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Existence results for nonlinear anisotropic elliptic partial differential equations with variable exponents

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Abstract. The focus of this paper will be on studying the existence of solutions in the sense of distribution, for a class of nonlinear partial differential equations defined by a variable exponent anisotropic elliptic operator with a growth conditions given by a strictly positive continuous real function. The functional setting involves variable exponents anisotropic Sobolev spaces.

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1. Introduction

Our goal is to prove the existence of at least one distributional solution to the $\overrightarrow{p}(\cdot)-$ nonlinear elliptic partial differential equations of the type :

$$\begin{cases} -\sum_{i=1}^{N} \partial_i\big(\sigma_i(x,u,\partial_i u)\big) + g(x,u) = f \ , \text{ in } \Omega, \\ u = 0 \qquad\qquad , \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Where, $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded open set with Lipschitz boundary $\partial\Omega$, $f \in L^{\overrightarrow{p}'(\cdot)}(\Omega)(= \bigcap_{i=1}^{N} L^{p_i'(\cdot)}(\Omega))$, with $p_i'(\cdot)(= \frac{p_i}{p_i-1}, i = 1,\dots,N)$ denotes the Hölder congugate of $p_i(\cdot)$, $\sigma_i : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $i = 1,\dots,N$, are Carathéodory functions

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fulfilling for almost everywhere $x \in \Omega$ and every $s, \eta, \eta' \in \mathbb{R}$, $(\eta, \eta') \neq (0,0)$, the following :

$$| \sigma_i(x,s,\eta) | \leq c_1 K(| s |) \left(| \eta | + | \vartheta_i |\right)^{p_i(x)-1}, \tag{1.2}$$

$$\sigma_i(x,s,\eta)\eta \geq c_2 K(| s |) | \eta |^{p_i(x)}, \tag{1.3}$$

$$\left(\sigma_i(x,s,\eta) - \sigma_i(x,s,\eta')\right)(\eta - \eta') \geq \Theta_i(x,\eta,\eta'), \tag{1.4}$$

$$\Theta_i(x,\eta,\eta') = \begin{cases} c_3 | \eta - \eta' |^{p_i(x)}, & \text{if } p_i(x) \geq 2 \\ c_4 \frac{|\eta-\eta'|^2}{(|\eta|+|\eta'|)^{2-p_i(x)}}, & \text{if } 1 < p_i(x) < 2 \end{cases}$$

where, c_l, $l = 1, \ldots, 4$ are positive constants, $\vartheta_i \in L^{p_i(\cdot)}(\Omega)$, $i = 1, \ldots, N$, and $K(\cdot) : \mathbb{R}_+ \longrightarrow \mathbb{R}_+^*$ is a continuous function such that,

$$K(| \xi |) \geq \alpha | \xi |^{r(x)}, \quad \text{for all } | \xi | \geq \lambda, \tag{1.5}$$

with, $\alpha > 0$, $\lambda > 0$ and $r(\cdot) \in \mathcal{C}(\overline{\Omega})$, where $r(\cdot) > 0$ in $\overline{\Omega}$.
For some $\beta > 0$

$$K(| s |) \geq \beta, \quad \text{for all } s \in \mathbb{R}. \tag{1.6}$$

$g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and satisfies ;
a.e. $x \in \Omega$ the following conditions:

$$g(x,s)(s - s') \geq 0, \quad \forall s, s' \in \mathbb{R},\ | s | = | s' |, \tag{1.7}$$

$$\sup_{|s| \leq t} | g(x,s) | \in L^1(\Omega), \quad \forall s \in \mathbb{R} \text{ and } \forall t > 0, \tag{1.8}$$

$$|g(x,s)| \leq c \sum_{i=1}^{N} |s|^{p_i(x)-1}, \quad \forall s \in \mathbb{R}. \tag{1.9}$$

As a typical example, we can consider the following model equation $(\sigma_i(x,u,\partial_i u) = K(| u |)|\partial_i u|^{p_i(x)-2}\partial_i u,\ g(x,u) = u \sum_{i=1}^{N} |u|^{p_i(x)-2})$:

$$\begin{cases} -\sum_{i=1}^{N} \partial_i\big(K(| u |)|\partial_i u|^{p_i(x)-2}\partial_i u\big) + u \sum_{i=1}^{N} |u|^{p_i(x)-2} & = f(x), \quad \text{in } \Omega, \\ & u = 0, \quad \text{on } \partial\Omega, \end{cases}$$

where $f \in L^{\overrightarrow{p}'(\cdot)}(\Omega)$, and the continuous function $K(\cdot)$ is defined for any fixed $x \in \Omega$ as follows:

$$\forall \eta \geq 0 : K(\eta) = \begin{cases} 1 + \lambda^{r(x)}, & \text{if } \eta < \lambda, \\ 1 + \eta^{r(x)}, & \text{if } \eta \geq \lambda, \end{cases}$$

where $\lambda > 0$, and $r(\cdot) \in \mathcal{C}(\overline{\Omega})$ with $r(\cdot) > 0$ in $\overline{\Omega}$.

Our boundary-value problems entails an $\overrightarrow{p}(x)$-nonlinear elliptic differential operartor wherein those sorts of operators have many makes use of withinside the carried out discipline of diverse sciences, amongst them modeling of image processing and electro-rheological fluids(see [9, 21, 3]). From the theoretical side related to the existence of solutions, we can refer , without limitation, to [11, 12, 13, 10, 14, 15, 16, 17, 18, 19, 20].

This paper seeks to prove the existence results of ditributional solutions for a class of anisotropic nonlinear elliptic problems with variable exponents and growth conditions given by a real positive continuous function, will provide us regular solutions in the anisotropic space $W_0^{1,\overrightarrow{q}(\cdot)}(\Omega)$ such that, $\overrightarrow{q}(\cdot) = (q_1(\cdot), \dots, q_N(\cdot))$, be restricted as in Theorem 3.2.

