Stud. Univ. Babeş-Bolyai Math. 70(2025), No. 2, 353–369
DOI: 10.24193/subbmath.2025.2.12

On the stability of KdV equation with time-dependent delay on the boundary feedback in presence of saturated source term

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Abstract. The current paper investigate the question of stabilizability of the Korteweg-de Vries equation with time-varying delay on the boundary feedback in the presence of a saturated source term. Under specific assumptions regarding the time-varying delay, we have established that the studied system is well-posed. Moreover, using an appropriate Lyapunov functional, we prove the exponential stability result. Finally, we give some conclusions.

Mathematics Subject Classification (2010): 93B05, 35R02, 93C20.

Keywords: KdV equation, stability, saturated source term, time-varying delay.

1. Introduction

In recent years a lot of work has come out on the study of Korteweg-de Vries equation with time-delay (see e.g. [25, 2, 36]). The Korteweg-de Vries equation (KdV)

$$u_t + u_x + u_{xxx} + uu_x = 0 \tag{1.1}$$

is a nonlinear one dimensional equation, more precisely the KdV equation is a mathematical model of waves on shallow water surfaces. In recent decades, the study of the Korteweg-de Vries equation has yielded intriguing results, particularly with regard to its controllability and stabilizability properties. This studies can be attributed to the efforts made by Russell and Zhang in [32]. Subsequently, significant research efforts have been dedicated to the examination of both controllability and stabilizability. For a comprehensive review of these studies, interested readers can refer to various works

Received 30 March 2024; Accepted 22 May 2024.

(such as [31, 37]), as well as the following references [3, 9, 5]. In the majority of these papers, the following system have been studied:

$$\begin{cases} u_t(t,x) + u_x(t,x) + u_{xxx}(t,x) + f = 0, & t \geq 0,\ x \in [0,L]; \\ u(t,0) = u(t,L) = u_x(t,L) = 0, & t \geq 0; \\ u(0,x) = u_0(x), \end{cases} \tag{1.2}$$

In a general context, the feedback control f in (1.2) is chosen to fulfill specific objectives. As a result, it must consistently adhere to predefined constraints. In particular, equation (1.2) has been the subject of investigation, with two distinct approaches studied in the literature: one involving distributed control (as examined in [29, 27]) and another involving boundary control (as discussed in [18, 4]). Notably, in [29], the authors demonstrate that the linear feedback control $f(t,x) = a(x)u(t,x)$, where $a = a(x)$ is a nonnegative function that satisfies certain conditions, makes the origin exponentially stable. It's worth mentioning as well that when a equals zero, the authors also prove that the linear Korteweg-de Vries (KdV) equation without control is exponentially stable under the conditions $L \notin \left\{ 2\pi\sqrt{\frac{k^2+n^2+kn}{3}} \mid n,k \in \mathbb{N}^* \right\}$.

One of the most well-known constraints that can affect the proper functioning of the control system is the saturation constraint, which has been discussed in various works (see, for instance, [20, 24, 16, 17, 10, 6]. The issue of input saturation in the control system is inevitable. Physical constraints or practical limitations can cause the restriction of input signal amplitudes, leading to unfavorable and even catastrophic outcomes for the control system.

In the literature, there are several articles that studies the stability result of KdV equation with input saturation (see e.g. [20, 34, 19]. In particular, [34] looks at the study of the following KdV equation:

$$\begin{cases} u_t(t,x) + u_x(t,x) + u_{xxx}(t,x) + sat(a(x)u(t,x) = 0, & t \geq 0,\ x \in [0,L]; \\ u(t,0) = u(t,L) = u_x(t,L) = 0, & t \geq 0; \\ u(0,x) = u_0(x), \end{cases} \tag{1.3}$$

where $a = a(x) \in L^\infty([0,L])$ satisfying $a_1 \geq a = a(x) \geq a_0 > 0$ on $\omega \subseteq [0,L]$ (ω is a nonempty open subset of $[0,L]$), and the saturation function $sat(.)$ is defined as follows

$$sat(t) = \begin{cases} t, & \text{if } \|t\|_{L^2(0,L)} \leq 1 \\ \dfrac{t}{\|t\|_{L^2(0,L)}}, & \text{if } \|t\|_{L^2(0,L)} \geq 1 \end{cases} \tag{1.4}$$

The well-posedness of the closed-loop system for the linear KdV equation (1.3) has been proved through the application of nonlinear semigroup theory. Moreover, the authors have demonstrated that the origin of the KdV equation (1.3) in closed-loop system with the saturated control (1.4) is exponentially stable. The asymptotic stability of KdV equation with a saturated internal control has been studied by [19]. In their work , they considered the system (1.2) with

$$f(t,x) = asat(u),$$

where a is a positive constant and the saturation function is defined as follows

$$sat(t) = \begin{cases} -u_0, & \text{if } t \leq -u_0 \\ t, & \text{if } -u_0 \leq t \leq u_0 \\ u_0, & \text{if } t \geq u_0 \end{cases} \tag{1.5}$$

where u_0 represent a positive constant. The authors prove the well-posedness by applying nonlinear semigroup theory. Additionally, using Lyapunov theory for infinite-dimensional systems, they also establish that the origin is asymptotically stable.

In this paper, we are interested in the study of time-varying delay on the boundary of the Korteweg-de Vries equation in the presence of a saturated source term. That is to say, we consider the same problem as in [34] with time-varying delay on the boundary feedback.

