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Threshold results of blow-up solutions to Kirchhoff equations with variable sources

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Abstract. This paper analyzes an initial boundary value problem for variable source Kirchhoff-type parabolic equations. We aim to derive a new sub-critical energy threshold for finite-time blow-up, a new blow-up condition, and estimates for lifespan and upper bounds for blow-up time across various initial energy cases.

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1. Introduction

In recent years, there has been a significant interest among numerous mathematical researchers in examining the blow-up time properties of solutions to equations used for describing the transverse vibrations of a stretched string while taking into account the change in the string length. These equations, proposed by Kirchhoff [19], [26] are widely employed in engineering disciplines like automotive, aerospace, and large-scale structures. The extensive applications of these materials have led to a growing desire among researchers to establish findings related to the presence and control of elasticity problems. Almeida Junior et al. [25] studied polynomial stability for the equations of porous elasticity in one-dimensional bounded domains. Iesan et al. [16, 17, 18] studied the theory of thermoelastic materials with voids. Santos et al. [30] considered a porous elastic system with porous dissipation In recent years, there has been a significant amount of research focused on developing mathematical models for nonlocal diffusion. These models are formulated by using parabolic equations that combine linear or nonlinear diffusion with a Kirchhoff term. The Kirchhoff problems are a type of problem

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that includes the term $M\left(\int_\Omega |\nabla u|^2 \mathrm{d}x\right)$, which causes the equation to no longer be a pointwise identity

$$\begin{cases} u_t - M\left(\int_\Omega |\nabla u|^2 \mathrm{d}x\right)\Delta u = g\left(x,u\right), & (x,t)\in\Omega\times(0,T),\\ u=0, & (x,t)\in\partial\Omega\times(0,T),\\ u(x,0)=u_0(x), & x\in\Omega, \end{cases}$$

The Kirchhoff problems are a type of problem that includes the term $M\left(\int_\Omega |\nabla u|^2 \mathrm{d}x\right)$, which causes the equation to no longer be a pointwise identity. The nonlinear Kirchhoff equation (NLKE) is a partial differential equation used to describe the transverse vibrations of a stretched string while taking into account the change in the string length [19]. It is also used to describe the movement of a semi-infinite string [26] and is an underlying equation of quantum mechanics. Partial differential equations have a wide range of applications, as listed in reference [33]. The study of Kirchhoff equations has a long history and was examined in detail in Lions research [23], where it became possible to investigate the existence, uniqueness, and regularity of the solutions in Kirchhoff's equations. For more information, interested readers can refer to [10, 11, 24] and the references therein. This paper studies a parabolic problem with a nonlocal diffusion coefficient, where a nonlinear source term modeled by an operator appears in the Kirchhoff equation.

$$\begin{cases} u_t - M\left(\int_\Omega |\nabla u|^2 \mathrm{d}x\right)\Delta u = |u|^{q(x)-1}u, & (x,t)\in\Omega\times(0,T),\\ u=0, & (x,t)\in\partial\Omega\times(0,T),\\ u(x,0)=u_0(x), & x\in\Omega, \end{cases} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^n$ $(n\geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$, we assume that $u_0\in H_0^1(\Omega)$ and $u_0(x)\not\equiv 0$, the diffusion coefficient has the specific form $M(s)=a+bs$ with positive parameters a,b, $\Omega\subset\mathbb{R}^n, q$ is constant and satisfy

$$(\mathcal{H}_1) 3<q_1\leq q\left(x\right)\leq q_2\leq \frac{n+2}{n-2}\text{if } \ n\geq 3,\ x\in\Omega; \tag{1.2}$$

$$(\mathcal{H}_2)\ \ 1<q_1\leq q\left(x\right)\leq q_2<3 \quad \text{if } n\geq 1,\ x\in\Omega.$$

We consider a mathematical model, where u_0 belongs to the Sobolev space $H_0^1(\Omega)$ and $-\Delta$ denotes the Laplace operator concerning the spatial variables. Our focus is on the explosion property in finite time. To this end, we use the potential well method and various inequality techniques to establish the the blow-up of weak solutions within a finite time and obtain a new blow-up criterion. Additionally, we determine the lifespan and an upper bounds for the blow-up time in different initial energy cases. It is important to note that the model (1.1) is called degenerate when $a=0$, and when $a>0$, we refer to it as a non-degenerate model. The exponent $q(.)$ is a measurable function on Ω that satisfies certain conditions.

$$1<q_1=ess\inf_{x\in\Omega} q\left(x\right)\leq q\left(x\right)\leq q_2=ess\sup_{x\in\Omega} q\left(x\right)<\infty, \tag{1.3}$$

and the following Zhikov–Fan uniform local continuity condition. There exist a constant $k>0$ such that for all points x, y in Ω with $0<|x-y|<\frac{1}{2}$, we have the inequality

$$|q(x) - q(y)| \leq k(|x-y|), \tag{1.4}$$

where $k(r)$ satisfies

$$\limsup_{r\to 0^+} k(r) \ln\left(\frac{1}{r}\right) = c < \infty.$$

This problem has its origin in the mathematical explanation of system in real world from the mathematical modeling for axially moving viscoelastic materials, they appear in numerous applications in the natural sciences, for instance models of flows of electro-rheological fluids or fluids with temperature-dependent viscosity, nonlinear viscoelasticity, filtration processes through a porous media [3, 4, 28], and the processing of digital images [2, 9, 22], and can all be linked with problem (1.1), further details on the subject can be seen in [5, 6, 29] and the other references contained therein. In recent years, the study of mathematical nonlinear models with variable exponent nonlinearity has attracted the attention of many researchers. Let us highlight some of these issues. For example, Pinasco [27] established the local existence of positive solutions for the parabolic problem.

$$\begin{cases} u_t - \Delta u = f(u) \text{ in} & \Omega \times (0,T) \\ u = 0, \text{ in} & \partial\Omega \times (0,T) \\ u(x,0) = u_0(x) \text{ in } \Omega \end{cases}$$

where the source term is of the form

$$f(u) = a(x)u^{p(x)} \quad \text{or} \quad f(u) = a(x)\int_\Omega u^{q(y)}dy.$$

He also proved that with sufficiently large initial data, solutions blow up in a finite time. Alaoui et al. [21] considered the following nonlinear heat equation,

$$u_t - div\left(|\nabla u(x)|^{m(x)-2}\nabla u\right) = |u|^{p(x)-2}u + f.$$

Under appropriate conditions on m and p, and with $f = 0$, they demonstrated that any solution with a nontrivial initial condition will experience a blow-up in finite time. Additionally, they provided numerical examples in two dimensions to illustrate their findings. Autuori et al. [7] investigated a nonlinear Kirchhoff system involving the $p(x,t)$-Laplace operator, a nonlinear force $f(t,x,u)$, and a nonlinear damping term $Q = Q(t,x,u,u\_t)$. They established a global nonexistence result under suitable conditions on f, Q, and p. In the classical case of constant exponent ($q(x)$ =constant= q), this equation has its origin in the nonlinear vibration of an elastic string, were the source term $u^{q-1}u$ forces the negative-energy solutions to explode in finite time. It's known that several authors have looked at problem (1.1) concerning the findings of the global existence and blow-up of solutions, and a powerful method for treating it is the "potential well method," which was founded by the first author Sattinger [31] in 1968 and later been enhanced by Liu and Zhao [32] by introducing the so-called family of potential wells which later became a significant technique for the study of nonlinear evolution equations and has also given many interesting results. Recently, authors of [14, 15] discussed in a bounded domain of $\mathbb{R}^n$ with $3 < q < \frac{n+2}{n-2}$ the global existence and finite time blow-up of solutions to problem

(1.1) when the initial data are at different energy levels $E(u_0) < d$, $E(u_0) = d$, and $E(u_0) > d$ respectively. If we know that the solutions of a given system explode in finite time, it is important to estimate the bounds of the explosion time from both above and below, which is the main goal of this work. We will expand the assumptions about the given q in the aforementioned works, assuming a new assumption on the critical exponent $q(.)$ such that $1 < q_1 \leq q(x) \leq q_2 < (n+2)/(n-2)$, under some sufficient conditions we giving a new blow-up criterion for problem (1.1) if the initial energy is not -negative, and derive the upper and lower bounds of this blow-up time. The table below provides a summary of the background for our work.

Table 1: Main results.

Main results	q	Initial data		Blow-up
Theorem 2	$(\mathcal{H}_1)$	$\left(\frac{B_1}{\sqrt{a}}\right)^{-q_1+1}$ (2.4) $\geq \begin{pmatrix} a\Vert\nabla u_0\Vert_2^2 \\ +\frac{b}{2}\Vert\nabla u_0\Vert_2^4 \end{pmatrix}^{\frac{q_1+1}{2}}$ $> \alpha_1$	$\mathrm{E}(u_0) < E_1$, E_1 as in (2.4)	Blow-up (2.6) $\lim\limits_{t\to\widehat{T}} \Vert u(t)\Vert_2^2 = \infty$
Theorem 3	$(\mathcal{H}_1)$	$u_0 \in H_0^1(\Omega)$, $u_0 \neq 0$	$\mathrm{E}(u_0) < 0$ $\mathrm{E}(u_0) \leq \mathrm{E}_d$, E_d as in (1.11) $0 \leq \mathrm{E}(u_0)$ $< C_0 \Vert u_0\Vert_2^2$ **(iii)**	Blow-up $\lim\limits_{t\to T^*} \int_0^t \Vert u(\tau)\Vert_2^2 \mathrm{d}\tau$ $= \infty$
Theorem 3	$(\mathcal{H}_2)$	$u_0 \in H_0^1(\Omega)$	$\mathrm{E}(u_0)$ $< -\frac{q_1+1}{q_1+5}\frac{b}{4\varepsilon}c(\varepsilon)$ (2.21),(2.22)	Blow-up $\lim\limits_{t\to T} \Vert u(t)\Vert_2^2 = \infty.$

Table 2: The estimate of blow-up time.

