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On a second order p-Laplacian impulsive boundary value problem on the half-line

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Abstract. In this article, we shall establish the existence of weak solutions for a p-Laplacian impulsive differential equation with Dirichlet boundary conditions on the half-line by using Browder theorem.

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1. Introduction

In this paper, we consider the following second-order p-Laplacian impulsive difierential equation with Dirichlet boundary conditions on the half-line

$$\begin{cases} -(\rho(x)|u'|^{p-2}u')' + |u|^{p-2}u &= f(x,u), \quad x \neq x_j, \quad \text{a.e. } x \geq 0, \\ \triangle(\rho(x_j)|u'(x_j)|^{p-2}u'(x_j)) &= g(x_j)I_j(u(x_j)), \quad j \in \mathbb{N}^*, \\ u(0) = u(\infty) &= 0, \end{cases} \tag{1.1}$$

where $p > 1$, $\rho : [0, \infty) \to (0, \infty)$ satisfies $\rho^{-\frac{1}{p-1}} \in L^1[0, \infty)$ and

$$M_0 = \left(\int_0^\infty \left(\int_x^\infty \rho^{-\frac{1}{p-1}}(s)ds \right) dx \right) < \infty.$$

The functions $f \in C([0, \infty) \times \mathbb{R}, \mathbb{R})$, $I_j : \mathbb{R} \to \mathbb{R}$ and $g : [0, \infty) \to (0, \infty)$ are assumed to be continuous with $\sum_{j=1}^\infty g(x_j) < \infty$, $0 = x_0 < x_1 < x_2 < ... < x_j < ... < x_m \to \infty$, as $m \to \infty$, are the impulse points, and

$$\triangle(\rho(x_j)|u'(x_j)|^{p-2}u'(x_j)) = \rho(x_j^+)|u'(x_j^+)|^{p-2}u'(x_j^+) - \rho(x_j^-)|u'(x_j^-)|^{p-2}u'(x_j^-),$$

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such that $u'(x_j^{\pm}) = \lim_{x\to x_j^{\pm}} u'(x)$ for $j \in \mathbb{N}^*$.

Recently, there is increasing interest in the existence and multiplicity of solutions for several types of differential equations with a p-Laplacian operator by applying variational methods and critical point theory. Meanwhile, some people begin to study p-Laplacian differential equations with impulsive effects, for example, see [1, 2, 4, 7, 8, 9] and the references therein.

Motivated by the works cited above, in this paper, we shall discuss the existence of solutions for problem (1.1) on the half-line by adoptting Browder theorem. The results obtained here improve some existing results in the literature.

2. Variational structure

Let define the following reflexive Banach space

$$X = \left\{ u \in W^{1,p}(0,\infty) : \quad u(0) = u(\infty) = 0, \quad \rho^{\frac{1}{p}} u' \in L^p(0,\infty) \right\},$$

equipped with the norm

$$\|u\| = \left(\int_0^{+\infty} \rho(x)|u'(x)|^p dx + \int_0^{+\infty} |u(x)|^p dx \right)^{\frac{1}{p}},$$

or the equivalent norm

$$\|u\|_X = \|\rho^{\frac{1}{p}} u'\|_p + \|u\|_p.$$

Also consider the space

$$C_0[0,+\infty) = \left\{ u \in C([0,+\infty), \mathbb{R}) : \quad \lim_{x\to\infty} u(x) = 0 \right\},$$

endowed with the norm

$$\|u\|_\infty = \sup_{x\in[0,+\infty)} |u(x)|.$$

In what follows, we shall convert the problem (1.1) into an integral equation. Multiply the two sides of the equality

$$-(\rho(x)|u'|^{p-2}u')' + |u|^{p-2}u = f(x,u),$$

by $v \in X$ and integrate from 0 to ∞, to obtain,

$$-\int_0^{+\infty} (\rho(x)|u'(x)|^{p-2}u'(x))'v(x)dx + \int_0^{+\infty} |u(x)|^{p-2}u(x)v(x)dx = \int_0^{+\infty} f(x,u(x))v(x)dx.$$

