Stud. Univ. Babeş-Bolyai Math. 70(2025), No. 3, 371–390
DOI: 10.24193/subbmath.2025.3.01

New developments of fractional integral inequalities and their applications

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Abstract. In this paper, we propose the so-called higher order strongly m-polynomial exponentially type convex functions. Some of its algebraic properties are given and a new fractional integral identity is established. Applying the class of higher order strongly m-polynomial exponentially type convex functions, we deduce some fractional integral inequalities using the basic identity. Furthermore, we offer some applications to demonstrate the efficiency of our results. Our results not only generalize the known results but also refine them.

Mathematics Subject Classification (2010): 26A33, 26A51, 26D07, 26D10, 26D15, 26D20.

Keywords: Hermite-Hadamard inequality, Hölder's inequality, power mean inequality, higher order strongly m-polynomial exponentially type convex functions, Bessel functions, bounded functions.

1. Introduction

A set $\mathrm{T} \subset \mathbb{R}$ ($\mathbb{R}$ represents the set of real numbers) is said to be convex, if

$$\vartheta \flat_1 + (1-\vartheta)\flat_2 \in \mathrm{T}, \quad \forall \flat_1, \flat_2 \in \mathrm{T} \text{ and } \vartheta \in [0,1].$$

A function $\hbar : \mathrm{T} \to \mathbb{R}$ is called convex, if

$$\hbar(\vartheta \flat_1 + (1-\vartheta)\flat_2) \leq \vartheta \hbar(\flat_1) + (1-\vartheta)\hbar(\flat_2), \quad \forall \flat_1, \flat_2 \in \mathrm{T} \text{ and } \vartheta \in [0,1]. \tag{1.1}$$

Moreover, $\hbar$ is concave whenever $-\hbar$ is convex.

Received 26 January 2025; Accepted 21 July 2025.

For the convex function the Hermite-Hadamard type integral inequality (H-H), is given by [5]:

$$\hbar\left(\frac{\flat_1+\flat_2}{2}\right) \leq \frac{1}{\flat_2-\flat_1}\int_{\flat_1}^{\flat_2} \hbar(x)dx \leq \frac{\hbar(\flat_1)+\hbar(\flat_2)}{2}. \tag{1.2}$$

The H-H integral inequality (1.2) has been applied to different types of convex functions (see [3, 4, 9, 12, 13]).

Now, we recall some definitions of convex type functions.

Definition 1.1. [2] *A function $\hbar : \mathrm{T} \subseteq \mathbb{R} \to \mathbb{R}$ is said to be exponentially convex, if*

$$\hbar(\vartheta\flat_1 + (1-\vartheta)\flat_2) \leq \vartheta\frac{\hbar(\flat_1)}{e^{\varsigma\flat_1}} + (1-\vartheta)\frac{\hbar(\flat_2)}{e^{\varsigma\flat_2}} \tag{1.3}$$

holds for all $\flat_1, \flat_2 \in \mathrm{T}$, $\vartheta \in [0,1]$ and $\varsigma \in \mathbb{R}$.

Toply *et al.* [10] introduced the class of m-polynomial convex functions as follows:

Definition 1.2. *Let $m \in \mathrm{N}$. A function $\hbar : \mathrm{T} \subseteq \mathbb{R} \to \mathbb{R}$ is said to be m-polynomial convex, if*

$$\hbar(\vartheta\flat_1 + (1-\vartheta)\flat_2) \leq \frac{1}{m}\sum_{\imath=1}^{m}\left[1-(1-\vartheta)^\imath\right]\hbar(\flat_1) + \frac{1}{m}\sum_{\imath=1}^{m}\left[1-\vartheta^\imath\right]\hbar(\flat_2) \tag{1.4}$$

holds for all $\flat_1, \flat_2 \in \mathrm{T}$ and $\vartheta \in [0,1]$.

With the help of the above definitions, we introduce the following definition.

Definition 1.3. *Let $m \in \mathrm{N}$ and $\varsigma \in \mathbb{R}$. The function $\hbar : \mathrm{T} \to \mathbb{R}$ is called higher order strongly m-polynomial exponentially type convex, if there exists a constant $\zeta > 0$, such that*

$$\begin{aligned}\hbar(\vartheta\flat_1 + (1-\vartheta)\flat_2) \leq{}& \frac{1}{m}\sum_{\imath=1}^{m}\left[1-(1-\vartheta)^\imath\right]\frac{\hbar(\flat_1)}{e^{\varsigma\flat_1}} + \frac{1}{m}\sum_{\imath=1}^{m}\left[1-\vartheta^\imath\right]\frac{\hbar(\flat_2)}{e^{\varsigma\flat_2}} \\ &- \zeta\left[\vartheta^p(1-\vartheta)+\vartheta(1-\vartheta)^p\right]|\flat_2-\flat_1|^p\end{aligned} \tag{1.5}$$

holds for every $\flat_1, \flat_2 \in \mathrm{T}$, $\vartheta \in [0,1]$ and $p \geq 1$.

Remark 1.4. From Definition 1.3, we can observe that:

1. If $m = 1$ and $\zeta \to 0^+$, then Definition 1.3 reduces to Definition 1.1.
2. If $\varsigma = 0$ and $\zeta \to 0^+$, then Definition 1.3 reduces to Definition 1.2.

Definition 1.5. *Let $\ell > 0$, $\flat_1 < \flat_2$ and $\hbar \in \mathcal{L}[\flat_1, \flat_2]$. Then the Riemann–Liouville fractional integrals (R-L) of order ℓ are defined by*

$$\mathcal{J}^{\ell}_{\flat_1^+}\hbar(x) = \frac{1}{\Gamma(\ell)}\int_{\flat_1}^{x}(x-\vartheta)^{\ell-1}\hbar(\vartheta)d\vartheta, \quad \flat_1 < x$$

and

$$\mathcal{J}^{\ell}_{\flat_2^-}\hbar(x) = \frac{1}{\Gamma(\ell)}\int_{x}^{\flat_2}(\vartheta-x)^{\ell-1}\hbar(\vartheta)d\vartheta, \quad \flat_2 > x,$$

where $\Gamma(\cdot)$ is the gamma function.

The H-H type integral inequalities are involved in fractional calculus models and they has been applied for different types of convex functions (see [1, 6, 7]).

Motivated from above literatures our paper is organized as follows: In Section 2, we introduce the higher order strongly m-polynomial exponentially type convex function as a new class of convex functions with its algebraic properties. In Section 3, we derive new integral inequality of H-H by using the new introduced definition. In Section 4, we derive a generalized fractional identity and some related inequalities for the higher order strongly m-polynomial exponentially type convex functions. In Section 5, we give some applications of the Bessel functions and bounded functions to support the main results from previous section. Finally, conclusions and future research are drawn in Section 6.

2. Algebraic properties

Here we derive some algebraic properties of our new defined convex function.

Theorem 2.1. *Let $m \in \mathrm{N}$ and $\varsigma \in \mathbb{R}$. Assume that $\hbar$, $\hbar_1$, $\hbar_2 : \mathrm{T} \to \mathbb{R}$ are three higher order strongly m-polynomial exponentially type convex functions with respect to the constants, ζ, ζ_1 and ζ_2, respectively, then*

1. *$\hbar_1 + \hbar_2$ is higher order strongly m-polynomial exponentially type convex function, with respect to the constant $\zeta_1 + \zeta_2$.*
2. *For nonnegative real number c, $c\hbar$ is higher order strongly m-polynomial exponentially type convex function, with respect to the constant $c\zeta$.*

Proof. The proof is evident, so we omit here. □

Theorem 2.2. *Let $m \in \mathrm{N}, \varsigma \in \mathbb{R}$ and $\mathcal{U} = \{\varpi \in [\flat_1, \flat_2] : \hbar(\varpi) < +\infty\}$. Assume that $\hbar_j : [\flat_1, \flat_2] \to \mathbb{R}$ is a family of higher order strongly m-polynomial exponentially type convex functions with respect to the constant $\zeta > 0$ and $\hbar(\varpi) := \sup_j \hbar_j(\varpi)$. Then, $\hbar$ is an higher order strongly m-polynomial exponentially type convex function with respect to the constant ζ on $\mathcal{U}$.*

Proof. Let $\flat_1, \flat_2 \in \mathcal{U}$ and $\vartheta \in [0, 1]$, then we have

$$\begin{aligned}
&\hbar(\vartheta\flat_1 + (1-\vartheta)\flat_2) = \sup_j \hbar_j(\vartheta\flat_1 + (1-\vartheta)\flat_2) \\
&\le \frac{1}{m}\sum_{\imath=1}^{m}[1-(1-\vartheta)^\imath]\,\frac{\sup_j \hbar_j(\flat_1)}{e^{\varsigma\flat_1}} + \frac{1}{m}\sum_{\imath=1}^{m}(1-\vartheta^\imath)\,\frac{\sup_j \hbar_j(\flat_2)}{e^{\varsigma\flat_2}} \\
&\quad - \zeta\left[\vartheta^p(1-\vartheta) + \vartheta(1-\vartheta)^p\right]|\flat_2 - \flat_1|^p \\
&= \frac{1}{m}\sum_{\imath=1}^{m}[1-(1-\vartheta)^\imath]\,\frac{\hbar(\flat_1)}{e^{\varsigma\flat_1}} + \frac{1}{m}\sum_{\imath=1}^{m}(1-\vartheta^\imath)\,\frac{\hbar(\flat_2)}{e^{\varsigma\flat_2}} \\
&\quad - \zeta\left[\vartheta^p(1-\vartheta) + \vartheta(1-\vartheta)^p\right]|\flat_2 - \flat_1|^p < +\infty,
\end{aligned}$$

which completes the proof. □

3. Main results

The aim of this section is to find some fractional integral inequalities of H-H type for higher order strongly m-polynomial exponentially type convex functions.

