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Existence, uniqueness and continuous dependence results of coupled system of Hilfer fractional stochastic pantograph equations with nonlocal integral conditions

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Abstract. This study explores the existence, uniqueness, and continuous dependence of solutions for coupled system of Hilfer fractional stochastic pantograph equations with nonlocal integral conditions. The existence of solutions is demonstrated using topological degree theory for condensing maps. The uniqueness is established via Banach's contraction principle. To address continuous dependence, the generalized Gronwall inequality is applied. Additionally, a numerical example is provided to illustrate and confirm the theoretical findings.

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1. Introduction

Fractional derivatives provide a flexible tool for modeling intricate processes in various fields by extending classical differentiation to non-integer orders. Several definitions of fractional derivatives exist, including the Riemann-Liouville (R-L) and Caputo derivatives. The R-L derivative offers a foundational approach to fractional differentiation [14], while the Caputo derivative is often used in practical applications due to its compatibility with standard initial conditions [14]. To unify and extend these approaches,

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Hilfer introduced a generalized fractional differential operator that combines the Caputo and R-L derivatives. This operator, called the fractional Hilfer derivative (HFD), has shown great promise in modeling systems with complex boundary conditions and temporal delays [3, 13, 20, 22, 30].

Fractional differential equations represent a substantial advancement in mathematical modeling, particularly in fields such as signal processing, biology, and engineering. By incorporating non-integer order derivatives, these equations capture complex system dynamics. A significant category within this domain is fractional pantograph delay differential equations, which integrate delays to model systems with memory effects. When combined with stochastic calculus, these equations evolve into stochastic fractional pantograph differential equations, which are valuable for describing systems influenced by both memory effects and random fluctuations [5, 7, 8, 21, 23, 24, 25, 28, 29, 31, 32, 34, 35].

The continuous dependence of stochastic fractional differential equations is crucial for ensuring that small changes in initial or nonlocal conditions lead to proportionally small variations in the solutions. Research has shown that mild solutions of mean-field stochastic functional differential equations exhibit sensitivity to initial data and coefficients within an appropriate topological framework [4, 27, 36, 37, 38]. Similarly, generalized Cauchy-type problems involving HFD demonstrate continuous dependence on the fractional order, supported by a generalization of Gronwall's inequality [1, 9, 11, 27, 33]. Solutions to random fractional-order differential equations with nonlocal conditions also maintain continuous dependence on initial conditions [15].

Coupled system with nonlocal conditions are particularly useful for modeling physical, chemical, or other processes that occur at multiple points within a domain rather than being restricted to boundary conditions. El-Sayed [16] explored the continuous dependence of solutions for stochastic differential equations with nonlocal conditions, while more recently, Arioui [6] studied the existence of coupled systems of fractional stochastic differential equations involving HFD. For more study about coupled systems, we refer to [2, 10, 17, 18, 19, 26, 39, 40]

To the best of our knowledge, no existing study has addressed the existence and continuous dependence of solutions for coupled systems of Hilfer fractional stochastic pantograph equations with nonlocal conditions. This paper aims to fill this gap by introducing a novel class of coupled system of Hilfer fractional stochastic pantograph equations with nonlocal integral conditions

$$\begin{cases} {}^{H}\mathfrak{D}^{p_1,q_1}_{0^+,\iota}\varrho(\iota) = \varpi_1\left(\iota, \varrho(\iota), \varrho\left(\kappa\iota\right), \xi(\iota)\right), \quad \iota \in J := (0, b], \\ {}^{H}\mathfrak{D}^{p_2,q_2}_{0^+,\iota}\xi(\iota) = \varpi_2\left(\iota, \varrho(\iota), \xi\left(\iota\right), \xi(\kappa\iota)\right)\dfrac{d\mathcal{W}(\iota)}{d\iota}, \\ \mathfrak{I}^{1-\gamma_1}_{0^+,\iota}\varrho(0) = \displaystyle\int_0^{\iota} g_1\left(s, \varrho(s), \xi(s)\right)d\mathcal{W}(s), \quad \gamma_1 = p_1 + q_1 - p_1 q_1, \\ \mathfrak{I}^{1-\gamma_2}_{0^+,\iota}\xi(0) = \displaystyle\int_0^{\iota} g_2\left(s, \varrho(s), \xi(s)\right)ds, \quad \gamma_2 = p_2 + q_2 - p_2 q_2, \end{cases} \tag{1.1}$$

where $\mathfrak{I}^{1-\gamma_j}_{0+,\iota}$ and ${}^{H}\mathfrak{D}^{p_j,q_j}_{0+,\iota}$ are the fractional integral of order $1-\gamma_j$ and the HFD of order p_j and type q_j, respectively, $j=1,2$. Here, $\frac{1}{2}<p_j<1, 0<q_j\leq 1$. Let $(W(\iota))_{\iota\geq 0}$ be 1-dimensional standard Brownian motion defined in the complete probability space $(\Omega,\mathcal{F}_\iota,\mathbb{P})$ with a normal filteration $(\mathcal{F}_\iota)_{\iota\geq 0}$. $\varpi_j, g_j : J\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are measurable functions and $0<\kappa<1$.

2. Preliminaries

Let $\mathbb{L}^2(\Omega,\mathcal{F}_\iota,\mathbb{R})=\mathbb{L}^2(\Omega,\mathbb{R})$ is the Hilbert space of real-valued random variables that are square-integrable with respect to the probability measure on $(\Omega,\mathcal{F}_\iota)$. Let $C(J,\mathbb{L}^2(\Omega,\mathbb{R}))$ is the space of continuous time stochastic processes that are square-integrable with the norm $\|\varrho\|^2=\sup\left\{\mathbb{E}\|\varrho(\iota)\|^2:\iota\in J\right\}$, where $\mathbb{E}$ is the mathematical expectation. On the other hand, define the Banach space

$$\begin{aligned}\mathcal{E}_j &= C_{1-\gamma_j}(J,L^2(\Omega,\mathbb{R}))\\ &=\left\{\varrho : J\to L^2(\Omega,\mathbb{R}) : \iota^{1-\gamma_j}\varrho(\iota)\in C(J,L^2(\Omega,\mathbb{R}))\right\}, 0<\gamma_j\leq 1, j=1,2,\end{aligned}$$

using the norm

$$\|\varrho\|^2_{\mathcal{E}_j}=\sup_{\iota\in J}\mathbb{E}\left\|\iota^{1-\gamma_j}\varrho(\iota)\right\|^2.$$

Furthermore, let $\mathcal{E}:=\mathcal{E}_1\times\mathcal{E}_2$ with the norm $\|(\varrho,\xi)\|_{\mathcal{E}}=\max\{\|\varrho\|_{\mathcal{E}_1},\|\xi\|_{\mathcal{E}_2}\}$. It is clear that $\mathcal{E}$ forms a Banach space.

Definition 2.1. [14] *For $p>0$, the fractional R-L integral with order p for a continuous function $\varrho:[a,\infty)\to\mathbb{R}$ can be written as*

$$\mathfrak{I}^{p}_{a+,\iota}\varrho(\iota)=\frac{1}{\Gamma(p)}\int_a^\iota(\iota-s)^{p-1}\varrho(s)ds.$$

Definition 2.2. [14] *For $n-1<p\leq n$, the fractional R-L derivative with order p for a continuous function ϱ is represented as*

$$\mathfrak{D}^{p}_{a+,\iota}\varrho(\iota)=D^n\mathfrak{I}^{n-p}_{a+,\iota}\varrho(\iota)=\frac{1}{\Gamma(n-p)}\left(\frac{d}{d\iota}\right)^n\int_a^\iota(\iota-s)^{n-p-1}\varrho(s)\mathrm{d}s.$$

Definition 2.3. [20] *For $n-1<p\leq n$, the HFD with order p and type $0\leq\ q\leq 1$ of ϱ is represented as*

$${}^{H}\mathfrak{D}^{p,q}_{a+,\iota}\varrho(\iota)=\mathfrak{I}^{q(n-p)}_{a+,\iota}D^n\mathfrak{I}^{(1-q)(n-p)}_{a+,\iota}\varrho(\iota)=\mathfrak{I}^{q(n-p)}_{a+,\iota}\mathfrak{D}^{\theta}_{a+,\iota}\varrho(\iota),$$

where $D=\frac{d}{d\iota}$ and $\theta=p+q(n-p)$.

