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The Minty-Browder theorem for nonlinear elliptic equations involving p-Laplacian with singular coefficients under form boundary conditions

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Abstract. We consider the elliptic parabolic partial differential equation with singular coefficients under the rather general form boundary conditions. We proved that the bounded operator associated with the elliptic equation satisfies monotony, coercivity, and semicontinuity conditions. Employing Minty-Browder arguments, we establish the existence and uniqueness of the weak solution to the elliptic equation with singular coefficients under form-boundary conditions.

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1. Introduction

In this article, we consider the existence of the weak solution to a quasi-linear elliptic differential equation in the divergent form

$$\lambda u\,|u|^{p-2} - \frac{d}{dx_i}a_i\,(x,\ u,\ \nabla u) + b\,(x,\ u,\ \nabla u) = 0$$

with a positive parameter λ, where the divergent term is given by

$$\frac{d}{dx_i}a_i\,(x,\ u,\ \nabla u) = \sum_{i=1,\ \ldots,\ l} \frac{\partial a_i\,(x,\ u,\ \nabla u)}{\partial x_i}$$

in domain $\Omega \subseteq \mathbb{R}^l,\ l \geq 3$. As a model example of the main term, we can consider the operator $\Delta_p u \equiv div\left(\nabla u\,|\nabla u|^{p-2}\right)$ and lower term $b\,(x,\ u,\ \nabla u) = c\,(x)\,u\,|u|^{p-2}$.

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Due to the plethora of applications of elliptic partial differential equations, the theory of the existence of solutions is well developed. There are many approaches to the solvability theory for elliptic equations, such as the mountain pass theorem, method of sub-super solutions, degree theory, and fixed point theory to name a few. A general version of the Minty-Browder theorem states that if an operator A from real, separable, reflexive Banach space X into its dual space X^* is semicontinuous, monotone, and coercive, then for each $\psi \in X^*$ there is a solution $f \in X$ to the equation $A(f) = \psi$. The classical results of the Minty-Browwder theorem can be found in [3 – 5, 15, 16], where the method of monotone operators was developed and its application to the Dirichlet problem for a quasi-linear elliptic partial differential equation in the divergence form was considered [16]. In [4, 5], nonlinear elliptic boundary value problems were considered in Hilbert spaces by the method of monotone operators, the semi-boundedness was employed instead of the positivity condition, also, the perturbation of such operators by compact operators was studied.

In this article, we consider elliptic differential equations in the divergent form under form-boundary conditions on its coefficients. The local singularities of the coefficients are supposed such that they belong to certain classes $PK(\beta)$.

Definition 1.1. *For a given number $\beta \in (0,1)$, the class of form-boundary functions $PK(\beta)$ consists of all functions $f \in L^1_{loc}(\Omega)$ such that the inequality*

$$\|f\phi\|^2_{L^2} \leq \beta \|\nabla\phi\|^2_{L^2} + c(\beta)\|\phi\|^2_{L^2}, \tag{1.1}$$

holds with a positive constant $c(\beta)$ and for all $\phi \in W^2_1(\Omega)$.

Some additional information on this type of form-boundary condition can be founded in [22, 23, 24].

From the definition of form-boundary class, assuming $\gamma \geq 0$ and $\gamma^{\frac{1}{2}} \in PK(\beta)$, we obtain

$$\int_\Omega \gamma |\phi|^p \, dx \leq \beta \frac{p^2}{4} \|\phi\|^{p-2}_{L^p} \|\nabla\phi\|^2_{L^p} + c(\beta)\|\phi\|^p_{L^p},$$

for all $\phi \in W^p_1(\Omega)$ and $p \geq 2$. Indeed, we estimate

$$\begin{aligned}
&\int_\Omega \gamma |\phi|^p \, dx = \int_\Omega \left(|\gamma|^{\frac{1}{2}} |\phi|^{\frac{p}{2}}\right)^2 dx = \left\| |\gamma|^{\frac{1}{2}} |\phi|^{\frac{p}{2}} \right\|^2_{L^2} \\
&\leq \beta \left\|\nabla\left(\phi^{\frac{p}{2}}\right)\right\|^2_{L^2} + c(\beta)\left\|\phi^{\frac{p}{2}}\right\|^2_{L^2} \\
&= \beta \int_\Omega \left(\nabla\left(\phi^{\frac{p}{2}}\right)\right)^2 dx + c(\beta)\int_\Omega \left(|\phi|^{\frac{p}{2}}\right)^2 dx \\
&= \beta \int_\Omega \left(\tfrac{p}{2}\phi^{\frac{p}{2}-1}\nabla\phi\right)^2 dx + c(\beta)\int_\Omega |\phi|^p \, dx \\
&= \beta \tfrac{p^2}{4} \int_\Omega \phi^{p-2}(\nabla\phi)^2 dx + c(\beta)\|\phi\|^p_{L^p}.
\end{aligned}$$

