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Depth and sdepth of powers of the path ideal of a cycle graph. II

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Abstract. Let $J_{n,m} := (x_1x_2\cdots x_m,\ x_2x_3\cdots x_{m+1},\ \ldots,\ x_{n-m+1}\cdots x_n,$ $x_{n-m+2}\cdots x_nx_1,\ldots,x_nx_1\cdots x_{m-1})$ be the m-path ideal of the cycle graph of length n, in the ring of polynomials $S=K[x_1,\ldots,x_n]$.

As a continuation of our previous paper [2], we prove several new results regarding $\operatorname{depth}(S/J_{n,m}^t)$ and $\operatorname{sdepth}(S/J_{n,m}^t)$, where $t\geq 1$.

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1. Introduction

Let $n\geq 1$ be an integer. We denote $[n]=\{1,2,\ldots,n\}$. Let K be a field and let $S=K[x_1,\ldots,x_n]$ be the polynomial ring in n variables over K. Given a simple graph $G=(V,E)$, on the vertex set $V=[n]$ with the edge set E, the edge ideal associated to G is

$$I(G)=(x_ix_j\ :\ \{i,j\}\in E)\subset S.$$

Note that $I(G)$ is a monomial ideal generated in degree 2. The study of the algebraic properties of $I(G)$ is a well established topic in combinatorial commutative algebra. Conca and De Negri generalized the definition of an edge ideal and first introduced the notion of a m-path ideal in [6], that is

$$I_m(G)=(x_{i_1}x_{i_2}\cdots x_{i_m}\ :\ i_1i_2\cdots i_m \text{ is a path in } G)\subset S.$$

In the recent years, several algebraic and combinatorial properties of path ideals have been studied. However, the study of powers of path ideals is a relatively new area of research.

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The path graph of length $n-1$ is $P_n = (V(P_n), E(P_n))$, where

$$V(P_n) = [n] \text{ and } E(P_n) = \{\{1,2\},\{2,3\},\ldots,\{n-1,n\}\}.$$

Let $1 \leq m \leq n$. One can easily check that the *m-path ideal of the path graph* of length $n-1$ is

$$I_{n,m} = (x_1x_2\cdots x_m,\ x_2x_3\cdots x_{m+1},\ \ldots,\ x_{n-m+1}\cdots x_n) \subset S.$$

The cycle graph of length n is $C_n = (V(C_n), E(C_n))$, where

$$V(C_n) = [n] \text{ and } E(C_n) = E(P_n) \cup \{\{1,n\}\}.$$

Also, for $2 \leq m < n$, the *m-path ideal of the cycle graph* of length n is

$$J_{n,m} = I_{n,m} + (x_{n-m+2}\cdots x_nx_1, x_{n-m+3}\cdots x_nx_1x_2, \ldots, x_nx_1\cdots x_{m-1}).$$

In [1] we studied the depth and Stanley depth of $S/I_{n,m}^t$, where $t \geq 1$. Also, in [2] we studied the depth and Stanley depth of $S/J_{n,m}^t$, where $t \geq 1$. Following [2], the aim of our paper is to further investigate the depth and Stanley depth of the quotient rings associated to powers of the m-path ideal of a cycle.

In Theorem 3.2, we reprove and also extend some results from [2]. As part of our new results, we show that for any $n \geq 5$:

$$\operatorname{sdepth}(S/J_{n,n-2}^t), \operatorname{depth}(S/J_{n,n-2}^t) > 0 \text{ if } n \text{ is odd, and } t < \frac{n-1}{2},$$
$$\operatorname{sdepth}(S/J_{n,n-2}^t), \operatorname{depth}(S/J_{n,n-2}^t) > 0 \text{ if } n \text{ is even, and } t \geq 1,$$
$$\frac{n}{2} \geq \operatorname{sdepth}(S/J_{n,n-2}^t) \geq \operatorname{depth}(S/J_{n,n-2}^t) = 1 \text{ if } n \text{ is even, and } t \geq n-1.$$

In Theorem 3.6, we show that if $n \geq 2m+1$ then

$$\operatorname{depth}(S/J_{n,m}^t) \leq \operatorname{depth}(S/I_{n,m}^t),$$

which improves the upper bound for $\operatorname{depth}(S/J_{n,m}^t)$ given in [2, Theorem 2.10].

Finally, in Section 4, we make some small steps in tackling the problem of computing $\operatorname{depth}(S/J_{n,m}^t)$, for any $t \geq 1$, see Proposition 4.2 and Proposition 4.3. Also, we illustrate, in two examples, the technical difficulties that arise, see Example 4.4 and Example 4.5.

2. Preliminaries

First, we recall the well known Depth Lemma, see for instance [12, Lemma 2.3.9].

Lemma 2.1. *(Depth Lemma) If $0 \to U \to M \to N \to 0$ is a short exact sequence of modules over a local ring S, or a Noetherian graded ring with S_0 local, then*

1. $\operatorname{depth} M \geq \min\{\operatorname{depth} N, \operatorname{depth} U\}$.
2. $\operatorname{depth} U \geq \min\{\operatorname{depth} M, \operatorname{depth} N + 1\}$.
3. $\operatorname{depth} N \geq \min\{\operatorname{depth} U - 1, \operatorname{depth} M\}$.

