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Certain subclass of close-to-convex univalent functions defined with q-derivative operator

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Abstract. The objective of this paper is to introduce a new subclass of strongly close-to-convex functions defined with q-derivative operator and by subordinating to generalized Janowski function. We establish several useful properties such as coefficient estimates, distortion theorem, argument theorem, inclusion relations and radius of convexity for this class. Some relevant connections of the results investigated here with those derived earlier are mentioned.

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1. Introduction

Let $\mathcal{A}$ be the class of analytic functions in the open unit disc $E = \{z :| z |< 1\}$ and having the Taylor-Maclaurin expansion of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Further, let $\mathcal{S}$ be the class of functions $f \in \mathcal{A}$ which are univalent in E.

By $\mathcal{U}$, we denote the class of Schwarzian functions w satisfying $w(0) = 0$ and $|w(z)| \leq 1$, which are analytic in E and of the form

$$w(z) = \sum_{n=1}^{\infty} c_n z^n, z \in E.$$

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For $0 \leq \alpha < 1$, $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ denote the classes of starlike functions and convex functions of order α respectively and are defined as

$$\mathcal{S}^*(\alpha) = \left\{ f : f \in \mathcal{A}, Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, z \in E \right\}$$

and

$$\mathcal{K}(\alpha) = \left\{ f : f \in \mathcal{A}, Re\left(\frac{(zf'(z))'}{f'(z)}\right) > \alpha, z \in E \right\}.$$

In particular, $\mathcal{S}^*(0) \equiv \mathcal{S}^*$ which is the class of starlike functions and $\mathcal{K}(0) = \mathcal{K}$, the class of convex functions. For $\alpha = \frac{1}{2}$, $\mathcal{S}^*(\frac{1}{2})$ is the class of starlike functions of order $\frac{1}{2}$.

For the functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ defined in (1) and $h(z) = z + \sum_{n=2}^{\infty} b_n z^n$, the Hadamard product(or convolution) of f and h is defined by

$$(f * h)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

A function $f \in \mathcal{A}$ is said to be close-to-convex function if there exists a function $g \in \mathcal{S}^*$ such that

$$Re\left(\frac{zf'(z)}{g(z)}\right) > 0 (z \in E).$$

The class of close-to-convex functions is denoted by $\mathcal{C}$ and was established by Kaplan [9].

Sakaguchi [18] established the class $\mathcal{S}_s^*$ of the functions $f \in \mathcal{A}$ which satisfy the condition

$$Re\left(\frac{2zf'(z)}{f(z) - f(-z)}\right) > 0.$$

The functions in the class $\mathcal{S}_s^*$ are called starlike functions with respect to symmetric points. Clearly, the class $\mathcal{S}_s^*$ is contained in the class $\mathcal{C}$ of close-to-convex functions, as $\dfrac{f(z) - f(-z)}{2}$ is a starlike function [3] in E.

Getting inspired from the class $\mathcal{S}_s^*$, Gao and Zhou [5] studied the class $\mathcal{K}_S$ given by

$$\mathcal{K}_s = \left\{ f : f \in \mathcal{A}, Re\left(\frac{-z^2 f'(z)}{g(z)g(-z)}\right) > 0, g \in \mathcal{S}^*\left(\frac{1}{2}\right), z \in E \right\}.$$

Kowalczyk and Les-Bomba [10] extended the class $\mathcal{K}_S$ by introducing the class $\mathcal{K}_S(\gamma) (0 \leq \gamma < 1)$ which is defined as

$$\mathcal{K}_s(\gamma) = \left\{ f : f \in \mathcal{A}, Re\left(\frac{-z^2 f'(z)}{g(z)g(-z)}\right) > \gamma, g \in \mathcal{S}^*\left(\frac{1}{2}\right), z \in E \right\}.$$

For $\gamma = 0$, the class $\mathcal{K}_S(\gamma)$ reduces to the class $\mathcal{K}_S$.

Later on, Prajapat [14] established that, a function $f \in \mathcal{A}$ is said to be in the class $\chi_t(\gamma)(|t| \leq 1, t \neq 0, 0 \leq \gamma < 1)$, if there exists a function $g \in \mathcal{S}^*\left(\frac{1}{2}\right)$, such that

$$Re\left[\frac{tz^2 f'(z)}{g(z)g(tz)}\right] > \gamma.$$

In particular $\chi_{-1}(\gamma) \equiv \mathcal{K}_S(\gamma)$ and $\chi_{-1}(0) \equiv \mathcal{K}_S$.

For $-1 \leq B < A \leq 1$ and $0 \leq \alpha < 1$, Polatoglu et al. [13] introduced the class $\mathcal{P}(A, B; \alpha)$, the subclass of $\mathcal{A}$ which consists of functions of the form $p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k$ such that $p(z) \prec \dfrac{1 + [B + (A - B)(1 - \alpha)]z}{1 + Bz}$. Also for $\alpha = 0$, the class $\mathcal{P}(A, B; \alpha)$ agrees with $\mathcal{P}(A, B)$, which is a subclass of $\mathcal{A}$ introduced by Janowski [8].