Next, applying the Holder inequality, we obtain

$$\begin{aligned}
&\int_\Omega \gamma |\phi|^p \, dx \leq \beta \tfrac{p^2}{4} \left\|\phi^{p-2}\right\|_{L^{\frac{p}{p-2}}} \left\|(\nabla\phi)^2\right\|_{L^{\frac{p}{2}}} + c(\beta)\|\phi\|^p_{L^p} \\
&= \beta \tfrac{p^2}{4} \|\phi\|^{p-2}_{L^p} \|\nabla\phi\|^2_{L^p} + c(\beta)\|\phi\|^p_{L^p}.
\end{aligned}$$

The form-boundary condition guarantees the coercitivity of the associated quadratic form in L^2, namely, the linear operator $-\Delta + \vec{f} \cdot \nabla$ is coercive in L^2 if $\left|\vec{f}\right| \in PK(\beta)$.

We proved that the operator $A : W_1^p(\Omega) \to W_1^p(\Omega)$ given in (2.1), satisfies the monotony, coercivity, and semi-continuity conditions. The existence and the uniqueness of the weak solution to the considered equation follow from the Minty-Browder theorem applied to the operator A.

2. An elliptic partial differential equation

2.1. Basic properties of Sobolev spaces

Let Ω be a smooth domain in R^l for $l \geq 3$. The Sobolev space $W_k^p(\Omega)$ is a Banach space consisting of all elements $u \in L^p(\Omega)$ such that for all multy-index α with $|\alpha| \leq k$, the distributional mixed partial derivative

$$D^\alpha u = u^{(\alpha)} = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} ... \partial x_l^{\alpha_l}}$$

exists and belongs to $L^p(\Omega)$, i.e., $\left\|u^{(\alpha)}\right\|_{L^p} < \infty$. The norm in $W_k^p(\Omega)$ is defined by

$$\|u\|_{W_k^p} = \left(\int_\Omega \left(|u|^p + \sum_{m=1,\ \ldots,\ k} \sum_{(m)} \left| D^{(m)} u \right|^p \right) dx \right)^{\frac{1}{p}},$$

or equivalent form in the sense of equivalence of norms

$$\|u\|_{W_k^p}^{\sim} = \|u\|_{L^p} + \sum_{m=1,\ \ldots,\ k} \sum_{(m)} \left\| D^{(m)} u \right\|_{L^p},$$

where the symbol $\sum_{(m)}$ means summation by all possible derivatives of u up to order m. For the domains Ω with smooth enough boundaries $\partial\Omega$, the space $W_k^p(\Omega)$ coincides with the closure of the set $C^\infty(\Omega)$ of all infinitely differentiable functions in $clos(\Omega)$. In particular, the norm of $W_1^p(\Omega)$ is given by

$$\|u\|_{W_1^p} = \left(\|u\|_{L^p}^p + \|\nabla u\|_{L^p}^p \right)^{\frac{1}{p}},$$

or equivalent form in the sense of equivalence of norms

$$\|u\|_{W_1^p}^{\sim} = \|u\|_{L^p} + \|\nabla u\|_{L^p}.$$

Property. *For $p \in (1, \infty)$ and for each integer $m \geq 0$, the Sobolev space $W_k^p(\Omega)$ is a reflexive separable Banach space with the dual $W_{-k}^q(\Omega)$, where $\frac{1}{p} + \frac{1}{q} = 1$. The set $C^\infty(clos(\Omega))$ is dense subset of $W_k^p(\Omega)$. The subspace $W_{k,0}^p(\Omega)$ is dense in $W_k^p(\Omega)$.*

In $W_{1,0}^p$, the following inequality holds true (Poincare inequality)

$$\|u\|_{L^p} \leq c \|\nabla u\|_{L^p},$$

for all $u \in W_{1,0}^p(\Omega)$, where the constant c depends only on the domain Ω and exponent p.

The Sobolev embedding theorem establishes that if $m \geq s$ and $m - \frac{l}{p} \geq s - \frac{l}{r}$ then the embedding $W_k^p(\Omega) \subseteq W_s^r(\Omega)$ is continuous, and moreover, when $m - \frac{l}{p} > s - \frac{l}{r}$ then the embedding is completely continuous, i.e., each relatively weakly compact subset maps into a relatively compact subset.

In this paper, we use the following Holder inequality

$$\int_\Omega |fg\varphi|\, dx \leq \|f\|_{L^p} \|g\|_{L^q} \|\varphi\|_{L^r},$$

where $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$. Also, for all $x, y \geq 0$, we use the Young inequality

$$xy \leq \frac{1}{a}(\varepsilon x)^a + \frac{1}{b}\left(\frac{y}{\varepsilon}\right)^b$$

for all $a, b \geq 1$ such that $\frac{1}{a} + \frac{1}{b} = 1$, and all $\varepsilon > 0$.