Theorem 3.1. *Let $\ell > 0$, $m \in \mathrm{N}$ and $\hbar : [\flat_1, \flat_2] \to \mathbb{R}$ be a higher order strongly m-polynomial exponentially type convex function with respect to the constant $\zeta > 0$. If $\hbar \in \mathcal{L}[\flat_1, \flat_2]$ and $\varsigma \in \mathbb{R}$, then we have*

$$m\left(\frac{2^m}{2^m(m-1)+1}\right)\left[\frac{1}{\ell}\hbar\left(\frac{\flat_1+\flat_2}{2}\right) - \frac{\zeta}{2^{p+\ell}}(\flat_2-\flat_1)^p\big(\beta(p+1,\ell)+\mathfrak{I}(p,\ell)\big)\right] \tag{3.1}$$

$$\leq \frac{1}{(\flat_2-\flat_1)^\ell}\left[A^\ell_{\hbar,1}(\varsigma;\flat_1,\flat_2) + A^\ell_{\hbar,2}(\varsigma;\flat_1,\flat_2)\right]$$

$$\leq \frac{1}{(\flat_2-\flat_1)^\ell}\Bigg\{\left[B^\ell_{1,m}(\varsigma;\flat_1,\flat_2) + B^\ell_{4,m}(\varsigma;\flat_1,\flat_2)\right]\frac{\hbar(\flat_1)}{e^{\varsigma\flat_1}}$$

$$+\left[B^\ell_{2,m}(\varsigma;\flat_1,\flat_2) + B^\ell_{3,m}(\varsigma;\flat_1,\flat_2)\right]\frac{\hbar(\flat_2)}{e^{\varsigma\flat_2}}\Bigg\}$$

$$-\frac{\zeta}{(\flat_2-\flat_1)^{\ell+1}}\left[C^{\ell,\varsigma}_1(\flat_2,\flat_1,p) + C^{\ell,\varsigma}_2(\flat_2,\flat_1,p)\right],$$

where

$$A^\ell_{\hbar,1}(\varsigma;\flat_1,\flat_2) := \int_{\flat_1}^{\flat_2}(\flat_2-x)^{\ell-1}\frac{\hbar(x)}{e^{\varsigma x}}dx, \quad A^\ell_{\hbar,2}(\varsigma;\flat_1,\flat_2) := \int_{\flat_1}^{\flat_2}(x-\flat_1)^{\ell-1}\frac{\hbar(x)}{e^{\varsigma x}}dx$$

and

$$\mathfrak{I}(p,\ell) := \int_0^1 \vartheta^p(1+\vartheta)^{\ell-1}d\vartheta,$$

$$B^\ell_{1,m}(\varsigma;\flat_1,\flat_2) := \frac{1}{m}\sum_{\imath=1}^m\int_{\flat_1}^{\flat_2}\frac{(\flat_2-x)^{\ell-1}}{e^{\varsigma x}}\left[1-\left(\frac{x-\flat_1}{\flat_2-\flat_1}\right)^\imath\right]dx,$$

$$B^\ell_{2,m}(\varsigma;\flat_1,\flat_2) := \frac{1}{m}\sum_{\imath=1}^m\int_{\flat_1}^{\flat_2}\frac{(\flat_2-x)^{\ell-1}}{e^{\varsigma x}}\left[1-\left(\frac{\flat_2-x}{\flat_2-\flat_1}\right)^\imath\right]dx,$$

$$B^\ell_{3,m}(\varsigma;\flat_1,\flat_2) := \frac{1}{m}\sum_{\imath=1}^m\int_{\flat_1}^{\flat_2}\frac{(x-\flat_1)^{\ell-1}}{e^{\varsigma x}}\left[1-\left(\frac{\flat_2-x}{\flat_2-\flat_1}\right)^\imath\right]dx,$$

$$B^\ell_{4,m}(\varsigma;\flat_1,\flat_2) := \frac{1}{m}\sum_{\imath=1}^m\int_{\flat_1}^{\flat_2}\frac{(x-\flat_1)^{\ell-1}}{e^{\varsigma x}}\left[1-\left(\frac{x-\flat_1}{\flat_2-\flat_1}\right)^\imath\right]dx,$$

$$C^{\ell,\varsigma}_1(\flat_1,\flat_2,p) := \int_{\flat_1}^{\flat_2}\frac{(x-\flat_1)^{\ell-1}}{e^{\varsigma x}}\Big[(x-\flat_1)(\flat_2-x)^p + (x-\flat_1)^p(\flat_2-x)\Big]dx,$$

$$C^{\ell,\varsigma}_2(\flat_1,\flat_2,p) := \int_{\flat_1}^{\flat_2}\frac{(\flat_2-x)^{\ell-1}}{e^{\varsigma x}}\Big[(x-\flat_1)^p(\flat_2-x) + (x-\flat_1)(\flat_2-x)^p\Big]dx.$$

Here, $\beta(\cdot,\cdot)$ is the beta function.

Proof. Let $x, y \in [\flat_1, \flat_2]$. Applying definition of higher order strongly m-polynomial exponentially type convex function with respect to the constant $\zeta > 0$ of $\hbar$ on $[\flat_1, \flat_2]$ and taking $\vartheta = 1/2$, we have

$$\hbar\left(\frac{x+y}{2}\right) \leq \frac{1}{m}\sum_{\imath=1}^{m}\left[1-\left(\frac{1}{2}\right)^{\imath}\right]\left[\frac{\hbar(x)}{e^{\varsigma x}}+\frac{\hbar(y)}{e^{\varsigma y}}\right]-\frac{\zeta}{2^p}|y-x|^p. \tag{3.2}$$

By making use of inequality (3.2) with $x = \vartheta\flat_2 + (1-\vartheta)\flat_1$ and $y = \vartheta\flat_1 + (1-\vartheta)\flat_2$, we get

$$\hbar\left(\frac{\flat_1+\flat_2}{2}\right) \leq \frac{1}{m}\sum_{\imath=1}^{m}\left[1-\left(\frac{1}{2}\right)^{\imath}\right]\left[\frac{\hbar(\vartheta\flat_2+(1-\vartheta)\flat_1)}{e^{\varsigma(\vartheta\flat_2+(1-\vartheta)\flat_1)}}+\frac{\hbar(\vartheta\flat_1+(1-\vartheta)\flat_2)}{e^{\varsigma(\vartheta\flat_1+(1-\vartheta)\flat_2)}}\right] \tag{3.3}$$
$$-\frac{\zeta}{2^p}(\flat_2-\flat_1)^p|1-2\vartheta|^p.$$

Multiplying both sides of (3.3) by $\vartheta^{\ell-1}$ and integrating the result with respect to ϑ over $[0,1]$, we obtain

$$\begin{aligned}
&\frac{1}{\ell}\hbar\left(\frac{\flat_1+\flat_2}{2}\right)\\
&\leq \frac{1}{m}\left(m-\frac{2^m-1}{2^m}\right)\left[\int_0^1 \vartheta^{\ell-1}\frac{\hbar(\vartheta\flat_2+(1-\vartheta)\flat_1)}{e^{\varsigma(\vartheta\flat_2+(1-\vartheta)\flat_1)}}d\vartheta+\int_0^1 \vartheta^{\ell-1}\frac{\hbar(\vartheta\flat_1+(1-\vartheta)\flat_2)}{e^{\varsigma(\vartheta\flat_1+(1-\vartheta)\flat_2)}}d\vartheta\right]\\
&\quad-\frac{\zeta}{2^p}(\flat_2-\flat_1)^p\int_0^1\vartheta^{\ell-1}|1-2\vartheta|^p d\vartheta\\
&=\frac{1}{m}\left(m-\frac{2^m-1}{2^m}\right)\frac{1}{(\flat_2-\flat_1)^{\ell}}\left[\int_{\flat_1}^{\flat_2}(\flat_2-x)^{\ell-1}\frac{\hbar(x)}{e^{\varsigma x}}dx+\int_{\flat_1}^{\flat_2}(x-\flat_1)^{\ell-1}\frac{\hbar(x)}{e^{\varsigma x}}dx\right]\\
&\quad-\frac{\zeta}{2^p}(\flat_2-\flat_1)^p\frac{1}{2^{\ell}}\left[\beta(p+1,\ell)+\Im(p,\ell)\right]\\
&=\frac{1}{m}\left(m-\frac{2^m-1}{2^m}\right)\frac{1}{(\flat_2-\flat_1)^{\ell}}\left[A^{\ell}_{\hbar,1}(\varsigma;\flat_1,\flat_2)+A^{\ell}_{\hbar,2}(\varsigma;\flat_1,\flat_2)\right]\\
&\quad-\frac{\zeta}{2^{p+\ell}}(\flat_2-\flat_1)^p\left[\beta(p+1,\ell)+\Im(p,\ell)\right],
\end{aligned}$$

which gives the left inequality of (3.5). In order to prove the right inequality of (3.5), we use the definition of higher order strongly m-polynomial exponentially type convex function with respect to the constant $\zeta > 0$ of $\hbar$ to get

$$\begin{aligned}
\frac{\hbar(\vartheta\flat_2+(1-\vartheta)\flat_1)}{e^{\varsigma(\vartheta\flat_2+(1-\vartheta)\flat_1)}} &\leq \frac{1}{e^{\varsigma(\vartheta\flat_2+(1-\vartheta)\flat_1)}}\\
&\times\left\{\frac{1}{m}\sum_{\imath=1}^{m}\left[1-\vartheta^{\imath}\right]\frac{\hbar(\flat_1)}{e^{\varsigma\flat_1}}+\frac{1}{m}\sum_{\imath=1}^{m}\left[1-(1-\vartheta)^{\imath}\right]\frac{\hbar(\flat_2)}{e^{\varsigma\flat_2}}\right.\\
&\left.-\zeta\left[\vartheta^p(1-\vartheta)+\vartheta(1-\vartheta)^p\right]|\flat_2-\flat_1|^p\right\}.
\end{aligned}$$

Similarly,

$$\begin{aligned}\frac{\hbar(\vartheta\flat_1+(1-\vartheta)\flat_2)}{e^{\varsigma(\vartheta\flat_1+(1-\vartheta)\flat_2)}} &\leq \frac{1}{e^{\varsigma(\vartheta\flat_1+(1-\vartheta)\flat_2)}}\\ &\times\left\{\frac{1}{m}\sum_{\imath=1}^{m}\left[1-\vartheta^{\imath}\right]\frac{\hbar(\flat_2)}{e^{\varsigma\flat_2}}+\frac{1}{m}\sum_{\imath=1}^{m}\left[1-(1-\vartheta)^{\imath}\right]\frac{\hbar(\flat_1)}{e^{\varsigma\flat_1}}\right.\\ &-\zeta\left[\vartheta^{p}(1-\vartheta)+\vartheta(1-\vartheta)^{p}\right]|\flat_2-\flat_1|^{p},\end{aligned}$$

where $\vartheta \in [0,1]$. Then, by adding the above inequalities, we have

$$\begin{aligned}&\frac{\hbar(\vartheta\flat_2+(1-\vartheta)\flat_1)}{e^{\varsigma(\vartheta\flat_2+(1-\vartheta)\flat_1)}}+\frac{\hbar(\vartheta\flat_1+(1-\vartheta)\flat_2)}{e^{\varsigma(\vartheta\flat_1+(1-\vartheta)\flat_2)}}\\ &\leq \frac{1}{e^{\varsigma(\vartheta\flat_2+(1-\vartheta)\flat_1)}}\\ &\quad\times\left\{\frac{1}{m}\sum_{\imath=1}^{m}\left[1-\vartheta^{\imath}\right]\frac{\hbar(\flat_1)}{e^{\varsigma\flat_1}}+\frac{1}{m}\sum_{\imath=1}^{m}\left[1-(1-\vartheta)^{\imath}\right]\frac{\hbar(\flat_2)}{e^{\varsigma\flat_2}}\right.\\ &\quad-\zeta\left[\vartheta^{p}(1-\vartheta)+\vartheta(1-\vartheta)^{p}\right]|\flat_2-\flat_1|^{p}+\frac{1}{e^{\varsigma(\vartheta\flat_1+(1-\vartheta)\flat_2)}}\\ &\quad\times\left\{\frac{1}{m}\sum_{\imath=1}^{m}\left[1-\vartheta^{\imath}\right]\frac{\hbar(\flat_2)}{e^{\varsigma\flat_2}}+\frac{1}{m}\sum_{\imath=1}^{m}\left[1-(1-\vartheta)^{\imath}\right]\frac{\hbar(\flat_1)}{e^{\varsigma\flat_1}}\right.\\ &\quad-\zeta\left[\vartheta^{p}(1-\vartheta)+\vartheta(1-\vartheta)^{p}\right]|\flat_2-\flat_1|^{p}.\end{aligned}\tag{3.4}$$