The following result, which will be used later on, is an easy application of the Depth Lemma:

Lemma 2.2. *Let $d \geq 1$ and $Z_1 \cup Z_2 \cup \cdots \cup Z_d = \{x_1, \ldots, x_n\}$ be a partition, i.e. $|Z_i| > 0$ and $Z_i \cap Z_j = \emptyset$ for all $i \neq j$. Let $P_i = (Z_i) \subset S$ for $1 \leq i \leq d$ and $U := P_1 \cap \cdots \cap P_d$. Then* $\operatorname{depth}(S/U) = d - 1$.

Now, we briefly recall the definition of the Stanley depth invariant, for a quotient of monomial ideals.

Let $0 \subset I \subsetneq J \subset S$ be two monomial ideals and $M = J/I$. A *Stanley decomposition* of M is the decomposition of M as a direct sum

$$\mathcal{D} : M = \bigoplus_{i=1}^{r} m_i K[Z_i],$$

of $\mathbb{Z}^n$-graded K-vector spaces, where $m_i \in S$ are monomials and $Z_i \subset \{x_1, \ldots, x_n\}$. We define $\operatorname{sdepth}(\mathcal{D}) = \min_{i=1,\ldots,r} |Z_i|$ and

$$\operatorname{sdepth}(M) = \max\{\operatorname{sdepth}(\mathcal{D}) |\ \mathcal{D} \text{ is a Stanley decomposition of } M\}.$$

The number $\operatorname{sdepth}(M)$ is called the *Stanley depth* of M.

Herzog, Vlădoiu and Zheng show in [9] that $\operatorname{sdepth}(M)$ can be computed in a finite number of steps. We say that M satisfies the Stanley inequality, if

$$\operatorname{sdepth}(M) \geq \operatorname{depth}(M).$$

Stanley [11] conjectured that any quotient of monomial ideals $M = J/I$ satisfies the Stanley inequality, a conjecture which proves to be false in general for $M = J/I$, where $I \neq 0$; see Duval et al. [8].

The explicit computation of the Stanley depth is a difficult task, both from a theoretical and practical point of view. Also, although the Stanley conjecture was disproved in the most general setting, it is interesting to find large classes of quotients of monomial ideals which satisfy the Stanley inequality.

In [10], Asia Rauf proved the analog of Lemma 2.1 for sdepth:

Lemma 2.3. *If $0 \to U \to M \to N \to 0$ is a short exact sequence of $\mathbb{Z}^n$-graded S-modules, then*

$$\operatorname{sdepth}(M) \geq \min\{\operatorname{sdepth}(U), \operatorname{sdepth}(N)\}.$$

We also recall the following well known results. See for instance [10, Corollary 1.3], [4, Proposition 2.7], [3, Theorem 1.1], [9, Lemma 3.6] and [10, Corollary 3.3].

Lemma 2.4. *Let $I \subset S$ be a monomial ideal and let $u \in S$ a monomial such that $u \notin I$. Then*

(1) $\operatorname{sdepth}(S/(I : u)) \geq \operatorname{sdepth}(S/I)$.
(2) $\operatorname{depth}(S/(I : u)) \geq \operatorname{depth}(S/I)$.

Lemma 2.5. *Let $I \subset S$ be a monomial ideal and let $u \in S$ a monomial such that $I = u(I : u)$. Then*

(1) $\operatorname{sdepth}(S/(I : u)) = \operatorname{sdepth}(S/I)$.
(2) $\operatorname{depth}(S/(I : u)) = \operatorname{depth}(S/I)$.

Lemma 2.6. *Let $I \subset S$ be a monomial ideal and $S' = S[x_{n+1}]$. Then*
(1) $\operatorname{sdepth}_{S'}(S'/IS') = \operatorname{sdepth}_S(S/I) + 1$,
(2) $\operatorname{depth}_{S'}(S'/IS') = \operatorname{depth}_S(S/I) + 1$.

Lemma 2.7. *Let $I \subset S$ be a monomial ideal. Then the following assertions are equivalent:*
(1) $\mathfrak{m} := (x_1, \ldots, x_n) \in \operatorname{Ass}(S/I)$.
(2) $\operatorname{depth}(S/I) = 0$.
(3) $\operatorname{sdepth}(S/I) = 0$.

We will use later on the following result from [5]:

Theorem 2.8. *(see* [5, Theorem 2.11]*) Let $1 \leq p \leq n-1$, $S' = K[x_1, \ldots, x_p]$ and $S'' = K[x_{p+1}, \ldots, x_n]$. Let $I \subset S'$ be a monomial ideal and let $L \subset S''$ be a complete intersection monomial ideal. We denote by $I+L$, the ideal $IS+LS$ of $S = S' \otimes_K S'' = K[x_1, \ldots, x_n]$. Then, for all $t \geq 1$, we have that*
(1) $\operatorname{depth}(S/(I+L)^t) = \min\limits_{1 \leq i \leq t}\{\operatorname{depth}_{S'}(S'/I^i)\} + \dim(S''/L)$.
(2) $p + \dim(S''/L) \geq \operatorname{sdepth}(S/(I+L)^t) \geq \min\limits_{1 \leq i \leq t}\{\operatorname{sdepth}_{S'}(S'/I^i)\} + \dim(S''/L)$.

