

Results on ϕ –like functions involving Hadamard product

Hardeep Kaur , Richa Brar  and Sukhwinder Singh Billing 

Abstract. In this paper, we derive a differential subordination theorem involving convolution of normalized analytic functions. By selecting different dominants to our main result, we find certain sufficient conditions for ϕ –likeness and parabolic ϕ –likeness of functions in class $\mathcal{A}$.

Mathematics Subject Classification (2010): 30C80; 30C45.

Keywords: Analytic function, differential subordination, parabolic ϕ –like function, ϕ –like function.

1. Introduction

A function f is said to be analytic at a point z in a domain $\mathbb{D}$ if it is differentiable not only at z but also in some neighbourhood of the point z . A function f is said to be analytic in a domain $\mathbb{D}$ if it is analytic at each point of $\mathbb{D}$. Let $\mathcal{H}$ be the class of analytic functions in the open unit disk $\mathbb{E} = \{z \in \mathbb{C} : |z| < 1\}$. For $a \in \mathbb{C}$ and $n \in \mathbb{N}$, let $\mathcal{H}[a, n]$ be the subclass of $\mathcal{H}$ consisting of the functions of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots$$

Let $\mathcal{A}$ be the class of functions f , analytic in the unit disk $\mathbb{E}$ and normalized by the conditions $f(0) = f'(0) - 1 = 0$.

Let $\mathcal{S}$ denote the class of all analytic univalent functions f defined in the open unit disk $\mathbb{E}$ which are normalized by the conditions $f(0) = f'(0) - 1 = 0$. The Taylor series expansion of any function $f \in \mathcal{S}$ is

$$f(z) = z + a_2 z^2 + a_3 z^3 + \dots$$

Received 25 February 2025; Accepted 31 May 2025.

© Studia UBB MATHEMATICA. Published by Babeş-Bolyai University

 This work is licensed under a Creative Commons Attribution-NonCommercial-NoDerivatives 4.0 International License.

Let the functions f and g be analytic in $\mathbb{E}$. We say that f is subordinate to g written as $f \prec g$ in $\mathbb{E}$, if there exists a Schwarz function ϕ in $\mathbb{E}$ (i.e. ϕ is regular in $|z| < 1$, $\phi(0) = 0$ and $|\phi(z)| \leq |z| < 1$) such that

$$f(z) = g(\phi(z)), \quad |z| < 1.$$

Let $\Phi : \mathbb{C}^2 \times \mathbb{E} \rightarrow \mathbb{C}$ be an analytic function, p an analytic function in $\mathbb{E}$ with $(p(z), zp'(z); z) \in \mathbb{C}^2 \times \mathbb{E}$ for all $z \in \mathbb{E}$ and h be univalent in $\mathbb{E}$. Then the function p is said to satisfy first order differential subordination if

$$\Phi(p(z), zp'(z); z) \prec h(z), \quad \Phi(p(0), 0; 0) = h(0). \quad (1.1)$$

A univalent function q is called dominant of the differential subordination (1.1) if $p(0) = q(0)$ and $p \prec q$ for all p satisfying (1.1). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (1.1), is said to be the best dominant of (1.1). The best dominant is unique up to a rotation of $\mathbb{E}$.

Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and $g(z) = \sum_{k=0}^{\infty} b_k z^k$ be two analytic functions, then the Hadamard product or convolution of f and g , written as $f * g$ is defined by

$$(f * g)(z) = \sum_{k=0}^{\infty} a_k b_k z^k.$$

Ronning [8] and Ma and Minda [6] studied the domain Ω and the function $q(z)$ defined below:

$$\Omega = \left\{ u + iv : u > \sqrt{(u-1)^2 + v^2} \right\}.$$

Clearly the function

$$q(z) = 1 + \frac{2}{\pi^2} \left(\log \left(\frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right)^2$$

maps the unit disk $\mathbb{E}$ onto the domain Ω . Let ϕ be analytic in a domain containing $f(\mathbb{E})$, $\phi(0) = 0$ and $\Re(\phi'(0)) > 0$. Then, the function $f \in \mathcal{A}$ is said to be ϕ -like in $\mathbb{E}$, if

$$\Re \left(\frac{zf'(z)}{\phi(f(z))} \right) > 0, \quad z \in \mathbb{E}.$$

This concept was introduced by Brickman [4]. He proved that an analytic function $f \in \mathcal{A}$ is univalent if and only if f is ϕ -like for some analytic function ϕ . Later, Ruscheweyh [9] investigated the following general class of ϕ -like functions:

Let ϕ be analytic in a domain containing $f(\mathbb{E})$, where $\phi(0) = 0$, $\phi'(0) = 1$ and $\phi(w) \neq 0$ for some $w \in f(\mathbb{E}) \setminus \{0\}$, then the function $f \in \mathcal{A}$ is called ϕ -like with respect to a univalent function q , $q(0) = 1$, if

$$\frac{zf'(z)}{\phi(f(z))} \prec q(z), \quad z \in \mathbb{E}.$$

A function $f \in \mathcal{A}$ is said to be parabolic ϕ -like in $\mathbb{E}$, if

$$\Re \left(\frac{zf'(z)}{\phi(f(z))} \right) > \left| \frac{zf'(z)}{\phi(f(z))} - 1 \right|, \quad z \in \mathbb{E}. \quad (1.2)$$

Equivalently, condition (1.2) can be written as:

$$\frac{zf'(z)}{\phi(f(z))} \prec q(z) = 1 + \frac{2}{\pi^2} \left(\log \left(\frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right)^2.$$

In 2007, Shanmugham et al. [10] proved the following result for ϕ -like functions.

Theorem 1.1. *Let $q(z) \neq 0$ be analytic and univalent in $\mathbb{E}$ with $q(0) = 1$ such that $\frac{zq'(z)}{q(z)}$ is starlike univalent in $\mathbb{E}$. Let $q(z)$ satisfy*

$$\Re \left[1 + \frac{\alpha q(z)}{\gamma} - \frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q'(z)} \right] > 0.$$

Let

$$\Psi(\alpha, \gamma, g; z) := \alpha \left\{ \frac{z(f * g)'(z)}{\phi(f * g)(z)} \right\} + \gamma \left\{ 1 + \frac{z(f * g)''(z)}{(f * g)'(z)} - \frac{z(\phi(f * g)(z))'}{\phi(f * g)(z)} \right\}.$$

If q satisfies

$$\Psi(\alpha, \gamma, g; z) \prec \alpha q(z) + \frac{\gamma zq'(z)}{q(z)},$$

then

$$\frac{z(f * g)'(z)}{\phi(f * g)(z)} \prec q(z)$$

and q is the best dominant.