Multiplying both sides of (3.4) by $\vartheta^{\ell-1}$ and integrating the obtained inequality with respect to ϑ from 0 to 1 and making the change of the variables, we get

$$\begin{aligned}&\int_0^1\vartheta^{\ell-1}\frac{\hbar(\vartheta\flat_2+(1-\vartheta)\flat_1)}{e^{\varsigma(\vartheta\flat_2+(1-\vartheta)\flat_1)}}d\vartheta+\int_0^1\vartheta^{\ell-1}\frac{\hbar(\vartheta\flat_2+(1-\vartheta)\flat_1)}{e^{\varsigma(\vartheta\flat_1+(1-\vartheta)\flat_2)}}d\vartheta\\ &\leq\int_0^1\vartheta^{\ell-1}\frac{1}{e^{\varsigma(\vartheta\flat_2+(1-\vartheta)\flat_1)}}\\ &\times\left\{\frac{1}{m}\sum_{\imath=1}^{m}\left(1-\vartheta^{\imath}\right)\frac{\hbar(\flat_1)}{e^{\varsigma\flat_1}}+\frac{1}{m}\sum_{\imath=1}^{m}\left[1-(1-\vartheta)^{\imath}\right]\frac{\hbar(\flat_2)}{e^{\varsigma\flat_2}}\right.\\ &-\zeta\left[\vartheta^{p}(1-\vartheta)+\vartheta(1-\vartheta)^{p}\right]|\flat_2-\flat_1|^{p}d\vartheta+\int_0^1\vartheta^{\ell-1}\frac{1}{e^{\varsigma(\vartheta\flat_1+(1-\vartheta)\flat_2)}}\\ &\times\left\{\frac{1}{m}\sum_{\imath=1}^{m}\left(1-\vartheta^{\imath}\right)\frac{\hbar(\flat_2)}{e^{\varsigma\flat_2}}+\frac{1}{m}\sum_{\imath=1}^{m}\left[1-(1-\vartheta)^{\imath}\right]\frac{\hbar(\flat_1)}{e^{\varsigma\flat_1}}\right.\\ &-\zeta\left[\vartheta^{p}(1-\vartheta)+\vartheta(1-\vartheta)^{p}\right]|\flat_2-\flat_1|^{p}d\vartheta.\end{aligned}$$

By simplifying it, we obtain

$$\begin{aligned}&\frac{1}{(\flat_2-\flat_1)^{\ell}}\left[A^{\ell}_{\hbar,1}(\varsigma;\flat_1,\flat_2)+A^{\ell}_{\hbar,2}(\varsigma;\flat_1,\flat_2)\right]\\ &\leq\frac{1}{(\flat_2-\flat_1)^{\ell}}\left\{\left[B^{\ell}_{1,m}(\varsigma;\flat_1,\flat_2)+B^{\ell}_{4,m}(\varsigma;\flat_1,\flat_2)\right]\frac{\hbar(\flat_1)}{e^{\varsigma\flat_1}}\right.\end{aligned}$$

$$+\left[B_{2,m}^{\ell}(\varsigma;\flat_1,\flat_2)+B_{3,m}^{\ell}(\varsigma;\flat_1,\flat_2)\right]\frac{\hbar(\flat_2)}{e^{\varsigma\flat_2}}\Bigg\}$$
$$-\frac{\zeta}{(\flat_2-\flat_1)^{\ell+1}}\left[C_1^{\ell,\varsigma}(\flat_1,\flat_2,p)+C_2^{\ell,\varsigma}(\flat_1,\flat_2,p)\right].$$

The proof of Theorem 3.1 is completed. □

Corollary 3.2. *Theorem 3.1 with* $\varsigma=0$ *becomes*

$$m\left(\frac{2^m}{2^m(m-1)+1}\right)\left[\frac{1}{\ell}\hbar\left(\frac{\flat_1+\flat_2}{2}\right)-\frac{\zeta}{2^{p+\ell}}(\flat_2-\flat_1)^p\big(\beta(p+1,\ell)+\Im(p,\ell)\big)\right] \tag{3.5}$$
$$\leq\frac{1}{(\flat_2-\flat_1)^{\ell}}\left[A_{\hbar,1}^{\ell}(\flat_1,\flat_2)+A_{\hbar,2}^{\ell}(\flat_1,\flat_2)\right]$$
$$\leq\frac{1}{(\flat_2-\flat_1)^{\ell}}\Bigg\{\left[B_{1,m}^{\ell}(\flat_1,\flat_2)+B_{4,m}^{\ell}(\flat_1,\flat_2)\right]\hbar(\flat_1)$$
$$+\left[B_{2,m}^{\ell}(\flat_1,\flat_2)+B_{3,m}^{\ell}(\flat_1,\flat_2)\right]\hbar(\flat_2)\Bigg\}$$
$$-\frac{\zeta}{(\flat_2-\flat_1)^{\ell+1}}\left[C_1^{\ell}(\flat_2,\flat_1,p)+C_2^{\ell}(\flat_2,\flat_1,p)\right],$$

where

$$A_{\hbar,1}^{\ell}(\flat_1,\flat_2):=\int_{\flat_1}^{\flat_2}(\flat_2-x)^{\ell-1}\hbar(x)dx,\quad A_{\hbar,2}^{\ell}(\flat_1,\flat_2):=\int_{\flat_1}^{\flat_2}(x-\flat_1)^{\ell-1}\hbar(x)dx$$

and

$$B_{1,m}^{\ell}(\flat_1,\flat_2):=\frac{1}{m}\sum_{\imath=1}^{m}\int_{\flat_1}^{\flat_2}(\flat_2-x)^{\ell-1}\left[1-\left(\frac{x-\flat_1}{\flat_2-\flat_1}\right)^{\imath}\right]dx,$$
$$B_{2,m}^{\ell}(\flat_1,\flat_2):=\frac{1}{m}\sum_{\imath=1}^{m}\int_{\flat_1}^{\flat_2}(\flat_2-x)^{\ell-1}\left[1-\left(\frac{\flat_2-x}{\flat_2-\flat_1}\right)^{\imath}\right]dx,$$
$$B_{3,m}^{\ell}(\flat_1,\flat_2):=\frac{1}{m}\sum_{\imath=1}^{m}\int_{\flat_1}^{\flat_2}(x-\flat_1)^{\ell-1}\left[1-\left(\frac{\flat_2-x}{\flat_2-\flat_1}\right)^{\imath}\right]dx,$$
$$B_{4,m}^{\ell}(\flat_1,\flat_2):=\frac{1}{m}\sum_{\imath=1}^{m}\int_{\flat_1}^{\flat_2}(x-\flat_1)^{\ell-1}\left[1-\left(\frac{x-\flat_1}{\flat_2-\flat_1}\right)^{\imath}\right]dx,$$
$$C_1^{\ell}(\flat_1,\flat_2,p):=\int_{\flat_1}^{\flat_2}(x-\flat_1)^{\ell-1}\Big[((x-\flat_1)(\flat_2-x))^p+(x-\flat_1)^p(\flat_2-x)\Big]dx,$$
$$C_2^{\ell}(\flat_1,\flat_2,p):=\int_{\flat_1}^{\flat_2}(\flat_2-x)^{\ell-1}\Big[(x-\flat_1)^p(\flat_2-x))+(x-\flat_1)(\flat_2-x)^p\Big]dx.$$

Corollary 3.3. *Theorem 3.1 with $\ell = 1$ leads to*

$$m\left(\frac{2^m}{2^m(m-1)+1}\right)\left[\hbar\left(\frac{\flat_1+\flat_2}{2}\right)-\frac{\zeta}{(p+1)2^p}(\flat_2-\flat_1)^p\right]$$
$$\leq \frac{1}{(\flat_2-\flat_1)}\left[A^1_{\hbar,1}(\varsigma;\flat_1,\flat_2)+A^1_{\hbar,2}(\varsigma;\flat_1,\flat_2)\right]$$
$$\leq \frac{1}{(\flat_2-\flat_1)}\Bigg\{\left[B^1_{1,m}(\varsigma;\flat_1,\flat_2)+B^1_{4,m}(\varsigma;\flat_1,\flat_2)\right]\frac{\hbar(\flat_1)}{e^{\varsigma\flat_1}}$$
$$+\left[B^1_{2,m}(\varsigma;\flat_1,\flat_2)+B^1_{3,m}(\varsigma;\flat_1,\flat_2)\right]\frac{\hbar(\flat_2)}{e^{\varsigma\flat_2}}\Bigg\}$$
$$-\frac{\zeta}{(\flat_2-\flat_1)^2}\left[C^{1,\varsigma}_1(\flat_2,\flat_1,p)+C^{1,\varsigma}_2(\flat_2,\flat_1,p)\right].$$

Corollary 3.4. *Letting $\zeta \to 0^+$ in Theorem 3.1, we have*

$$\frac{m}{\ell}\left(\frac{2^m}{2^m(m-1)+1}\right)\hbar\left(\frac{\flat_1+\flat_2}{2}\right)$$
$$\leq \frac{1}{(\flat_2-\flat_1)^\ell}\left[A^\ell_{\hbar,1}(\varsigma;\flat_1,\flat_2)+A^\ell_{\hbar,2}(\varsigma;\flat_1,\flat_2)\right]$$
$$\leq \frac{1}{(\flat_2-\flat_1)^\ell}\Bigg\{\left[B^\ell_{1,m}(\varsigma;\flat_1,\flat_2)+B^\ell_{4,m}(\varsigma;\flat_1,\flat_2)\right]\frac{\hbar(\flat_1)}{e^{\varsigma\flat_1}}$$
$$+\left[B^\ell_{2,m}(\varsigma;\flat_1,\flat_2)+B^\ell_{3,m}(\varsigma;\flat_1,\flat_2)\right]\frac{\hbar(\flat_2)}{e^{\varsigma\flat_2}}\Bigg\}.$$

Corollary 3.5. *Theorem 3.1 with $\varsigma = 0, \ell = 1$ and $\zeta \to 0^+$ becomes* [10, Theorem 4].

4. Further results

We need the following lemma in order to proceed with our next results.

