Stud. Univ. Babeş-Bolyai Math. 70(2025), No. 4, 573–584
DOI: 10.24193/subbmath.2025.4.03

Local fractal functions on Orlicz-Sobolev spaces

Waldo Arriagada

Abstract. In these notes we consider a class of iterated function systems whose attractors are the graphs of local fractal functions belonging to Orlicz and to Orlicz-Sobolev spaces. We prove that these maps are in correspondence with the fixed points of the Read-Bajraktarević operator. Our method extends a number of outcomes on the existence of local fractal functions of the Lebesgue and Sobolev classes, to more general function spaces where the role of the norm is now played by a Young function. The existence of local fractal functions of the Orlicz and of the Orlicz-Sobolev classes is demonstrated through an intermediate result. The realization of a contractive iterated function system in the (previously untreated) multidimensional case is obtained via a stronger version of the finite increments theorem. Our results somewhat show that it would be natural to extend the Read-Bajraktarević functional to other function spaces on subdomains of differentiable and real analytic manifolds. Other questions, such as the existence of fixed points in higher-orders etc., remain open as well. Our generalizations may be useful in applications.

Mathematics Subject Classification (2010): 28A80, 35J60, 37C70, 37G35, 46E30.

Keywords: Fractal; attractor; IFS; Orlicz-Sobolev space; Read-Bajraktarević operator; contractive map.

1. Introduction

An iterated function system (IFS) consists of a finite set S of continuous functions defined on a complete metric space, with the images in the same space. Hutchinson and Mandelbrot introduced IFSs on the real plane in the literature in the early 1980s and their applications were widely popularized by Barnsley in the 1990s. A result due to Hutchinson [11] asserts that there exists a unique nonempty compact subset of the plane which is equal to the union of its images under the corresponding elements in S.

Received 26 February 2025; Accepted 5 April 2025.

Such subset is called an attractor of the IFS. Conversely, Barnsley demonstrated that it is possible to construct an IFS with prescribed attractor. The method is known as the *random iteration algorithm,* dubbed the chaos game by Barnsley himself [5]. During the last decades, the theory has been extended to other spaces and contracting maps [20].

Iterated function systems are tightly connected with the Read-Bajraktarević equation, introduced by M. Bajraktarević in the 1950s [4]. In its elementary form the Read-Bajraktarević equation corresponds to $u(x) = \nu(x, u(b(x)))$, where $b : I \to I$ and $\nu : I \times \mathbb{R} \to \mathbb{R}$, together with the unknown function $u : I \to \mathbb{R}$, are defined on a closed real interval I. Similarly, the Read-Bajraktarević operator

$$\mathbf{T}u(x) = \nu(x, u(b(x))) \tag{1.1}$$

appeared later (1980s) in the works of C.J. Read in the context of the invariant subspace problem [19]. Initially defined on the space $C^\infty(I)$ of infinitely differentiable functions on I, the operator (1.1) is closely related to Bajraktarević equation, as follows. It is known that if b, ν are continuous, b is surjective, and if there exists $c \in (0, 1)$ such that $|\nu(x, y_1) - \nu(x, y_2)| \le c|y_1 - y_2|$ for $x \in I$ and $y_1, y_2 \in \mathbb{R}$, then (1.1) is contractive [16]. It follows from the Banach theorem that $\mathbf{T}$ admits a unique fixed point $u^* \in C^\infty(I)$, which is actually the limit of the iterates $u_k(x) = \nu(x, u_{k-1}(b(x)))$, $k = 1, 2, \ldots$, where the initial condition u_0 is any element in $C^\infty(I)$. The fixed point u^* is called a *smooth local fractal function*, or a local fractal function of the smooth class. Different choices of the source space of (1.1) give rise to a number of corresponding categories of fixed points. Local fractal functions in the bounded, Hölder, smooth, Lebesgue and Sobolev spaces of functions (on a subset X of the real line) are well characterized and their properties are extensively examined in [17, 16]. For example, it is known that the graph of a local fractal function in any of these categories is the local attractor (a self-similar fractal) of an associated local IFS, and that the set of the discontinuities of a bounded local fractal function in 1 dimension is at most countably infinite [6].

These notes are strongly based on the pioneering paper by Massopust [17]. However, a remark made in the introduction of the reference [6] leads us to believe that the multidimensional case has not been previously treated. The purpose of this note is then twofold. On the one hand, we examine the multidimensional case, in which the underlying set X is assumed to be a nonempty connected and bounded subset of $\mathbb{R}^N$, $N \ge 1$. We employ a stronger version of the finite increments theorem in $\mathbb{R}^N$ to solve the *realization problem.* More specifically, we construct explicit contractive local IFSs whose attractors are, respectively, the graphs of local fractal functions in the Orlicz and in the (order one) Orlicz-Sobolev spaces of functions on X. On the other hand, we prove that local fractal functions in these categories are exactly the fixed points of the Read-Bajraktarević functional. Our results thus generalize previous theorems on the existence of local fractal functions in the Lebesgue and in the Sobolev classes, to a broader category of spaces, where the role of the norm is played now by a Young function. The problem whether our conclusions actually extend to higher orders is still open. The generalized Faà di Bruno formula [9] seems to be instrumental in this direction.