Let $1 \leq m \leq n$ be two integers. The m-path ideal of the path graph P_n is

$$I_{n,m} = (x_1 \cdots x_m,\ x_2 \cdots x_{m+1},\ \ldots,\ x_{n-m+1} \cdots x_n) \subset S.$$

We denote

$$\varphi(n,m,t) := \begin{cases} n-t+2-\left\lfloor \frac{n-t+2}{m+1} \right\rfloor - \left\lceil \frac{n-t+2}{m+1} \right\rceil, & t \leq n+1-m \\ m-1, & t > n+1-m \end{cases}.$$

We recall the main result of [1]:

Theorem 2.9. *(See* [1, Theorem 2.6]*) With the above notations, we have that*
(1) $\operatorname{sdepth}(S/I_{n,m}^t) \geq \operatorname{depth}(S/I_{n,m}^t) = \varphi(n,m,t)$, *for all* $t \geq 1$.
(2) $\operatorname{sdepth}(S/I_{n,m}^t) \leq \operatorname{sdepth}(S/I_{n,m}) = \varphi(n,m,1)$.

Let $2 \leq m < n$ be two integers. The m-path ideal of the cycle graph C_n is

$$J_{n,m} = I_{n,m} + (x_{n-m+2} \cdots x_n x_1,\ x_{n-m+3} \cdots x_n x_1 x_2,\ \ldots,\ x_n x_1 \cdots x_{m-1}).$$

Let $d = \gcd(n,m)$ and let $t_0 := t_0(n,m)$ be the maximal integer such that $t_0 \leq n-1$ and there exists a positive integer α such that

$$mt_0 = \alpha n + d.$$

Let $t \geq t_0$ be an integer. Let $w = (x_1 x_2 \cdots x_n)^\alpha$, $w_t = w \cdot (x_1 \cdots x_m)^{t-t_0}$, $r := \frac{n}{d}$ and $s := \frac{m}{d}$. If $d > 1$, we consider the ideal

$$U_{n,d} = (x_1, x_{d+1}, \cdots, x_{d(r-1)+1}) \cap (x_2, x_{d+2}, \cdots, x_{d(r-1)+2}) \cap \cdots \cap (x_d, x_{2d}, \ldots, x_{rd}).$$

We recall the following results from [2]:

Lemma 2.10. *(*[2, Lemma 2.2]*) With the above notations, we have:*
(1) *If $d = 1$ then $(J_{n,m}^t : w_t) = \mathfrak{m}$ for all $t \geq t_0$.*
(2) *If $d > 1$ then $(J_{n,m}^t : w_t) = U_{n,d}$ for all $t \geq t_0$.*

We also recall the following result of [2]:

Theorem 2.11. *([2, Theorem 2.10]) With the above notations, we have that*

$$\operatorname{depth}(S/J_{n,m}^t) \leq \varphi(n-1,m,t)+1, \text{ for all } t \geq 1.$$

3. Main results

At the beginning of this section, we prove the following lemma:

Lemma 3.1. *We have that:*

(1) $\mathfrak{m} \in \operatorname{Ass}(S/J_{n,n-1}^t)$, *for all* $n \geq 2$ *and* $t \geq n-1$.
(2) *If* $n \geq 3$ *is odd then* $\mathfrak{m} \notin \operatorname{Ass}(S/J_{n,n-2}^t)$, *for all* $t < \frac{n-1}{2}$.
(3) *If* $n \geq 3$ *is odd then* $\mathfrak{m} \in \operatorname{Ass}(S/J_{n,n-2}^t)$, *for all* $t \geq \frac{n-1}{2}$.
(4) *If* $n \geq 3$ *is even then* $\mathfrak{m} \notin \operatorname{Ass}(S/J_{n,n-2}^t)$, *for all* $t \geq 1$.

Proof. (1) The result follows from Lemma 2.10(1). However, we present here a new proof: Let $w_t := x_1^{t-1}\cdots x_{n-1}^{t-1}x_n^{n-2}$. Note that $J_{n,n-1}^t$ is minimally generated by monomials of degree $(n-1)t$, while $\deg(w_t) = (n-1)t-1$. Thus, $w_t \notin J_{n,n-1}^t$. We claim that

$$(J_{n,n-1}^t : w_t) = \mathfrak{m}. \tag{3.1}$$

Since $(x_1\cdots x_{n-1})^{t-n+1} \in J_{n,n-1}^{t-n+1}$, $w_t = (x_1\cdots x_{n-1})^{t-n+1}w_{n-1}$ and $w_t \notin J_{n,n-1}^t$ it follows that

$$(J_{n,n-1}^{n-1} : w_{n-1}) \subseteq (J_{n,n-1}^t : w_t) \subsetneq S.$$

Therefore, as $\mathfrak{m}$ is maximal, it is enough to prove (3.1) for $t = n-1$.

Note that $G(J_{n,n-1}) = \{u_1,\ldots,u_n\}$, where $u_i = \prod_{j\neq i} x_j$ for all $1 \leq j \leq n$. It is easy to see that:

$$x_jw_{n-1} = \prod_{k\neq j} u_k \in J_{n,n-1}^{n-1} \text{ for all } 1 \leq j \leq n,$$

hence (3.1) is true.

(2) Assume by contradiction that $\mathfrak{m} \in \operatorname{Ass}(S/J_{n,n-2}^t)$. Then there exists a monomial $w \in S$ with $w \notin J_{n,n-2}^t$ such that $(J_{n,n-2}^t : w) = \mathfrak{m}$. For degree reasons, we must have $\deg(w) \geq t(n-2)-1$. Without any loss of generality, we may assume that $w = x_1^{a_1}\cdots x_n^{a_n}$ with $a_1 \geq a_2 \geq \cdots \geq a_n$. Then, we deduce that $a_1 \geq t$. Since $x_1w \in J_{n,n-2}^t$ it follows that $w \in J_{n,n-2}^t$, a contradiction.