Lemma 4.1. *Let $\hbar : \mathrm{T} \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on T with $\flat_1, \flat_2 \in \mathrm{T}$ and $\flat_1 < \flat_2$. Also, let $\ell > 0$ and $m \in \mathbb{N}$. If $\hbar' \in \mathcal{L}[\flat_1, \flat_2]$, then we have*

$$\mathrm{Q}^\ell_m(\hbar;\flat_1,\flat_2) := \left(\frac{\flat_2-\flat_1}{4m}\right)\Gamma(\ell+1)$$
$$\times\sum_{\jmath=0}^{m-1}\Bigg\{\left(\frac{2m}{\flat_2-\flat_1}\right)^{\ell+1}\mathcal{J}^\ell_{\left(\frac{(2(m-\jmath)-1)\flat_1+(2\jmath+1)\flat_2}{2m}\right)^-}\hbar\left(\frac{(m-\jmath)\flat_1+\jmath\flat_2}{m}\right)$$
$$-\left(\frac{2m}{\flat_1-\flat_2}\right)^{\ell+1}\mathcal{J}^\ell_{\left(\frac{(2(m-\jmath)-1)\flat_1+(2\jmath+1)\flat_2}{2m}\right)^-}\hbar\left(\frac{(m-\jmath-1)\flat_1+(\jmath+1)\flat_2}{m}\right)\Bigg\}$$
$$-\sum_{\jmath=0}^{m-1}\hbar\left(\frac{(2(m-\jmath)-1)\flat_1+(2\jmath+1)\flat_2}{2m}\right)$$

$$= \left(\frac{\flat_2-\flat_1}{4m}\right) \cdot \sum_{\jmath=0}^{m-1} \left\{ \int_0^1 \vartheta^\ell \hbar' \left(\frac{\vartheta}{2}\frac{(m-\jmath)\flat_1+\jmath\flat_2}{m} + \frac{(2-\vartheta)}{2}\frac{(m-\jmath-1)\flat_1+(\jmath+1)\flat_2}{m}\right) d\vartheta - \int_0^1 \vartheta^\ell \hbar' \left(\frac{\vartheta}{2}\frac{(m-\jmath-1)\flat_1+(\jmath+1)\flat_2}{m} + \frac{(2-\vartheta)}{2}\frac{(m-\jmath)\flat_1+\jmath\flat_2}{m}\right) d\vartheta \right\}. \tag{4.1}$$

Proof. Setting

$$\mathcal{J}_1 := \int_0^1 \vartheta^\ell \hbar' \left(\frac{\vartheta}{2}\frac{(m-\jmath)\flat_1+\jmath\flat_2}{m} + \frac{(2-\vartheta)}{2}\frac{(m-\jmath-1)\flat_1+(\jmath+1)\flat_2}{m}\right) d\vartheta, \tag{4.2}$$

and

$$\mathcal{J}_2 := \int_0^1 \vartheta^\ell \hbar' \left(\frac{\vartheta}{2}\frac{(m-\jmath-1)\flat_1+(\jmath+1)\flat_2}{m} + \frac{(2-\vartheta)}{2}\frac{(m-\jmath)\flat_1+\jmath\flat_2}{m}\right) d\vartheta. \tag{4.3}$$

By applying integration by parts on equality (4.2), we have

$$\begin{aligned}\mathcal{J}_1 &= \left(\frac{2m}{\flat_1-\flat_2}\right)\left[\vartheta^\ell \hbar \left(\frac{\vartheta}{2}\frac{(m-\jmath)\flat_1+\jmath\flat_2}{m} + \frac{(2-\vartheta)}{2}\frac{(m-\jmath-1)\flat_1+(\jmath+1)\flat_2}{m}\right)\Bigg|_0^1 \right.\\ &\quad \left. - \ell \int_0^1 \vartheta^{\ell-1} \hbar \left(\frac{\vartheta}{2}\frac{(m-\jmath)\flat_1+\jmath\flat_2}{m} + \frac{(2-\vartheta)}{2}\frac{(m-\jmath-1)\flat_1+(\jmath+1)\flat_2}{m}\right) d\vartheta\right]\\ &= \left(\frac{2m}{\flat_1-\flat_2}\right)\left[\hbar\left(\frac{(2(m-\jmath)-1)\flat_1+(2\jmath+1)\flat_2}{2m}\right)\right.\\ &\quad \left. - \left(\frac{2m}{\flat_1-\flat_2}\right)^\ell \Gamma(\ell+1)\, \mathcal{J}^\ell_{\left(\frac{(2(m-\jmath)-1)\flat_1+(2\jmath+1)\flat_2}{2m}\right)^-} \hbar\left(\frac{(m-\jmath-1)\flat_1+(\jmath+1)\flat_2}{m}\right)\right].\end{aligned} \tag{4.4}$$

Similarly, from equality (4.3), we obtain

$$\begin{aligned}\mathcal{J}_2 &= \left(\frac{2m}{\flat_2-\flat_1}\right)\left[\hbar\left(\frac{(2(m-\jmath)-1)\flat_1+(2\jmath+1)\flat_2}{2m}\right)\right.\\ &\quad \left. - \left(\frac{2m}{\flat_2-\flat_1}\right)^\ell \Gamma(\ell+1)\, \mathcal{J}^\ell_{\left(\frac{(2(m-\jmath)-1)\flat_1+(2\jmath+1)\flat_2}{2m}\right)^-} \hbar\left(\frac{(m-\jmath)\flat_1+\jmath\flat_2}{m}\right)\right],\end{aligned} \tag{4.5}$$

for all $\jmath = 0, 1, 2, \ldots, m-1$. Then, by subtracting equality (4.5) from (4.4), multiplying by the factor $\left(\frac{\flat_2-\flat_1}{4m}\right)$ and summing over $\jmath$ from 0 to $m-1$, we can easily attain the desired identity (4.1). □

Remark 4.2. Lemma 4.1 with $m = 1$ leads to

$$\frac{2^{\ell-1}\Gamma(\ell+1)}{(\flat_2-\flat_1)^\ell}\left\{\mathcal{J}^\ell_{\left(\frac{\flat_1+\flat_2}{2}\right)^+}\hbar(\flat_2) + \mathcal{J}^\ell_{\left(\frac{\flat_1+\flat_2}{2}\right)^-}\hbar(\flat_1)\right\} - \hbar\left(\frac{\flat_1+\flat_2}{2}\right)$$

$$= \frac{(\flat_2-\flat_1)}{4}\left\{\int_0^1 \vartheta^\ell \hbar'\left(\frac{\vartheta}{2}\flat_1+\frac{(2-\vartheta)}{2}\flat_2\right)d\vartheta-\int_0^1 \vartheta^\ell \hbar'\left(\frac{\vartheta}{2}\flat_2+\frac{(2-\vartheta)}{2}\flat_1\right)d\vartheta\right\}, \tag{4.6}$$

which is established in [8, Lemma 3].

Throughout the rest of this study, we consider

$$\mathbf{v}_{m,\jmath} := \frac{(m-\jmath)\flat_1+\jmath\flat_2}{m} \quad \text{and} \quad \mathbf{v}_{m,\jmath+1} := \frac{(m-\jmath-1)\flat_1+(\jmath+1)\flat_2}{m}.$$

Theorem 4.3. *Let $\ell > 0$, $m \in \mathrm{N}$ and $\hbar : [\flat_1,\flat_2] \to \mathbb{R}$ be a differentiable function on $(\flat_1,\flat_2)$ such that $\hbar' \in \mathcal{L}[\flat_1,\flat_2]$. If $|\hbar'|$ is a higher order strongly m-polynomial exponentially type convex function with respect to the constant $\zeta > 0$ on $[\flat_1,\flat_2]$ and $\varsigma \in \mathbb{R}$, then we have*

$$\left|\mathrm{Q}_m^\ell(\hbar;\flat_1,\flat_2)\right| \le \left(\frac{\flat_2-\flat_1}{4m}\right)[T_{m,\ell}+M_{m,\ell}]\sum_{\jmath=0}^{m-1}\left[\frac{|\hbar'(\boldsymbol{v}_{m,\jmath})|}{e^{\varsigma \boldsymbol{v}_{m,\jmath}}}+\frac{|\hbar'(\boldsymbol{v}_{m,\jmath+1})|}{e^{\varsigma \boldsymbol{v}_{m,\jmath+1}}}\right] \tag{4.7}$$

$$-\frac{m\zeta}{2^p}\left(\frac{\ell+p+3}{(\ell+p+1)(\ell+p+2)}-2^{\ell+p+2}B_{\frac{1}{2}}(\ell+2,p+1)\right)\left|\frac{\flat_2-\flat_1}{m}\right|^p,$$

where

$$T_{m,\ell} := \frac{1}{m}\sum_{\imath=1}^m \int_0^1 \vartheta^\ell\left[1-\left(1-\frac{\vartheta}{2}\right)^\imath\right]d\vartheta, \quad M_{m,\ell} := \frac{1}{m}\sum_{\imath=1}^m \int_0^1 \vartheta^\ell\left[1-\left(\frac{\vartheta}{2}\right)^\imath\right]d\vartheta.$$

Here, $B_x(\cdot,\cdot)$ is the incomplete beta function for all $0 < x \le 1$.

Proof. By making use of Lemma 4.1 and properties of modulus, we can deduce

$$\left|\mathrm{Q}_m^\ell(\hbar;\flat_1,\flat_2)\right| \le \left(\frac{\flat_2-\flat_1}{4m}\right)$$

$$\times\sum_{\jmath=0}^{m-1}\left\{\int_0^1 \vartheta^\ell\left|\hbar'\left(\frac{\vartheta}{2}\frac{(m-\jmath)\flat_1+\jmath\flat_2}{m}+\frac{(2-\vartheta)}{2}\frac{(m-\jmath-1)\flat_1+(\jmath+1)\flat_2}{m}\right)\right|d\vartheta\right.$$

$$\left.+\int_0^1 \vartheta^\ell\left|\hbar'\left(\frac{\vartheta}{2}\frac{(m-\jmath-1)\flat_1+(\jmath+1)\flat_2}{m}+\frac{(2-\vartheta)}{2}\frac{(m-\jmath)\flat_1+\jmath\flat_2}{m}\right)\right|d\vartheta\right\}.$$

Using the definition of higher order strongly m-polynomial exponentially type convex function with respect to the constant $\zeta > 0$ of $|\hbar'|$, we get

$$\left|\mathrm{Q}_m^\ell(\hbar;\flat_1,\flat_2)\right| \le \left(\frac{\flat_2-\flat_1}{4m}\right)\sum_{\jmath=0}^{m-1}\left\{\int_0^1 \vartheta^\ell\left[\frac{1}{m}\sum_{\imath=1}^m\left[1-\left(1-\frac{\vartheta}{2}\right)^\imath\right]\frac{|\hbar'(\mathbf{v}_{m,\jmath})|}{e^{\varsigma\mathbf{v}_{m,\jmath}}}\right.\right.$$

$$\left.+\frac{1}{m}\sum_{\imath=1}^m\left[1-\left(\frac{\vartheta}{2}\right)^\imath\right]\frac{|\hbar'(\mathbf{v}_{m,\jmath+1})|}{e^{\varsigma\mathbf{v}_{m,\jmath+1}}}-\zeta\left[(\frac{\vartheta}{2})^p(1-\frac{\vartheta}{2})+\frac{\vartheta}{2}(1-\frac{\vartheta}{2})^p\right]\left|\frac{\flat_2-\flat_1}{m}\right|^p\right]d\vartheta$$

$$+\int_0^1 \vartheta^\ell\left[\frac{1}{m}\sum_{\imath=1}^m\left[1-\left(1-\frac{\vartheta}{2}\right)^\imath\right]\frac{|\hbar'(\mathbf{v}_{m,\jmath+1})|}{e^{\varsigma\mathbf{v}_{m,\jmath+1}}}\right.$$

