of analytic functions f in the open unit disk

$$\mathcal{U} := \{z \in \mathbb{C} : |z| < 1\},$$

which are given by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

for all $z \in \mathcal{U}$, that satisfy the following normalization condition

$$f(0) = f'(0) - 1 = 0.$$

Furthermore, let us denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ that consists of univalent functions (i.e., injective functions).

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In the paper [11], J. Liu and D. Yang introduced the class $\mathcal{S}(\alpha)$ as follows (see [11, Relations (1.4)-(1.5)]).

Definition 1.1. *For $\alpha \in (0,2]$, define the class $\mathcal{S}(\alpha)$ as the class of all functions $f \in \mathcal{A}$ that satisfy the following conditions:*

(i) $f(z) \neq 0$, *for* $0 < |z| < 1$ *and*

(ii)

$$\left| \left(\frac{z}{f(z)} \right)'' \right| \leq \alpha,$$

for all $z \in \mathcal{U}$.

We also have the following result (see [11, Theorem 3, Relation (3.14)]).

Lemma 1.2. *Let $\alpha \in (0,2]$. If $f \in \mathcal{S}(\alpha)$ then*

$$\left| \frac{zf'(z)}{f^2(z)} - 1 \right| \leq \alpha |z|^2,$$

for all $z \in \mathcal{U}$.

Next, we recall the notions of starlikeness of order k and convexity of order k (see, e.g., [8, Def. 2.3.1]).

We say that a function $f \in \mathcal{A}$ belongs to the class $\mathcal{S}^*(k)$ (i.e., the class of starlike functions of order k in $\mathcal{U}$), for $k \in [0,1)$, if it satisfies the inequality

$$\operatorname{Re}\left[\frac{zf'(z)}{f(z)} \right] > k,$$

for $z \in \mathcal{U}$.

For $k \in [0,1)$, we denote by $\mathcal{K}(k)$ the class of convex functions f of order k in $\mathcal{U}$, i.e., the class of analytic functions that satisfy

$$\operatorname{Re}\left[\frac{zf''(z)}{f'(z)} + 1 \right] > k,$$

for $z \in \mathcal{U}$.

A function $f \in \mathcal{A}$ is said to belong to the class $\mathcal{R}(k)$, for $k \in [0,1)$, if

$$\operatorname{Re}|f'(z)| > k,$$

for $z \in \mathcal{U}$.

Frasin and Jahangiri (see [7, Relation (1.5)]) have studied the class $\mathcal{B}(\mu, k)$, for $\mu \geq 0$ and $k \in [0,1)$. This class contains all functions $f \in \mathcal{A}$, that satisfy the condition

$$\left| f'(z) \left(\frac{z}{f(z)} \right)^{\mu} - 1 \right| < 1 - k,$$

for $z \in \mathcal{U}$.

The class $\mathcal{B}(\mu, k)$ is a comprehensive class of normalized analytic functions in $\mathcal{U}$ that contains other classes of analytic and univalent functions in $\mathcal{U}$, such as

$$\mathcal{B}(1,k) = \mathcal{S}^*(k),\ \mathcal{B}(0,k) = \mathcal{R}(k).$$

A great deal of researchers have devoted themselves to the study of sufficient conditions for the univalence of various integral operators (see, e.g., [1], [2], [3], [4], [5], [6], [8], [9], [10], [12], [17]).

The goal of this present paper is to study the univalence conditions for the integral operators

$$F_{n,\beta}(z) := \left[\beta \int_0^z t^{\beta-1} \prod_{i=1}^{n} \frac{e^{f_i(t)}}{g_i'(t)} \mathrm{d}t\right]^{\frac{1}{\beta}}, \tag{1.1}$$

for $\beta \in \mathbb{C} \setminus \{0\}$, $f_i, g_i \in \mathcal{A}$, $i = \overline{1,n}$, and

$$G_{\beta,\gamma}(f,g)(z) := \left[\beta \int_0^z t^{\beta-1} \left[\frac{e^{f(t)}}{g'(t)}\right]^{\gamma} \mathrm{d}t\right]^{\frac{1}{\beta}}, \tag{1.2}$$

for $\beta \in \mathbb{C} \setminus \{0\}$, $\gamma \in \mathbb{C}$, $f, g \in \mathcal{A}$.