Let f and g be two analytic functions in E. Then f is said to be subordinate to g (symbolically $f \prec g$) if there exists a Schwarzian function $w \in \mathcal{U}$ such that $f(z) = g(w(z))$. Further, if g is univalent in E, then $f \prec g$ is equivalent to $f(0) = g(0)$ and $f(E) \subset g(E)$. Recently some subordination properties for certain classes of analytic functions were studied in [16].

Using the concept of subordination, Singh et al. [20] introduced the class $\chi_t(A, B)(|t| \leq 1, t \neq 0)$, which consists of functions $f \in \mathcal{A}$ with the conditions

$$\frac{tz^2 f'(z)}{g(z)g(tz)} \prec \frac{1 + Az}{1 + Bz}, -1 \leq B < A \leq 1, z \in E,$$

where $g \in \mathcal{S}^*\left(\frac{1}{2}\right)$. The following observations are obvious:
(i) $\chi_t(1 - 2\gamma, -1) \equiv \chi_t(\gamma)$.
(ii)$\chi_{-1}(1 - 2\gamma, -1) \equiv \mathcal{K}_S(\gamma)$.
(iii)$\chi_{-1}(1, -1) \equiv \mathcal{K}_S$.

Raina et al. [15] defined the class of strongly close-to-convex functions of order β, as below:

$$\mathcal{C}'_\beta = \left\{f : f \in \mathcal{A}, \left|arg\left\{\frac{zf'(z)}{g(z)}\right\}\right| < \frac{\beta\pi}{2}, g \in \mathcal{K}, 0 < \beta \leq 1, z \in E\right\},$$

or equivalently

$$\mathcal{C}'_\beta = \left\{f : f \in \mathcal{A}, \frac{zf'(z)}{g(z)} \prec \left(\frac{1 + z}{1 - z}\right)^\beta, g \in \mathcal{K}, 0 < \beta \leq 1, z \in E\right\}.$$

Quantum calculus is ordinary classical calculus which introduces q-calculus, where q stands for quantum. Nowadays, q-calculus has attracted many researchers as it is widely useful in various branches of Mathematics and Physics. The application of q-calculus was initiated by Jackson [6, 7] and he developed q-integral and

q-derivative in a systematic way. For $0 < q < 1$, Jackson [6] defined the q-derivative of a function $f \in \mathcal{A}$ as

$$D_q f(z) = \begin{cases} \frac{f(z)-f(qz)}{(1-q)z} & \text{for } z \neq 0, \\ f'(0) & \text{for } z = 0, \end{cases} \tag{1.2}$$

where $D_q^2 f(z) = D_q(D_q f(z))$.
From (1.2), it is obvious that

$$D_q f(z) = 1 + \sum_{k=2}^{\infty} [k]_q a_k z^{k-1},$$

where $[k]_q = \frac{1-q^k}{1-q} = 1 + q + q^2 + ... + q^{k-1}$ and $[0]_q = 0$. If $q \to 1^-$, then $[k]_q \to k$. Further $D_q z^k = [k]_q z^{k-1}$ and $\lim_{q \to 1^-} D_q f(z) = f'(z)$. Recently a new subclass of analytic functions defined with q-derivative operator is studied in [22].

The q-shifted factorial is given by

$$(a;q)_n = \begin{cases} 1 & \text{for } n = 0, \\ (1-a)(1-aq)...(1-aq^{n-1}) & \text{for } n = 1, 2, ... \end{cases}$$

As a generalization of the hypergeometric series, Heine established the q-hypergeometric series as

$${}_2F_1[a, b; c : q, z] = \sum_{n=0}^{\infty} \frac{(a;q)_n (b;q)_n}{(q;q)_n (c;q)_n} z^n.$$

Generalising the Heine's series, we define the basic hypergeometric series ${}_r\phi_s$ as below:

$${}_r\phi_s(a_1, a_2, ..., a_r; b_1, b_2, ..., b_s : q, z) = \sum_{n=0}^{\infty} \frac{(a_1;q)_n (a_2;q)_n ... (a_r;q)_n}{(q;q)_n (b_1;q)_n ... (b_s;q)_n} \left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} z^n, \tag{1.3}$$

where $\binom{n}{2} = \frac{n(n-1)}{2}$, and $q \neq 0$ when $r > s + 1$. In (1.3), it is supposed that the parameters $b_1, b_2, ..., b_s$ are such that the denominator factors in the terms of the series are never zero. In basic hypergeometric series, q is a fixed parameter with $q \in \mathbb{C}$ and $|q| < 1$.