2. Iterated function systems

Let (X, d) denote a complete metric space and $\{w_i : X \to X\}_{i=1}^n$ a set of n continuous maps. Then the tuple $\mathcal{F} = (X, w_1, \ldots, w_n)$ is called an n-map iterated function system, or IFS.[1]. We say that the functions w_i belong to the IFS $\mathcal{F}$, and we write $w_i \in \mathcal{F}$, $1 \le i \le n$. These structures were introduced in the works of Hutchinson (1981), Mandelbrot (1982) and Barnsley (1993), as follows. Let $\mathcal{H}(X)$ denote the set of nonempty compact subsets of X. Associated with the IFS $\mathcal{F}$ is the set-valued map $\mathbf{w} : \mathcal{H}(X) \to \mathcal{H}(X)$ defined by

$$\mathbf{w}(S) = \bigcup_{i=1}^n w_i(S) \quad S \subset \mathcal{H}(X).$$

The IFS $\mathcal{F}$ is called contractive if there exists a metric $\tilde{d}$, which is equivalent to d, such that each $w_i \in \mathcal{F}$ is a contraction with respect to $\tilde{d}$. That is, for each $1 \le i \le n$ there exists $c_i \in [0, 1)$ such that $\tilde{d}(w_i(x), w_i(y)) \le c_i \, \tilde{d}(x, y)$ for all $x, y \in X$. Hutchinson demonstrated [11] that in this case $\mathbf{w}$ is itself a contraction on $\mathcal{H}(X)$:

$$d_{\mathcal{H}}(\mathbf{w}(A), \mathbf{w}(B)) \le c \, d_{\mathcal{H}}(A, B),$$

where

$$A, B \in \mathcal{H}(X), c = \max_{1 \le i \le n} \{c_i\}$$

and

$$d_{\mathcal{H}}(A, B) = \max \left\{ \max_{a \in A} \min_{b \in B} d(a, b), \max_{b \in B} \min_{a \in A} d(a, b) \right\}$$

is the Hausdorff distance between compact sets. In this case, the unique fixed point (Banach theorem) $A \in \mathcal{H}(X)$ is called the attractor of the IFS. By definition the attractor satisfies $A = \mathbf{w}(A)$ and is self-similar, since it may be expressed as a union of n contracted copies of itself. It is also known that non-contractive IFSs (i.e., such that the maps w_i are not contractions with respect to any topologically equivalent metric in X) can yield attractors [14]. For more details and examples we refer the reader to [5, 13].

The following notion is due to Barnsley and Hurd [7]. Let $\{X_1, \ldots, X_n\} \subset (X, d)$ be a family of nonempty subsets, equipped with a family of continuous maps $w_i : X_i \to X$, $1 \le i \le n$. Then $\mathcal{F}_{\text{loc}} = \{(X_1, w_1), \ldots, (X_n, w_n)\}$ is called a local iterated function system. A local IFS is called contractive if there exists a metric, equivalent to d, for which every $w \in \mathcal{F}_{\text{loc}}$ is contractive. Associated with any local IFS $\mathcal{F}_{\text{loc}} = \{(X_1, w_1), \ldots, (X_n, w_n)\}$ is the operator $\mathbf{w}_{\text{loc}} : 2^X \to 2^X$,

$$\mathbf{w}_{\text{loc}}(S) = \bigcup_{i=1}^n w_i(S \cap X_i), \tag{2.1}$$

where 2^X is the power set of X.

[1]The letter N is commonly used in the literature to denote the number of maps in the definition of the IFS. We will use n instead, and we will employ N to rather denote the dimension of the domain that appears later.

Definition 2.1. *A subset $A \in 2^X$ is called a local attractor of $\mathcal{F}_{\rm loc}$ if $A = \mathbf{w}_{\rm loc}(A)$.*

The local attractor of a contractive local IFS is unique. Moreover, suppose that $\mathcal{F} = (X, w_1, \dots, w_n)$ and $\mathcal{F}_{\rm loc} = \{(X_1, w_1), \dots, (X_n, w_n)\}$ are both contractive, where $X_1, \dots, X_n \subset X$ are nonempty. It is well known [17, Proposition 1] that if X is compact and X_i is closed for $1 \le i \le n$, then the attractor A of $\mathcal{F}_{\rm loc}$ is a subset of the attractor of $\mathcal{F}$.

3. Orlicz and Orlicz-Sobolev spaces

This section is brief summary on Orlicz and Orlicz-Sobolev spaces. For further details we refer the reader to [8, 18]. A function $\Phi : [0, \infty) \to [0, \infty)$ which is increasing, continuous, unbounded and such that $\Phi(0) = 0$ is called a φ-function [15, p.11]. If any such a Φ is, moreover, convex then it has the integral representation

$$\Phi(t) = \int_0^t \phi(s)\, ds, \tag{3.1}$$

where $\phi : [0, \infty) \to [0, \infty)$ is a nondecreasing, right-continuous function satisfying $\phi(t) = 0$ if and only if $t = 0$ and $\lim_{t\to\infty} \phi(t) = \infty$. The function ϕ is called the right derivative of Φ. A convex φ-function Φ satisfying

$$\frac{\Phi(t)}{t} \to 0 \text{ as } t \to 0 \quad \text{and} \quad \frac{\Phi(t)}{t} \to \infty \text{ as } t \to \infty,$$

is denominated a Young function [15, p.47]. Young functions are sometimes called N-functions; however, to avoid confusion we will not employ that denomination in this article[2]. For example, if $\phi(t) = pt^{p-1}$, $t \ge 0$ and $1 \le p < \infty$, then $\Phi(t) = t^p$ and

$$\|u\|_p = \Phi^{-1}\left(\int_X \Phi(|u(x)|) dx\right)$$

for $u \in L^p(X)$ and where $\Phi^{-1}(t) = t^{1/p}$ is the inverse function.