Remark 1.3. In the case $f_i = f$, $g_i = g$, for $i = \overline{1,n}$, then

$$F_{n,\beta} = G_{\beta,n}(f,g).$$

Remark 1.4. For $\beta = 1$, the operator (1.2) reduces to the operator $G_\gamma(f,g)$, given by

$$G_\gamma(f,g)(z) := \int_0^z \left[\frac{e^{f(t)}}{g'(t)}\right]^{\gamma} \mathrm{d}t.$$

For additional details on the operator $G_\gamma(f,g)$, we refer the reader to [2].

In order to derive our main results, we recall the following univalence criteria (see also, [14]).

Theorem 1.5. ([15, Theorem 2]) *Let $f \in \mathcal{A}$ and $\beta \in \mathbb{C}$. If $\mathrm{Re}(\beta) > 0$ and*

$$\frac{1-|z|^{2\mathrm{Re}(\beta)}}{\mathrm{Re}(\beta)} \cdot \left|\frac{zf''(z)}{f'(z)}\right| \leq 1.$$

for all $z \in \mathcal{U}$, then the function $\mathcal{F}_\beta$ defined by

$$\mathcal{F}_\beta(z) := \left[\beta \int_0^z t^{\beta-1} f'(t) \mathrm{d}t\right]^{\frac{1}{\beta}}, \tag{1.3}$$

belongs to the class $\mathcal{S}$ of analytic and univalent functions in $\mathcal{U}$.

Theorem 1.6. ([16]) *Let $c, \alpha \in \mathbb{C}$ such that $\operatorname{Re}(\alpha) > 0$ and $|c| \leq 1$, $c \neq -1$. If the function $f \in \mathcal{A}$,*

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

for all $z \in \mathcal{U}$, satisfies the condition

$$\left| c|z|^{2\alpha} + (1 - |z|^{2\alpha}) \frac{zf''(z)}{\alpha f'(z)} \right| \leq 1,$$

for $z \in \mathcal{U}$, then we have that the function $\mathcal{F}_\alpha$ given by relation (1.3) belongs to the class $\mathcal{S}$.

In addition, we recall the generalized Schwarz lemma (see, e.g., [8, p. 35], [9], [13]).

Lemma 1.7. *(Generalized Schwarz Lemma) Let $R > 0$ be a constant. Let f be an analytic function in the disk*

$$\mathcal{U}_R := \{z \in \mathbb{C} : |z| < R\},$$

satisfies the property that there is a fixed constant $M > 0$ such that

$$|f(z)| < M,$$

for $|z| < R$. If f has at $z = 0$ a root with its multiplicity order greater than some m, then

$$|f(z)| \leq \frac{M}{R^m} |z|^m, \tag{1.4}$$

for all $z \in \mathcal{U}_R$.

The equality in relation (1.4) holds for $z \neq 0$ if

$$f(z) = e^{i\theta} \frac{M}{R^m} z^m,$$

for all $z \in \mathcal{U}_R$, where θ is a constant.

2. Main results

In this section, we aim to use the univalence criteria presented in the previous section (see Theorem 1.5, Theorem 1.6) in order to give univalence conditions for the integral operators $F_{n,\beta}$ and $G_{\beta,\gamma}$, given by relations (1.1) and (1.2), respectively.

In the latter, let us denote the set of non-negative real numbers by $\mathbb{R}_+$.

The following theorem gives us sufficient conditions for the univalence of the integral operators $F_{n,\beta}$ (see relation (1.1)) and $G_{\beta,\gamma}$ (see relation (1.2)), whenever the functions involved in the definitions of these integral operators belong to the class $\mathcal{A}$.

