Stud. Univ. Babeş-Bolyai Math. 70(2025), No. 3, 427–440
DOI: 10.24193/subbmath.2025.3.04

On the coefficient estimates for a subclass of m-fold symmetric bi-univalent functions

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Abstract. In this work, we introduce and investigate a subclass $\mathcal{G}_{\Sigma_m}^{h,p}(\lambda,\gamma)$ of analytic and bi-univalent functions when both f and f^{-1} are m-fold symmetric in the open unit disk $\mathbb{U}$. Moreover, we find upper bounds for the initial coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ for functions belonging to this subclass $\mathcal{G}_{\Sigma_m}^{h,p}(\lambda,\gamma)$. The results presented in this paper would generalize and improve those that were given in several recent works.

Mathematics Subject Classification (2010): 30C45, 30C50.

Keywords: Analytic functions, bi-univalent functions, coefficient estimates, m-fold symmetric bi-univalent functions.

1. Introduction

Let $\mathcal{A}$ denote the class of functions of the following normalized form:

$$f(z) = z + \sum_{j=2}^{\infty} a_j z^j \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$.

Also, we denote by $\mathcal{S}$ the class of all functions in the normalized analytic function class $f \in \mathcal{A}$ which are univalent in $\mathbb{U}$.

Since univalent functions are one-to-one, they are invertible and the inverse functions need not be defined on the entire unit disk $\mathbb{U}$. The *Koebe One-Quarter Theorem* [4] ensures that the image of $\mathbb{U}$ under every univalent function $f \in \mathcal{S}$ contains

Received 12 November 2024; Accepted 31 May 2025.

a disk of radius $\frac{1}{4}$. Hence, every function $f \in \mathcal{S}$ has an inverse f^{-1}, which is defined by

$$f^{-1}(f(z)) = z \ (z \in \mathbb{U}),$$

and

$$f(f^{-1}(w)) = w \ \left(|w| < r_0(f); r_0(f) \geq \frac{1}{4}\right),$$

where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots . \tag{1.2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$, if both f and f^{-1} are univalent in $\mathbb{U}$. The class consisting of bi-univalent functions are denoted by Σ.

Determination of the bounds for the coefficients a_n is an important problem in geometric function theory as they give information about the geometric properties of these functions. For example, the bound for the second coefficient a_2 of functions $f \in \mathcal{S}$ gives the growth and distortion bounds as well as covering theorems.

Lewin [8] investigated the class Σ of bi-univalent functions and showed that $|a_2| < 1.51$ for the functions belonging to Σ. Subsequently, Brannan and Clunie [2] conjectured that $|a_2| \leq \sqrt{2}$. Kedzierawski [7] proved this conjecture for a special case when the function f and f^{-1} are starlike functions. Tan [14] obtained the bound for $|a_2|$ namely $|a_2| \leq 1.485$ which is the best known estimate for functions in the class Σ. Recently there are interest to study the bi-univalent functions class Σ (see [5, 6, 16, 17]) and obtain non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$. The coefficient estimate problem i.e. bound of $|a_n|$ $(n \in \mathbb{N} - \{1, 2\})$ for each $f \in \Sigma$ given by (1.1) is still an open problem. For each function $f \in \mathcal{S}$ the function $h(z)$ given by

$$h(z) = \sqrt[m]{f(z^m)} \qquad (z \in \mathbb{U}, m \in \mathbb{N})$$

is univalent and maps the unit disk $\mathbb{U}$ into a region with m-fold symmetry. A function is called m-fold symmetric (see[11, 12, 13]) if the function $f(z)$ has the following normalized form:

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1} \qquad (z \in \mathbb{U}, m \in \mathbb{N}) \tag{1.3}$$

We denote by $\mathcal{S}_m$ the class of m-fold symmetric univalent functions in $\mathbb{U}$, which are normalized by the series expansion (1.3). In fact, the functions in the class $\mathcal{S}$ are one-fold symmetric, that is

$$\mathcal{S}_1 = \mathcal{S}$$

Analogous to the concept of m-fold symmetric univalent functions, we now introduce the concept of m-fold symmetric bi-univalent functions. Each function $f \in \Sigma$ generates an m-fold symmetric bi-univalent function for each integer$f \in N$. The normalized form of f is given as in (1.3). Furthermore, the series expansion for f^{-1} , which was recently proven by Srivastava et al. [13], is given as follows:

$$g(w) = w - a_{m+1}w^{m+1} + [(m+1)a_{m+1}^2 - a_{2m+1}]w^{2m+1} -$$
$$[\frac{1}{2}(m+1)(3m+2)a_{m+1}^3 - (3m+2)a_{m+1}a_{2m+1} + a_{3m+1}]w^{3m+1} + \cdots,$$

where $g = f^{-1}$.
We denote by Σ_m the class of m-fold symmetric bi-univalent functions in $\mathbb{U}$. In the special case when $m = 1$, the formula (1.4) for the class Σ_m coincides with the formula (1.2) for the class Σ. Some examples of m-fold symmetric bi-univalent functions are given below:

$$\left(\frac{z^m}{1-z^m}\right)^{\frac{1}{m}} \qquad \text{and} \qquad [-\log(1-z^m)]^{\frac{1}{m}}$$

with the corresponding inverse functions given by

$$\left(\frac{w^m}{1-w^m}\right)^{\frac{1}{m}} \qquad \text{and} \qquad \left(\frac{e^{w^m}-1}{e^{w^m}}\right)^{\frac{1}{m}}$$

respectively.