For complex parameteres $a_1, a_2, ..., a_r$ and $b_1, b_2, ..., b_s$, $(b_j \in \mathbb{C} \setminus \mathbb{Z}_0^- = 0, -1, -2, ...; j = 1, 2, ..., s)$, the generalized q-hypergeometric function ${}_r\psi_s(a_1, a_2, ..., a_r; b_1, b_2, ..., b_s : q, z)$ is defined by

$${}_r\psi_s(a_1, a_2, ..., a_r; b_1, b_2, ..., b_s; q, z) = \sum_{n=0}^{\infty} \frac{(a_1;q)_n (a_2;q)_n ... (a_r;q)_n}{(q;q)_n (b_1;q)_n ... (b_s;q)_n} z^n,$$

where $r = s + 1; r, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, z \in E$.

The function $\mathcal{G}_{r,s}(a_i, b_j; q, z)(i = 1, 2, ..., r; j = 1, 2, ..., s)$ is defined by

$$\mathcal{G}_{r,s}(a_i, b_j; q, z) := z_r\psi_s(a_1, a_2, ..., a_r; b_1, b_2, ..., b_s : q, z).$$

Now we define the operator $\mathcal{J}_\lambda^m(a_1, b_1; q, z)f : E \to E$ as

$$\mathcal{J}_\lambda^0(a_1, b_1; q, z)f(z) = f(z) * \mathcal{G}_{r,s}(a_i, b_j; q, z),$$

$$\mathcal{J}_\lambda^1(a_1, b_1; q, z)f(z) = (1-\lambda)(f(z) * \mathcal{G}_{r,s}(a_i, b_j; q, z)) + \lambda z D_q(f(z) * \mathcal{G}_{r,s}(a_i, b_j; q, z), \tag{1.4}$$

$$\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z) = \mathcal{J}_\lambda^1(\mathcal{J}_\lambda^{m-1}(a_1, b_1; q, z)f(z)). \tag{1.5}$$

For $f \in \mathcal{A}$, it can be easily deduced from (1.4) and (1.5), that

$$\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z) = z + \sum_{n=2}^{\infty}[1 - \lambda + [n]_q\lambda]^m \Gamma_n a_n z^n,$$

where $\Gamma_n = \frac{(a_1;q)_{n-1}(a_2;q)_{n-1}\cdots(a_r;q)_{n-1}}{(q;q)_{n-1}(b_1;q)_{n-1}\cdots(b_s;q)_{n-1}}$ and $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \lambda \geq 0$.
In particular
(i) For $m = 0$, the operator $\mathcal{J}_\lambda^m(a_1, b_1; q, z)$ agrees with the q-analogue of Dziok-Srivastava operator [4].
(ii) For $r = 2, s = 1, a_1 = b_1, a_2 = q, \lambda = 1$, the operator $\mathcal{J}_\lambda^m(a_1, b_1; q, z)$ reduces to the well known Sãlãgean operator [19].

Motivated by the above mentioned work, now we introduce the following subclass of close-to-convex functions defined by subordinating to generalized Janowski function.

Definition 1.1. *Let $\mathcal{S}_{q,t}^{\lambda,m}(a_1, b_1; A, B; \alpha; \beta)(|t| \leq 1, t \neq 0, 0 \leq \alpha < 1, 0 < \beta \leq 1)$ denote the class of functions $f \in \mathcal{A}$ which satisfy the conditions,*

$$\frac{tz^2[\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)]'}{[\mathcal{J}_\lambda^m(a_1, b_1; q, z)g(z)][\mathcal{J}_\lambda^m(a_1, b_1; q, z)g(tz)]} \prec \left(\frac{1 + [B + (A - B)(1 - \alpha)]z}{1 + Bz}\right)^\beta,$$

where $\mathcal{J}_\lambda^m(a_1, b_1; q, z)g(z) \in \mathcal{S}^*\left(\frac{1}{2}\right)$, $-1 \leq B < A \leq 1$ and $z \in E$.