$E(u_0)$	Upper bound estimate	Lower bound estimate
$\mathrm{E}(u_0) < E_1$	$\surd$	
$\mathrm{E}(u_0) < 0$	$\surd$	
$\mathrm{E}(u_0) = 0$	?	?
$\mathrm{E}(u_0) < E_d$	$\surd$	
$0 \leq \mathrm{E}(u_0) < C_0 \Vert u_0\Vert_2^2$	$\surd$	
$\mathrm{E}(u_0) < -\frac{q_1+1}{q_1+5}\frac{b}{4\varepsilon}c(\varepsilon)$	$\surd$	

1.1. Modified potential wells

For $u \in H_0^1(\Omega)$, we define the functionals

$$\begin{aligned} \mathrm{E}(u\,(t)) =: \mathrm{E}(t) =& \frac{a}{2}\|\nabla u\|_2^2 + \frac{b}{4}\|\nabla u\|_2^4 - \int_\Omega \frac{1}{q(x)+1}|u|^{q(x)+1}\mathrm{d}x, \\ I(u\,(t)) =& a\|\nabla u\|_2^2 + b\|\nabla u\|_2^4 - \int_\Omega |u|^{q(x)+1}\mathrm{d}x. \qquad (1.5) \\ \mathrm{M}(u\,(t)) =: \mathrm{M}(t) =& \frac{1}{2}\|u\,(t)\,\|_2^2. \end{aligned}$$

and testing (1.1) by u_{t} we have $E(t)$ is nonincreasing, i.e.,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{E(t)} = -\,\|u_t(t)\|_2^2 \le 0, \tag{1.6}$$

and

$$\mathrm{E}(t) + \int_0^t \|u_t(t)\|_2^2\,\mathrm{d}s \le \mathrm{E}(u_0) \quad \text{a.e. } t \in (0,T), \tag{1.7}$$

$$L'(t) = -I(u\,(t)) \quad \text{a.e. } t \in (0,T). \tag{1.8}$$

We then have the following lemma.

Lemma 1.1. *For $q(x)$ be (1.4) and $u \in H_0^1(\Omega)\backslash\{0\}$. Let $F : [0,+\infty) \to \mathbb{R}$ the Euler functional defined by*

$$F(\lambda) = \frac{\lambda^2}{2}a\|\nabla u\|^2 + \frac{\lambda^4}{4}b\|\nabla u\|^4 - \int_\Omega \frac{\lambda^{q(x)+1}}{q(x)+1}|u|^{q(x)+1}\mathrm{d}x,$$

then, F keeps the following properties:

(i) . $\lim_{\lambda\to 0^+} F(\lambda) = 0$ *and* $\lim_{\lambda\to+\infty} F(\lambda) = -\infty$.

(ii). *There is at least one solution to the equation $F'(\lambda) = 0$ on the interval $[\lambda_1, \lambda_2]$, where*

$$\lambda_1 = \min\left[\rho\,(u)^{\frac{-1}{1-q_2}}, \rho\,(u)^{\frac{-1}{3-q_1}}\right],\ \lambda_2 = \max\left[\rho\,(u)^{\frac{-1}{1-q_2}}, \rho\,(u)^{\frac{-1}{3-q_1}}\right], \tag{1.9}$$

and

$$\rho\,(u) := \frac{a\|\nabla u\|^2 + b\|\nabla u\|^4}{\int_\Omega |u|^{q(x)+1}\mathrm{d}x}.$$

(iii) . *There exists a $\lambda^* = \lambda^*(u) > 0$ such that $F(\lambda)$ gets its maximum at $\lambda = \lambda^*$. Furthermore, we have that $0 < \lambda^* < 1$, $\lambda^* = 1$ and $\lambda^* > 1$ provided $I(u) < 0$, $I(u) = 0$ and $I(u) > 0$, respectively.*

Proof. Since $q(x) \in C_+(\bar\Omega) = \left\{p \in C(\bar\Omega) : \inf_{x\in\bar\Omega} q\,(x) > 3\right\}$, the assertion (i) is shown by the following:

$$F(\lambda) \le \frac{\lambda^2}{2}a\|\nabla u\|^2 + \frac{\lambda^4}{4}b\|\nabla u\|^4 - \min\left\{\lambda^{q_1+1}, \lambda^{q_2+1}\right\}\int_\Omega \frac{1}{q(x)+1}|u|^{q(x)+1}\mathrm{d}x,$$

and

$$F(\lambda) \ge \frac{\lambda^2}{2}a\|\nabla u\|^2 + \frac{\lambda^4}{4}b\|\nabla u\|^4 - \max\left\{\lambda^{q_1+1}, \lambda^{q_2+1}\right\}\int_\Omega \frac{1}{q(x)+1}|u|^{q(x)+1}\mathrm{d}x,$$

For (ii). We have

$$F'(\lambda) = \lambda a \int_\Omega |\nabla u(x)|^2 \mathrm{d}x + \lambda^3 b \|\nabla u\|^4 - \int_\Omega \lambda^{q(x)} |u|^{q(x)+1} \mathrm{d}x,$$

which implies that $F'(\lambda)$ lies in the following two inequalities

$$\begin{aligned} F'(\lambda) &\geq \lambda a \int_\Omega |\nabla u(x)|^2 \mathrm{d}x + \lambda^3 b \|\nabla u\|^4 - \max\{\lambda^{q_1}, \lambda^{q_2}\} \int_\Omega |u|^{q(x)+1} \mathrm{d}x \\ &= \max\{\lambda^{q_1}, \lambda^{q_2}\} \left[\begin{array}{r} \min\left\{\lambda^{1-q_1}, \lambda^{1-q_2}\right\} a \int_\Omega |\nabla u(x)|^2 \mathrm{d}x \\ + \min\left\{\lambda^{3-q_1}, \lambda^{3-q_2}\right\} a \int_\Omega |\nabla u(x)|^4 \mathrm{d}x - \int_\Omega |u|^{q(x)+1} \mathrm{d}x \end{array} \right], \end{aligned}$$

and

$$\begin{aligned} F'(\lambda) &\leq \lambda a \int_\Omega |\nabla u(x)|^2 \mathrm{d}x + \lambda^3 b \|\nabla u\|^4 - \min\{\lambda^{q_1}, \lambda^{q_2}\} \int_\Omega |u|^{q(x)+1} \mathrm{d}x \\ &= \min\{\lambda^{q_1}, \lambda^{q_2}\} \left[\begin{array}{r} \max\left\{\lambda^{1-q_1}, \lambda^{1-q_2}\right\} a \int_\Omega |\nabla u(x)|^2 \mathrm{d}x \\ + \max\left\{\lambda^{3-q_1}, \lambda^{3-q_2}\right\} a \int_\Omega |\nabla u(x)|^4 \mathrm{d}x - \int_\Omega |u|^{q(x)+1} \mathrm{d}x \end{array} \right], \end{aligned}$$

Since $q_2 \geq q_1 > 3$, we signify that $F'(\lambda)$ has at least one zero point λ satisfying (1.9). So we get (ii). The definition of λ^* and the relation $I(\lambda u) = \lambda F'(\lambda)$ and

$$F'(\lambda) \leq (\lambda - \lambda^{q_2})\, a \int_\Omega |\nabla u(x)|^2 \mathrm{d}x + \left(\lambda^3 - \lambda^{q_2}\right) b \int_\Omega |\nabla u(x)|^4 \mathrm{d}x + \lambda^{q_2} I(u), \text{ for } \lambda \in (0,1),$$

and

$$F'(\lambda) \geq (\lambda - \lambda^{q_2})\, a \int_\Omega |\nabla u(x)|^2 \mathrm{d}x + \left(\lambda^3 - \lambda^{q_2}\right) b \int_\Omega |\nabla u(x)|^4 \mathrm{d}x + \lambda^{q_2} I(u), \text{ for } \lambda \in (1,\infty),$$

lead to the last claim (iii). Completeness of the proof. □

1.2. Assumptions and main results

As E is the Fréchet-differentiable functional with derivative E', let suppose that $u \neq 0$ is a critical point of E, i.e., $E'(u) = 0$. Then necessarily u is contained in the set

$$\mathcal{N} = \left\{u \in H_0^1(\Omega) \backslash \{0\} : I(u) = \langle \mathrm{E}'(u), u \rangle = 0\right\},$$

so $\mathcal{N}$ is a natural constraint for the problem of finding nontrivial critical points of E, $\mathcal{N}$ is called the Nehari manifold associated with the energy functional E. By Lemma 1.1 we know that $\mathcal{N}$ is not empty set. It is clear that $E(u)$ is coercive on $\mathcal{N}$. The depth of the potential well, denoted as d, characterized by

$$d = \inf_{u \in \mathcal{N}} \mathrm{E}(u). \tag{1.10}$$

Under the appropriate conditions we have d is a positive finite number and is therefore well-defined. For E_d is a constant given by

$$\mathrm{E}_d = \frac{q_1 - 1}{q_1 + 1} \frac{q_2 + 1}{q_2 - 1} d \leq d, \tag{1.11}$$

we define the modified stable and unstable sets as follows

$$\mathcal{W}=\left\{u\in H_0^1(\Omega):\mathrm{E}(u)<\mathrm{E}_d,\ I(u)>0\right\}\cup\{0\},$$
$$\mathcal{U}=\left\{u\in H_0^1(\Omega):\mathrm{E}(u)<\mathrm{E}_d,\ I(u)<0\right\}.$$

2. Blow-up and bounds of blow-up time

In this section, we get new bounds for the blow-up time to problem (1.1) if the variable exponent $q(.)$ and the initial data satisfy some conditions. Before stating our main results, without proof, we preferably give the following theorem of existence and uniqueness, as well as the regularity:

Definition 2.1 (Weak solution). [20]*A function $u(x,t)$ is said to be a weak solution of problem* (1.1) *defined on the time interval* $[0,T]$, *provide that* $u(x,t)\in L^{\infty}\left(0,T;H_0^1(\Omega)\right)$ *with* $u_t\in L^2\left(0,T;L^2(\Omega)\right)$, *if for every test-function* $\eta\in H_0^1(\Omega)$ *and a.e.* $t\in[0,T]$, *the following identity holds:*

$$(u_t,\eta)_\Omega+\left(a+b\|\nabla u\|_2^2\right)(\nabla u,\nabla\eta)_\Omega=\left(|u|^{q(x)-1}u,\eta\right)_\Omega,\ \ a.e.\ t\in(0,T),\tag{2.1}$$

with $u(x,0)=u_0\in H_0^1(\Omega)$.