2.2. A nonlinear elliptic partial differential equation involving p-Laplace operator

Let Ω be a smooth domain in $\mathbb{R}^l$ for $l \geq 3$, which can coincide with whole $\mathbb{R}^l$. For some $\lambda > 0$, we consider a nonlinear elliptic partial differential equation

$$A(u) \equiv \lambda u |u|^{p-2} - div(a_i(x,\ u,\ \nabla u)) + b(x,\ u,\ \nabla u) = 0, \tag{2.1}$$

where $u(x)$ an unknown function in $\Omega \subseteq \mathbb{R}^l$.

Functions $a_i(x,\ u,\ \xi)$ and $b(x,\ u,\ \xi)$ are defined for all $x \in clos(\Omega)$ and all $u \in \mathbb{R},\ \xi \in \mathbb{R}^l$; $a_i(x,\ u,\ \xi)$ and $b(x,\ u,\ \xi)$ are continuous at u and ξ.

We assume the following conditions

$$\sum_i a_i(x,\ u,\ \xi)\xi_i \geq \nu |\xi|^p, \tag{2.2}$$

$$\sum_i (a_i(x,\ u,\ \xi) - a_i(x,\ v,\ \eta))(\xi_i - \eta_i) > \nu_1 |\xi - \eta|^p > 0, \tag{2.3}$$

$$|a_i(x,\ u,\ \xi)| \leq \mu |\xi|^{p-1} + \gamma_1(x)|u|^{p-1} + \gamma_2(x), \tag{2.4}$$

$$|a_i(x,\ u,\ \xi) - a_i(x,\ v,\ \eta)| \leq \mu_3 |\xi - \eta|^{p-1} + \gamma_6(x)|u - v|^{p-1}, \tag{2.5}$$

$$|b(x,\ u,\ \xi)| \leq \mu_1 |\xi|^{p-1} + \gamma_3(x)|u|^{p-1} + \gamma_4(x), \tag{2.6}$$

$$|b(x,\ u,\ \xi) - b(x,\ v,\ \eta)| \leq \mu_2 |\xi - \eta|^{p-1} + \gamma_5(x)|u - v|^{p-1}, \tag{2.7}$$

for all $\xi \in \mathbb{R}^l$. We assume

$$\gamma_1{}^{\frac{q}{2}},\ \gamma_3{}^{\frac{q}{2}},\ \gamma_5{}^{\frac{q}{2}}, \gamma_6{}^{\frac{q}{2}} \in PK(\beta),\ \gamma_3{}^{\frac{1}{2}},\ \gamma_5{}^{\frac{1}{2}} \in PK(\beta),$$

and

$$\gamma_4 \in L^q(\Omega).$$

We remark that the inequalities

$$\left(\xi|\xi|^{p-2} - \eta|\eta|^{p-2},\ \xi - \eta\right) \geq c(p)|\xi - \eta|^p$$

and

$$\left|x|x|^{p-2} - y|y|^{p-2}\right| \leq (p-1)|x - y|\left(|x|^{p-2} + |y|^{p-2}\right)$$

hold for all $\xi,\ \eta \in \mathbb{R}^l$ and $x,\ y \in \mathbb{R}$ with the constant $c(p) = 2^{2-p}$. Employing this estimate, we obtain that p-Laplacian $a(u) = \Delta_p(u) = div\left(\nabla u\,|\nabla u|^{p-2}\right)$ satisfies our conditions.

Definition 2.1. *The function $u(x,\ t)$ is called a weak solution to the equation* (2.1) *if $u \in W_1^p(\Omega)$ and the identity*

$$\lambda \int_\Omega |u|^{p-2}\,\phi dx + \int_\Omega a_i(x,\ u,\ \nabla u)\,\nabla_i \phi dx + \int_\Omega b(x,\ u,\ \nabla u)\,\phi dx = 0 \tag{2.8}$$

holds for all $\phi \in W_{1,\,0}^p(\Omega)$. The solution u is called a bounded weak solution to the equation (2.1) *if $ess\max_\Omega |u| < \infty$.*

Definition 2.2. *The operator $A\ :\ W_1^p(\Omega) \to W_{-1}^q(\Omega)$ is called monotone if the inequality*

$$\langle A(u) - A(v)\,,\ (u - v)\rangle \geq 0 \tag{2.9}$$

holds for all $u,\ v \in W_{1,0}^p(\Omega)$.

The operator $A\ :\ W_1^p(\Omega) \to W_{-1}^q(\Omega)$ is called strictly monotone if the inequality

$$\langle A(u) - A(v)\,,\ u - v\rangle > 0 \tag{2.10}$$

holds for all $u,\ v \in W_{1,0}^p(\Omega)$, $u \neq v$.