Quite recently, Wanas and Páll-Szabó [15] introduced two new general subclasses $\mathcal{AS}_{\Sigma_m}(\gamma, \lambda; \alpha)$ and $\mathcal{AS}^*_{\Sigma_m}(\gamma, \lambda; \beta)$ of the m-fold symmetric bi-univalent function class Σ_m consisting of analytic and m-fold symmetric bi-univalent functions in $\mathbb{U}$ and derived the coefficient bounds for $|a_{m+1}|$ and $|a_{2m+1}|$ for functions in each of these new subclasses.

Definition 1.1. [15] *A function $f \in \Sigma_m$ given by* (1.3) *is said to be in the class $\mathcal{AS}_{\Sigma_m}(\gamma, \lambda; \alpha)$ if it satisfies the following conditions:*

$$\left|arg\left[(1-\lambda)\frac{zf'(z)}{f(z)} + \lambda\left(1 + \frac{zf''(z)}{f'(z)}\right)\right]^{\gamma}\right| < \frac{\alpha\pi}{2}$$

and

$$\left|arg\left[(1-\lambda)\frac{wg'(w)}{g(w)} + \lambda\left(1 + \frac{wg''(w)}{g'(w)}\right)\right]^{\gamma}\right| < \frac{\alpha\pi}{2},$$

where $z, w \in \mathbb{U},\ 0 < \gamma \leq 1,\ 0 \leq \lambda \leq 1,\ 0 < \alpha \leq 1,\ m \in \mathbb{N}$ *and* $g = f^{-1}$.

Theorem 1.2 ([15]). *Let $f \in \mathcal{AS}_{\Sigma_m}(\gamma, \lambda; \alpha)$ be given by* (1.3). *Then*

$$|a_{m+1}| \leq \frac{2\alpha}{m\sqrt{2\alpha\gamma(1+\lambda m) + \gamma(\gamma-\alpha)(1+\lambda m)^2}}$$

and

$$|a_{2m+1}| \leq \frac{2\alpha^2(m+1)}{m^2\gamma^2(1+\lambda m)^2} + \frac{\alpha}{m\gamma(1+2\lambda m)}.$$

Definition 1.3. [15] *A function $f \in \Sigma_m$ given by* (1.3) *is said to be in the class ${\mathcal{AS}^*}_{\Sigma_m}(\gamma, \lambda; \beta)$, if it satisfies the following conditions:*

$$\Re\left[(1-\lambda)\frac{zf'(z)}{f(z)} + \lambda\left(1 + \frac{zf''(z)}{f'(z)}\right)\right]^{\gamma} > \beta$$

and

$$\Re\left[(1-\lambda)\frac{wg'(w)}{g(w)}+\lambda\left(1+\frac{wg''(w)}{g'(w)}\right)\right]^{\gamma}>\beta,$$

where $z,w\in\mathbb{U},\ 0<\gamma\le 1,\ 0\le\lambda\le 1,\ 0\le\beta<1,\ m\in\mathbb{N}$ *and* $g=f^{-1}$.

Theorem 1.4 ([15]). *Let* $f\in\mathcal{AS}^*{}_{\Sigma_m}(\gamma,\lambda;\beta)$ *be given by* (1.3)*. Then*

$$|a_{m+1}|\le\frac{2}{m}\sqrt{\frac{1-\beta}{2\gamma(1+\lambda m)+\gamma(\gamma-1)(1+\lambda m)^2}}$$

and

$$|a_{2m+1}|\le\frac{2(m+1)(1-\beta)^2}{m^2\gamma^2(1+\lambda m)^2}+\frac{1-\beta}{m\gamma(1+2\lambda m)}.$$

The main objective of this paper is to present an elegant formula for computing the coefficients of the inverse functions for the class Σ_m of m-fold symmetric functions by means of the residue calculus. As an application, we introduce a new subclass of bi-univalent functions in which both f and f^{-1} are m-fold symmetric analytic functions and obtain upper bounds for the coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ for functions in this subclass. Our results for the bi-univalent function class $\mathcal{G}^{h,p}_{\Sigma_m}(\gamma,\lambda)$, which we shall introduce in section 2, would generalize and improve some recent works by Wanas and Páll-Szabó [15] and some of other researchers[1, 9, 10]

2. Coefficient Estimates

In this section, we introduce and investigate the general subclass $\mathcal{G}^{h,p}_{\Sigma_m}(\gamma,\lambda)$.