The following observations are obvious:
(i) For $\alpha = 0, \beta = 1$, the class $\mathcal{S}_{q,t}^{\lambda,m}(a_1, b_1; A, B; \alpha; \beta)$ reduces to $\mathcal{S}_{q,t}^{\lambda,m}(a_1, b_1; A, B)$, the class studied by Murugusundaramoorthy and Reddy [12].
(ii) For $m = 0, r = 2, s = 1, a_1 = b_1, a_2 = q, \alpha = 0, \beta = 1$ and $q \to 1^-$, the class $\mathcal{S}_{q,t}^{\lambda,m}(a_1, b_1; A, B; \alpha; \beta)$ reduces to $\chi_t(A, B)$, the class studied by Singh et al. [20].
(iii) For $m = 0, r = 2, s = 1, a_1 = b_1, a_2 = q, \alpha = 0, \beta = 1, A = 1 - 2\gamma, B = -1$ and $q \to 1^-$, the class $\mathcal{S}_{q,t}^{\lambda,m}(a_1, b_1; A, B; \alpha; \beta)$ agrees with $\chi_t(\gamma)$, the class established by Prajapat [14].
(iv) For $m = 0, r = 2, s = 1, a_1 = b_1, a_2 = q, \alpha = 0, \beta = 1, A = 1, B = -1, t = -1$ and $q \to 1^-$, the class $\mathcal{S}_{q,t}^{\lambda,m}(a_1, b_1; A, B; \alpha; \beta)$ reduces to K_s, the class introduced by Gao and Zhou [5].
(v) For $m = 0, r = 2, s = 1, a_1 = b_1, a_2 = q, \alpha = 0, \beta = 1, A = 1 - 2\gamma, B = -1, t = -1$ and $q \to 1^-$, the class $\mathcal{S}_{q,t}^{\lambda,m}(a_1, b_1; A, B; \alpha; \beta)$ agrees with $K_s(\gamma)$, the class studied by

Kowalczyk and Les Bomba [10].

As $f \in \mathcal{S}_{q,t}^{\lambda;m}(a_1, b_1; A, B; \alpha; \beta)$, therefore by the principle of subordination, it follows that

$$\frac{tz^2[\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)]'}{[\mathcal{J}_\lambda^m(a_1, b_1; q, z)g(z)][\mathcal{J}_\lambda^m(a_1, b_1; q, z)g(tz)]} = \left(\frac{1 + [B + (A - B)(1 - \alpha)]w(z)}{1 + Bw(z)}\right)^\beta, \tag{1.6}$$

where $w \in \mathcal{U}$.

In the present investigation, we obtain the coefficient estimates, inclusion relation, distortion theorem, argument theorem and radius of convexity for the functions in class $\mathcal{S}_{q,t}^{\lambda;m}(a_1, b_1; A, B; \alpha; \beta)$. Our results extend the known results due to various authors.
Throughout our discussion, we assume that $-1 \leq B < A \leq 1, 0 < |t| \leq 1, t \neq 0, 0 \leq \alpha < 1, 0 < \beta \leq 1, m \in \mathbb{N}_0, \lambda \geq 0, z \in E$.

2. Preliminary lemmas

Lemma 2.1. [1, 17] *Let,*

$$\left(\frac{1 + [B + (A - B)(1 - \alpha)]w(z)}{1 + Bw(z)}\right)^\beta = (P(z))^\beta = 1 + \sum_{n=1}^{\infty} p_n z^n, \tag{2.1}$$

then

$$|p_n| \leq \beta(1 - \alpha)(A - B), n \geq 1.$$

Lemma 2.2. [21] *Let* $g \in S^*\left(\frac{1}{2}\right)$, *then* $\frac{g(z)g(tz)}{tz} \in S^*$.

On the lines of Lemma 2.2, the following result is obvious.

Lemma 2.3. *Let* $\mathcal{J}_\lambda^m(a_1, b_1; q, z)g \in S^*\left(\frac{1}{2}\right)$, *then for*

$$G(z) = \frac{[\mathcal{J}_\lambda^m(a_1, b_1; q, z)g(z)][\mathcal{J}_\lambda^m(a_1, b_1; q, z)g(tz)]}{tz} = z + \sum_{n=2}^{\infty} d_n z^n \in S^*, \tag{2.2}$$

we have, $|d_n| \leq n$.

Lemma 2.4. [15] *Let* $-1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1$ *and* $0 < \beta \leq 1$, *then*

$$\left(\frac{1 + A_1 z}{1 + B_1 z}\right)^\beta \prec \left(\frac{1 + A_2 z}{1 + B_2 z}\right)^\beta.$$

3. Main results

Theorem 3.1. *If* $f \in \mathcal{S}_{q,t}^{\lambda,m}(a_1, b_1; A, B; \alpha; \beta)$, *then*

$$|a_n| \leq \frac{1}{|\Gamma_n|[1-\lambda+[n]_q\lambda]^m}\left[1+\frac{\beta(1-\alpha)(n-1)(A-B)}{2}\right]. \tag{3.1}$$

The result is sharp.