Definition 2.3. *The operator $A\ :\ W_1^p(\Omega) \to W_{-1}^q(\Omega)$ is called coercive if the inequality*

$$\frac{\langle A(u)\,,\ u\rangle}{\|u\|_{W_1^p}} \overset{\|u\|_{W_1^p}\to\infty}{\longrightarrow} \infty. \tag{2.11}$$

Definition 2.4. *The operator $A\ :\ W_1^p(\Omega) \to W_{-1}^q(\Omega)$ is called semicontinuous if the mapping $t \mapsto \langle A(u+tv)\,,\ w\rangle$ is continuous for all $u,\ v,\ w \in W_{1,0}^p(\Omega)$.*

Below, we assume that the operator $A\ :\ W_1^p(\Omega) \to W_{-1}^q(\Omega)$ is associated with the elliptic equation (2.1).

Lemma 2.5. *Let $p \geq 2$ and let q be its conjugate, i.e., $\frac{1}{p} + \frac{1}{q} = 1$. Assume the conditions* (2.2)-(2.8) *are satisfied. Then, the operator*

$$A\ :\ W_1^p(\Omega) \to W_{-1}^q(\Omega)$$

is bounded.

Proof. For all $u,\ v \in W_{1,0}^p(\Omega)$, we have

$$\begin{aligned}
&|\langle A(u)\,,\ v\rangle| \leq \lambda \int_\Omega \lambda\,|u|^{p-1}\,|v|\,dx \\
&+ \int_\Omega \left(\mu\,|\nabla u|^{p-1} + \gamma_1(x)\,|u|^{p-1} + \gamma_2(x)\right)|\nabla v|\,dx \\
&+ \int_\Omega \left(\mu_1\,|\nabla u|^{p-1} + \gamma_3(x)\,|u|^{p-1} + \gamma_4(x)\right)|v|\,dx \\
&\leq \lambda\,\|u\|_{L^p}^{p-1}\,\|v\|_{L^p} + \mu\,\|\nabla u\|_{L^p}^{p-1}\,\|\nabla v\|_{L^p} + \left\|\gamma_1{}^{\frac{1}{p-1}} u\right\|_{L^p}^{p-1}\,\|\nabla v\|_{L^p} \\
&+ \|\gamma_2\|_{L^q}\,\|\nabla v\|_{L^p} + \mu_1\,\|\nabla u\|_{L^p}^{p-1}\,\|v\|_{L^p} \\
&+ \left\|\gamma_3{}^{\frac{1}{p-1}} u\right\|_{L^p}^{p-1}\,\|v\|_{L^p} + \|\gamma_4\|_{L^q}\,\|v\|_{L^p}\,.
\end{aligned}$$

Applying the Young inequality for $a=\frac{p}{p-2}$ and $b=\frac{p}{2}$, and the form-boundary condition, we have

$$\int_\Omega {\gamma_1}^q |u|^p\,dx \le \beta\int_\Omega \left(\nabla\left(|u|^{\frac{p}{2}}\right)\right)^2 dx + c(\beta)\int_\Omega |u|^p\,dx$$
$$\le \beta\frac{p^2}{4}\left(\frac{2}{\varepsilon^{\frac{p}{2}}p}\|\nabla u\|_{L^p}^p + (p-2)\frac{\varepsilon^{\frac{p}{p-2}}}{p}\|u\|_{L^p}^p\right) + c(\beta)\|u\|_{L^p}^p,$$

and similarly, we obtain

$$\int_\Omega {\gamma_3}^q |u|^p\,dx$$
$$\le \beta\frac{p^2}{4}\left(\frac{2}{\varepsilon^{\frac{p}{2}}p}\|\nabla u\|_{L^p}^p + (p-2)\frac{\varepsilon^{\frac{p}{p-2}}}{p}\|u\|_{L^p}^p\right) + c(\beta)\|u\|_{L^p}^p,$$

so we conclude

$$\begin{aligned}
&|\langle A(u),\ v\rangle| \le \lambda\|u\|_{L^p}^{p-1}\|v\|_{L^p} + \mu\|\nabla u\|_{L^p}^{p-1}\|\nabla v\|_{L^p}\\
&+\begin{pmatrix} \beta\frac{p^2}{4}\left(\frac{2}{\varepsilon^{\frac{p}{2}}p}\|\nabla u\|_{L^p}^p + (p-2)\frac{\varepsilon^{\frac{p}{p-2}}}{p}\|u\|_{L^p}^p\right)+ \\ +c(\beta)\|u\|_{L^p}^p \end{pmatrix}^{p-1}\|\nabla v\|_{L^p}\\
&+\|\gamma_2\|_{L^q}\|\nabla v\|_{L^p} + \mu_1\|\nabla u\|_{L^p}^{p-1}\|v\|_{L^p}\\
&+\begin{pmatrix} \beta\frac{p^2}{4}\left(\frac{2}{\varepsilon^{\frac{p}{2}}p}\|\nabla u\|_{L^p}^p + (p-2)\frac{\varepsilon^{\frac{p}{p-2}}}{p}\|u\|_{L^p}^p\right)+ \\ +c(\beta)\|u\|_{L^p}^p \end{pmatrix}^{p-1}\|v\|_{L^p}\\
&+\|\gamma_4\|_{L^q}\|v\|_{L^p},
\end{aligned}$$

thus, the operator A is bounded.