Definition 2.1. *Let* $h,p:\mathbb{U}\to\mathbb{C}$ *be analytic functions and*

$$\min\{\Re(h(z)),\Re(p(z))\}>0\ (z\in\mathbb{U})\ \text{and}\ h(0)=p(0)=1.$$

A function f *given by* (1.3) *is said to be in the class* $\mathcal{G}^{h,p}_{\Sigma_m}(\gamma,\lambda)$*, if the following conditions are satisfied:*

$$\left[(1-\lambda)\frac{zf'(z)}{f(z)}+\lambda\left(1+\frac{zf''(z)}{f'(z)}\right)\right]^{\gamma}\in h(\mathbb{U}) \tag{2.1}$$

and

$$\left[(1-\lambda)\frac{wg'(w)}{g(w)}+\lambda\left(1+\frac{wg''w)}{g'(w)}\right)\right]^{\gamma}\in p(\mathbb{U}) \tag{2.2}$$

where $z,w\in\mathbb{U}$, $0<\gamma\le 1$, $0\le\lambda\le 1$, $m\in\mathbb{N}$ *and* $g=f^{-1}$.

Remark 2.2. There are many choices of the functions h,p which would provide interesting subclasses of the general class $\mathcal{G}^{h,p}_{\Sigma_m}(\gamma,\lambda)$. For example, if we set $\gamma=1$, the subclass $\mathcal{G}^{h,p}_{\Sigma_m}(\gamma,\lambda)$ reduces to the subclass $f\in\mathcal{M}^{h,p}_{\Sigma_m}(\lambda,1)$ which was introduced by Motamednezhad et al. [10].

If we let

$$h(z) = p(z) = \left(\frac{1+z^m}{1-z^m}\right)^\alpha = 1 + 2\alpha z^m + 2\alpha^2 z^{2m} + \cdots (0 < \alpha \leq 1),$$

it can easily be verified that the functions $h(z)$ and $p(z)$ satisfy the hypotheses of Definition 2.1. Thus, if we have $f \in \mathcal{G}^{h,p}_{\Sigma_m}(\gamma, \lambda)$, then

$$\left| arg \left[(1-\lambda)\frac{zf'(z)}{f(z)} + \lambda \left(1 + \frac{zf''(z)}{f'(z)} \right) \right]^\gamma \right| < \frac{\alpha\pi}{2}$$

and

$$\left| arg \left[(1-\lambda)\frac{wg'(w)}{g(w)} + \lambda \left(1 + \frac{wg''(w)}{g'(w)} \right) \right]^\gamma \right| < \frac{\alpha\pi}{2}.$$

In this case we say that f belongs to the subclass $f \in \mathcal{AS}_{\Sigma_m}(\gamma, \lambda; \alpha)$. If we put $h(z) = p(z) = \left(\frac{1+z^m}{1-z^m}\right)^\alpha$ and $\gamma = 1$, the subclass $\mathcal{G}^{h,p}_{\Sigma_m}(\gamma, \lambda)$ reduces to the subclass $\mathcal{M}_{\Sigma_m}(\alpha, \lambda, 1)$ which was considered by Motamednezad et al. [10].

Also, for $h(z) = p(z) = \left(\frac{1+z^m}{1-z^m}\right)^\alpha$, $\gamma = 1$ and $\lambda = 0$, the subclass $\mathcal{G}^{h,p}_{\Sigma_m}(\gamma, \lambda)$ reduces to the subclass $\mathcal{S}^\alpha_{\Sigma_m}$ which was considered by Altinkaya and Yalcin [1].

On the other hand, if we take

$$h(z) = p(z) = \frac{1 + (1-2\beta)z^m}{1 - z^m} = 1 + 2(1-\beta)z^m + 2(1-\beta)z^{2m} + \cdots (0 \leq \beta < 1).$$

then the conditions of Definition 2.1 are satisfied for both functions $h(z)$ and $p(z)$. Thus, if $f \in \mathcal{G}^{h,p}_{\Sigma_m}(\gamma, \lambda)$; then

$$\Re \left[(1-\lambda)\frac{zf'(z)}{f(z)} + \lambda \left(1 + \frac{zf''(z)}{f'(z)} \right) \right]^\gamma > \beta$$

and

$$\Re \left[(1-\lambda)\frac{wg'(w)}{g(w)} + \lambda \left(1 + \frac{wg''(w)}{g'(w)} \right) \right]^\gamma > \beta.$$

In this case we say that f belonges to the subclass $f \in \mathcal{AS^*}_{\Sigma_m}(\gamma, \lambda; \beta)$. If we put $h(z) = p(z) = \frac{1+(1-2\beta)z^m}{1-z^m}$ and $\gamma = 1$, the subclass $\mathcal{G}^{h,p}_{\Sigma_m}(\gamma, \lambda)$ reduces to the subclass $\mathcal{M}_{\Sigma_m}(\beta, \lambda, 1)$ which was considered by Motamednezad et al. [10].