Let consider the first term

$$-\int_0^{+\infty} (\rho(x)|u'(x)|^{p-2}u'(x))'v(x)dx = \sum_{j=0}^{\infty} \int_{x_j^+}^{x_{j+1}^-} -(\rho(x)|u'(x)|^{p-2}u'(x))'v(x)$$

$$= \int_0^{+\infty} \rho(x)|u'(x)|^{p-2}u'(x)v'(x)dx + \sum_{j=1}^{\infty} \Big[\rho(x_j^+)|u'(x_j^+)|^{p-2}u'(x_j^+) - \rho(x_j^-)|u'(x_j^-)|^{p-2}u'(x_j^-)\Big] v(x_j)$$

$$= \int_0^{+\infty} \rho(x)|u'(x)|^{p-2}u'(x)v'(x)dx + \sum_{j=1}^{\infty} \triangle(\rho(x_j)|u'(x_j)|^{p-2}u'(x_j))v(x_j)$$

$$= \int_0^{+\infty} \rho(x)|u'(x)|^{p-2}u'(x)v'(x)dx + \sum_{j=1}^{\infty} g(x_j)I_j(u(x_j))v(x_j),$$

and then, we have

$$\int_0^{+\infty} \rho(x)|u'(x)|^{p-2}u'(x)v'(x)dx + \int_0^{+\infty} |u(x)|^{p-2}u(x)v(x)dx + \sum_{j=1}^{\infty} g(x_j)I_j(u(x_j))v(x_j) = \int_0^{+\infty} f(x,u(x))v(x)dx.$$

This leads us to introduce the following concept for the solution for (1.1).

Definition 2.1. *We say that a function $u \in X$ is a weak solution of the impulsive problem (1.1) if u satisfies*

$$\int_0^{+\infty} \rho(x)|u'(x)|^{p-2}u'(x)v'(x)dx + \int_0^{+\infty} |u(x)|^{p-2}u(x)v(x)dx + \sum_{j=1}^{\infty} g(x_j)I_j(u(x_j))v(x_j) - \int_0^{+\infty} f(x,u(x))v(x)dx = 0.$$

Concerning the previous spaces, we have the following vital embeddings.

Lemma 2.2. *Let $u \in X$. Then*

$$\|u\|_p^p \le M_0 \|u\|^p, \tag{2.1}$$

where

$$M_0 = \int_0^{\infty} \left(\int_x^{\infty} \rho^{-\frac{1}{p-1}}(s)ds \right) dx.$$

Proof. For $u \in X$, we find

$$|u(x)| = \Big| \int_x^{\infty} u'(s)ds \Big| = \Big| \int_x^{\infty} \rho^{\frac{1}{p}}(s)u'(s)\rho^{-\frac{1}{p}}(s)ds \Big|.$$

Then, by the *Hölder* inequality, we obtain

$$\begin{aligned}|u(x)|^p &\le \left(\int_x^\infty \rho(s)|u'(s)|^p ds\right)\left(\int_x^\infty \rho^{-\frac{1}{p-1}}(s)ds\right)\\ &\le \left(\int_0^\infty \rho(s)|u'(s)|^p ds\right)\left(\int_x^\infty \rho^{-\frac{1}{p-1}}(s)ds\right).\end{aligned}$$

Hence,

$$\int_0^\infty |u(x)|^p dx \le \left(\int_0^\infty \left(\int_x^\infty \rho^{-\frac{1}{p-1}}(s)ds\right)dx\right)\left(\int_0^\infty \rho(s)|u'(s)|^p ds\right).$$

As a result we obtain (2.1). □

Lemma 2.3. *Let $u \in X$. Then*

$$\|u\|_\infty \le M\|u\|,$$

where $M = \|\rho^{-\frac{1}{p-1}}\|_1^{\frac{p-1}{p}}$.

Proof. For $u \in X$, we get

$$\begin{aligned}|u(x)| &= \left|\int_0^x u'(s)ds\right|\\ &\le \int_0^x \rho^{-\frac{1}{p}}(s)\rho^{\frac{1}{p}}(s)|u'(s)|ds\\ &\le \left(\int_0^\infty \rho^{-\frac{1}{p-1}}(s)ds\right)^{\frac{p-1}{p}}\left(\int_0^\infty \rho(s)|u'(s)|^p ds\right)^{\frac{1}{p}}\\ &\le \|\rho^{-\frac{1}{p-1}}\|_1^{\frac{p-1}{p}}\|u\|.\end{aligned}$$

Hence, $\|u\|_\infty \le M\|u\|$. □

To prove that X embeds compactly in $C_0[0,+\infty)$ we need the following Corduneanu compactness criterion.