The proof of our main result requires proving the existence of a sequence of suitable approximate solutions (u_n) by applying the main Theorem of pseudo-monotone operators and the results obtained in [12, 13]. Prior estimates are then used to show the boundedness of the solutions u_n and the almost everywhere convergence of their partial derivatives $\partial_i u_n$, $i = 1, \dots, N$, which can be converted into strong L^1-convergence. Through this, we can pass to the limit by L^1-strongly sense for $\sigma_i(x, T_n(u_n), \partial_i u_n)$, and for $g(x, u_n)$, then we conclude the convergence of u_n to the solution of (1.1).

The paper is divided into several sections, in Section 2 we discuss variable exponents anisotropic Lebesgue-Sobolev spaces and their key characteristics, as well as mentioning some embedding theorems. The main theorem and its proof can be found in Section 3.

2. Preliminaries and basic concepts

In this section, we will learn about anisotropic Lebesgue-Sobolev spaces with variable exponent and their most important distinctive properties, as explained, for example, in the papers [6, 4, 5].

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded open subset, and let the following set be:

$$\mathcal{C}_+(\overline{\Omega}) = \{\text{continuous function} \quad p(\cdot) : \overline{\Omega} \longmapsto \mathbb{R}, \quad p^-(= \min_{x \in \overline{\Omega}} p(x)) > 1\}.$$

Assume $p(\cdot) \in \mathcal{C}_+(\overline{\Omega})$. The variable exponent Lebesgue reflexive Banach space $\big(L^{p(\cdot)}(\Omega), \|\cdot\|_{p(\cdot)}\big)$ defined by

$$L^{p(\cdot)}(\Omega) := \{\text{measurable functions } u : \Omega \mapsto \mathbb{R}, \int_\Omega |u(x)|^{p(x)} dx < \infty\},$$

under the Luxemburg norm

$$u \mapsto \|u\|_{p(\cdot)} := \|u\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{s > 0 : \varrho_{p(\cdot)}(\frac{u}{s}) \leq 1\right\}.$$

The function

$$\varrho_{p(\cdot)} : u \mapsto \int_\Omega |u(x)|^{p(x)} dx \text{ is called the convex modular.}$$

The variable exponents Sobolev Banach space $\big(W^{1,p(\cdot)}(\Omega), \|\cdot\|_{1,p(\cdot)}\big)$ defined as fellows

$$W^{1,p(\cdot)}(\Omega) := \left\{u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega)\right\},$$

such that

$$u \mapsto \|u\|_{1,p(\cdot)} := \|\nabla u\|_{p(\cdot)}. \tag{2.1}$$

We define also the reflexive and separable Banach space $\left(W_0^{1,p(\cdot)}(\Omega), \|\cdot\|_{1,p(\cdot)}\right)$ by

$$W_0^{1,p(\cdot)}(\Omega) := \overline{C_0^\infty(\Omega)}^{W^{1,p(\cdot)}(\Omega)}.$$

The following Hölder type inequality holds :

$$\left|\int_\Omega uv\,dx\right| \leq \left(\frac{1}{p^-}+\frac{1}{p'^-}\right)\|u\|_{p(\cdot)}\|v\|_{p'(\cdot)} \leq 2\|u\|_{p(\cdot)}\|v\|_{p'(\cdot)},$$

where, $p'(\cdot)$ denotes the Hölder conjugate of $p(\cdot)$ (i.e. $\frac{1}{p(\cdot)}+\frac{1}{p'(\cdot)}=1$ in $\overline{\Omega}$).
Next results(see [4, 5]) we need to use them later. Let $u \in L^{p(\cdot)}(\Omega)$, then:

$$\min\left(\varrho_{p(\cdot)}^{\frac{1}{p^+}}(u), \rho_{p(\cdot)}^{\frac{1}{p^-}}(u)\right) \leq \|u\|_{p(\cdot)} \leq \max\left(\varrho_{p(\cdot)}^{\frac{1}{p^+}}(u), \varrho_{p(\cdot)}^{\frac{1}{p^-}}(u)\right), \tag{2.2}$$

$$\min\left(\|u\|_{p(\cdot)}^{p^-}, \|u\|_{p(\cdot)}^{p^+}\right) \leq \varrho_{p(\cdot)}(u) \leq \max\left(\|u\|_{p(\cdot)}^{p^-}, \|u\|_{p(\cdot)}^{p^+}\right). \tag{2.3}$$

We will now define the variable exponents anisotropic Sobolev spaces $W^{1,\overrightarrow{p}(\cdot)}(\Omega)$.
Let $p_i(\cdot) \in C\big(\overline{\Omega}, [1,+\infty)\big)$, $i \in \{1,\ldots,N\}$, and $\forall x \in \overline{\Omega}$ we set that

$$\overrightarrow{p}(x) = (p_1(x),\ldots,p_N(x)), \quad p_+(x) = \max_{1\leq i\leq N} p_i(x),\ p_-(x) = \min_{1\leq i\leq N} p_i(x),$$

$$\frac{1}{\overline{p}(x)} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{p_i(x)}.$$

The Banach space $W^{1,\overrightarrow{p}(\cdot)}(\Omega)$ is defined by

$$W^{1,\overrightarrow{p}(\cdot)}(\Omega) = \left\{u \in L^{p_+(\cdot)}(\Omega) \text{ and } \partial_i u \in L^{p_i(\cdot)}(\Omega),\ i \in \{1,\ldots,N\}\right\},$$

equipped with the following norm :

$$u \mapsto \|u\|_{\overrightarrow{p}(\cdot)} = \|u\|_{p_+(\cdot)} + \sum_{i=1}^{N}\|\partial_i u\|_{p_i(\cdot)}. \tag{2.4}$$

The Banach space $\left(W_0^{1,\overrightarrow{p}(\cdot)}(\Omega), \|\cdot\|_{\overrightarrow{p}(\cdot)}\right)$ defined as follows

$$W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) = \overline{C_0^\infty(\Omega)}^{W^{1,\overrightarrow{p}(\cdot)}(\Omega)},$$