In general, the presence of delay in scientific phenomena is a multifaceted consideration. It is widely acknowledged that even a minor delay in the feedback mechanism can potentially induce instability in a system, as discussed in various references [12, 7, 21]. Alternatively, delays can be used as a tool to improve performance by introducing beneficial phase shifts to optimize system behavior, as studied in references such as [1, 30]. When delays become time-varying, the complexity of analyzing system stability significantly increases. Several studies have examined the stability of partial differential equations (PDEs) involving time-varying delay, with notable references including [22, 8, 26, 13].

In recent years, researchers have shown increasing interest in solving stability and robustness problems related to constant delay for the Korteweg-de Vries equation. Notable contributions have been made by researchers such as Baudouin et al. and Parada et al., as mentioned in [2, 23], where they studied the Korteweg-de Vries equation with time-delay feedback, establishing the well-posedness and proving exponential stability through the use of the observability inequality. For more details on the KdV equation with time-delay, the readers can find more details in [35, 11, 15]. Concerning the Korteweg-de Vries equation with time-varying delay, there is a notably singular study conducted by Parada et al. [25]. This study examined the issue of time-varying delay both on the boundary or internal feedback. With specific assumptions concerning time-varying delay, they proved the well-posedness and the stability results is analyzed, using an appropriate Lyapunov functional. However, in the literature to the best of our knowledge, there has been no prior work addressing this issue in the context of the Korteweg-de Vries equation with a saturated source term.

In our paper, we focus on the Korteweg-de Vries equation with time-varying delay on the boundary feedback in presence of saturated source term. The equation under investigation is given as follows:

$$\begin{cases} u_t(x,t) + u_x(x,t) + u_{xxx}(x,t) = f(x,t), \quad t > 0, \quad x \in [0,L]; \\ u(0,t) = u(L,t) = 0, \quad t > 0; \\ u_x(L,t) = \alpha u_x(0,t) + \beta u_x(0,t-\theta(t)), \quad t > 0; \\ u(x,0) = u_0(x), \quad x \in [0,L]; \\ u_x(0,t-\theta(0)) = z_0(t-\theta(0)), \quad 0 < t < \theta(0), \end{cases} \tag{1.6}$$

with

$$f(x,t) = -sat(a(x)u(t;x)), \tag{1.7}$$

where $a(\cdot) \in L^\infty([0,L])$ is a nonnegative function satisfying some conditions, and $sat(.)$ is the same given by (1.4). The main contribution of this paper is to study the well-posedness and exponential stability of the linear KdV equation with time-varying delay on the boundary feedback, as given in equation (1.6)-(1.7). The well-posedness of the system (1.6)-(1.7) is proven under some conditions. By using an appropriate Lyapunov functional, we demonstrate that the KdV equation (1.6)-(1.7) with the saturated source term (1.4) is exponentially stable.

Our paper is organized as follows. In the next section, we formulate our problem. In section 3, we examine the well-posedness of (1.6)-(1.7). Section 4 is dedicated to the exponential stabilization of (1.6)-(1.7). Finally, we present some conclusions in section 5.

2. Problem statement

The aim of this paper is to study the following KdV equation with time-varying delay

$$\begin{cases} u_t(x,t) + u_x(x,t) + u_{xxx}(x,t) = -sat(a(x)u(x,t)), \quad t > 0, \quad x \in [0,L]; \\ u(0,t) = u(L,t) = 0, \quad t > 0; \\ u_x(L,t) = \alpha u_x(0,t) + \beta u_x(0,t-\theta(t)), \quad t > 0; \\ u(x,0) = u_0(x), \quad x \in [0,L]; \\ u_x(0,t-\theta(0)) = z_0(t-\theta(0)), \quad 0 < t < \theta(0), \end{cases} \tag{2.1}$$

where where $a = a(x) \in L^\infty[0,L]$ satisfying

$$\begin{cases} a = a(x) \geq a_0 > 0 \quad \text{on} \quad \omega \subseteq [0,L], \\ \omega \quad \text{is a nonempty open subset of} \quad [0,L], \end{cases} \tag{2.2}$$

Moreover, suppose that the delay $\theta(\cdot) \in W^{2,\infty}[0,T]$ for all $T > 0$ and satisfies the following conditions

$$0 < \theta_0 \leq \theta(t) \leq K, \text{ for all } t \geq 0, \tag{2.3}$$

and

$$\dot{\theta}(t) \leq d \leq 1, \text{ for all } t \geq 0, \tag{2.4}$$

where $d \geq 0$.
Furthermore, we define the matrix M_1 by

$$M_1 = \begin{pmatrix} \alpha^2 - 1 + |\beta| & \alpha\beta \\ \alpha\beta & \beta^2 + |\beta|(d-1) \end{pmatrix} \tag{2.5}$$

Where α, β and d are real constants that satisfy the following inequality

$$|\alpha| + |\beta| + d < 1. \tag{2.6}$$

If (2.6) is satisfied, then the matrix M_1 is definite negative according to [25].
In this context, we introduce a new variable $z(\mu,t) = u_x(0, t-\theta(t)\mu)$ for $\mu \in [0,1]$ and $t>0$. Then, $z(\cdot,\cdot)$ satisfies the following system

$$\begin{cases} \theta(t)z_t(\mu,t) + (1-\dot{\theta}(t)\mu)z_\mu(\mu,t) = 0, & t>0, \mu\in[0,1]; \\ z(0,t) = u_x(0,t), & t>0; \\ z(\mu,0) = z_0(-\theta(0)\mu), & \mu\in[0,1]. \end{cases} \tag{2.7}$$