$$+\frac{1}{m}\sum_{\imath=1}^{m}\left[1-\left(\frac{\vartheta}{2}\right)^{\imath}\right]\frac{|\hbar'(\mathbf{v}_{m,\jmath})|}{e^{\varsigma\mathbf{v}_{m,\jmath}}}-\zeta\left[(\frac{\vartheta}{2})^{p}(1-\frac{\vartheta}{2})+\frac{\vartheta}{2}(1-\frac{\vartheta}{2})^{p}\right]\left|\frac{\flat_2-\flat_1}{m}\right|^{p}\Bigg]d\vartheta\Bigg\}$$

$$=\left(\frac{\flat_2-\flat_1}{4m}\right)\left[T_m^{\ell}+M_m^{\ell}\right]\sum_{\jmath=0}^{m-1}\left[\frac{|\hbar'(\mathbf{v}_{m,\jmath})|}{e^{\varsigma\mathbf{v}_{m,\jmath}}}+\frac{|\hbar'(\mathbf{v}_{m,\jmath+1})|}{e^{\varsigma\mathbf{v}_{m,\jmath+1}}}\right]$$

$$-\frac{m\zeta}{2^{p}}\left(\frac{\ell+p+3}{(\ell+p+1)(\ell+p+2)}-2^{\ell+p+2}B_{\frac{1}{2}}\left(\ell+2,p+1\right)\right)\left|\frac{\flat_2-\flat_1}{m}\right|^{p},$$

which completes the proof. □

Corollary 4.4. *Theorem 4.3 with $\varsigma=0$ leads to*

$$\left|\mathrm{Q}_m^{\ell}(\hbar;\flat_1,\flat_2)\right|\leq\left(\frac{\flat_2-\flat_1}{4m}\right)\left[T_{m,\ell}+M_{m,\ell}\right]\sum_{\jmath=0}^{m-1}\left[|\hbar'(\boldsymbol{v}_{m,\jmath})|+|\hbar'(\boldsymbol{v}_{m,\jmath+1})|\right] \tag{4.8}$$

$$-\frac{m\zeta}{2^{p}}\left(\frac{\ell+p+3}{(\ell+p+1)(\ell+p+2)}-2^{\ell+p+2}B_{\frac{1}{2}}\left(\ell+2,p+1\right)\right)\left|\frac{\flat_2-\flat_1}{m}\right|^{p}.$$

Corollary 4.5. *Theorem 4.3 with $m=1$ leads to*

$$\begin{aligned}&\left|\frac{2^{\ell-1}\Gamma(\ell+1)}{(\flat_2-\flat_1)^{\ell}}\left\{\mathcal{J}^{\ell}_{\left(\frac{\flat_1+\flat_2}{2}\right)^{+}}\hbar(\flat_2)+\mathcal{J}^{\ell}_{\left(\frac{\flat_1+\flat_2}{2}\right)^{-}}\hbar(\flat_1)\right\}-\hbar\left(\frac{\flat_1+\flat_2}{2}\right)\right|\\&\leq\left(\frac{\flat_2-\flat_1}{4(\ell+1)}\right)\left[\frac{|\hbar'(\flat_1)|}{e^{\varsigma\flat_1}}+\frac{|\hbar'(\flat_2)|}{e^{\varsigma\flat_2}}\right]\\&\quad-\frac{\zeta}{2^{p}}\left(\frac{\ell+p+3}{(\ell+p+1)(\ell+p+2)}-2^{\ell+p+2}B_{\frac{1}{2}}\left(\ell+2,p+1\right)\right)|\flat_2-\flat_1|^{p}.\end{aligned} \tag{4.9}$$

Moreover, if $\varsigma=0$ and $\zeta\to0^{+}$, we get

$$\left|\frac{2^{\ell-1}\Gamma(\ell+1)}{(\flat_2-\flat_1)^{\ell}}\left\{\mathcal{J}^{\ell}_{\left(\frac{\flat_1+\flat_2}{2}\right)^{+}}\hbar(\flat_2)+\mathcal{J}^{\ell}_{\left(\frac{\flat_1+\flat_2}{2}\right)^{-}}\hbar(\flat_1)\right\}-\hbar\left(\frac{\flat_1+\flat_2}{2}\right)\right|$$
$$\leq\left(\frac{\flat_2-\flat_1}{4(\ell+1)}\right)\left[|\hbar'(\flat_1)|+|\hbar'(\flat_2)|\right],$$

which is established in the first step of proof of [8, Theorem 5].

Theorem 4.6. *Let $\ell>0$, $m\in\mathrm{N}$ and $\hbar:[\flat_1,\flat_2]\to\mathbb{R}$ be a differentiable function on $(\flat_1,\flat_2)$ such that $\hbar'\in\mathcal{L}[\flat_1,\flat_2]$. If $|\hbar'|^{q}$ is higher order strongly m-polynomial exponentially type convex function with respect to the constant $\zeta>0$ on $[\flat_1,\flat_2]$ and $\varsigma\in\mathbb{R}$, then for $q>1$, and $\frac{1}{q}+\frac{1}{r}=1$, we have*

$$\left|\mathrm{Q}_m^{\ell}(\hbar;\flat_1,\flat_2)\right|\leq\left(\frac{\flat_2-\flat_1}{4m}\right)\left(\frac{1}{\ell r+1}\right)^{\frac{1}{r}}$$
$$\times\sum_{\jmath=0}^{m-1}\left\{\left(R_m\frac{|\hbar'(\boldsymbol{v}_{m,\jmath})|^{q}}{e^{\varsigma\boldsymbol{v}_{m,\jmath}}}+S_m\frac{|\hbar'(\boldsymbol{v}_{m,\jmath+1})|^{q}}{e^{\varsigma\boldsymbol{v}_{m,\jmath+1}}}-\frac{\zeta}{(p+1)(p+2)}\left|\frac{\flat_2-\flat_1}{m}\right|^{p}\right)^{\frac{1}{q}}\right.$$

$$+\left(R_m\frac{|\hbar'(\boldsymbol{v}_{m,\jmath+1})|^q}{e^{\varsigma \boldsymbol{v}_{m,\jmath+1}}}+S_m\frac{|\hbar'(\boldsymbol{v}_{m,\jmath})|^q}{e^{\varsigma \boldsymbol{v}_{m,\jmath}}}-\frac{\zeta}{(p+1)(p+2)}\left|\frac{\flat_2-\flat_1}{m}\right|^p\right)^{\frac{1}{q}}\Bigg\}, \quad (4.10)$$

where

$$R_m:=\frac{1}{m}\sum_{\imath=1}^{m}\int_0^1\left[1-\left(1-\frac{\vartheta}{2}\right)^{\imath}\right]d\vartheta=1+\frac{2}{m}\sum_{\imath=1}^{m}\frac{1}{\imath+1}\left(\frac{1}{2^{\imath+1}}-1\right)$$

and

$$S_m:=\frac{1}{m}\sum_{\imath=1}^{m}\int_0^1\left[1-\left(\frac{\vartheta}{2}\right)^{\imath}\right]d\vartheta=1-\frac{1}{m}\sum_{\imath=1}^{m}\frac{1}{2^{\imath}(\imath+1)}.$$

Proof. By making use of Lemma 4.1, Hölder's inequality and properties of modulus, we can deduce

$$\begin{aligned}\left|\mathrm{Q}_m^{\ell}(\hbar;\flat_1,\flat_2)\right| &\le \left(\frac{\flat_2-\flat_1}{4m}\right)\sum_{\jmath=0}^{m-1}\Bigg\{\int_0^1\vartheta^{\ell}\left|\hbar'\left(\frac{\vartheta}{2}\mathbf{v}_{m,\jmath}+\frac{(2-\vartheta)}{2}\mathbf{v}_{m,\jmath+1}\right)\right|d\vartheta\\
&\quad+\int_0^1\vartheta^{\ell}\left|\hbar'\left(\frac{(2-\vartheta)}{2}\mathbf{v}_{m,\jmath}+\frac{\vartheta}{2}\mathbf{v}_{m,\jmath+1}\right)\right|d\vartheta\Bigg\}\\
&\le \left(\frac{\flat_2-\flat_1}{4m}\right)\left(\int_0^1\vartheta^{r\ell}d\vartheta\right)^{\frac{1}{r}}\\
&\quad\cdot\sum_{\jmath=0}^{m-1}\Bigg\{\left(\int_0^1\left|\hbar'\left(\frac{\vartheta}{2}\mathbf{v}_{m,\jmath}+\frac{(2-\vartheta)}{2}\mathbf{v}_{m,\jmath+1}\right)\right|^q d\vartheta\right)^{\frac{1}{q}}\\
&\quad+\left(\int_0^1\left|\hbar'\left(\frac{(2-\vartheta)}{2}\mathbf{v}_{m,\jmath}+\frac{\vartheta}{2}\mathbf{v}_{m,\jmath+1}\right)\right|^q d\vartheta\right)^{\frac{1}{q}}\Bigg\}.\end{aligned}$$

Applying the definition of higher order strongly m-polynomial exponentially type convex function with respect to the constant $\zeta>0$ of $|\hbar'|^q$, we get

$$\begin{aligned}\left|\mathrm{Q}_m^{\ell}(\hbar;\flat_1,\flat_2)\right| &\le \left(\frac{\flat_2-\flat_1}{4m}\right)\left(\frac{1}{\ell r+1}\right)^{\frac{1}{r}}\\
&\quad\times\sum_{\jmath=0}^{m-1}\Bigg\{\Bigg(\int_0^1\Bigg[\frac{1}{m}\sum_{\imath=1}^{m}\left[1-\left(1-\frac{\vartheta}{2}\right)^{\imath}\right]\frac{|\hbar'(\mathbf{v}_{m,\jmath})|^q}{e^{\varsigma\mathbf{v}_{m,\jmath}}}\\
&\quad+\frac{1}{m}\sum_{\imath=1}^{m}\left[1-\left(\frac{\vartheta}{2}\right)^{\imath}\right]\frac{|\hbar'(\mathbf{v}_{m,\jmath+1})|^q}{e^{\varsigma\mathbf{v}_{m,\jmath+1}}}\\
&\quad-\zeta\left[\left(\frac{\vartheta}{2}\right)^p\left(1-\frac{\vartheta}{2}\right)+\frac{\vartheta}{2}\left(1-\frac{\vartheta}{2}\right)^p\right]\left|\frac{\flat_2-\flat_1}{m}\right|^p\Bigg]d\vartheta\Bigg)^{\frac{1}{q}}\\
&\quad+\Bigg(\int_0^1\Bigg[\frac{1}{m}\sum_{\imath=1}^{m}\left[1-\left(1-\frac{\vartheta}{2}\right)^{\imath}\right]\frac{|\hbar'(\mathbf{v}_{m,\jmath+1})|^q}{e^{\varsigma\mathbf{v}_{m,\jmath+1}}}\end{aligned}$$

$$+\frac{1}{m}\sum_{\imath=1}^{m}\left[1-\left(\frac{\vartheta}{2}\right)^{\imath}\right]\frac{|\hbar'(\mathbf{v}_{m,\jmath})|^q}{e^{\varsigma \mathbf{v}_{m,\jmath}}}$$