Lemma 2.4. [20] *For $n-1<p\leq n$, $f\in L^1(a,b)$, $0\leq\beta\leq 1$, and $\mathfrak{I}^{(1-q)(n-p)}_{a^+,\iota}\varrho\in AC^k[a,b]$, then*

$$\mathfrak{I}^{p}_{a+,\iota}\,{}^{H}\mathfrak{D}^{p,q}_{a+,\iota}\varrho(\iota)=\varrho(\iota)-\sum_{k=1}^{n}\frac{(\iota-a)^{\theta-k}}{\Gamma(\theta+1-k)}\cdot\lim_{\iota\longrightarrow+a}\frac{d^k}{\mathrm{d}\iota^k}\mathfrak{I}^{(1-q)(n-p)}_{a^+,\iota}\varrho(\iota).$$

Lemma 2.5. [20] *Let $p > 0$ and $q > 0$. Following that $\forall\iota \in J$ there is*

$$\left[\mathfrak{I}^{p}_{a^+,\iota}(\iota)^{q-1}\right](\iota) = \frac{\Gamma(q)}{\Gamma(q+p)}\iota^{q+p-1},$$

and

$$\left[\mathfrak{D}^{p}_{a^+,\iota}(\iota)^{p-1}\right](\iota) = 0, \quad 0 < p < 1.$$

Lemma 2.6. *A stochastic process $(\varrho, \xi) \in \mathcal{E}$ is called a solution of problem (1.1) if (ϱ, ξ) satisfies the following stochastic integral equation*

$$\begin{aligned}\varrho(\iota) =& \frac{\iota^{\gamma_1-1}}{\Gamma(\gamma_1)}\int_0^b g_1\left(s, \varrho(s), \xi(s)\right) d\mathcal{W}(s)\\ &+ \frac{1}{\Gamma(p_1)}\int_0^\iota (\iota - s)^{p_1-1}\varpi_1\left(s, \varrho(s), \varrho\left(\kappa s\right), \xi(s)\right) ds\end{aligned}$$

and

$$\begin{aligned}\xi(\iota) =& \frac{\iota^{\gamma_2-1}}{\Gamma(\gamma_2)}\int_0^b g_2\left(s, \varrho(s), \xi\left(s\right)\right) ds\\ &+ \frac{1}{\Gamma(p_2)}\int_0^\iota (\iota - s)^{p_2-1}\varpi_2\left(s, \varrho(s), \xi\left(s\right), \xi(\kappa s)\right) d\mathcal{W}(s).\end{aligned}$$

Definition 2.7. [12] *Let $\mathcal{A} : \mathcal{S} \longrightarrow \mathcal{E}$ be a bounded continuous map, where $\mathcal{S} \subseteq \mathcal{E}$. Then $\mathcal{A}$ is*

(i) *ϑ-Lipschitz if there exists $r \geq 0$ such that $\vartheta(\mathcal{A}(\mathcal{K})) \leq r\vartheta(\mathcal{K})$ for all bounded subsets $\mathcal{K} \subseteq \mathcal{S}$;*
(ii) *Strict ϑ-contraction if there exists $0 \leq r < 1$ such that $\vartheta(\mathcal{A}(\mathcal{K})) \leq r\vartheta(\mathcal{K})$;*
(iii) *ϑ-condensing if $\vartheta(\mathcal{A}(\mathcal{K})) < \vartheta(\mathcal{K})$ for all bounded subsets $\mathcal{K} \subseteq \mathcal{S}$ with $\vartheta(\mathcal{K}) > 0$,*

where ϑ is the Kuratowski measure of non-compactness.

Proposition 2.8. [21] *If $\mathcal{A}, \mathcal{B} : \mathcal{S} \longrightarrow \mathcal{E}$ are ϑ-Lipschitz with respective constants r_1 and r_2, then $\mathcal{A} + \mathcal{B}$ is ϑ-lipschitz with constant $r_1 + r_2$.*

Proposition 2.9. [21] *If $\mathcal{A} : \mathcal{S} \longrightarrow \mathcal{E}$ is Lipschitz with constant r, then $\mathcal{A}$ is ϑ-lipschitz with the same constant r.*

Proposition 2.10. [21] *If $\mathcal{A} : \mathcal{S} \longrightarrow \mathcal{E}$ is compact, then $\mathcal{Z}$ is ϑ-lipschitz with constant $r = 0$.*

Theorem 2.11. [21] *Let $\mathcal{C} : \mathcal{S} \longrightarrow \mathcal{E}$ is ϑ-condensing and*

$$\Gamma_\delta = \{\varrho \in \mathcal{C} : \text{there exists } 0 \leq \delta \leq 1 \text{ such that } \varrho = \delta\mathcal{C}\varrho\}.$$

If Γ_δ is a bounded set in $\mathcal{E}$, then there exists $a > 0$ such that $\Gamma_\delta \subset B_a(0)$ and

$$Deg\left(I - \delta\mathcal{C}, B_a(0), 0\right) = 1 \text{ for all } \delta \in [0, 1].$$

Thus, $\mathcal{C}$ has at least one fixed point, and the set of all fixed points of $\mathcal{C}$ lies in $B_a(0)$.

3. Existence and uniqueness results

In this part we will use the degree theory to prove the existence of solutions to the problem (1.1).
First, we give the following essential hypotheses:
(H_1): For arbitrary $(\varrho_1, \varrho_2), (\xi_1, \xi_2) \in \mathcal{E}_1 \times \mathcal{E}_2$, there exist positive constants $\mathcal{L}_{\varpi_1}, l_{\varpi_1}$, m_{ϖ_1} and $q_1, q_2 \in (0,1)$ such that

$$\begin{aligned}
&\|\varpi_1(\iota, \varrho_1(\iota), \varrho_1(\kappa\iota), \varrho_2(\iota)) - \varpi_1(\iota, \xi_1(\iota), \xi_1(\kappa\iota), \xi_2(\iota))\|^2 \\
&\leq \mathcal{L}_{\varpi_1}\left(\iota^{2(1-\gamma_1)}2\|\varrho_1 - \xi_1\|^2 + \iota^{2(1-\gamma_2)}\|\varrho_2 - \xi_2\|^2\right), \\
&\|\varpi_1(\iota, \varrho_1(\iota), \varrho_1(\kappa\iota), \varrho_2(\iota))\|^2 \\
&\leq l_{\varpi_1}\left(\iota^{2q_1(1-\gamma_1)}2\|\varrho_1\|^{2q_1} + \iota^{2q_2(1-\gamma_2)}\|\varrho_2\|^{2q_2}\right) + m_{\varpi_1}.
\end{aligned}$$

(H_2): For arbitrary $(\varrho_1, \varrho_2), (\xi_1, \xi_2) \in \mathcal{E}_1 \times \mathcal{E}_2$, there exist positive constants $\mathcal{L}_{\varpi_2}, l_{\varpi_2}$, m_{ϖ_2} and $q_1, q_2 \in (0,1)$ such that

$$\begin{aligned}
&\|\varpi_2(\iota, \varrho_1(\iota), \varrho_2(\iota), \varrho_2(\kappa\iota)) - \varpi_2(\iota, \xi_1(\iota), \xi_2(\iota), \xi_2(\kappa\iota))\|^2 \\
&\leq \mathcal{L}_{\varpi_2}\left(\iota^{2(1-\gamma_1)}\|\varrho_1 - \xi_1\|^2 + 2\iota^{2(1-\gamma_2)}\|\varrho_2 - \xi_2\|^2\right), \\
&\|\varpi_2(\iota, \varrho_1(\iota), \varrho_2(\iota), \varrho_2(\kappa\iota))\|^2 \\
&\leq l_{\varpi_2}\left(\iota^{2q_1(1-\gamma_1)}\|\varrho_1\|^{2q_1} + \iota^{2q_2(1-\gamma_2)}2\|\varrho_2\|^{2q_2}\right) + m_{\varpi_2}.
\end{aligned}$$