Proof. As $f \in \mathcal{S}_{q,t}^{\lambda,m}(a_1, b_1; A, B; \alpha; \beta)$, therefore from (1.6) and (2.1), we obtain

$$\frac{tz^2[\mathcal{J}_\lambda^m(a_1,b_1;q,z)f(z)]'}{[\mathcal{J}_\lambda^m(a_1,b_1;q,z)g(z)][\mathcal{J}_\lambda^m(a_1,b_1;q,z)g(tz)]} = (P(z))^\beta. \tag{3.2}$$

Using (2.2), (3.2) takes the form

$$\frac{z[\mathcal{J}_\lambda^m(a_1,b_1;q,z)f(z)]'}{G(z)} = (P(z))^\beta. \tag{3.3}$$

On expanding (3.3), it yields

$$1+\sum_{n=2}^{\infty}[1-\lambda+[n]_q\lambda]^m n\Gamma_n a_n z^{n-1} = \left(1+\sum_{n=2}^{\infty} d_n z^{n-1}\right)\left(1+\sum_{n=1}^{\infty} p_n z^n\right). \tag{3.4}$$

Equating the coefficients of z^{n-1} in (3.4), we have

$$n[1-\lambda+[n]_q\lambda]^m\Gamma_n a_n = d_n + d_{n-1}p_1 + d_{n-2}p_2 + \ldots + d_2 p_{n-2} + p_{n-1}. \tag{3.5}$$

Using Lemma 2.1, Lemma 2.3 and applying triangle inequality in (3.5), it gives

$$n[1-\lambda+[n]_q\lambda]^m|\Gamma_n||a_n| \leq n+\beta(1-\alpha)(A-B)[(n-1)+(n-2)+\ldots+2+1]. \tag{3.6}$$

After simplification, (3.1) can be easily obtained from (3.6).

Equality in (3.1) is attained for the function f defined by

$$\frac{tz^2[\mathcal{J}_\lambda^m(a_1,b_1;q,z)f(z)]'}{[\mathcal{J}_\lambda^m(a_1,b_1;q,z)g(z)][\mathcal{J}_\lambda^m(a_1,b_1;q,z)g(tz)]} = \left(\frac{1+[B+(A-B)(1-\alpha)]z}{1+Bz)}\right)^\beta.$$

□

For $\alpha = 0, \beta = 1$, Theorem 3.1 gives the following result due to Murugusundaramoorthy and Reddy [12].

Remark 3.2. If $f \in \mathcal{S}_{q,t}^{\lambda,m}(a_1, b_1; A, B)$, then

$$|a_n| \leq \frac{1}{|\Gamma_n|[1-\lambda+[n]_q\lambda]^m}\left[1+\frac{(n-1)(A-B)}{2}\right].$$

For $m = 0, r = 2, s = 1, a_1 = b_1, a_2 = q, \alpha = 0, \beta = 1$ and $q \to 1^-$, Theorem 3.1 agrees with the following result by Singh et al. [20].

Remark 3.3. If $f \in \chi_t(A,B)$, then

$$|a_n| \leq 1 + \frac{(n-1)(A-B)}{2}.$$

For $m = 0, r = 2, s = 1, a_1 = b_1, a_2 = q, \alpha = 0, \beta = 1, A = 1 - 2\gamma, B = -1$ and $q \to 1^-$, Theorem 3.1 yields the below mentioned result established by Prajapat [14].

Remark 3.4. If $f \in \chi_t(\gamma)$, then

$$|a_n| \leq 1 + (n-1)(1-\gamma).$$

For $m = 0, r = 2, s = 1, a_1 = b_1, a_2 = q, \alpha = 0, \beta = 1, A = 1, B = -1, t = -1$ and $q \to 1^-$, Theorem 3.1 gives the following result for the class $\mathcal{K}_s$.

Remark 3.5. If $f \in \mathcal{K}_s$, then

$$|a_n| \leq n.$$

Theorem 3.6. *If $-1 \leq B_2 = B_1 < A_1 \leq A_2 \leq 1$ and $0 \leq \alpha_2 \leq \alpha_1 < 1$, then*

$$\mathcal{S}_{q,t}^{\lambda,m}(a_1, b_1; A_1, B_1; \alpha_1; \beta) \subset \mathcal{S}_{q,t}^{\lambda,m}(a_1, b_1; A_2, B_2; \alpha_2; \beta).$$

Proof. As $f \in \mathcal{S}_{q,t}^{\lambda,m}(a_1, b_1; A_1, B_1; \alpha_1; \beta)$, so

$$\frac{tz^2[\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)]'}{[\mathcal{J}_\lambda^m(a_1, b_1; q, z)g(z)][\mathcal{J}_\lambda^m(a_1, b_1; q, z)g(tz)]} \prec \left(\frac{1 + [B_1 + (A_1 - B_1)(1-\alpha_1)]z}{1 + B_1 z}\right)^\beta.$$

As $-1 \leq B_2 = B_1 < A_1 \leq A_2 \leq 1$ and $0 \leq \alpha_2 \leq \alpha_1 < 1$, we have

$$-1 \leq B_1 + (1-\alpha_1)(A_1 - B_1) \leq B_2 + (1-\alpha_2)(A_2 - B_2) \leq 1.$$

Thus by Lemma 2.4, it yields

$$\frac{tz^2[\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)]'}{[\mathcal{J}_\lambda^m(a_1, b_1; q, z)g(z)][\mathcal{J}_\lambda^m(a_1, b_1; q, z)g(tz)]} \prec \left(\frac{1 + [B_2 + (A_2 - B_2)(1-\alpha_2)]z}{1 + B_2 z}\right)^\beta,$$

which implies $f \in \mathcal{S}_{q,t}^{\lambda,m}(a_1, b_1; A_2, B_2; \alpha_2; \beta)$.