(3) The result follows from Lemma 2.10(1), since $\gcd(n,n-2) = 1$ and $t_0(n,n-2) = \frac{n-1}{2}$ for n odd.

(4) First, note that $d = \gcd(n,n-2) = 2$. Assume by contradiction that there exists a monomial $w \in S$ with $w \notin J_{n,n-2}^t$ such that $(J_{n,n-2}^t : w) = \mathfrak{m}$. It follows that $x_jw \in J_{n,n-2}^t$ for all $1 \leq j \leq n$. Since $x_1w \in J_{n,n-2}^t$ and $w \notin J_{n,n-2}^t$ it follows that $w = u_1\cdots u_{t-1}v$, where $u_j \in G(J_{n,n-2})$, $v \notin J_{n,n-2}$ and $x_1v \in J_{n,n-2}$. This implies

(i) $\operatorname{supp}(v) = \{x_1,x_2,\ldots,x_n\} \setminus \{x_1,x_j,x_{j+1}\}$, where $2 \leq j \leq n-1$, or
(ii) $\operatorname{supp}(v) = \{x_1,x_2,\ldots,x_n\} \setminus \{x_1,x_j\}$, where $3 \leq j \leq n-1$.

Note that, if $x_2(u_1 \cdots u_{t-1}) = x_\ell(u'_1 \cdots u'_{t-1})$ for some $u'_j \in G(J_{n,n-2})$, then it follows that ℓ is even, since u_j's and u'_j's are products of $\frac{n-2}{2}$ variables with odd indices and $\frac{n-2}{2}$ variables with even indices. Therefore, if $x_2 w \in J^t_{n,n-2}$ then $x_\ell v \in J_{n,n-2}$ for some even index ℓ.

In the case (i), it follows that $\operatorname{supp}(x_\ell v) = \{x_1, x_2, \ldots, x_n\} \setminus \{x_1, x_j\}$ with $j \neq 1$ and j odd. But this contradicts the fact that $x_\ell v \in J_{n,n-2}$.

In the case (ii), if j is odd, then $\operatorname{supp}(x_\ell v) = \{x_1, x_2, \ldots, x_n\} \setminus \{x_1, x_j\}$ and again we get a contradiction. It follows that

$$\operatorname{supp}(v) = \{x_1, x_2, \ldots, x_n\} \setminus \{x_1, x_{2k}\}, \text{ where } 2 \leq k \leq \frac{n-2}{2}. \tag{3.2}$$

We claim that $w \in J^t_{n,n-2}$. Indeed, since $x_2 w \in J^t_{n,n-2}$, from (3.2) it follows that there exist $u'_j \in G(J_{n,n-2})$ with $1 \leq j \leq t-1$ such that $x_2(u_1 \cdots u_{t-1}) = x_{2k}(u'_1 \cdots u'_{t-1})$. Therefore,

$$w = (u_1 \cdots u_{t-1})v = x_2(u_1 \cdots u_{t-1})\frac{v}{x_2} = x_{2k}(u'_1 \cdots u'_{t-1})\frac{v}{x_2}.$$

Denoting $v' = v \cdot \frac{x_{2k}}{x_2}$, it is easy to see that $\operatorname{supp}(v') = \{x_3, \ldots, x_n\}$. Hence, $v' \in J_{n,n-2}$ and thus $w \in J^t_{n,n-2}$, a contradiction. $\square$

Theorem 3.2. *We have that:*

1. $\operatorname{sdepth}(S/J^t_{n,n-1}) = \operatorname{depth}(S/J^t_{n,n-1}) = 0$*, for all* $n \geq 2$ *and* $t \geq n-1$.
2. *If* $n \geq 3$ *is odd then* $\operatorname{sdepth}(S/J^t_{n,n-2}) = \operatorname{depth}(S/J^t_{n,n-2}) = 0$*, for all* $t \geq \frac{n-1}{2}$.
3. *If* $n \geq 3$ *is odd then* $\operatorname{sdepth}(S/J^t_{n,n-2}), \operatorname{depth}(S/J^t_{n,n-2}) > 0$*, for all* $t < \frac{n-1}{2}$.
4. *If* $n \geq 3$ *is even then* $\operatorname{sdepth}(S/J^t_{n,n-2}), \operatorname{depth}(S/J^t_{n,n-2}) > 0$*, for all* $t \geq 1$.
5. *If* $n \geq 3$ *is even then* $\frac{n}{2} \geq \operatorname{sdepth}(S/J^t_{n,n-2}) \geq \operatorname{depth}(S/J^t_{n,n-2}) = 1$*, for all* $t \geq n-1$.

Proof. (1), (2), (3) and (4) follows immediately from Lemma 3.1 and Lemma 2.7.

(5) follows from (4) and [2, Corollary 2.8]. $\square$

Remark 3.3. Note that (1) from Theorem 3.2 was proved in [2, Theorem 3.1] and (2) from Theorem 3.2 was proved in [2, Corollary 2.8(1)]. The results (3), (4) and (5) from Theorem 3.2 are new.

