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Application of Riemann-Liouville fractional integral to fuzzy differential subordination of analytic univalent functions

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Abstract. This paper focuses on geometric function theory, a subfield of complex analysis that has been adapted for fuzzy set analysis. We construct new operator denoted by $\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}$, formed by applying Riemann-Liouville fractional integral to the linear combination of the Pascal and Catas operator. Using this operator, we describe a specific fuzzy class of analytic univalent functions, presented by $\mathcal{DN}Y^F(n,\eta,\sigma,b,v,\vartheta,\alpha,\varsigma)$ in the open unit disk. A number of novel findings that are applicable to this class are found by applying the concept of fuzzy differential subordination. Interesting corollaries are discovered using specific functions, and an example illustrates the practical usage of the results.

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1. Introduction

Lotfi A. Zadeh established the concept of fuzzy sets in 1965 [36], and it has seen remarkable development to become employed in numerous areas of science and technology nowadays. The constantly concerns of mathematicians about incorporating the concept of fuzzy sets into mathematical theories that were already well-established led to the combination of fuzzy sets theory and geometric function theory. The authors highlight Lotfi A. Zadeh's scholarly contributions in their 2017 review article [10] by going over the progress of the idea of a fuzzy set and its applications in numerous

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fields.

Differential subordination was first proposed by S.S. Miller and P.T. Mocanu in [17, 19]. These approaches made it easier to verify the conclusions that had previously been produced and inspired a great deal of new research using techniques specific to this theory. The book written by S.S. Miller and P.T. Mocanu[17] and released in 2000 contains the essential elements of the theory of differential subordination. It is effectively developed over subsequent decades by other authors [18, 12, 5, 6, 20]. There are a few instances of differential subordination in utilization [9, 33, 4].

The fuzzy differential subordination theory is based on the general theory of differential subordination and it evolves by incorporating the majority of the classical theory's concepts to provide novel outcomes. The notion of differential subordination was newly extended from fuzzy set theory to geometric function theory by authors G. I. Oros and Gh. Oros[22, 23, 24]. Numerous authors have further expanded it [32, 11, 25, 26, 14, 15, 21, 3, 13, 27], and they have produced findings using fuzzy differential subordination. The progress made possible by the incorporation of quantum calculus and elements of fractional calculus into geometric function theory.

Let $\mathscr{U} = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{H}(\mathscr{U})$ denote the class of analytic functions in $\mathscr{U}$. Denote

$$\mathcal{H}[c, n] = \{\mathfrak{t} : \mathfrak{t} \in \mathcal{H}(\mathscr{U}) \textit{ and } \mathfrak{t}(z) = c + c_n z^n + \cdots, z \in \mathscr{U}\},$$

$$\mathcal{A}_n = \{\mathfrak{t} : \mathfrak{t} \in \mathcal{H}(\mathscr{U}) \textit{ and } \mathfrak{t}(z) = z + c_{n+1} z^{n+1} + \cdots, z \in \mathscr{U}\} \text{ and } \mathcal{A}_1 = \mathcal{A}.$$

Definition 1.1. [23] *Consider, $\mathcal{X}$ be a non-empty set. An application $F : \mathcal{X} \to [0, 1]$ is called fuzzy subset. An alternate definition, more precise would be the following: A pair $(\mathcal{S}, F_{\mathcal{S}})$, where $F_{\mathcal{S}} : \mathcal{X} \to [0, 1]$ and $\mathcal{S} = \{x \in \mathcal{X} : 0 < F_{\mathcal{S}}(x) \leq 1\}$ is called fuzzy subset. The function $F_{\mathcal{S}}$ is called membership function of the fuzzy subset $(\mathcal{S}, F_{\mathcal{S}})$.*

Definition 1.2. [16] *Let $\mathcal{D}$ is a set in $\mathbb{C}$, $z_0 \in \mathcal{D}$ is a fixed point and let the functions $f, g \in \mathcal{H}(\mathcal{D})$. The function f is named a fuzzy subordinate to g and written as $f \prec_F g$ if*

1. $f(z_0) = g(z_0)$
2. $F_{f(\mathcal{D})} f(z) \leq F_{g(\mathcal{D})} g(z), z \in \mathcal{D}$.

Remark 1.3.
1. Let $\mathcal{D} \subset \mathbb{C}$, $z_0 \in \mathcal{D}$ be a fixed point, and the functions $f, g \in \mathcal{H}(\mathcal{D})$. If g is univalent function in $\mathcal{D}$ then $f \prec_F g$ if and only if $f(z_0) = g(z_0)$ and $f(\mathcal{D}) \subset g(\mathcal{D})$.
2. A function $F : \mathbb{C} \to [0, 1]$, can be defined as, for example $F(z) = \frac{|z|}{1+|z|}, F(z) = \frac{1}{1+|z|}, |\sin|z||, |\cos|z||$.
3. If $\mathcal{D} = \mathscr{U}$ then the conditions become $f(0) = g(0)$ and $f(\mathscr{U}) \subset g(\mathscr{U})$ which is same as the classical definition of subordination.

Definition 1.4. [35] *Let h be univalent in $\mathscr{U}$ and $\Psi : \mathbb{C}^3 \times \mathcal{U} \to \mathbb{C}$. If $\mathcal{P}$ is analytic in $\mathscr{U}$ and satisfies the fuzzy differential subordination*

$$F_{\Psi(\mathbb{C}^3 \times \mathscr{U})}(\Psi(\mathcal{P}(z), z\mathcal{P}'(z), z^2\mathcal{P}''(z); z)) \leq F_{h(\mathscr{U})} h(z) \tag{1.1}$$

$$i.e. \Psi(\mathcal{P}(z), z\mathcal{P}'(z), z^2\mathcal{P}''(z); z) \prec_F h(z), z \in \mathscr{U}$$

then $\mathcal{P}$ is called a fuzzy solution of the fuzzy differential subordination. The univalent function q is called a fuzzy dominant of the fuzzy solutions of the fuzzy differential subordination, or more simple a fuzzy dominant, if $\mathcal{P} \prec_F q$ for all $\mathcal{P}$ satisfying (1.1). A fuzzy dominant $\tilde{q}$ that satisfies $\tilde{q}(z) \prec_F q(z), z \in \mathscr{U}$ for all fuzzy dominant q of (1.1) is said to be the best fuzzy dominant of (1.1).