Lemma 2.6. *Let $p\ge 2$ and let q be its conjugate, i.e., $\frac{1}{p}+\frac{1}{q}=1$. Assume the conditions (3)-(8) are satisfied. Then, the operator*

$$A\ :\ W_1^p(\Omega)\to W_{-1}^q(\Omega)$$

is monotone.

Proof. For all $u,\ v,\ w\in W_{1,0}^p(\Omega)$, we have

$$\begin{aligned}
&\langle A(u)-A(v),\ u-v\rangle\\
&=\lambda\int_\Omega\left(u|u|^{p-2}-v|v|^{p-2}\right)(u-v)\,dx\\
&\quad+\int_\Omega\left(a_i(x,\ u,\ \nabla u)-a_i(x,\ v,\ \nabla v)\right)(\nabla_i u-\nabla_i v)\,dx\\
&\quad+\int_\Omega\left(b(x,\ u,\ \nabla u)-b(x,\ v,\ \nabla v)\right)(u-v)\,dx\\
&\ge\lambda c(p)\|u-v\|_{L^p}^p+\nu_1\|\nabla(u-v)\|_{L^p}^p\\
&\quad-\int_\Omega\left(\mu_2|\nabla(u-v)|^{p-1}+\gamma_5(x)|u-v|^{p-1}\right)(u-v)\,dx\\
&\ge\lambda c(p)\|u-v\|_{L^p}^p+\nu_1\|\nabla(u-v)\|_{L^p}^p\\
&\quad-\left(\mu_2\frac{1}{\varepsilon^p p}\|u-v\|_{L^p}^p+\mu_2\frac{\varepsilon^q}{q}\|\nabla(u-v)\|_{L^p}^p\right)\\
&\quad-\int_\Omega\gamma_5(x)|u-v|^p\,dx.
\end{aligned}$$

We assume $\gamma_5{}^{\frac{1}{2}} \in PK(\beta)$. Applying the form-boundary condition, we have

$$\begin{aligned}
&\int_\Omega \gamma_5(x)\,|u-v|^p\,dx \\
&\le \beta \int_\Omega \left(\nabla\left(|u-v|^{\frac{p}{2}}\right)\right)^2 dx + c(\beta)\int_\Omega |u-v|^p\,dx \\
&\le \beta\frac{p^2}{4}\left(\frac{2}{\varepsilon^{\frac{p}{2}}p}\,\|\nabla(u-v)\|^p_{L^p} + (p-2)\,\frac{\varepsilon^{\frac{p}{p-2}}}{p}\,\|u-v\|^p_{L^p}\right) \\
&\quad + c(\beta)\,\|u-v\|^p_{L^p},
\end{aligned}$$

so, we conclude

$$\begin{aligned}
&\langle A(u) - A(v),\ u-v\rangle \\
&\ge \left(\lambda c(p) - \mu_2\frac{1}{\varepsilon^p p} - \beta(p-2)\,\varepsilon^{\frac{p}{p-2}}\frac{p}{4} - c(\beta)\right)\|u-v\|^p_{L^p} \\
&\quad + \left(\nu_1 - \mu_2\frac{\varepsilon^q}{q} - \beta\frac{p}{\varepsilon^{\frac{p}{2}}2}\right)\|\nabla(u-v)\|^p_{L^p} > 0,
\end{aligned}$$

thus, the operator A is strictly monotone.

Lemma 2.7. *Let $p \ge 2$ and let q be its conjugate, i.e., $\frac{1}{p}+\frac{1}{q} = 1$. Assume the conditions (3)-(8) are satisfied. Then, the operator*

$$A\ :\ W^p_1(\Omega) \to W^q_{-1}(\Omega)$$

is coercive.