Later in 2018, Brar and Billing [3] obtained the following result.

Theorem 1.2. *Let $q(z) \neq 0$, be a univalent function in $\mathbb{E}$ such that*

$$(i) \quad \Re \left[1 + \frac{zq''(z)}{q'(z)} + (\gamma - 1) \frac{zq'(z)}{q(z)} \right] > 0 \text{ and}$$

$$(ii) \quad \Re \left[1 + \frac{zq''(z)}{q'(z)} + (\gamma - 1) \frac{zq'(z)}{q(z)} + \frac{\beta(1 - \alpha)}{\alpha} (q(z))^{\beta - \gamma} + \gamma \right] > 0.$$

If f and $g \in \mathcal{A}$ satisfy

$$\begin{aligned} (1 - \alpha) \left[\frac{z(f * g)'(z)}{\phi(f * g)(z)} \right]^\beta + \alpha \left[\frac{z(f * g)'(z)}{\phi(f * g)(z)} \right]^\gamma & \left[2 + \frac{z(f * g)''(z)}{(f * g)'(z)} - \frac{z(\phi((f * g)(z)))'}{\phi((f * g)(z)))} \right] \\ & \prec (1 - \alpha)(q(z))^\beta + \alpha(q(z))^\gamma \left(1 + \frac{zq'(z)}{q(z)} \right), \end{aligned}$$

then

$$\frac{z(f * g)'(z)}{\phi(f * g)(z)} \prec q(z), \quad z \in \mathbb{E},$$

where α, β, γ are complex numbers such that $\alpha \neq 0$, and $q(z)$ is the best dominant.

In 2019, Adegani et al. [1] established sufficient subordination conditions for functions to be close-to-convex.

Moreover, this study is also motivated by the findings of Cho et al. [5] and Adegani et al. [2] who explored subordination conditions in geometric function theory.

The aim of the present investigation is to find sufficient conditions for parabolic ϕ -likeness and ϕ -likeness of analytic functions.

To prove our main result, we shall use the following lemma of Miller and Mocanu.

Lemma 1.3. ([7], Theorem 3.4h, p.132). *Let q be univalent in $\mathbb{E}$ and let θ and φ be analytic in a domain $\mathbb{D}$ containing $q(\mathbb{E})$, with $\varphi(w) \neq 0$, when $w \in q(\mathbb{E})$. Set $Q(z) = zq'(z)\varphi[q(z)]$, $h(z) = \theta[q(z)] + Q(z)$ and suppose that either*

1. *h is convex, or*

2. *Q is starlike.*

In addition, assume that

3. *$\Re \left(\frac{zh'(z)}{Q(z)} \right) > 0$ for all $z \in \mathbb{E}$.*

If p is analytic in $\mathbb{E}$, with $p(0) = q(0)$, $p(\mathbb{E}) \subset \mathbb{D}$ and

$$\theta[p(z)] + zp'(z)\varphi[p(z)] \prec \theta[q(z)] + zq'(z)\varphi[q(z)], \quad z \in \mathbb{E},$$

then $p(z) \prec q(z)$ and q is the best dominant.

2. A subordination theorem

In what follows, all the powers taken are principal ones.

Theorem 2.1. *Let β and γ be complex numbers such that $\beta \neq 0$. Let $q(z) \neq 0$, be a univalent function in $\mathbb{E}$ such that*

$$(i) \Re \left[1 + \frac{zq''(z)}{q'(z)} + \left(\frac{\gamma}{\beta} - 1 \right) \frac{zq'(z)}{q(z)} \right] > 0 \text{ and}$$

$$(ii) \Re \left[1 + \frac{zq''(z)}{q'(z)} + \left(\frac{\gamma}{\beta} - 1 \right) \frac{zq'(z)}{q(z)} + \frac{a}{c} \left(\frac{\gamma}{\beta} + 1 \right) q(z) + \frac{b}{c} \left(\frac{\gamma}{\beta} + 2 \right) q^2(z) \right] > 0, \text{ where}$$

*a , b and c are real numbers with $c \neq 0$. Let ϕ be analytic function in the domain containing $(f * g)(\mathbb{E})$ such that $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in (f * g)(\mathbb{E}) \setminus \{0\}$.*

*If $f, g \in \mathcal{A}$, $\frac{z(f * g)'(z)}{\phi((f * g)(z))} \neq 0$, $z \in \mathbb{E}$, satisfy*

$$\left[\frac{z(f * g)'(z)}{\phi((f * g)(z))} \right]^\gamma \cdot \left\{ a \frac{z(f * g)'(z)}{\phi((f * g)(z))} + b \left[\frac{z(f * g)'(z)}{\phi((f * g)(z))} \right]^2 + c \left[1 + \frac{z(f * g)''(z)}{(f * g)'(z)} - \frac{z(\phi((f * g)(z)))'}{\phi((f * g)(z))} \right] \right\}^\beta \prec [q(z)]^\gamma \left[aq(z) + bq^2(z) + c \frac{zq'(z)}{q(z)} \right]^\beta, \quad (2.1)$$

then

$$\frac{z(f * g)'(z)}{\phi((f * g)(z))} \prec q(z), \quad z \in \mathbb{E},$$

and $q(z)$ is the best dominant.

Proof. Define the function p by

$$p(z) = \frac{z(f * g)'(z)}{\phi((f * g)(z))}, \quad z \in \mathbb{E}.$$

Then the function p is analytic in $\mathbb{E}$ and $p(0) = 1$.