Definition 1.5. [14] *Let $f(\mathcal{D}) = supp(f(\mathcal{D}), F_{f(\mathcal{D})})$=$\{z \in \mathcal{D} : 0 < F_{f(\mathcal{D})} \leq 1\}$, where $F_{f(\mathcal{D})}$ is the membership function of the fuzzy subset $f(\mathcal{D})$ associated to the function f.*
The membership function of the fuzzy set $(\mu f)(\mathcal{D})$ associated to the function μf coincides with the membership function of the fuzzy set $f(\mathcal{D})$ associated to the function f, i.e. $F_{(\mu f)\mathcal{D}} = F_{f(\mathcal{D})}, z \in \mathcal{D}$.
The membership function of the fuzzy set $(g + h)(\mathcal{D})$ associated to the function $g + h$ coincide with the half sum of the membership functions of the fuzzy sets $g(\mathcal{D})$, respectively $h(\mathcal{D})$, associated to the function g, respectively h,
i.e. $F_{(g+h)(\mathcal{D})}((g + h)z) = \dfrac{F_{g(\mathcal{D})}g(z) + F_{h(\mathcal{D})}h(z)}{2}, z \in \mathcal{D}$.

Definition 1.6. [34] *Let $\mathfrak{t} \in \mathcal{A}$ then a Pascal operator $Y_\vartheta^\eta : \mathcal{A} \to \mathcal{A}$ is given by*

$$Y_\vartheta^\eta \mathfrak{t}(z) = z + \sum_{r=2}^{\infty} \binom{r+\eta-2}{\eta-1} \vartheta^{r-1} c_r z^r;$$

$$(z \in \mathscr{U},\ \eta \geq 1, 0 \leq \vartheta < 1).$$

Definition 1.7. [7] *For $\mathfrak{t} \in \mathcal{A}$, Catas defined the operator as follow:*

$$\mathcal{N}_{b,v}^n \mathfrak{t}(z) = z + \sum_{r=2}^{\infty} \left\{ \frac{1+v+b(r-1)}{1+v} \right\}^n c_r z^r;$$

$$(n \in \mathbb{N}_0,\ z \in \mathscr{U},\ b, v \geq 0).$$

Now we define the linear operator $\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma} : \mathcal{A} \to \mathcal{A}$ as

$$\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma} \mathfrak{t}(z) = (1-\sigma)\mathcal{N}_{b,v}^n \mathfrak{t}(z) + \sigma Y_\vartheta^\eta \mathfrak{t}(z).$$

In series form, it is able to shown as

$$\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma} \mathfrak{t}(z) = z + \sum_{r=2}^{\infty} \Xi_r(n,\eta,\sigma,b,v,\vartheta) c_r z^r,$$

with $\Xi_r(n,\eta,\sigma,b,v,\vartheta) = \left[(1-\sigma) \left\{ \dfrac{1+v+b(r-1)}{1+v} \right\}^n + \sigma \dbinom{r+\eta-2}{\eta-1} \vartheta^{r-1} \right]$

$$(z \in \mathscr{U},\ \eta \geq 1, 0 \leq \vartheta < 1, n \in \mathbb{N}_0,\ b, v, \sigma \geq 0).$$

Definition 1.8. [8]*(see also* [1, 2]*) Given an analytical function $\mathfrak{t}$, the Riemann-Liouville fractional integral of order α is*

$$\mathfrak{D}_z^{-\alpha} \mathfrak{t}(z) = \frac{1}{\Gamma(\alpha)} \int_0^z \frac{\mathfrak{t}(t)}{(z-t)^{1-\alpha}} dt, \qquad \alpha > 0.$$

where $\Gamma(\alpha) = \int_0^\infty e^{-t}t^{\alpha-1}dt$ *with* $\Gamma(1) = 1, \Gamma(\alpha+1) = \alpha\Gamma(\alpha)$, *and* $\mathfrak{t}$ *is an analytic function in a simply connected region of the z-plane containing the origin and the multiplicity of* $(z-t)^{1-\alpha}$ *is removed by requiring* $\log(z-t)$ *to be real when* $z - t > 0$.

Applying the Riemann-Liouville fractional integral of order α to the linear operator $\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}$ yields the following:

$$\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z) = \frac{1}{\Gamma(\alpha)}\int_0^z \frac{\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(t)}{(z-t)^{1-\alpha}}dt.$$

After simple calculation which yields the series form

$$\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z) = \frac{z^{(1+\alpha)}}{\Gamma(2+\alpha)} + \sum_{r=2}^{\infty}\Xi_r(n,\eta,\sigma,b,v,\vartheta)\frac{\Gamma(r+1)}{\Gamma(r+\alpha+1)}c_r z^{(r+\alpha)}.$$

This study focuses on recent work in fuzzy differential subordination that introduces new operators to construct and study a novel fuzzy class. Here, we discuss multiple findings related to fuzzy differential subordination connected to the Riemann-Liouville fractional integral from the linear combination of Pascal operator and Catas operator. Fuzzy differential subordinations have been obtained in order to identify the fuzzy best dominants. Specific functions are used to derive some corollaries of the primary findings. A few examples are provided to illustrate the main findings.

Previous studies [28, 29, 30, 31] served as inspiration for this work.

To support our primary findings, we shall use the following Lemmas.

Lemma 1.9. [17] *Let* $k \in \mathcal{A}$. *If* $\Re\{1 + \frac{zk''(z)}{k'(z)}\} > \frac{-1}{2}, z \in \mathscr{U}$, *then* $\frac{1}{z}\int_0^z k(t)dt$ *is convex function.*

Lemma 1.10. [23] *Let* h *be a convex function with* $h(0) = a$ *and* $\rho \in \mathbb{C}^*$ *such that* $\Re(\rho) \geq 0$. *If* $\mathcal{P} \in \mathcal{H}[a,n]$ *with* $\mathcal{P}(0) = a$ *and* $\Psi : \mathbb{C}^2 \times \mathscr{U} \to \mathbb{C}$, $\Psi(\mathcal{P}(z), z\mathcal{P}'(z))$ *is analytic in* $\mathscr{U}$, *then*

$$F_{\Psi(\mathbb{C}^2\times\mathscr{U})}\left[\mathcal{P}(z) + \frac{1}{\rho}z\mathcal{P}'(z)\right] \leq F_{h(\mathscr{U})}h(z),$$

implies

$$F_{\mathcal{P}(\mathscr{U})}\mathcal{P}(z) \leq F_{g(\mathscr{U})}g(z) \leq F_{h(\mathscr{U})}h(z)$$

with the convex function $g(z) = \frac{\rho}{nz^{\frac{\rho}{n}}}\int_0^z h(t)t^{\frac{\rho}{n}-1}dt, z \in \mathscr{U}$ *as the fuzzy best dominant.*