For more detail about a new variable z that takes into account $\theta(\cdot)$ (see [21, 22]).
Therefore, we investigate the following semi-linear system

$$\begin{cases} u_t(x,t) + u_x(x,t) + u_{xxx}(x,t) = -sat(a(x)u(x,t)), & t>0, x\in[0,L]; \\ \theta(t)z_t(\mu,t) + (1-\dot{\theta}(t)\mu)z_\mu(\mu,t) = 0, & t>0, \mu\in[0,1]; \\ u(0,t) = u(L,t) = 0, & t>0 \\ u_x(L,t) = \alpha u_x(0,t) + \beta u_x(0,t-\theta(t)), & t>0; \\ u(x,0) = u_0(x), & x\in[0,L]; \\ u_x(0,t-\theta(0)) = z_0(t-\theta(0)), & 0<t<\theta(0). \\ z(0,t) = u_x(0,t), & t>0; \\ z(\mu,0) = z_0(-\theta(0)\mu), & \mu\in[0,1]. \end{cases} \tag{2.8}$$

Let $Y = \begin{pmatrix} u \\ z \end{pmatrix}$, then the system (2.8) can be rewritten as the following first-order system

$$\begin{cases} Y_t = A(t)Y(t) + \begin{pmatrix} -sat(a(x)u) \\ 0 \end{pmatrix}, & t>0, \\ Y(0) = \begin{pmatrix} u_0, & z_0(-\theta(0)) \end{pmatrix}^T. \end{cases} \tag{2.9}$$

Where the operator $A(t)$ is defined by

$$\begin{aligned} D(A(t)) &= \{(u,z)\in H^3([0,L])\times H^1([0,L]), u(0)=u(L)=0 \\ &\qquad z(0)=u_x(0), u_x(L) = \alpha u_x(0,t) + \beta u_x(0,t-\theta(t))\} \end{aligned}$$

$$A(t)\begin{pmatrix} u \\ z \end{pmatrix} = \begin{pmatrix} -u_x - u_{xxx} \\ \frac{\dot{\theta}(t)\mu-1}{\theta(t)} z_\mu \end{pmatrix} \text{ for all } \begin{pmatrix} u \\ z \end{pmatrix} \in D(A(t)). \tag{2.10}$$

It has been proved in [25] that the domain of operator $A(t)$ is independent of t, i.e.

$$D(A(t)) = D(A(0)).$$

Let the Hilbert space $H = L^2[0,L]\times L^2[0,1]$ equipped with the following usual inner product

$$\left\langle \begin{pmatrix} u \\ z \end{pmatrix}, \begin{pmatrix} u_1 \\ z_1 \end{pmatrix} \right\rangle_H = \int_0^L uu_1\,dx + \int_0^1 zz_1\,d\mu,$$

and its norm

$$\left\| \begin{pmatrix} u \\ z \end{pmatrix} \right\|_H^2 = \int_0^L u^2\,dx + \int_0^1 z^2\,d\mu$$

We introduce a new inner product on H. This inner product is dependent to time t and define as follows

$$\left\langle \begin{pmatrix} u \\ z \end{pmatrix}, \begin{pmatrix} u_1 \\ z_1 \end{pmatrix} \right\rangle_t = \int_0^L uu_1\,dx + |\beta|\theta(t)\int_0^1 zz_1\,d\mu,$$

with associated norm denoted by $\|\cdot\|_t$. Using (2.3), the norm $\|\cdot\|_t$ and $\|\cdot\|_H$ are equivalent in H. Indeed, for all $t \geq 0$, and all $\begin{pmatrix} u \\ z \end{pmatrix} \in H$, we have

$$(1+|\beta|\theta_0)\left\|\begin{pmatrix} u \\ z \end{pmatrix}\right\|_H^2 \leq \left\|\begin{pmatrix} u \\ z \end{pmatrix}\right\|_t^2 \leq (1+|\beta|K)\left\|\begin{pmatrix} u \\ z \end{pmatrix}\right\|_H^2 \tag{2.11}$$

Now, we recall the definition of mild solution.
Let us consider the abstract system in a Hilbert space Z

$$\begin{cases} \dot{u}(t) = \mathcal{A}u(t) + f(t), \quad t > 0, \\ u(0) = u_0, \end{cases} \tag{2.12}$$

where $\mathcal{A}$ is an infinitesimal generator of linear C_0-semigroup $(T(t))_{t\geq 0}$ defined on its domain $D(\mathcal{A}) \subseteq H$, where Z is a Hilbert space and $f \in L^1_{loc}([0,T], Z)$.

Definition 2.1. [28, Definition 2.3] Let $\mathcal{A}$ be the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$. Let $u_0 \in Z$ and $f \in L^1(0,T,Z)$. Then the function $u \in \mathcal{C}([0,T], Z)$ given by

$$u(t) = T(t)u_0 + \int_0^t T(t-s)f(s)ds \quad 0 \leq t \leq T, \tag{2.13}$$

is the unique mild solution of the initial value problem (2.12) on $[0,T]$.

We recall that the saturation function is Lipschitzian in $L^2[0,L]$.