Also, for $h(z) = p(z) = \frac{1+(1-2\beta)z^m}{1-z^m}$, $\gamma = 1$ and $\lambda = 0$, the subclass $\mathcal{G}^{h,p}_{\Sigma_m}(\gamma, \lambda)$ reduces to the subclass $\mathcal{S}^\beta_{\Sigma_m}$ which was considered by Altinkaya and Yalcin [1].

Remark 2.3. For one-Fold symmetric bi-univalent functions, we denote the subclass $\mathcal{G}^{h,p}_{\Sigma_m}(\gamma, \lambda) = \mathcal{G}^{h,p}_{\Sigma}(\gamma, \lambda)$. Special cases of this subclass illustrated below:

(A) By putting $h(z) = p(z) = \left(\frac{1+z}{1-z}\right)^\alpha$ and $\gamma = 1$, then the subclass $\mathcal{G}^{h,p}_{\Sigma}(\lambda, \gamma)$ reduces to the subclass $M_\Sigma(\alpha, \lambda)$ studied by Li and Wang [9].

(B) By putting $h(z) = p(z) = \left(\frac{1+z}{1-z}\right)^\alpha$, $\gamma = 1$ and $\lambda = 0$, then the subclass $\mathcal{G}^{h,p}_{\Sigma}(\lambda, \gamma)$ reduces to the subclass $S^*_\Sigma(\alpha)$ studied by Brannan and Taha[3].

(C) By putting $h(z) = p(z) = \left(\frac{1+z}{1-z}\right)^{\alpha}$, and $\lambda = \gamma = 1$, then the subclass $\mathcal{G}_{\Sigma}^{h,p}(\lambda,\gamma)$ reduces to the subclass $M_{\Sigma}(\alpha, 1)$ studied by Li and Wang [9].

(D) By putting $h(z) = p(z) = \frac{1+(1-2\beta)z}{1-z}$ and $\gamma = 1$, then the subclass $\mathcal{G}_{\Sigma}^{h,p}(\lambda,\gamma)$ reduces to the subclass $B_{\Sigma}(\beta, \lambda)$ studied by Li and Wang [9].

(E) By putting $h(z) = p(z) = \frac{1+(1-2\beta)z}{1-z}$, $\gamma = 1$ and $\lambda = 0$, then the subclass $\mathcal{G}_{\Sigma}^{h,p}(\lambda,\gamma)$ reduces to the subclass $S_{\Sigma}^{*}(\beta)$ of bi-starlike functions of order $\beta(0 \leq \beta < 1)$ studied by Brannan and Taha[3].

(F) By putting $h(z) = p(z) = \frac{1+(1-2\beta)z}{1-z}$ and $\lambda = \gamma = 1$, then the subclass $\mathcal{G}_{\Sigma}^{h,p}(\lambda,\gamma)$ reduces to the subclass $B_{\Sigma}(\beta, 1)$ of bi-convex functions of order $\beta(0 \leq \beta < 1)$ studied by Li and Wang [9].

We are now ready to express the bounds for the coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ for the subclass $\mathcal{G}_{\Sigma_m}^{h,p}(\gamma,\lambda)$ of the normalized bi-univalent function class Σ_m.

Theorem 2.4. *Let the function f given by (1.3) be in the class $\mathcal{G}_{\Sigma_m}^{h,p}(\gamma,\lambda)$. Then*

$$|a_{m+1}| \leq min\left\{\sqrt{\frac{|h_{2m}| + |p_{2m}|}{m^2\gamma[2(1+\lambda m) + (\gamma-1)(1+\lambda m)^2]}},\right.$$
$$\left.\sqrt{\frac{|h_m|^2 + |p_m|^2}{2[m\gamma(1+\lambda m)]^2}}\right\} \tag{2.3}$$

and

$$|a_{2m+1}| \leq min\left\{\frac{|h_{2m}| + |p_{2m}|}{4\gamma m(1+2\lambda m)} + \frac{(m+1)(|h_m|^2 + |p_m|^2)}{4\gamma^2 m^2(1+\lambda m)^2},\right.$$
$$\frac{\left|2m(1+\lambda m) + 2(m+1)(1+2\lambda m) + m(\gamma-1)(1+\lambda m)^2\right|}{4m^2\gamma(1+2\lambda m)[2(1+\lambda m) + (\gamma-1)(1+\lambda m)^2]}|h_{2m}| +$$
$$\left.\frac{\left|2(m+1)(1+2\lambda m) - 2m(1+\lambda m) - m(\gamma-1)(1+\lambda m)^2\right|}{4m^2\gamma(1+2\lambda m)[2(1+\lambda m) + (\gamma-1)(1+\lambda m)^2]}|p_{2m|}\right\}. \tag{2.4}$$

