Stud. Univ. Babeş-Bolyai Math. 70(2025), No. 2, 335–352
DOI: 10.24193/subbmath.2025.2.11

An elastic-viscoplastic contact problem with internal state variable, normal damped response and unilateral constraint

Lamia Chouchane and Dounia Bouchelil

Abstract. In this manuscript, we study a contact problem between an elastic-viscoplastic body and an obstacle. The contact is quasistatic and it is described with a normal damped response condition with friction and unilateral constraint. Moreover, we use an elastic-viscoplastic constitutive law with internal state variable to model the material's behavior. We present the classical problem then we derive its variational formulation. Finally, we prove that the associated variational problem has a unique solution. The proof is based on arguments of quasivariational inequalities and fixed points.

Mathematics Subject Classification (2010): 74M15, 74C05, 74M10, 35J87, 35Q74.

Keywords: Contact problem, elastic-viscoplastic material, internal state variable, normal damped response, unilateral constraint, friction, quasivariational inequality.

1. Introduction

Contact problems represent an important topic both in Applied Mathematics and Engineering Sciences. References in the field include [1, 5, 6, 11, 13, 12, 14, 15, 16, 17]. In this work, we deal with a model of the frictional contact between an elastic-viscoplastic body and an obstacle named foundation, for the purpose of modelling and establishing variational analysis of this one. This analysis is done within the infinitesimal strain theory. We model the material's behavior with the following

Received 07 August 2024; Accepted 09 October 2024.

constitutive law with internal state variable

$$\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)) + \mathcal{B}\boldsymbol{\varepsilon}(\mathbf{u}(t)) + \int_0^t \mathcal{G}(\boldsymbol{\sigma}(s) - \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(s)), \boldsymbol{\varepsilon}(\mathbf{u}(s)), \mathbf{k}(s))\, ds, \tag{1.1}$$

in which the viscosity operator $\mathcal{A}$ and the elasticity operator $\mathcal{B}$ are assumed to be nonlinear and $\mathcal{G}$ represents a nonlinear function. Also, $\mathbf{u}$ denotes the displacement field, $\boldsymbol{\sigma}$ represents the stress tensor and $\boldsymbol{\varepsilon}(\mathbf{u})$ is the linearized strain tensor. The internal state variable $\mathbf{k}$ is a vector-valued function whose evolution is governed by the following differential equation

$$\dot{\mathbf{k}}(t) = \varphi\left(\boldsymbol{\sigma}(t) - \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)), \boldsymbol{\varepsilon}(\mathbf{u}(t)), \mathbf{k}(t)\right), \tag{1.2}$$

in which φ is a nonlinear constitutive function with values in $\mathbb{R}^m$, m being a positive integer. Elastic-viscoplastic models can be found in [3, 4, 9, 10]. In particular, the reader can refer to [7, 8, 16] where he finds a detailed analysis of elastic-viscoplastic contact problems with internal state variables.

In this paper, we assume that the part of the body's boundary which will be in contact with the foundation is covered by a thin lubricant layer. Lubricants make sliding of rubbing surfaces easier by interposing a smooth film between these parts. We can find examples of lubrication in many fields such as oil rigs and car mechanics. To model lubrication, we usually use a normal damped response contact condition in which the normal stress on the contact surface depends on the normal velocity, see [1, 2]. However, in this manuscript, we model the contact with normal damped response and unilateral constraint for the velocity field, associated with a version of Coulomb's law of dry friction. These boundary conditions model the contact with a foundation in such a way that the normal velocity is restricted by a unilateral constraint. Also, when the body moves towards the obstacle, the contact is described with a normal damped response condition associated with the friction law. On the other hand, when the body moves in the opposite direction then the reaction of the foundation vanishes. More details on the normal damped response boundary condition with friction and unilateral constraint can be found in [1].

The main novelty of this paper is to describe a frictional contact with the normal damped response and unilateral constraint in velocity for elastic-viscoplastic materials with internal state variable.

The rest of the paper is divided into three sections. Section 2 contains both notations and preliminary material. In section 3, we list assumptions on the data that are required to solve the variational problem derived in the same section. Section 4 deals with different steps taken to prove the main existence and uniqueness result, Theorem 4.1.