Lemma 2.4. [5] *Let $D \subset C_0([0,+\infty),\mathbb{R})$ be a bounded set. Then D is relatively compact if the following conditions hold:*
(a) D is equicontinuous on any compact sub-interval of $\mathbb{R}^+$, i.e.

$$\begin{aligned}&\forall J \subset [0,+\infty) \ \text{compact}, \forall \varepsilon > 0, \exists \delta > 0, \forall x_1, x_2 \in J:\\ &\qquad |x_1 - x_2| < \delta \Longrightarrow |u(x_1) - u(x_2)| \le \varepsilon, \forall u \in D;\end{aligned}$$

(b) D is equiconvergent at $+\infty$ i.e.,

$$\begin{aligned}&\forall \varepsilon > 0, \exists T = T(\varepsilon) > 0 \ \text{such that}\\ &\forall x : x \ge T(\varepsilon) \Longrightarrow |u(x) - u(+\infty)| \le \varepsilon, \forall u \in D.\end{aligned}$$

Lemma 2.5. *The embedding $X \hookrightarrow C_0[0,\infty)$ is compact.*

Proof. Let $D \subset X$ be a bounded set. Then, D is bounded in $C_0[0,\infty)$ by Lemma 2.3. Let $R > 0$ be such that $\|u\| \leq R$ for all $u \in D$. We will apply Lemma 2.4.

(a) D is equicontinuous on every compact interval of $[0,+\infty)$. Let $u \in D$ and $x_1, x_2 \in J \subset [0,+\infty)$ where J is a compact sub-interval. Using *Hölder* inequality, we have

$$\begin{aligned}
|u(x_1) - u(x_2)| &= \Big| \int_{x_1}^{x_2} u'(s)ds \Big| \\
&= \Big| \int_{x_1}^{x_2} \rho^{-\frac{1}{p}}(s)\rho^{\frac{1}{p}}(s)u'(s)ds \Big| \\
&\leq \Big(\int_{x_1}^{x_2} \rho^{-\frac{1}{p-1}}(s)ds \Big)^{\frac{p-1}{p}} \Big(\int_{x_1}^{x_2} \rho(s)|u'(s)|^p ds \Big)^{\frac{1}{p}} \\
&\leq \Big(\int_{x_1}^{x_2} \rho^{-\frac{1}{p-1}}(s)ds \Big)^{\frac{p-1}{p}} \|u\| \leq R \Big(\int_{x_1}^{x_2} \rho^{-\frac{1}{p-1}}(s)ds \Big)^{\frac{p-1}{p}} \longrightarrow 0,
\end{aligned}$$

as $|x_1 - x_2| \to 0$.

(b) D is equiconvergent at $+\infty$. For $x \in [0,+\infty)$ and $u \in D$, using the fact that $u(\infty) = 0$ and by *Hölder* inequality, we have

$$\begin{aligned}
|u(x) - u(\infty)| &= |u(x)| \\
&= \Big| \int_{x}^{\infty} u'(s)ds \Big| \\
&\leq \Big(\int_{x}^{\infty} \rho^{\frac{1}{p}}(s)|u'(s)|ds \Big) \Big(\int_{x}^{\infty} \rho^{-\frac{1}{p-1}}(s) \Big)^{\frac{p-1}{p}} \\
&\leq \Big(\int_{x}^{\infty} \rho^{-\frac{1}{p-1}}(s)ds \Big)^{\frac{p-1}{p}} \|u\| \\
&\leq R \Big(\int_{x}^{\infty} \rho^{-\frac{1}{p-1}}(s)ds \Big)^{\frac{p-1}{p}} \xrightarrow[x \to \infty]{} 0.
\end{aligned}$$

□

Finally, we present the Browder Theorem which will be needed in our argument.