Proof. From the definition, we obtain

$$\begin{aligned}
&\langle A(u),\ u\rangle = \lambda\int_\Omega |u|^p\,dx \\
&+ \int_\Omega a_i(x,\ u,\ \nabla u)\,\nabla_i u dx + \int_\Omega b(x,\ u,\ \nabla u)\,u dx \\
&\ge \lambda\,\|u\|^p_{L^p} + \nu\,\|\nabla u\|^p_{L^p} - \int_\Omega \left(\mu_1\,|\nabla u|^{p-1} + \gamma_3(x)\,|u|^{p-1} + \gamma_4(x)\right) u dx \\
&\ge \left(\lambda - \mu_1\frac{1}{\varepsilon_1{}^p p}\right)\|u\|^p_{L^p} + \left(\nu - \mu_1\frac{\varepsilon_1{}^q}{q}\right)\|\nabla u\|^p_{L^p} \\
&\quad - \int_\Omega \gamma_3(x)\,|u|^p\,dx - \int_\Omega \gamma_4\,|u|\,dx,
\end{aligned}$$

for all $u \in W^p_{1,0}(\Omega)$. By form-boundary condition, we have

$$\int_\Omega \gamma_3(x)\,|u|^p\,dx \le \beta\frac{p^2}{4}\left(\frac{2}{\varepsilon^{\frac{p}{2}}p}\,\|\nabla u\|^p_{L^p} + (p-2)\,\frac{\varepsilon^{\frac{p}{p-2}}}{p}\,\|u\|^p_{L^p}\right) + c(\beta)\,\|u\|^p_{L^p},$$

thus, it follows that

$$\begin{aligned}
\langle A(u),\ u\rangle \ge &\left(\lambda - \beta\frac{p}{4}(p-2)\,\varepsilon^{\frac{p}{p-2}} - \frac{\varepsilon^p}{p} - \mu_1\frac{1}{\varepsilon_1{}^p p}\right)\|u\|^p_{L^p} \\
&+ \left(\nu - \mu_1\frac{\varepsilon_1{}^q}{q} - \beta\frac{p}{\varepsilon^{\frac{p}{2}}2}\right)\|\nabla u\|^p_{L^p} - \frac{\varepsilon^q}{q}\,\|\gamma_4\|^q_{L^q},
\end{aligned}$$

and

$$\begin{aligned}
\frac{\langle A(u),\,u\rangle}{\|u\|_{L^p}} \ge &\left(\lambda - \beta\frac{p}{4}(p-2)\,\varepsilon^{\frac{p}{p-2}} - \frac{\varepsilon^p}{p} - \mu_1\frac{1}{\varepsilon_1{}^p p}\right)\|u\|^{p-1}_{L^p} \\
&+ \left(\nu - \mu_1\frac{\varepsilon_1{}^q}{q} - \beta\frac{p}{\varepsilon^{\frac{p}{2}}2}\right)\frac{\|\nabla u\|^p_{L^p}}{\|u\|_{L^p}} - \frac{\varepsilon^q}{q}\,\frac{\|\gamma_4\|^q_{L^q}}{\|u\|_{L^p}} \overset{\|u\|_{L^p}\to\infty}{\longrightarrow} \infty,
\end{aligned}$$

so, A is a coercive operator.

Lemma 2.8. *Let $p \geq 2$ and let q be its conjugate, i.e., $\frac{1}{p}+\frac{1}{q}=1$. Assume the conditions (3)-(8) are satisfied. Then, the operator*

$$A \,:\, W_1^p(\Omega) \to W_{-1}^q(\Omega)$$

is semicontinuous.

Proof. Due to the arbitrarity of the element $v \in W_{1,0}^p(\Omega)$ in the definition of semicontinuity, we conclude that the operator $A \,:\, W_1^p(\Omega) \to W_{-1}^q(\Omega)$ is semicontinuous if the limit

$$\langle A(u+tv),\ w\rangle \overset{t\to 0}{\longrightarrow} \langle A(u),\ w\rangle$$

holds for all $u,\ v,\ w \in W_{1,0}^p(\Omega)$. So, it is sufficient to show that

$$|\langle A(u+tv) - A(u),\ w\rangle| \overset{t\to 0}{\longrightarrow} 0,$$

for all $u,\ v,\ w \in W_{1,0}^p(\Omega)$.

For $u,\ v,\ w \in W_{1,0}^p(\Omega)$, we calculate

$$\begin{aligned}
&|\langle A(u+tv) - A(u),\ w\rangle| \\
&= \lambda \int_\Omega \left((u+tv)|u+tv|^{p-2} - u|u|^{p-2}\right) w dx \\
&\quad + \int_\Omega (a_i(x,\ u+tv,\ \nabla(u+tv)) - a_i(x,\ u,\ \nabla u)) \nabla_i w dx \\
&\quad + \int_\Omega (b(x,\ u+tv,\ \nabla(u+tv)) - b(x,\ u,\ \nabla u)) w dx \\
&\leq \lambda t(p-1) \int_\Omega |v| \left(|u+tv|^{p-2} + |u|^{p-2}\right) w dx \\
&\quad + t\int_\Omega \left(\mu_2 |\nabla v|^{p-1} + \gamma_6(x)|v|^{p-1}\right) |\nabla w| dx \\
&\quad + t\int_\Omega \left(\mu_3 |\nabla v|^{p-1} + \gamma_5(x)|v|^{p-1}\right) |w| dx.
\end{aligned}$$