(H_3): For arbitrary $(\varrho_1, \varrho_2), (\xi_1, \xi_2) \in \mathcal{E}_1 \times \mathcal{E}_2$, there exist positive constants $\mathcal{L}_{g_j}, l_{g_j}$, m_{g_j} $(j = 1, 2)$ and $q_1, q_2 \in (0,1)$ such that

$$\begin{aligned}
&\|g_j(\iota, \varrho_1(\iota), \varrho_2(\iota)) - g_j(\iota, \xi_1(\iota), \xi_2(\iota))\|^2 \\
&\leq \mathcal{L}_{g_j}\left(\iota^{2(1-\gamma_1)}\|\varrho_1 - \xi_1\|^2 + \iota^{2(1-\gamma_2)}\|\varrho_2 - \xi_2\|^2\right), \\
&\|g_j(\iota, \varrho_1(\iota), \varrho_2(\iota))\|^2 \leq l_{g_j}\left(\iota^{2q_1(1-\gamma_1)}\|\varrho_1\|^{2q_1} + \iota^{2q_2(1-\gamma_2)}\|\varrho_2\|^{2q_2}\right) + m_{g_j}.
\end{aligned}$$

To make clarity, we set the following notations:

$$\begin{aligned}
\Delta_{1_j} &= \frac{2b\mathcal{L}_{g_j}}{\Gamma^2(\gamma_j)}, j = 1, 2, \\
\Delta_{2_j} &= \frac{2bl_{g_j}}{\Gamma^2(\gamma_j)}, j = 1, 2, \\
\Delta_{3_j} &= \frac{bm_{g_j}}{\Gamma^2(\gamma_j)}, \\
\Delta_{4_j} &= \frac{3\mathcal{L}_{\varpi_j} b^{2-2\gamma_j+2p_j} l_{\varpi_j}}{\Gamma^2(p_j)2p_j - 1}, j = 1, 2, \\
\Delta &= \max\{\Delta_{1_1}, \Delta_{1_2}\}, \quad \bar{\Delta} = \max\{\Delta_{4_1}, \Delta_{4_2}\}.
\end{aligned}$$

Based on Lemma 2.6, we let the operators $\mathcal{A},\mathcal{B},\mathcal{C}:\mathcal{E}_1\times\mathcal{E}_2\longrightarrow\mathcal{E}_1\times\mathcal{E}_2$ defined by

$$\mathcal{A}(\varrho,\xi)(\iota)=(\mathcal{A}_1(\varrho,\xi)(\iota),\mathcal{A}_2(\varrho,\xi)(\iota)),\quad \mathcal{B}(\varrho,\xi)(\iota)=(\mathcal{B}_1(\varrho,\xi)(\iota),\mathcal{B}_2(\varrho,\xi)(\iota)),$$
$$\mathcal{C}(\varrho,\xi)(\iota)=\mathcal{A}(\varrho,\xi)(\iota)+\mathcal{B}(\varrho,\xi)(\iota),$$

where

$$\begin{cases}\mathcal{A}_1(\varrho,\xi)(\iota)=\dfrac{\iota^{\gamma_1-1}}{\Gamma(\gamma_1)}\displaystyle\int_0^b g_1\left(s,\varrho(s),\xi(s)\right)d\mathcal{W}(s),\\ \mathcal{A}_2(\varrho,\xi)(\iota)=\dfrac{\iota^{\gamma_2-1}}{\Gamma(\gamma_2)}\displaystyle\int_0^b g_2\left(s,\varrho(s),\xi\left(s\right)\right)ds\end{cases}$$

and

$$\begin{cases}\mathcal{B}_1(\varrho,\xi)(\iota)=\dfrac{1}{\Gamma(p_1)}\displaystyle\int_0^\iota(\iota-s)^{p_1-1}\varpi_1\left(s,\varrho(s),\varrho\left(\kappa s\right),\xi(s)\right)ds,\\ \mathcal{B}_2(\varrho,\xi)(\iota)=\dfrac{1}{\Gamma(p_2)}\displaystyle\int_0^\iota(\iota-s)^{p_2-1}\varpi_2\left(s,\varrho(s),\xi\left(s\right),\xi(\kappa s)\right)d\mathcal{W}(s).\end{cases}$$

We shall now prove, step-by-step, that the proposed operators satisfy the conditions of Theorem 2.11.

Lemma 3.1. *The operator $\mathcal{A}$ is ϑ-Lipschitz with a constant Δ. Furthermore, $\mathcal{A}$ adheres to the inequality presented below*

$$\begin{aligned}&\|\mathcal{A}(\varrho,\xi)\|_{\mathcal{E}}^2\leq\Lambda+\bar{\Lambda}\|(\varrho,\xi)\|_{\mathcal{E}}^{2q},\ \text{where}\\ &\Lambda=\max\{\Delta_{3_1},\Delta_{3_2}\}\ \text{and}\\ &\bar{\Lambda}=\max\{\Delta_{2_1},\Delta_{2_2}\}.\end{aligned}\tag{3.1}$$

Proof. Let $(\varrho_1,\varrho_2),(\xi_1,\xi_2)\in\mathcal{E}_1\times\mathcal{E}_2$, we have

$$\begin{aligned}&\mathbb{E}\left\|\iota^{1-\gamma_1}\left(\mathcal{A}_1\left(\varrho_1,\varrho_2\right)(\iota)-\mathcal{A}_1\left(\xi_1,\xi_2\right)(\iota)\right)\right\|^2\\ \leq&\frac{1}{\Gamma^2(\gamma_1)}\mathbb{E}\left\|\int_0^b\left[g_1\left(s,\varrho_1(s),\varrho_2(s)\right)-g_1\left(s,\xi_1(s),\xi_2(s)\right)\right]ds\right\|^2.\end{aligned}$$

By applying Ito isometry and (H_3), we arrive at

$$\begin{aligned}&\mathbb{E}\left\|\iota^{1-\gamma_1}\left(\mathcal{A}_1\left(\varrho_1,\varrho_2\right)(\iota)-\mathcal{A}_1\left(\xi_1,\xi_2\right)(\iota)\right)\right\|^2\\ \leq&\frac{\mathcal{L}_{g_1}}{\Gamma^2(\gamma_1)}\int_0^b\left[s^{2(1-\gamma)}\mathbb{E}\left\|\varrho_1(s)-\xi_1(s)\right\|^2+s^{2(1-\gamma_2)}\mathbb{E}\left\|\varrho_2(s)-\xi_2(s)\right\|^2\right]ds.\end{aligned}$$

Therefore,

$$\mathbb{E}\left\|\iota^{1-\gamma_1}\left(\mathcal{A}_1\left(\varrho_1,\varrho_2\right)(\iota)-\mathcal{A}_1\left(\xi_1,\xi_2\right)(\iota)\right)\right\|^2\leq\frac{b\mathcal{L}_{g_1}}{\Gamma^2(\gamma_1)}\left(\|\varrho_1-\xi_1\|_{\mathcal{E}_1}^2+\|\varrho_2-\xi_2\|_{\mathcal{E}_2}^2\right).$$

Consequently,

$$\|\mathcal{A}_1\left(\varrho_1,\varrho_2\right)-\mathcal{A}_1\left(\xi_1,\xi_2\right)\|_{\mathcal{E}_1}^2\leq\Delta_{1_1}\|\left(\varrho_1,\varrho_2\right)-\left(\xi_1,\xi_2\right)\|_{\mathcal{E}}^2.$$

Similar by the Cauchy-Schwartz (C-S) inequality, we can obtain

$$\|\mathcal{A}_2\left(\varrho_1,\varrho_2\right)-\mathcal{A}_2\left(\xi_1,\xi_2\right)\|_{\mathcal{E}_2}^2\leq\Delta_{1_2}\|\left(\varrho_1,\varrho_2\right)-\left(\xi_1,\xi_2\right)\|_{\mathcal{E}}^2.$$