Let $p(\cdot) \in C_+(\overline{\Omega})$, the variable exponent Marcinkiewicz space $\mathcal{M}^{p(\cdot)}(\Omega)$ is defineed by

$$\mathcal{M}^{p(\cdot)}(\Omega) := \{\text{measurable functions } u : \Omega \mapsto \mathbb{R};\ \exists M > 0 : \int_{\{|u|>s\}} t^{p(x)}\,dx \leq M,\ \forall s > 0\}.$$

The truncation function $\forall t > 0$, $T_t : \mathbb{R} \longrightarrow \mathbb{R}$ is defined as

$$T_t(s) := \begin{cases} s, & \text{if } |s| \leq t, \\ \frac{s}{|s|}t, & \text{if } |s| > t, \end{cases} \tag{2.5}$$

and its derivative (see [11]) given by

$$(DT_t)(s) = \begin{cases} 1, & |s| < t, \\ 0, & |s| > t. \end{cases} \tag{2.6}$$

3. Statement of results and proof

Definition 3.1. *The function $u : \Omega \longrightarrow \mathbb{R}$ is a distributional solution for (1.1) if and only if $u \in W_0^{1,1}(\Omega)$, and $\forall \varphi \in C_c^\infty(\Omega)$:*

$$\int_\Omega \sum_{i=1}^N \sigma_i(x, u, \partial_i u)\partial_i \varphi \, dx + \int_\Omega g(x, u)\varphi \, dx = \int_\Omega f(x)\varphi \, dx.$$

The main result is that.

Theorem 3.2. *Let $p_i(\cdot) \in C\big(\overline{\Omega},]1, +\infty)\big)$, $i = 1, \ldots, N$, such that $\overline{p} < N$. Assume that $f \in L^{\overrightarrow{p}'(\cdot)}(\Omega)$, and g, σ_i, $i = 1, \ldots, N$, be Carathéodory functions that satisfy (1.2)-(1.4), and (1.7)-(1.9). If we have*

$$\frac{\overline{p}(\cdot)(N-1)}{N(r(\cdot)+1)(\overline{p}(\cdot)-1)} < p_i(\cdot) < \frac{\overline{p}(\cdot)(N-1)}{(r(\cdot)+1)(N-\overline{p}(\cdot))}, \text{ in } \overline{\Omega}, \; i = 1, \ldots, N \tag{3.1}$$

where, $r(\cdot)$ defined in (1.5) and satisfies

$$r(\cdot) < \frac{N - \overline{p}(\cdot)}{N(\overline{p}(\cdot)-1)} \text{ in } \overline{\Omega}. \tag{3.2}$$

Then (1.1) has at least one solution u in the sense of distributions in $W_0^{1,\overrightarrow{q}(\cdot)}(\Omega)$, where $\overrightarrow{q}(\cdot) = (q_1(\cdot), \ldots, q_N(\cdot))$, and

$$\max\{1, (r(\cdot)+1)p_i(\cdot) - 1\} < q_i(\cdot) < \frac{Np_i(\cdot)(\overline{p}(\cdot)-1)(r(\cdot)+1)}{\overline{p}(\cdot)(N-1)} \text{ in } \overline{\Omega}, \; i = 1, \ldots, N. \tag{3.3}$$

Remark 3.3. The assumption (3.2) ensure that

$$\frac{\overline{p}(\cdot)(N-1)}{N(r(\cdot)+1)(\overline{p}(\cdot)-1)} > 1 \text{ in } \overline{\Omega}, \; i = 1, \ldots, N.$$

Remark 3.4. The upper bound in (3.1) implies that

$$\frac{Np_i(\cdot)(\overline{p}(\cdot)-1)(r(\cdot)+1)}{\overline{p}(\cdot)(N-1)} > (r(\cdot)+1)p_i(\cdot) - 1 \text{ in } \overline{\Omega}, \; i = 1, \ldots, N. \tag{3.4}$$

3.1. Existence of approximate solutions

Let (f_n) be a sequence of bounded functions defined in Ω which converges to f in $L^{\overrightarrow{p}'(\cdot)}(\Omega)$.

It should be noted here that:

Since $f_n \in L^{\overrightarrow{p}'(\cdot)}(\Omega)$, then from (2.2), we obtain

$$\|f_n\|_{p_i'(\cdot)} \le 1 + \rho_{p_i'(\cdot)}^{\frac{1}{p'^-_i(\cdot)}}(f_n) \le 2 + \rho_{p_i}^{\frac{1}{p'^-}}(f_n) < \infty.$$

Through this, we conclude that

$$f_n \text{ is bounded in } L^{p_i'(\cdot)}(\Omega),\ i = 1, \dots, N. \tag{3.5}$$

Lemma 3.5. *Let $p_i(\cdot) \in C(\overline{\Omega},]1, +\infty))$, $i = 1, \dots, N$, such that $\overline{p} < N$, and let f is in $L^{\overrightarrow{p}'(\cdot)}(\Omega)$. Let g, σ_i, $i = 1, \dots, N$, be Carathéodory functions satisfying* (1.2)-(1.4), *and* (1.7)-(1.9). *Then, there exists at least one weak solution $u_n \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ to the approximated problems*

$$\begin{aligned} -\sum_{i=1}^{N} \partial_i\big(\sigma_i(x, T_n(u_n), \partial_i u_n)\big) + g(x, u_n)) = f_n, \quad in\ \Omega, \\ u_n = 0, \quad on\ \partial\Omega, \end{aligned} \tag{3.6}$$

in the sense that; for every $\varphi \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$

$$\sum_{i=1}^{N} \int_\Omega \sigma_i(x, T_n(u_n), \partial_i u_n)\partial_i \varphi \, dx + \int_\Omega g(x, u_n)\varphi \, dx = \int_\Omega f_n \varphi \, dx. \tag{3.7}$$