Lemma 3.4. *Let* $n > m \geq 2$ *and* $t \geq 1$ *be some integers. Then*

$$(J^t_{n,m} : (x_{n-m+1}x_{n-m+2} \cdots x_{n-1})^t) = \begin{cases} (x_{n-m}, x_n)^t, & n \leq 2m \\ (I_{n-m-1,m}S + (x_{n-m}, x_n))^t, & n \geq 2m+1 \end{cases}.$$

Proof. It is easy to check that

$$(J_{n,m} : (x_{n-m+1} \cdots x_{n-1})) = \begin{cases} (x_{n-m}, x_n), & n \leq 2m \\ I_{n-m-1,m}S + (x_{n-m}, x_n), & n \geq 2m+1 \end{cases}. \tag{3.3}$$

Since $(J_{n,m} : (x_{n-m+1} \cdots x_{n-1}))^t \subset (J^t_{n,m} : (x_{n-m+1} \cdots x_{n-1})^t)$, in order to complete the proof, by (3.3), it suffices to show that for any monomial v with

$(x_{n-m+1}\cdots x_{n-1})^t v\in J^t_{n,m}$, there exists some monomials

$$v_1,\ldots,v_t\in\begin{cases}(x_{n-m},x_n), & n\leq 2m\\ I_{n-m-1,m}S+(x_{n-m},x_n), & n\geq 2m+1\end{cases},$$

such that $v_1\cdots v_t\mid v$.

Indeed, let v be a monomial as above. Let $a=\deg_{x_{n-m}}(v)$ and $b=\deg_{x_n}(v)$. If $a\geq t$ then we can choose $v_1=\cdots=v_t=x_{n-m}$ and we are done. Also, if $a<t$ and $a+b\geq t$, we can choose $v_1=\cdots=v_a=x_{n-m}$, $v_{a+1}=\cdots=v_t=x_n$ and we are also done.

Now, assume $a+b<t$. Let $v_1=\cdots=v_a=x_{n-m}$ and $v_{a+1}=\cdots=v_{a+b}=x_n$. Since $(x_{n-m+1}\cdots x_{n-1})^t v\in J^t_{n,m}$, there are $g_1,\ldots,g_t\in G(J_{n,m})$ such that $g_1\cdots g_t\mid (x_{n-m+1}\cdots x_{n-1})^t v$. It is clear that at most $a+b$ of the monomials $g_1,\ldots,g_t$ are divisible by x_{n-m} or x_n. Hence, there are $t-a-b$ such monomials, let's say $g_{a+b+1},\ldots,g_t$ which are divisible neither by x_{n-m}, neither by x_n. In particular, it follows that

$$g_{a+b+1},\ldots,g_t\in K[x_1,\ldots,x_{n-m-1},x_{n-m+1},\ldots,x_{n-1}],$$

which leads to a contradiction if $n\leq 2m$. On the other hand, if $n\geq 2m+1$, then we get $g_{a+b+1},\ldots,g_t\in G(I_{n-m-1,m})$. We let $v_{a+b+1}=g_{a+b+1},\ldots,v_t=g_t$ and we are done. □

Lemma 3.5. *Let $n>m\geq 2$ and $t\geq 1$ be some integers with $n\geq 2m+1$. Let $V:=(J^t_{n,m}:(x_{n-m+1}x_{n-m+2}\cdots x_{n-1})^t)$. Then:*

$$\operatorname{sdepth}(S/V)\geq\operatorname{depth}(S/V)=\begin{cases}\varphi(n,m,t), & t\leq n-2m\\ 2(m-1), & t\geq n-2m+1\end{cases}.$$

Proof. Let $S':=K[x_1,\ldots,x_{n-m-1}]$ and $S'':=K[x_{n-m},\ldots,x_n]$. We consider the ideals $I=I_{n-m-1,m}\subset S'$ and $L=(x_{n-m},x_n)\subset S''$. According to Lemma 3.4, we have that $V=(IS+LS)^t$. It is clear that

$$\dim(S''/L)=m+1-2=m-1. \tag{3.4}$$

Also, from Theorem 2.9, we have that

$$\operatorname{sdepth}(S'/I^t)\geq\operatorname{depth}(S'/I^t)=\varphi(n-m-1,m,t). \tag{3.5}$$

Since $V=(IS+LS)^t$, from Theorem 2.8, (3.4), (3.5) and the fact that $t\mapsto\varphi(n-m-1,m,t)$ is nonincreasing, it follows that

$$\operatorname{sdepth}(S/V)\geq\operatorname{depth}(S/V)=\varphi(n-m-1,m,t)+m-1.$$

The required formula follows by straightforward computations. □

The following result is an improvement of Theorem 2.11 for $t\leq n-2m$.