Lemma 1.11. [23] *Suppose that* g *be a convex function in* $\mathscr{U}$ *and* $h(z) = g(z) + n\lambda zg'(z), n \in \mathbb{N}, \lambda > 0$. *If* $\mathcal{P} \in \mathcal{H}[g(0),n]$ *and* $\Psi : \mathbb{C}^2 \times \mathscr{U} \to \mathbb{C}$, $\Psi(\mathcal{P}(z), z\mathcal{P}'(z)) = \mathcal{P}(z) + \lambda z\mathcal{P}'(z)$ *is analytic in* $\mathscr{U}$, *then*

$$F_{\Psi(\mathbb{C}^2\times\mathscr{U})}\left[\mathcal{P}(z) + \lambda z\mathcal{P}'(z)\right] \leq F_{h(\mathscr{U})}h(z),$$

implies sharp result,

$$F_{\mathcal{P}(\mathscr{U})}\mathcal{P}(z) \leq F_{g(\mathscr{U})}g(z), \qquad z \in \mathscr{U}$$

and g *is fuzzy best dominant.*

We are going to define a new fuzzy class of univalent and analytic functions.

Definition 1.12. *If* $\mathfrak{t} \in \mathcal{A}$ *satisfies the following criteria, it is said to be in the class* $\mathcal{DNY}^F(n, \eta, \sigma, b, v, \vartheta, \alpha, \varsigma)$

$$F_{(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t})'(\mathscr{U})}\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)}{z^\alpha}\right)' > \varsigma;$$

$$(z \in \mathscr{U},\ \eta \geq 1, 0 \leq \vartheta < 1, n \in \mathbb{N}_0,\ b, v, \sigma \geq 0, \alpha > 0, \varsigma \in [0,1)).$$

Remark 1.13. In particular, $t(z) = z \in \mathcal{A}$ with $F(z) = \frac{1}{1+|z|}$ belongs to the class $\mathcal{DNY}^F(n, \eta, \sigma, b, v, \vartheta, \alpha, 0)$.

2. Main results

Theorem 2.1. *The class* $\mathcal{DNY}^F(n, \eta, \sigma, b, v, \vartheta, \alpha, \varsigma)$ *is a convex set.*

Proof. Consider

$$\mathfrak{t}_j(z) = z + \sum_{r=2}^{\infty} c_{jr} z^r, \qquad j = 1, 2,$$

belongs to the class $\mathcal{DNY}^F(n, \eta, \sigma, b, v, \vartheta, \alpha, \varsigma)$. We have to show that the function $h(z) = \beta_1\mathfrak{t}_1(z) + \beta_2\mathfrak{t}_2(z)$, $\beta_1, \beta_2 \geq 0, \beta_1 + \beta_2 = 1$, belongs to the class $\mathcal{DNY}^F(n, \eta, \sigma, b, v, \vartheta, \alpha, \varsigma)$.

Now, $h'(z) = \beta_1\mathfrak{t}_1'(z) + \beta_2\mathfrak{t}_2'(z)$ and $\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}h(z)}{z^\alpha}\right)'$
$= \beta_1\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}_1(z)}{z^\alpha}\right)' + \beta_2\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}_2(z)}{z^\alpha}\right)'.$

We have

$$\begin{aligned}
&F_{(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}h)'\mathscr{U}}\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}h(z)}{z^\alpha}\right)' \\
&= F_{(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}(\beta_1\mathfrak{t}_1(z)+\beta_2\mathfrak{t}_2))'\mathscr{U}}\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}(\beta_1\mathfrak{t}_1(z)+\beta_2\mathfrak{t}_2(z))}{z^\alpha}\right)' \\
&= F_{(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}(\beta_1\mathfrak{t}_1+\beta_2\mathfrak{t}_2))'\mathscr{U}}\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\beta_1\mathfrak{t}_1(z)}{z^\alpha}\right)' \\
&\quad + F_{(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}(\beta_1\mathfrak{t}_1+\beta_2\mathfrak{t}_2))'\mathscr{U}}\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\beta_2\mathfrak{t}_2(z)}{z^\alpha}\right)' \\
&= \frac{F_{(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}_1)'\mathscr{U}}\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}_1(z)}{z^\alpha}\right)' + F_{(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}_2)'\mathscr{U}}\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}_2(z)}{z^\alpha}\right)'}{2}.
\end{aligned}$$

As $\mathfrak{t}_1, \mathfrak{t}_2 \in \mathcal{DNY}^F(n, \eta, \sigma, b, v, \vartheta, \alpha, \varsigma)$, we have

$$\varsigma < F_{(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}_1)'\mathscr{U}}\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}_1(z)}{z^\alpha}\right)' \leq 1,$$

$$\varsigma < F_{(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}_2)'\mathscr{U}}\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}_2(z)}{z^\alpha}\right)' \leq 1.$$

This implies,

$$\varsigma < \frac{F_{(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}_1)'\mathscr{U}}\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}_1(z)}{z^\alpha}\right)' + F_{(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}_2)'\mathscr{U}}\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}_2(z)}{z^\alpha}\right)'}{2}$$
$$\leq 1$$

i.e.

$$\varsigma < F_{(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}h)'\mathscr{U}}\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}h(z)}{z^\alpha}\right)' \leq 1.$$

□

Theorem 2.2. *Considering g as a convex function and $h(z) = g(z) + \frac{1}{c+2}zg'(z), c > 0$. If $\mathfrak{t} \in \mathcal{DN}Y^F(n,\eta,\sigma,b,v,\vartheta,\alpha,\varsigma)$ and $G(z) = \frac{c+2}{z^{c+1}}\int_0^z t^c\mathfrak{t}(t)dt$, then*

$$F_{(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t})'\mathscr{U}}\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)}{z^\alpha}\right)' \leq F_{h(\mathscr{U})}h(z) \tag{2.1}$$

implies the next sharp result

$$F_{(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}G)'\mathscr{U}}\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}G(z)}{z^\alpha}\right)' \leq F_{g(\mathscr{U})}g(z),$$

and g is fuzzy best dominant.