Theorem 2.1. *Let* $f_i \in \mathcal{A}$, $i = \overline{0,n}$, *such that*

$$\left| \frac{z^2 f_i'(z)}{f_i^2(z)} - 1 \right| \leq 1, \tag{2.1}$$

for $i = \overline{0,n}$, *and for all* $z \in \mathcal{U}$.

Let $M_i, N_i \in \mathbb{R}_+$, $i = \overline{0,n}$ *and let* $g_i \in \mathcal{A}$, $i = \overline{0,n}$, *such that*

$$|f_i(z)| < M_i, \quad \left| \frac{g_i''(z)}{g_i'(z)} \right| \leq N_i, \tag{2.2}$$

for $i = \overline{0,n}$, *and for all* $z \in \mathcal{U}$.

Let $\gamma \in \mathbb{C}$ *and let* $\beta \in \mathbb{C}$ *such that* $\operatorname{Re}(\beta) =: a > 0$.

Denote

$$c := \frac{2}{2a+1} \left(\frac{1}{2a+1} \right)^{\frac{1}{2a}}. \tag{2.3}$$

(i) If

$$c \sum_{i=1}^{n} (2M_i^2 + N_i) \leq 1, \tag{2.4}$$

then the integral operator $F_{n,\beta}$ *given by relation* (*1.1*) *belongs to the class* $\mathcal{S}$.

(ii) If

$$c|\gamma|(2M_0^2 + N_0) \leq 1, \tag{2.5}$$

then the integral operator $G_{\beta,\gamma}(f_0, g_0)$ *given by relation* (*1.2*) *belongs to the class* $\mathcal{S}$.

Proof. We will prove statement (i).

To achieve this, we consider the function $h_n : \mathcal{U} \to \mathbb{C}$, defined by

$$h_n(z) := \int_0^z \prod_{i=1}^{n} \frac{e^{f_i(t)}}{g_i'(t)} \mathrm{d}t. \tag{2.6}$$

The function h_n is holomorphic in the unit disk. By differentiation, we get

$$h_n'(z) = \prod_{i=1}^{n} \frac{e^{f_i(z)}}{g_i'(z)}, \tag{2.7}$$

and

$$h_n''(z) = \sum_{i=1}^{n} \frac{e^{f_i(z)} f_i'(z) g_i'(z) - e^{f_i(z)} g_i''(z)}{g_i^2(z)} \cdot \prod_{k=1, k\neq i}^{n} \frac{e^{f_k(z)}}{g_k'(z)}. \tag{2.8}$$

Using relations (2.7) and (2.8) we obtain

$$\frac{h_n''(z)}{h_n'(z)} = \sum_{i=1}^{n} \left(f_i'(z) - \frac{g_i''(z)}{g_i'(z)} \right). \tag{2.9}$$

We multiply the modulus of (2.9) by $\frac{1-|z|^{2\mathrm{Re}(\beta)}}{\mathrm{Re}(\beta)} > 0$ and by $|z|$ and we get

$$\begin{aligned}\frac{1-|z|^{2\mathrm{Re}(\beta)}}{\mathrm{Re}(\beta)}\left|\frac{zh_n''(z)}{h_n'(z)}\right| &= \frac{1-|z|^{2\mathrm{Re}(\beta)}}{\mathrm{Re}(\beta)}|z|\sum_{i=1}^{n}\left|f_i'(z)-\frac{g_i''(z)}{g_i'(z)}\right| \\ &\le \frac{1-|z|^{2a}}{a}|z|\sum_{i=1}^{n}\left(|f_i'(z)|+\left|\frac{g_i''(z)}{g_i'(z)}\right|\right).\end{aligned} \tag{2.10}$$