Proof. Consider the problem

$$\begin{aligned} -\sum_{i=1}^{N} \partial_i\big(\sigma_i(x, T_n(u_{n_k}), \partial_i u_{n_k})\big) + g_k(x, u_{n_k}) = f_n, \quad \text{in } \Omega, \\ u_{n_k} = 0, \quad \text{on } \partial\Omega, \end{aligned} \tag{3.8}$$

where,

$$g_k(x, \xi) = \frac{g(x, \xi)}{1 + \frac{|g(x,\xi)|}{k}}, \quad \forall k \in \mathbb{N}^*.$$

Note that,

$$| g_k(x, \xi) | \le | g(x, \xi) |, \text{ and } \ | g_k(x, \xi) | \le k.$$

In a similar manner to the results obtained in [12] or in [13] by applying the main Theorem on pseudo-monotone operators, we conclude that there exists a solution $u_{n_k} \in W_0^{1,\vec{p}(\cdot)}(\Omega)$ to problem (3.8), which satisfies

$$\begin{aligned} \sum_{i=1}^{N} \int_\Omega \sigma_i(x, T_n(u_{n_k}), \partial_i u_{n_k})\partial_i \varphi \, dx + \int_\Omega g_k(x, u_{n_k})\varphi \, dx \\ = \int_\Omega f_n \varphi \, dx, \quad \forall \varphi \in W_0^{1,\vec{p}(\cdot)}(\Omega), \end{aligned} \tag{3.9}$$

And also in a similar way, we can obtain (3.7) by passing to the limit in (3.9). □

3.1.1. A priori estimates.

Lemma 3.6. *Let $u_n \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$ be a solution to problem* (3.6). *Then, there exists a constant C, such that*

$$\sum_{i=1}^{N} \int_{\{|u_n| \le t\}} | \partial_i u_n |^{p_i(x)} \, dx \le C(t + 1), \quad \forall t > 0, \tag{3.10}$$

$$\int_\Omega |\ g(x,u_n)\ |\ dx \leq C, \tag{3.11}$$

$$\sum_{i=1}^{N} \int_{\{K(|u_n|)<t\}} (t+1)^{-1} \mid \partial_i(K(\mid u_n \mid)) \mid^{p_i(x)}\ dx \leq C, \quad \forall t > 0. \tag{3.12}$$

Proof. By choosing $\varphi = T_t(u_n)$ in (3.7) and use (1.3), (1.6), (2.5), and (2.6), we find that, for all $t > 0$

$$C\sum_{i=1}^{N} \int_{|u_n|\leq t} \mid \partial_i u_n \mid^{p_i(x)}\ dx + t\int_{|u_n|>t} \frac{u_n}{\mid u_n \mid} g(x,u_n)\,dx$$
$$\leq ct + \int_{|u_n|<t} \mid u_n \mid\mid g(x,u_n) \mid\ dx. \tag{3.13}$$

Then, from (1.8), and the fact that

$$\frac{u_n}{\mid u_n \mid} g(x,u_n) \geq\mid g(x,u_n) \mid . \tag{3.14}$$

Which is produced by the following: due (1.7) we get
$\frac{u_n}{|u_n|} g(x,u_n)- \mid g(x,u_n) \mid= \frac{1}{|u_n|} g(x,u_n)\big(u_n - \mid u_n \mid \frac{g(x,u_n)}{|g(x,u_n)|}\big) \geq 0]$, we obtain

$$C\sum_{i=1}^{N} \int_{|u_n|\leq t} \mid \partial_i u_n \mid^{p_i(x)}\ dx + t\int_{|u_n|>t} \mid g(x,u_n) \mid\ dx \leq c't. \tag{3.15}$$

So, (3.15) give us (3.10), and also, for all $t > 0$

$$\int_{|u_n|>t} \mid g(x,u_n) \mid\ dx \leq c'. \tag{3.16}$$

From (3.16) and (1.8) we get (3.11).
In order to prove (3.12), we choose $\varphi = T_t(K(\mid u_n \mid))$ in (3.7) with the use of (1.3), (1.6), (2.5), and (2.6), we can get for all $t > 0$

$$c_2\beta \sum_{i=1}^{N} \int_{\{K(|u_n|)<t\}} \mid \partial_i(K(\mid u_n \mid)) \mid^{p_i(x)}\ dx + t\int_{K(|u_n|)>t} g(x,u_n)\,dx$$
$$+ \beta\int_{K(|u_n|)\leq t} g(x,u_n)\,dx \leq ct. \tag{3.17}$$

Then, we get

$$c_2\beta \sum_{i=1}^{N} \int_{\{K(|u_n|)<t\}} \mid \partial_i(K(\mid u_n \mid)) \mid^{p_i(x)}\ dx$$
$$\leq ct + t\int_{\{(K(|u_n|)>t\}} \mid g(x,u_n) \mid\ dx + \beta\int_{\{K(|u_n|)\leq t\}} \mid g(x,u_n) \mid\ dx. \tag{3.18}$$

Let's simplify the second side to (3.18).
By (1.8) and (3.16), we obtain

$$\begin{aligned}\int_{\{K(|u_n|)>t\}} \mid g(x,u_n) \mid \, dx &= \int_{\{K(|u_n|)>t\}\cap\{|u_n|>t\}} \mid g(x,u_n) \mid \, dx \\ &+ \int_{\{K(|u_n|)>t\}\cap\{|u_n|\leq t\}} \mid g(x,u_n) \mid \, dx \\ &\leq \int_{\{|u_n|>t\}} \mid g(x,u_n) \mid \, dx + \int_{\{|u_n|\leq t\}} \mid g(x,u_n) \mid \, dx \leq C,\end{aligned}$$

and

$$\begin{aligned}\int_{\{(K(|u_n|)<t\}} \mid g(x,u_n) \mid \, dx &= \int_{\{(K(|u_n|)<t\}\cap\{|u_n|>t\}} \mid g(x,u_n) \mid \, dx \\ &+ \int_{\{(K(|u_n|)<t\}\cap\{|u_n|\leq t\}} \mid g(x,u_n) \mid \, dx \\ &\leq \int_{\{|u_n|>t\}} \mid g(x,u_n) \mid \, dx + \int_{\{|u_n|\leq t\}} \mid g(x,u_n) \mid \, dx \leq C'.\end{aligned}$$