Proof. The main idea in the proof of Theorem 2.4 is to get the desired bounds for the coefficient $|a_{m+1}|$ and $|a_{2m+1}|$. Indeed, by considering the relations (2.1) and (2.2), we have

$$\left[(1-\lambda)\frac{zf'(z)}{f(z)} + \lambda\left(1 + \frac{zf''(z)}{f'(z)}\right)\right]^{\gamma} = h(z) \tag{2.5}$$

and

$$\left[(1-\lambda)\frac{wg'(w)}{g(w)} + \lambda\left(1 + \frac{wg''(w)}{g'(w)}\right)\right]^{\gamma} = p(z), \tag{2.6}$$

where each of the functions h and p satisfies the conditions of Definition 1.3. In light of the following Taylor-Maclaurin series expansions for the functions h and p, we get

$$h(z) = 1 + h_m z^m + h_{2m} z^{2m} + h_{3m} z^{3m} + \cdots \tag{2.7}$$

and

$$p(w) = 1 + p_m w^m + p_{2m} w^{2m} + p_{3m} w^{3m} + \cdots . \tag{2.8}$$

By substituting the relations (2.7) and (2.8) into (2.5) and (2.6), respectively, we get

$$m\gamma(1+\lambda m)a_{m+1} = h_m, \tag{2.9}$$

$$\gamma m\left[\frac{(\gamma-1)}{2}m(1+\lambda m)^2 - (\lambda m^2+2\lambda m+1)\right]a_{m+1}^2 + 2m\gamma(1+2\lambda m)a_{2m+1} = h_{2m}, \tag{2.10}$$

$$-m\gamma(1+\lambda m)a_{m+1} = p_m \tag{2.11}$$

and

$$\gamma m\left[\left(3\lambda m^2+2(\lambda+1)m+1\right) + \frac{(\gamma-1)}{2}m(1+\lambda m)^2\right]a_{m+1}^2 - 2m\gamma(1+2\lambda m)a_{2m+1} = p_{2m}. \tag{2.12}$$

Comparing the coefficients (2.9) and (2.11), we obtain

$$h_m = -p_m \tag{2.13}$$

and

$$2m^2\gamma^2(1+\lambda m)^2 a_{m+1}^2 = h_m^2 + p_m^2. \tag{2.14}$$

Now, if we add (2.10) and (2.12), we get the following relation

$$m^2\gamma\left[2(1+\lambda m) + (\gamma-1)(1+\lambda m)^2\right]a_{m+1}^2 = h_{2m} + p_{2m}. \tag{2.15}$$

Therefore, from (2.14) and (2.15), we have

$$a_{m+1}^2 = \frac{h_m^2 + p_m^2}{2[m\gamma(1+\lambda m)]^2} \tag{2.16}$$

and

$$a_{m+1}^2 = \frac{h_{2m} + p_{2m}}{m^2\gamma[2(1+\lambda m) + (\gamma-1)(1+\lambda m)^2]}, \tag{2.17}$$

respectively.

Therefore, we find from the equations (2.16) and (2.17) that

$$|a_{m+1}|^2 \le \frac{|h_m|^2 + |p_m|^2}{2\gamma^2 m^2(1+\lambda m)^2}$$

and

$$|a_{m+1}|^2 \le \frac{|h_{2m}| + |p_{2m}|}{m^2\gamma[2(1+\lambda m) + (\gamma-1)(1+\lambda m)^2]},$$

respectively. We have thus derived the desired bound on the coefficient $|a_{m+1}|$.

The proof is completed by finding the bound on the coefficient $|a_{2m+1}|$. Upon subtracting (2.12) from (2.10), we get

$$a_{2m+1} = \frac{h_{2m} - p_{2m}}{4\gamma m(1+2\lambda m)} + \frac{(m+1)}{2}a_{m+1}^2. \tag{2.18}$$

Putting the value of a_{m+1}^2 from (2.16) into (2.18), it follows that

$$a_{2m+1} = \frac{h_{2m} - p_{2m}}{4\gamma m(1+2\lambda m)} + \frac{(m+1)h_{2m} + p_{2m}}{m^2\gamma[2(1+\lambda m) + (\gamma-1)(1+\lambda m)^2]}.$$

Therefore, we conclude the following bound:

$$|a_{2m+1}| \le \frac{|h_{2m}| + |p_{2m}|}{4\gamma m(1+2\lambda m)} + \frac{(m+1)(|h_m|^2 + |p_m|^2)}{4[\gamma m(1+\lambda m)]^2}. \tag{2.19}$$

By substituting the value of a_{m+1}^2 from (2.17) into (2.18), we obtain

$$a_{2m+1} = \frac{m[2(1+\lambda m) + (\gamma-1)(1+\lambda m)^2](h_{2m} - p_{2m}) + (m+1)(1+2\lambda m)}{4m^2\gamma(1+2\lambda m)[2(1+\lambda m) + (\gamma-1)(1+\lambda m)^2]} \tag{2.20}$$

$$\frac{(h_{2m} + p_{2m})}{4m^2\gamma(1+2\lambda m)[2(1+\lambda m) + (\gamma-1)(1+\lambda m)^2]}$$

which readily yields

$$|a_{2m+1}| \le \frac{\left|2m(1+\lambda m) + 2(m+1)(1+2\lambda m) + m(\gamma-1)(1+\lambda m)^2\right|}{4m^2\gamma(1+2\lambda m)[2(1+\lambda m) + (\gamma-1)(1+\lambda m)^2]}|h_{2m}| + \tag{2.21}$$

$$\frac{\left|2(m+1)(1+2\lambda m) - 2m(1+\lambda m) - m(\gamma-1)(1+\lambda m)^2\right|}{4m^2\gamma(1+2\lambda m)[2(1+\lambda m) + (\gamma-1)(1+\lambda m)^2]}|p_{2m|}. \tag{2.22}$$