Theorem 3.6. *Let $n>m\geq 2$ and $t\geq 1$ be some integers. If $n\geq 2m+1$ then*

$$\operatorname{depth}(S/J^t_{n,m})\leq\begin{cases}\varphi(n,m,t), & t\leq n-2m\\ \varphi(n-1,m,t)+1, & n-2m+1\leq t\leq n-m\\ m, & t\geq n-m+1\end{cases}.$$

Proof. From Lemma 3.4, Lemma 3.5 and Lemma 2.4(2) it follows that

$$\operatorname{depth}(S/J_{n,m}^t) \leq \begin{cases} \varphi(n,m,t), & t \leq n-2m \\ 2(m-1), & t \geq n-2m+1 \end{cases}. \tag{3.6}$$

On the other hand, according to Theorem 2.11 we have that

$$\operatorname{depth}(S/J_{n,m}^t) \leq \varphi(n-1,m,t)+1. \tag{3.7}$$

Also, it is easy to check that

$$\varphi(n-1,m,t)+1 \leq 2m-2 \text{ for all } t \geq n-2m+1 \text{ and}$$

$$\varphi(n-1,m,t) = m-1 \text{ for all } t \geq n-m+1. \tag{3.8}$$

The required conclusion follows from (3.6), (3.7) and (3.8). □

Remark 3.7. If $m < n \leq 2m$ and $t \geq 1$ then Lemma 3.4 and Lemma 2.4 imply

$$\operatorname{depth}(S/J_{n,m}^t) \leq \operatorname{depth}(S/(x_{n-m},x_n)^t) = n-2 \text{ and}$$
$$\operatorname{sdepth}(S/J_{n,m}^t) \leq \operatorname{sdepth}(S/(x_{n-m},x_n)^t) = n-2.$$

However, the above inequalities are trivial, since the ideal $J_{n,m}^t$ is not principal.

4. Remarks in the general case

Let $n > m \geq 2$ be two integers. In [2] we studied the functions $t \mapsto \operatorname{depth}(S/J_{n,m}^t)$ and $t \mapsto \operatorname{sdepth}(S/J_{n,m}^t)$ for $t \geq t_0$, where t_0 was defined in Section 2. However, for $2 \leq t \leq t_0 - 1$, this problem is much harder. In the following, we present a possible way of tackling it and we point out the difficulties which appear.

For convenience, we introduce the following notations: We let $J = J_{n,m} \subset S$, $S' = K[x_1,\ldots,x_{n-1}]$, $J' = (J : x_n) \cap S'$ and $I = I_{n-1,m} \subset S'$.

Lemma 4.1. *With the above notations, we have that:*

1. $(J^t, x_n^k) = (I^{t+1-k}J^{k-1}, x_n^k)$, *for all* $1 \leq k \leq t$.
2. $((J^t : x_n^{k-1}), x_n) = (I^{t+1-k}J'^{k-1}, x_n)$, *for all* $1 \leq k \leq t$.
3. $(J^t, x_n) = (I^t, x_n)$ *and* $(J^t : x_n^t) = J'^t S$.
4. $\operatorname{depth}(S/J^t) \leq \operatorname{depth}(S'/J'^t)+1$ *and* $\operatorname{sdepth}(S/J^t) \leq \operatorname{sdepth}(S'/J'^t)+1$

Proof. (1) Since $IS \subset J$, the inclusion "$\supseteq$" is clear. In order to prove the other inclusion, it is enough to note that if $u \in G(J^t)$ such that $x_n^k \nmid u$, then $u \in G(I^{t+1-k}J^{k-1})$.

(2) Since $((J^t : x_n^{k-1}), x_n) = ((J^t, x_n^k) : x_n^{k-1})$, the conclusion follows from (1).

(3) $(J^t, x_n) = (I^t, x_n)$ follows from (1), for $k = 1$. The second equality is trivial.

(4) From Lemma 2.4 and Lemma 2.6 and (3) it follows that

$$\operatorname{depth}(S/J^t) \leq \operatorname{depth}(S/(J^t : x_n)) \leq \cdots \leq \operatorname{depth}(S/(J^t : x_n^t)) = \operatorname{depth}(S'/J'^t)+1 \text{ and}$$
$$\operatorname{sdepth}(S/J^t) \leq \operatorname{sdepth}(S/(J^t : x_n)) \leq \cdots \leq \operatorname{sdepth}(S/(J^t : x_n^t)) = \operatorname{sdepth}(S'/J'^t)+1.$$

Hence, we get the required conclusion. □

With the notations from Lemma 4.1, let

$$d_k := \operatorname{depth}(S'/I^{t+1-k}J'^{k-1}) \text{ and } s_k = \operatorname{sdepth}(S'/I^{t+1-k}J'^{k-1}) \text{ for } 1 \le k \le t.$$

Note that, according to Theorem 2.9, we have that $s_1 \ge d_1 = \varphi(n-1,m,t)$.

Proposition 4.2. *We have that:*

(1) $d_k \ge \operatorname{depth}(S/(J^t : x_n^{k-1})) - 1$ *for all* $1 \le k \le t$.
(2) *If* $\operatorname{depth}(S/(J^t : x_n^k)) > \operatorname{depth}(S/(J^t : x_n^{k-1}))$ *then* $d_k = \operatorname{depth}(S/(J^t : x_n^{k-1}))$.
(3) *If* $\operatorname{sdepth}(S/(J^t : x_n^k)) > \operatorname{sdepth}(S/(J^t : x_n^{k-1}))$ *then* $s_k \le \operatorname{sdepth}(S/(J^t : x_n^{k-1}))$.