It follows that

$$\|\mathcal{A}\left(\varrho_1,\varrho_2\right)-\mathcal{A}\left(\xi_1,\xi_2\right)\|_{\mathcal{E}}^2 \leq \Delta\|\left(\varrho_1,\varrho_2\right)-\left(\xi_1,\xi_2\right)\|_{\mathcal{E}}^2.$$

Thus, $\mathcal{A}$ satisfies the Lipschitz condition with the constant Δ. By Proposition 2.9, $\mathcal{A}$ is also ϑ-Lipschitz with the same constant Δ.
For the growth condition.
Let $(\varrho,\xi)\in\mathcal{E}_1\times\mathcal{E}_2$. Under the Ito isometry and (H_3), we have

$$\begin{aligned}\mathbb{E}\left\|\iota^{1-\gamma_1}\mathcal{A}_1\left(\varrho,\xi\right)(\iota)\right\|^2 \leq & \frac{1}{\Gamma^2(\gamma_1)}\int_0^\iota\Big[m_{g_1}+l_{g_1}s^{2q_1(1-\gamma_1)}\mathbb{E}\left\|\varrho(s)\right\|^{2q_1} \\ & +l_{g_1}s^{2q_2(1-\gamma_2)}\mathbb{E}\left\|\xi(s)\right\|^{2q_2}\Big]\,ds.\end{aligned}$$

Therefore,

$$\mathbb{E}\left\|\iota^{1-\gamma_1}\mathcal{A}_1\left(\varrho,\xi\right)(\iota)\right\|^2 \leq \frac{b}{\Gamma^2(\gamma_1)}\left(m_{g_1}+l_{g_1}\|\varrho\|_{\mathcal{E}_1}^{2q_1}+l_{g_1}\|\xi\|_{\mathcal{E}_2}^{2q_2}\right).$$

Consequently,

$$\|\mathcal{A}_1\left(\varrho,\xi\right)\|_{\mathcal{E}_1}^2 \leq \Delta_{3_1}+\Delta_{2_1}\|\left(\varrho,\xi\right)\|_{\mathcal{E}}^{2q},$$

where $q=\max\{q_1,q_2\}$.
Similarly, we find that

$$\|\mathcal{A}_2\left(\varrho,\xi\right)\|_{\mathcal{E}_2}^2 \leq \Delta_{3_2}+\Delta_{2_2}\|\left(\varrho,\xi\right)\|_{\mathcal{E}}^{2q}.$$

It follows that

$$\|\mathcal{A}(\varrho,\xi)\|_{\mathcal{E}}^2\leq\Lambda+\bar{\Lambda}\|(\varrho,\xi)\|_{\mathcal{E}}^{2q}.$$

□

Lemma 3.2. *The operator $\mathcal{B}$ is continuous. Furthermore, $\mathcal{B}$ satisfies the inequality*

$$\begin{aligned}&\|\mathcal{B}(\varrho,\xi)\|_{\mathcal{E}}^2 \leq \Xi+\bar{\Xi}\|(\varrho,\xi)\|_{\mathcal{E}}^{2q}, \text{ where}\\ &\Xi=\max\{\frac{b^{2-2\gamma_1+2p_1}m_{\varpi_1}}{\Gamma^2(p_1)2p_1-1},\frac{b^{2-2\gamma_2+2p_2}m_{\varpi_2}}{\Gamma^2(p_2)2p_2-1}\} \text{ and}\\ &\bar{\Xi}=\max\{\frac{3b^{2-2\gamma_1+2p_1}l_{\varpi_1}}{\Gamma^2(p_1)2p_1-1},\frac{3b^{2-2\gamma_2+2p_2}l_{\varpi_2}}{\Gamma^2(p_2)2p_2-1}\}.\end{aligned} \tag{3.2}$$

Proof. For the continuity of $\mathcal{B}$, let $(\varrho_n,\xi_n)\to(\varrho,\xi)$ in $\mathcal{E}$. From the fact that ϖ_1 and ϖ_2 are continuous functions linking with the Lebesgue dominated convergence theorem, we can obtain

$$\|\mathcal{B}(\varrho_n,\xi_n)-\mathcal{B}(\varrho,\xi)\|_{\mathcal{E}}^2\to 0 \quad \text{as} \quad n\to\infty.$$

Moreover, $\mathcal{B}$ satisfies the growth condition.
Using the C-S inequality, we have

$$\begin{aligned}&\mathbb{E}\left\|\iota^{1-\gamma_1}\mathcal{B}_1(\varrho,\xi)(\iota)\right\|^2\\ \leq&\iota^{2(1-\gamma_1)}\frac{\iota}{\Gamma^2(p_1)}\int_0^\iota(\iota-s)^{2(p_1-1)}\mathbb{E}\left\|\varpi_1\left(s,\varrho(s),\varrho(\kappa s),\xi\left(s\right)\right)\right\|^2ds.\end{aligned}$$

Based on the assumptions (H_1), it can be concluded that

$$\mathbb{E}\left\|\iota^{1-\gamma_1}\mathcal{B}_1(\varrho,\xi)(\iota)\right\|^2 \leq \frac{t^{2-2\gamma_1+2p_1}}{\Gamma^2(p_1)2p_1-1}\left(m_{\varpi_1}+2l_{\varpi_1}\|\varrho\|_{\mathcal{E}_1}^{2q_1}+l_{\varpi_1}\|\xi\|_{\mathcal{E}_2}^{2q_2}\right).$$

Subsequently,

$$\|\mathcal{B}_1(\varrho,\xi)\|_{\mathcal{E}_1}^2 \leq \frac{b^{2-2\gamma_1+2p_1}m_{\varpi_1}}{\Gamma^2(p_1)2p_1-1}+\frac{3b^{2-2\gamma_1+2p_1}l_{\varpi_1}}{\Gamma^2(p_1)2p_1-1}\|(\varrho,\xi)\|_{\mathcal{E}}^{2q}.$$

Similarly,

$$\|\mathcal{B}_2(\varrho,\xi)\|_{\mathcal{E}_2}^2 \leq \frac{b^{2-2\gamma_2+2p_2}m_{\varpi_2}}{\Gamma^2(p_2)2p_2-1}+\frac{3b^{2-2\gamma_2+2p_2}l_{\varpi_2}}{\Gamma^2(p_2)2p_2-1}\|(\varrho,\xi)\|_{\mathcal{E}}^{2q}.$$

Thus,

$$\|\mathcal{B}(\varrho,\xi)\|_{\mathcal{E}}^2 \leq \Xi+\bar{\Xi}\|(\varrho,\xi)\|_{\mathcal{E}}^{2q}, \text{ where}$$
$$\Xi=\max\{\frac{b^{2-2\gamma_1+2p_1}m_{\varpi_1}}{\Gamma^2(p_1)2p_1-1},\frac{b^{2-2\gamma_2+2p_2}m_{\varpi_2}}{\Gamma^2(p_2)2p_2-1}\} \text{ and}$$
$$\bar{\Xi}=\max\{\frac{3b^{2-2\gamma_1+2p_1}l_{\varpi_1}}{\Gamma^2(p_1)2p_1-1},\frac{3b^{2-2\gamma_2+2p_2}l_{\varpi_2}}{\Gamma^2(p_2)2p_2-1}\}.$$

□

Lemma 3.3. *$\mathcal{B}$ is compact; consequently, $\mathcal{B}$ is ϑ-Lipschitz with a zero constant.*

Proof. Let $B_\tau=\{(\varrho,\xi)\in\mathcal{E}_1\times\mathcal{E}_2:\|(\varrho,\xi)\|\leq\tau\}$ and consider a bounded set $\mathcal{K}$ such that $\mathcal{K}\subset\mathcal{B}_\tau$. It remains to demonstrate that $\mathcal{B}(\mathcal{K})$ is relatively compact in $\mathcal{E}$. For this purpose, let $(\varrho,\xi)\in\mathcal{K}\subset\mathcal{B}_\tau$ and by (3.2), we derive

$$\|\mathcal{B}(\varrho,\xi)\|_{\mathcal{E}}^2 \leq \Xi+\bar{\Xi}\|(\varrho,\xi)\|_{\mathcal{E}}^{2q}:=\Upsilon.$$