Without proof, we give the local existence of a solution of (1.1) that can be obtained by the Faedo-Galerkin methods together with the Banach fixed point theorem [1, 8].

Theorem 2.2. *Assume that* (1.3)-(1.4) *hold. Then the problem* (1.1) *for given* $u_0\in H_0^1(\Omega)$ *admits a unique local solution*

$$u\in C\left([0,T_{\max});H_0^1(\Omega)\right),\ u_t\in C\left([0,T_{\max});L^2(\Omega)\right),$$

where $T_{\max}>0$ is the maximal existence time of $u(t)$.

2.1. Function spaces and lemmas

In this section, we present some preliminary concepts and notations that we shall employ in our further analysis. Let us start by introducing the variable-order Lebesgue space $L^{p(.)}(\Omega)$, which is defined for all $p:\Omega\to[1,+\infty]$ a measurable function as

$$L^{p(.)}(\Omega):=\left\{u:\Omega\to\mathbb{R}\text{ measurable }:\int_\Omega|u(x)|^{p(x)}\,\mathrm{d}x<+\infty\right\}.$$

We then know that $L^{p(.)}(\Omega)$ is a Banach space, equipped with the Luxemburg-type norm

$$\|u\|_{p(.)}:=\inf\left\{\lambda>0,\quad\int_\Omega\left|\frac{u(x)}{\lambda}\right|^{p(x)}\mathrm{d}x\le1\right\}.$$

Next, we define the variable-order Sobolev space $W^{1,p(.)}(\Omega)$ as

$$W^{1,p(.)}(\Omega):=\left\{u\in L^{p(.)}(\Omega)\ :\ \nabla u\in L^{p(.)}(\Omega)\right\},$$

equipped with the norm

$$\|u\|_{W^{1,p(.)}(\Omega)}=\|u\|_{p(.)}^2+\|\nabla u\|_{p(.)}^2.\tag{2.2}$$

Moreover, in what follows we will need the following embedding result from [12, 13].

Lemma 2.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded regular domain. It holds the following.*

1. *If $p \in C(\overline{\Omega})$ and $q : \Omega \to [1, +\infty)$ is a measurable function such that*
$$\underset{x\in\Omega}{\text{ess inf}}\left(p^*(x) - q(x)\right) > 0,$$
with p^* defined as in (1.2)*, then $W_0^{1,p(.)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ with continuous and compact embedding.*
2. *If p satisfy* (1.3)*, then $\|u\|_{p(.)} \leq C\|\nabla u\|_{p(.)}$ for all $u \in W_0^{1,p(.)}(\Omega)$. In particular, $\|u\|_{1,p(.)} = \|\nabla u\|_{p(.)}$ defines a norm on $W_0^{1,p(.)}(\Omega)$ which is equivalent to* (2.2)*.*

It is not difficult to set up the following lemma's, so we will ignore its proof here.

Lemma 2.4. *Allow* (1.3)-(1.4) *to apply. Let $u(t) := u(x, t)$ be a local solution to problem* (1.1)*. Then, the following assertions hold:*

- **(i).** *If there is a time $t_0 \in [0, T_{\max})$ such that $u(t_0) \in \mathcal{W}$ and $\mathrm{E}(t_0) < d$, then $u(t)$ stays within the set $\mathcal{W}$ for all $t \in [t_0, T_{\max})$.*
- **(ii).** *If there is a time $t_0 \in [0, T_{\max})$ such that $u(t_0) \in \mathcal{U}$ and $\mathrm{E}(t_0) < d$, then $u(t)$ stays within the set $\mathcal{U}$ for all $t \in [t_0, T_{\max})$.*

Lemma 2.5. *Suppose that a positive, twice-differentiable function $\varphi(t)$ satisfies on $t \geq 0$ the inequality*
$$\varphi''\varphi - (1 + \alpha)(\varphi')^2 \geq 0, \ \alpha > 0.$$
If
$$\varphi(0) > 0, \ \text{and } \varphi'(0) > 0,$$
then, then there exists $t_1 \in \left(0, \frac{\varphi(0)}{\alpha\varphi'(0)}\right)$ such that
$$\varphi(t) \to \infty \ \text{as } t \to t_1.$$

Lemma 2.6. *Let Ω be a bounded domain of $\mathbb{R}^n$, $q(.)$ satisfies* (1.2) *and* (1.4)*, then*
$$B\|\nabla u\|_2 \geq \|u\|_{q(.)+1}, \ \text{for all } u \in W_0^{1,2}(\Omega). \tag{2.3}$$
where the optimal constant of Sobolev embedding B is depends on $q_{1,2}$ and $|\Omega|$.

Lemma 2.7. *Assuming (u_0, u_1) are in $H_0^1(\Omega) \times L^2(\Omega)$ and that u_0 is an element of $\mathcal{U}$, the following holds:*
$$d \leq \left(\frac{1}{2} - \frac{1}{q_2+1}\right) a\|\nabla u(t)\|_2^2 + \left(\frac{1}{4} - \frac{1}{q_2+1}\right) b\|\nabla u(t)\|_2^4, \quad \text{for } t \in [0, T_{\max}).$$

Proof. Because $u_0 \in \mathcal{U}$, according to Lemma 2.4 $u(t) \in \mathcal{U}$ for $t \in [0, T_{\max})$ and thus $I(u(t)) < 0$. By Lemma 1.1 there exists $\lambda^* \in (0, 1)$ such that $I(\lambda^* u) = 0$, i.e.
$$\int_\Omega (\lambda^*)^{q(x)+1} |u(t)|^{q(x)+1} \mathrm{d}x = a(\lambda^*)^2 \|\nabla u\|_2^2 + b(\lambda^*)^4 \|\nabla u\|_2^4$$

Thanks to $\lambda^* < 1$ we can derive from the definition of d

$$\begin{aligned} d \leq \mathrm{E}\left(\lambda^* u(t)\right) &= a\frac{\left(\lambda^*\right)^2}{2}\|\nabla u(t)\|_2^2 + b\frac{\left(\lambda^*\right)^4}{4}\|\nabla u\|_2^4 - \int_\Omega \frac{\left(\lambda^*\right)^{q(x)+1}}{q(x)+1}|u(t)|^{p(x)}\mathrm{d}x \\ &\leq a\frac{\left(\lambda^*\right)^2}{2}\|\nabla u(t)\|_2^2 + b\frac{\left(\lambda^*\right)^4}{4}\|\nabla u\|_2^4 - \frac{1}{q_2+1}\int_\Omega \left(\lambda^*\right)^{q(x)+1}|u(t)|^{q(x)+1}\mathrm{d}x \\ &= \left(\frac{1}{2} - \frac{1}{q_2+1}\right)\left(\lambda^*\right)^2 a\|\nabla u(t)\|_2^2 + \left(\frac{1}{4} - \frac{1}{q_2+1}\right)\left(\lambda^*\right)^4 b\|\nabla u(t)\|_2^4 + \frac{1}{q_2+1}I\left(\lambda^* u(t)\right) \\ &\leq \left(\frac{1}{2} - \frac{1}{q_2+1}\right) a\|\nabla u(t)\|_2^2 + \left(\frac{1}{4} - \frac{1}{q_2+1}\right) b\|\nabla u(t)\|_2^4. \end{aligned}$$

The proof is completed. □

Suppose there are positive constants B_1, α_1, α_0, and E_1 that satisfy the following argument:

$$\begin{aligned} B_1 = \max\left(1, B\right),\ \alpha_0 = \sqrt{a\|\nabla u_0\|_2^2 + \frac{b}{2}\|\nabla u_0\|_2^4}, \\ \alpha_1 = \left(\frac{B_1^2}{a}\right)^{-\frac{q_1+1}{2(q_1-1)}}, E_1 = \left(\frac{1}{2} - \frac{1}{q_1+1}\right)\alpha_1^2. \end{aligned} \tag{2.4}$$

Based on equations (2.3) and (2.1), we can come to a conclusion that

$$\begin{aligned} \mathrm{E}\left(t\right) \geq& \frac{a}{2}\|\nabla u\|_2^2 + \frac{b}{4}\|\nabla u\|_2^4 - \frac{1}{q_1+1}\max\left(\|u\|_{q(.)+1}^{q_2+1}, \|u\|_{q(.)+1}^{q_1+1}\right) \\ \geq& \frac{a}{2}\|\nabla u\|_2^2 + \frac{b}{4}\|\nabla u\|_2^4 - \frac{1}{q_1+1}\max\left(\left(B_1^2\|\nabla u\|_2^2\right)^{\frac{q_2+1}{2}}, \left(B_1^2\|\nabla u\|_2^2\right)^{\frac{q_1+1}{2}}\right) \\ \geq& \frac{1}{2}\alpha^2 - \frac{1}{q_1+1}\max\left(\left(\frac{B_1^2}{a}\right)^{\frac{q_2+1}{2}}\alpha^{q_2+1}, \left(\frac{B_1^2}{a}\right)^{\frac{q_1+1}{2}}\alpha^{q_1+1}\right) := g\left(\alpha\right)\ \forall\alpha \geq 0, \end{aligned} \tag{2.5}$$

where $\alpha = \sqrt{a\|\nabla u\|_2^2 + \frac{b}{2}\|\nabla u\|_2^4}$.