Through this we find that, (3.18) gives us

$$\sum_{i=1}^{N} \int_{\{K(|u_n|)<t\}} \mid \partial_i(K(\mid u_n \mid)) \mid^{p_i(x)} \, dx \leq c(t+1). \tag{3.19}$$

Then, from (3.17) we obtain (3.12). □

Remark 3.7. (3.10) implies that

$$\int_{\{|u_n|\leq t\}} (t+1)^{-1} \mid \partial_i u_n \mid^{p_i(x)} \, dx \leq C, \quad \forall t > 0. \tag{3.20}$$

Remark 3.8. The relationship (3.14) implies that

$$u_n g(x,u_n) \geq 0. \tag{3.21}$$

We need the following technical Lemma that came in [11] and scalar case in [2]

Lemma 3.9. *(see* [2, 11]*) Let $\overrightarrow{p}(\cdot) = (p_1(\cdot),\ldots,p_N(\cdot)) \in \left(C_+(\overline{\Omega})\right)^N$ with $\overline{p}(\cdot) < N$, and let f be a nonnegative function in $W_0^{1,\vec{p}(\cdot)}(\Omega)$. Suppose that there exists a constant c such that*

$$\sum_{i=1}^{N} \int_{\{f\leq t\}} \mid \partial_i f \mid^{p_i(x)} \, dx \leq c(t+1), \quad \forall t > 0. \tag{3.22}$$

Then there exists a constant C, depending on c, such that

$$\int_{\{f>t\}} t^{h(x)} \, dx \leq C, \quad \forall t > 0, \quad h(x) = \frac{N(\overline{p}(x)-1)}{N-\overline{p}(x)}, \quad \forall x \in \overline{\Omega}. \tag{3.23}$$

Lemma 3.10. *Let $u_n \in W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$ be a solution to problem* (3.6)*. Then,*

$$\partial_i(K(| u_n |)) \text{ is bounded in } \mathcal{M}^{\frac{Np_i(\cdot)(\overline{p}(\cdot)-1)}{\overline{p}(\cdot)(N-1)}}(\Omega),\ i=1,\dots,N \tag{3.24}$$

$$\partial_i u_n \text{ is bounded in } \mathcal{M}^{\frac{Np_i(\cdot)(\overline{p}(\cdot)-1)(r(\cdot)+1)}{\overline{p}(\cdot)(N-1)}}(\Omega),\ i=1,\dots,N. \tag{3.25}$$

Proof. For all $i=1,\dots,N$ setting $\alpha_i(\cdot)=\frac{p_i(\cdot)}{h(\cdot)+1}$ where $h(x)=\frac{N(\overline{p}(x)-1)}{N-\overline{p}(x)}$, $\forall x\in\overline{\Omega}$, then we have:
If $0<t<1$, we have trivially that

$$\int_{\{|\partial_i(K(|u_n|))|^{\alpha_i(x)}>t\}} t^{h(x)}\,dx \le |\,\Omega\,|\,.$$

If $t\ge 1$, using (3.12), Lemma 3.9 (due (3.10), and the fact that $|\,\partial_i\,|\,u_n\,||\le|\,\partial_i u_n\,|$ $u_n\neq 0$, $i=1,\dots,N$), we get that

$$\begin{aligned}
\int_{\{|\partial_i(K(|u_n|))|^{\alpha_i(x)}>t\}} t^{h(x)}\,dx \le & \int_{\{|\partial_i(K(|u_n|))|^{\alpha_i(x)}>t\}\cap\{K(|u_n|)<t\}} t^{h(x)}\,dx \\
& + \int_{\{|\partial_i(K(|u_n|))|^{\alpha_i(x)}>t\}\cap\{K(|u_n|)\ge t\}} t^{h(x)}\,dx \\
\le & \int_{\{|\partial_i(K(|u_n|))|^{\alpha_i(x)}>t\}\cap\{K(|u_n|)<t\}\cap\{|u_n|\le t\}} t^{h(x)}\,dx \\
& + \int_{\{|\partial_i(K(|u_n|))|^{\alpha_i(x)}>t\}\cap\{K(|u_n|)<t\}\cap\{|u_n|>t\}} t^{h(x)}\,dx \\
& + \int_{\{|\partial_i(K(|u_n|))|^{\alpha_i(x)}>t\}\cap\{K(|u_n|)\ge t\}\cap\{|u_n|\le t\}} t^{h(x)}\,dx \\
& + \int_{\{|\partial_i(K(|u_n|))|^{\alpha_i(x)}>t\}\cap\{K(|u_n|)\ge t\}\cap\{|u_n|>t\}} t^{h(x)}\,dx \\
\le & 2\int_{\{K(|u_n|)<t\}} t^{h(x)}\,dx + 2\int_{\{|u_n|>t\}} t^{h(x)}\,dx \\
\le & 2\int_{\{K(|u_n|)<t\}} t^{h(x)}\left(\frac{|\,\partial_i(K(|\,u_n\,|))\,|^{\alpha_i(x)}}{t}\right)^{\frac{p_i(x)}{\alpha_i(x)}} dx \\
& + 2\int_{\{|u_n|>t\}} t^{h(x)}\,dx \\
\le & 2\int_{\{K(|u_n|)<t\}} t^{-1}\,|\,\partial_i(K(|\,u_n\,|))\,|^{p_i(x)}\,dx + c \\
\le & 4\int_{\{K(|u_n|)<t\}} (2t)^{-1}\,|\,\partial_i(K(|\,u_n\,|))\,|^{p_i(x)}\,dx + c \\
\le & 4\int_{\{K(|u_n|)<t\}} (t+1)^{-1}\,|\,\partial_i(K(|\,u_n\,|))\,|^{p_i(x)}\,dx + c \le C'.
\end{aligned}$$