Therefore, from equation (2.1), we get:

$$[p(z)]^\gamma \left[ap(z) + bp^2(z) + c \frac{zp'(z)}{p(z)} \right]^\beta \prec [q(z)]^\gamma \left[aq(z) + bq^2(z) + c \frac{zq'(z)}{q(z)} \right]^\beta$$

or

$$\begin{aligned} a[p(z)]^{\frac{\gamma}{\beta}+1} + b[p(z)]^{\frac{\gamma}{\beta}+2} + c[p(z)]^{\frac{\gamma}{\beta}-1} zp'(z) \\ \prec a[q(z)]^{\frac{\gamma}{\beta}+1} + b[q(z)]^{\frac{\gamma}{\beta}+2} + c[q(z)]^{\frac{\gamma}{\beta}-1} zq'(z) \end{aligned}$$

Let the functions θ and φ be defined as:

$$\theta(w) = aw^{\frac{\gamma}{\beta}+1} + bw^{\frac{\gamma}{\beta}+2} \text{ and } \varphi(w) = cw^{\frac{\gamma}{\beta}-1}$$

Clearly, the functions θ and φ are analytic in domain $\mathbb{D} = \mathbb{C} \setminus \{0\}$ and $\varphi(w) \neq 0$ in $\mathbb{D}$. Therefore,

$$Q(z) = \varphi[q(z)]zq'(z) = c[q(z)]^{\frac{\gamma}{\beta}-1}zq'(z)$$

and

$$h(z) = \theta[q(z)] + Q(z) = a[q(z)]^{\frac{\gamma}{\beta}+1} + b[q(z)]^{\frac{\gamma}{\beta}+2} + c[q(z)]^{\frac{\gamma}{\beta}-1}zq'(z)$$

On differentiating, we get

$$\frac{zh'(z)}{Q(z)} = 1 + \frac{zq''(z)}{q'(z)} + \left(\frac{\gamma}{\beta} - 1 \right) \frac{zq'(z)}{q(z)}$$

and

$$\frac{zh'(z)}{Q(z)} = 1 + \frac{zq''(z)}{q'(z)} + \left(\frac{\gamma}{\beta} - 1 \right) \frac{zq'(z)}{q(z)} + \frac{a}{c} \left(\frac{\gamma}{\beta} + 1 \right) q(z) + \frac{b}{c} \left(\frac{\gamma}{\beta} + 2 \right) q^2(z).$$

In view of the given conditions (i) and (ii), we see that Q is starlike and

$$\Re \left(\frac{zh'(z)}{Q(z)} \right) > 0.$$

Therefore, the proof, now follows from Lemma [1.3]. $\square$

For $g(z) = \frac{z}{1-z}$ in Theorem 2.1, we have

Theorem 2.2. Let β and γ be complex numbers such that $\beta \neq 0$. Let $q(z) \neq 0$, be a univalent function in $\mathbb{E}$ which satisfy conditions (i) and (ii) of Theorem 2.1. Let ϕ be analytic function in the domain containing $f(\mathbb{E})$ such that $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in f(\mathbb{E}) \setminus \{0\}$. If $f \in \mathcal{A}$, $\frac{zf'(z)}{\phi(f(z))} \neq 0$, $z \in \mathbb{E}$, satisfy

$$\begin{aligned} \left\{ \frac{zf'(z)}{\phi(f(z))} \right\}^\gamma \left\{ a \frac{zf'(z)}{\phi(f(z))} + b \left(\frac{zf'(z)}{\phi(f(z))} \right)^2 + c \left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(f(z)))'}{\phi(f(z))} \right) \right\}^\beta \\ \prec (q(z))^\gamma \left\{ aq(z) + bq^2(z) + c \frac{zq'(z)}{q(z)} \right\}^\beta, \end{aligned}$$

where a , b and c are real numbers with $c \neq 0$, then

$$\frac{zf'(z)}{\phi(f(z))} \prec q(z), \quad z \in \mathbb{E},$$

and $q(z)$ is the best dominant.

3. Applications to parabolic ϕ -like functions

Remark 3.1. Selecting $q(z) = 1 + \frac{2}{\pi^2} \left(\log \left(\frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right)^2$, $\beta = \gamma = 1$ in Theorem 2.2, then after having some calculations,

$$\begin{aligned} q'(z) &= \frac{4}{\pi^2 \sqrt{z}(1-z)} \log \left(\frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \\ \frac{q'(z)}{q(z)} &= \frac{\frac{4}{\pi^2 \sqrt{z}(1-z)} \left[\log \left(\frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right]}{1 + \frac{2}{\pi^2} \left[\log \left(\frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right]^2} \\ \frac{q''(z)}{q'(z)} &= \frac{3z-1}{2z(1-z)} + \frac{1}{\sqrt{z}(1-z) \log \left(\frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right)}. \end{aligned}$$

Thus the conditions (i) and (ii) of Theorem 2.1 becomes

$$1 + \frac{zq''(z)}{q'(z)} + \left(\frac{\gamma}{\beta} - 1 \right) \frac{zq'(z)}{q(z)} = 1 + \frac{zq''(z)}{q'(z)} = \frac{1+z}{2(1-z)} + \frac{\sqrt{z}}{(1-z) \log \left(\frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right)}$$

and

$$\begin{aligned} 1 + \frac{zq''(z)}{q'(z)} + \left(\frac{\gamma}{\beta} - 1 \right) \frac{zq'(z)}{q(z)} + \frac{a}{c} \left(\frac{\gamma}{\beta} + 1 \right) q(z) + \frac{b}{c} \left(\frac{\gamma}{\beta} + 2 \right) q^2(z) \\ = 1 + \frac{zq''(z)}{q'(z)} + \frac{2a}{c} q(z) + \frac{3b}{c} q^2(z) \\ = \frac{1+z}{2(1-z)} + \frac{\sqrt{z}}{(1-z) \log \left(\frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right)} + \frac{2a}{c} \left[1 + \frac{2}{\pi^2} \left(\log \left(\frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right)^2 \right] \\ + \frac{3b}{c} \left[1 + \frac{2}{\pi^2} \left(\log \left(\frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) \right)^2 \right]^2. \end{aligned}$$

Therefore, for real numbers a , b , c with $c \neq 0$ and $\frac{a}{c}, \frac{b}{c} \geq 0$, we notice that $q(z)$ satisfy conditions (i) and (ii) of Theorem 2.1. Thus, we derive the following result from Theorem 2.2.

Theorem 3.2. Let ϕ be analytic function in the domain containing $f(\mathbb{E})$ such that $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in f(\mathbb{E}) \setminus \{0\}$. If $f \in \mathcal{A}$, $\frac{zf'(z)}{\phi(f(z))} \neq 0$, $z \in \mathbb{E}$, satisfy

$$\begin{aligned} & a \left(\frac{zf'(z)}{\phi(f(z))} \right)^2 + b \left(\frac{zf'(z)}{\phi(f(z))} \right)^3 \\ & + cz \left\{ \frac{[\phi(f(z))] [zf''(z) + f'(z)] - zf'(z) [\phi(f(z))]' }{[\phi(f(z))]^2} \right\} \\ & \prec a \left[1 + \frac{2}{\pi^2} \left(\log \left(\frac{1+\sqrt{z}}{1-\sqrt{z}} \right) \right)^2 \right]^2 + b \left[1 + \frac{2}{\pi^2} \left(\log \left(\frac{1+\sqrt{z}}{1-\sqrt{z}} \right) \right)^2 \right]^3 \\ & + \frac{4c\sqrt{z}}{\pi^2(1-z)} \log \left(\frac{1+\sqrt{z}}{1-\sqrt{z}} \right), \end{aligned}$$

where a, b, c are real numbers such that $c \neq 0$ and $\frac{a}{c}, \frac{b}{c} \geq 0$, then

$$\frac{zf'(z)}{\phi(f(z))} \prec 1 + \frac{2}{\pi^2} \left(\log \left(\frac{1+\sqrt{z}}{1-\sqrt{z}} \right) \right)^2, \quad z \in \mathbb{E}.$$

Hence f is parabolic ϕ -like.