Given a Young function Φ with the integral representation (3.1), the right-inverse function of ϕ is defined for $s \ge 0$ by $\psi(s) = \sup\{t : \phi(t) \le s\}$. If ϕ is continuous and increasing then ψ is the ordinary inverse ϕ^{-1}. The function ψ has the same properties as ϕ : it is positive for $s > 0$, right-continuous for $s \ge 0$ and satisfies $\psi(0) = 0$ and $\psi(s) \to \infty$ as $s \to \infty$. Hence the integral

$$\int_0^t \psi(s)\, ds$$

is a Young function as well, called the conjugate (or complementary) of Φ. The complementary Young function is usually denoted by $\overline{\Phi}(t)$.

Let $X \subset \mathbb{R}^N$ (with $N \ge 1$) be a bounded subset and let Φ be a Young function. The Orlicz class $\mathcal{L}_\Phi(X)$ is the set of (equivalence classes of) real-valued measurable functions u such that $\Phi(u) \in L^1(X)$. In general, $\mathcal{L}_\Phi(X)$ is not a vector space [12].

[2]This definition is ambiguous. Some texts denominate $\Phi : [0, \infty) \to [0, \infty]$ a Young function if Φ is not identically zero, convex and $\lim_{t\to 0^+} \Phi(t) = \Phi(0) = 0$.

However, the linear hull $L_\Phi(X)$ of the Orlicz class $\mathcal{L}_\Phi(X)$ is a Banach space when equipped with the Luxemburg norm

$$\|u\|_\Phi = \inf\left\{\tau > 0 : \int_X \Phi\left(\frac{|u|}{\tau}\right) dx \le 1\right\} \quad u \in L_\Phi(X).$$

We denote by $E_\Phi(X)$ the closure (for the norm-topology) of $L^\infty(X)$ in $L_\Phi(X)$. The space $E_\Phi(X)$ is separable and Banach for the inherited norm. In general, $E_\Phi(X) \subset \mathcal{L}_\Phi(X) \subset L_\Phi(X)$ and $E_\Phi(X) = L_\Phi(X)$ if and only if Φ satisfies a Δ_2-condition (at infinity). This means that for $r > 1$ there exists $\gamma(r) > 0$ such that

$$\Phi(rt) \le \gamma(r)\,\Phi(t) \quad t \ge T,$$

where T is also positive. (The Δ_2-condition is global if $T \ge 0$). It is known that if Φ and $\overline{\Phi}$ satisfy a Δ_2-condition at infinity then $L_\Phi(X)$ and $L_{\overline{\Phi}}(X)$ are reflexive and separable. It follows that one can identify the dual space of $E_\Phi(X)$ with $L_{\overline{\Phi}}(X)$ and the dual space of $E_{\overline{\Phi}}(X)$ with $L_\Phi(X)$, cf. [1, 8].

The Orlicz-Sobolev space $W^1L_\Phi(X)$. The Orlicz-Sobolev space $W^1L_\Phi(X)$ is the vector subspace of functions in $L_\Phi(X)$ with first distributional derivatives in $L_\Phi(X)$. Likewise, the Orlicz-Sobolev space $W^1E_\Phi(X)$ is the vector subspace of functions in $E_\Phi(X)$ with first distributional derivatives in $E_\Phi(X)$. The spaces $W^1L_\Phi(X)$ and $W^1E_\Phi(X)$ are Banach when endowed with the norm

$$\|u\|_{1,\Phi} = \|u\|_\Phi + \|\nabla u\|_\Phi = \|u\|_\Phi + \sum_{i=1}^{N} \|\partial_{x_i} u\|_\Phi, \tag{3.2}$$

where ∂_{x_i} is the partial derivative $\partial/\partial x_i$. Usually, $W^1L_\Phi(X)$ and $W^1E_\Phi(X)$ are identified with subspaces of the products $L_\Phi(X)^{N+1} = \Pi L_\Phi(X)$ and $E_\Phi(X)^{N+1} = \Pi E_\Phi(X)$, respectively. The natural embedding of $W^1E_\Phi(X)$ into $E_\Phi(X)^{N+1}$ proves that $W^1E_\Phi(X)$ is separable since $E_\Phi(X)$ is itself separable. The space $W^1L_\Phi(X)$ is not separable in general, and $W^1L_\Phi(X) = W^1E_\Phi(X)$ if Φ satisfies a Δ_2-condition.

4. Local fractal functions of Orlicz and Orlicz-Sobolev classes

Let $X \subset \mathbb{R}^N$ $(N \ge 1)$ be a nonempty connected and bounded subset and let $\Phi : [0,\infty) \to [0,\infty)$ be a Young function with the integral form (3.1). We assume that there exist p_Φ, $q_\Phi > 0$ such that

$$p_\Phi \le \frac{t\,\phi(t)}{\Phi(t)} \le q_\Phi < \infty \quad t \ne 0. \tag{4.1}$$

These numbers appear in the characterization of the properties of Young functions in connection with Orlicz and Orlicz-Sobolev spaces [2, 3, 10]. For example, these inequalities ensure that Φ satisfies a global Δ_2-condition [1]. The complementary $\overline{\Phi}$ satisfies a global Δ_2-condition if and only if $p_\Phi > 1$, etc. A quick integration yields

$$\min\{\rho^{p_\Phi}, \rho^{q_\Phi}\}\,\Phi(t) \le \Phi(\rho t) \le \max\{\rho^{p_\Phi}, \rho^{q_\Phi}\}\,\Phi(t) \quad \rho, t \ge 0. \tag{4.2}$$

Let $\{X_i\}_{i=1}^n$ be a family of nonempty and connected subsets of X. In what follows $\{\alpha_i : X_i \to X\}_{i=1}^n$ will denote a collection of bounded diffeomorphisms such that

$$\alpha_i(X_i) \cap \alpha_{i'}(X_{i'}) = \emptyset \;\text{ for } i \neq i', \text{ and } \quad X = \dot{\bigcup}_{i=1}^n \alpha_i(X_i). \tag{4.3}$$

For $1 \le j \le N$ let ∂_{x_j} be the x_j-th partial derivative $\partial/\partial x_j$, and let $\partial_x \alpha_i$ denote the Jacobian matrix of α_i at $x \in X_i$.