Taking into account the inequality

$$\frac{1-|z|^{2a}}{a}|z| \le \frac{2}{2a+1}\left(\frac{1}{2a+1}\right)^{\frac{1}{2a}} = c, \tag{2.11}$$

for all $|z|<1$, and the relation (2.10), we obtain

$$\frac{1-|z|^{2\mathrm{Re}(\beta)}}{\mathrm{Re}(\beta)}\left|\frac{zh_n''(z)}{h_n'(z)}\right| \le c\sum_{i=1}^{n}\left(|f_i'(z)|+\left|\frac{g_i''(z)}{g_i'(z)}\right|\right). \tag{2.12}$$

Now, we apply Lemma 1.7 to the functions f_i, $i=\overline{1,n}$, and we get

$$|f_i(z)| \le M_i|z|, \tag{2.13}$$

for all $i=\overline{1,n}$ and for all $z\in\mathcal{U}$.

Using relations (2.12) and (2.13) we deduce that

$$\begin{aligned}\frac{1-|z|^{2\mathrm{Re}(\beta)}}{\mathrm{Re}(\beta)}\left|\frac{zh_n''(z)}{h_n'(z)}\right| &\le c\sum_{i=1}^{n}\left(\left|\frac{z^2f_i'(z)}{f_i^2(z)}\right|\left|\frac{f_i(z)}{z}\right|^2+\left|\frac{g_i''(z)}{g_i'(z)}\right|\right) \\ &\le c\sum_{i=1}^{n}\left(\left|\frac{z^2f_i'(z)}{f_i^2(z)}-1+1\right|M_i^2+N_i\right) \\ &\le c\sum_{i=1}^{n}(2M_i^2+N_i) \le 1.\end{aligned} \tag{2.14}$$

Now, by applying Theorem 1.5, it follows that $F_{n,\beta}\in\mathcal{S}$.

The proof of statement (ii) follows similar arguments to those presented in the proof of statement (i).

In the case of statement (ii), one considers the function $h:\mathcal{U}\to\mathbb{C}$,

$$h(z) := \int_0^z\left(\frac{e^{f_0(t)}}{g_0'(t)}\right)^{\gamma}\mathrm{d}t. \tag{2.15}$$

The proof of this statement is omitted for the sake of brevity.

This concludes our proof. □

In the following result, we show that the operators $F_{n,\beta}$ (see relation (1.1)) and $G_{\beta,\gamma}(f_0,g_0)$ (see relation (1.2)) belong to the class of univalent functions in $\mathcal{U}$, in the case that the functions $f_i\in\mathcal{S}(\alpha_i)$, for $\alpha_i\in(0,2]$, $i=\overline{0,n}$ (see Definition 1.1).

Theorem 2.2. *Let $\alpha_i \in (0,2]$ and let $f_i \in \mathcal{S}(\alpha_i)$, for $i = \overline{0,n}$. Let $M_i, N_i \in \mathbb{R}_+$ and $g_i \in \mathcal{A}$ for $i = \overline{0,n}$, such that*

$$|f_i(z)| < M_i, \quad \left|\frac{g_i''(z)}{g_i'(z)}\right| \leq N_i, \tag{2.16}$$

for $i = \overline{0,n}$ and for all $z \in \mathcal{U}$.

Let $\gamma \in \mathbb{C}$ and let $\beta \in \mathbb{C}$ such that $\mathrm{Re}(\beta) =: a > 0$.

Denote

$$c := \frac{2}{2a+1}\left(\frac{1}{2a+1}\right)^{\frac{1}{2a}}. \tag{2.17}$$

(i) If

$$c\sum_{i=1}^{n}((\alpha_i+1)M_i^2 + N_i) \leq 1, \tag{2.18}$$

then the integral operator $F_{n,\beta}$ given by relation (1.1) belongs to the class $\mathcal{S}$.