Remark 3.3. Selecting $q(z) = 1 + \frac{2}{\pi^2} \left(\log \left(\frac{1+\sqrt{z}}{1-\sqrt{z}} \right) \right)^2$, $\beta = 1$ and $\gamma = 0$ in Theorem 2.2, then after having some calculations, we have

$$\begin{aligned} q'(z) &= \frac{4}{\pi^2 \sqrt{z}(1-z)} \log \left(\frac{1+\sqrt{z}}{1-\sqrt{z}} \right) \\ \frac{q'(z)}{q(z)} &= \frac{\frac{4}{\pi^2 \sqrt{z}(1-z)} \left[\log \left(\frac{1+\sqrt{z}}{1-\sqrt{z}} \right) \right]}{1 + \frac{2}{\pi^2} \left[\log \left(\frac{1+\sqrt{z}}{1-\sqrt{z}} \right) \right]^2} \\ \frac{q''(z)}{q'(z)} &= \frac{3z-1}{2z(1-z)} + \frac{1}{\sqrt{z}(1-z) \log \left(\frac{1+\sqrt{z}}{1-\sqrt{z}} \right)}. \end{aligned}$$

Thus the conditions (i) and (ii) of Theorem 2.1 becomes

$$\begin{aligned} 1 + \frac{zq''(z)}{q'(z)} + \left(\frac{\gamma}{\beta} - 1 \right) \frac{zq'(z)}{q(z)} &= 1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} \\ &= \frac{1+z}{2(1-z)} + \frac{\sqrt{z}}{(1-z) \log \left(\frac{1+\sqrt{z}}{1-\sqrt{z}} \right)} - \frac{\frac{4\sqrt{z}}{\pi^2(1-z)} \left[\log \left(\frac{1+\sqrt{z}}{1-\sqrt{z}} \right) \right]}{1 + \frac{2}{\pi^2} \left[\log \left(\frac{1+\sqrt{z}}{1-\sqrt{z}} \right) \right]^2} \end{aligned}$$

and

$$\begin{aligned}
 & 1 + \frac{zq''(z)}{q'(z)} + \left(\frac{\gamma}{\beta} - 1\right) \frac{zq'(z)}{q(z)} + \frac{a}{c} \left(\frac{\gamma}{\beta} + 1\right) q(z) + \frac{b}{c} \left(\frac{\gamma}{\beta} + 2\right) q^2(z) \\
 &= 1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} + \frac{a}{c} q(z) + \frac{2b}{c} q^2(z) \\
 &= \frac{1+z}{2(1-z)} + \frac{\sqrt{z}}{(1-z) \log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)} - \frac{\frac{4\sqrt{z}}{\pi^2(1-z)} \left[\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right]}{1 + \frac{2}{\pi^2} \left[\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right]^2} \\
 &\quad + \frac{a}{c} \left[1 + \frac{2}{\pi^2} \left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right] + \frac{2b}{c} \left[1 + \frac{2}{\pi^2} \left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right]^2.
 \end{aligned}$$

Therefore, for real numbers a, b, c with $c \neq 0$ and $\frac{a}{c}, \frac{b}{c} \geq 0$, we notice that $q(z)$ satisfy conditions (i) and (ii) of Theorem 2.1. Thus, we derive the following result from Theorem 2.2.

Theorem 3.4. Let ϕ be analytic function in the domain containing $f(\mathbb{E})$ such that $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in f(\mathbb{E}) \setminus \{0\}$. If $f \in \mathcal{A}$, $\frac{zf'(z)}{\phi(f(z))} \neq 0$, $z \in \mathbb{E}$, satisfy

$$\begin{aligned}
 & a \frac{zf'(z)}{\phi(f(z))} + b \left(\frac{zf'(z)}{\phi(f(z))}\right)^2 + c \left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(f(z)))'}{\phi(f(z))}\right) \\
 & \prec a \left[1 + \frac{2}{\pi^2} \left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right] + b \left[1 + \frac{2}{\pi^2} \left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right]^2 \\
 & \quad + \frac{\frac{4c\sqrt{z}}{\pi^2(1-z)} \left[\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right]}{1 + \frac{2}{\pi^2} \left[\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right]^2},
 \end{aligned}$$

where a, b, c are real numbers such that $c \neq 0$ and $\frac{a}{c}, \frac{b}{c} \geq 0$, then

$$\frac{zf'(z)}{\phi(f(z))} \prec 1 + \frac{2}{\pi^2} \left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2, \quad z \in \mathbb{E}.$$

Hence f is parabolic ϕ -like.

4. Applications to ϕ -like functions

Remark 4.1. By taking $q(z) = 1 + tz$, $0 < t \leq 1$, $\beta = \gamma = 1$ in Theorem 2.2, then after having some calculations we have

$$1 + \frac{zq''(z)}{q'(z)} = 1$$

and

$$1 + \frac{zq''(z)}{q'(z)} + \frac{2a}{c}q(z) + \frac{3b}{c}q^2(z) = 1 + \frac{2a}{c}(1+tz) + \frac{3b}{c}(1+tz)^2.$$

Thus for real numbers a , b and c ($\neq 0$) such that $0 \leq \frac{a}{c} \leq 1$,

$0 \leq \frac{b}{c} \leq 1$, we observe that $q(z)$ satisfy conditions (i) and (ii) of Theorem 2.1. Therefore, we immediately, arrive at the following result from Theorem 2.2.