Proof. We fix some k with $1 \le k \le t$. We have the short exact sequence

$$0 \to S/(J^t : x_n^k) \to S/(J^t : x_n^{k-1}) \to S/((J^t : x_n^{k-1}), x_n) \cong S'/I^{t+1-k}J'^{k-1} \to 0, \tag{4.1}$$

where the isomorphism is given by Lemma 4.1(2). Thus, from (4.1), Lemma 2.1 (Depth lemma), Lemma 2.4 and Lemma 2.3 it follows that

$$\begin{aligned}
&\operatorname{depth}(S/(J^t : x_n^k)) \ge \operatorname{depth}(S/(J^t : x_n^{k-1})),\\
&\operatorname{depth}(S/(J^t : x_n^{k-1})) \ge \min\{\operatorname{depth}(S/(J^t : x_n^k)), d_k\},\\
&d_k \ge \min\{\operatorname{depth}(S/(J^t : x_n^k)) - 1, \operatorname{depth}(S/(J^t : x_n^{k-1}))\} \text{ and}\\
&\operatorname{sdepth}(S/(J^t : x_n^{k-1})) \ge \min\{\operatorname{sdepth}(S/(J^t : x_n^k)), s_k\}.
\end{aligned}$$

Now, we get the required conclusions (1-3). □

Proposition 4.3. *With the above notations, we have that:*

(1) $\operatorname{depth}(S/(J^t, x_n^t)) \ge \min\{\varphi(n-1,m,t), d_2, \ldots, d_t\}$.
(2) $\operatorname{sdepth}(S/(J^t, x_n^t)) \ge \min\{\varphi(n-1,m,t), s_2, \ldots, s_t\}$.
(3) $\operatorname{depth}(S/J^t) \le \operatorname{depth}(S/(J^t, x_n^t)) + 1$.
(4) *If* $\operatorname{depth}(S/(J^t : x_n^t)) > \operatorname{depth}(S/J^t)$ *then* $\operatorname{depth}(S/J^t) \ge \operatorname{depth}(S/(J^t, x_n^t))$ *and similarly for the Stanley depth.*

Proof. (1) We consider the following short exact sequences

$$0 \to S/((J^t : x_n^{k-1}), x_n) \to S/(J^t, x_n^k) \to S/(J^t, x_n^{k-1}) \to 0 \text{ for } 2 \le k \le t. \tag{4.2}$$

Since, by Lemma 4.1, we have that $S'/I^{t+1-k}J'^{k-1} \cong S/((J^t : x_n^{k-1}), x_n)$, the conclusion follows from (4.2), Lemma 2.1 (Depth lemma) and Theorem 2.9.

(2) The proof is similar to the proof of (1), using Lemma 2.3 instead of Lemma 2.1.

(3) follows from Lemma 2.1, Lemma 2.4 and the short exact sequence

$$0 \to S/(J^t : x_n^t) \to S/J^t \to S/(J^t, x_n^t) \to 0. \tag{4.3}$$

(4) follows from (4.3), Lemma 2.1 and Lemma 2.3. □

The following examples illustrate the computational difficulties which appear, when we apply the previous results in order to compute $\operatorname{depth}(S/J^t)$ and $\operatorname{sdepth}(S/J^t)$.

Example 4.4. Let $J = J_{6,3} \subset S = K[x_1, \ldots, x_6]$. Let $I = I_{5,3} \subset S' = K[x_1, \ldots, x_5]$. We have

$$J' = (J : x_6) \cap S' = (x_1x_2, x_2x_3x_4, x_4x_5, x_5x_1).$$

As in the proof of Proposition 4.2, we have the short exact sequences:

$$0 \to S/(J^2 : x_6) \to S/J^2 \to S/(J^2, x_6) \cong S'/I^2 \to 0 \text{ and} \tag{4.4}$$

$$0 \to S/(J^2 : x_6^2) \to S/(J^2 : x_6) \to S/((J^2 : x_6), x_6) \cong S'/IJ' \to 0. \tag{4.5}$$

From Theorem 2.9 it follows that:

$$\operatorname{sdepth}(S'/I^2) \geq \operatorname{depth}(S'/I^2) = \varphi(5, 3, 2) = 2.$$

Note that $IJ' = x_3L$, where $L = (x_1x_2, x_2x_4, x_4x_5)(x_1x_2, x_2x_3x_4, x_4x_5, x_5x_1) \subset S'$, and $(J^2 : x_6^2) = J'^2S = (x_1x_2, x_2x_3x_4, x_4x_5, x_5x_1)^2S$. From Lemma 2.5, it follows that

$$\operatorname{depth}(S'/IJ') = \operatorname{depth}(S'/L) \text{ and } \operatorname{sdepth}(S'/IJ') = \operatorname{sdepth}(S'/L). \tag{4.6}$$

We have $(L : x_2x_3x_4) = (x_1, x_4) \cap (x_2, x_5)$. By straightforward computations we get:

$$\operatorname{sdepth}(S'/(L : x_2x_3x_4)) = \operatorname{depth}(S'/(L : x_2x_3x_4)) = 2.$$

Also, we have that

$$W := (L, x_2x_3x_4) = (x_2x_3x_4, x_1x_2x_4x_5, x_1^2x_2x_5, x_1^2x_2^2, x_2x_4^2x_5, x_1x_2^2x_4, x_4^2x_5^2, x_1x_4x_5^2).$$

Note that $(W : x_2x_4x_5) = (x_3, x_4, x_1)$. Therefore we get

$$\operatorname{sdepth}(S'/(W : x_2x_4x_5)) = \operatorname{depth}(S'/(W : x_2x_4x_5)) = 2.$$