$$-\zeta\left[\left(\frac{\vartheta}{2}\right)^{p}\left(1-\frac{\vartheta}{2}\right)+\frac{\vartheta}{2}\left(1-\frac{\vartheta}{2}\right)^{p}\right]\left|\frac{\flat_2-\flat_1}{m}\right|^{p}\Bigg]d\vartheta\Bigg)^{\frac{1}{q}}\Bigg\}$$

$$=\left(\frac{\flat_2-\flat_1}{4m}\right)\left(\frac{1}{\ell r+1}\right)^{\frac{1}{r}}$$

$$\times\sum_{\jmath=0}^{m-1}\Bigg\{\left(R_m\frac{|\hbar'(\mathbf{v}_{m,\jmath})|^q}{e^{\varsigma \mathbf{v}_{m,\jmath}}}+S_m\frac{|\hbar'(\mathbf{v}_{m,\jmath+1})|^q}{e^{\varsigma \mathbf{v}_{m,\jmath+1}}}-\frac{\zeta}{(p+1)(p+2)}\left|\frac{\flat_1-1}{m}\right|^{p}\right)^{\frac{1}{q}}$$

$$+\left(R_m\frac{|\hbar'(\mathbf{v}_{m,\jmath+1})|^q}{e^{\varsigma \mathbf{v}_{m,\jmath+1}}}+S_m\frac{|\hbar'(\mathbf{v}_{m,\jmath})|^q}{e^{\varsigma \mathbf{v}_{m,\jmath}}}-\frac{\zeta}{(p+1)(p+2)}\left|\frac{\flat_2-\flat_1}{m}\right|^{p}\right)^{\frac{1}{q}}\Bigg\},$$

which ends our proof. □

Corollary 4.7. *Theorem 4.6 with* $\varsigma = 0$ *leads to*

$$\left|\mathrm{Q}_m^{\ell}(\hbar;\flat_1,\flat_2)\right|\le\left(\frac{\flat_2-\flat_1}{4m}\right)\left(\frac{1}{\ell r+1}\right)^{\frac{1}{r}} \tag{4.11}$$

$$\times\sum_{\jmath=0}^{m-1}\Bigg\{\left(R_m|\hbar'(\boldsymbol{v}_{m,\jmath})|^q+S_m|\hbar'(\boldsymbol{v}_{m,\jmath+1})|^q-\frac{\zeta}{(p+1)(p+2)}\left|\frac{\flat_2-\flat_1}{m}\right|^{p}\right)^{\frac{1}{q}} \tag{4.12}$$

$$+\left(R_m|\hbar'(\boldsymbol{v}_{m,\jmath+1})|^q+S_m|\hbar'(\boldsymbol{v}_{m,\jmath})|^q-\frac{\zeta}{(p+1)(p+2)}\left|\frac{\flat_2-\flat_1}{m}\right|^{p}\right)^{\frac{1}{q}}\Bigg\}. \tag{4.13}$$

Corollary 4.8. *Theorem 4.6 with* $m = 1$ *leads to*

$$\left|\frac{2^{\ell-1}\Gamma(\ell+1)}{(\flat_2-\flat_1)^{\ell}}\left[\mathcal{J}^{\ell}_{\left(\frac{\flat_1+\flat_2}{2}\right)^+}\hbar(\flat_2)+\mathcal{J}^{\ell}_{\left(\frac{\flat_1+\flat_2}{2}\right)^-}\hbar(\flat_1)\right]-\hbar\left(\frac{\flat_1+\flat_2}{2}\right)\right| \tag{4.14}$$

$$\le\left(\frac{\flat_2-\flat_1}{4}\right)\left(\frac{1}{\ell r+1}\right)^{\frac{1}{r}}\left(\frac{1}{4}\right)^{\frac{1}{q}}\Bigg\{\left(\frac{|\hbar'(\flat_1)|^q}{e^{\varsigma\flat_1}}+3\frac{|\hbar'(\flat_2)|^q}{e^{\varsigma\flat_2}}-\frac{\zeta}{(p+1)(p+2)}|\flat_2-\flat_1|^p\right)^{\frac{1}{q}} \tag{4.15}$$

$$+\left(3\frac{|\hbar'(\flat_1)|^q}{e^{\varsigma\flat_1}}+\frac{|\hbar'(\flat_2)|^q}{e^{\varsigma\flat_2}}-\frac{\zeta}{(p+1)(p+2)}|\flat_2-\flat_1|^p\right)^{\frac{1}{q}}\Bigg\}. \tag{4.16}$$

Moreover, if $\varsigma = 0$ *and* $\zeta \to 0^+$, *we get*

$$\left|\frac{2^{\ell-1}\Gamma(\ell+1)}{(\flat_2-\flat_1)^{\ell}}\left\{\mathcal{J}^{\ell}_{\left(\frac{\flat_1+\flat_2}{2}\right)^+}\hbar(\flat_2)+\mathcal{J}^{\ell}_{\left(\frac{\flat_1+\flat_2}{2}\right)^-}\hbar(\flat_1)\right\}-\hbar\left(\frac{\flat_1+\flat_2}{2}\right)\right|$$

$$\le\left(\frac{\flat_2-\flat_1}{4}\right)\left(\frac{1}{\ell r+1}\right)^{\frac{1}{r}}\left(\frac{1}{4}\right)^{\frac{1}{q}}\left\{\left(|\hbar'(\flat_1)|^q+3|\hbar'(\flat_2)|^q\right)^{\frac{1}{q}}+\left(3|\hbar'(\flat_1)|^q+|\hbar'(\flat_2)|^q\right)^{\frac{1}{q}}\right\},$$

which is established in [8, Theorem 6].

Theorem 4.9. *Let $\ell > 0$, $m \in \mathrm{N}$ and $\hbar : [\flat_1, \flat_2] \to \mathbb{R}$ be a differentiable function on $(\flat_1, \flat_2)$ such that $\hbar' \in \mathcal{L}[\flat_1, \flat_2]$. If $|\hbar'|^q$ is higher order strongly m-polynomial exponentially type convex function with respect to the constant $\zeta > 0$ on $[\flat_1, \flat_2]$ and $\varsigma \in \mathbb{R}$, then for $q \geq 1$, we have*

$$\begin{aligned}
\left|\mathrm{Q}_m^{\ell}(\hbar; \flat_1, \flat_2)\right| \leq \left(\frac{\flat_2 - \flat_1}{4m}\right)\left(\frac{1}{\ell+1}\right)^{1-\frac{1}{q}} \sum_{\jmath=0}^{m-1} \Bigg\{ \Bigg[T_{m,\ell}\frac{|\hbar'(\boldsymbol{v}_{m,\jmath})|^q}{e^{\varsigma \boldsymbol{v}_{m,\jmath}}} + M_{m,\ell}\frac{|\hbar'(\boldsymbol{v}_{m,\jmath+1})|^q}{e^{\varsigma \boldsymbol{v}_{m,\jmath+1}}} \\
- \frac{\zeta}{2^{p+1}}\left|\frac{\flat_2 - \flat_1}{m}\right|^p \left(\frac{\ell+p+3}{(\ell+p+1)(\ell+p+2)} - 2^{\ell+p+2} B_{\frac{1}{2}}(\ell+2, p+1)\right)\Bigg]^{\frac{1}{q}} \\
+ \Bigg[T_{m,\ell}\frac{|\hbar'(\boldsymbol{v}_{m,\jmath+1})|^q}{e^{\varsigma \boldsymbol{v}_{m,\jmath+1}}} + M_{m,\ell}\frac{|\hbar'(\boldsymbol{v}_{m,\jmath})|^q}{e^{\varsigma \boldsymbol{v}_{m,\jmath}}} \\
- \frac{\zeta}{2^{p+1}}\left|\frac{\flat_2 - \flat_1}{m}\right|^p \left(\frac{\ell+p+3}{(\ell+p+1)(\ell+p+2)} - 2^{\ell+p+2} B_{\frac{1}{2}}(\ell+2, p+1)\right)\Bigg]^{\frac{1}{q}} \Bigg\}, \quad (4.17)
\end{aligned}$$

where $T_{m,\ell}$ and $M_{m,\ell}$ are as given in Theorem 4.3.

Proof. By making use of Lemma 4.1, the power mean inequality and properties of modulus, we have

$$\begin{aligned}
\left|\mathrm{Q}_m^{\ell}(\hbar; \flat_1, \flat_2)\right| &\leq \left(\frac{\flat_2 - \flat_1}{4m}\right) \sum_{\jmath=0}^{m-1} \left\{ \int_0^1 \vartheta^{\ell} \left|\hbar'\left(\frac{\vartheta}{2}\mathbf{v}_{m,\jmath} + \frac{(2-\vartheta)}{2}\mathbf{v}_{m,\jmath+1}\right)\right| d\vartheta \right. \\
&\qquad \left. + \int_0^1 \vartheta^{\ell} \left|\hbar'\left(\frac{(2-\vartheta)}{2}\mathbf{v}_{m,\jmath} + \frac{\vartheta}{2}\mathbf{v}_{m,\jmath+1}\right)\right| d\vartheta \right\} \\
&\leq \left(\frac{\flat_2 - \flat_1}{4m}\right)\left(\int_0^1 \vartheta^{\ell} d\vartheta\right)^{1-\frac{1}{q}} \sum_{\jmath=0}^{m-1} \left\{ \left(\int_0^1 \vartheta^{\ell} \left|\hbar'\left(\frac{\vartheta}{2}\mathbf{v}_{m,\jmath} + \frac{(2-\vartheta)}{2}\mathbf{v}_{m,\jmath+1}\right)\right|^q d\vartheta\right)^{\frac{1}{q}} \right. \\
&\qquad \left. + \left(\int_0^1 \vartheta^{\ell} \left|\hbar'\left(\frac{(2-\vartheta)}{2}\mathbf{v}_{m,\jmath} + \frac{\vartheta}{2}\mathbf{v}_{m,\jmath+1}\right)\right|^q d\vartheta\right)^{\frac{1}{q}} \right\}.
\end{aligned}$$