(ii) If

$$c|\gamma|((\alpha_0+1)M_0^2 + N_0) \leq 1, \tag{2.19}$$

then the integral operator $G_{\beta,\gamma}(f_0, g_0)$ given by relation (1.2) belongs to the class $\mathcal{S}$.

Proof. We will prove statement (i). To this end, we consider the function $h_n : \mathcal{U} \to \mathbb{C}$ given by relation (2.6). Recall that the function h_n is holomorphic in the unit disk.

Recall that, in the proof of Theorem 2.1, we have obtained the inequality (2.14).

Now, let $|z| < 1$. If we take into account relations (2.11), (2.14) and Lemma 1.7 and Lemma 1.2, we obtain

$$\begin{aligned}
\frac{1-|z|^{2\mathrm{Re}(\beta)}}{\mathrm{Re}(\beta)}\left|\frac{zh_n''(z)}{h_n'(z)}\right| &\leq c\sum_{i=1}^{n}\left(\left|\frac{z^2 f_i'(z)}{f_i^2(z)} - 1\right| M_i^2 + M_i^2 + N_i\right) \\
&\leq c\sum_{i=1}^{n}(\alpha_i|z|^2 M_i^2 + M_i^2 + N_i) \\
&\leq c\sum_{i=1}^{n}((\alpha_i+1)M_i^2 + N_i) \\
&\leq 1.
\end{aligned} \tag{2.20}$$

By applying Theorem 1.5, we deduce that $F_{n,\beta} \in \mathcal{S}$.

The proof of statement (ii) is similar to that of statement (i). In this case, one must consider the function $h : \mathcal{U} \to \mathbb{C}$ given by relation (2.15). We omit the proof for the sake of brevity.

This concludes our proof. □

Next, we analyze univalence conditions for the operators $F_{n,\beta}$ (see relation (1.1)) and $G_{\beta,\gamma}$ (see relation (1.2)) in the case that the functions $f_i \in \mathcal{B}(\mu_i, \alpha_i)$, $\mu_i \geq 0$, $\alpha_i \in [0,1)$, for $i = \overline{0,n}$.

Theorem 2.3. *Let $\mu_i \geq 0$, $\alpha_i \in [0,1)$, and let $f_i \in \mathcal{B}(\mu_i, \alpha_i)$, for $i = \overline{0,n}$.*

Let $M_i, N_i \in \mathbb{R}_+$ and $g_i \in \mathcal{A}$, for $i = \overline{0,n}$, such that

$$M_i \geq 1, \tag{2.21}$$

for $i = \overline{0,n}$ and

$$|f_i(z)| < M_i, \quad \left|\frac{g_i''(z)}{g_i'(z)}\right| \leq N_i, \tag{2.22}$$

for $i = \overline{0,n}$ and for all $z \in \mathcal{U}$.

Let $\gamma \in \mathbb{C}$ and let $\beta \in \mathbb{C}$ such that $\mathrm{Re}(\beta) =: a > 0$.

Denote

$$c := \frac{2}{2a+1}\left(\frac{1}{2a+1}\right)^{\frac{1}{2a}}. \tag{2.23}$$

(i) If

$$c\sum_{i=1}^{n}((2-\alpha_i)M_i^{\mu_i} + N_i) \leq 1, \tag{2.24}$$

then the integral operator $F_{n,\beta}$ given by relation (1.1) belongs to the class $\mathcal{S}$.

(ii) If

$$c|\gamma|((2-\alpha_0)M_0^{\mu_0} + N_0) \leq 1, \tag{2.25}$$

then the integral operator $G_{\beta,\gamma}(f_0, g_0)$ given by relation (1.2) belongs to the class $\mathcal{S}$.

Proof. In order to prove statement (i), we consider the function $h_n : \mathcal{U} \to \mathbb{C}$, given by relation (2.6).

We also recall the fact that, in the proof of Theorem 2.1, we have established relation (2.10).