Then, for all $i = 1, \dots, N$, $| \partial_i (K(| u_n |)) |^{\alpha_i(\cdot)}$ is bounded in $\mathcal{M}^{h(\cdot)}(\Omega)$.
This gives us, for all $i = 1, \dots, N$, $| \partial_i (K(| u_n |)) |$ is bounded in $\mathcal{M}^{h(\cdot)\alpha_i(\cdot)}(\Omega)$ where

$$h(\cdot)\alpha_i(\cdot) = \frac{p_i(\cdot)h(\cdot)}{h(\cdot)+1} = \frac{Np_i(\cdot)(\overline{p}(\cdot)-1)}{\overline{p}(\cdot)(N-1)}.$$

Now we will prove (3.25), For $\alpha_i(\cdot)$, $i = 1,,N$ and $h(\cdot)$ defined previously, we find that
If $0 < t < 1$, we get that

$$\int_{\{||\partial_i u_n|))|^{\alpha_i(x)(r(x)+1)} > t\}} t^{h(x)})\, dx \leq | \Omega | \, .$$

If $t \geq 1$, by (3.12), Lemma 3.9 (due (3.10), and the fact that $| \partial_i \mid u_n || \leq | \partial_i u_n |, \; u_n \neq 0, \, i = 1, \dots, N)$, we obtain that

$$\begin{aligned}
\int_{\{|\partial_i u_n|^{\alpha_i(x)(r(x)+1)} > t\}} t^{h(x)}\, dx &\leq \int_{\{|\partial_i u_n|^{\alpha_i(x)(r(x)+1)} > t\} \cap \{|u_n| \leq t\}} t^{h(x)}\, dx \\
&\quad + \int_{\{\{|\partial_i u_n|^{\alpha_i(x)(r(x)+1)} > t\} \cap \{|u_n| > t\}} t^{h(x)}\, dx \\
&\leq \int_{\{|u_n| \leq t\}} t^{h(x)} \left(\frac{| \partial_i u_n |^{\alpha_i(x)(r(x)+1)}}{t} \right)^{\frac{p_i(x)}{\alpha_i(x)(r(x)+1)}} dx \\
&\quad + \int_{\{|u_n| > t\}} t^{h(x)}\, dx \\
&\leq \int_{\{|u_n| \leq t\}} t^{h(x)r(x)-1} \, | \partial_i u_n |^{p_i(x)} \; dx + c. \qquad (3.26)
\end{aligned}$$

By noting that the assumption (3.2) is equivalent to:

$$h(\cdot)r(\cdot) - 1 < 0, \quad \text{in } \overline{\Omega},$$

and through the positivity of $r(\cdot)$ and $h(\cdot)$, we find that

$$h(\cdot)r(\cdot) - 1 \geq -1, \quad \text{in } \overline{\Omega}.$$

So, we get that

$$(h(\cdot)r(\cdot) - 1) \in [-1, 0), \quad \text{in } \overline{\Omega}. \qquad (3.27)$$

By using (3.27) in (3.26), and thanks to (3.20), we can obtain that

$$\begin{aligned}
\int_{\{|\partial_i u_n|^{\alpha_i(x)(r(x)+1)} > t\}} t^{h(x)}\, dx &\leq \int_{\{|u_n| \leq t\}} t^{-1} \, | \partial_i u_n |^{p_i(x)} \; dx + c \\
&\leq 2 \int_{\{|u_n| \leq t\}} (2t)^{-1} \, | \partial_i u_n |^{p_i(x)} \; dx + c \\
&\leq 2 \int_{\{|u_n| \leq t\}} (t+1)^{-1} \, | \partial_i u_n |^{p_i(x)} \; dx + c \leq C. \qquad (3.28)
\end{aligned}$$

Hence, we can obtain,

$$| \partial_i u_n |^{\alpha_i(\cdot)(r(x)+1)} \;\; \text{is bounded in } \mathcal{M}^{h(\cdot)}(\Omega) \;\; i = 1, \dots, N.$$

From this we conclude that

$$| \partial_i u_n | \text{ is bounded in } \mathcal{M}^{h(\cdot)\alpha_i(\cdot)(r(\cdot)+1)}(\Omega) \;\; i = 1, \ldots, N,$$

where,

$$h(\cdot)\alpha_i(\cdot)(r(\cdot)+1) = \frac{p_i(\cdot)h(\cdot)(r(\cdot)+1)}{h(\cdot)+1} = \frac{Np_i(\cdot)(\overline{p}(\cdot)-1)(r(\cdot)+1)}{\overline{p}(\cdot)(N-1)}.$$

□

We need the following Lemma (see [22]) to prove the Lemma after it

Lemma 3.11 ([22]). *Let $v(\cdot), w(\cdot) \in C(\overline{\Omega})$, such that $w^- > 0, (v-w)^- > 0$.*
If $u \in \mathcal{M}^{v(\cdot)}(\Omega)$, then $| u |^{w(\cdot)} \in L^1(\Omega)$.
In addition to that, $\mathcal{M}^{v(\cdot)}(\Omega) \subset L^{w(\cdot)}(\Omega)$ for all $v(\cdot),\ w(\cdot) \geq 1$.

Lemma 3.12. *Let f, g and p_i, σ_i, $i = 1, \ldots, N$ be restricted as in Theorem 3.2. Then, for all $i = 1, \ldots, N$,*

$$u_n \text{ is bounded in } L^{q_i(x)}(\Omega), \tag{3.29}$$

$$\partial_i u_n \text{ is bounded in } L^{q_i(x)}(\Omega), \tag{3.30}$$

where $q_i(\cdot)$, $i = 1, \ldots, N$ satisfying (3.3).