Definition 2.6. [6] *Let X be a reflexive real Banach space and X^* its dual. The operator $\mathcal{L} : X \to X^*$ is called to be demicontinuous if $\mathcal{L}$ maps strongly convergent sequences in X to weakly convergent sequences in X^*.*

Lemma 2.7 (Browder theorem). [3], [6] *Let X be a reflexive real Banach space. Moreover, Let $\mathcal{L} : X \to X^*$ be an operator satisfying the following conditions:*
(i) $\mathcal{L}$ is bounded and demicontinuous;
(ii) $\mathcal{L}$ is coercive, that is, $\lim_{\|u\|\to\infty} \dfrac{\langle \mathcal{L}(u), u\rangle}{\|u\|} = +\infty$;
(iii) $\mathcal{L}$ is monotone on the space X; that is; for all $u, v \in X$; one has

$$\langle \mathcal{L}(u) - \mathcal{L}(v), u - v\rangle \geq 0. \tag{2.2}$$

Then the equation $\mathcal{L}(u) = f^*$ has at least one solution $u \in X$ for every $f^* \in X^*$.
If, moreover, the inequality (2.2) is strict for all $u, v \in X$, $u \neq v$, then the equation $\mathcal{L}(u) = f^*$ has precisely one solution $u \in X$ for all $f^* \in X^*$.

3. Results

Suppose the following hypotheses hold:

(H1) The function $f(x,u)$ is decreasing about u, uniformly in $x \in [0,\infty)$; and $I_j(u)$ ($j \in \mathbb{N}^*$) are increased functions with u.

(H2) There exist $\alpha_j, \beta_j > 0$ and $\gamma \in [1,p)$ with $\sum_{j=1}^{\infty} \alpha_j g(x_j) < \infty$, $\sum_{j=1}^{\infty} \beta_j g(x_j) < \infty$, such that

$$|I_j(u)| \leq \alpha_j + \beta_j |u|^{\gamma-1}, \quad \text{for all } u \in \mathbb{R} \text{ and } j \in \mathbb{N}^*.$$

(H3) There exist positive functions $c_1, c_2 \in L^{\frac{p}{p-1}}[0,\infty)$ and a constant $\mu \in (0, p-1)$ such that

$$|f(x,u)| \leq c_1(x) + c_2(x)|u|^{\mu}, \quad \forall (x,u) \in [0,\infty) \times \mathbb{R}.$$

Let $\mathcal{L}$ be the operator defined from X into X^* by

$$\langle \mathcal{L}(u), v\rangle = \langle L_1(u), v\rangle + \langle L_2(u), v\rangle - \langle L_3(u), v\rangle, \qquad \forall u, v \in X,$$

where

$$\langle L_1(u), v\rangle = \int_0^{+\infty} \rho(x)|u'(x)|^{p-2}u'(x)v'(x)dx + \int_0^{+\infty} |u(x)|^{p-2}u(x)v(x)dx,$$

$$\langle L_2(u), v\rangle = \sum_{j=1}^{\infty} g(x_j) I_j(u(x_j))v(x_j),$$

$$\langle L_3(u), v\rangle = \int_0^{+\infty} f(x, u(x))v(x)dx.$$

We search for a weak solution of problem (1.1) which is a solution for the operator equation $\mathcal{L}(u) = 0$.

Theorem 3.1. *Assume that $(H1)$-$(H3)$ hold. Then (1.1) has a unique weak solution.*

Proof. The proof consists of four steps:
Claim 1. $\mathcal{L}$ is bounded and demicontinuous.
It is sufficient to show that the operators $L_i (i = 1,2,3)$ are bounded and continuous. Firstly, we prove that $\mathcal{L}$ is bounded.
Using *Hölder* inequality, together with the following result

$$\forall a,b,c,d > 0 \quad \forall \beta \in (0,1): \quad (a+b)^{\beta}(c+d)^{1-\beta} \geq a^{\beta}c^{1-\beta} + b^{\beta}d^{1-\beta},$$

we obtain for all $u, v \in X$, (see [7]),

$$|\langle L_1(u), v\rangle| = \Big| \int_0^{+\infty} \rho(x)|u'(x)|^{p-2}u'(x)v'(x)dx + \int_0^{+\infty} |u(x)|^{p-2}u(x)v(x)dx \Big|$$

$$
\begin{aligned}
&\leq \left(\int_0^{+\infty} \rho(x)|u'(x)|^p dx + \int_0^{+\infty} |u(x)|^p dx \right)^{\frac{p-1}{p}} \\
&\times \left(\int_0^{+\infty} \rho(x)|v'(x)|^p dx + \int_0^{+\infty} |v(x)|^p dx \right)^{\frac{1}{p}} \\
&\leq \|u\|^{p-1}\|v\| \\
&< \infty,
\end{aligned}
$$

as a result, L_1 is bounded.