Proof. Let $G(z) = \dfrac{c+2}{z^{c+1}}\displaystyle\int_0^z t^c\mathfrak{t}(t)dt.$
Differentiating w.r.t. z, we get

$$(c+1)G(z) + zG'(z) = (c+2)\mathfrak{t}(z),$$

$$(c+1)\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}G(z)}{z^\alpha}\right) + z\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}G(z)}{z^\alpha}\right)'$$
$$= (c+2)\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)}{z^\alpha}\right).$$

Again differentiating w.r.t z, we obtain

$$\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}G(z)}{z^\alpha}\right)' + \frac{1}{c+2}z\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}G(z)}{z^\alpha}\right)''$$
$$= \left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)}{z^\alpha}\right)'.$$

Now, the Inequality (2.1) becomes

$$F_{(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}G)'\mathscr{U}}\left[\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}G(z)}{z^\alpha}\right)'+\frac{1}{c+2}z\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}G(z)}{z^\alpha}\right)''\right] \leq F_{g(\mathscr{U})}\left[g(z)+\frac{1}{c+2}zg'(z)\right].$$

Consider, $p(z)=F_{(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}G)'\mathscr{U}}\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}G(z)}{z^\alpha}\right)'$.

Here, $p\in\mathcal{H}[1,1]$ and we obtain

$$F_{p(\mathscr{U})}\left[p(z)+\tfrac{1}{c+2}zp'(z)\right]\leq F_{g(\mathscr{U})}\left[g(z)+\tfrac{1}{c+2}zg'(z)\right].$$

Employing Lemma 1.3, we have

$$F_{(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}G)'\mathscr{U}}\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}G(z)}{z^\alpha}\right)'\leq F_{g(\mathscr{U})}g(z),$$

and g is fuzzy best dominant. □

Theorem 2.3. *Consider that* $h(z)=\frac{1+(2\varsigma-1)z}{1+z}$ *and* $G(z)=\frac{c+2}{z^{c+1}}\int_0^z t^c\mathfrak{t}(t)dt$, $\varsigma\in[0,1), c>0$ *then*

$$G[\mathcal{DN}Y^F(n,\eta,\sigma,b,v,\vartheta,\alpha,\varsigma)]\subset\mathcal{DN}Y^F(n,\eta,\sigma,b,v,\vartheta,\alpha,\varsigma^*),$$

where $\varsigma^*=(2\varsigma-1)+2(c+2)(1-\varsigma)\int_0^1\frac{t^{c+1}}{t+1}dt$.

Proof. Given that $h(z)=\frac{1+(2\varsigma-1)z}{1+z}$ is convex function and following the same steps from Theorem (2.2), we conclude the following fuzzy differential subordination

$$F_{p(\mathscr{U})}\left[p(z)+\tfrac{1}{c+2}zp'(z)\right]\leq F_{h(\mathscr{U})}h(z)$$

where, $p(z)=\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}G(z)}{z^\alpha}\right)'$.

From Lemma 1.10, we may conclude that

$$F_{(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}G)'\mathscr{U}}\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}G(z)}{z^\alpha}\right)'\leq F_{g(\mathscr{U})}g(z)\leq F_{h(\mathscr{U})}h(z),$$

where

$$g(z)=\frac{c+2}{z^{c+2}}\int_0^z t^{c+1}\left[\frac{1+(2\varsigma-1)t}{1+t}\right]dt=(2\varsigma-1)+\frac{(c+2)(2-2\varsigma)}{z^{c+2}}\int_0^z\frac{t^{c+1}}{t+1}dt.$$

Since, g is convex function and $g(\mathscr{U})$ is symmetric with respect to real axis, we obtain

$$F_{(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}G)'\mathscr{U}}\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}G(z)}{z^\alpha}\right)'\geq\min_{|z|=1}F_{g(\mathscr{U})}g(z)=F_{g(\mathscr{U})}g(1)$$

and $\varsigma^* = g(1) = 2\varsigma - 1 + (c+2)(2-2\varsigma)\displaystyle\int_0^1 \frac{t^{c+1}}{t+1}dt.$ □

Theorem 2.4. *Let's take g be a convex function such that $g(0) = 1$ and $h(z) = g(z) + zg'(z)$. If $\mathfrak{t} \in \mathcal{A}$, the fuzzy differential subordination is satisfied*

$$F_{(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t})'\mathscr{U}}\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)}{z^{\alpha}}\right)' \leq F_{h(\mathscr{U})}h(z) \tag{2.2}$$

implies the sharp result

$$F_{(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t})\mathscr{U}}\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)}{z^{1+\alpha}}\right) \leq F_{g(\mathscr{U})}g(z),$$

and g is fuzzy best dominant.

Proof. The function $p(z) = \left(\dfrac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)}{z^{1+\alpha}}\right)$ belongs to $\mathcal{H}[1,1]$. Furthermore, we may write

$$zp(z) = \left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)}{z^{\alpha}}\right).$$

Now, differentiating w.r.t. z, we have

$$p(z) + zp'(z) = \left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)}{z^{\alpha}}\right)'.$$

The Inequality (2.2), becomes

$$F_{p(\mathscr{U})}[p(z) + zp'(z)] \leq F_{h(\mathscr{U})}h(z).$$

Lemma 1.11 is applied, and we find that

$$F_{(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t})\mathscr{U}}\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)}{z^{1+\alpha}}\right) \leq F_{g(\mathscr{U})}g(z),$$

and g is fuzzy best dominant. □

Example 2.5. Take $g(z) = \frac{1-z}{1+z}$ ans is convex in $\mathscr{U}$, with $g(0) = 1$, $g'(z) = \dfrac{-2}{(1+z)^2}$.

Now $h(z) = g(z) + zg'(z) = \dfrac{1 - z^2 - 2z}{(1+z)^2}$.