Theorem 4.2. Let ϕ be analytic function in the domain containing $f(\mathbb{E})$ such that $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in f(\mathbb{E}) \setminus \{0\}$. If $f \in \mathcal{A}$, $\frac{zf'(z)}{\phi(f(z))} \neq 0$, $z \in \mathbb{E}$, satisfy

$$\begin{aligned} & a \left(\frac{zf'(z)}{\phi(f(z))} \right)^2 + b \left(\frac{zf'(z)}{\phi(f(z))} \right)^3 \\ & + cz \left\{ \frac{[\phi(f(z))][zf''(z) + f'(z)] - zf'(z)[\phi(f(z))']}{[\phi(f(z))]^2} \right\} \\ & \prec a(1+tz)^2 + b(1+tz)^3 + ctz, \end{aligned}$$

where a , b , c are real numbers such that $c \neq 0$, $0 \leq \frac{a}{c} \leq 1$ and $0 \leq \frac{b}{c} \leq 1$, then

$$\frac{zf'(z)}{\phi(f(z))} \prec 1 + tz, \quad 0 < t \leq 1, \quad z \in \mathbb{E}.$$

Therefore, f is ϕ -like in $\mathbb{E}$.

Remark 4.3. When we select $q(z) = e^z$, $\beta = \gamma = 1$ in Theorem 2.2, a little calculation yields that

$$1 + \frac{zq''(z)}{q'(z)} = 1 + z$$

and

$$1 + \frac{zq''(z)}{q'(z)} + \frac{2a}{c}q(z) + \frac{3b}{c}q^2(z) = 1 + z + \frac{2a}{c}e^z + \frac{3b}{c}e^{2z}.$$

For real numbers a , b , c such that $c \neq 0$, $\frac{a}{c} \geq 0.4$ and $\frac{b}{c} = 1$, we see that $q(z)$ satisfy conditions (i) and (ii) of Theorem 2.1. Hence, we obtain the following result from Theorem 2.2.

Theorem 4.4. Let ϕ be analytic function in the domain containing $f(\mathbb{E})$ such that $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in f(\mathbb{E}) \setminus \{0\}$. If $f \in \mathcal{A}$, $\frac{zf'(z)}{\phi(f(z))} \neq 0$, $z \in \mathbb{E}$, satisfy

$$\begin{aligned} & a \left(\frac{zf'(z)}{\phi(f(z))} \right)^2 + b \left(\frac{zf'(z)}{\phi(f(z))} \right)^3 \\ & + cz \left\{ \frac{[\phi(f(z))][zf''(z) + f'(z)] - zf'(z)[\phi(f(z))']}{[\phi(f(z))]^2} \right\} \\ & \prec ae^{2z} + be^{3z} + cze^z, \end{aligned}$$

where a, b, c are real numbers such that $c \neq 0$, $\frac{a}{c} \geq 0.4$ and $\frac{b}{c} = 1$, then

$$\frac{zf'(z)}{\phi(f(z))} \prec e^z, \quad z \in \mathbb{E},$$

i.e. f is ϕ -like.

Remark 4.5. By selecting $q(z) = 1 + \frac{2}{3}z^2$, $\beta = \gamma = 1$ in Theorem 2.2, we have

$$1 + \frac{zq''(z)}{q'(z)} = 2$$

and

$$1 + \frac{zq''(z)}{q'(z)} + \frac{2a}{c}q(z) + \frac{3b}{c}q^2(z) = 2 + \frac{2a}{c}\left(1 + \frac{2}{3}z^2\right) + \frac{3b}{c}\left(1 + \frac{2}{3}z^2\right)^2.$$

For real numbers a, b, c such that $c \neq 0$, $\frac{a}{c} \geq -0.6$ and $\frac{b}{c} \geq 0$, we notice that $q(z)$ satisfy conditions (i) and (ii) of Theorem 2.1. Hence, we obtain the following result from Theorem 2.2.

Theorem 4.6. Let ϕ be analytic function in the domain containing $f(\mathbb{E})$ such that $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in f(\mathbb{E}) \setminus \{0\}$. If $f \in \mathcal{A}$, $\frac{zf'(z)}{\phi(f(z))} \neq 0$, $z \in \mathbb{E}$, satisfy

$$\begin{aligned} & a \left(\frac{zf'(z)}{\phi(f(z))} \right)^2 + b \left(\frac{zf'(z)}{\phi(f(z))} \right)^3 \\ & + cz \left\{ \frac{[\phi(f(z))][zf''(z) + f'(z)] - zf'(z)[\phi(f(z))']}{[\phi(f(z))]^2} \right\} \\ & \prec a \left(1 + \frac{2}{3}z^2 \right)^2 + b \left(1 + \frac{2}{3}z^2 \right)^3 + \frac{4}{3}cz^2, \end{aligned}$$

where a, b, c are real numbers such that $c \neq 0$, $\frac{a}{c} \geq -0.6$ and $\frac{b}{c} \geq 0$, then

$$\frac{zf'(z)}{\phi(f(z))} \prec 1 + \frac{2}{3}z^2, \quad z \in \mathbb{E}.$$

Thus f is ϕ -like.

Remark 4.7. By taking $q(z) = \left(\frac{1+z}{1-z}\right)^\delta$; $0 < \delta \leq 1$, $\beta = \gamma = 1$ in Theorem 2.2, we get

$$1 + \frac{zq''(z)}{q'(z)} = \frac{1 + 2\delta z + z^2}{1 - z^2}$$

and

$$1 + \frac{zq''(z)}{q'(z)} + \frac{2a}{c}q(z) + \frac{3b}{c}q^2(z) = \frac{1 + 2\delta z + z^2}{1 - z^2} + \frac{2a}{c}\left(\frac{1+z}{1-z}\right)^\delta + \frac{3b}{c}\left(\frac{1+z}{1-z}\right)^{2\delta}.$$

For real numbers a, b, c such that $c \neq 0, b = 0$ and $\frac{a}{c} \geq 0$, we notice that $q(z)$ satisfy conditions (i) and (ii) of Theorem 2.1. Hence, we obtain the following result from Theorem 2.2.