Finally, from (2.19) and (2.21), we get the desired estimate on the coefficient $|a_{2m+1}|$ as asserted in Theorem 2.4. The proof of Theorem 2.4 is thus completed. □

3. Corollaries and Consequences

If we put

$$h(z) = p(z) = \left(\frac{1+z^m}{1-z^m}\right)^\alpha = 1 + 2\alpha z^m + 2\alpha^2 z^{2m} + \cdots,$$

in Theorem 2.4, then it can be obtained the following result.

Corollary 3.1. *Let the function f given by* (1.3) *be in the class $\mathcal{AS}_{\Sigma_m}(\gamma, \lambda; \alpha)$. Then*

$$|a_{m+1}| \le min\left\{\frac{2\alpha}{m\gamma(1+\lambda m)}, \frac{2\alpha}{m\sqrt{\gamma[2(1+\lambda m) + (\gamma-1)(1+\lambda m)^2]}}\right\}$$

and

$$|a_{2m+1}| \leq min\Bigg\{\frac{\alpha^2}{m\gamma(1+2\lambda m)} + \frac{2(m+1)\alpha^2}{\gamma^2 m^2(1+\lambda m)^2},$$
$$\left|\frac{2m(1+\lambda m)+2(m+1)(1+2\lambda m)+m(\gamma-1)(1+\lambda m)^2}{2m^2\gamma(1+2\lambda m)[2(1+\lambda m)+(\gamma-1)(1+\lambda m)^2]}\right|\alpha^2 +$$
$$\left|\frac{2(m+1)(1+2\lambda m)-2m(1+\lambda m)-m(\gamma-1)(1+\lambda m)^2}{2m^2\gamma(1+2\lambda m)[2(1+\lambda m)+(\gamma-1)(1+\lambda m)^2]}\right|\alpha^2\Bigg\}.$$

Remark 3.2. For the coefficient $|a_{2m+1}|$ it is easily seen that

$$\frac{\alpha^2}{m\gamma(1+2\lambda m)} + \frac{2(m+1)\alpha^2}{\gamma^2 m^2(1+\lambda m)^2} \leq \frac{\alpha}{m\gamma(1+2\lambda m)} + \frac{2(m+1)\alpha^2}{\gamma^2 m^2(1+\lambda m)^2}.$$

Therefore, clearly, Corollary3.1 provides an improvement over Theorem1.2.

By setting $\gamma = 1$ in Corollary 3.1, we conclude the following result.

Corollary 3.3. *Let the function f given by* (1.3) *be in the subclass $\mathcal{M}_{\Sigma_m}(\alpha, \lambda, 1)$. Then*

$$|a_{m+1}| \leq min\left\{\frac{2\alpha}{m(1+\lambda m)}, \frac{2\alpha}{m\sqrt{2(1+\lambda m)}}\right\} = \begin{cases} \frac{2\alpha}{m\sqrt{2(1+\lambda m)}}, & 0 \leq \lambda \leq \frac{1}{m} \\ \frac{2\alpha}{m(1+\lambda m)}, & \frac{1}{m} \leq \lambda < 1 \end{cases}$$

and

$$|a_{2m+1}| \leq min\left\{\frac{m(1+\lambda m)^2+2(m+1)(1+2\lambda m)}{m^2(1+2\lambda m)(1+\lambda m)^2}\alpha^2, \frac{(m+1)}{m^2(1+\lambda m)}\alpha^2\right\}.$$

By setting $\lambda = 0$ in Corollary 3.3, we conclude the following result.

Corollary 3.4. *Let the function f given by* (1.3) *be in the subclass $\mathcal{S}^{\alpha}_{\Sigma_m}$. Then*

$$|a_{m+1}| \leq min\left\{\frac{2\alpha}{m}, \frac{\sqrt{2}\alpha}{m}\right\} = \frac{\sqrt{2}\alpha}{m}$$

and

$$|a_{2m+1}| \leq min\left\{\frac{(3m+2)\alpha^2}{m^2}, \frac{(m+1)\alpha^2}{m^2}\right\} = \frac{(m+1)\alpha^2}{m^2}.$$

Remark 3.5. The bounds on $|a_{m+1}|$ and $|a_{2m+1}|$ given in Corollary 3.4 are better than those given by Altinkaya and Yalcin [1, Corollary 6], because of

$$\frac{\sqrt{2}\alpha}{m} \leq \frac{2\alpha}{m\sqrt{\alpha+1}}$$

and

$$\frac{(m+1)\alpha^2}{m^2} \leq \frac{\alpha}{m} + \frac{2(m+1)\alpha^2}{m^2}.$$

By setting $\gamma = 1$ and $m = 1$ in Corollary 3.1, we conclude the following result.