Thus, $\mathcal{B}(\mathcal{K})\subset\mathcal{B}_\tau$, and as a result, $\mathcal{B}(\mathcal{K})$ is bounded.
It remains to prove the equicontinuity of $\mathcal{B}$.
Let $0\leq\epsilon_1<\epsilon_2\leq b$ and $(\varrho,\xi)\in B_\tau$, then

$$\begin{aligned}&\mathbb{E}\left\|\epsilon_2^{1-\gamma_1}\left(\mathcal{B}_1(\varrho,\xi)\right)(\epsilon_2)-\epsilon_1^{1-\gamma_1}\left(\mathcal{B}_1(\varrho,\xi)\right)(\epsilon_1)\right\|^2\\ \leq&2\mathbb{E}\left\|\frac{1}{\Gamma(p_1)}\int_0^{\epsilon_1}\left[\epsilon_2^{2(1-\gamma_1)}(\epsilon_2-s)^{p_1-1}-\epsilon_1^{2(1-\gamma_1)}(\epsilon_1-s)^{p_1-1}\right]\right.\\ &\times\varpi_1(s,\varrho(s),\varrho(\kappa s),\xi(s))ds\|^2\\ &+2\mathbb{E}\left\|\frac{1}{\Gamma(p_1)}\int_{\epsilon_1}^{\epsilon_2}\epsilon_2^{2(1-\gamma_1)}(\epsilon_2-s)^{p_1-1}\varpi_1(s,\varrho(s),\varrho(\kappa s),\xi(s))ds\right\|^2.\end{aligned}$$

By C-S inequality (H_1), we obtain

$$\begin{aligned}&\mathbb{E}\left\|\epsilon_2^{1-\gamma_1}\left(\mathcal{B}_1(\varrho,\xi)\right)(\epsilon_2)-\epsilon_1^{1-\gamma_1}\left(\mathcal{B}_1(\varrho,\xi)\right)(\epsilon_1)\right\|^2\\ \leq&\frac{2\epsilon_1\left(m_{\varpi_1}+3l_{\varpi_1}\tau^q\right)}{\Gamma^2(p_1)}\int_0^{\epsilon_1}\left[\epsilon_2^{2(1-\gamma_1)}(\epsilon_2-s)^{p_1-1}-\epsilon_1^{2(1-\gamma_1)}(\epsilon_1-s)^{p_1-1}\right]^2ds\\ &+\frac{2(\epsilon_2-\epsilon_1)\left(m_{\varpi_1}+3l_{\varpi_1}\tau^q\right)\epsilon_2^{2(1-\gamma_1)}}{\Gamma^2(p_1)2p_1-1}(\epsilon_2-\epsilon_1)^{2p_1-1}\longrightarrow 0,\quad\text{as}\quad\epsilon_1\longrightarrow\epsilon_2.\end{aligned}$$

Similarly, we get

$$\begin{aligned}
&\mathbb{E}\left\|\epsilon_2^{1-\gamma_2}\left(\mathcal{B}_2(\varrho,\xi)\right)(\epsilon_2)-\epsilon_1^{1-\gamma_2}\left(\mathcal{B}_2(\varrho,\xi)\right)(\epsilon_1)\right\|^2\\
\leq&\frac{2\epsilon_1\left(m_{\varpi_2}+3l_{\varpi_2}\tau^q\right)}{\Gamma^2(p_2)}\int_0^{\epsilon_1}\left[\epsilon_2^{2(1-\gamma_2)}(\epsilon_2-s)^{p_1-1}-\epsilon_1^{2(1-\gamma_2)}(\epsilon_1-s)^{p_2-1}\right]^2ds\\
&+\frac{2(\epsilon_2-\epsilon_1)\left(m_{\varpi_2}+3l_{\varpi_2}\tau^q\right)\epsilon_2^{2(1-\gamma_2)}}{\Gamma^2(p_2)2p_2-1}(\epsilon_2-\epsilon_1)^{2p_2-1}\longrightarrow 0,\quad \text{as}\quad \epsilon_1\longrightarrow\epsilon_2.
\end{aligned}$$

Therefore, we find that

$$\begin{aligned}
&\mathbb{E}\left\|\epsilon_2^{1-\gamma_1}\left(\mathcal{B}_1(\varrho,\xi)\right)(\epsilon_2)-\epsilon_1^{1-\gamma_1}\left(\mathcal{B}_1(\varrho,\xi)\right)(\epsilon_1)\right\|^2,\\
&\mathbb{E}\left\|\epsilon_2^{1-\gamma_2}\left(\mathcal{B}_2(\varrho,\xi)\right)(\epsilon_2)-\epsilon_1^{1-\gamma_2}\left(\mathcal{B}_2(\varrho,\xi)\right)(\epsilon_1)\right\|^2
\end{aligned}$$

approaches zero as $\epsilon_1\to\epsilon_2$. By applying the Arzelà-Ascoli theorem, it can be concluded that the operator $\mathcal{B}$ is compact. As a result of Proposition 2.10, $\mathcal{B}$ is ϑ-Lipschitz with a zero constant. □

Theorem 3.4. *Assume that* $(H_1)-(H_3)$ *hold and* $0<\Delta<1$. *Then the problem (1.1) has at least one solution on* $\mathcal{E}$. *Moreover, the set of the solutions of the problem (1.1) is bounded in* $\mathcal{E}$.

Proof. By Lemma 3.1, the operator $\mathcal{A}$ is shown to be ϑ-Lipschitz with a constant $\Delta\in(0,1)$. Similarly, from Lemma 3.3, the operator $\mathcal{B}$ is ϑ-Lipschitz with a constant equal to zero. Consequently, based on Proposition 2.8 and Definition 2.7, the operator $\mathcal{C}$ qualifies as a ϑ-contraction with the constant Δ. This implies that $\mathcal{C}$ is ϑ-condensing. Now, consider the following set

$$\Gamma_\delta=\{(\varrho,\xi)\in\mathcal{E}_1\times\mathcal{E}_2:(\varrho,\xi)=\delta\mathcal{C}\left(\varrho,\xi\right),\quad\text{for}\quad 0\leq\delta<1\}.$$

We need to demonstrate that Γ_δ is bounded in $\mathcal{E}_1\times\mathcal{E}_2$. Let $(\varrho,\xi)\in\Gamma_\delta$. Then, by Lemma 3.1 and 3.2, it follows that

$$\begin{aligned}
\|\left(\varrho,\xi\right)\|_{\mathcal{E}}^2&=\delta^2\|\mathcal{A}\left(\varrho,\xi\right)+\mathcal{B}\left(\varrho,\xi\right)\|_{\mathcal{E}}^2\\
&\leq 2\delta^2\left(\|\mathcal{A}\left(\varrho,\xi\right)\|_{\mathcal{E}}^2+\|\mathcal{B}\left(\varrho,\xi\right)\|_{\mathcal{E}}^2\right)\\
&\leq 2\left(\Lambda+\Xi\right)+2\left(\bar{\Lambda}+\bar{\Xi}\right)\|\left(\varrho,\xi\right)\|_{\mathcal{E}}^{2q}.
\end{aligned}$$

Thus, the set Γ_δ is bounded in $\mathcal{E}$. If this is not true, by dividing the above inequality by $\theta:=\|\left(\varrho,\xi\right)\|_{\mathcal{E}}^2\to\infty$, we obtain

$$1\leq\lim_{\theta\to\infty}\frac{1}{\theta}\left[2\left(\Lambda+\Xi\right)+2\left(\bar{\Lambda}+\bar{\Xi}\right)\theta^q\right]=0,$$

which is a contradiction. Consequently, Theorem 2.11 ensures that $\mathcal{C}$ has at least one fixed point. Therefore, our problem (1.1) has at least one solution. □

Theorem 3.5. *Assume assumptions* $(H_1)-(H_3)$ *hold and* $0<2\left(\Delta+\bar{\Delta}\right)<1$, *it follows that the problem (1.1) has a unique solution.*