Taking $n = 0, \sigma = 0, \eta = 1$ and $\mathfrak{t}(z) = z + z^2$, then we find that $\mathcal{N}Y_{b,v,\vartheta}^{0,1,0}\mathfrak{t}(z) = z + z^2$.

$$\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{0,1,0}\mathfrak{t}(z) = \frac{1}{\Gamma(\alpha)}\int_0^z \frac{\mathcal{N}Y_{b,v,\vartheta}^{0,1,0}\mathfrak{t}(z)}{(z-t)^{1-\alpha}}dt = \frac{1}{\Gamma(\alpha)}\int_0^z \frac{z+z^2}{(z-t)^{1-\theta}}dt$$
$$= \frac{z^{1+\alpha}}{\Gamma(2+\alpha)} + \frac{2z^{2+\alpha}}{\Gamma(3+\alpha)}.$$

Implies,

$$\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{0,1,0}\mathfrak{t}(z)}{z^{\alpha}} = z + \frac{2z^2}{(2+\alpha)}.$$

After differentiation, we get

$$\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{0,1,0}\mathfrak{t}(z)}{z^{\alpha}}\right)' = 1 + \frac{4z}{(2+\alpha)}.$$

Using Theorem 2.4 now, we can derive that the fuzzy subordination that follows

$$1 + \frac{4z}{2+\alpha} \prec_F \frac{1-z^2-2z}{(1+z)^2}$$

implies that

$$1 + \frac{2z}{2+\alpha} \prec_F \frac{1-z}{1+z}.$$

Theorem 2.6. *Let h be a analytic in $\mathscr{U}$ with $h(0) = 1$ and $\Re\left(1 + \frac{zh''(z)}{h'(z)}\right) > \frac{-1}{2}$. If $\mathfrak{t} \in \mathcal{A}$, the fuzzy differential subordination*

$$F_{(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t})'\mathscr{U}}\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)}{z^{\alpha}}\right)' \leq F_{h(\mathscr{U})}h(z) \tag{2.3}$$

implies that

$$F_{(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t})\mathscr{U}}\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)}{z^{1+\alpha}}\right) \leq F_{q(\mathscr{U})}q(z),$$

where $q(z) = \frac{1}{z}\int_0^z h(t)dt$ is convex and it is fuzzy best dominant.

Proof. Given that $\Re\left(1 + \frac{zh''(z)}{h'(z)}\right) > -\frac{1}{2}, z \in \mathscr{U}$, and from Lemma 1.9, we find that $q(z) = \frac{1}{z}\int_0^z h(t)dt$ is convex function and it is solution of Fuzzy differential subordination (2.3), $h(z) = q(z) + zq'(z)$, so it is fuzzy best dominant.

Let $zp(z) = \left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)}{z^{\theta}}\right)$

Differentiating w.r.t z, we get

$$p(z) + zp'(z) = \left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)}{z^{\alpha}}\right)'$$

The fuzzy differential subordination (2.3) is transformed into

$$F_{p(\mathscr{U})}[p(z) + zp'(z)] \leq F_{h(\mathscr{U})}h(z).$$

Using Lemma 1.11, we find that

$$F_{(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t})\mathscr{U}}\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)}{z^{1+\alpha}}\right) \leq F_{q(\mathscr{U})}q(z).$$

□

Corollary 2.7. *Assuming that* $h(z) = \frac{1+(2\xi-1)z}{1+z}, \xi \in [0,1)$ *is convex function in* $\mathscr{U}$. *If* $\mathfrak{t} \in \mathcal{A}$, *the following fuzzy differential subordination*

$$F_{(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t})'\mathscr{U}}\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)}{z^{\alpha}}\right)' \le F_{h(\mathscr{U})}h(z) \tag{2.4}$$

implies that

$$F_{(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t})\mathscr{U}}\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)}{z^{1+\alpha}}\right) \le F_{q(\mathscr{U})}q(z),$$

where $q(z) = (2\xi-1) + 2(1-\xi)\dfrac{\ln(1+z)}{z}$ *is convex and fuzzy best dominant.*

Proof. Given $h(z) = \frac{1+(2\xi-1)z}{1+z}$ with $h(0) = 1, h'(z) = \frac{2(\beta-1)}{(1+z)^2}, h''(z) = \frac{-4(\xi-1)}{(1+z)^3}$.
Consider
$\Re\left(1+\frac{zh''(z)}{h'(z)}\right) = \Re\left(\frac{1-z}{1+z}\right) = \Re\left(\frac{1-r\cos\phi - ir\sin\phi}{1+r\cos\phi+ir\sin\phi}\right) = \frac{1-r^2}{1+2r\cos\phi+r^2} > 0 > -\frac{1}{2}$.
Following the same steps from Theorem 2.6 with $p(z) = \left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)}{z^{1+\alpha}}\right)$, the Inequality (2.4) becomes

$$F_{p(\mathscr{U})}[p(z)+zp'(z)] \le F_{h(\mathscr{U})}h(z).$$

Employing Lemma 1.10 with $n = \rho = 1$, we deduce that

$$F_{(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t})\mathscr{U}}\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)}{z^{1+\alpha}}\right) \le F_{q(\mathscr{U})}q(z),$$

where $q(z) = \dfrac{1}{z}\displaystyle\int_0^z \frac{1+(2\xi-1)t}{1+t}dt = (2\xi-1)+2(1-\xi)\frac{\ln(1+z)}{z}$. □

Example 2.8. Consider $h(z) = \dfrac{1-z}{1+z}$ is convex in $\mathscr{U}$.
Taking $n = 0, \sigma = 0, \eta = 1$ and $\mathfrak{t}(z) = z + z^2$, then we find that $\mathcal{N}Y_{b,v,\vartheta}^{0,1,0}\mathfrak{t}(z) = z + z^2$.

$$\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{0,1,0}\mathfrak{t}(z) = \frac{1}{\Gamma(\alpha)}\int_0^z \frac{\mathcal{N}Y_{b,v,\vartheta}^{0,1,0}\mathfrak{t}(z)}{(z-t)^{1-\alpha}}dt = \frac{1}{\Gamma(\alpha)}\int_0^z \frac{z+z^2}{(z-t)^{1-\theta}}dt$$
$$= \frac{z^{1+\alpha}}{\Gamma(2+\alpha)} + \frac{2z^{2+\alpha}}{\Gamma(3+\alpha)}.$$