Finally, let $\{\lambda_i : X_i \to \mathbb{R}\}_{i=1}^n \subset L_\Phi(X)$, and choose $S_1, \dots, S_n$ with $S_i \in (L^\infty(X_i), \|\cdot\|_{i,\infty})$. Define the Read-Bajraktarević functional $\mathbf{T} : L_\Phi(X) \to \mathbb{R}^X$ by

$$\mathbf{T}u(x) = \sum_{i=1}^n \Big((\lambda_i \circ \alpha_i^{-1})(x) + (S_i \circ \alpha_i^{-1})(x)(u|_{X_i} \circ \alpha_i^{-1})(x) \Big) \mathbb{1}_{\alpha_i(X_i)}(x), \tag{4.4}$$

where $\mathbb{1}_{\alpha_i(X_i)}$ is the characteristic function of $\alpha_i(X_i)$ (i.e., $\mathbb{1}_{\alpha_i(X_i)}(x) = 1$ if $x \in \alpha_i(X_i)$ and it is zero otherwise) and $u|_{X_i}$ is the restriction of $u \in L_\Phi(X)$ to X_i. Note that if u^* is a fixed point of $\mathbf{T}$ then

$$u^* \circ \alpha_i = \lambda_i + S_i\, u^*|_{X_i}, \tag{4.5}$$

for all indices $1 \le i \le n$.

4.1. Existence of fixed points

In this paragraph we tackle the problem on the existence of fixed points of (4.4), and of its restriction to the Orlicz-Sobolev space $W^1 L_\Phi(X)$. We set for every $1 \le i \le n$

$$a_i = \sup_{x \in X_i} |\det \partial_x \alpha_i| \quad \text{and} \quad r_i = \max\{1, \|S_i\|_{i,\infty}\}. \tag{4.6}$$

Lemma 4.1. *The Read-Bajraktarević operator is well defined on $L_\Phi(X)$ and sends this space into itself. Moreover,*

$$\|\mathbf{T}u - \mathbf{T}v\|_\Phi \le \Big(\sum_{i=1}^n a_i r_i^{q_\Phi} \Big) \|u - v\|_\Phi$$

for all $u, v \in L_\Phi(X)$.

Proof. Since α_i is a diffeomorphism and $\lambda_i \in L_\Phi(X)$, $1 \le i \le n$, the functional (4.4) is well defined and indeed $\mathbf{T}(L_\Phi(X)) \subset L_\Phi(X)$. Let τ be positive and $u, v \in L_\Phi(X)$. A change of coordinates $y = \alpha_i^{-1}(x)$ and subsequent re-labeling $x \mapsto y$ produce

$$\begin{aligned} \int_X \Phi\Big(\frac{1}{\tau}|\mathbf{T}u(x) - \mathbf{T}v(x)|\Big) dx &\le \sum_{i=1}^n \int_{\alpha_i(X_i)} \Phi\Big(\frac{1}{\tau}|S_i \circ \alpha_i^{-1}(x)| \big|u|_{X_i} \circ \alpha_i^{-1}(x) - v|_{X_i} \circ \alpha_i^{-1}(x)\big|\Big) dx \\ &= \sum_{i=1}^n \int_{X_i} \Phi\Big(\frac{1}{\tau}|S_i(x)| \big|u|_{X_i}(x) - v|_{X_i}(x)\big|\Big) |\det \partial_x \alpha_i| dx. \end{aligned}$$

Therefore, (4.2) implies

$$\int_X \Phi\Big(\frac{1}{\tau}|\mathbf{T}u(x) - \mathbf{T}v(x)|\Big) dx \le \Big(\sum_{i=1}^n a_i r_i^{q_\Phi} \Big) \int_X \Phi\Big(\frac{1}{\tau}|u(x) - v(x)|\Big) dx.$$

The conclusion follows by the definition of the Luxemburg norm. □

In particular, the functional (4.4) is a contraction on $L_\Phi(X)$ provided

$$\sum_{i=1}^{n} a_i r_i^{q_\Phi} < 1. \tag{4.7}$$

In this case, the fixed point of $\mathbf{T} : L_\Phi(X) \to L_\Phi(X)$ is called a local fractal function of the Orlicz class. A local fractal function evidently depends on the choice of the functions λ_i, S_i and on the diffeomorphisms $\alpha_1, \ldots, \alpha_n$. Define

$$b_i = \max\left\{1, \sup_{x\in X_i} \max_{1\le k,j\le N} \left|\left[(\partial_{\alpha_i^{-1}(x)}\alpha_i)^{-1}\right]_{kj}\right|\right\} \quad 1 \le i \le n, \tag{4.8}$$

where $\left[(\partial_{\alpha_i^{-1}(x)}\alpha_i)^{-1}\right]_{kj}$ is the (k,j)-entry of the inverse of the Jacobian matrix of α_i at $y = \alpha_i^{-1}(x)$, and $y = (y_1, \ldots, y_N)$ parametrizes the source space of α_i. The next result asserts that if (4.7) is slightly modified then the contraction of (4.4) happens on $W^1L_\Phi(X)$.