Theorem 4.8. *Let ϕ be analytic function in the domain containing $f(\mathbb{E})$ such that $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in f(\mathbb{E}) \setminus \{0\}$. If $f \in \mathcal{A}$, $\frac{zf'(z)}{\phi(f(z))} \neq 0, z \in \mathbb{E}$, satisfy*

$$\begin{aligned} & a \left(\frac{zf'(z)}{\phi(f(z))} \right)^2 + b \left(\frac{zf'(z)}{\phi(f(z))} \right)^3 \\ & + cz \left\{ \frac{[\phi(f(z))][zf''(z) + f'(z)] - zf'(z)[\phi(f(z))]' }{[\phi(f(z))]^2} \right\} \\ & \prec a \left(\frac{1+z}{1-z} \right)^{2\delta} + b \left(\frac{1+z}{1-z} \right)^{3\delta} + cz \left(\frac{2\delta}{1-z^2} \right) \left(\frac{1+z}{1-z} \right)^\delta, \end{aligned}$$

where a, b, c are real numbers such that $c \neq 0, b = 0$ and $\frac{a}{c} \geq 0$, then

$$\frac{zf'(z)}{\phi(f(z))} \prec \left(\frac{1+z}{1-z} \right)^\delta; \quad 0 < \delta \leq 1, \quad z \in \mathbb{E}.$$

Remark 4.9. When we put $q(z) = \frac{1 + (1-2\eta)z}{1-z}; 0 \leq \eta < 1, \beta = \gamma = 1$ in Theorem 2.2, a little calculation yields that

$$1 + \frac{zq''(z)}{q'(z)} = \frac{1+z}{1-z}$$

and

$$\begin{aligned} 1 + \frac{zq''(z)}{q'(z)} + \frac{2a}{c}q(z) + \frac{3b}{c}q^2(z) &= \frac{1+z}{1-z} + \frac{2a}{c} \left[\frac{1 + (1-2\eta)z}{1-z} \right] \\ &+ \frac{3b}{c} \left[\frac{1 + (1-2\eta)z}{1-z} \right]^2. \end{aligned}$$

For real numbers a, b, c such that $c \neq 0, b = 0$ and $\frac{a}{c} \geq 0$, we see that $q(z)$ satisfy conditions (i) and (ii) of Theorem 2.1. Therefore, we obtain the following result from Theorem 2.2.

Theorem 4.10. *Let ϕ be analytic function in the domain containing $f(\mathbb{E})$ such that $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in f(\mathbb{E}) \setminus \{0\}$. If $f \in \mathcal{A}$, $\frac{zf'(z)}{\phi(f(z))} \neq 0, z \in \mathbb{E}$, satisfy*

$$\begin{aligned} & a \left(\frac{zf'(z)}{\phi(f(z))} \right)^2 + b \left(\frac{zf'(z)}{\phi(f(z))} \right)^3 \\ & + cz \left\{ \frac{[\phi(f(z))][zf''(z) + f'(z)] - zf'(z)[\phi(f(z))]' }{[\phi(f(z))]^2} \right\} \end{aligned}$$

$$\prec a \left[\frac{1 + (1 - 2\eta)z}{1 - z} \right]^2 + b \left[\frac{1 + (1 - 2\eta)z}{1 - z} \right]^3 + cz \left[\frac{2(1 - \eta)}{(1 - z)^2} \right],$$

where a, b, c are real numbers such that $c \neq 0$, $b = 0$ and $\frac{a}{c} \geq 0$, then

$$\frac{zf'(z)}{\phi(f(z))} \prec \frac{1 + (1 - 2\eta)z}{1 - z}, \quad z \in \mathbb{E}, \quad 0 \leq \eta < 1,$$

i.e. f is ϕ -like in $\mathbb{E}$.

Remark 4.11. When we select $q(z) = \frac{\alpha'(1 - z)}{\alpha' - z}$; $\alpha' > 1$, $\beta = \gamma = 1$ in Theorem 2.2, after a little calculation, we obtain

$$1 + \frac{zq''(z)}{q'(z)} = \frac{\alpha' + z}{\alpha' - z}$$

and

$$1 + \frac{zq''(z)}{q'(z)} + \frac{2a}{c}q(z) + \frac{3b}{c}q^2(z) = \frac{\alpha' + z}{\alpha' - z} + \frac{2a}{c} \left[\frac{\alpha'(1 - z)}{\alpha' - z} \right] + \frac{3b}{c} \left[\frac{\alpha'(1 - z)}{\alpha' - z} \right]^2.$$

For real numbers a, b, c such that $c \neq 0$, $b = 0$ and $\frac{a}{c} \geq 0$, we see that $q(z)$ satisfy conditions (i) and (ii) of Theorem 2.1. Thus, we get the following Theorem from Theorem 2.2.

Theorem 4.12. Let ϕ be analytic function in the domain containing $f(\mathbb{E})$ such that $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in f(\mathbb{E}) \setminus \{0\}$. If $f \in \mathcal{A}$, $\frac{zf'(z)}{\phi(f(z))} \neq 0$, $z \in \mathbb{E}$, satisfy

$$a \left(\frac{zf'(z)}{\phi(f(z))} \right)^2 + b \left(\frac{zf'(z)}{\phi(f(z))} \right)^3 + cz \left\{ \frac{[\phi(f(z))][zf''(z) + f'(z)] - zf'(z)[\phi(f(z))']}{[\phi(f(z))]^2} \right\} \prec a \left[\frac{\alpha'(1 - z)}{\alpha' - z} \right]^2 + b \left[\frac{\alpha'(1 - z)}{\alpha' - z} \right]^3 + cz \left[\frac{\alpha'(1 - \alpha')}{(\alpha' - z)^2} \right],$$

where a, b, c are real numbers such that $c \neq 0$, $b = 0$ and $\frac{a}{c} \geq 0$, then

$$\frac{zf'(z)}{\phi(f(z))} \prec \frac{\alpha'(1 - z)}{\alpha' - z}, \quad z \in \mathbb{E}, \quad \alpha' > 1,$$

i.e. f is ϕ -like.

Remark 4.13. By taking $q(z) = 1 + tz$, $0 < t \leq 0.8$, $\beta = 1$ and $\gamma = 0$ in Theorem 2.2, then after having some calculations we have

$$1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} = \frac{1}{1 + tz}$$

and

$$1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} + \frac{a}{c}q(z) + \frac{2b}{c}q^2(z) = \frac{1}{1+tz} + \frac{a}{c}(1+tz) + \frac{2b}{c}(1+tz)^2.$$

Thus for real numbers a, b, c such that $c \neq 0$ and $\frac{a}{c}, \frac{b}{c} \geq 0$, we observe that $q(z)$ satisfy conditions (i) and (ii) of Theorem 2.1. Therefore, we immediately, arrive at the following result from Theorem 2.2.