By the definition of higher order strongly m-polynomial exponentially type convex function with respect to the constant $\zeta > 0$ of $|\hbar'|^q$, we have

$$\begin{aligned}
&\left|\mathrm{Q}_m^{\ell}(\hbar; \flat_1, \flat_2)\right| \leq \left(\frac{\flat_2 - \flat_1}{4m}\right)\left(\frac{1}{\ell+1}\right)^{1-\frac{1}{q}} \\
&\times \sum_{\jmath=0}^{m-1} \left\{ \left(\int_0^1 \vartheta^{\ell} \left[\frac{1}{m}\sum_{\imath=1}^{m}\left[1 - \left(1 - \frac{\vartheta}{2}\right)^{\imath}\right] \frac{|\hbar'(\mathbf{v}_{m,\jmath})|^q}{e^{\varsigma \mathbf{v}_{m,\jmath}}} + \frac{1}{m}\sum_{\imath=1}^{m}\left[1 - \left(\frac{\vartheta}{2}\right)^{\imath}\right] \frac{|\hbar'(\mathbf{v}_{m,\jmath+1})|^q}{e^{\varsigma \mathbf{v}_{m,\jmath+1}}} \right.\right.\right. \\
&\left.\left. - \zeta\left[\left(\frac{\vartheta}{2}\right)^p\left(1 - \frac{\vartheta}{2}\right) + \frac{\vartheta}{2}\left(1 - \frac{\vartheta}{2}\right)^p\right]\left|\frac{\flat_2 - \flat_1}{m}\right|^p \right] d\vartheta \right)^{\frac{1}{q}}
\end{aligned}$$

$$+\left(\int_0^1 \vartheta^\ell\left[\frac{1}{m}\sum_{\imath=1}^{m}\left[1-\left(1-\frac{\vartheta}{2}\right)^\imath\right]\frac{|\hbar'(\mathbf{v}_{m,\jmath+1})|^q}{e^{\varsigma\mathbf{v}_{m,\jmath+1}}}+\frac{1}{m}\sum_{\imath=1}^{m}\left[1-\left(\frac{\vartheta}{2}\right)^\imath\right]\frac{|\hbar'(\mathbf{v}_{m,\jmath})|^q}{e^{\varsigma\mathbf{v}_{m,\jmath}}}\right.\right.$$

$$\left.\left.-\zeta\left[\left(\frac{\vartheta}{2}\right)^p\left(1-\frac{\vartheta}{2}\right)+\frac{\vartheta}{2}\left(1-\frac{\vartheta}{2}\right)^p\right]\left|\frac{\flat_2-\flat_1}{m}\right|^p\right]d\vartheta\right)^{\frac{1}{q}}\Bigg\}$$

$$=\left(\frac{\flat_2-\flat_1}{4m}\right)\left(\frac{1}{\ell+1}\right)^{1-\frac{1}{q}}\sum_{\jmath=0}^{m-1}\Bigg\{\Bigg[T_{m,\ell}\frac{|\hbar'(\mathbf{v}_{m,\jmath})|^q}{e^{\varsigma\mathbf{v}_{m,\jmath}}}+M_{m,\ell}\frac{|\hbar'(\mathbf{v}_{m,\jmath+1})|^q}{e^{\varsigma\mathbf{v}_{m,\jmath+1}}}$$

$$-\frac{\zeta}{2^{p+1}}\left|\frac{\flat_2-\flat_1}{m}\right|^p\left(\frac{\ell+p+3}{(\ell+p+1)(\ell+p+2)}-2^{\ell+p+2}B_{\frac{1}{2}}(\ell+2,p+1)\right)\Bigg]^{\frac{1}{q}}$$

$$+\Bigg[T_{m,\ell}\frac{|\hbar'(\mathbf{v}_{m,\jmath+1})|^q}{e^{\varsigma\mathbf{v}_{m,\jmath+1}}}+M_{m,\ell}\frac{|\hbar'(\mathbf{v}_{m,\jmath})|^q}{e^{\varsigma\mathbf{v}_{m,\jmath}}}$$

$$-\frac{\zeta}{2^{p+1}}\left|\frac{\flat_2-\flat_1}{m}\right|^p\left(\frac{\ell+p+3}{(\ell+p+1)(\ell+p+2)}-2^{\ell+p+2}B_{\frac{1}{2}}(\ell+2,p+1)\right)\Bigg]^{\frac{1}{q}}\Bigg\},$$

which completes the proof. □

Corollary 4.10. *Theorem 4.9 with $\varsigma = 0$ leads to*

$$\left|\mathrm{Q}_m^\ell(\hbar;\flat_1,\flat_2)\right|\leq\left(\frac{\flat_2-\flat_1}{4m}\right)\left(\frac{1}{\ell+1}\right)^{1-\frac{1}{q}} \tag{4.18}$$

$$\times\sum_{\jmath=0}^{m-1}\Bigg\{\Bigg[T_{m,\ell}|\hbar'(\boldsymbol{v}_{m,\jmath})|^q+M_{m,\ell}|\hbar'(\boldsymbol{v}_{m,\jmath+1})|^q$$

$$-\frac{\zeta}{2^{p+1}}\left|\frac{\flat_2-\flat_1}{m}\right|^p\left(\frac{\ell+p+3}{(\ell+p+1)(\ell+p+2)}-2^{\ell+p+2}B_{\frac{1}{2}}(\ell+2,p+1)\right)\Bigg]^{\frac{1}{q}}$$

$$+\Bigg[T_{m,\ell}|\hbar'(\boldsymbol{v}_{m,\jmath+1})|^q+M_{m,\ell}|\hbar'(\boldsymbol{v}_{m,\jmath})|^q$$

$$-\frac{\zeta}{2^{p+1}}\left|\frac{\flat_2-\flat_1}{m}\right|^p\left(\frac{\ell+p+3}{(\ell+p+1)(\ell+p+2)}-2^{\ell+p+2}B_{\frac{1}{2}}(\ell+2,p+1)\right)\Bigg]^{\frac{1}{q}}\Bigg\}.$$

Corollary 4.11. *Theorem 4.9 with $m = 1$ leads to*

$$\left|\frac{2^{\ell-1}\Gamma(\ell+1)}{(\flat_2-\flat_1)^\ell}\left\{\mathcal{J}^\ell_{\left(\frac{\flat_1+\flat_2}{2}\right)^+}\hbar(\flat_2)+\mathcal{J}^\ell_{\left(\frac{\flat_1+\flat_2}{2}\right)^-}\hbar(\flat_1)\right\}-\hbar\left(\frac{\flat_1+\flat_2}{2}\right)\right|$$

$$\leq\left(\frac{\flat_2-\flat_1}{4}\right)\left(\frac{1}{\ell+1}\right)^{1-\frac{1}{q}}\Bigg\{\Bigg(\frac{1}{2(\ell+2)}\frac{|\hbar'(\flat_1)|^q}{e^{\varsigma\flat_1}}+\frac{\ell+3}{2(\ell+1)(\ell+2)}\frac{|\hbar'(\flat_2)|^q}{e^{\varsigma\flat_2}}$$

$$
- \frac{\zeta}{2^{p+1}} |\flat_2 - \flat_1|^p \left(\frac{\ell+p+3}{(\ell+p+1)(\ell+p+2)} - 2^{\ell+p+2} B_{\frac{1}{2}}(\ell+2, p+1) \right) \Bigg)^{\frac{1}{q}}
+ \Bigg(\frac{1}{2(\ell+2)} \frac{|\hbar'(\flat_2)|^q}{e^{\varsigma \flat_2}} + \frac{\ell+3}{2(\ell+1)(\ell+2)} \frac{|\hbar'(\flat_1)|^q}{e^{\varsigma \flat_1}}
- \frac{\zeta}{2^{p+1}} |\flat_2 - \flat_1|^p \left(\frac{\ell+p+3}{(\ell+p+1)(\ell+p+2)} - 2^{\ell+p+2} B_{\frac{1}{2}}(\ell+2, p+1) \right) \Bigg)^{\frac{1}{q}} \Bigg\}. \quad (4.19)
$$

Moreover, if $\varsigma = 0$ *and* $\zeta \to 0^+$, *we get*

$$
\left| \frac{2^{\ell-1}\Gamma(\ell+1)}{(\flat_2 - \flat_1)^\ell} \left\{ \mathcal{J}^\ell_{\left(\frac{\flat_1+\flat_2}{2}\right)^+} \hbar(\flat_2) + \mathcal{J}^\ell_{\left(\frac{\flat_1+\flat_2}{2}\right)^-} \hbar(\flat_1) \right\} - \hbar\left(\frac{\flat_1+\flat_2}{2} \right) \right|
\leq \left(\frac{\flat_2 - \flat_1}{4(\ell+1)} \right) \left\{ \left(\frac{\ell+1}{2(\ell+2)} |\hbar'(\flat_1)|^q + \frac{\ell+3}{2(\ell+2)} |\hbar'(\flat_2)|^q \right)^{\frac{1}{q}}
+ \left(\frac{\ell+1}{2(\ell+2)} |\hbar'(\flat_2)|^q + \frac{\ell+3}{2(\ell+2)} |\hbar'(\flat_1)|^q \right)^{\frac{1}{q}} \right\}, \quad (4.20)
$$

which is established in [8, Theorem 5].

5. Applications

5.1. Bessel functions

Consider the function $\mathrm{B}_\sigma : (0, +\infty) \to [1, +\infty)$ with $\sigma > -1$, given by

$$
\mathrm{B}_\sigma(x) := 2^\sigma \Gamma(\sigma+1) x^{-\sigma} \mathcal{P}_\sigma(x),
$$

where $\mathcal{P}_\sigma$ is the modified Bessel function of the first kind defined by (see [11, on page 77]):

$$
\mathcal{P}_\sigma(x) = \sum_{m=0}^{+\infty} \frac{\left(\frac{x}{2}\right)^{\sigma+2m}}{m!\Gamma(\sigma+1+m)}, \quad x \in \mathbb{R}.
$$

Following [11], we have

$$
\mathrm{B}'_\sigma(x) = \frac{x}{2(\sigma+1)} \mathrm{B}_{\sigma+1}(x), \quad (5.1)
$$

$$
\mathrm{B}''_\sigma(x) = \frac{x^2 \mathrm{B}_{\sigma+2}(x)}{4(\sigma+1)(\sigma+2)} + \frac{\mathrm{B}_{\sigma+1}(x)}{2(\sigma+1)}. \quad (5.2)
$$

Assume that all assumptions of the used corollaries in the following examples are satisfied.

Example 5.1. Let $0 < \flat_1 < \flat_2$ and $\sigma > -1$. Then, by using Corollary 4.5 with $\ell = 1$ for $\hbar(x) = \mathrm{B}'_\sigma(x)$ and the identities (5.1) and (5.2), we have

$$\begin{aligned}\left|\frac{\mathrm{B}_\sigma(\flat_2)-\mathrm{B}_\sigma(\flat_1)}{\flat_2-\flat_1}-\frac{(\flat_1+\flat_2)}{4(\sigma+1)}\mathrm{B}_{\sigma+1}\left(\frac{\flat_1+\flat_2}{2}\right)\right| &\le \left(\frac{\flat_2-\flat_1}{16(\sigma+1)}\right)\\ &\times\left[\frac{1}{e^{\varsigma\flat_1}}\left(\frac{\flat_1^2\mathrm{B}_{\sigma+2}(\flat_1)}{2(\sigma+2)}+\mathrm{B}_{\sigma+1}(\flat_1)\right)\right.\\ &\left.+\frac{1}{e^{\varsigma\flat_2}}\left(\frac{\flat_2^2\mathrm{B}_{\sigma+2}(\flat_2)}{2(\sigma+2)}+\mathrm{B}_{\sigma+1}(\flat_2)\right)\right]\\ &-\frac{\zeta}{2^p}\left(\frac{p+4}{(p+2)(p+3)}-2^{p+3}B_{\frac{1}{2}}(3,p+1)\right)|\flat_2-\flat_1|^p.\end{aligned}$$