Applying Holder inequality and form-boundary condition, we estimate

$$\begin{aligned}
&\int_\Omega \left(\mu_2 |\nabla v|^{p-1} + \gamma_6(x)|v|^{p-1}\right) |\nabla w| dx \\
&\quad + \int_\Omega \left(\mu_3 |\nabla v|^{p-1} + \gamma_5(x)|v|^{p-1}\right) |w| dx \\
&\leq \mu_2 \|\nabla v\|_{L^p}{}^{p-1} \|\nabla w\|_{L^p} \\
&\quad + \left(\beta \frac{p^2}{4}\left(\frac{2}{\varepsilon^{\frac{p}{2}} p} \|\nabla v\|_{L^p}^p + (p-2)\frac{\varepsilon^{\frac{p}{p-2}}}{p}\|v\|_{L^p}^p\right) + c(\beta)\|v\|_{L^p}^p\right)^{\frac{1}{q}} \|\nabla w\|_{L^p} \\
&\quad + \mu_3 \|\nabla v\|_{L^p}{}^{p-1} \|w\|_{L^p} \\
&\quad + \left(\beta \frac{p^2}{4}\left(\frac{2}{\varepsilon^{\frac{p}{2}} p} \|\nabla v\|_{L^p}^p + (p-2)\frac{\varepsilon^{\frac{p}{p-2}}}{p}\|v\|_{L^p}^p\right) + c(\beta)\|v\|_{L^p}^p\right)^{\frac{1}{q}} \|w\|_{L^p}.
\end{aligned}$$

Using the Holder inequality, we deduce

$$\begin{aligned}
&|\langle A(u+tv)-A(u),\ w\rangle| \\
&\le \lambda t(p-1)\left(\|v\|_{L^p}\|u+tv\|_{L^p}{}^{p-2}\|w\|_{L^p}+\|v\|_{L^p}\|u\|_{L^p}{}^{p-2}\|w\|_{L^p}\right) \\
&\quad +t\mu_2\|\nabla v\|_{L^p}{}^{p-1}\|\nabla w\|_{L^p}+t\mu_3\|\nabla v\|_{L^p}{}^{p-1}\|w\|_{L^p} \\
&\quad +t\left(\beta\frac{p^2}{4}\left(\frac{2}{\varepsilon^{\frac{p}{2}}p}\|\nabla v\|_{L^p}^p+(p-2)\frac{\varepsilon^{\frac{p}{p-2}}}{p}\|v\|_{L^p}^p\right)+c(\beta)\|v\|_{L^p}^p\right)^{\frac{1}{q}}\|\nabla w\|_{L^p} \\
&\quad +t\left(\beta\frac{p^2}{4}\left(\frac{2}{\varepsilon^{\frac{p}{2}}p}\|\nabla v\|_{L^p}^p+(p-2)\frac{\varepsilon^{\frac{p}{p-2}}}{p}\|v\|_{L^p}^p\right)+c(\beta)\|v\|_{L^p}^p\right)^{\frac{1}{q}}\|w\|_{L^p} \\
&\overset{t\to 0}{\longrightarrow} 0.
\end{aligned}$$

3. The Minty-Browder theorem

To complete our investigation, we present the scheme of the proof of the existence and uniqueness of the solution. The proof is based on Minty's ideas [9, 16] and on a variant of the Galerkin method, which are applied to the operator $A\ :\ W_1^p(\Omega)\to W_{-1}^q(\Omega)$, and employment of limits in weak topology.

Theorem 3.1 (Minty-Browder). *Let $p\ge 2$ and q its conjugate i.e., $\frac{1}{p}+\frac{1}{q}=1$. Assume conditions (3) – (8) are satisfied. Then the elliptic equation (2) has a unique weak solution in the Sobolev space $W_1^p(\Omega)$.*

Proof. Let

$$A\ :\ W_1^p(\Omega)\to W_{-1}^q(\Omega)$$

be the operator associated with equation (2.1). Under conditions (2.2)-(2.8), the operator

$$A\ :\ W_1^p(\Omega)\to W_{-1}^q(\Omega)$$

is a bounded, monotone, coercive, and semi-continuous operator. Thus, we are going to show that there exists a solution $u\in W_1^p(\Omega)$ to the operator equation $A(u)=\varphi$ for each fixed $\varphi\in W_{-1,0}^q$.

Let $\{v_i\}$ be a basis in $W_{1,0}^p$ and X_k be a linear span of $\{v_1,\ ...,\ v_k\}$. We compose the nonlinear Galerkin approximation system

$$\langle A(u_k)-\varphi,\ v_i\rangle=0,$$

where $u_k\in X_k$, $i=1,\ ...,\ k$ so we denote $u_k=\sum_{i=1,\dots,k}c_{ik}v_i$ with coefficients c_{ik} to be calculated.