To the best of our knowledge, no evidence has been found regarding the blow-up of solutions to this equation in $\mathbb{R}^n$, given the initial data at a high energy level. This paper aims to investigate this matter by examining the finite-time explosion of weak solutions in the initial boundary value problem provided.

In the following sections, we will present our main theorems

For $3 < q_1 \leq q\left(x\right) \leq q_2 \leq \frac{n+2}{n-2}$, we have the following result

2.2. Results on the blow-up time

Theorem 2.8. *Supposed that q satisfies $(\mathcal{H}_1)$. If $u_0 \neq 0$ is chosen in such a way that $\mathrm{E}(u_0) < E_1$ and $\left(\frac{B_1}{\sqrt{a}}\right)^{-q_1+1} \geq \left(a\|\nabla u_0\|_2^2 + \frac{b}{2}\|\nabla u_0\|_2^4\right)^{\frac{q_1+1}{2}} > \alpha_1$. Then the solution of the problem (1.1) will eventually blow-up in finite time T. Moreover, the blow-up* *time T can be estimated from above by $\widehat{T}$, where*

$$\widehat{T} = \max\left(\frac{(q_1+1)\,|\Omega|^{\frac{q_1-2}{2}}\left(\int_\Omega u_0^2 \mathrm{d}x\right)^{\frac{1-q_1}{2}}}{(q_1-1)\,(q_1-3)\left(1-\left((q_1+1)\left(\frac{1}{2}-\frac{\mathrm{E}(u_0)}{\alpha_1^2}\right)\right)^{\frac{-q_1-1}{q_1-1}}\right)}, \frac{(q_1+1)\,|\Omega|^{\frac{q_1-2}{2}}\left(\int_\Omega u_0^2 \mathrm{d}x\right)^{\frac{1-q_2}{2}}}{(q_2-1)\,(q_1-3)\left(1-\left((q_1+1)\left(\frac{1}{2}-\frac{\mathrm{E}(u_0)}{\alpha_1^2}\right)\right)^{\frac{-q_1-1}{q_1-1}}\right)}. \right) \tag{2.6}$$

Lemma 2.9. *Let define $h : [0, +\infty) \to \mathbb{R}$ as*

$$h(\alpha) = \frac{1}{2}\alpha^2 - \frac{1}{q_1+1}\left(\frac{B_1^2}{a}\right)^{\frac{q_1+1}{2}} \alpha^{q_1+1}. \tag{2.7}$$

Then, under the assumptions of Theorem 2.8, the following properties hold :

(i). *h is increasing for $0 < \alpha \le \alpha_1$ and decreasing for $\alpha \ge \alpha_1$;*
(ii). *$\lim_{\alpha\to+\infty} h(\alpha) = -\infty$ and $h(\alpha_1) = E_1$.*

Proof. By the assumption that $B_1 > 1$ and $p_1 > 1$, $h(\alpha) = g(\alpha)$, for $0 < \alpha \le \left(\frac{B_1}{\sqrt{a}}\right)^{-q_1+1}$. Moreover, $h(\alpha)$ is continuous and differentiable in $[0, +\infty)$.

$$h'(\alpha) = \alpha - \left(\frac{B_1^2}{a}\right)^{\frac{q_1+1}{2}} \alpha^{q_1},\ 0 \le \alpha < \left(\frac{B_1}{\sqrt{a}}\right)^{-q_1+1}.$$

Then **(i)** follows. Since $q_1 - 1 > 0$, we have $\lim_{\alpha\to+\infty} h(\alpha) = -\infty$. A typical computation yields to $h(\alpha_1) = E_1$. This means that (**ii**) is true. □

Lemma 2.10. *According to Theorem 2.8, it can be assumed that there is a positive constant $\alpha_2 > \alpha_1$ such that*

$$\sqrt{a\|\nabla u\|_2^2 + \frac{b}{2}\|\nabla u\|_2^4} \ge \alpha_2, \quad t \ge 0, \tag{2.8}$$

$$\int_\Omega \frac{1}{q(x)+1}|u(x,t)|^{q(x)+1} \mathrm{d}x \ge \frac{1}{q_1+1}\left(\frac{B_1^2}{a}\right)^{\frac{q_1+1}{2}} \alpha_2^{q_1+1}, \tag{2.9}$$

and

$$\frac{\alpha_2}{\alpha_1} \ge \left((q_1+1)\left(\frac{1}{2} - \frac{\mathrm{E}(u_0)}{\alpha_1^2}\right)\right)^{\frac{1}{q_1-1}} > 1. \tag{2.10}$$

Proof. According to Lemma 2.9, since $\mathrm{E}(u_0) < E_1$, there must be a positive constant $\alpha_2 > \alpha_1$ such that $\mathrm{E}(u_0) = h(\alpha_2)$. Using equation (2.5), we can see that $h(\alpha_0) = g(\alpha_0) \le \mathrm{E}(u_0) = h(\alpha_2)$. With the help of Lemma 2.9**(i)**, we can conclude that $\alpha_0 \ge \alpha_2$, which proves that (2.8) holds for $t = 0$. Now, to prove (2.8) by contradiction, let's assume that there exists a $t^* > 0$ with $\sqrt{a\|\nabla u(t^*)\|_2^2 + \frac{b}{2}\|\nabla u(t^*)\|_2^4} < \alpha_2$. By the continuity of $\sqrt{a\|\nabla u\,(.,t^*)\,\|_2^2+\frac{b}{2}\|\nabla u\,(.,t^*)\,\|_2^4}$ and $\alpha_2>\alpha_1$, we may take t^* such that $\alpha_2>\sqrt{a\|\nabla u\,(t^*)\,\|_2^2+\frac{b}{2}\|\nabla u\,(t^*)\,\|_2^4}>\alpha_1$, then it follows from (2.5) and (2.7) that

$$\mathrm{E}(u_0)=h(\alpha_2)<h\left(\sqrt{a\|\nabla u\,(t^*)\,\|_2^2+\frac{b}{2}\|\nabla u\,(t^*)\,\|_2^4}\right)\leq \mathrm{E}\,(t^*)\,,$$

which contradicts to (1.6), and (2.8) follows. By (2.1) and (??), we obtain

$$\begin{aligned}\int_\Omega \frac{1}{q\,(x)+1}\,|u\,(x,t)|^{q(x)+1}\,\mathrm{d}x \geq &\frac{a}{2}\|\nabla u\|_2^2+\frac{b}{4}\|\nabla u\|_2^4-\mathrm{E}(u_0)\\ \geq&\frac{1}{2}\alpha_2^2-h(\alpha_2)=\frac{1}{q_1+1}\left(\frac{B_1^2}{a}\right)^{\frac{q_1+1}{2}}\alpha^{q_1+1}\end{aligned}\tag{2.11}$$

and (2.9) follows. Since $\mathrm{E}(u_0)<E_1$, by a simple calculation, we can check

$$\left((q_1+1)\left(\frac{1}{2}-\frac{\mathrm{E}(u_0)}{\alpha_1^2}\right)\right)^{\frac{2}{q_1-1}}>1,$$

then the second inequality in (2.10) holds, and we only need to show the first inequality. Denote $\beta=\frac{\alpha_2}{\alpha_1}$, then $\beta>1$ by the fact that $\alpha_2>\alpha_1$. So it results from $\mathrm{E}(u_0)=h(\alpha_2)$, $B_1>1$ and (2.4) that

$$\begin{aligned}\mathrm{E}(u_0)=h(\alpha_2)=h\,(\beta\alpha_1)=&\alpha_1^2\left(\frac{1}{2}\beta^2-\frac{1}{q_1+1}\frac{1}{a^{\frac{q_1+1}{2}}}B_1^{q_1+1}\beta^{q_1+1}\alpha_1^{q_1-1}\right)\\ =\alpha_1^2\beta^2\left(\frac{1}{2}-\frac{1}{q_1+1}\beta^{q_1-1}\right)\geq&\alpha_1^2\left(\frac{1}{2}-\frac{1}{q_1+1}\beta^{q_1-1}\right),\end{aligned}$$

which implies that the first inequality in (2.10) holds. □

Consider $H(t)=E_1-\mathrm{E}(t)$ for $t\geq 0$, the following lemma holds.

Lemma 2.11. *According to Theorem 2.8, the functional $H(t)$ mentioned earlier has the following estimates:*

$$0<H(0)\leq H(t)\leq\int_\Omega\frac{1}{q\,(x)+1}\,|u\,(x,t)|^{q(x)+1}\,\mathrm{d}x,\quad t\geq 0.\tag{2.12}$$

Proof. By (1.6), $H(t)$ is nondecreasing in t. Thus

$$H(t)\geq H(0)=E_1-\mathrm{E}(u_0)>0,\quad t\geq 0.\tag{2.13}$$

Combining (2.1), (2.4), (2.8) and $\alpha_2>\alpha_1$, we have

$$\begin{aligned}H(t)-\int_\Omega\frac{1}{q\,(x)+1}\,|u\,(x,t)|^{q(x)+1}\,\mathrm{d}x=&E_1-\frac{1}{2}\left(a\|\nabla u\|_2^2+\frac{b}{2}\|\nabla u\|_2^4\right)\\ \leq&\left(\frac{1}{2}-\frac{1}{q_1+1}\right)\alpha_1^2-\frac{1}{2}\alpha_1^2<0,\ t\geq 0.\end{aligned}\tag{2.14}$$

(2.12) follows from (2.13) and (2.14). □

With the three lemmas presented above, we can give the proof of the Theorem 2.8.