Now, we prove the boundedness of L_2 and L_3 respectively. Using Lemma 2.3 and $(H2)$, gives

$$
\begin{aligned}
|\langle L_2(u), v\rangle| &= \Big| \sum_{j=1}^{\infty} g(x_j) I_j(u(x_j)) v(x_j) \Big| \\
&\leq \sum_{j=1}^{\infty} g(x_j) |I_j(u(x_j))| |v(x_j)| \\
&\leq \sum_{j=1}^{\infty} g(x_j)(\alpha_j + \beta_j |u(x_j)|^{\gamma-1}) |v(x_j)| \\
&\leq \sum_{j=1}^{\infty} (\alpha_j g(x_j) + \beta_j g(x_j) \|u\|_\infty^{\gamma-1}) \|v\|_\infty \\
&\leq M \Big(\sum_{j=1}^{\infty} \alpha_j g(x_j) + M^{\gamma-1} \|u\|^{\gamma-1} \sum_{j=1}^{\infty} \beta_j g(x_j) \Big) \|v\| \\
&< \infty, \quad \forall u, v \in X,
\end{aligned}
$$

that implies L_2 is bounded.

From the condition (H_3), we get

$$
|\langle L_3(u), v\rangle| = \Big| \int_0^{+\infty} f(x, u(x)) v(x) dx \Big| \leq \int_0^{+\infty} (c_1(x) + c_2(x)|u(x)|^{\mu}) |v(x)| dx,
$$

by the *Hölder* inequality, Lemma 2.2 and Lemma 2.3, we arrive immediately at

$$
\begin{aligned}
|\langle L_3(u), v\rangle| &\leq \Big(\int_0^{+\infty} |c_1(x)|^{\frac{p}{p-1}} dx \Big)^{\frac{p-1}{p}} \Big(\int_0^{+\infty} |v(x)|^p dx \Big)^{\frac{1}{p}} + \|u\|_\infty^{\mu} \\
&\Big(\int_0^{+\infty} |c_2(x)|^{\frac{p}{p-1}} dx \Big)^{\frac{p-1}{p}} \Big(\int_0^{+\infty} |v(x)|^p dx \Big)^{\frac{1}{p}} \\
&\leq \|c_1\|_{\frac{p}{p-1}} \|v\|_p + \|u\|_\infty^{\mu} \|c_2\|_{\frac{p}{p-1}} \|v\|_p \\
&\leq M_0^{\frac{1}{p}} \left(\|c_1\|_{\frac{p}{p-1}} + M^{\mu} \|c_2\|_{\frac{p}{p-1}} \|u\|^{\mu} \right) \|v\| \\
&< \infty,
\end{aligned}
$$

as a consequence, L_3 is bounded. We deduce that $\mathcal{L}$ is a bounded operator.
Secondly, we prove that $\mathcal{L}$ is demicontinuous.
For $u_n \to u$ in X, we have

$$|\langle L_1(u_n) - L_1(u), u_n - u\rangle| \leq \Big(\int\limits_0^{+\infty} \rho(x)\Big(|u_n'(x)|^{p-2}u_n'(x) - |u'(x)|^{p-2}u'(x)\Big)^{\frac{p}{p-1}} dx$$

$$+ \int\limits_0^{+\infty} \Big(|u_n(x)|^{p-2}u_n(x) - |u(x)|^{p-2}u(x)\Big)^{\frac{p}{p-1}} dx\Big)^{\frac{p-1}{p}} \|u_n - u\|.$$

Since $\lim_{n\to\infty} \|u_n - u\| = 0$, the last integral tends to zero. We see that L_1 is continuous.
To show the continuity of L_2, we prove that L_2 is strongly continuous, that is, if $u_n \rightharpoonup u$ in X then $L_2(u_n) \to L_2(u)$, as $n \to \infty$.
Assume $u_n \rightharpoonup u$ in X, Lemma 2.5 guarantees that (u_n) converges uniformly to u on $[0, \infty)$, as $n \to \infty$. Since I_j are continuous, then

$$I_j(u_n(x_j)) \to I_j(u(x_j)), \quad n \to \infty, \quad j \in \mathbb{N}^*,$$

moreover, from $(H2)$ we get

$$\sum_{j=1}^{\infty} g(x_j) I_j(u_n(x_j)) < \infty,$$

by applying Lebesgue's dominated convergence theorem, we obtain

$$\sum_{j=1}^{\infty} g(x_j) I_j(u_n(x_j)) \to \sum_{j=1}^{\infty} g(x_j) I_j(u(x_j)) \quad \text{as} \quad n \to \infty,$$