Lemma 2.2. [33, Theorem 5.1] *For all* $(u,v) \in L^2[0,L]$, *we have*

$$\|sat(u) - sat(v)\|_{L^2[0,L]} \leq 3\|u-v\|_{L^2[0,L]}$$

3. Well-posedness

Before stating the well-posedness result system (2.9), we recall the result of well-posedness of the following linear system without source term which has been treated by Parada et al. [25]

$$\begin{cases} u_t(x,t) + u_x(x,t) + u_{xxx}(x,t) = 0, & t > 0, x \in [0,L]; \\ \theta(t)z_t(\mu,t) + (1-\theta_t(t)\mu)z_\mu(\mu,t) = 0, & t > 0, \mu \in [0,1]; \\ u(0,t) = u(L,t) = 0, & t > 0 \\ u_x(L,t) = \alpha u_x(0,t) + \beta u_x(0,t-\theta(t)), & t > 0; \\ u(x,0) = u_0(x), & x \in [0,L]; \\ u_x(0,t-\theta(0)) = z_0(t-\theta(0)), & 0 < t < \theta(0). \\ z(0,t) = u_x(0,t), & t > 0; \\ z(\mu,0) = z_0(-\theta(0)\mu), & \mu \in [0,1]. \end{cases} \tag{3.1}$$

As previously, let $Y = \begin{pmatrix} u \\ z \end{pmatrix}$, then the system (3.1) can be rewritten as the following first-order system

$$\begin{cases} Y_t = A(t)Y(t), & t > 0, \\ Y(0) = \begin{pmatrix} u_0, & z_0(-\theta(0)) \end{pmatrix}^T. \end{cases} \tag{3.2}$$

Where $A(t)$ is given by (2.10). The well-posedness of (3.2), is proved in [25, Theorem 2.2]. To prove the well-posedness of (3.2), they used the following Theorem

Theorem 3.1. *Assume that*

1. *$D(A(0))$, is a dense subset of H.*
2. *$D(A(t)) = D(A(0)) \quad \forall t \geq 0$.*
3. *For all $t \in [0,T]$ $A(t)$ generates a strongly continuous semigroup on H and the family $A = \{A(t) : t \in [0,T]\}$ is stable with stability constant C and m independent of t, i.e. the semigroup $(T_t(s))_{s\geq 0}$ generated by $A(t)$ satisfies*

$$\|T_t(s)Y\|_H \leq Ce^{ms}\|Y\|_H, \text{ for all } Y \in H, \text{ and } s \geq 0.$$

4. *$\partial_t A(t)$ belong to $L^\infty_*([0,T], B(D(A(0))))$, the space of equivalent of essentially classes bounded strongly measure functions from $[0,T]$ into the set $B(D(A(0)), H)$ of bounded operator from $D(A(0)$ to H.*

Then the system (3.2) *has a unique solution $Y \in \mathcal{C}([0,T], D(A(0))) \cap \mathcal{C}^1([0,T], H)$*

More precisely, in [25], the authors demonstrated that, if (2.3)-(2.6) holds, the operator $A(t)$ satisfy all assumptions of Theorem 3.1 and the system (3.2) has a unique solution $u \in \mathcal{C}([0,+\infty[, H)$. Moreover if $Y_0 \in D(A(0))$, then $Y \in \mathcal{C}([0,+\infty[, D(A(0))) \cap \mathcal{C}^1([0,+\infty[, H)$.

The following result gives the conditions for the existences and the uniqueness of the solution of (2.9).

Theorem 3.2. *Let $(u_0, z_0) \in H$ and suppose that* (2.3)-(2.6) *holds. Assume also that $a = a(x) \in L^\infty[0,L]$ satisfying* (2.2) *and $u \in L^2(0,T,H^1[0,L])$. Then, there exists a unique solution $Y = (u,z) \in \mathcal{C}([0,+\infty[, H)$ of* (2.9).

Proof. Let $G(u) = \begin{pmatrix} -sat(a(x)u) \\ 0 \end{pmatrix}$. By assumption $u \in L^2(0,T,H^1[0,L])$, hence $\begin{pmatrix} -sat(a(x)u) \\ 0 \end{pmatrix} \in L^1(0,T,H)$. Indeed, let $u_1, u_2 \in L^2(0,T,H^1[0,L])$, by using the Holder inequality, ([20, Proposition 3.4]) and ([33, Theorem 5.1]), we get

$$\begin{aligned}
\|G(u_1) - G(u_2)\|_{L^1(0,T,H)} &= \int_0^T \|G(u_1) - G(u_2)\|_H dt \\
&= \int_0^T \|sat(au_1) - sat(au_2)\|_{L^2[0,L])} + \|0\|_{L^2[0,1]} dt \\
&= \int_0^T \|sat(au_1) - sat(au_2)\|_{L^2[0,L])} dt \\
&\leq 3\int_0^T \|au_1 - au_2\|_{L^2[0,L])} dt \\
&\leq 3\|a\|_{L^\infty[0,L]} \int_0^T \|u_1 - u_2\|_{L^2[0,L])} dt \\
&\leq 3\|a\|_{L^\infty[0,L]} \sqrt{T}\sqrt{L}\|u_1 - u_2\|_{L^2(0,T,H^1[0,L])} < +\infty.
\end{aligned}$$

Therefore, $\begin{pmatrix} -sat(a(x)u) \\ 0 \end{pmatrix} \in L^1(0,T,H)$. Moreover, from [25, Theorem 2.2] the operator $A(t)$ satisfy all assumption of Theorem 3.1. Thus, since $\begin{pmatrix} -sat(a(x)u) \\ 0 \end{pmatrix} \in L^1(0,T,H)$, then if $Y_0 \in H$, the system (3.2) has a unique solution $Y = (u,z) \in \mathcal{C}([0,+\infty[,H)$, according to [14, Theorem 2].
Furthermore, $sat(\tilde{a}(x)u) \in L^1(0,T,L^2[0,L])$, hence if $\begin{pmatrix} u_0 \\ z_0 \end{pmatrix} \in D(A(0))$, then from [2, Proposition 2] the solution of (2.9) is a regular solution. □

4. Exponential stability

Consider the following energy

$$E(t) = \frac{1}{2}\int_0^L u^2(x,t)dx + \frac{\|\beta\|}{2}\theta(t)\int_0^1 u_x^2(0,t-\theta(t)\mu)d\mu. \tag{4.1}$$

The following lemma proves that the energy (4.1) does not increase.