Also, $(W, x_2x_4x_5) = (x_2x_4x_5, x_2x_3x_4, x_1x_4x_5^2, x_4^2x_5^2, x_1x_2^2x_4, x_1^2x_2^2, x_1^2x_2x_5)$. By continuing the computations, we can deduce that

$$\operatorname{sdepth}(S'/(W, x_2x_4x_5)) = \operatorname{depth}(S'/(W, x_2x_4x_5)) = 2.$$

From the short exact sequence

$$0 \to S'/(W : x_2x_4x_5) \to S'/W \to S'/(W, x_2x_4x_5) \to 0,$$

using Lemma 2.1, Lemma 2.3 and Lemma 2.4, we deduce that

$$\operatorname{depth}(S'/W) = \operatorname{sdepth}(S'/W) = 2.$$

From the short exact sequence

$$0 \to S'/(L : x_2x_3x_4) \to S'/L \to S'/W \to 0,$$

and (4.6), using Lemma 2.1, Lemma 2.3 and Lemma 2.4, we deduce that:

$$d_2 := \operatorname{depth}(S'/IJ') = \operatorname{depth}(S'/L) = 2 \text{ and}$$
$$s_2 := \operatorname{sdepth}(S'/IJ') = \operatorname{sdepth}(S'/L) = 2.$$

Using similar computations, as above, one can deduce that

$$d_1 = \operatorname{depth}(S'/J'^2) \geq 2 \text{ and } s_1 := \operatorname{sdepth}(S'/J'^2) \geq 2.$$

As in the proof of Proposition 4.2, we can deduce from the short exact sequences (4.4) that:

$$\operatorname{depth}(S/J^2) \geq 2 \text{ and } \operatorname{sdepth}(S/J^2) \geq 2.$$

We mention that, according to Cocoa [7], $\operatorname{sdepth}(S/J^2) = \operatorname{depth}(S/J^2) = 3$.

Example 4.5. Let $J = J_{6,4} \subset S = K[x_1, \ldots, x_6]$. Let $I = I_{5,4} \subset S' = K[x_1, \ldots, x_5]$. We have $J' = (J : x_6) \cap S' = (x_1x_2x_3, x_3x_4x_5, x_4x_5x_1, x_5x_1x_2)$, and the short exact sequences

$$0 \to S/(J^2 : x_6) \to S/J^2 \to S/(J^2, x_6) \cong S'/I^2 \to 0 \text{ and} \tag{4.7}$$

$$0 \to S/(J^2 : x_6^2) \to S/(J^2 : x_6) \to S/((J^2 : x_6), x_6) \cong S'/IJ' \to 0. \tag{4.8}$$

From Theorem 2.9 it follows that

$$\operatorname{sdepth}(S'/I^2) \geq \operatorname{depth}(S'/I^2) = \varphi(5, 4, 2) = 3.$$

Note that $IJ' = x_2x_3x_4L$, where $L = (x_1, x_5)(x_1x_2x_3, x_3x_4x_5, x_4x_5x_1, x_5x_1x_2) \subset S'$. From Lemma 2.5, it follows that

$$\operatorname{depth}(S'/IJ') = \operatorname{depth}(S'/L) \text{ and } \operatorname{sdepth}(S'/IJ') = \operatorname{sdepth}(S'/L). \tag{4.9}$$

We have $(L : x_3) = (x_1, x_5)(x_1x_2, x_4x_5)$ and $(L, x_3) = x_1x_5(x_1, x_5)(x_2, x_4) + (x_3)$. By straightforward computation we have

$$\operatorname{sdepth}\left(\frac{K[x_1, x_2, x_4, x_5]}{(x_1, x_5)(x_2, x_4)}\right) = \operatorname{depth}\left(\frac{K[x_1, x_2, x_4, x_5]}{(x_1, x_5)(x_2, x_4)}\right) = 1$$
$$\operatorname{sdepth}\left(\frac{K[x_1, x_2, x_4, x_5]}{(x_1, x_5)(x_1x_2, x_4x_5)}\right) = \operatorname{depth}\left(\frac{K[x_1, x_2, x_4, x_5]}{(x_1, x_5)(x_1x_2, x_4x_5)}\right) = 2.$$

Therefore, using Lemma 2.5, we obtain $\operatorname{sdepth}(S/(L, x_3)) = \operatorname{depth}(S/(L, x_3)) = 1$. Also $\operatorname{sdepth}(S/(L : x_3)) = \operatorname{depth}(S/(L : x_3)) = 3$. From the short exact sequence

$$0 \to S/(L : x_3) \to S/L \to S/(L, x_3) \to 0$$

we deduce that $\operatorname{depth}(S/L), \operatorname{sdepth}(S/L) \geq 1$.

Hence, from (4.9) it follows that $\operatorname{depth}(S'/IJ'), \operatorname{sdepth}(S'/IJ') \geq 1$. Using similar computations, one can deduce that $\operatorname{depth}(S'/J'^2), \operatorname{sdepth}(S'/J'^2) \geq 2$.

Therefore, from (4.7), we deduce that $\operatorname{depth}(S/J^2), \operatorname{sdepth}(S/J^2) \geq 1$. Note that

$$(J^2 : x_1x_2x_3x_4x_5x_6) = (x_1, x_3, x_5) \cap (x_2, x_4, x_6).$$

Therefore, according to Lemma 2.2, it follows that

$$\operatorname{depth}(S/(J^2 : x_1x_2x_3x_4x_5x_6)) = 1.$$

Consequently, we get $\operatorname{depth}(S/J^2) = 1$.

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