Furthermore, if we now apply similar arguments to those in the proof of Theorem 2.2 and take into account the inequality (2.11), the hypotheses of Theorem 2.3, Lemma 1.7 (for f_i, $i = \overline{1,n}$), we obtain

$$\frac{1-|z|^{2\mathrm{Re}(\beta)}}{\mathrm{Re}(\beta)}\left|\frac{zh_n''(n)}{h_n'(z)}\right| \leq c\sum_{i=1}^{n}((2-\alpha_i)M_i^{\mu_i} + N_i) \leq 1. \tag{2.26}$$

It remains now to apply Theorem 1.5 in order to show that, indeed, $F_{n,\beta} \in \mathcal{S}$.

Now, to prove statement (ii), we consider the function $h : \mathcal{U} \to \mathbb{C}$ given by relation (2.15). Using the same reasoning as in the proof of Theorem 2.1, one is able to show that

$$\frac{1-|z|^{2\mathrm{Re}(\beta)}}{\mathrm{Re}(\beta)}\left|\frac{zh_n''(n)}{h_n'(z)}\right| \leq \frac{1-|z|^{2\mathrm{Re}(\beta)}}{\mathrm{Re}(\beta)}|z||\gamma|\left(|f_0'(z)| + \left|\frac{g_0''(z)}{g_0'(z)}\right|\right). \tag{2.27}$$

We proceed with the computations and we get

$$\begin{aligned}\frac{1-|z|^{2\mathrm{Re}(\beta)}}{\mathrm{Re}(\beta)}\left|\frac{zh_n''(n)}{h_n'(z)}\right| &\leq \frac{1-|z|^{2\mathrm{Re}(\beta)}}{\mathrm{Re}(\beta)}|z||\gamma| \cdot \\ \left(\left|f_0'(z)\left(\frac{z}{f_0(z)}\right)^{\mu} - 1\right|\left|\left(\frac{f_0(z)}{z}\right)^{\mu}\right| + \left|\left(\frac{f_0(z)}{z}\right)^{\mu}\right| + \left|\frac{g_0''(z)}{g_0'(z)}\right|\right).\end{aligned} \tag{2.28}$$

We make use now of the inequality (2.11), the hypotheses of Theorem 2.3 and Lemma 1.7 applied to the function f_0 and we get

$$\frac{1-|z|^{2\mathrm{Re}(\beta)}}{\mathrm{Re}(\beta)}\left|\frac{zh_n''(n)}{h_n'(z)}\right| \le c|\gamma|((2-\alpha_0)M_0^{\mu_0}+N_0) \le 1. \tag{2.29}$$

Now, by Theorem 1.5, it follows that $G_{\beta,\gamma}(f_0, g_0) \in \mathcal{S}$.
This concludes our proof. □

We end this section by providing a theorem that gives different conditions for the univalence of the operators $F_{n,\beta}$ and $G_{\beta,\gamma}$ (see relations (1.1) and (1.2)), for $f_i \in \mathcal{B}(\mu_i, \alpha_i)$, $\mu_i \ge 0$, $\alpha_i \in [0, 1)$, for $i = \overline{0, n}$.

Theorem 2.4. *Let $\mu_i \ge 0$, $\alpha_i \in [0,1)$, and let $f_i \in \mathcal{B}(\mu_i, \alpha_i)$, for $i = \overline{0,n}$.*
Let $M_i, N_i \in \mathbb{R}_+$ and $g_i \in \mathcal{A}$, for $i = \overline{0,n}$, such that

$$M_i \ge 1, \tag{2.30}$$

for $i = \overline{0,n}$ and

$$|f_i(z)| < M_i, \quad \left|\frac{g_i''(z)}{g_i'(z)}\right| \le N_i, \tag{2.31}$$

for $i = \overline{0,n}$ and for all $z \in \mathcal{U}$.
Let $\gamma \in \mathbb{C}$ and let $\beta \in \mathbb{C}$ such that $\mathrm{Re}(\beta) > 0$.
In addition, let $c \in \mathbb{C}$ such that $|c| \le 1$, $c \ne -1$.