Lemma 4.1. *Assume that assumptions* (2.3), (2.4) *and* (2.6) *are satisfied. Moreover suppose also that* $u \in L^2(0,T,H^1[0,L])$ *and* $a = a(x) \in L^\infty[0,L]$, *satisfying* (2.2). *Then, for any regular solution of* (2.9), *the energy* (4.1) *satisfies the following inequality*

$$\begin{aligned} \frac{d}{dt}E(t) \leq & \begin{pmatrix} u_x(0,t) \\ z(1,t) \end{pmatrix}^T \left(\frac{1}{2}M_1\right) \begin{pmatrix} u_x(0,t) \\ z(1,t) \end{pmatrix} \\ \leq & 0. \end{aligned} \tag{4.2}$$

Proof. Let u a regular solution of (2.1). By definition $z(\mu,t) = u_x(0,t-\theta(t)\mu)$, hence we rewrite the energy (4.1) as follows

$$E(t) = \frac{1}{2}\int_0^L u^2(x,t)dx + \frac{\|\beta\|}{2}\theta(t)\int_0^1 z^2(\mu,t)d\mu.$$

Differentiating $E(\cdot)$, we get

$$\begin{aligned} \frac{d}{dt}E(t) =& \int_0^L uu_t dx + \frac{|\beta|}{2}\dot{\theta}(t)\int_0^1 z^2 d\mu + |\beta|\theta(t)\int_0^1 zz_t d\mu \\ =& -\int_0^L uu_x dx - \int_0^L uu_{xxx}dx - \int_0^L sat(au)udx \\ &+\frac{|\beta|}{2}\dot{\theta}(t)\int_0^1 z^2 d\mu + |\beta|\theta(t)\int_0^1 zz_t d\mu \end{aligned} \tag{4.3}$$

After some integrations by parts, we obtain

$$-\int_0^L uu_x dx = 0; \quad -\int_0^L uu_{xxx}dx = \frac{1}{2}u_x^2(L,t) - \frac{1}{2}u_x^2(0,t),$$

and

$$|\beta|\theta(t)\int_0^1 zz_t d\mu = |\beta|\theta(t)\int_0^1 \frac{\dot{\theta}(t)\mu - 1}{\theta(t)} zz_\mu d\mu$$
$$= |\beta|\dot{\theta}(t)\int_0^1 \mu zz_\mu d\mu - |\beta|\int_0^1 zz_\mu d\mu.$$

Thus

$$\begin{aligned} |\beta|\dot{\theta}(t)\int_0^1 \mu zz_\mu d\mu &= \frac{|\beta|}{2}\dot{\theta}(t)\left[\mu z^2(\mu,t)\right]_0^1 - \frac{|\beta|}{2}\dot{\theta}(t)\int_0^1 z^2(\mu,t)d\mu \\ &= \frac{|\beta|}{2}\dot{\theta}(t)z^2(1,t) - \frac{|\beta|}{2}\dot{\theta}(t)\int_0^1 z^2(\mu,t)d\mu \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} -|\beta|\int_0^1 zz_\mu d\mu &= -\frac{|\beta|}{2}\left[z^2(\mu,t)\right]_0^1 \\ &= -\frac{|\beta|}{2}\left[z^2(1,t) - z^2(0,t)\right] \\ &= \frac{|\beta|}{2}u_x^2(0,t) - \frac{|\beta|}{2}z^2(1,t). \end{aligned} \tag{4.5}$$

Using (2.8), (4.3), (4.4) and (4.5), we get

$$\begin{aligned} \frac{d}{dt}E(t) &= \frac{1}{2}(\alpha u_x(0,t) + \beta z(1,t))^2 - \frac{1}{2}u_x^2(0,t) - \int_0^L sat(au)udx \\ &+ \frac{|\beta|}{2}\dot{\theta}(t)\int_0^1 z^2 d\mu + |\beta|\dot{\theta}(t)\int_0^1 \mu zz_\mu d\mu - |\beta|\int_0^1 zz_\mu d\mu \\ &= \frac{1}{2}\alpha^2 u_x^2(0,t) + \alpha\beta u_x(0,t)z(1,t) + \frac{1}{2}\beta^2 z^2(1,t) - \frac{1}{2}u_x^2(0,t) \\ &- \int_0^L sat(au)udx + \frac{|\beta|}{2}\dot{\theta}(t)\int_0^1 z^2 d\mu + \frac{|\beta|}{2}\dot{\theta}(t)z^2(1,t) \\ &- \frac{|\beta|}{2}\dot{\theta}(t)\int_0^1 z^2 d\mu + \frac{|\beta|}{2}u_x^2(0,t) - \frac{|\beta|}{2}z^2(1,t) \\ &= \frac{1}{2}(\alpha^2 - 1 + |\beta|)u_x^2(0,t) + \alpha\beta u_x(0,t)z(1,t) \\ &+ \frac{1}{2}\left(\beta^2 + |\beta|(\dot{\theta}(t) - 1)\right)z^2(1,t) - \int_0^L sat(au)udx. \end{aligned}$$