Example 5.2. Let $0<\flat_1<\flat_2$ and $\sigma>-1$. Then, by applying Corollary 4.8 with $\ell=1$, $\hbar(x)=\mathrm{B}'_\sigma(x)$ and the identities (5.1) and (5.2), we get

$$\begin{aligned}&\left|\frac{\mathrm{B}_\sigma(\flat_2)-\mathrm{B}_\sigma(\flat_1)}{\flat_2-\flat_1}-\frac{(\flat_1+\flat_2)}{4(\sigma+1)}\mathrm{B}_{\sigma+1}\left(\frac{\flat_1+\flat_2}{2}\right)\right| \le \left(\frac{\flat_2-\flat_1}{4\sqrt[q]{4}}\right)\left(\frac{1}{r+1}\right)^{\frac{1}{r}}\\ &\times\left\{\left[\frac{1}{e^{\varsigma\flat_1}}\left(\frac{\flat_1^2\mathrm{B}_{\sigma+2}(\flat_1)}{4(\sigma+1)(\sigma+2)}+\frac{\mathrm{B}_{\sigma+1}(\flat_1)}{2(\sigma+1)}\right)^q+\frac{3}{e^{\varsigma\flat_2}}\left(\frac{\flat_2^2\mathrm{B}_{\sigma+2}(\flat_2)}{4(\sigma+1)(\sigma+2)}+\frac{\mathrm{B}_{\sigma+1}(\flat_2)}{2(\sigma+1)}\right)^q\right.\right.\\ &\left.-\frac{\zeta}{(p+1)(p+2)}|\flat_2-\flat_1|^p\right]^{\frac{1}{q}}\\ &+\left[\frac{3}{e^{\varsigma\flat_1}}\left(\frac{\flat_1^2\mathrm{B}_{\sigma+2}(\flat_1)}{4(\sigma+1)(\sigma+2)}+\frac{\mathrm{B}_{\sigma+1}(\flat_1)}{2(\sigma+1)}\right)^q+\frac{1}{e^{\varsigma\flat_2}}\left(\frac{\flat_2^2\mathrm{B}_{\sigma+2}(\flat_2)}{4(\sigma+1)(\sigma+2)}+\frac{\mathrm{B}_{\sigma+1}(\flat_2)}{2(\sigma+1)}\right)^q\right.\\ &\left.\left.-\frac{\zeta}{(p+1)(p+2)}|\flat_2-\flat_1|^p\right]^{\frac{1}{q}}\right\}.\end{aligned}$$

Example 5.3. Let $0<\flat_1<\flat_2$ and $\sigma>-1$. Then, by using Corollary 4.11 with $\ell=1$, $\hbar(x)=\mathrm{B}'_\sigma(x)$ and the identities (5.1) and (5.2), we obtain

$$\begin{aligned}&\left|\frac{\mathrm{B}_\sigma(\flat_2)-\mathrm{B}_\sigma(\flat_1)}{\flat_2-\flat_1}-\frac{(\flat_1+\flat_2)}{4(\sigma+1)}\mathrm{B}_{\sigma+1}\left(\frac{\flat_1+\flat_2}{2}\right)\right| \le \left(\frac{\flat_2-\flat_1}{4}\right)\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\\ &\times\left\{\left[\frac{1}{6e^{\varsigma\flat_1}}\left(\frac{\flat_1^2\mathrm{B}_{\sigma+2}(\flat_1)}{4(\sigma+1)(\sigma+2)}+\frac{\mathrm{B}_{\sigma+1}(\flat_1)}{2(\sigma+1)}\right)^q+\frac{1}{3e^{\varsigma\flat_2}}\left(\frac{\flat_2^2\mathrm{B}_{\sigma+2}(\flat_2)}{4(\sigma+1)(\sigma+2)}+\frac{\mathrm{B}_{\sigma+1}(\flat_2)}{2(\sigma+1)}\right)^q\right.\right.\\ &\left.-\frac{\zeta}{2^{p+1}}|\flat_2-\flat_1|^p\left(\frac{p+4}{(p+2)(p+3)}-2^{p+3}B_{\frac{1}{2}}(3,p+1)\right)\right]^{\frac{1}{q}}\\ &+\left[\frac{1}{3e^{\varsigma\flat_1}}\left(\frac{\flat_1^2\mathrm{B}_{\sigma+2}(\flat_1)}{4(\sigma+1)(\sigma+2)}+\frac{\mathrm{B}_{\sigma+1}(\flat_1)}{2(\sigma+1)}\right)^q+\frac{1}{6e^{\varsigma\flat_2}}\left(\frac{\flat_2^2\mathrm{B}_{\sigma+2}(\flat_2)}{4(\sigma+1)(\sigma+2)}+\frac{\mathrm{B}_{\sigma+1}(\flat_2)}{2(\sigma+1)}\right)^q\right.\\ &\left.\left.-\frac{\zeta}{2^{p+1}}|\flat_2-\flat_1|^p\left(\frac{p+4}{(p+2)(p+3)}-2^{p+3}B_{\frac{1}{2}}(3,p+1)\right)\right]^{\frac{1}{q}}\right\}.\end{aligned}$$

5.2. Bounded functions

Proposition 5.4. *Let $\ell > 0$, $m \in \mathrm{N}$, $\varsigma \in \mathbb{R}$ and $\hbar : [\flat_1, \flat_2] \to \mathbb{R}$ be a differentiable function on $(\flat_1, \flat_2)$ such that $\hbar' \in \mathcal{L}[\flat_1, \flat_2]$. If $|\hbar'|$ is a higher order strongly m-polynomial exponentially type convex function, with respect to the constant $\zeta > 0$ and $|\hbar'| \leq \mathcal{K}$ on $[\flat_1, \flat_2]$, then we have*

$$\begin{aligned} \left|\mathrm{Q}_m^{\ell}(\hbar; \flat_1, \flat_2)\right| &\leq \mathcal{K}\left(\frac{\flat_2 - \flat_1}{4m}\right)[T_{m,\ell} + M_{m,\ell}] \sum_{\jmath=0}^{m-1}\left[\frac{1}{e^{\varsigma \boldsymbol{v}_{m,\jmath}}} + \frac{1}{e^{\varsigma \boldsymbol{v}_{m,\jmath+1}}}\right] \\ &- \frac{m\zeta}{2^p}\left(\frac{\ell + p + 3}{(\ell + p + 1)(\ell + p + 2)} - 2^{\ell + p + 2} B_{\frac{1}{2}}(\ell + 2, p + 1)\right)\left|\frac{\flat_2 - \flat_1}{m}\right|^p, \end{aligned} \tag{5.3}$$

where $T_{m,\ell}$ and $M_{m,\ell}$ are as given in Theorem 4.3.

Proposition 5.5. *Let $\ell > 0$, $m \in \mathrm{N}$, $\varsigma \in \mathbb{R}$ and $\hbar : [\flat_1, \flat_2] \to \mathbb{R}$ be a differentiable function on $(\flat_1, \flat_2)$ such that $\hbar' \in \mathcal{L}[\flat_1, \flat_2]$. If $|\hbar'|^q$ is higher order strongly m-polynomial exponentially type convex function with respect to the constant $\zeta > 0$ and $|\hbar'| \leq \mathcal{K}$ on $[\flat_1, \flat_2]$, then for $q > 1$ and $\frac{1}{q} + \frac{1}{r} = 1$, we have*

$$\begin{aligned} \left|\mathrm{Q}_m^{\ell}(\hbar; \flat_1, \flat_2)\right| &\leq \mathcal{K}\left(\frac{\flat_2 - \flat_1}{4m}\right)\left(\frac{1}{\ell r + 1}\right)^{\frac{1}{r}} \\ &\times \sum_{\jmath=0}^{m-1}\left\{\left(\frac{R_m}{e^{\varsigma \boldsymbol{v}_{m,\jmath}}} + \frac{S_m}{e^{\varsigma \boldsymbol{v}_{m,\jmath+1}}} - \frac{\zeta}{\mathcal{K}^q (p+1)(p+2)}\left|\frac{\flat_2 - \flat_1}{m}\right|^p\right)^{\frac{1}{q}}\right. \\ &\left. + \left(\frac{R_m}{e^{\varsigma \boldsymbol{v}_{m,\jmath+1}}} + \frac{S_m}{e^{\varsigma \boldsymbol{v}_{m,\jmath}}} - \frac{\zeta}{\mathcal{K}^q (p+1)(p+2)}\left|\frac{\flat_2 - \flat_1}{m}\right|^p\right)^{\frac{1}{q}}\right\}, \end{aligned} \tag{5.4}$$

where R_m and S_m are as given in Theorem 4.6.

Proposition 5.6. *Let $\ell > 0$, $m \in \mathrm{N}, \varsigma \in \mathbb{R}$ and $\hbar : [\flat_1, \flat_2] \to \mathbb{R}$ be a differentiable function on $(\flat_1, \flat_2)$ such that $\hbar' \in \mathcal{L}[\flat_1, \flat_2]$. If $|\hbar'|^q$ is higher order strongly m-polynomial exponentially type convex function with respect to the constant $\zeta > 0$ and $|\hbar'| \leq \mathcal{K}$ on $[\flat_1, \flat_2]$, then for $q \geq 1$, we have*

$$\begin{aligned} \left|\mathrm{Q}_m^{\ell}(\hbar; \flat_1, \flat_2)\right| &\leq \mathcal{K}\left(\frac{\flat_2 - \flat_1}{4m}\right)\left(\frac{1}{\ell + 1}\right)^{1 - \frac{1}{q}} \times \sum_{\jmath=0}^{m-1}\left\{\left[\frac{T_{m,\ell}}{e^{\varsigma \boldsymbol{v}_{m,\jmath}}} + \frac{M_{m,\ell}}{e^{\varsigma \boldsymbol{v}_{m,\jmath+1}}}\right.\right. \\ &\left.- \frac{\zeta}{\mathcal{K}^q 2^{p+1}}\left|\frac{\flat_2 - \flat_1}{m}\right|^p\left(\frac{\ell + p + 3}{(\ell + p + 1)(\ell + p + 2)} - 2^{\ell + p + 2} B_{\frac{1}{2}}(\ell + 2, p + 1)\right)\right]^{\frac{1}{q}} \\ &+ \left[\frac{T_{m,\ell}}{e^{\varsigma \boldsymbol{v}_{m,\jmath+1}}} + \frac{M_{m,\ell}}{e^{\varsigma \boldsymbol{v}_{m,\jmath}}}\right. \\ &\left.\left.- \frac{\zeta}{\mathcal{K}^q 2^{p+1}}\left|\frac{\flat_2 - \flat_1}{m}\right|^p\left(\frac{\ell + p + 3}{(\ell + p + 1)(\ell + p + 2)} - 2^{\ell + p + 2} B_{\frac{1}{2}}(\ell + 2, p + 1)\right)\right]^{\frac{1}{q}}\right\}, \end{aligned} \tag{5.5}$$

where $T_{m,\ell}$ and $M_{m,\ell}$ are as given in Theorem 4.3.

6. Conclusion

In this article, we proposed the higher order strongly m-polynomial exponentially type convex functions and some of its algebraic properties are given. Furthermore, we deduced some fractional integral inequalities using the basic identity for the new class of function. Moreover, we demonstrated the efficiency of our results via some applications. Our results not only generalized the previous known results but also refined them. For future research in this direction, we will offer several new inequalities pertaining to Hölder-İşcan, Chebyshev, Markov, Young and Minkowski type inequalities for this generic class of convex functions in fractional and quantum calculus.

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