Hence,

$$\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{0,1,0}\mathfrak{t}(z)}{z^{\alpha}} = z + \frac{2z^2}{(2+\alpha)}.$$

After differentiation, we get

$$\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{0,1,0}\mathfrak{t}(z)}{z^{\alpha}}\right)' = 1 + \frac{4z}{(2+\alpha)}.$$

Also, $q(z) = \frac{1}{z}\int_0^z \frac{1-t}{1+t}dt = \frac{2\ln(1+z)}{z} - 1.$
Utilizing Theorem 2.6, we now possess the fuzzy differential subordination

$$1 + \frac{4z}{2+\alpha} \prec_F \frac{1-z}{1+z}$$

implies the result

$$1 + \frac{2z}{2+\alpha} \prec_F \frac{2\ln(1+z)}{z} - 1.$$

Theorem 2.9. *Letting g be a convex function and consider that $g(0) = 1$. If $\mathfrak{t} \in \mathcal{A}$, the fuzzy differential subordination*

$$F_{(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t})'\mathscr{U}}\left[\frac{z\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n+1,\eta,\sigma}\mathfrak{t}(z)}{\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)}\right]' \leq F_{h(\mathscr{U})}h(z) \tag{2.5}$$

implies the sharp result

$$F_{(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t})\mathscr{U}}\left[\frac{\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n+1,\eta,\sigma}\mathfrak{t}(z)}{\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)}\right] \leq F_{g(\mathscr{U})}g(z),$$

and g is fuzzy best dominant.

Proof. Suppose $p(z) = \left[\frac{\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n+1,\eta,\sigma}\mathfrak{t}(z)}{\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)}\right]$.
Differentiating w.r.t. z, we have the relation

$$p(z) + zp'(z) = \left[\frac{z\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n+1,\eta,\sigma}\mathfrak{t}(z)}{\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)}\right]'.$$

Consequently, fuzzy differential subordination (2.5) turns into

$$F_{p(\mathscr{U})}[p(z) + zp'(z)] \leq F_{h(\mathscr{U})}h(z) = F_{g(\mathscr{U})}[g(z) + zg'(z)].$$

Now, applying Lemma 1.11, we have

$$F_{(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t})\mathscr{U}}\left[\frac{\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n+1,\eta,\sigma}\mathfrak{t}(z)}{\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)}\right] \leq F_{g(\mathscr{U})}g(z),$$

and g is fuzzy best dominant. □

Theorem 2.10. *Let g be a convex function and consider that $g(0) = 1$ and $h(z) = g(z) + \gamma zg'(z), \gamma, \lambda > 0$. If $\mathfrak{t} \in \mathcal{A}$ and the fuzzy differential subordination*

$$F_{(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t})\mathscr{U}}\left[\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)}{z^{1+\alpha}}\right)^{\lambda-1}\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)}{z^{\alpha}}\right)'\right] \leq F_{h(\mathscr{U})}h(z) \tag{2.6}$$

implies the following sharp result

$$F_{(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t})\mathscr{U}}\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)}{z^{1+\alpha}}\right)^{\lambda} \leq F_{g(\mathscr{U})}g(z),$$

and g is fuzzy best dominant.

Proof. Let $p(z) = \left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)}{z^{1+\alpha}}\right)^{\lambda}$ belongs to $\mathcal{H}[1,1]$.
Differentiating w.r.t. z, we obtain

$$p'(z) = \lambda\left[\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)}{z^{1+\alpha}}\right]^{\lambda-1}\left[\frac{\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)}{z^{\alpha}}\right)}{z}\right]'.$$

Following a little computation, we have

$$p(z)+\frac{1}{\lambda}zp'(z) = \left[\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)}{z^{1+\alpha}}\right)^{\lambda-1}\left(\frac{\Gamma(2+\theta)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)}{z^{\alpha}}\right)'\right].$$

Therefore, fuzzy differential subordination (2.6), becomes

$$F_{p(\mathscr{U})}[p(z)+\frac{1}{\lambda}zp'(z)] \leq F_{h(\mathscr{U})}h(z) = F_{g(\mathscr{U})}[g(z)+\gamma zg'(z)].$$

Applying Lemma 1.11, we obtain that

$$F_{(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t})\mathscr{U}}\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)}{z^{1+\alpha}}\right)^{\lambda} \leq F_{g(\mathscr{U})}g(z),$$

and g is fuzzy best dominant. □

Example 2.11. Suppose $g(z) = \frac{1-z}{1+z}$ and $h(z) = g(z)+zg'(z) = \frac{1-2z-z^2}{(1+z)^2}$.
Take $n=0, \sigma=0, \eta=1$ and $\mathfrak{t}(z) = z+z^2$, then we find that $\mathcal{N}Y_{b,v,\vartheta}^{0,1,0}\mathfrak{t}(z) = z+z^2$.

$$\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{0,1,0}\mathfrak{t}(z) = \frac{1}{\Gamma(\alpha)}\int_0^z \frac{\mathcal{N}Y_{b,v,\vartheta}^{0,1,0}\mathfrak{t}(z)}{(z-t)^{1-\alpha}}dt = \frac{1}{\Gamma(\alpha)}\int_0^z \frac{z+z^2}{(z-t)^{1-\theta}}dt$$
$$= \frac{z^{1+\alpha}}{\Gamma(2+\alpha)} + \frac{2z^{2+\alpha}}{\Gamma(3+\alpha)}.$$

Thus, we have

$$\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{0,1,0}\mathfrak{t}(z)}{z^{\alpha}} = z + \frac{2z^2}{(2+\alpha)}.$$

After differentiation, we get

$$\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{0,1,0}\mathfrak{t}(z)}{z^{\alpha}}\right)' = 1 + \frac{4z}{(2+\alpha)}.$$

The following fuzzy differential subordination is obtained by using Theorem 2.10

$$\left(1+\frac{2z}{2+\alpha}\right)^{\lambda-1}\left(1+\frac{4z}{2+\alpha}\right)' \prec_F \frac{1-2z-z^2}{(1+z)^2}$$

implies that

$$\left(1+\frac{2z}{2+\alpha}\right)^{\lambda} \prec_F \frac{1-z}{1+z}.$$

Theorem 2.12. *Considering h as a convex function with $h(0)=1, \lambda>0$. If $\mathfrak{t}\in\mathcal{A}$, the fuzzy differential subordination*

$$F_{(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t})\mathscr{U}}\left[\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)}{z^{1+\alpha}}\right)^{\lambda-1}\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)}{z^{\alpha}}\right)'\right]$$
$$\leq F_{h(\mathscr{U})}h(z) \tag{2.7}$$

implies the result

$$F_{(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t})\mathscr{U}}\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)}{z^{1+\alpha}}\right)^{\lambda} \leq F_{g(\mathscr{U})}g(z),$$

where $g(z)=\frac{1}{z}\int_0^z h(t)dt$ is convex and fuzzy best dominant.