Therefore using (2.4), we obtain

$$\begin{aligned}
\frac{d}{dt}E(t) &+ \begin{pmatrix} u_x(0,t) \\ z(1,t) \end{pmatrix}^T \left(-\frac{1}{2}M_1\right) \begin{pmatrix} u_x(0,t) \\ z(1,t) \end{pmatrix} \\
&\leq \frac{1}{2}(\alpha^2 - 1 + |\beta|)u_x^2(0,t) + \alpha\beta u_x(0,t)z(1,t) \\
&+ \frac{1}{2}\left(\beta^2 + |\beta|(\dot{\theta}(t) - 1)\right) z^2(1,t) - \int_0^L sat(au)udx \\
&+ \begin{pmatrix} u_x(0,t) \\ z(1,t) \end{pmatrix}^T \left(-\frac{1}{2}M_1\right) \begin{pmatrix} u_x(0,t) \\ z(1,t) \end{pmatrix} \\
&= -\int_0^L sat(au)udx \\
&\leq 0.
\end{aligned}$$

Because $\int_0^L sat(au)udx \geq 0$, indeed, if $\|au\|_{L^2} \leq 1$, then

$$sat(au)u = au^2 \geq 0.$$

If $\|au\|_{L^2} \geq 1$,

$$sat(au)u = \frac{au}{\|au\|_{L^2}}u = \frac{au^2}{\|au\|_{L^2}} \geq 0,$$

where $a = a(x)$ is a nonnegative function. Consequently, using (2.6), we have

$$\frac{d}{dt}E(t) \leq \begin{pmatrix} u_x(0,t) \\ z(1,t) \end{pmatrix}^T \left(\frac{1}{2}M_1\right) \begin{pmatrix} u_x(0,t) \\ z(1,t) \end{pmatrix} \leq 0. \qquad \square$$

The following lemmas play an important role to prove the exponential stability of (2.8). Before that, consider the following lyapunov function

$$V(t) = E(t) + \lambda V_1(t) + \gamma V(t)_2, \tag{4.6}$$

where $\lambda, \gamma \geq 0$, $E(\cdot)$ is given by (4.1), and

$$V_1(t) = \int_0^L xu^2(x,t)dx \tag{4.7}$$

$$V_2(t) = \theta(t)\int_0^1 (1-\mu)u^2(x, t - \theta(t)\mu)d\mu. \tag{4.8}$$

Lemma 4.2. *Assume that $a = a(ax) \in L^\infty[0,L]$ satisfies (2.2), $\begin{pmatrix} u_0 \\ z_0 \end{pmatrix} \in D(A(0))$ and $u \in L^2(0,T,H^1[0,L])$, then for any regular solution of (2.1), the following equation is satisfied*

$$\begin{aligned}
\dot{V}_1(t) =& L(\alpha^2 u_x^2(0,t) + 2\alpha\beta u_x(0,t)u_x(0,t-\theta(t)) + \beta^2 u_x^2(0,t-\theta(t)) \\
&+ \int_0^L u^2dx - 3\int_0^L u_x^2dx - \int_0^L xsat(au)udx
\end{aligned} \tag{4.9}$$

Proof. Let us consider a regular solution, then Differentiate $V_1(\cdot)$ we have

$$\begin{aligned} \dot{V}_1(t) &= 2\int_0^L xuu_t dx \\ &= -2\int_0^L xuu_x dx - 2\int_0^L xuu_{xxx} dx - 2\int_0^L xsat(au)udx \end{aligned}$$

After some integrations by parts, we obtain

$$-2\int_0^L xuu_x dx = \int_0^L u^2 dx;$$

$$\begin{aligned} -2\int_0^L xuu_{xxx} dx &= Lu^2(L,t) - 3\int_0^L u_x^2 dx \\ &= L(\alpha u_x(0,t) + \beta u_x(0,t-\theta(t)))^2 - 3\int_0^L u_x^2 dx \end{aligned}$$

Using the last equations, we get

$$\begin{aligned} \dot{V}_1(t) &= \int_0^L u^2 dx + L(\alpha u_x(0,t) + \beta u_x(0,t-\theta(t)))^2 \\ &- 3\int_0^L u_x^2 dx - 2\int_0^L xsat(au)udx \\ &= L(\alpha^2 u_x^2(0,t) + 2\alpha\beta u_x(0,t)u_x(0,t-\theta(t)) + \beta^2 u_x^2(0,t-\theta(t))) \\ &+ \int_0^L u^2 dx - 3\int_0^L u_x^2 dx - 2\int_0^L xsat(au)udx \end{aligned}$$

□

Lemma 4.3. *Assume that* (2.4) *is satisfied. Suppose also* $\begin{pmatrix} u_0 \\ z_0 \end{pmatrix} \in D(A(0))$ *and* $u \in L^2(0,T,H^1[0,L])$, *then for any regular solution of* (2.1), *the following inequality is satisfied*

$$\dot{V}_2(t) \le -(1-d)\int_0^1 u_x^2(0,t-\theta(t)\mu)d\mu + u_x^2(0,t). \tag{4.10}$$

Proof. Consider a regular solution, then differentiate $V_2(\cdot)$, we have

$$\begin{aligned} \dot{V}_2(t) =&\dot{\theta}(t)\int_0^1 (1-\mu)u_x^2(0,t-\theta(t)\mu)d\mu \\ &+2\theta(t)\int_0^1 (1-\mu)\partial_t u_x(0,t-\theta(t)\mu)u_x(0,t-\theta(t)\mu)d\mu \\ =&\dot{\theta}(t)\int_0^1 (1-\mu)u_x^2(0,t-\theta(t)\mu)d\mu + 2\int_0^1 \theta(t)\partial_t u_x(0,t-\theta(t)\mu)u_x(0,t-\theta(t)\mu)d\mu \\ &-2\int_0^1 \mu\theta(t)\partial_t u_x(0,t-\theta(t)\mu)u_x(0,t-\theta(t)\mu)d\mu \end{aligned} \tag{4.11}$$