Theorem 4.2. *Assume that $S_i(x) = s_i \in \mathbb{R}$, $1 \le i \le n$, $x \in X_i$ and $\{\lambda_1, \ldots, \lambda_n\} \subset W^1L_\Phi(X)$. Then the restriction of the Read-Bajraktarević operator to $W^1L_\Phi(X)$ is well defined and sends this space into itself. Moreover, if the condition*

$$\sum_{i=1}^{n} a_i (r_i b_i)^{q_\Phi} < \frac{1}{N^{\frac{q_\Phi}{2}+1}} \tag{4.9}$$

is fulfilled then (4.4) is a contraction on $W^1L_\Phi(X)$.

Proof. Since $\lambda_i \in W^1L_\Phi(X)$, $1 \le i \le n$, the functional (4.4) is well defined on $W^1L_\Phi(X)$ and clearly sends this space into itself. Note that if $u \in W^1L_\Phi(X)$ then the composite $u \circ \alpha_i^{-1}$ is x_j-differentiable and $\partial_{x_j}(u \circ \alpha_i^{-1}) = \nabla(u \circ \alpha_i^{-1}) \cdot \partial_{x_j}\alpha_i^{-1}$, for $1 \le j \le N$. The chain rule yields for $u, v \in W^1L_\Phi(X)$ and $\tau > 0$,

$$\begin{aligned}\int_X \Phi\left(\frac{1}{\tau}\left|\partial_{x_j}\mathbf{T}u(x) - \partial_{x_j}\mathbf{T}v(x)\right|\right)dx &\le \sum_{i=1}^{n}\int_{\alpha_i(X_i)} \Phi\left(\frac{|s_i|}{\tau}\left|\partial_{x_j}\{u|_{X_i}\circ\alpha_i^{-1}(x) - v|_{X_i}\circ\alpha_i^{-1}(x)\}\right|\right)dx \\ &= \sum_{i=1}^{n}\int_{\alpha_i(X_i)} \Phi\left(\frac{|s_i|}{\tau}\left|\nabla(u|_{X_i} - v|_{X_i})(\alpha_i^{-1}(x)) \cdot \partial_{x_j}\alpha_i^{-1}(x)\right|\right)dx.\end{aligned}$$

Cauchy-Schwarz inequality implies that the integrand on the right is bounded by

$$\Phi\left(\frac{r_i}{\tau}\left|\nabla(u|_{X_i} - v|_{X_i})(\alpha_i^{-1}(x))\right| \sqrt{N} \max_{1\le k,j\le N} \left|\left[(\partial_{\alpha_i^{-1}(x)}\alpha_i)^{-1}\right]_{kj}\right|\right)dx. \tag{4.10}$$

The maximum above is bounded by b_i. The change of coordinates $y = \alpha_i^{-1}(x)$ (and re-labeling $x \mapsto y$) along with (4.2) applied with $\rho = \sqrt{N} r_i b_i$ altogether yield

$$\int_X \Phi\left(\frac{1}{\tau}\left|\partial_{x_j}\mathbf{T}u(x) - \partial_{x_j}\mathbf{T}v(x)\right|\right)dx \le N^{\frac{q_\Phi}{2}}\left(\sum_{i=1}^{n} a_i (r_i b_i)^{q_\Phi}\right)\int_X \Phi\left(\frac{1}{\tau}\left|\nabla u(x) - \nabla v(x)\right|\right)dx,$$

for $1 \le j \le N$. The definition of the Luxemburg norm and (3.2) thus produce

$$\|\nabla\mathbf{T}u - \nabla\mathbf{T}v\|_\Phi \le N^{\frac{q_\Phi}{2}+1}\left(\sum_{i=1}^{n} a_i (r_i b_i)^{q_\Phi}\right)\|\nabla u - \nabla v\|_\Phi.$$

Since $b_i^{q_\Phi} \geq 1$ for $1 \leq i \leq n$, Lemma 4.1 implies

$$\|\mathbf{T}u - \mathbf{T}v\|_{1,\Phi} = \|\mathbf{T}u - \mathbf{T}v\|_\Phi + \|\nabla\mathbf{T}u - \nabla\mathbf{T}v\|_\Phi \leq N^{\frac{q_\Phi}{2}+1}\Big(\sum_{i=1}^n a_i(r_ib_i)^{q_\Phi}\Big)\|u-v\|_{1,\Phi}.$$

The theorem is proved. □

If (4.9) is satisfied then the fixed point u^* of $\mathbf{T} : W^1L_\Phi(X) \to W^1L_\Phi(X)$ is called a local fractal function of the Orlicz-Sobolev class $W^1L_\Phi(X)$.

Observation. A coarser (more restrictive) condition substitutes (4.9) if the convexity of Φ is utilized. Indeed, we note that bound (4.10) can be replaced by

$$\Phi\bigg(\frac{r_i}{\tau}\max_{1\leq k,j\leq N}\big|\big[(\partial_{\alpha_i^{-1}(x)}\alpha_i)^{-1}\big]_{kj}\big|\sum_{k=1}^N|\partial_{y_k}(u|_{X_i}-v|_{X_i})(\alpha_i^{-1}(x))|\bigg)dx.$$

Jensen inequality ensures that this term is in turn less than or equal to

$$\frac{1}{N}\sum_{k=1}^N\Phi\bigg(\frac{1}{\tau}Nr_ib_i\big|\partial_{y_k}(u|_{X_i}-v|_{X_i})(\alpha_i^{-1}(x))\big|\bigg)dx.$$

Therefore, (4.2) applied with $\rho = Nr_ib_i$ yields

$$\int_X\Phi\bigg(\frac{1}{\tau}|\partial_{x_j}\mathbf{T}u(x)-\partial_{x_j}\mathbf{T}v(x)|\bigg)dx \leq N^{q_\Phi}\Big(\sum_{i=1}^n a_i(r_ib_i)^{q_\Phi}\Big)\int_X\Phi\bigg(\frac{1}{\tau}\big|\nabla u(x)-\nabla v(x)\big|\bigg)dx,$$

which entails $\|\nabla\mathbf{T}u - \nabla\mathbf{T}v\|_\Phi \leq N^{q_\Phi+1}\Big(\sum_{i=1}^n a_i(r_ib_i)^{q_\Phi}\Big)\|\nabla u - \nabla v\|_\Phi$.