Proof. By applying the Banach contraction theorem, for any $(\varrho_1, \varrho_2), (\xi_1, \xi_2) \in \mathcal{E}_1 \times \mathcal{E}_2$, it follows from the arguments presented in the proof of Lemma 3.3 that

$$\|\mathcal{A}(\varrho_1, \varrho_2) - \mathcal{A}(\xi_1, \xi_2)\|_{\mathcal{E}}^2 \leq \Delta \|(\varrho_1, \varrho_2) - (\xi_1, \xi_2)\|_{\mathcal{E}}^2.$$

Next, by the C-S inequality, we obtain

$$\begin{aligned}
&\mathbb{E}\left\|\iota^{1-\gamma_1}\left(\mathcal{B}_1(\varrho_1, \varrho_2)(\iota) - \mathcal{B}_1(\xi_1, \xi_2)(\iota)\right)\right\|^2 \\
\leq &\iota^{2(1-\gamma_1)} \frac{\iota}{\Gamma^2(p_1)} \int_0^\iota (\iota - s)^{2(p_1-1)} \mathbb{E}\left\|\varpi_1(s, \varrho_1(s), \varrho_1(\kappa s), \varrho_2(s))\right. \\
&\left. - \varpi_1(s, \xi_1(s), \xi_1(\kappa s), \xi_2(s))\right\|^2 ds.
\end{aligned}$$

Based on the assumptions (H_1), it can be concluded that

$$\begin{aligned}
&\mathbb{E}\left\|\iota^{1-\gamma_1}\left(\mathcal{B}_1(\varrho_1, \varrho_2)(\iota) - \mathcal{B}_1(\xi_1, \xi_2)(\iota)\right)\right\|^2 \\
\leq &\frac{\mathcal{L}_{\varpi_1} t^{2-2\gamma_1+2p_1}}{\Gamma^2(p_1)2p_1 - 1}\left(2\|\varrho_1 - \xi_1\|_{\mathcal{E}_1}^2 + \|\varrho_2 - \xi_2\|_{\mathcal{E}_2}^2\right).
\end{aligned}$$

Subsequently,

$$\|\mathcal{B}_1(\varrho_1, \varrho_2) - \mathcal{B}_1(\xi_1, \xi_2)\|_{\mathcal{E}_1}^2 \leq \frac{3\mathcal{L}_{\varpi_1} b^{2-2\gamma_1+2p_1}}{\Gamma^2(p_1)2p_1 - 1}\|(\varrho_1, \varrho_2) - (\xi_1, \xi_2)\|_{\mathcal{E}}^2.$$

Similarly,

$$\|\mathcal{B}_2(\varrho_1, \varrho_2) - \mathcal{B}_2(\xi_1, \xi_2)\|_{\mathcal{E}_2}^2 \leq \frac{3\mathcal{L}_{\varpi_2} b^{2-2\gamma_2+2p_2}}{\Gamma^2(p_2)2p_2 - 1}\|(\varrho_1, \varrho_2) - (\xi_1, \xi_2)\|_{\mathcal{E}}^2.$$

Therefore,

$$\|\mathcal{B}(\varrho_1, \varrho_2) - \mathcal{B}(\xi_1, \xi_2)\|_{\mathcal{E}}^2 \leq \bar{\Delta}\|(\varrho_1, \varrho_2) - (\xi_1, \xi_2)\|_{\mathcal{E}}^2.$$

Thus,

$$\begin{aligned}
\|\mathcal{C}(\varrho_1, \varrho_2) - \mathcal{C}(\xi_1, \xi_2)\|_{\mathcal{E}}^2 &\leq 2\left(\|\mathcal{A}(\varrho_1, \varrho_2) - \mathcal{A}(\xi_1, \xi_2)\|_{\mathcal{E}}^2 + \|\mathcal{B}(\varrho_1, \varrho_2) - \mathcal{B}(\xi_1, \xi_2)\|_{\mathcal{E}}^2\right) \\
&\leq 2\left(\Delta + \bar{\Delta}\right)\|(\varrho_1, \varrho_2) - (\xi_1, \xi_2)\|_{\mathcal{E}}^2.
\end{aligned}$$

This implies that $\mathcal{C}$ is a contraction. As a result, the problem (1.1) has a unique solution.

□

4. Continuous dependence of solutions

Now, we study the continuous dependence on the nonlocal conditions of the solutions of problem (1.1).

Definition 4.1. *The solution $(\varrho, \xi) \in \mathcal{E}_1 \times \mathcal{E}_2$ of problem (1.1) is said to be continuously dependent on the nonlocal conditions g_1 and g_2 if for all $\epsilon > 0$, $\exists \delta > 0$ such that $\left\|g_j(s, \cdot, \cdot) - g_j^*(s, \cdot, \cdot)\right\|^2 \leq \delta$, $j = 1, 2$ implies that $\|(\varrho, \xi) - (\bar{\varrho}, \bar{\xi})\|_{\mathcal{E}}^2 \leq \epsilon$.*

Theorem 4.2. *Assume hypotheses (H_1)-(H_3) are fulfilled, then the solution of the problem (1.1) is continuously dependent on g_1 and g_2.*

Proof. Let $(\varrho,\xi),(\bar{\varrho},\bar{\xi})$ be the solutions of problem (1.1) such that

$$\begin{cases}\varrho(\iota)=\dfrac{\iota^{\gamma_1-1}}{\Gamma(\gamma_1)}\displaystyle\int_0^\iota g_1\left(s,\varrho(s),\xi(s)\right)d\mathcal{W}(s)\\ \qquad+\dfrac{1}{\Gamma(p_1)}\displaystyle\int_0^\iota(\iota-s)^{p_1-1}\varpi_1\left(s,\varrho(s),\varrho\left(\kappa s\right),\xi(s)\right)ds\\ \xi(\iota)=\dfrac{\iota^{\gamma_2-1}}{\Gamma(\gamma_2)}\displaystyle\int_0^\iota g_2\left(s,\varrho(s),\xi\left(s\right)\right)ds\\ \qquad+\dfrac{1}{\Gamma(p_2)}\displaystyle\int_0^\iota(\iota-s)^{p_2-1}\varpi_2\left(s,\varrho(s),\xi\left(s\right),\xi(\kappa s)\right)d\mathcal{W}(s),\end{cases}$$

and

$$\begin{cases}\bar{\varrho}(\iota)=\dfrac{\iota^{\gamma_1-1}}{\Gamma(\gamma_1)}\displaystyle\int_0^\iota g_1^*\left(s,\bar{\varrho}(s),\bar{\xi}(s)\right)d\mathcal{W}(s)\\ \qquad+\dfrac{1}{\Gamma(p_1)}\displaystyle\int_0^\iota(\iota-s)^{p_1-1}\varpi_1\left(s,\bar{\varrho}(s),\bar{\varrho}\left(\kappa s\right),\bar{\xi}(s)\right)ds\\ \bar{\xi}(\iota)=\dfrac{\iota^{\gamma_2-1}}{\Gamma(\gamma_2)}\displaystyle\int_0^\iota g_2^*\left(s,\bar{\varrho}(s),\bar{\xi}\left(s\right)\right)ds\\ \qquad+\dfrac{1}{\Gamma(p_2)}\displaystyle\int_0^\iota(\iota-s)^{p_2-1}\varpi_2\left(s,\bar{\varrho}(s),\bar{\xi}\left(s\right),\bar{\xi}(\kappa s)\right)d\mathcal{W}(s),\end{cases}$$

where

$$\left\|g_j\left(s,\cdot,\cdot\right)-g_j^*\left(s,\cdot,\cdot\right)\right\|^2\le\delta,\quad j=1,2.$$