Corollary 3.6. *Let the function f given by* (1.1) *be in the subclass $\mathcal{M}_{\Sigma}(\alpha,\lambda)$. Then*

$$|a_2| \leq min\left\{\frac{2\alpha}{1+\lambda}, \alpha\sqrt{\frac{2}{1+\lambda}}\right\} = \alpha\sqrt{\frac{2}{1+\lambda}}$$

and

$$|a_3| \leq min\left\{\frac{\lambda^2+10\lambda+5}{(1+2\lambda)(1+\lambda)^2}\alpha^2, \frac{2\alpha^2}{1+\lambda}\right\} = \frac{2\alpha^2}{1+\lambda}.$$

Remark 3.7. The bounds on $|a_2|$ and $|a_3|$ given in Corollary 3.6 are better than those given by Li and Wang [9, Theorem 2.2].

By setting $\lambda = 0$ in Corollary 3.6, we conclude the following result.

Corollary 3.8. *Let the function f given by* (1.1) *be in the subclass $S^*_{\Sigma}(\alpha)$. Then*

$$|a_2| \leq \sqrt{2}\alpha \qquad and \qquad |a_3| \leq 2\alpha^2.$$

Remark 3.9. The bounds on $|a_2|$ and $|a_3|$ given in Corollary 3.10 are better than those given by Brannan and Taha [3].

By setting $\lambda = 1$ in Corollary 3.6, we conclude the following result.

Corollary 3.10. *Let the function f given by* (1.1) *be in the subclass $\mathcal{M}_{\Sigma}(\alpha,1)$. Then*

$$|a_2| \leq \alpha \qquad and \qquad |a_3| \leq \alpha^2.$$

Remark 3.11. The bound on $|a_3|$ given in Corollary 3.8 are better than those given by Li and Wang [9, Theorem 2.2] for $\lambda = 1$.

By letting

$$h(z) = p(z) = \frac{1+(1-2\beta)z^m}{1-z^m} = 1 + 2(1-\beta)z^m + 2(1-\beta)z^{2m} + \cdots (0 \leq \beta < 1).$$

in Theorem 2.4, we deduce the following corollary.

Corollary 3.12. *Let the function f given by* (1.3) *be in the class $f \in \mathcal{AS}^*{}_{\Sigma_m}(\gamma,\lambda;\beta)$. Then*

$$|a_{m+1}| \leq min\left\{\frac{2(1-\beta)}{m\gamma(1+\lambda m)}, \frac{2}{m}\sqrt{\frac{(1-\beta)}{\gamma[2(1+\lambda m)+(\gamma-1)(1+\lambda m)^2]}}\right\}$$

and

$$|a_{2m+1}| \leq min\left\{\frac{1-\beta}{m\gamma(1+2\lambda m)} + \frac{2(m+1)(1-\beta)^2}{\gamma^2 m^2(1+\lambda m)^2},\right.$$

$$\left|\frac{2m(1+\lambda m)+2(m+1)(1+2\lambda m)+m(\gamma-1)(1+\lambda m)^2}{2m^2\gamma(1+2\lambda m)[2(1+\lambda m)+(\gamma-1)(1+\lambda m)^2]}\right|(1-\beta) +$$

$$\left.\left|\frac{2(m+1)(1+2\lambda m)-2m(1+\lambda m)-m(\gamma-1)(1+\lambda m)^2}{2m^2\gamma(1+2\lambda m)[2(1+\lambda m)+(\gamma-1)(1+\lambda m)^2]}\right|(1-\beta)\right\}.$$

Remark 3.13. Clearly, Corollary 3.12 provides an improvement over Theorem1.4.

By setting $\gamma = 1$ in Corollary 3.12, we conclude the following result.

Corollary 3.14. *Let the function f given by (1.3) be in the subclass $\mathcal{M}_{\Sigma_m}(\alpha, \lambda, 1)$. Then*

$$|a_{m+1}| \leq min\left\{\frac{2(1-\beta)}{m(1+\lambda m)}, \frac{2}{m}\sqrt{\frac{(1-\beta)}{2(1+\lambda m)}}\right\}$$

and

$$|a_{2m+1}| \leq min\left\{\frac{1-\beta}{m(1+2\lambda m)} + \frac{2(m+1)(1-\beta)^2}{m^2(1+\lambda m)^2}, \frac{m+1}{m^2(1+\lambda m)}(1-\beta)\right\}.$$

By setting $\lambda = 0$ in Corollary 3.14, we conclude the following result.