(i) If

$$\mathrm{Re}(\beta) \ge \sum_{i=1}^{n} ((2-\alpha_i)M_i^{\mu_i} + N_i), \tag{2.32}$$

and

$$|c| \le 1 - \frac{1}{\mathrm{Re}(\beta)} \sum_{i=1}^{n} ((2-\alpha_i)M_i^{\mu_i} + N_i) \tag{2.33}$$

then the integral operator $F_{n,\beta}$ given by relation (1.1) belongs to the class $\mathcal{S}$.

(ii) If

$$\mathrm{Re}(\beta) \ge |\gamma| ((2-\alpha_0)M_0^{\mu_0} + N_0), \tag{2.34}$$

and

$$|c| \le 1 - \frac{|\gamma|}{\mathrm{Re}(\beta)} ((2-\alpha_0)M_0^{\mu_0} + N_0) \tag{2.35}$$

then the integral operator $G_{\beta,\gamma}(f_0, g_0)$ given by relation (1.2) belongs to the class $\mathcal{S}$.

Proof. In order to prove statement (i), we use the function $h_n : \mathcal{U} \to \mathbb{C}$ given by relation (2.6). We start by recalling that we have shown that relation (2.9) holds.

For $c \in \mathbb{C}$, $|c| \leq 1$, $c \neq -1$, we have

$$\left|c|z|^{2\beta} + (1-|z|^{2\beta})\frac{zh_n''(z)}{\beta h_n'(z)}\right| = \left|c|z|^{2\beta} + (1-|z|^{2\beta})\frac{z}{\beta}\sum_{i=1}^{n}\left(f_i'(z) - \frac{g_i''(z)}{g_i'(z)}\right)\right|$$
$$\leq |c| + \frac{1}{|\beta|}\sum_{i=1}^{n}\left(|f_i'(z)| + \left|\frac{g_i''(z)}{g_i'(z)}\right|\right)$$
$$\leq |c| + \frac{1}{|\beta|}\sum_{i=1}^{n}\left(\left|f_i'(z)\left(\frac{z}{f_i(z)}\right)^{\mu_i} - 1\right|\left|\left(\frac{f_i(z)}{z}\right)^{\mu_i}\right| + \left|\left(\frac{f_i(z)}{z}\right)^{\mu_i}\right| + \left|\frac{g_i''(z)}{g_i'(z)}\right|\right) \quad (2.36)$$

Now, we take into account the hypotheses of Theorem 2.4, Lemma 1.7 (applied to the functions f_i, for $i = \overline{1,n}$) and relation (2.36) becomes

$$\begin{aligned}\left|c|z|^{2\beta} + (1-|z|^{2\beta})\frac{zh_n''(z)}{\beta h_n'(z)}\right| &\leq |c| + \frac{1}{|\beta|}\sum_{i=1}^{n}((2-\alpha_i)M_i^{\mu_i} + N_i) \\ &\leq |c| + \frac{1}{\mathrm{Re}(\beta)}\sum_{i=1}^{n}((2-\alpha_i)M_i^{\mu_i} + N_i) \\ &\leq 1.\end{aligned} \quad (2.37)$$

Now, we apply Theorem 1.6, which shows that $F_{n,\beta} \in \mathcal{S}$.

For the second part, namely statement (ii), one considers the function $h : \mathcal{U} \to \mathbb{C}$ given by relation (2.15) and uses similar arguments to those in the proof of statement (i) in order to show that $G_{\beta,\gamma}(f_0, g_0) \in \mathcal{S}$. The proof is omitted for the sake of brevity.

This concludes the proof of our result. □

3. Conclusion

In this work, we have considered two new general integral operators. We have employed established univalence criteria to determine conditions under which our new integral operators are univalent. These univalence conditions were obtained for different choices of the functions that are involved in the definitions of the considered integral operators.

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