Proof of Theorem 2.8. Let define the function

$$\varphi(t) = \frac{1}{2}\int_\Omega u(x,t)^2\,\mathrm{d}x, \tag{2.15}$$

According to the definitions of E(t) and H(t), the derivative of $\varphi'(t)$ meets the requirements

$$\begin{aligned}
\varphi'(t) &= \int_\Omega u(x,t)\,u_t(x,t)\,\mathrm{d}x \\
&= \int_\Omega u(x,t)\left(M\left(\int_\Omega |\nabla u|^2 \mathrm{d}x\right)\Delta u + |u|^{q(x)-1}u\right)\mathrm{d}x \\
&= -a\int_\Omega |\nabla u(x,t)|^2\,\mathrm{d}x - b\int_\Omega |\nabla u(x,t)|^4\,\mathrm{d}x + \int_\Omega |u|^{q(x)+1}\,\mathrm{d}x \\
&\geq \left(-4\mathrm{E}(t) + a\|\nabla u\|_2^2 - 4\int_\Omega \frac{1}{q(x)+1}|u|^{q(x)+1}\mathrm{d}x\right) + \int_\Omega |u|^{q(x)+1}\,\mathrm{d}x \\
&\geq -4\left(E_1 - H(t)\right) + \left(1 - \frac{4}{q_1+1}\right)\int_\Omega |u(x,t)|^{q(x)+1}\,\mathrm{d}x \\
&\geq -4E_1 + 2H(t) + \frac{q_1-3}{q_1+1}\int_\Omega |u(x,t)|^{q(x)+1}\,\mathrm{d}x
\end{aligned} \tag{2.16}$$

By (2.4) and (2.8), we see

$$\begin{aligned}
4E_1 &= 4\frac{q_1-1}{2(q_1+1)}\left(\frac{B_1^2}{a}\right)^{-\frac{q_1+1}{q_1-1}} = 2\frac{q_1-1}{q_1+1}\left(\frac{B_1^2}{a}\right)^{\frac{q_1+1}{2}}\alpha_1^{q_1+1} \\
&= 2\frac{q_1-1}{q_1+1}\left(\frac{\alpha_1}{\alpha_2}\right)^{q_1+1}\left(\left(\frac{B_1^2}{a}\right)^{\frac{q_1+1}{2}}\alpha_2^{q_1+1}\right) \\
&\leq 2\frac{q_1-1}{q_1+1}\left(\frac{\alpha_1}{\alpha_2}\right)^{q_1+1}\int_\Omega |u(x,t)|^{q(x)+1}\,\mathrm{d}x \\
&\leq \frac{q_1-3}{q_1+1}\left(\frac{\alpha_1}{\alpha_2}\right)^{q_1+1}\int_\Omega |u(x,t)|^{q(x)+1}\,\mathrm{d}x.
\end{aligned} \tag{2.17}$$

According to Lemmas 2.11, (2.16) and (2.17), this result

$$\varphi'(t) \geq \gamma\int_\Omega |u(x,t)|^{q(x)+1}\,\mathrm{d}x \tag{2.18}$$

where

$$\gamma = \frac{q_1-3}{q_1+1}\left(1 - \left(\frac{\alpha_1}{\alpha_2}\right)^{q_1+1}\right) > 0$$

According to Hölder's inequality we have

$$\begin{aligned}
\varphi^{\frac{q_1+1}{2}}(t) &\leq C_1\int_\Omega |u|^{q_1+1}\,\mathrm{d}x, \\
\varphi^{\frac{q_2+1}{2}}(t) &\leq C_2\int_\Omega |u|^{q_2+1}\,\mathrm{d}x
\end{aligned} \tag{2.19}$$

where

$$C_1 = |\Omega|^{\frac{q_1-2}{2}} \left(\frac{1}{2}\right)^{\frac{q_1+1}{2}}, \text{ and } C_2 = |\Omega|^{\frac{q_2-1}{2}} \left(\frac{1}{2}\right)^{\frac{q_2+1}{2}}.$$

$|\Omega|$ is the Lebesgue measure of Ω. Then it follows from (2.18) and (2.19) that

$$\begin{aligned}\varphi'(t) \geq & \gamma \min\left(\int_\Omega |u(x,t)|^{q_1+1} \mathrm{d}x, \int_\Omega |u(x,t)|^{q_2+1} \mathrm{d}x\right) \\ \geq & \gamma \min\left(\frac{\varphi^{\frac{q_2+1}{2}}(t)}{C_2}, \frac{\varphi^{\frac{q_1+1}{2}}(t)}{C_1}\right),\end{aligned}$$

this implies

$$\varphi(t) \geq \min\left\{\begin{aligned} & \left(\left(\frac{1}{2}\int_\Omega u_0^2 \mathrm{d}x\right)^{\frac{1-q_1}{2}} - \frac{\gamma(q_1-1)}{2C_1} t\right)^{\frac{-2}{q_1-1}} \\ & , \left(\left(\frac{1}{2}\int_\Omega u_0^2 \mathrm{d}x\right)^{\frac{1-q_2}{2}} - \frac{\gamma(q_2-1)}{2C_2} t\right)^{\frac{-2}{q_2-1}} \end{aligned}\right\}.$$

Now, let

$$0 < T^* := \max\left(\frac{2^{\frac{q_1}{2}} C_1}{\gamma(q_1-1)} \left(\int_\Omega u_0^2 \mathrm{d}x\right)^{\frac{1-q_1}{2}}, \frac{2^{\frac{q_2}{2}} C_2}{\gamma(q_2-1)} \left(\int_\Omega u_0^2 \mathrm{d}x\right)^{\frac{1-q_2}{2}}\right) < \infty, \tag{2.20}$$

then $\varphi(t)$ blows up at time T^*. Hence, $u(x,t)$ discontinues at some finite time $T \leq T^*$, that is to means, $u(x,t)$ blows up at a finite time T. Next, we estimate T. By (2.10) and the values of γ, C_1, C_2, we have

$$\frac{2^{\frac{q_1}{2}} C_1}{\gamma(q_1-1)} \leq \frac{(q_1+1)|\Omega|^{\frac{q_1-2}{2}}}{(q_1-1)(q_1-3)\left(1-\left((q_1+1)\left(\frac{1}{2}-\frac{\mathrm{E}(u_0)}{\alpha_1^2}\right)\right)^{\frac{-q_1-1}{q_1-1}}\right)},$$

$$\frac{2^{\frac{p_2}{2}} C_2}{\gamma(q_2-1)} \leq \frac{(q_1+1)|\Omega|^{\frac{q_1-2}{2}}}{(q_2-1)(q_1-3)\left(1-\left((q_1+1)\left(\frac{1}{2}-\frac{\mathrm{E}(u_0)}{\alpha_1^2}\right)\right)^{\frac{-q_1-1}{q_1-1}}\right)}.$$

The pair of inequalities shown above coupling (2.20) imply that $T \leq T^* \leq \widehat{T}$, with $\widehat{T}$ being a fixed in (2.6). ☐

For $1 < q_1 \leq q(x) \leq q_2 \leq \frac{n+2}{n-2}$, we have the following blow-up results

Theorem 2.12. *Let* $u(x,t)$ *the weak solution to problem* (1.1) *with the initial data* $u_0 \in H_0^1(\Omega)$ *are such that* $u_0 \neq 0$.

1. *Let* q *satisfy* $(\mathcal{H}_1)$. *Suppose that one of the following claims holds:*
   (i). $\mathrm{E}(u_0) < 0$,
   (ii). $E(0) \leq \mathrm{E}_d$,

(iii). $0 \le \mathrm{E}(u_0) < C_0 \|u_0\|_2^2 \triangleq \min\left(\frac{a(q_1-1)}{q_1+1}\lambda_1, \frac{b(q_1-3)}{2(q_1+1)}\lambda_1^2\|u_0\|_2^2\right)\|u_0\|_2^2$, *where $\lambda_1 > 0$ is the first eigenvalue of $-\Delta$ in Ω with homogeneous Dirichlet boundary condition.*

Then $u(x,t)$ blows up in finite time. Moreover, the upper bound for T has the following proprieties:

In case (i), $T \le \frac{\|u_0\|_2^2}{(1-q_1^2)\mathrm{E}(u_0)}$.

In case (ii), when $\mathrm{E}(u_0) < \mathrm{E}_d$, *then the T can be bounded above as:*

$$T \le \frac{4q_1 \|u_0\|_2^2}{(q_1-1)^2(q_1+1)(\mathrm{E}_d - \mathrm{E}(0))}.$$

In case (iii), $T \le \frac{4q_1\|u_0\|_2^2}{(q_1-1)^2\left(\min\left(a(q_1-1)\lambda_1, \frac{b(q_1-3)}{2}\lambda_1^2\|u_0\|_2^2\right)\|u_0\|_2^2-(q_1+1)\mathrm{E}(u_0)\right)}$.