□

Theorem 3.7. *If $f \in \mathcal{S}_{q,t}^{\lambda,m}(a_1, b_1; A, B; \alpha; \beta)$, then for $|z| = r, 0 < r < 1$, we have*

$$\left(\frac{1-[B+(A-B)(1-\alpha)]r}{1-Br}\right)^\beta \cdot \frac{1}{(1+r)^2}$$

$$\leq |[\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)]'| \leq \left(\frac{1 + [B + (A-B)(1-\alpha)]r}{1 + Br}\right)^\beta \cdot \frac{1}{(1-r)^2} \tag{3.7}$$

and

$$\int_0^r \left(\frac{1-[B+(A-B)(1-\alpha)]t}{1-Bt}\right)^\beta \cdot \frac{1}{(1+t)^2} dt$$

$$\leq |\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)| \leq \int_0^r \left(\frac{1 + [B + (A-B)(1-\alpha)]t}{1 + Bt}\right)^\beta \cdot \frac{1}{(1-t)^2} dt. \tag{3.8}$$

Proof. From (3.3), we have

$$|[\mathcal{J}_\lambda^m(a_1,b_1;q,z)f(z)]'| = \frac{|G(z)|}{|z|}(P(z))^\beta. \tag{3.9}$$

Aouf [2] proved that

$$\frac{1-[B+(A-B)(1-\alpha)]r}{1-Br} \le |P(z)| \le \frac{1+[B+(A-B)(1-\alpha)]r}{1+Br},$$

which implies

$$\left(\frac{1-[B+(A-B)(1-\alpha)]r}{1-Br}\right)^\beta \le |P(z)|^\beta \le \left(\frac{1+[B+(A-B)(1-\alpha)]r}{1+Br}\right)^\beta. \tag{3.10}$$

By Lemma 2.3, G is a starlike function and so due to Mehrok [11], we have

$$\frac{r}{(1+r)^2} \le |G(z)| \le \frac{r}{(1-r)^2}. \tag{3.11}$$

Using (3.10) and (3.11) in (3.9), (3.7) can be easily obtained. On integrating (3.7) from 0 to r, (3.8) follows.

□

On putting $\alpha = 0, \beta = 1$ in Theorem 3.7, the following result due to Murugusundaramoorthy and Reddy [12] can be easily obtained.

Remark 3.8. If $f \in \mathcal{S}_{q,t}^{\lambda,m}(a_1,b_1;A,B)$, then for $|z| = r, 0 < r < 1$, we have

$$\left(\frac{1-Ar}{1-Br}\right)\cdot\frac{1}{(1+r)^2} \le |[\mathcal{J}_\lambda^m(a_1,b_1;q,z)f(z)]'| \le \left(\frac{1+Ar}{1+Br}\right)\cdot\frac{1}{(1-r)^2}$$

and

$$\int_0^r \left(\frac{1-At}{1-Bt}\right)\cdot\frac{1}{(1+t)^2}dt \le |\mathcal{J}_\lambda^m(a_1,b_1;q,z)f(z)| \le \int_0^r \left(\frac{1+At}{1+Bt}\right)\cdot\frac{1}{(1-t)^2}dt.$$

For $m = 0, r = 2, s = 1, a_1 = b_1, a_2 = q, \alpha = 0, \beta = 1$ and $q \to 1^-$, Theorem 3.7 gives the following result due to Singh et al. [20].

Remark 3.9. If $f \in \chi_t(A,B)$, then for $|z| = r, 0 < r < 1$, we have

$$\frac{1-Ar}{(1-Br)(1+r)^2} \le |f'(z)| \le \frac{1+Ar}{(1+Br)(1-r)^2}$$

and

$$\int_0^r \frac{1-At}{(1-Bt)(1+t)^2}dt \le |f(z)| \le \int_0^r \frac{1+At}{(1+Bt)(1-t)^2}dt.$$

For $m = 0, r = 2, s = 1, a_1 = b_1, a_2 = q, \alpha = 0, \beta = 1, A = 1-2\gamma, B = -1$ and $q \to 1^-$, Theorem 3.7 agrees with the following result established by Prajapat [14].