Proof. From (3.24), thanks to Lemma 3.11, we deduce that

$$\partial_i(K(| u_n |)) \text{ is bounded in } L^{h_i(\cdot)}(\Omega),\ i = 1, \ldots, N, \tag{3.31}$$

where, $1 < h_i(\cdot) < \frac{Np_i(\cdot)(\overline{p}(\cdot)-1)}{\overline{p}(\cdot)(N-1)}$, $i = 1, \ldots, N$.
So, we can obtain

$$K(| u_n |) \text{ is bounded in } L^{h_i(\cdot)}(\Omega),\ i = 1, \ldots, N, \tag{3.32}$$

where, $1 < h_i(\cdot) < \frac{Np_i(\cdot)(\overline{p}(\cdot)-1)}{\overline{p}(\cdot)(N-1)}$, $i = 1, \ldots, N$.
By condition (1.5) we can get

$$C \mid K(| u_n |) \mid \geq \mid u_n \mid^{r(\cdot)+1},\ \mid u_n \mid \geq \lambda,\ i = 1, \ldots, N. \tag{3.33}$$

Then, through (3.33) and (3.32) we obtain

$$\begin{aligned} \int_\Omega | u_n |^{h_i(\cdot)(r(\cdot)+1)}\, dx &= \int_{\{|u_n| \geq \lambda\}} | u_n |^{h_i(\cdot)(r(\cdot)+1)}\, dx \\ &\quad + \int_{\{|u_n| < \lambda\}} | u_n |^{h_i(\cdot)(r(\cdot)+1)}\, dx \\ &\leq c \int_\Omega | K(| u_n |) |^{h_i(\cdot)}\, dx + (1 + \lambda^{h_{i+}^{+}(r_+^+ + 1)}) \mid \Omega \mid \leq C. \end{aligned} \tag{3.34}$$

Then, (3.34) implies that, for all $i = 1, \ldots, N$

$$u_n \in L^{q_i(\cdot)}(\Omega), \tag{3.35}$$

where, $q_i(\cdot)$, $i = 1, \ldots, N$ satisfying (3.3).
From (2.2) and (3.35) we deduce (3.29).
Finally, by (3.25), Lemma 3.11, and (2.2), we can get (3.30). □

Remark 3.13. Lemma 3.12 implies that

$$u_n \text{ is bounded in } W_0^{1,\overrightarrow{q}(\cdot)}(\Omega), \tag{3.36}$$

where, $\overrightarrow{q}(\cdot) = (q_1(\cdot), \ldots, q_N(\cdot))$, such that $q_i(\cdot)$, $i = 1, \ldots, N$ satisfying (3.3).

Lemma 3.14. *For all* $i = 1, \ldots, N$

$$\lim_{n \longrightarrow +\infty} \Delta_{i,n} = 0, \tag{3.37}$$

where,

$$\Delta_{i,n} = \int_\Omega (\sigma_i(x, T_n(u_n), \partial_i u_n) - \sigma_i(x, T_n(u_n), \partial_i u)) (\partial_i u_n - \partial_i u)\, dx.$$

Proof. From (3.36), we can conclude that the sequence (u_n) is bounded in

$$W_0^{1,q_-^-}(\Omega), \text{ where } q_-^- = \min_{1 \le i \le N} \min_{x \in \overline{\Omega}} q_i(x).$$

So, a sequence (still denoted by (u_n)) can be extracted from them, such that

$$u_n \longrightarrow u \quad \text{strongly in } W_0^{1,q_-^-}(\Omega) \text{ and a.e in } \Omega, \tag{3.38}$$

$$\text{and, } \partial_i u_n \rightharpoonup \partial_i u \text{ weakly in } L^{p_i(x)}(\Omega), \quad i = 1, \ldots, N. \tag{3.39}$$

Note that, for all $i = 1, \ldots, N$,

$$\Delta_{i,n} = \Delta_{i,n}^{(1)} - \Delta_{i,n}^{(2)}$$

where

$$\Delta_{i,n}^{(1)} = \int_\Omega \sigma_i(x, T_n(u_n), \partial_i u_n)(\partial_i u_n - \partial_i u)\, dx$$

$$\Delta_{i,n}^{(2)} = \int_\Omega \sigma_i(x, T_n(u_n), \partial_i u)(\partial_i u_n - \partial_i u)\, dx.$$

First, let's prove for all $i = 1, \ldots, N$

$$\lim_{n \longrightarrow +\infty} \Delta_{i,n}^{(1)} = 0. \tag{3.40}$$

Choose $\varphi = u_n - u$ as a test function in (3.7), we get

$$\sum_{i=1}^N \int_\Omega \sigma_i(x, T_n(u_n), \partial_i u_n)(\partial_i u_n - \partial_i u)\, dx + \int_\Omega g(x, u_n)(u_n - u)\, dx = \int_\Omega f_n(u_n - u)\, dx. \tag{3.41}$$

By using (1.9), and that $u_n \in L^{p_i(\cdot)}(\Omega)$, $i = 1, \ldots, N$, we can get

$$\int_\Omega |\, g(x, u_n) \,|^{p_i'(x)} \; dx \le \int_\Omega |\, u_n \,|^{p_i(x)} \; dx \le c, \tag{3.42}$$

then (3.42) implies that,

$$(g(x, u_n)) \text{ is bounded in } L^{p_i'(\cdot)}(\Omega). \tag{3.43}$$

So, from (3.43), and (3.38), we get

$$\lim_{n\longrightarrow+\infty}\int_\Omega g(x,u_n)(u_n-u)\,dx\,dx=0. \tag{3.44}$$

Now, by (1.2), Young's inequality, and that $\partial_i u_n$, $K(T_n(u_n))\in L^{p_i(\cdot)}(\Omega)$, $i=1,\dots,N$, we deduce that

$$\begin{aligned}\int_\Omega \mid \sigma_i(x,T_n(u_n),\partial_i u_n)\mid^{p_i'(\cdot)}\;dx &\le c\int_\Omega \mid K(T_n(u_n))\mid^{p_i(x)}\;dx\\ &+c'\int_\Omega \mid \partial_i u_n\mid^{p_i(x)}\;dx+c''\le C,\end{aligned}$$

and this implies the boudedness of $(\sigma_i(x,T_n(u_n),\partial_i u_n))$ in $L^{p_i'(\cdot)}(\Omega)$.
Thanks to this and (3.39) we get

$$\lim_{n\longrightarrow+\infty}\int_\Omega \sigma_i(x,T_n(u_n),\partial_i u_n)(\partial_i u_n-\partial_i u)\,dx=0. \tag{3.45}$$