2. *Let q satisfy $(\mathcal{H}_2)$. Suppose that the following claim holds:* $\mathrm{E}(u_0) < -\frac{q_1+1}{q_1+5}\frac{b}{4\varepsilon}c(\varepsilon)$, *where*

$$0 < c(\varepsilon) = \max\begin{pmatrix} \frac{3-q_1}{4}\left(B\varepsilon^{-\frac{q_1+1}{4}}\frac{q_1+1}{4}\right)^{\frac{4}{3-q_1}}, \\ \frac{3-q_2}{4}\left(B\varepsilon^{-\frac{q_2+1}{4}}\frac{q_2+1}{4}\right)^{\frac{4}{3-q_2}} \end{pmatrix}, \tag{2.21}$$

and

$$0 < \varepsilon \le \frac{b(q_1+1)^2}{16} \tag{2.22}$$

. Then $T < +\infty$, which implies that $u(x,t)$ blows up in finite time. Moreover, the upper bound for T has the following form

$$T \le \frac{\|u_0\|_2^2}{(1-q_1^2)\left(\frac{q_1+5}{q_1+1}\mathrm{E}(u_0) + \frac{b}{4\varepsilon}c(\varepsilon)\right)}.$$

Proof. 1. **(I)** Set

$$\mathrm{M}(t) = \frac{1}{2}\|u(t)\|_2^2, \quad \mathrm{J}(t) = -\mathrm{E}(u(t)) \triangleq -\mathrm{E}(u(x,t)),$$

then $\mathrm{M}(0) > 0$, $\mathrm{J}(0) > 0$. By (1.7) we have $\mathrm{J}'(t) = -\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{E}(u(t)) = \|u_t(t)\|_2^2 \ge 0$, which infers that $\mathrm{J}(t) \ge \mathrm{J}(0) > 0$ for all $t \in [0,T)$. Evoking (1.5), (1.8) and the fact that $q_1 > 3$, we gain, for any $t \in [0,T)$, that

$$\begin{aligned}\mathrm{M}'(t) = -I(u(t)) &\ge -(q_1+1)\mathrm{E}(u) + (q_1-1)\frac{a}{2}\|\nabla u\|_2^2 + \frac{b}{4}(q_1-3)\|\nabla u\|_2^4 \\ &\ge (q_1+1)\mathrm{J}(t),\end{aligned} \tag{2.23}$$

This, when combined with the Cauchy-Schwarz inequality, results

$$\begin{aligned}\mathrm{M}(t)\mathrm{J}'(t) =&\frac{1}{2}\|u(t)\|_2^2\|u_t(t)\|_2^2 \ge \frac{1}{2}\|u(t)\|_2^2\|u_t(t)\|_2^2 \\ \ge&\frac{1}{2}(u,u_t)^2 = \frac{1}{2}(\mathrm{M}'(t))^2 \ge \frac{q_1+1}{2}\mathrm{M}'(t)\mathrm{J}(t).\end{aligned} \tag{2.24}$$

Based on direct calculations, it can be inferred from (2.24) that

$$\left(\mathrm{J}(t)\mathrm{M}^{-\frac{q_1+1}{2}}(t)\right)' = \mathrm{M}^{-\frac{q_1+3}{2}}(t)\left(\mathrm{J}'(t)\mathrm{M}(t) - \frac{q_1+1}{2}\mathrm{J}(t)\mathrm{M}'(t)\right) \geq 0.$$

Therefore,

$$\begin{aligned} 0 \quad &< \quad \kappa := \mathrm{J}(0)\mathrm{M}^{-\frac{q_1+1}{2}}(0) \leq \mathrm{J}(t)\mathrm{M}^{-\frac{q_1+1}{2}}(t) \\ &\leq \quad \frac{1}{q_1+1}\mathrm{M}'(t)L^{-\frac{q_1+1}{2}}(t) = \frac{2}{1-q_1^2}\left(\mathrm{M}^{\frac{1-q_1}{2}}(t)\right)'. \end{aligned} \tag{2.25}$$

By integrating (2.25) over the interval $[0,t]$, where t belongs to the open interval $(0,T)$, and taking into consideration that $q_1 > 3$, we can derive the following result

$$\kappa t \leq \frac{2}{1-q_1^2}\left(\mathrm{M}^{\frac{1-q_1}{2}}(t) - \mathrm{M}^{\frac{1-q_1}{2}}(0)\right),$$

or equivalently

$$0 \leq \mathrm{M}^{\frac{1-q_1}{2}}(t) \leq \mathrm{M}^{\frac{1-q_1}{2}}(0) - \frac{q_1^2-1}{2}\kappa t, \quad t \in (0,T). \tag{2.26}$$

It is clear that (2.26) cannot hold for all $t > 0$, implying $T < +\infty$. Furthermore, it can be deduced from (2.26) that

$$T \leq \frac{2}{(q_1^2-1)\,\kappa}\mathrm{M}^{\frac{1-q_1}{2}}(0) = \frac{\|u_0\|_2^2}{(1-q_1^2)\,\mathrm{E}\,(u_0)}.$$

(II) Assuming the existence of $u(t)$ globally, we will use contradiction and define the following function:

$$\theta(t) = \int_0^t \|u(s)\|_2^2 \mathrm{d}s + (T_0 - t)\,\|u_0\|_2^2 + \beta\,(t+t_0)^2\,,\ t \in [0,T_0],\ t_0 > 0. \tag{2.27}$$

where t_0, T_0 and β are positive constants to be determined later. Then we have

$$\begin{aligned} \theta'(t) =& \|u(t)\|_2^2 - \|u_0\|_2^2 + 2\beta\,(t+t_0) \\ =& \int_0^t \frac{\mathrm{d}}{\mathrm{d}s}\|u(s)\|_2^2 \mathrm{d}s + 2\beta\,(t+t_0) \\ =& 2\int_0^t \int_\Omega u_t(s)u(s)\mathrm{d}x\mathrm{d}s + 2\beta\,(t+t_0)\,, \end{aligned} \tag{2.28}$$

and

$$\theta''(t) = 2\int_\Omega u_t(t)u(t)\mathrm{d}x + 2\beta. \tag{2.29}$$

Using (1.1), and (2.29) we deduce that

$$\theta''(t) = -a\|\nabla u\|_2^2 - b\|\nabla u\|_2^4 + \int_\Omega |u|^{q(x)+1}\mathrm{d}x + 2\beta. \tag{2.30}$$

Based on (2.27), (2.28) and (2.30), it can be concluded that

$$\begin{aligned}\theta''(t)\theta(t)-&\frac{q_1+1}{2}\left(\theta'(t)\right)^2\\ =&2\theta(t)\left[-a\|\nabla u\|_2^2-b\|\nabla u\|_2^4+\int_\Omega |u|^{q(x)+1}\mathrm{d}x+\beta\right]\\ &-\frac{q_1+1}{2}\left(2\int_0^t\int_\Omega u_t(s)u(s)\mathrm{d}x\mathrm{d}s+2\beta\left(t+t_0\right)\right)^2\\ =&2\theta(t)\left[-a\|\nabla u\|_2^2-b\|\nabla u\|_2^4+\int_\Omega |u|^{q(x)+1}\mathrm{d}x+\beta\right]\\ &+2\left(q_1+1\right)\left[\eta(t)-\left(\theta(t)-(T-t)\|u_0\|_2^2\right)\left(\beta+\int_0^t\|u_t(s)\|_2^2\mathrm{d}s\right)\right]\end{aligned} \tag{2.31}$$

where $\eta:[0,T]\to\mathbb{R}$ is the function given by

$$\begin{aligned}\eta(t)\quad=\quad&\left(\beta\left(t+t_0\right)^2+\int_0^t\|u(s)\|_2^2\mathrm{d}s\right)\left(\beta+\int_0^t\|u_t(s)\|_2^2\,\mathrm{d}s\right)\\ &-\left(\beta\left(t+t_0\right)+\int_0^t\int_\Omega u_t(s)u(s)\mathrm{d}x\mathrm{d}s\right)^2.\end{aligned} \tag{2.32}$$

By utilizing the Cauchy-Schwarz and Young's inequalities, we can ensure that:

$$\begin{aligned}\left(\int_\Omega u(t)u_t(t)\mathrm{d}x\right)^2\leq&\|u(t)\|_2^2\,\|u_t(t)\|_2^2\,,\\ 2\beta\left(t+\sigma\right)\int_0^t\int_\Omega u_t(s)u(s)\mathrm{d}x\mathrm{d}s\leq&\beta\left(t+t_0\right)^2\int_0^t\|u_t(s)\|_2^2\,\mathrm{d}s+\beta\int_0^t\|u(s)\|_2^2\mathrm{d}s\end{aligned} \tag{2.33}$$

By (2.33), we get

$$\begin{aligned}\eta(t)\geq\beta\left(t+t_0\right)^2&\int_0^t\|u_t(s)\|_2^2\,\mathrm{d}s+\beta\int_0^t\|u(s)\|_2^2\mathrm{d}s+\int_0^t\|u_t(s)\|_2^2\,\mathrm{d}s\int_0^t\|u(s)\|_2^2\mathrm{d}s\\ -2\beta\left(t+t_0\right)&\int_0^t\int_\Omega u_t(s)u(s)\mathrm{d}x\mathrm{d}s-\left(\int_0^t\int_\Omega u_t(s)u(s)\mathrm{d}x\mathrm{d}s\right)^2\geq 0,\quad\forall t\in[0,T].\end{aligned} \tag{2.34}$$

From (2.31) and (2.34) we obtain

$$\theta''(t)\theta(t)-\frac{q_1+1}{2}\left(\theta'(t)\right)^2\geq\theta(t)\zeta(t), \tag{2.35}$$

where $\zeta(t)$ is given by

$$\zeta(t)=-2a\|\nabla u\|_2^2-2b\|\nabla u\|_2^4+2\int_\Omega |u|^{q(x)+1}\mathrm{d}x+2\beta-2\left(q_1+1\right)\left(\beta+\int_0^t\|u_t(s)\|_2^2\mathrm{d}s\right) \tag{2.36}$$

We will now make an estimation of $\zeta(t)$, using equations (1.7), and (2.36) yields

$$\begin{aligned}
\zeta(t) =& -2a\|\nabla u\|_2^2 - 2b\|\nabla u\|_2^4 + 2\int_\Omega |u|^{q(x)+1}\mathrm{d}x \\
&+ 2\left(q_1+1\right)\mathrm{E}\left(u\right) - 2\left(q_1+1\right)\mathrm{E}(u_0) - 2q_1\beta \\
\geq& -2a\|\nabla u\|_2^2 - 2b\|\nabla u\|_2^4 + \left(q_1+1\right)a\|\nabla u\|_2^2 \\
&+ \frac{b}{2}\left(q_1+1\right)\|\nabla u\|_2^4 - 2\left(q_1+1\right)\mathrm{E}(u_0) - 2q_1\beta \\
=& \left(q_1-1\right)a\|\nabla u\|_2^2 + \frac{q_1-3}{2}b\|\nabla u\|_2^4 - 2\left(q_1+1\right)\mathrm{E}(u_0) - 2q_1\beta \\
=& 2\left(q_1+1\right)\left[\left(\frac{1}{2}-\frac{1}{q_1+1}\right)a\|\nabla u\|_2^2 + \left(\frac{1}{4}-\frac{1}{q_1+1}\right)b\|\nabla u\|_2^4 - \mathrm{E}(u_0) - \frac{q_1}{q_1+1}\beta\right]
\end{aligned} \tag{2.37}$$