4.2. Realization of a local contractive IFS in N dimensions

Suppose that $\mathbf{T} : L_\Phi(X) \to \mathbb{R}^X$ has a fixed point $u^* \in W^1L_\Phi(X)$. We would like to construct a contractive local IFS whose attractor is exactly the graph of u^*, namely $G(u^*) = \{(x, u^*(x)) : x \in X\}$. In dimension $N = 1$ the solution to this problem is due to Massopust *et al.* and can be found in [6]. We adapt the procedure to the case of N dimensions via a general version of the finite increments theorem.

Recall that if $x', x'' \in \mathbb{R}^N$ then $[x', x''] = \{(1-t)x' + tx'' : 0 \leq t \leq 1\}$ is the closed segment from x' to x'', and the set $(x', x'') = \{(1-t)x' + tx'' : 0 < t < 1\}$ is the corresponding open segment.

Lemma 4.3. *Let* $\operatorname{int}(\Omega)$ *denote the interior of a nonempty set* $\Omega \subset \mathbb{R}^N$ *and choose* $x', x'' \in \Omega$ *such that* $[x', x''] \subset \operatorname{int}(\Omega)$. *Let* $\alpha : \Omega \to \mathbb{R}^N$ *be continuous on* $[x', x'']$ *and differentiable on* (x', x''). *Then*

$$\|\alpha(x'') - \alpha(x')\|_1 \leq N\Big(\max_{1\leq k,j\leq N}\sup_{x\in(x',x'')}|[\partial_x\alpha]_{kj}|\Big)\|x''-x'\|_1,$$

where $[\partial_x\alpha]_{kj}$ *is the* (k, j)*-entry of the Jacobian matrix of* α *at* $x \in \Omega$ *and*

$$\|x\|_1 = \sum_{k=1}^N |x_k|.$$

This result follows from the finite increments theorem for real-valued functions.[3] With the maps and notation from the previous section, we assume that X is a closed subset of a complete metric space and that X_i is convex, $1 \le i \le n$. We require that

$$M = \max_{1\le i\le n} \|S_i\|_{i,\infty} < 1.$$

In this case the functions $\nu_i : X_i \times \mathbb{R} \to \mathbb{R}$,

$$\nu_i(x,y) = \lambda_i(x) + S_i(x)y \quad 1 \le i \le n, \tag{4.11}$$

are uniformly contractive in the second variable: $|\nu_i(x,y_1) - \nu_i(x,y_2)| \le M|y_1 - y_2|$, $x \in X_i$ and $y_1, y_2 \in \mathbb{R}$. Furthermore, we assume that for $1 \le i \le n$ the function λ_i is so chosen that ν_i is uniformly Lipschitz continuous in the first variable. That is, there exists $\tilde{M} > 0$ such that

$$|\nu_i(x_1,y) - \nu_i(x_2,y)| \le \tilde{M}\|x_1 - x_2\|_1, \quad x_1, x_2 \in X_i \text{ and } y \in \mathbb{R}.$$

(This happens e.g., when the λ_i are uniformly Lipschitz continuous and the S_i are constants). Finally, for $1 \le i \le n$ define $w_i : X_i \times \mathbb{R} \to X \times \mathbb{R}$ by

$$w_i(x,y) = (\alpha_i(x), \nu_i(x,y))$$

and let $d_\theta : (X \times \mathbb{R}) \times (X \times \mathbb{R}) \to \mathbb{R}$ be the map

$$d_\theta((x_1,y_1),(x_2,y_2)) = \|x_1 - x_2\|_1 + \theta|y_1 - y_2|,$$

where $\theta = (1 - Nm)/2\tilde{M}$, and

$$m = \max_{\substack{1\le i\le n \\ 1\le k,j\le N}} \sup_{x\in X_i} |[\partial_x \alpha_i]_{kj}|.$$

(Evidently, d_θ is a metric on the product space, provided $m < 1/N$. In dimension $N = 1$ the latter occurs when each α_i is contractive).

Theorem 4.4. *If $m < 1/N$ then $\mathcal{F}_{\rm loc} = \{(X_i \times \mathbb{R}, w_i)\}_{i=1}^n$ is a contractive local* IFS *for the metric d_θ, and $G(u^*)$ is a local attractor of $\mathcal{F}_{\rm loc}$, i.e. $\mathbf{w}_{\rm loc}(G(u^*)) = G(u^*)$, where $\mathbf{w}_{\rm loc}$ is the map* (2.1) *associated to $\mathcal{F}_{\rm loc}$.*

Proof. Let $i \in \{1, \cdots, n\}$ and choose two points (x_1,y_1) and (x_2,y_2) in the product $X_i \times \mathbb{R}$. Since X_i is convex, Lemma 4.3 implies

$$\begin{aligned} d_\theta(w_i(x_1,y_1),w_i(x_2,y_2)) &\le Nm\|x_1 - x_2\|_1 + \theta|\nu_i(x_1,y_1) - \nu_i(x_2,y_1)| + \theta|\nu_i(x_2,y_1) - \nu_i(x_2,y_2)| \\ &\le (Nm + \theta\tilde{M})\|x_1 - x_2\|_1 + \theta M|y_1 - y_2| \\ &\le \epsilon\, d_\theta((x_1,y_1),(x_2,y_2)), \end{aligned}$$

where $\epsilon = \max\{Nm + \theta\tilde{M}, M\} < 1$, and the first assertion follows.