concequently,

$$|\langle L_2(u_n) - L_2(u)\rangle| \to 0 \quad \text{as} \quad n \to \infty,$$

that means L_2 is strongly continuous and therefore it is continuous.
In what follows, we discuss the continuity of L_3.
Let (u_n) be such that $u_n \rightharpoonup u$ in X. So (u_n) is bounded in X and by Lemma 2.5, we have that (u_n) is bounded in $C_0[0, +\infty)$. By Lemma 2.5, $u_n \to u$ in $C_0[0, +\infty)$. We have

$$\begin{aligned}
\|L_3(u_n) - L_3(u)\|_{X^*} &= \sup_{\|v\|\leq 1} |\langle L_3(u_n) - L_3(u)\rangle| \\
&= \sup_{\|v\|\leq 1} \Big| \int\limits_0^{+\infty} \Big[f(x, u_n(x)) - f(x, u(x))\Big] dx\Big| \\
&\leq \sup_{\|v\|\leq 1} \Big(\int\limits_0^{+\infty} |f(x, u_n(x))v(x)| dx\Big)
\end{aligned}$$

$$
\begin{aligned}
&+\sup_{\|v\|\leq 1}\Big(\int_0^{+\infty}|f(x,u(x))v(x)|dx\Big)\\
&\leq \sup_{\|v\|\leq 1}\Big(\int_0^{+\infty}(c_1(x)|v(x)|+c_2(x)|u_n(x)|^{\mu}|v(x)|)dx\Big)\\
&+\sup_{\|v\|\leq 1}\Big(\int_0^{+\infty}(c_1(x)|v(x)|+c_2(x)|u(x)|^{\mu}|v(x)|dx\Big)\\
&\leq 2\|c_1\|_{\frac{p}{p-1}}+\Big(\|u_n\|_{\infty}^{\mu}+\|u\|_{\infty}^{\mu}\Big)\\
&\leq 2\|c_1\|_{\frac{p}{p-1}}+M^{\mu}\|c_2\|_{\frac{p}{p-1}}\Big(\|u_n\|^{\mu}+\|u\|^{\mu}\Big)\\
&\leq 2\|c_1\|_{\frac{p}{p-1}}+CM^{\mu}\|c_2\|_{\frac{p}{p-1}}.
\end{aligned}
$$

for some constant $C>0$. Since $u_n\to u$, $n\to\infty$ in $C_0[0,+\infty)$, we obtain

$$
\int_0^{+\infty}\left(f(x,u_n(x))-f(x,u(x))\right)v(x)dx\to 0 \quad \text{as} \quad n\to\infty,
$$

this implies that L_3 is continuous. Thus the operator $\mathcal{L}$ is continuous and hence it is demicontinuous. So assumption (i) of Lemma 2.7 holds.

Claim 2. $\mathcal{L}$ is monotone.

By $(H1)$, for all $u,v\in X$, we have

$$
\begin{aligned}
\langle\mathcal{L}(u)-\mathcal{L}(v),u-v\rangle &= \int_0^{+\infty}\rho(x)\Big[|u'(x)|^{p-2}u'(x)-|v'(x)|^{p-2}v'(x)\Big](u'(x)-v'(x))dx\\
&+\int_0^{+\infty}\Big[|v(x)|^{p-2}v(x)-|v(x)|^{p-2}v(x)\Big](u(x)-v(x))dx\\
&-\int_0^{+\infty}\Big[f(x,u(x))-f(x,v(x))\Big](u(x)-v(x))dx\\
&+\sum_{j=1}^{\infty}\Big[g(x_j)I_j(u(x_j))-g(x_j)I_j(v(x_j))\Big](u(x_j)-v(x_j))\\
&\geq \int_0^{+\infty}\rho(x)\Big[|u'(x)|^{p-2}u'(x)-|v'(x)|^{p-2}v'(x)\Big](u'(x)-v'(x))dx\\
&+\int_0^{+\infty}\Big[|v(x)|^{p-2}v(x)-|v(x)|^{p-2}v(x)\Big](u(x)-v(x))dx
\end{aligned}
$$

$$\geq \Big(\|u\|^{p-1} - \|v\|^{p-1}\Big)\Big(\|u\| - \|v\|\Big)$$
$$\geq 0,$$

so, $\mathcal{L}$ is monotone.