Let β be a positive value such that $\beta \in \left(0, \frac{q_1+1}{q_1}\left(\mathrm{E}_d - \mathrm{E}(u_0)\right)\right]$, and since $u_0 \in \mathcal{U}$ by Lemma 2.7, we have:

$$d \leq \left(\frac{1}{2}-\frac{1}{q_2+1}\right)a\|\nabla u(t)\|_2^2 + \left(\frac{1}{4}-\frac{1}{q_2+1}\right)b\|\nabla u(t)\|_2^4. \tag{2.38}$$

And by assuming $\mathrm{E}(u_0) < \mathrm{E}_d$ we get

$$\begin{aligned}
\mathrm{E}(u_0) <& \frac{q_1-1}{q_1+1}\frac{q_2+1}{q_2-1}d \leq d \\
&\leq \left(\frac{1}{2}-\frac{1}{q_2+1}\right)a\|\nabla u(t)\|_2^2 + \left(\frac{1}{4}-\frac{1}{q_2+1}\right)b\|\nabla u(t)\|_2^4.
\end{aligned} \tag{2.39}$$

If we connect (2.37) and (2.39) we obtain

$$\zeta(t) > \rho > 0. \tag{2.40}$$

From (2.35) and (2.40), we reach at

$$\theta''(t)\theta(t) - \frac{q_1+1}{2}\left(\theta'(t)\right)^2 \geq \rho\theta(t). \tag{2.41}$$

By the continuity of θ and equation (2.38), we can infer that there exists a positive constant c such that $\theta(t) \geq c$ for t in the interval $[0, T]$. Therefore, equation (2.41) produces

$$\theta''(t)\theta(t) - \frac{q_1+1}{2}\left(\theta'(t)\right)^2 \geq c\rho. \tag{2.42}$$

In this case, we prove that T cannot be infinite, meaning there is no weak solution at all times. We use Lemma 2.5 to infer that $\theta(t) \to \infty$ as $t \to T_*$, where

$$T_* \leq \frac{\theta(0)}{(q_1-1)\theta'(0)} = \frac{T_0\|u_0\|_2^2 + \beta t_0^2}{(q_1-1)\beta t_0},$$

there exists a $T^* < T_*$ which

$$\lim_{t\to T^*}\int_0^t \|u(s)\|_2^2 ds + (T_0 - t)\|u_0\|_2^2 + \beta(t+t_0)^2 = +\infty.$$

Let's choose appropriate values for t_0 and T_0. We can set t_0 to any number that depends only on q_1, $d - E(0)$ and u_0

$$t_0 > \frac{\|u_0\|_2^2}{(q_1-1)\beta}$$

If t_0 is fixed, then T_0 can be chosen as

$$T_0 = \frac{T_0\|u_0\|_2^2 + \beta t_0^2}{(q_1-1)\beta t_0},$$

so that

$$T_0 = \frac{\beta t_0^2}{(q_1-1)\beta t_0 - \|u_0\|_2^2}.$$

The lifespan of the solution $u(x,t)$ is bounded by a certain number as

$$T_0 = \inf_{t\geq t_0}\frac{\beta t^2}{\left((q_1-1)\beta t - \|u_0\|_2^2\right)} = \frac{4\|u_0\|_2^2}{(q_1-1)^2\beta} = \frac{4q_1\|u_0\|_2^2}{(q_1-1)^2(q_1+1)\left(\mathrm{E}_d - \mathrm{E}(u_0)\right)}.$$

Due to the arbitrariness of $T_0 < T$ it follows that

$$T \leq \frac{4q_1\|u_0\|_2^2}{(q_1-1)^2(q_1+1)\left(\mathrm{E}_d - \mathrm{E}(u_0)\right)}.$$

(III) To deal with the case $0 \leq \mathrm{E}(u_0) < C_0\|u_0\|_2^2$, first, it follows from the definitions of $I(u)$, $\mathrm{E}(u)$ and the assumption (ii) that

$$\begin{aligned} I(u_0) &= (q_1+1)\mathrm{E}(u_0) - \frac{a(q_1-1)}{2}\|\nabla u_0\|_2^2 - \frac{b(q_1-3)}{4}\|\nabla u_0\|_2^4 \\ &= (q_1+1)\left(\mathrm{E}(u_0) - C_0\|u_0\|_2^2\right) - \frac{a(q_1-1)}{2}\left(\|\nabla u_0\|_2^2 - \lambda_1\|u_0\|_2^2\right) \\ &- \frac{b(q_1-3)}{4}\|\nabla u_0\|_2^4 < 0. \end{aligned}$$

We claim that for all $t \in [0,T)$, $I(u(t)) < 0$. Otherwise, there would exist a $t_0 \in (0,T)$ such that $I(u(t)) < 0$ for all $t \in [0,t_0)$ and $I(u(t_0)) = 0$. By (2.23), we have that $\|u(t)\|_2^2$ and $\|u(t)\|_2^4$ are strictly increasing in t for $t \in [0,t_0)$, and therefore

$$0 \leq \mathrm{E}(u_0) < C_0\|u_0\|_2^2 < C_0\|u(t_0)\|_2^2. \tag{2.43}$$

On the other hand, we can deduce from the monotonicity of $\mathrm{E}(u(t))$ and (1.5)

$$\begin{aligned}\mathrm{E}\left(u_0\right) \geq & \mathrm{E}\left(u\left(t_0\right)\right)=\frac{a(q_1-1)}{2(q_1+1)}\left\|\nabla u\left(t_0\right)\right\|_2^2+\frac{b(q_1-3)}{4(q_1+1)}\left\|\nabla u\left(t_0\right)\right\|_2^4+\frac{1}{q_1+1} I\left(u\left(t_0\right)\right) \\ & \geq \min \left(\frac{a(q_1-1)}{(q_1+1} \lambda_1, \frac{b(q_1-3)}{2(q_1+1)} \lambda_1^2\|u_0\|_2^2\right)\left\|u\left(t_0\right)\right\|_2^2=C_0\left\|u\left(t_0\right)\right\|_2^2,\end{aligned}$$

Therefore, since (2.43) is contradictory, we have $I(u(t)<0$ for all $t \in [0,T)$. Then, $\|u(t)\|_2^2$ is strictly increasing on $[0,T)$ and $\|u(t)\|_2^4$ is also strictly increasing on $[0,T)$. For any $T_0 \in (0,T)$, $\beta > 0$, and $t_0 > 0$, we define

$$F(t)=\int_0^t\|u(\tau)\|_2^2 \mathrm{d} \tau-(T_0-t)\left\|u_0\right\|_2^2+\beta(t+t_0)^2, \quad t \in\left[0, T_0\right]. \tag{2.44}$$

Through a direct calculations

$$\begin{aligned} F'(t)= & \|u(t)\|_2^2-\left\|u_0\right\|_2^2+2 \beta(t+t_0)=\int_0^t \frac{\mathrm{d}}{\mathrm{d} \tau}\|u(\tau)\|_2^2 \mathrm{d} \tau+2 \beta(t+t_0) \\ = & 2 \int_0^t\left(u, u_\tau\right) \mathrm{d} \tau+2 \beta(t+t_0), \\ F''(t)= & 2\left(u, u_t\right)+2 \beta=-2 I(u(t))+2 \beta \\ = & -2(q_1+1) \mathrm{E}(u(t))+a(q_1-1)\|\nabla u(t)\|_2^2+\frac{b(q_1-3)}{2}\|\nabla u(t)\|_2^4+2 \beta \\ = & -2(q_1+1) \mathrm{E}\left(u_0\right)+2(q_1+1) \int_0^t\left\|u_\tau(\tau)\right\|_2^2 \mathrm{d} \tau+a(q_1-1)\|\nabla u(t)\|_2^2 \\ & +\frac{b(q_1-3)}{2}\|\nabla u(t)\|_2^4+2 \beta . \end{aligned} \tag{2.45}$$

For $t \in [0,T_0]$, set

$$\begin{aligned} \theta(t)= & \left(\int_0^t\|u(\tau)\|_2^2 \mathrm{d} \tau+\beta(t+t_0)^2\right)\left(\int_0^t\left\|u_\tau\right\|_2^2 \mathrm{d} \tau+\beta\right) \\ & -\left(\int_0^t\left(u, u_\tau\right) \mathrm{d} \tau+\beta(t+t_0)\right)^2 . \end{aligned}$$

By applying Cauchy-Schwarz and Hölder's inequalities, we can show that $F(t)$ is non-negative on the interval $[0,T_0]$. As a result, we can use equation (2.44)-(2.45) and the monotonicity of $\|u(t)\|_2^2$ and $\|u(t)\|_2^4$ to conclude

$$
\begin{aligned}
&F(t)F''(t)-\frac{q_1+1}{2}\left(F'(t)\right)^2\\
=&F(t)F''(t)-2(q_1+1)\left(\int_0^t (u,u_\tau)\,\mathrm{d}\tau+\beta(t+t_0)\right)^2\\
=&F(t)F''(t)+2(q_1+1)\left[\theta(t)-\left(F-(T-t)\left\|u_0\right\|_2^2\right)\left(\int_0^t \|u_\tau\|_2^2\,\mathrm{d}\tau+\beta\right)\right]\\
\geq&F(t)F''(t)-2(q_1+1)F(t)\left(\int_0^t \|u_\tau\|_2^2\,\mathrm{d}\tau+\beta\right)\\
=&F(t)\Big[-2(q_1+1)\mathrm{E}\left(u_0\right)+2(q_1+1)\int_0^t \|u_\tau\|_2^2\,\mathrm{d}\tau+a(q_1-1)\|\nabla u(t)\|_2^2\\
&+\frac{b(q_1-3)}{2}\|\nabla u(t)\|_2^4+2\beta-2(q_1+1)\int_0^t \|u_\tau\|_2^2\,\mathrm{d}\tau-2(q_1+1)\beta\Big]\\
\geq&F(t)\left[-2(q_1+1)\mathrm{E}\left(u_0\right)+a(q_1-1)\lambda_1\|u(t)\|_2^2+\frac{b(q_1-3)}{2}\lambda_1^2\|u(t)\|_2^4-2q_1\beta\right]\\
\geq&F(t)\left[-2(q_1+1)\mathrm{E}\left(u_0\right)+\min\left(a(q_1-1)\lambda_1,\frac{b(q_1-3)}{2}\lambda_1^2\|u_0\|_2^2\right)\|u_0\|_2^2-2q_1\beta\right]\\
=&2(q_1+1)F(t)\left[C_0\left\|u_0\right\|_2^2-\mathrm{E}\left(u_0\right)-\frac{q_1\beta}{q_1+1}\right]\geq 0.
\end{aligned}
\tag{2.46}
$$