Theorem 4.14. *Let ϕ be analytic function in the domain containing $f(\mathbb{E})$ such that $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in f(\mathbb{E}) \setminus \{0\}$. If $f \in \mathcal{A}$, $\frac{zf'(z)}{\phi(f(z))} \neq 0$, $z \in \mathbb{E}$, satisfy*

$$a \frac{zf'(z)}{\phi(f(z))} + b \left(\frac{zf'(z)}{\phi(f(z))} \right)^2 + c \left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(f(z)))'}{\phi(f(z))} \right) \prec a(1+tz) + b(1+tz)^2 + \frac{ctz}{1+tz},$$

where a, b, c are real numbers such that $c \neq 0$ and $\frac{a}{c}, \frac{b}{c} \geq 0$, then

$$\frac{zf'(z)}{\phi(f(z))} \prec 1+tz, \quad 0 < t \leq 0.8, \quad z \in \mathbb{E}.$$

Therefore, f is ϕ -like.

Remark 4.15. When we select $q(z) = e^z$, $\beta = 1$ and $\gamma = 0$ in Theorem 2.2, a little calculation yields that

$$1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} = 1$$

and

$$1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} + \frac{a}{c}q(z) + \frac{2b}{c}q^2(z) = 1 + \frac{a}{c}e^z + \frac{2b}{c}e^{2z}.$$

For real numbers a, b, c such that $c \neq 0$, $\frac{a}{c} \geq 0$ and $0 \leq \frac{b}{c} \leq 0.8$, we see that $q(z)$ satisfy conditions (i) and (ii) of Theorem 2.1. Hence, we obtain the following result from Theorem 2.2.

Theorem 4.16. *Let ϕ be analytic function in the domain containing $f(\mathbb{E})$ such that $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in f(\mathbb{E}) \setminus \{0\}$. If $f \in \mathcal{A}$, $\frac{zf'(z)}{\phi(f(z))} \neq 0$, $z \in \mathbb{E}$, satisfy*

$$a \frac{zf'(z)}{\phi(f(z))} + b \left(\frac{zf'(z)}{\phi(f(z))} \right)^2 + c \left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(f(z)))'}{\phi(f(z))} \right) \prec ae^z + be^{2z} + cz,$$

where a, b, c are real numbers such that $c \neq 0$, $\frac{a}{c} \geq 0$ and $0 \leq \frac{b}{c} \leq 0.8$, then

$$\frac{zf'(z)}{\phi(f(z))} \prec e^z, \quad z \in \mathbb{E},$$

i.e. f is ϕ -like.

Remark 4.17. By selecting $q(z) = 1 + \frac{2}{3}z^2$, $\beta = 1$ and $\gamma = 0$ in Theorem 2.2, we have

$$1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} = \frac{6}{3 + 2z^2}$$

and

$$1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} + \frac{a}{c}q(z) + \frac{2b}{c}q^2(z) = \frac{6}{3 + 2z^2} + \frac{a}{c}\left(1 + \frac{2}{3}z^2\right) + \frac{2b}{c}\left(1 + \frac{2}{3}z^2\right)^2.$$

For real numbers a, b, c such that $c \neq 0$, $\frac{a}{c} \geq 0.6$ and $0 \leq \frac{b}{c} \leq 0.7$, we notice that $q(z)$ satisfy conditions (i) and (ii) of Theorem 2.1. Hence, we obtain the following result from Theorem 2.2.

Theorem 4.18. Let ϕ be analytic function in the domain containing $f(\mathbb{E})$ such that $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in f(\mathbb{E}) \setminus \{0\}$. If $f \in \mathcal{A}$, $\frac{zf'(z)}{\phi(f(z))} \neq 0$, $z \in \mathbb{E}$, satisfy

$$\begin{aligned} a \frac{zf'(z)}{\phi(f(z))} + b \left(\frac{zf'(z)}{\phi(f(z))} \right)^2 + c \left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(f(z)))'}{\phi(f(z))} \right) \\ \prec a \left(1 + \frac{2}{3}z^2 \right) + b \left(1 + \frac{2}{3}z^2 \right)^2 + \frac{4cz^2}{3 + 2z^2}, \end{aligned}$$

where a, b, c are real numbers such that $c \neq 0$, $\frac{a}{c} \geq 0.6$ and

$0 \leq \frac{b}{c} \leq 0.7$, then

$$\frac{zf'(z)}{\phi(f(z))} \prec 1 + \frac{2}{3}z^2, \quad z \in \mathbb{E}.$$

Thus f is ϕ -like.

Remark 4.19. By taking $q(z) = \left(\frac{1+z}{1-z} \right)^\delta$; $0 < \delta \leq 0.5$, $\beta = 1$ and $\gamma = 0$ in Theorem 2.2, we get

$$1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} = \frac{1+z^2}{1-z^2}$$

and

$$1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} + \frac{a}{c}q(z) + \frac{2b}{c}q^2(z) = \frac{1+z^2}{1-z^2} + \frac{a}{c}\left(\frac{1+z}{1-z}\right)^\delta + \frac{2b}{c}\left(\frac{1+z}{1-z}\right)^{2\delta}.$$

For real numbers a, b, c such that $c \neq 0$ and $\frac{a}{c}, \frac{b}{c} \geq 0$, we notice that $q(z)$ satisfy conditions (i) and (ii) of Theorem 2.1. Hence, we obtain the following result from Theorem 2.2.

Theorem 4.20. Let ϕ be analytic function in the domain containing $f(\mathbb{E})$ such that $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in f(\mathbb{E}) \setminus \{0\}$. If $f \in \mathcal{A}$, $\frac{zf'(z)}{\phi(f(z))} \neq 0$, $z \in \mathbb{E}$, satisfy

$$a \frac{zf'(z)}{\phi(f(z))} + b \left(\frac{zf'(z)}{\phi(f(z))} \right)^2 + c \left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(f(z)))'}{\phi(f(z))} \right) \\ \prec a \left(\frac{1+z}{1-z} \right)^\delta + b \left(\frac{1+z}{1-z} \right)^{2\delta} + \frac{2\delta cz}{1-z^2},$$

where a , b , c are real numbers such that $c \neq 0$ and $\frac{a}{c}$, $\frac{b}{c} \geq 0$, then

$$\frac{zf'(z)}{\phi(f(z))} \prec \left(\frac{1+z}{1-z} \right)^\delta; \quad 0 < \delta \leq 0.5, \quad z \in \mathbb{E}.$$

Remark 4.21. When we put $q(z) = \frac{1+(1-2\eta)z}{1-z}$; $0 \leq \eta < 1$, $\beta = 1$ and $\gamma = 0$ in Theorem 2.2, a little calculation yields that

$$1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} = \frac{1+z}{1-z} - \frac{2z(1-\eta)}{(1-z)[1+(1-2\eta)z]}$$

and

$$1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} + \frac{a}{c}q(z) + \frac{2b}{c}q^2(z) = \frac{1+z}{1-z} - \frac{2z(1-\eta)}{(1-z)[1+(1-2\eta)z]} \\ + \frac{a}{c} \left[\frac{1+(1-2\eta)z}{1-z} \right] + \frac{2b}{c} \left[\frac{1+(1-2\eta)z}{1-z} \right]^2.$$

For real numbers a , b , c such that $c \neq 0$, $b = 0$ and $\frac{a}{c} \geq 0$, we see that $q(z)$ satisfy conditions (i) and (ii) of Theorem 2.1. Therefore, we obtain the following result from Theorem 2.2.