After some integration by parts and using the following equation

$$-\theta(t)\partial_t u_x(0,t-\theta(t)\mu) = (1-\dot{\theta}(t)\mu)\partial_\mu u_x(0,t-\theta(t)\mu),$$

we obtain

$$\begin{aligned} 2\int_0^1 \theta(t)\partial_t u_x(0,t-\theta(t)\mu)u_x(0,t-\theta(t)\mu)d\mu =& u_x^2(0,t) - (1-\dot{\theta}(t))u_x^2(0,t-\theta(t)) \\ & -\dot{\theta}(t)\int_0^1 u_x^2(0,t-\theta(t)\mu)d\mu \end{aligned} \tag{4.12}$$

and

$$\begin{aligned} -2\int_0^1 \mu\theta(t)\partial_t u_x(0,t-\theta(t)\mu)u_x(0,t-\theta(t)\mu)d\mu =& (1-\dot{\theta}(t))u_x^2(0,t-\theta(t)) \\ & -\int_0^1 u_x^2(0,t-\theta(t)\mu)d\mu \\ & +2\int_0^1 \mu\dot{\theta}(t)u_x^2(0,t-\theta(t)\mu)d\mu \end{aligned} \tag{4.13}$$

We deduce from (4.11), (4.12), (4.13) and (2.4) that

$$\begin{aligned} \dot{V}_2(t) &= -\int_0^1 u_x^2(0,t-\theta(t)\mu)d\mu \\ &+ \dot{\theta}(t)\int_0^1 \mu u_x^2(0,t-\theta(t)\mu)d\mu + u_x^2(0,t) \\ &\leq -\int_0^1 u_x^2(0,t-\theta(t)\mu)d\mu + d\int_0^1 \mu u_x^2(0,t-\theta(t)\mu)d\mu + u_x^2(0,t) \\ &= -(1-d)\int_0^1 u_x^2(0,t-\theta(t)\mu)d\mu + u_x^2(0,t) \end{aligned}$$

□

Now, we are able to state and prove the main result of this section.

Theorem 4.4. *Assume that $a = a(x) \in L^\infty[0,L]$ satisfying (2.2), and $L < \pi\sqrt{3}$. Moreover suppose that the assumptions (2.3) , (2.4) and (2.6) are satisfied. Then, there exists $r > 0$ such that for every $(u_0, z_0) \in \mathcal{H}$ satisfying $\|(u_0,z_0)\|_{\mathcal{H}} \leq r$, there exists $\delta > 0$ and $M > 0$ such that*

$$E(t) \leq Me^{-2\delta t}E(0), \quad \forall t > 0. \tag{4.14}$$

where for λ and γ sufficiently small, the two positive constants δ and M satisfy the following inequality:

$$\delta \leq \min\left\{ \frac{(9\pi^2 - 3L^2 - 2L^{\frac{3}{2}}r\pi^2)}{3L^2(1+2L\lambda)}\lambda, \frac{\gamma}{h(2\gamma+|\beta|)} \right\} \tag{4.15}$$

and

$$M \leq 1 + \max\left\{L\lambda, \frac{2\gamma}{|\beta|}\right\}.$$

Where λ *and* γ*, satisfying the following inequality*

$$\lambda \leq \left\{ \begin{array}{l} \frac{(1-|\beta|)(1-|\beta|-d)+\alpha^2(d-1)+2\gamma(|\beta|+d-1)}{2L(|\beta|-\alpha^2(d-1)-\beta^2-2\gamma|\beta|)} \\ \frac{1-\alpha^2-\beta^2-|\beta|d+2\gamma}{2L(\alpha^2+\beta^2)} \end{array} \right\}. \tag{4.16}$$

and

$$\gamma \leq \left\{ \begin{array}{l} \frac{1-\alpha^2-\beta^2-|\beta|d}{2} \\ \frac{(1-|\beta|)(1-|\beta|-d)+\alpha^2(d-1)}{2(1-|\beta|-d)} \\ \frac{|\beta|-\alpha^2(d-1)-\beta^2}{2\beta} \end{array} \right\}. \tag{4.17}$$

Remark 4.5. The Lyapunov function $V(\cdot)$ and the energy $E(\cdot)$ are equivalent. Indeed,

$$E(t) \leq V(t) \leq M_1 E(t) \quad \forall t > 0, \tag{4.18}$$

where $M_1 = 1 + \max\left\{L\lambda, \frac{2\gamma}{|\beta|}\right\} > 0$. Thanks to inequality (4.18), in order to prove the exponential stability of system (2.1), it is sufficient to show that for all $\delta > 0$,

$$\frac{d}{dt}V(t) + 2\delta V(t) \leq 0.$$

Proof. Let $\begin{pmatrix} u_0 \\ z_0 \end{pmatrix} \in D(A(0))$ such that $\| \begin{pmatrix} u_0 \\ z_0 \end{pmatrix} \|_0 \leq r$, with $r > 0$ chosen later. Using (4.2), (4.9) and (4.10), we get