Since the operator A is coercive there exists a number $R>0$ such that

$$\langle A(u)-\varphi,\ u\rangle>0$$

for all $u\in W_{1,0}^p,\quad \|u\|\ge R$. Therefore, we have the system

$$\left\langle A\left(\sum_{i=1,\dots,k}c_{ik}v_i\right)-\varphi,\ v_j\right\rangle=0$$

with an unknown real vector $\{c_{1k},\ ...,\ c_{kk}\}$. The function

$$u\mapsto\langle A(u)-\varphi,\ v_i\rangle$$

is continuous on $W^p_{1,0}$ with respect to variables $\{c_{1k},\ ...,\ c_{kk}\}$. The system

$$\sum_{j=1,\dots,k}\left\langle A^{p(\cdot)}_{\lambda}\left(\sum_{i=1,\dots,k}c_{ik}v_i\right)-\varphi,\ v_j\right\rangle c_{jk}>0$$

has a solution for all $u_k\in X_k$, $i=1,\ ...,\ k$ such that $u_k\in W^p_{1,0},\quad \|u_k\|_{W^p_1}=R$.

The fixed point theorem states that: let function

$$\left\langle A\left(\sum_{i=1,\dots,k}c_{ik}v_i\right)-\varphi,\ v_j\right\rangle\ :\ clos\left(B\left(0,\ R\right)\right)\to\mathbb{R}$$

be continuous for each $j=1,\ ...,\ k$ and

$$\sum_{j=1,\dots,k}\left\langle A\left(\sum_{i=1,\dots,k}c_{ik}v_i\right)-\varphi,\ v_j\right\rangle c_{jk}>0$$

for all $u_k\in W^p_{1,0}$, $\|u_k\|_{W^p_1}=R$; then the system

$$\left\langle A\left(\sum_{i=1,\dots,k}c_{ik}v_i\right)-\varphi,\ v_j\right\rangle=0$$

has a solution for all $u_k\in W^p_{1,0},\quad \|u_k\|_{W^p_1}\le R$. □

From

$$\langle A\left(u\right)-\varphi,\ u\rangle=0$$

and statement that

$$\langle A\left(u\right)-\varphi,\ u\rangle>0$$

for all $u\in W^p_{1,0},\quad \|u\|_{W^p_1}\ge R$, we deduce that $\|u\|_{W^p_1}\le R$, which provides us with a priori solution estimate.

The sequences $\{u_k\}$ and $\{A\left(u_k\right)\}$ are bounded since the operator A is bounded, therefore, $A\left(u_k\right)\overset{weakly}{\longrightarrow}\varphi$ in $W^q_{-1,0}$ and there exists a subsequence $\left\{u_{\tilde{k}}\right\}\subset\{u_k\}$ such that $u_{\tilde{k}}\overset{weakly}{\longrightarrow}u$. Thus, we have

$$\left\langle A\left(u_{\tilde{k}}\right),\ u_{\tilde{k}}\right\rangle=\left\langle\varphi,\ u_{\tilde{k}}\right\rangle\overset{\tilde{k}\to\infty}{\longrightarrow}\langle\varphi,\ u\rangle\,.$$

In finite-dimensional Banach spaces, the strong and weak convergences coincide. We choose subsequence $\left\{u_{\tilde{k}}\right\}\subset\{u_k\}$ such that $u_{\tilde{k}}\overset{weakly}{\longrightarrow}u$ and

$$A\left(u_{\tilde{k}}\right)\overset{weakly}{\longrightarrow}\varphi,$$

and

$$\left\langle A\left(u_{\tilde{k}}\right),\ u_{\tilde{k}}\right\rangle\overset{\tilde{k}\to\infty}{\longrightarrow}\langle\varphi,\ u\rangle\,.$$

So, we have

$$u_{\tilde{k}}\overset{weakly}{\longrightarrow}u$$

and

$$A\left(u_{\tilde{k}}\right) \overset{weakly}{\longrightarrow} A\left(u\right).$$

Thus, the equation $A\left(u\right) = 0$ has a solution in $W_1^p\left(\Omega\right)$, which proves the existence of a weak solution to elliptic equation (2.1) under the conditions (2.2)-(2.8).

Let $u \in W_1^p\left(\Omega\right)$ and $v \in W_1^p\left(\Omega\right)$ be two different solution to (2.1) so that $A\left(u\right) = 0$ and $A\left(v\right) = 0$. On another hand, the strict monotony yields that from

$$\langle A\left(u\right) - A\left(v\right),\ u - v\rangle = 0$$

follows $u = v$, thus we have proved the uniqueness of the solution.

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