Theorem 4.22. Let ϕ be analytic function in the domain containing $f(\mathbb{E})$ such that $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in f(\mathbb{E}) \setminus \{0\}$. If $f \in \mathcal{A}$, $\frac{zf'(z)}{\phi(f(z))} \neq 0$, $z \in \mathbb{E}$, satisfy

$$a \frac{zf'(z)}{\phi(f(z))} + b \left(\frac{zf'(z)}{\phi(f(z))} \right)^2 + c \left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(f(z)))'}{\phi(f(z))} \right) \\ \prec a \left[\frac{1+(1-2\eta)z}{1-z} \right] + b \left[\frac{1+(1-2\eta)z}{1-z} \right]^2 + cz \left[\frac{2(1-\eta)}{(1-z)(1+(1-2\eta)z)} \right],$$

where a , b , c are real numbers such that $c \neq 0$, $b = 0$ and $\frac{a}{c} \geq 0$, then

$$\frac{zf'(z)}{\phi(f(z))} \prec \frac{1+(1-2\eta)z}{1-z}, \quad z \in \mathbb{E}, \quad 0 \leq \eta < 1,$$

i.e. f is ϕ -like.

Remark 4.23. When we select $q(z) = \frac{\alpha'(1-z)}{\alpha' - z}$; $\alpha' > 1$, $\beta = 1$ and $\gamma = 0$ in Theorem 2.2, after a little calculation, we obtain

$$1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} = \frac{\alpha' - z^2}{(1-z)(\alpha' - z)}$$

and

$$1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} + \frac{a}{c}q(z) + \frac{2b}{c}q^2(z) = \frac{\alpha' - z^2}{(1-z)(\alpha' - z)} + \frac{a}{c} \left[\frac{\alpha'(1-z)}{\alpha' - z} \right] + \frac{2b}{c} \left[\frac{\alpha'(1-z)}{\alpha' - z} \right]^2.$$

For real numbers a , b , c such that $c \neq 0$, $\frac{a}{c} \geq 0$ and $\frac{b}{c} \geq 0$, we see that $q(z)$ satisfy conditions (i) and (ii) of Theorem 2.1. Thus, we get the following Theorem from Theorem 2.2.

Theorem 4.24. Let ϕ be analytic function in the domain containing $f(\mathbb{E})$ such that $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in f(\mathbb{E}) \setminus \{0\}$. If $f \in \mathcal{A}$, $\frac{zf'(z)}{\phi(f(z))} \neq 0$, $z \in \mathbb{E}$, satisfy

$$a \frac{zf'(z)}{\phi(f(z))} + b \left(\frac{zf'(z)}{\phi(f(z))} \right)^2 + c \left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(f(z)))'}{\phi(f(z))} \right) \prec a \left[\frac{\alpha'(1-z)}{\alpha' - z} \right] + b \left[\frac{\alpha'(1-z)}{\alpha' - z} \right]^2 + \frac{(1-\alpha')cz}{(1-z)(\alpha' - z)},$$

where a , b , c are real numbers such that $c \neq 0$, $\frac{a}{c} \geq 0$ and $\frac{b}{c} \geq 0$, then

$$\frac{zf'(z)}{\phi(f(z))} \prec \frac{\alpha'(1-z)}{\alpha' - z}, \quad z \in \mathbb{E}, \quad \alpha' > 1,$$

i.e. f is ϕ -like.

5. Conclusion

Using the differential subordination technique involving convolution, we derived new conditions under which normalized analytic functions exhibit ϕ -likeness and parabolic ϕ -likeness. These results contribute to a deeper understanding of geometric function theory and open pathways for further applications.

References

- [1] Adegani, E.A., Bulboaca, T. and Motamednezhad, A., *Simple sufficient subordination conditions for close-to-convexity*, Mathematics, **7(3)**(2019), 241.
- [2] Adegani, E.A., Motamednezhad, A., Bulboaca, T. and Lecko A., *On a logarithmic coefficients inequality for the class of close-to-convex functions*, Proceedings of the Romanian Academy, Series A: Mathematics, 2023.

- [3] Brar, R. and Billing, S. S., *Certain results on ϕ -like and starlike functions in a parabolic region*, Int. J. Open Problems Complex Analysis, **9**(2)(2017), 30-41.
- [4] Brickman, L., *ϕ -like analytic functions*, I, Bull. Amer. Math. Soc., **79**(1973), 555-558.
- [5] Cho, N.E., Srivastava, H.M., Adegani, E.A. and Motamednezhad, A., *Criteria for a certain class of the Carathéodory functions and their applications*, J. Inq. and App., **85**(2020).
- [6] Ma, W. C. and Minda, D., *Uniformly convex functions*, Ann. Polon. Math., **57**(2)(1992), 165-175.
- [7] Miller, S. S. and Mocanu, P. T., *Differential Subordinations: Theory and Applications*, Marcel Dekker, New York and Basel (2000).
- [8] Ronning, F., *Uniformly convex functions and a corresponding class of starlike functions*, Proc. Amer. Math. Soc., **118**(1993), 189-196.
- [9] Ruscheweyh, St., *A subordination theorem for ϕ -like functions*, J. London Math. Soc., **2**(13)(1976), 275-280.
- [10] Shanmugam, T. N., Sivasubramanian, S. and Darus, Maslina, *Subordination and superordination results for ϕ -like functions*, J. Inq. Pure and App. Math., **8**(1)(2007).

Hardeep Kaur 

“Department of Mathematics” (UIS),
Chandigarh University,
Mohali – 140413, Punjab, India.
e-mail: kaurhardeep959@gmail.com

Richa Brar 

“Sri Guru Granth Sahib World” University,
Department of Mathematics
Fatehgarh Sahib,
140407, Punjab, India
e-mail: richabrar4@gmail.com

Sukhwinder Singh Billing 

“Sri Guru Granth Sahib World” University,
Department of Mathematics
Fatehgarh Sahib,
140407, Punjab, India
e-mail: ssbilling@gmail.com