By the Ito isometry linking with the C-S inequality, we get

$$\begin{aligned}&\mathbb{E}\left\|\iota^{1-\gamma_1}\left(\varrho(\iota)-\bar{\varrho}(\iota)\right)\right\|^2\\ \le&\frac{2}{\Gamma^2(\gamma_1)}\int_0^\iota\mathbb{E}\left\|g_1\left(s,\varrho(s),\xi(s)\right)-g_1^*\left(s,\bar{\varrho}(s),\bar{\xi}(s)\right)\right\|^2ds\\ &+\frac{2\iota^{2(1-\gamma_1)}\iota}{\Gamma^2(\gamma_1)}\int_0^\iota(\iota-s)^{2(p_1-1)}\mathbb{E}\left\|\varpi_1\left(s,\varrho(s),\varrho\left(\kappa s\right),\xi(s)\right)\right.\\ &\left.-\varpi_1\left(s,\bar{\varrho}(s),\bar{\varrho}\left(\kappa s\right),\bar{\xi}(s)\right)\right\|^2ds\\ \le&\frac{4}{\Gamma^2(\gamma_1)}\int_0^\iota\mathbb{E}\left(\left\|g_1\left(s,\varrho(s),\xi(s)\right)-g_1^*\left(s,\varrho(s),\xi(s)\right)\right\|^2\right.\\ &\left.+\left\|g_1^*\left(s,\varrho(s),\xi(s)\right)-g_1^*\left(s,\bar{\varrho}(s),\bar{\xi}(s)\right)\right\|^2\right)ds\\ &+\frac{2\iota^{2(1-\gamma_1)}\iota}{\Gamma^2(\gamma_1)}\int_0^\iota(\iota-s)^{2(p_1-1)}\mathbb{E}\left\|\varpi_1\left(s,\varrho(s),\varrho\left(\kappa s\right),\xi(s)\right)\right.\\ &\left.-\varpi_1\left(s,\bar{\varrho}(s),\bar{\varrho}\left(\kappa s\right),\bar{\xi}(s)\right)\right\|^2ds.\end{aligned}$$

By applying (H_1) and (H_3), we arrive at

$$\begin{aligned}
&\mathbb{E}\left\|\iota^{1-\gamma_1}\left(\varrho(\iota)-\bar{\varrho}(\iota)\right)\right\|^2\\
\leq&\frac{4b}{\Gamma^2(\gamma_1)}\delta+\frac{4\mathcal{L}_{g_1}}{\Gamma^2(\gamma_1)}\int_0^\iota\left[s^{2(1-\gamma_1)}\mathbb{E}\left\|\varrho(s)-\bar{\varrho}(s)\right\|^2+s^{2(1-\gamma_2)}\mathbb{E}\left\|\xi(s)-\bar{\xi}(s)\right\|^2\right]ds\\
&+\frac{2b^{3-2\gamma_1}\mathcal{L}_{\varpi_1}}{\Gamma^2(\gamma_1)}\int_0^\iota(\iota-s)^{2(p_1-1)}\left[s^{2(1-\gamma_1)}\mathbb{E}\left\|\varrho(s)-\bar{\varrho}(s)\right\|^2+s^{2(1-\gamma_1)}\mathbb{E}\left\|\varrho(\kappa s)-\bar{\varrho}(\kappa s)\right\|^2\right.\\
&\left.+s^{2(1-\gamma_2)}\mathbb{E}\left\|\xi(s)-\bar{\xi}(s)\right\|^2\right]ds.
\end{aligned}$$

Let
$\varphi_1(\iota)=\sup_{s\in(0,\iota)}\mathbb{E}\left\|\iota^{1-\gamma_1}\left(\varrho(s)-\bar{\varrho}(s)\right)\right\|^2$ and
$\varphi_2(\iota)=\sup_{s\in(0,\iota)}\mathbb{E}\left\|\iota^{1-\gamma_2}\left(\xi(s)-\bar{\xi}(s)\right)\right\|^2$, for $\iota\in J$.
We have

$$\begin{cases}\mathbb{E}\left\|\iota^{1-\gamma_1}\left(\varrho(s)-\bar{\varrho}(s)\right)\right\|^2\leq\varphi_1(s)\\ \mathbb{E}\left\|\iota^{1-\gamma_2}\left(\xi(s)-\bar{\xi}(s)\right)\right\|^2\leq\varphi_2(s),\end{cases}$$

and

$$\begin{cases}\mathbb{E}\left\|\iota^{1-\gamma_1}\left(\varrho(\kappa s)-\bar{\varrho}(\kappa s)\right)\right\|^2\leq\varphi_1(s)\\ \mathbb{E}\left\|\iota^{1-\gamma_2}\left(\xi(\kappa s)-\bar{\xi}(\kappa s)\right)\right\|^2\leq\varphi_2(s).\end{cases}$$

Then, for $\iota\in J$, we get

$$\begin{aligned}\mathbb{E}\left\|\iota^{1-\gamma_1}\left(\varrho(\iota)-\bar{\varrho}(\iota)\right)\right\|^2\leq&\frac{4b}{\Gamma^2(\gamma_1)}\delta+\frac{4\mathcal{L}_{g_1}}{\Gamma^2(\gamma_1)}\int_0^\iota\left(\varphi_1(s)+\varphi_2(s)\right)ds\\&+\frac{2b^{3-2\gamma_1}\mathcal{L}_{\varpi_1}}{\Gamma^2(\gamma_1)}\int_0^\iota(\iota-s)^{2(p_1-1)}\left(2\varphi_1(s)+\varphi_2(s)\right)ds.\end{aligned}$$

Then

$$\begin{aligned}\varphi_1(\iota)\leq&\frac{4b}{\Gamma^2(\gamma_1)}\delta+\frac{4\mathcal{L}_{g_1}}{\Gamma^2(\gamma_1)}\int_0^\iota\left(\varphi_1(s)+\varphi_2(s)\right)ds\\&+\frac{2b^{3-2\gamma_1}\mathcal{L}_{\varpi_1}}{\Gamma^2(\gamma_1)}\int_0^\iota(\iota-s)^{2(p_1-1)}\left(2\varphi_1(s)+\varphi_2(s)\right)ds.\end{aligned}$$

Similar, we find that

$$\begin{aligned}\varphi_2(\iota)\leq&\frac{4b}{\Gamma^2(\gamma_2)}\delta+\frac{4\mathcal{L}_{g_2}}{\Gamma^2(\gamma_2)}\int_0^\iota\left(\varphi_1(s)+\varphi_2(s)\right)ds\\&+\frac{2b^{3-2\gamma_2}\mathcal{L}_{\varpi_2}}{\Gamma^2(\gamma_2)}\int_0^\iota(\iota-s)^{2(p_2-1)}\left(\varphi_1(s)+2\varphi_2(s)\right)ds.\end{aligned}$$

Take now $\varphi=\max\{\varphi_1,\varphi_2\}$, it follows that

$$\varphi(\iota)\leq\wp\delta+\Re\int_0^\iota\varphi(s)ds+\Im\int_0^\iota(\iota-s)^{2(p-1)}\varphi(s)ds,$$

where

$$\begin{aligned}&\wp = \max\{\tfrac{4b}{\Gamma^2(\gamma_1)}, \tfrac{4b}{\Gamma^2(\gamma_2)}\}, \quad \Re = \max\{\tfrac{8\mathcal{L}_{g_1}}{\Gamma^2(\gamma_1)}, \tfrac{8\mathcal{L}_{g_2}}{\Gamma^2(\gamma_2)}\},\\ &\Im = \max\{\frac{6b^{3-2\gamma_1}\mathcal{L}_{\varpi_1}}{\Gamma^2(\gamma_1)}, \frac{6b^{3-2\gamma_2}\mathcal{L}_{\varpi_2}}{\Gamma^2(\gamma_2)}\}.\\ &p = \max\{p_1, p_2\}.\end{aligned}$$

So,

$$\varphi(\iota) \leq \wp\delta + \Re\int_0^\iota \varphi(s)ds + \Im\int_0^\iota (\iota - s)^{2(p-1)}\varphi(s)ds.$$

By Generalised Gronwall inequality, we obtain

$$\begin{aligned}\varphi(\iota) &\leq \left(\wp\delta + \Re\int_0^\iota \varphi(s)ds\right) E_{2p-1}\left(\Im\Gamma(2p-1)\iota^{2p-1}\right)\\ &\leq \aleph\delta + \hbar\int_0^\iota \varphi(s)ds,\end{aligned}$$

where

$$\aleph = \wp E_{2p-1}\left(\Im\Gamma(2p-1)b^{2p-1}\right), \quad \hbar = \Re E_{2p-1}\left(\Im\Gamma(2p-1)b^{2p-1}\right).$$

By Gronwall inequality, we obtain

$$\varphi(\iota) \leq \aleph\delta e^{\hbar\iota}.$$

Hence,

$$\max\{\|\varrho - \bar{\varrho}\|^2_{\mathcal{E}_1}, \|\xi - \bar{\xi}\|^2_{\mathcal{E}_2}\} \leq \aleph\delta e^{\hbar b} = \epsilon.$$

We conclude that the solution of the problem (1.1) is continuously dependent on g_1 and g_2.