Choosing $0<\beta<\frac{q_1+1}{q_1}\left(C_0\left\|u_0\right\|_2^2-\mathrm{E}\left(u_0\right)\right)$. Then using Lemma 2.5, to infer $F(t)\to\infty$ as $t\to T^*$, where

$$T^*\leq\frac{F(0)}{(q_1-1)F'(0)}=\frac{T_0\left\|u_0\right\|_2^2+\beta t_0^2}{(q_1-1)\beta t_0}. \tag{2.47}$$

Let's choose appropriate values for t_0 and T_0. We can set t_0 to any number that only depends on q_1, $d-E(0)$ and u_0 as

$$t_0>\frac{\|u_0\|_2^2}{(q_1-1)\beta}.$$

Fix t_0, then T_0 can be picking a

$$T_0=\frac{T_0\left\|u_0\right\|_2^2+\beta t_0^2}{(q_1-1)\beta t_0},$$

so that

$$T_0=\frac{\beta t_0^2}{(q_1-1)\beta t_0-\left\|u_0\right\|_2^2}.$$

Therefore, the lifespan of the solution $u(x,t)$ is bounded by

$$T_0=\inf_{t\geq t_0}\frac{\beta t^2}{(q_1-1)\beta t-\left\|u_0\right\|_2^2}=\frac{4q_1\left\|u_0\right\|_2^2}{(q_1-1)^2\left(q_1+1\right)\left(C_0\left\|u_0\right\|_2^2-\mathrm{E}\left(u_0\right)\right)},$$

due to the arbitrariness of $T_0 < T$ it follows that

$$T_0 \leq \frac{4q_1 \left\|u_0\right\|_2^2}{(q_1-1)^2\,(q_1+1)\left(\min\left(a(q_1-1)\lambda_1, \frac{b(q_1-3)}{2}\lambda_1^2\|u_0\|_2^2\right)\|u_0\|_2^2 - \mathrm{E}\,(u_0)\right)}.$$

2. To handle the case where $1 < q_1 \leq q(x) \leq q_2 \leq 3$, we modify the energy functional E by setting

$$\begin{aligned} \mathrm{M}(t) &= \frac{1}{2}\|u(t)\|_2^2, \quad \mathrm{J}(t) = -\mathrm{E}(u(t)) - \left(\frac{4}{q_1+1}E(0) + \frac{b}{4\varepsilon}c\,(\varepsilon)\right) \\ &\triangleq -\mathrm{E}(u(x,t)) - \left(\frac{4}{q_1+1}\mathrm{E}(u_0) + \frac{b}{4\varepsilon}c\,(\varepsilon)\right), \end{aligned}$$

then $\mathrm{M}(0) > 0$, $\mathrm{J}(0) > 0$. By (1.7) we also have

$$\mathrm{J}'(t) = -\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{E}(u(t)) = \|u_t(t)\|_2^2 \geq 0.$$

It implies that $\mathrm{J}(t) \geq \mathrm{J}(0)$ for all $t \in [0,T)$. Additionally, Lemma 2.6 states that for any $\varepsilon > 0$

$$\begin{aligned} \int_\Omega |u|^{q(x)+1}\mathrm{d}x &\leq B\max\left(\|\nabla u\|_2^{q_1+1}, \|\nabla u\|_2^{q_2+1}\right) \\ &\leq \max\begin{pmatrix} \varepsilon\|\nabla u\|_2^4 + \frac{3-q_1}{4}\left(\frac{B}{\varepsilon^{\frac{q_1+1}{4}}}\frac{q_1+1}{4}\right)^{\frac{4}{3-q_1}}, \\ \varepsilon\|\nabla u\|_2^4 + \frac{3-q_2}{4}\left(\frac{B}{\varepsilon^{\frac{q_2+1}{4}}}\frac{q_2+1}{4}\right)^{\frac{4}{3-q_2}} \end{pmatrix} \\ &\leq \varepsilon\|\nabla u\|_2^4 + c\,(\varepsilon), \end{aligned}$$

which give

$$\|\nabla u\|_2^4 \geq \frac{1}{\varepsilon}\int_\Omega |u|^{q(x)+1}\mathrm{d}x - \frac{1}{\varepsilon}c\,(\varepsilon),$$

and from (1.5)

$$\begin{aligned} E(u) &= \frac{a}{2}\|\nabla u\|_2^2 + \frac{b}{4}\|\nabla u\|_2^4 - \int_\Omega \frac{1}{q(x)+1}|u|^{q(x)+1}\mathrm{d}x \\ &\geq \frac{a}{2}\|\nabla u\|_2^2 + \frac{b}{4}\frac{1}{\varepsilon}\int_\Omega |u|^{q(x)+1}\mathrm{d}x - \frac{1}{q_1+1}\int_\Omega |u|^{q(x)+1}\mathrm{d}x - \frac{b}{4\varepsilon}c\,(\varepsilon), \end{aligned}$$

also using (1.5), and $(1.5)_2$ we have

$$b\|\nabla u\|_2^4 \leq 4E(0) + \frac{4}{q_1+1}\int_\Omega |u|^{q(x)+1}\mathrm{d}x,$$

in which $(1.5)_2$ becomes

$$\begin{aligned} I(u) \leq a\|\nabla u\|_2^2 + 4\mathrm{E}(u_0) + \frac{4}{q_1+1}\int_\Omega |u|^{q(x)+1}\mathrm{d}x \\ - \int_\Omega |u|^{q(x)+1}\mathrm{d}x + \frac{a(q_1-1)}{2}\|\nabla u_0\|_2^2 \end{aligned}$$

thus we obtain, for any $t \in [0, T)$, that

$$\begin{aligned}
& (q_1+1)\,\mathrm{E}(u) - I(u) \\
\geq\;& \frac{q_1-1}{2} a\|\nabla u\|_2^2 + \frac{b\,(q_1+1)}{4\varepsilon}\int_\Omega |u|^{q(x)+1}\mathrm{d}x - \int_\Omega |u|^{q(x)+1}\mathrm{d}x \\
& -4\mathrm{E}(u_0) - \frac{b\,(q_1+1)}{4\varepsilon}c\,(\varepsilon) - \frac{4}{q_1+1}\int_\Omega |u|^{q(x)+1}\mathrm{d}x + \int_\Omega |u|^{q(x)+1}\mathrm{d}x \\
\geq\;& \left(\frac{b\,(q_1+1)}{4\varepsilon} - \frac{4}{q_1+1}\right)\int_\Omega |u|^{q(x)+1}\mathrm{d}x - 4\mathrm{E}(u_0) - \frac{b\,(q_1+1)}{4\varepsilon}c\,(\varepsilon) \\
\geq\;& -4\mathrm{E}(u_0) - \frac{b\,(q_1+1)}{4\varepsilon}c\,(\varepsilon)\,,
\end{aligned}$$

this meaning that

$$\mathrm{M}'(t) = -I(u) \geq -\,(q_1+1)\,\mathrm{E}(u) - 4\mathrm{E}(u_0) - \frac{b\,(q_1+1)}{4\varepsilon}c\,(\varepsilon) = -\,(q_1+1)\,\mathrm{J}(t),$$

which, together with Cauchy-Schwarz inequality, yields

$$\mathrm{M}(t)\mathrm{J}'(t) = \frac{1}{2}\|u(t)\|_2^2\,\|u_t(t)\|_2^2 \geq \frac{1}{2}\,(u, u_t)^2 = \frac{1}{2}\,(\mathrm{M}'(t))^2 \geq \frac{q+1}{2}\mathrm{M}'(t)\mathrm{J}(t).$$

By direct computations as previously, it follows that

$$0 \leq \mathrm{M}^{\frac{1-q_1}{2}}(t) \leq \mathrm{M}^{\frac{1-q_1}{2}}(0) - \frac{q_1^2-1}{2}\mathrm{J}(0)\mathrm{M}^{-\frac{q_1+1}{2}}(0)t, \quad t \in (0, T). \tag{2.48}$$

It is obvious to see that (2.48) cannot hold for all $t > 0$. Therefore, $T < +\infty$. Moreover, it can be inferred that

$$T \leq \frac{2}{(q_1^2-1)\,\mathrm{J}(0)L^{-\frac{q_1+1}{2}}(0)}\mathrm{M}^{\frac{1-q_1}{2}}(0) = \frac{\|u_0\|_2^2}{(1-q_1^2)\left(\frac{q_1+5}{q_1+1}\mathrm{E}(u_0) + \frac{b}{4\varepsilon}c\,(\varepsilon)\right)}.$$

□

Remark 2.13. It is not possible to compare the conditions in Theorem2.8 and Theorem2.12, which use E_d, E_1, and $\mathrm{E}(u_0)$. However, when $1 < q_1 \leq q\,(x) \leq q_2 \leq 3$, instead of $3 < q_1 \leq q\,(x) \leq q_2 \leq 2^*$, and $n \geq 3$, three new blow-up criteria are obtained which have not been addressed before in [14, 15].

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