Corollary 3.15. *Let the function f given by (1.3) be in the subclass $S^{\beta}_{\Sigma_m}$. Then*

$$|a_{m+1}| \leq min\left\{\frac{2(1-\beta)}{m}, \frac{\sqrt{2(1-\beta)}}{m}\right\} = \begin{cases} \frac{\sqrt{2(1-\beta)}}{m}, & 0 \leq \beta \leq \frac{1}{2} \\ \frac{2(1-\beta)}{m}, & \frac{1}{2} \leq \beta < 1 \end{cases}$$

and

$$|a_{2m+1}| \leq min\left\{\frac{m(1-\beta)+2(m+1)(1-\beta)^2}{m^2}, \frac{m+(1-\beta)}{m^2}\right\}$$

$$= \begin{cases} \frac{m+(1-\beta)}{m^2}, & 0 \leq \beta \leq \frac{1+2m}{2(1+m)} \\ \frac{m(1-\beta)+2(m+1)(1-\beta)^2}{m^2}, & \frac{1+2m}{2(1+m)} \leq \beta < 1. \end{cases}$$

Remark 3.16. Clearly,the bounds on $|a_{m+1}|$ and $|a_{2m+1}|$ given in Corollary 3.15 are better than those given by Altinkaya and Yalcin [1, Corolary 7].

By setting $\gamma = 1$ and $m = 1$ in Corollary 3.12, we conclude the following result.

Corollary 3.17. *Let the function f given by (1.1) be in the subclass $\mathcal{B}_{\Sigma}(\beta, \lambda)$. Then*

$$|a_2| \leq min\left\{\frac{2(1-\beta)}{1+\lambda}, \sqrt{\frac{2(1-\beta)}{1+\lambda}}\right\} = \begin{cases} \sqrt{\frac{2(1-\beta)}{1+\lambda}}, & 0 \leq \beta \leq \frac{1-\lambda}{2} \\ \frac{2(1-\beta)}{1+\lambda}, & \frac{1-\lambda}{2} \leq \beta < 1 \end{cases}$$

and

$$|a_3| \leq min\left\{\frac{1-\beta}{1+2\lambda} + \frac{4(1-\beta)^2}{(1+\lambda)^2}, \frac{2(1-\beta)}{1+\lambda}\right\}$$

$$= \begin{cases} \frac{2(1-\beta)}{1+\lambda}, & 0 \leq \beta \leq \frac{3+4\lambda-3\lambda^2}{4(1+2\lambda)} \\ \frac{1-\beta}{1+2\lambda} + \frac{4(1-\beta)^2}{(1+\lambda)^2}, & \frac{3+4\lambda-3\lambda^2}{4(1+2\lambda)} \leq \beta < 1. \end{cases}$$

.

Remark 3.18. The bounds on $|a_2|$ and $|a_3|$ given in Corollary 3.17 is better than that given by Li and Wang [9, Theorem 3.2].

By setting $\lambda = 0$ in Corollary 3.17, we conclude the following result.

Corollary 3.19. *Let the function f given by* (1.1) *be in the subclass $S^*_\Sigma(\beta)$. Then*

$$|a_2| \leq min\left\{2(1-\beta), \sqrt{2(1-\beta)}\right\} = \begin{cases} \sqrt{2(1-\beta)} \ , & 0 \leq \beta \leq \frac{1}{2} \\ 2(1-\beta) \ , & \frac{1}{2} \leq \beta < 1 \end{cases}$$

and

$$|a_3| \leq min\left\{(1-\beta)(5-4\beta), 2(1-\beta)\right\} = \begin{cases} 2(1-\beta) \ , \ 0 \leq \beta \leq \frac{3}{4} \\ (1-\beta)(5-4\beta) \ , \ \frac{3}{4} \leq \beta < 1. \end{cases}$$

Remark 3.20. The bounds on $|a_2|$ and $|a_3|$ given in Corollary 3.19 are better than those given by Brannan and Taha [3].

By setting $\lambda = 1$ in Corollary 3.17, we conclude the following result.

Corollary 3.21. *Let the function f given by* (1.1) *be in the subclass $B_\Sigma(\beta, 1)$. Then*

$$|a_2| \leq min\left\{1-\beta, \sqrt{1-\beta}\right\} = 1-\beta$$

and

$$|a_3| \leq min\left\{\frac{(1-\beta)+3(1-\beta)^2}{3}, 1-\beta\right\} = \begin{cases} 1-\beta \ , & 0 \leq \beta \leq \frac{1}{3} \\ \frac{(1-\beta)+3(1-\beta)^2}{3} \ , & \frac{1}{3} \leq \beta < 1. \end{cases}$$

Remark 3.22. The bounds on $|a_2|$ and $|a_3|$ given in Corollary 3.21 are better than those given by Li and Wang [9, Theorem 3.2] for $\lambda = 1$.

4. Conclusions

In this paper, we introduce a new subclass $\mathcal{G}^{h,p}_{\Sigma_m}(\gamma, \lambda)$ of analytic functions, characterized by m-fold symmetric as a foundational framework. It is worth noting that this subclass is a generalization of many well-known or new subclasses, mentioned in section 2. Moreover, by Theorem 2.4, we obtained sharp bounds of the coefficients for many well-known subclasses as consequences. That in certain cases our data has improved the results of others.

Declaration of authorship

All authors contributed equally to this work and share first authorship.

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