[3]If $\alpha(x) = (\alpha^1(x), \ldots, \alpha^N(x))$, $\alpha^j : \mathbb{R}^N \to \mathbb{R}$, then for $1 \le j \le N$ there exists $c_j \in (x', x'')$ such that $\alpha^j(x'') - \alpha^j(x') = \sum_{k=1}^N \partial_{x_k}\alpha^j(c_j)(x''_k - x'_k)$. Hence,

$$\max_{1\le j\le N} |\alpha^j(x'') - \alpha^j(x')| \le \max_{1\le k,j\le N} \sup_{x\in(x',x'')} |[\partial_x \alpha]_{kj}|\, \|x'' - x'\|_1.$$

On the other hand, from (4.5) and (4.11) the fixed point of (4.4) satisfies the equation $u^*(\alpha_i(x)) = \nu_i(x, u^*(x))$, for all $x \in X_i$. Therefore,

$$\mathbf{w}_{\mathrm{loc}}(G(u^*)) = \bigcup_i \{(\alpha_i(x), \nu_i(x, u^*(x))) : x \in X_i\} = \bigcup_i \{(\alpha_i(x), u^*(\alpha_i(x))) : x \in X_i\},$$

which is equal to $G(u^*)$. The two equalities above follow from (4.3). □

5. Examples

The following cases were treated in [6, 17]. In one dimension these examples are easily retrieved from Lemma 4.1 and Theorem 4.2, respectively, as we demonstrate below.

5.1. Lebesgue space $L^p[0,1]$ with $p \in (0,\infty]$.

Let $\{X_i\}_{i=1}^n$ be a family of connected semi-open intervals of $[0,1]$ and let $\{x_0 = 0 < \cdots < x_n = 1\}$ be a partition of the closed unit interval. The map $\alpha_i : X_i \to [0,1]$ is chosen to be linear, from X_i onto $[x_{i-1}, x_i)$, $1 \le i \le n$, and where $\alpha_i^{-1}([x_{n-1}, x_n) \cup \{x_n\})$ is denoted by X_n^+. Then (4.3) is satisfied in this case. If we choose $\lambda_i \in L^p(X_i)$ and $S_i \in L^\infty(X_i)$, $1 \le i \le n$, then the image of $\mathbf{T} : L^p[0,1] \to \mathbb{R}^{[0,1]}$ belongs to $L^p[0,1]$. Let d_i denote the ordinary derivative (slope) of α_i. If

$$\begin{cases} \displaystyle\sum_{i=1}^n d_i \|S_i\|_{\infty,X_i}^p < 1, & p > 0; \\ \displaystyle\max_{1\le i\le n} \|S_i\|_{\infty,X_i} < 1, & p = \infty, \end{cases} \tag{5.1}$$

are fulfilled, then (4.4) is contractive on $L^p[0,1]$ and there exists a unique local fractal function $u^* \in L^p[0,1]$ satisfying (4.5). Evidently, this function depends on the choice of λ_i, S_i and α_i, $1 \le i \le n$. Note that for values $1 < p < \infty$, the first condition in (5.1) is a particular case of (4.7). The existence of a fractal function itself is consequence of Lemma 4.1 applied with the Young function $\Phi(t) = t^p$ in (3.1) and therefore $p_\Phi = q_\Phi = p$ in (4.1) while $a_i = d_i$ and $b_i = 1/d_i$ in (4.6) and (4.8), respectively.

5.2. The Sobolev space $W^{m,p}(0,1)$ with $1 < p \le \infty$.

This case is also well documented in the literature. Let $X_1, \ldots, X_n$ be nonempty connected open subintervals of $X = (0,1)$ and let $\{x_1 < \cdots < x_{n-1}\}$ be a partition of X. The map $\alpha_i : X_i \to [0,1]$ is chosen linear and increasing such that $\alpha_i(X_i) = (x_{i-1}, x_i)$, where $x_0 = 0$ and $x_n = 1$. Let m be a positive integer. We choose $S_i \equiv s_i \in \mathbb{R}$ and $\lambda_i \in W^{m,p}(X_i)$, $1 \le i \le n$. The operator $\mathbf{T} : W^{m,p}(0,1) \to \mathbb{R}^{(0,1)}$ is then well defined and sends $W^{m,p}(0,1)$ into itself. Let $d_i > 0$ denote the slope of α_i. If

$$\begin{cases} \displaystyle\max_{q=0,1,\ldots,m} \sum_{i=1}^n |s_i|^p / d_i^{qp-1} < 1, & 1 \le p < \infty; \\ \displaystyle\max_{q=0,1,\ldots,m} \sum_{i=1}^n |s_i| / d_i^q < 1, & p = \infty, \end{cases} \tag{5.2}$$

then $\mathbf{T}$ is contractive on $W^{m,p}(0,1)$. The unique fixed point $u^* \in W^{m,p}(0,1)$ is called a local fractal function of class $W^{m,p}(0,1)$. In the case $N = m = 1$ and for values

$1 < p < \infty$ the first condition in (5.2) is a particular case of our general requirement (4.9). In this case, Theorem 4.2 is applied to the Young function $\Phi(t) = t^p$ so that again, $p_\Phi = q_\Phi = p$ in (4.1) while $a_i = d_i$ in (4.6) and $b_i = 1/d_i$ in (4.8).