□

5. An example

Consider the following coupled system of Hilfer fractional stochastic pantograph equations with nonlocal integral conditions

$$\begin{cases} {}^H\mathfrak{D}^{0.75,0.5}_{0^+,\iota}\varrho(\iota) = \varpi_1\left(\iota, \varrho(\iota), \varrho\left(0.5\iota\right), \xi(\iota)\right), \quad \iota \in (0,1],\\ {}^H\mathfrak{D}^{0.85,0.6}_{0^+,\iota}\xi(\iota) = \varpi_2\left(\iota, \varrho(\iota), \xi\left(\iota\right), \xi(0.5\iota)\right)\dfrac{d\mathcal{W}(\iota)}{d\iota},\\ \mathfrak{I}^{1-\gamma_1}_{0^+,\iota}\varrho(0) = \displaystyle\int_0^\iota g_1\left(\iota, \varrho(\iota), \xi(\iota)\right)d\mathcal{W}(s),\\ \mathfrak{I}^{1-\gamma_2}_{0^+,\iota}\xi(0) = \displaystyle\int_0^\iota g_2\left(\iota, \varrho(\iota), \xi(\iota)\right)ds, \end{cases} \tag{5.1}$$

where

$$\varpi_1\left(\iota, \varrho(\iota), \varrho\left(0.5\iota\right), \xi(\iota)\right) = \frac{e^{-\pi\iota}}{\sqrt{13}+\iota} + \frac{1}{\sqrt{50}+\iota}\left(|\varrho(\iota)| + |\varrho(\kappa\iota)| + |\sin(\xi(\iota))|\right),$$

$$\varpi_2\left(\iota, \varrho(\iota), \xi\left(\iota\right), \xi(0.5\iota)\right) = \frac{1}{14} + \frac{e^{-\iota}}{4\sqrt{5}}\left(|\varrho(\iota)| + |\cos(\xi(\iota))| + \sqrt{|\xi(0.5\iota)|}\right),$$

$$g_1(\iota, \varrho(\iota), \xi(\iota)) = \frac{1}{17} + \frac{1}{4\sqrt{2}}\left(|\varrho(\iota)| + |\sin(\xi(\iota))|\right),$$

$$g_1(\iota, \varrho(\iota), \xi(\iota)) = \frac{e^{-\iota^2}}{3\iota+\sqrt{21}} + \frac{1}{\iota^2+\sqrt{31}}\left(|\cos(\varrho(\iota))| + |\xi(\iota)|\right).$$

Here, $p_1 = 0.75$, $p_2 = 0.85$, $q_1 = 0.5$, $q_2 = 0.6$, $\kappa = 0.5$, $\gamma_1 = 0.875$, $\gamma_2 = 0.94$. The assumptions (H_1), (H_2) and (H_3) are satisfied with $\mathcal{L}_{\varpi_1} = l_{\varpi_1} = \frac{3}{25}$, $\mathcal{L}_{\varpi_2} = l_{\varpi_2} = \frac{3}{40}$, $m_{\varpi_1} = \frac{2}{13}$, $m_{\varpi_2} = \frac{1}{7}$, $\mathcal{L}_{g_1} = l_{g_1} = \frac{1}{8}$, $\mathcal{L}_{g_2} = l_{g_2} = \frac{4}{31}$, $m_{g_1} = \frac{2}{17}$ and $m_{g_2} = \frac{2}{21}$. Additionally, we find $\Delta = 0.239329 < 1$. Theorem 3.4 shows that problem (5.1) has at least one solution. Further, $2\left(\Delta + \bar{\Delta}\right) = 0.80914 < 1$. Thus by Theorem 3.5 the problem has a unique solution.
Next, we plot the approximate solution $(\varrho(\iota), \xi(\iota))$ of problem (5.1) for different values of p_1, p_2, q_1 and q_2.

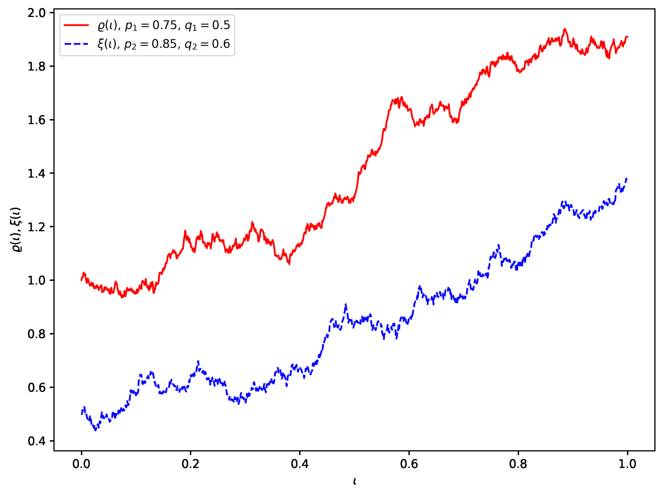

FIGURE 1. Solution $(\varrho(\iota), \xi(\iota))$ for $p_1 = 0.75$, $q_1 = 0.5$, $p_2 = 0.85$, $q_2 = 0.6$.

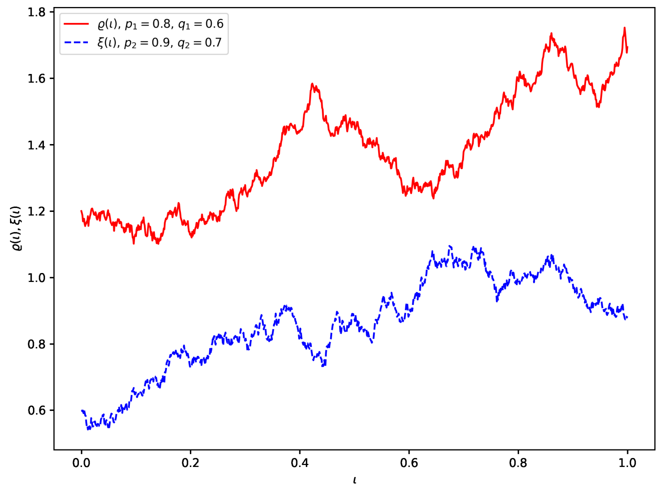

FIGURE 2. Solution $(\varrho(\iota), \xi(\iota))$ for $p_1 = 0.8$, $q_1 = 0.6$, $p_2 = 0.9$, $q_2 = 0.7$.

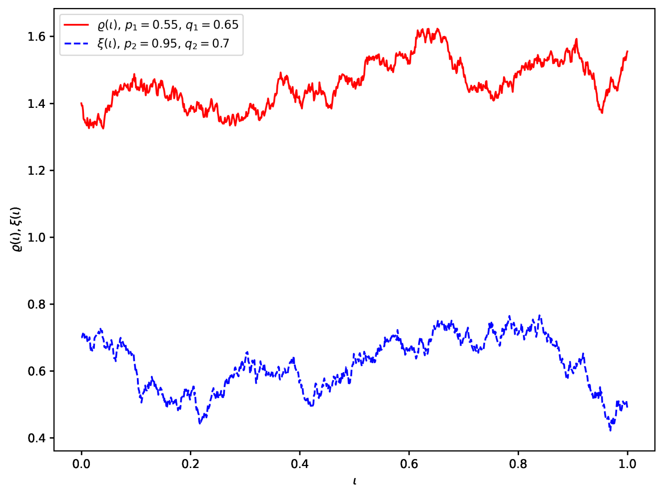

FIGURE 3. Solution $(\varrho(\iota), \xi(\iota))$ for $p_1 = 0.55$, $q_1 = 0.65$, $p_2 = 0.95$, $q_2 = 0.7$.

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