Proof. Following the same technique of Theorem 2.10 and taking
$p(z)=\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)}{z^{1+\alpha}}\right)^{\lambda}$, we have

$$F_{p(\mathscr{U})}\left[p(z)+\frac{1}{\lambda}zp'(z)\right] \leq F_{h(\mathscr{U})}h(z).$$

Using Lemma 1.10, we deduce that

$$F_{(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t})\mathscr{U}}\left(\frac{\Gamma(2+\alpha)\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)}{z^{1+\alpha}}\right)^{\lambda} \leq F_{g(\mathscr{U})}g(z),$$

where $g(z)=\frac{1}{z}\int_0^z h(t)dt$ is convex and fuzzy best dominant. □

Example 2.13. Considering $h(z)=\dfrac{1-z}{1+z}$ with $h(0)=1$ and it is convex function in $\mathscr{U}$.
Take $n=0, \sigma=0, \eta=1$ and $\mathfrak{t}(z)=z+z^2$, then we obtain $\mathcal{N}Y_{b,v,\vartheta}^{0,1,0}\mathfrak{t}(z)=z+z^2$, then we find that

$$\mathcal{N}Y_{b,v,\vartheta}^{0,1,0}\mathfrak{t}(z)=z+z^2.$$

Now,

$$\mathcal{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{0,1,0}\mathfrak{t}(z)=\frac{z^{1+\alpha}}{\Gamma(2+\alpha)}+\frac{2z^{2+\alpha}}{\Gamma(3+\alpha)}.$$

Thus,

$$\frac{\Gamma(2+\alpha)\mathcal{N}Y_{b,v,\vartheta}^{0,1,0}\mathfrak{t}(z)}{z^{\alpha}} = z + \frac{2z^2}{(2+\alpha)}.$$

After differentiation, we obtain

$$\left(\frac{\Gamma(2+\alpha)\mathcal{N}Y_{b,v,\vartheta}^{0,1,0}\mathfrak{t}(z)}{z^{\alpha}}\right)' = 1 + \frac{4z}{(2+\alpha)}.$$

Additionally,

$$g(z) = \frac{1}{z}\int_0^z h(t)dt = \frac{2\ln(1+z)}{z} - 1.$$

We have the fuzzy differential subordination described below using Theorem 2.12

$$\left(1+\frac{2z}{2+\alpha}\right)^{\lambda-1}\left(1+\frac{4z}{2+\alpha}\right) \prec_F \frac{1-z}{1+z}$$

implies the result

$$\left(1+\frac{2z}{2+\alpha}\right)^{\lambda} \prec_F \frac{2\ln(1+z)}{z} - 1.$$

Theorem 2.14. *Let g is a convex function with $g(0) = 1$ and $h(z) = g(z) + zg'(z)$. If $\mathfrak{t} \in \mathcal{A}$, the fuzzy differential subordination*

$$F_{\left(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}\right)\mathscr{U}}\left[1 - \frac{\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)\left(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)\right)''}{\left[\left(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)\right)'\right]^2}\right] \leq F_{h(\mathscr{U})}h(z), \quad (2.8)$$

implies sharp result,

$$F_{\left(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}\right)\mathscr{U}}\left[\frac{\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)}{z\left(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)\right)'}\right] \leq F_{g(\mathscr{U})}g(z),$$

and g is fuzzy best dominant.

Proof. Suppose $p(z) = \dfrac{\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)}{z\left(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)\right)'}$ belongs to $\mathcal{H}[1,1]$ and $z \in \mathscr{U}$.
Differentiating w.r.t. z, we have the relation

$$p(z) + zp'(z) = 1 - \frac{\left(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)\right)\left(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)\right)''}{\left[\left(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)\right)'\right]^2}.$$

Inequality (2.8), becomes

$$F_{p(\mathscr{U})}[p(z) + zp'(z)] \leq F_{h(\mathscr{U})}h(z),$$

We now get the sharp fuzzy differential subordination using Lemma 1.11,

$$F_{\left(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}\right)\mathscr{U}}\left[\frac{\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)}{z\left(\mathfrak{D}_z^{-\alpha}\mathcal{N}Y_{b,v,\vartheta}^{n,\eta,\sigma}\mathfrak{t}(z)\right)'}\right] \leq F_{g(\mathscr{U})}g(z),$$

and g is fuzzy best dominant. □

3. Conclusion

At this point, we discussed a number of fuzzy differential subordination results of analytic functions that are connected to the Riemann-Liouville fractional integral and the linear combination of the Pascal and Catas operator. A new fuzzy class was also developed, and fuzzy differential subordination results and a few examples were inferred.

With reference to this operator, further subclasses of analytic functions can be created, and some of their features, including coefficient estimates, distortion theorems, and closure theorems, can be examined. Also, New fuzzy class identification, fuzzy superordination results, higher-dimensional results extension, and the use of fuzzy differential subordination to address practical issues are important topics for development.

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References

[1] Alb Lupaş, A., *Applications of the Fractional Calculus in Fuzzy Differential Subordinations and Superordinations*, Mathematics, **9**(2021), no. 20, 2601-2601.

[2] Alb Lupaş, A., *On Special Fuzzy Differential Subordinations Obtained for Riemann–Liouville Fractional Integral of Ruscheweyh and Sălăgean Operators*, Axioms, **11**(2022), no. 9, 428-428.

[3] Ali, E. E., Oros, G. I., El-Ashwah, R. M., Albalahi, A. M., *Applications of fuzzy differential subordination theory on analytic p-valent functions connected with $\mathfrak{q}$-calculus operator*, AIMS Mathematics, **9**(2024), no. 8, 21239-21254.

[4] Analouei Adegani, E., Bulboaca, T., Motamednezhad, A., *Simple Sufficient Subordination Conditions for Close-to-Convexity*, Mathematics, **7**(2019), 3, 241-241.