Claim 3. $\mathcal{L}$ is coercive.

For all $u, v \in X$, we have

$$\langle \mathcal{L}(u), u\rangle = \|u\|^p + \sum_{j=1}^{\infty} g(x_j) I_j(u(x_j))u(x_j) - \int_0^{+\infty} f(x, u(x))u(x)dx.$$

From Lemma 2.2 and Lemma 2.3, combining assumption $(H2)$ and $(H3)$, we find

$$\begin{aligned}\langle \mathcal{L}(u), u\rangle &\geq \|u\|^p - \sum_{j=1}^{\infty} g(x_j) I_j(u(x_j))u(x_j) - \int_0^{+\infty} f(x, u(x))u(x)dx \\ &\geq \|u\|^p - \sum_{j=1}^{\infty} g(x_j)\Big(\alpha_j + \beta_j |u(x_j)|^{\gamma-1}\Big)u(x_j) \\ &\quad - \int_0^{+\infty} \Big(c_1(x) + c_2(x)|u(x)|^{\mu}\Big)|u(x)|dx\end{aligned}$$

hence,

$$\begin{aligned}\langle \mathcal{L}(u), u\rangle &\geq \|u\|^p - \Big(\sum_{j=1}^{\infty} \alpha_j g(x_j) + \sum_{j=1}^{\infty} \beta_j g(x_j)\|u\|_{\infty}^{\gamma-1}\Big)\|u\|_{\infty} - \int_0^{+\infty} c_1(x)|u(x)|dx \\ &\quad - \int_0^{+\infty} c_2(x)|u(x)|^{\mu}|u(x)|dx \\ &\geq \|u\|^p - \Big(\sum_{j=1}^{\infty} \alpha_j g(x_j) + \sum_{j=1}^{\infty} \beta_j g(x_j)\|u\|_{\infty}^{\gamma-1}\Big)\|u\|_{\infty} - \|c_1\|_{\frac{p}{p-1}}\|u\|_p \\ &\quad - \|u\|_{\infty}^{\mu}\|c_2\|_{\frac{p}{p-1}}\|u\|_p \\ &\geq \|u\|^p - \Big(M\sum_{j=1}^{\infty} \alpha_j g(x_j) + M_0^{\frac{1}{p}}\|c_1\|_{\frac{p}{p-1}}\Big)\|u\| - \Big(M^{\gamma}\sum_{j=1}^{\infty} \beta_j g(x_j)\Big)\|u\|^{\gamma} \\ &\quad - M_0^{\frac{1}{p}} M^{\mu}\|c_2\|_{\frac{p}{p-1}}\|u\|^{\mu+1},\end{aligned}$$

so $\lim_{\|u\|\to\infty} \frac{\langle \mathcal{L}(u), u\rangle}{\|u\|} = +\infty$.

Lemma 2.7 guarantees that problem (1.1) has a weak solution.

Claim 4. Uniqueness.

For all $u, v \in X$, $u \neq v$, we have

$$\langle \mathcal{L}(u) - \mathcal{L}(v), u - v\rangle \geq \Big(\|u\|^{p-1} - \|v\|^{p-1}\Big)\Big(\|u\| - \|v\|\Big) > 0,$$

so $\mathcal{L}$ is strictly monotone. $\square$

Example 3.2. Let $p = 4$ and $\gamma = \frac{5}{2}$. Consider the problem

$$\begin{cases} -(e^{3x}|u'|u')' + |u|u &= e^{-x} - 2e^{-3x}u^2, \qquad \text{a.e. } x \neq x_j,\ x \geq 0, \\ \triangle(e^{3j}u'(j)) &= e^{-j}\left(\frac{1}{j} + \frac{1}{j^2}|u(j)|^{\frac{3}{2}}\right), \qquad j \in \mathbb{N}^*, \\ u(0) = u(\infty) &= 0, \end{cases}$$

where $c_1(x) = e^{-x}$, $c_2(x) = 2e^{-3x}$ and $g(x) = e^{-x}$.
It's clear that $(H1) - (H3)$ hold true. Hence, we may apply Lemma 2.7 and conclude that (1.1) has precisely a weak solution.