$$\begin{aligned} \dot{V}(t) \leq & \frac{1}{2} Y M_1 Y + L\lambda\alpha^2 u_x^2(0,t) + 2L\lambda\alpha\beta u_x(0,t)u_x(0,t-\theta(t)) \\ & + L\beta^2 u_x^2(0,t-\theta(t)) + \lambda\int_0^L u^2 dx - 3\lambda\int_0^L u_x^2 dx \\ & - \int_0^L x sat(au)u dx - \gamma(1-d)\int_0^1 u_x^2(0,t-\theta(t)\mu)d\mu + \gamma u_x^2(0,t) \\ = & Y^T\left[\frac{1}{2}M_1 + M_2\right]Y + \lambda\int_0^L u^2 dx - 3\lambda\int_0^L u_x^2 dx \\ & - \int_0^L x sat(au)u dx - \gamma(1-d)\int_0^1 u_x^2(0,t-\theta(t)\mu)d\mu, \end{aligned}$$

where $Y = \begin{pmatrix} u_x(0,t) \\ u_x(0,t-\theta(t)) \end{pmatrix}$ and $M_2 = \begin{pmatrix} L\lambda\alpha^2+\gamma & L\lambda\alpha\beta \\ L\lambda\alpha\beta & L\lambda\beta^2 \end{pmatrix}$ and the matrix M_1 is given by (2.5).

Since $x \in [0, L]$ and $sat(au)u \geq 0$, we deduce that $\int_0^L xsat(au)udx \geq 0$. Consequently we deduce that

$$\begin{aligned}\dot{V}(t) \leq & Y^T \left[\frac{1}{2}M_1 + M_2\right] Y + \lambda \int_0^L u^2 dx - 3\lambda \int_0^L u_x^2 dx \\ & -\gamma(1-d) \int_0^1 u_x^2(0, t-\theta(t)\mu)d\mu.\end{aligned} \tag{4.19}$$

Now, we calculate $2\delta V(t)$, using (2.3), we have

$$\begin{aligned}2\delta V(t) =& 2\delta E(t) + 2\delta\lambda V_1(t) + 2\delta\gamma V_2(t) \\ =& \delta \int_0^L u^2 dx + \delta|\beta|\theta(t) \int_0^1 u_x^2(0, t-\theta(t)\mu)d\mu + 2\delta\lambda \int_0^L xu^2 dx \\ & + 2\delta\gamma\theta(t) \int_0^1 u_x^2(0, t-\theta(t)\mu)d\mu - 2\delta\gamma\theta(t) \int_0^1 \mu u_x^2(0, t-\theta(t)\mu)d\mu \\ \leq & \delta \int_0^L u^2 dx + \delta|\beta|K \int_0^1 u_x^2(0, t-\theta(t)\mu)d\mu \\ & + 2\delta\lambda L \int_0^L u^2 dx + 2\delta\gamma K \int_0^1 u_x^2(0, t-\theta(t)\mu)d\mu\end{aligned} \tag{4.20}$$

According to [25, Theorem 3.2], for λ and γ small enough, the matrix $\frac{1}{2}M_1 + M_2$ is definite negative, and from (4.19) and (4.20) we deduce that

$$\begin{aligned}\dot{V}(t) + 2\delta V(t) \leq & Y^T \left[\frac{1}{2}M_1 + M_2\right] Y + (\lambda + \delta + 2L\lambda\delta) \int_0^L u^2 dx - 3\lambda \int_0^L u_x^2 dx \\ & + (\delta|\beta|K + 2\gamma\delta K - \gamma(1-d)) \int_0^1 u_x^2(0, t-\theta(t)\mu)d\mu \\ \leq & (\lambda + \delta + 2L\lambda\delta) \int_0^L u^2 dx - 3\lambda \int_0^L u_x^2 dx \\ & + (\delta|\beta|K + 2\gamma\delta K - \gamma(1-d)) \int_0^1 u_x^2(0, t-\theta(t)\mu)d\mu\end{aligned} \tag{4.21}$$

By using the Poincaré inequality, we get

$$\begin{aligned}\dot{V}(t) + 2\delta V(t) \leq & \left(\frac{L^2}{\pi^2}(\lambda + \delta + 2L\lambda\delta) - 3\lambda\right) \int_0^L u_x^2 dx \\ & + (\delta|\beta|K + 2\gamma\delta K - \gamma(1-d)) \int_0^1 u_x^2(0, t-\theta(t)\mu)d\mu.\end{aligned} \tag{4.22}$$

□

By assumption $L < \pi\sqrt{3}$, then from [2], it is possible to choose r small enough to have $r < \dfrac{3(\pi^2 - L^2)}{2L^{\frac{3}{2}}\pi^2}$. Consequently, we can choose $\delta > 0$ such that (4.15) holds in order to obtain that

$$\frac{L^2}{\pi^2}(\lambda + \delta + 2L\lambda\delta) - 3\lambda \leq 0,$$

and

$$\delta|\beta|K + 2\gamma\delta K - \gamma(1-d) \leq 0,$$

therefore

$$\dot{V}(t) + 2\delta V(t) \leq 0 \quad \forall t \geq 0.$$

Hence, we deduce that

$$V(t) \leq Ce^{-2\delta t}V(0) \quad \forall t \geq 0.$$

By (4.18), we get

$$E(t) \leq Ce^{-2\delta t}E(0) \quad \forall t \geq 0.$$

Using the density of $D(A(0))$, we conclude the proof by extending the result to any initial condition within H.

5. Conclusion

In this work, we investigated the linear Korteweg-de Vries equation with a time-varying delay on the boundary feedback in the presence of a saturated source term. This study has illustrated that the incorporation of a time-varying delay in the Korteweg-de Vries equation, along with a saturated source term, leads to a well-posed system under some conditions. Using a suitable Lyapunov functional, we prove that the system (2.1) is locally exponentially stable. An inserting topic for further research is the analysis of exponential stability of the non-linear KdV equation with time-variyng delay in presence of non-linear source term.

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