Remark 3.10. If $f \in \chi_t(\gamma)$, then for $|z| = r, 0 < r < 1$, we have

$$\frac{1-(1-2\gamma)r}{(1+r)^3} \leq |f'(z)| \leq \frac{1+(1-2\gamma)r}{(1-r)^3}$$

and

$$\int_0^r \frac{1-(1-2\gamma)t}{(1+t)^3} dt \leq |f(z)| \leq \int_0^r \frac{1+(1-2\gamma)t}{(1-t)^3} dt.$$

For $m = 0, r = 2, s = 1, a_1 = b_1, a_2 = q, \alpha = 0, \beta = 1, A = 1, B = -1, t = -1$ and $q \to 1^-$, Theorem 3.7 gives the following result for the class $\mathcal{K}_s$.

Remark 3.11. If $f \in \mathcal{K}_s$, then for $|z| = r, 0 < r < 1$, we have

$$\frac{1-r}{(1+r)^3} \leq |f'(z)| \leq \frac{1+r}{(1-r)^3}$$

and

$$\int_0^r \frac{1-t}{(1+t)^3} dt \leq |f(z)| \leq \int_0^r \frac{1+t}{(1-t)^3} dt.$$

Theorem 3.12. *If $f \in \mathcal{S}_{q,t}^{\lambda,m}(a_1, b_1; A, B; \alpha; \beta)$, then for $|z| = r, 0 < r < 1$, we have*

$$|arg[\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)]'| \leq \beta sin^{-1}\left(\frac{(A-B)(1-\alpha)r}{1-[B+(A-B)(1-\alpha)]Br^2}\right) + 2sin^{-1}r.$$

Proof. From (3.3), we have

$$[\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)]' = \frac{G(z)}{z}(P(z))^\beta,$$

which implies

$$|arg[\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)]'| \leq \beta|argP(z)| + \left|arg\frac{G(z)}{z}\right|. \quad (3.12)$$

As G is a starlike function and so due to Mehrok [11], we have

$$\left|arg\frac{G(z)}{z}\right| \leq 2sin^{-1}r. \quad (3.13)$$

Aouf [1], established that,

$$|argP(z)| \leq sin^{-1}\left(\frac{(A-B)(1-\alpha)r}{1-[B+(A-B)(1-\alpha)]Br^2}\right). \quad (3.14)$$

Using (3.13) and (3.14) in (3.12), the proof is obvious.

□

On putting $\alpha = 0, \beta = 1$ in Theorem 3.12, the following result for the class $\mathcal{S}_{q,t}^{\lambda,m}(a_1, b_1; A, B)$ can be easily obtained.

Remark 3.13. If $f \in \mathcal{S}_{q,t}^{\lambda,m}(a_1, b_1; A, B)$, then for $|z| = r, 0 < r < 1$, we have

$$|arg[\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)]'| \leq sin^{-1}\left(\frac{(A-B)r}{1-ABr^2}\right) + 2sin^{-1}r.$$

For $m = 0, r = 2, s = 1, a_1 = b_1, a_2 = q, \alpha = 0, \beta = 1$ and $q \to 1^-$, Theorem 3.12 gives the following result due to Singh et al. [20].

Remark 3.14. If $f \in \chi_t(A, B)$, then for $|z| = r, 0 < r < 1$, we have

$$|argf'(z)| \leq sin^{-1}\left(\frac{(A-B)r}{1-ABr^2}\right) + 2sin^{-1}r.$$

For $m = 0, r = 2, s = 1, a_1 = b_1, a_2 = q, \alpha = 0, \beta = 1, A = 1 - 2\gamma, B = -1$ and $q \to 1^-$, Theorem 3.12 gives the following result for the class $\chi_t(\gamma)$.

Remark 3.15. If $f \in \chi_t(\gamma)$, then for $|z| = r, 0 < r < 1$, we have

$$|argf'(z)| \leq sin^{-1}\left(\frac{2(1-\gamma)r}{1+(1-2\gamma)r^2}\right) + 2sin^{-1}r.$$

For $m = 0, r = 2, s = 1, a_1 = b_1, a_2 = q, \alpha = 0, \beta = 1, A = 1, B = -1, t = -1$ and $q \to 1^-$, Theorem 3.12 gives the following result for the class $\mathcal{K}_s$.