Then, from (3.38), (3.44), (3.45), and (3.5), we get (3.40).
Now, from (1.2), and that $\partial_i u\in L^{p_i(\cdot)}$, we obtain for all $i=1,\dots,N$

$$\int_\Omega \mid \sigma_i(x,T_n(u_n),\partial_i u)\mid^{p_i'(\cdot)}\;dx\le c\int_\Omega \mid \partial_i u\mid^{p_i(x)}\;dx+c'\le C.$$

And therefore

$$(\sigma_i(x,T_n(u_n),\partial_i u))\;\text{ is bounded in } L^{p_i'(\cdot)}(\Omega),\quad i=1,\dots,N. \tag{3.46}$$

Then, (3.46) and (3.39) implies that

$$\lim_{n\longrightarrow+\infty}\Delta^{(2)}_{i,n}=0. \tag{3.47}$$

So, by (3.40) and (3.47), we derive (3.37). □

Lemma 3.15. *For all* $i=1,\dots,N$

$$\partial_i u_n\longrightarrow\partial_i u,\quad a.e.\ in\ \overline{\Omega}. \tag{3.48}$$

Proof. Through (1.4) we conclude that, for all $i=1,\dots,N$

$$\left(\sigma_i(x,T_n(u_n),\partial_i u_n)-\sigma_i(x,T_n(u_n),\partial_i u)\right)(\partial_i u_n-\partial_i u)>0. \tag{3.49}$$

Then, (3.49) and (3.37) gives us, for all $i=1,\dots,N$

$$\left(\sigma_i(x,T_n(u_n),\partial_i u_n)-\sigma_i(x,T_n(u_n),\partial_i u)\right)(\partial_i u_n-\partial_i u)\longrightarrow 0,\text{strongly in } L^1(\Omega). \tag{3.50}$$

Extracting a subsequence (still denoted by (u_n)), we have for all $i=1,\dots,N$

$$\left(\sigma_i(x,T_n(u_n),\partial_i u_n)-\sigma_i(x,T_n(u_n),\partial_i u)\right)(\partial_i u_n-\partial_i u)\longrightarrow 0\quad\text{a.e. in }\Omega. \tag{3.51}$$

Then, by the same techniques used in [11, 8] we can obtain (3.48). □

3.2. Proof of the Theorem 3.2 :

By (3.38), we have

$$g(x,u_n) \longrightarrow g(x,u) \quad \text{a.e. in } \Omega. \tag{3.52}$$

Let $E \subset \Omega$ be any measurable set, we write

$$\int_E \mid g(x,u_n) \mid \, dx = \int_{E\cap\{|u_n|\le t\}} \mid g(x,u_n) \mid \, dx + \int_{E\cap\{|u_n|>t\}} \mid g(x,u_n) \mid \, dx.$$

Let $0 < M < t$, and observe that

$$|T_t(u_n)| \le |T_t(u_n)|\mathbf{1}_{\{|u_n|\le M\}} + |T_t(u_n)|\mathbf{1}_{\{|u_n|>M\}} \le M + t\mathbf{1}_{\{|u_n|>M\}}. \tag{3.53}$$

Then, after taking $\varphi = T_t(u_n)$ in (3.7) and using (3.53), yields

$$t\int_{\{|u_n|>t\}} |g(x,u_n)|\,dx \le M\int_\Omega |f_n|\,dx + t\int_{\{|u_n|>M\}} |f_n|\,dx. \tag{3.54}$$

From (3.54) and (3.11), we conclude the equi-integrability of $g(x,u_n)$ in $L^1(\Omega)$. Through this, (3.52), and Vitali's theorem we get

$$g(x,u_n) \longrightarrow g(x,u) \quad \text{strongly in } L^1(\Omega). \tag{3.55}$$

From (3.29) and (3.48), we have

$$\sigma_i\big(x,T_n(u_n),\partial_i u_n\big) \longrightarrow \sigma_i\big(x,u,\partial_i u\big) \;\; \text{a.e. in } \Omega. \tag{3.56}$$

Now, we prove that

$$\sigma_i\left(x,T_n(u_n),\partial_i u_n\right) \longrightarrow \sigma_i\left(x,u,\partial_i u\right) \quad \text{strongly in } L^{\frac{q_i(\cdot)}{p_i(\cdot)-1}}(\Omega),$$

where $q_i(\cdot)$, $i = 1,\dots,N$ are a continuous functions on $\overline{\Omega}$ satisfying (3.3). Then, we have, for all $x \in \overline{\Omega}$

$$1 < \frac{q_i(x)}{p_i(x)-1} < \frac{N(\overline{p}(x)-1)p_i(x)(r(x)+1)}{\overline{p}(x)(N-1)(p_i(x)-1)}, \quad i = 1,\dots,N. \tag{3.57}$$

The choice of $\frac{q_i(x)}{p_i(x)-1} > 1$ is possible since we have (3.4).
Using (1.2), and (3.30), we get that,

$$\int_\Omega \mid \sigma_i(x,T_n(u_n),\partial_i u_n) \mid^{\frac{q_i(x)}{p_i(x)-1}} \, dx \le c\int_\Omega \mid \partial_i u_n \mid^{q_i(x)} + C\,dx \le C',\; i = 1,\dots,N. \tag{3.58}$$

Then, by (3.58) and using (2.2), we conclude that, for all $i = 1,\dots,N$,

$$\big(\sigma_i\big(x,T_n(u_n),\partial_i u_n\big)\big) \quad \text{uniformly bounded in} \quad L^{\frac{q_i(\cdot)}{p_i(\cdot)-1}}(\Omega).$$

So, by (3.56) and Vitali's theorem, we derive, for all $i = 1,\dots,N$,

$$\sigma_i\big(x,T_n(u_n),\partial_i u_n\big) \longrightarrow \sigma_i\big(x,u,\partial_i u\big) \;\; \text{strongly in } L^1(\Omega). \tag{3.59}$$

So, by passing to the limit in (3.7), we have completed the proof.

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