[5] Bulboaca, T., *Classes of first-order differential superordinations*, Demonstratio Mathematica, **35**(2002), no. 2, 287-292.

[6] Bulboaca, T., *A class of superordination-preserving integral operators*, Indagationes Mathematicae, **13**(2002), no. 3, 301-311.

[7] Catas A., *On certain classes of p-valent functions defined by multiplier trasformations*, In.Proc.Book of international symposium on geometric function theory and applications, Istanbul, Turkey, 2007, 241-250.

[8] Cho, N. E., Aouf, M. K., *Some applications of fractional calculus operators to a certain subclass of analytic functions with negative coefficients*, Turkish Journal of Mathematics, **20**(1996), no. 4, 553-562.

[9] Cho, N. E., Srivastava, H. M., Analouei Adegani, E., Motamednezhad, A., *Criteria for a certain class of the Carathéodory functions and their applications*, J. Inequal. Appl., **85**(2020), 1-14.

[10] Dzitac, I., Filip, F. G., Manolescu, M. J., *Fuzzy logic is not fuzzy: world-renowned computer scientist Lotfi A. Zadeh*, International journal of computers communication & control, **12**(2017), no. 6, 748–789.

[11] El-Deeb, S.M., Oros, G.I., *Fuzzy differential subordinations connected with the linear operator*, Mathematica Bohemica, **146**(2021), no. 4, 397-406.

[12] Ibrahim, R. W., Darus, M., *On a univalent class involving differential subordination with applications*, J. Math. Statistics, **7**(2011), no. 2, 137-143.

[13] Khan, S., Ro, J. S., Tchier, F., Khan, N., *Applications of fuzzy differential subordination for a new subclass of analytic functions*, Axioms, **12**(2023), no. 8, 745-745.

[14] Lupas, A. A., Oros, G. I., *New applications of Salagean and Ruscheweyh operators for obtaining fuzzy differential subordinations*, Mathematics, **9**(2021), no. 16, 2000-2000.

[15] Lupas, A. A., Oros, G., *On special fuzzy differential subordinations using Sălăgean and Ruscheweyh operators*, Applied Mathematics and Computation, **261**(2015), 119-127.

[16] Lupas, A. A., *A note on special fuzzy differential subordinations using generalized Salagean operator and Ruscheweyh derivative*, J. Comput. Anal. Appl, **15**(2013), 1476-1483.

[17] Miller, S. S., Mocanu, P. T., *Differential subordinations: theory and applications*, Marcel Dekkar, Inc. New York, USA, Basel, Switzerland, 2000.

[18] Miller, S. S., Mocanu, P. T., *Differential subordinations and univalent functions*, Michigan Mathematical Journal, **28**(1981), no. 2, 157-172.

[19] Miller, S. S., Mocanu, P. T., *Second order differential inequalities in the complex plane*, Journal of Mathematical Analysis and Applications, **65**(1978), no. 2 , 289-305.

[20] Miller, S. S., Mocanu, P. T., *Subordinants of differential superordinations*, Complex variables, **48**(2003), no. 10, 815-826.

[21] Noor, K. I., *Fuzzy differential subordination involving generalized Noor-Salagean operator*, Inf. Sci. Lett, **11**(2022), 1-7.

[22] Oros, G.I., Oros, Gh., *Fuzzy differential subordination*, Acta Univ. Apulensis, **3**, 2012, 55-64.

[23] Oros, G.I., Oros, Gh., *Dominants and best dominants in fuzzy differential subordinations*, Stud. Univ. Babes-Bolyai Math, **57**(2012), no. 2, 239-248.

[24] Oros, G.I., Oros, Gh., *The notion of subordination in fuzzy sets theory*, General Mathematics, **19**(2011), no. 4, 97-103.

[25] Oros, G.I., Dzitac, S., *Applications of subordination chains and fractional integral in fuzzy differential subordinations*, Mathematics, **10**(2022), no. 10, 1690-1690.

[26] Oros, G.I., *Fuzzy differential subordinations obtained using a hypergeometric integral operator*, Mathematics, **9**(2021), no. 20, 2539-2539.

[27] Oros, G. I., Oros, G., *Briot-Bouquet fuzzy differential subordination*, An. Univ. Oradea Fasc. Mat, **19**(2012), 83-87.

[28] Shelake, G. D., Nilapgol S. K., *Estimates on initial coefficient bounds of subclasses of bi-univalent functions defined by generalized differential operator*, South East Asian Journal of Mathematics & Mathematical Sciences, **20**(2024), no. 1, 59-70.

[29] Shelake, G. D., Nilapgol, S. K., *Initial coefficient Estimates for subclasses of Bi-univalent Functions Involving q-Derivative Operator*, Ganita, **74**(2024), no, 1, 151-160.

[30] Shelake G. D., Joshi S. B., *On subclass of uniformly convex functions involving certain fractional calculus operator*, Indian Journal of Mathematics, **58**(2016), no. 1, 43-57.

[31] Shelake, G. D., Nilapgol, S. K., Joshi S. B., *Bi-concave and Ozaki-type bi-close-to-convex functions associated with Miller Ross type Poisson distribution*, Journal of Indian academy of Mathematics, **47**(2025), no. 1, 113-131.

[32] Srivastava, H. M., El-Deeb, S. M., *Fuzzy differential subordinations based upon the Mittag-Leffler type Borel distribution*, Symmetry, **13**(2021), no. 6, 1023-1023.

[33] Srivastava, H. M., Motamednezhad, A., Adegani, E.A., *Faber Polynomial Coefficient Estimates for Bi-Univalent Functions Defined by Using Differential Subordination and a Certain Fractional Derivative Operator*, Mathematics, **8**(2020), no. 2, 172-172.

[34] Thirupathi, G., *Coefficient estimates for subclsses of bi-univalent functions with Pascal operator*, **15**(2024), no. 1, 1-9.

[35] Wanas, A. K., Bulut, S., *Some results for fractional derivative associated with fuzzy differential subordinations*, Journal of Al-Qadisiyah for computer science and mathematics, **12**(2020), no. 3, 27-36.

[36] Zadeh, L. A., *Fuzzy Sets*, Inf. Control, **8**, 1965, 338–353.

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