6. Concluding remarks and future research

In one dimension the two examples from the previous section also hold true under the assumption that each function α_i is either a smooth bounded diffeomorphism, or a bounded invertible real-analytic map, from X_i onto the semi-open interval $[x_{i-1}, x_i)$. It would be natural to extend these results to underlying sets X belonging to subdomains of differentiable and real analytic manifolds. The geometric properties of these spaces (such as the existence of Riemann-Schwarz reflections and complex conjugations in some sense) may eventually reflect in the particular form of the Read-Bajraktarević operator itself. The problem whether this functional extends to these finer spaces is, to our knowledge, untreated. In addition, it seems that the proof of Theorem 4.2 can be modified in the obvious way so that this result be valid as well in the Orlicz-Sobolev space $W^m L_\Phi(X)$, $m \geq 2$. The generalization would require an expression for the higher-order weak derivatives of the composite $(u \circ \alpha_i^{-1})$. Arguably, condition (4.9) should be modified as well so as to ensure that (4.4) be a contraction on $W^m L_\Phi(X)$. These generalizations may be useful in applications. We look forward to addressing these questions in future publications.

Acknowledgments

The author is grateful to an unknown referee for providing useful, constructive and helpful remarks.

References

[1] Adams, R.A., Fournier, J.F., *Sobolev Spaces,* 2nd edition, Acad. Press, 2003.

[2] Arriagada, W., *Matuszewska-Orlicz indices of the Sobolev conjugate Young function,* Partial Differ. Equ. Appl. Math., vol. **3**(2021).

[3] Arriagada, W., Huentutripay, J., *Regularity, positivity and asymptotic vanishing of solutions of a ϕ-Laplacian,* An. Ştiinţ. Univ. "Ovidius" Constanţa Ser. Mat., vol. **25**(2017), no. 3, 59–72.

[4] Bajraktarević, M., *Sur une équation fonctionnelle,* Glas Mat. Fiz. Astr. **12**(1957), 201–205.

[5] Barnsley, M., *Fractals Everywhere,* (2nd Ed.) San Diego, CA: Acad. Press. 1993.

[6] Barnsley, M., Hegland, M., Massopust, P., *Numerics and fractals,* Bull. Inst. Math. Acad. Sin. (N.S.), vol. **9**(2014), no. 3, 389–430.

[7] Barnsley, M., Hurd, L.P., *Fractal Image Compression,* A K Peters/CRC Press; 1st edition (January 18, 1993).

[8] Gossez, J., *Nonlinear elliptic boundary value problems for equations with rapidly (or slowly) increasing coefficients,* Trans. Amer. Math. Soc., **190**(1974), 163–205.

[9] Hernández Encinas, L., Muñoz Masqué, J., *A short proof of the Generalized Faà di Bruno's formula,* Appl. Math. Lett., **16**(2003), 975–979.

[10] Huentutripay, J., Manásevich, R., *Nonlinear eigenvalues for a Quasilinear Elliptic System in Orlicz-Sobolev Spaces,* J. Dynam. Differential Equations, **18**(2006), 901–921.

[11] Hutchinson, J., *Fractals and self-similarity,* Indiana Univ. Math. J., **30**(1981), 713–747.

[12] Krasnosel'skii, M., Rutic'kii, J., *Convex Functions and Orlicz Spaces,* Noordhoff, Groningen, 1961.

[13] Kunze, H., La Torre, D., Mendivil, F., Vrscay, ER., *Fractal-based Methods in Analysis,* New York, NY: Springer, 2012.

[14] Leśniak, K., Snigireva, N., Strobin, F., Vince, A., *Highly Non-contractive Iterated Function Systems on Euclidean Space Can Have an Attractor,* J. Dynam. Differential Equations, (2024). https://doi.org/10.1007/s10884-024-10367-6

[15] Maligranda, L., *Indices and interpolation,* Seminars in Math., Univ. of Campinas, Campinas SP: Brazil; 81–95, 1989.

[16] Massopust, P.R., *Fractal Functions, Fractal Surfaces, and Wavelets,* Acad. Press, Inc., San Diego, CA, 1994.

[17] Massopust, P.R., *Local Fractal Functions and Function Spaces,* in Bandt, C., Barnsley, M., Devaney, R., Falconer, K., Kannan, V., Vinod Kumar, P.B., (eds) *Fractals, Wavelets, and their Applications,* Springer Proc. Math. Stat., vol. 92. Springer, Cham. 2014.

[18] Pick, L., Kufner, A., John, O., Fučik, S., *Function spaces, Vol. 1,* 2nd Edition, De Gruyter Ser. Nonlinear Anal. Appl., **14**(2012).

[19] Read, C.J., *A solution to the invariant subspace problem,* Bull. Lond. Math. Soc. **16**(1984), no. 4, 337–401.

[20] Vrscay, E.R., *Moment and Collage Methods for the Inverse Problem of Fractal Construction with Iterated Function Systems,* in Peitgen HO, Henriques JM, Penedo LF, eds. Fractals in the Fundamental and Applied Sciences. North-Holland, 1991, 443–461.

Waldo Arriagada
Department of Mathematics, New Uzbekistan University, Movarounnahr 1, Mirzo Ulugbek, Tashkent, Uzbekistan.
e-mail: w.arriagada@newuu.uz