Next, we consider the limit case $\mu = p - 1$.

Theorem 3.3. *Assume that* $(H1)$ *and* $(H2)$ *are hold both with*

(H4) *There exist positive functions* $c_1, c_2 \in L^{\frac{p}{p-1}}[0,\infty)$ *such that*

$$|f(x,u)| \leq c_1(x) + c_2(x)|u|^{p-1}, \quad \forall (x,u) \in [0,\infty) \times \mathbb{R}.$$

with

$$M_0^{\frac{1}{p}} M^{p-1} \|c_2\|_{\frac{p}{p-1}} < 1.$$

Then (1.1) has a unique weak solution.

Proof. Arguing as in the proof of Theorem 3.1, we prove that $\mathcal{L}$ is bounded, demicontinuous and monotone.
We check that $\mathcal{L}$ is a coercive. Indeed, under $(H2)$, $(H4)$, in view of Lemma 2.2 and Lemma 2.3, it is easy to verify that

$$\begin{aligned} \langle \mathcal{L}(u), u\rangle &\geq \|u\|^p - \Big(\sum_{j=1}^{\infty} \alpha_j g(x_j) + \sum_{j=1}^{\infty} \beta_j g(x_j)\|u\|_{\infty}^{\gamma-1}\Big)\|u\|_{\infty} - \int_0^{+\infty} c_1(x)|u(x)|dx \\ &\quad - \int_0^{+\infty} c_2(x)|u(x)|^{p-1}|u(x)|dx \\ &\geq \|u\|^p - \Big(\sum_{j=1}^{\infty} \alpha_j g(x_j) + \sum_{j=1}^{\infty} \beta_j g(x_j)\|u\|_{\infty}^{\gamma-1}\Big)\|u\|_{\infty} \\ &\quad - \|c_1\|_{\frac{p}{p-1}}\|u\|_p - \|u\|_{\infty}^{p-1}\|c_2\|_{\frac{p}{p-1}}\|u\|_p \\ &\geq \|u\|^p - M\Big(\sum_{j=1}^{\infty} \alpha_j g(x_j) + M^{\gamma-1}\|u\|^{\gamma-1}\sum_{j=1}^{\infty} \beta_j g(x_j)\Big)\|u\| - M_0^{\frac{1}{p}}\|c_1\|_{\frac{p}{p-1}}\|u\| \\ &\quad - M_0^{\frac{1}{p}} M^{p-1}\|c_2\|_{\frac{p}{p-1}}\|u\|^p \\ &\geq \Big(1 - M_0^{\frac{1}{p}} M^{p-1}\|c_2\|_{\frac{p}{p-1}}\Big)\|u\|^p - \Big(M\sum_{j=1}^{\infty} \alpha_j g(x_j) + M_0^{\frac{1}{p}}\|c_1\|_{\frac{p}{p-1}}\Big)\|u\| \end{aligned}$$

$$- \Big(M^{\gamma}\sum_{j=1}^{\infty}\beta_j g(x_j)\Big)\|u\|^{\gamma},$$

we conclude that $\lim_{\|u\|\to\infty}\frac{\langle\mathcal{L}(u),u\rangle}{\|u\|} = +\infty$.
Theorem 3.3 guarantees that problem (1.1) has a unique weak solution. □

Example 3.4. Let $p = 2$ and $\gamma = \frac{1}{2}$. Consider the problem

$$\begin{cases} -(e^x|u'|u')' + |u|u &= e^{-\frac{1}{2}x} - e^{-x}|u|, \quad \text{a.e. } x \neq x_j,\ x \geq 0, \\ \triangle(e^j u'(j)) &= e^{-j}\left(\frac{1}{j} + \frac{1}{j^2}|u(j)|^{\frac{1}{2}}\right), \qquad j \in \mathbb{N}^*, \\ u(0) = u(\infty) &= 0, \end{cases}$$

where $c_1(x) = e^{-\frac{1}{2}x}$, $c_2(x) = e^{-x}$, $\|c_2\|_2 = \frac{1}{\sqrt{2}}$, $g(x) = e^{-x}$ and $M = M_0 = 1$.
It's clear that $(H1), (H2)$ and $(H4)$ hold true. Hence, from Lemma 2.7 we find that problem (1.1) has precisely a weak solution.

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