Remark 3.16. If $f \in \mathcal{K}_s$, then for $|z| = r, 0 < r < 1$, we have

$$|argf'(z)| \leq sin^{-1}\left(\frac{2r}{1+r^2}\right) + 2sin^{-1}r.$$

Theorem 3.17. *Let $f \in \mathcal{S}_{q,t}^{\lambda,m}(a_1, b_1; A, B; \alpha; \beta)$, then $\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)$ is convex in $|z| < r_1$, where r_1 is the smallest positive root in $(0, 1)$ of the equation*

$$B[B + (A-B)(1-\alpha)]r^3 - [B(B-2) + (A-B)(1-\alpha)(B-1-\beta)]r^2$$
$$-[(1-\beta)(A-B)(1-\alpha) + (2B-1)]r - 1 = 0. \quad (3.15)$$

Proof. As $f \in \mathcal{S}_{q,t}^{\lambda,m}(a_1, b_1; A, B; \alpha; \beta)$, we have

$$z[\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)]' = G(z)(P(z))^\beta.$$

On differentiating it logarithmically, we get

$$\frac{(z[\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)]')'}{[\mathcal{J}_\lambda^m(a_1, b_1; q, z)f(z)]'} = \frac{zG'(z)}{G(z)} + \beta\frac{zP'(z)}{P(z)}. \quad (3.16)$$

As $G \in \mathcal{S}^*$, from [11], we have

$$Re\left(\frac{zG'(z)}{G(z)}\right) \geq \frac{1-r}{1+r}. \quad (3.17)$$

Also it can be easily verified that

$$\left|\frac{zP'(z)}{P(z)}\right| \leq \frac{r(A-B)(1-\alpha)}{(1+Br)(1+[B+(A-B)(1-\alpha)]r)}. \quad (3.18)$$

(3.16) can be expressed as

$$Re\left(\frac{(z[\mathcal{J}_{\lambda}^{m}(a_1,b_1;q,z)f(z)]')'}{[\mathcal{J}_{\lambda}^{m}(a_1,b_1;q,z)f(z)]'}\right)\geq Re\left(\frac{zG'(z)}{G(z)}\right)-\beta\left|\frac{zP'(z)}{P(z)}\right|. \tag{3.19}$$

Using (3.17) and (3.18), (3.19) yields

$$Re\left(\frac{(z[\mathcal{J}_{\lambda}^{m}(a_1,b_1;q,z)f(z)]')'}{[\mathcal{J}_{\lambda}^{m}(a_1,b_1;q,z)f(z)]'}\right)\geq\frac{1-r}{1+r}-\beta\frac{r(A-B)(1-\alpha)}{(1+Br)(1+[B+(A-B)(1-\alpha)]r)}. \tag{3.20}$$

After some simplification, (3.20) takes the form

$$Re\left(\frac{(z[\mathcal{J}_{\lambda}^{m}(a_1,b_1;q,z)f(z)]')'}{[\mathcal{J}_{\lambda}^{m}(a_1,b_1;q,z)f(z)]'}\right)\geq\frac{-B[B+(A-B)(1-\alpha)]r^3}{(1+r)(1+Br)(1+[B+(A-B)(1-\alpha)]r)}$$
$$+\frac{[B(B-2)+(A-B)(1-\alpha)(B-1-\beta)]r^2}{(1+r)(1+Br)(1+[B+(A-B)(1-\alpha)]r)}$$
$$+\frac{[(1-\beta)(A-B)(1-\alpha)-(2B-1)]r+1}{(1+r)(1+Br)(1+[B+(A-B)(1-\alpha)]r)}.$$

Hence, the function $\mathcal{J}_{\lambda}^{m}(a_1,b_1;q,z)f(z)$ is convex in $|z|<r_1$, where r_1 is the smallest positive root in $(0,1)$ of the equation

$$B[B+(A-B)(1-\alpha)]r^3-[B(B-2)+(A-B)(1-\alpha)(B-1-\beta)]r^2$$
$$-[(1-\beta)(A-B)(1-\alpha)+(2B-1)]r-1=0.$$

□

On putting $\alpha=0,\beta=1$ in Theorem 3.17, the following result due to Murugusundaramoorthy and Reddy [12] can be easily obtained.

Remark 3.18. If $f\in\mathcal{S}_{q,t}^{\lambda,m}(a_1,b_1;A,B)$, then $\mathcal{J}_{\lambda}^{m}(a_1,b_1;q,z)f(z)$ is convex in $|z|<r_2$, where r_2 is the smallest positive root in $(0,1)$ of the equation

$$ABr^3-A(B-2)r^2-(2B-1)r-1=0.$$

For $m=0,r=2,s=1,a_1=b_1,a_2=q,\alpha=0,\beta=1$ and $q\to 1^-$, Theorem 3.17 gives the following result due to Singh et al. [20].

Remark 3.19. If $f\in\chi_t(A,B)$, then $f(z)$ is convex in $|z|<r_3$, where r_3 is the smallest positive root in $(0,1)$ of the equation

$$ABr^3-A(B-2)r^2-(2B-1)r-1=0.$$

For $m=0,r=2,s=1,a_1=b_1,a_2=q,\alpha=0,\beta=1,A=1-2\gamma,B=-1$ and $q\to 1^-$, Theorem 3.17 gives the following result due to Prajapat [14].

Remark 3.20. If $f\in\chi_t(\gamma)$, then $f(z)$ is convex in $|z|<r_4=2-\sqrt{3}$.

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