2. Notations and preliminaries

In this short section, we make an overview of the notation we shall use and some preliminary material. The notation $\mathbb{N}$ is used to represent the set of positive integers. For $d \in \mathbb{N}$, we denote by $\mathbb{S}^d$ the space of second-order symmetric tensors on $\mathbb{R}^d$ $(d = 2, 3)$. We define the inner products and norms of $\mathbb{R}^d$ and $\mathbb{S}^d$ by

$$\mathbf{u} \;.\mathbf{v} = u_i v_i \;,\;\; \|\mathbf{v}\| = (\mathbf{v} \;.\mathbf{v})^{\frac{1}{2}} \;\; \forall \mathbf{u}, \mathbf{v} \in \mathbb{R}^d,$$
$$\boldsymbol{\sigma} \;.\boldsymbol{\tau} = \sigma_{ij}\tau_{ij} \;,\;\; \|\boldsymbol{\tau}\| = (\boldsymbol{\tau} \;.\boldsymbol{\tau})^{\frac{1}{2}} \;\; \forall \boldsymbol{\sigma}, \boldsymbol{\tau} \in \mathbb{S}^d.$$

Note that the indices i and j run between 1 to d and that the summation convention over repeated indices is used. Also, an index that follows a comma represents the partial derivative with respect to the corresponding component of the spatial variable, e.g. $u_{i,j} = \delta u_i / \delta x_j$.

Let $\Omega \subset \mathbb{R}^d$ $(d = 1, 2, 3)$ be a bounded domain with a Lipschitz continuous boundary Γ and let Γ_1 be a measurable part of Γ such that *meas* $(\Gamma_1) > 0$. We use $\mathbf{x} = (x_i)$ for a generic point in $\Omega \cup \Gamma$ and we denote by $\boldsymbol{\nu} = (\nu_i)$ the outward unit normal at Γ. We use the standard notation for the Lebesgue and Sobolev spaces associated with Ω and Γ; moreover, we consider the spaces

$$\begin{cases} H = \left\{\mathbf{u} = (u_i)/u_i \in L^2(\Omega)\right\}, \\ \mathcal{H} = \left\{\boldsymbol{\sigma} = (\sigma_{ij})/\sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\right\}, \\ H_1 = \left\{\mathbf{u} = (u_i)/u_i \in H^1(\Omega)\right\}, \\ \mathcal{H}_1 = \{\boldsymbol{\sigma} \in \mathcal{H}/Div\boldsymbol{\sigma} \in H\}. \end{cases}$$

The spaces $H, \mathcal{H}, H_1$ and $\mathcal{H}_1$ are real Hilbert spaces with the inner products

$$\begin{cases} (\mathbf{u}, \mathbf{v})_H = \int_\Omega u_i v_i \, d\mathbf{x}, \\ (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} = \int_\Omega \sigma_{ij}\tau_{ij} \, d\mathbf{x}, \\ (\mathbf{u}, \mathbf{v})_{H_1} = (\mathbf{u}, \mathbf{v})_H + (\boldsymbol{\varepsilon}(\mathbf{u}), \varepsilon(\mathbf{v}))_{\mathcal{H}}, \\ (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}_1} = (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} + (Div\;\boldsymbol{\sigma}, Div\;\boldsymbol{\tau})_H, \end{cases}$$

respectively, where $\boldsymbol{\varepsilon} : H_1 \longrightarrow \mathcal{H}$ and $Div : \mathcal{H}_1 \longrightarrow H$ are respectively the deformation and the divergence operators defined by

$$\boldsymbol{\varepsilon}(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u}))\;, \qquad \varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}(u_{i,j} + u_{j,i})\;, \qquad Div\,\boldsymbol{\sigma} = (\sigma_{ij,j}).$$

The associated norms on $H, \mathcal{H}, H_1$ and $\mathcal{H}_1$ are denoted by $\|.\|_H, \|.\|_{\mathcal{H}}, \|.\|_{H_1}$ and $\|.\|_{\mathcal{H}_1}$ respectively.

Next, for the displacement field, we introduce the closed subspace V of H_1 defined as follows

$$V = \{\mathbf{v} \in H_1/\mathbf{v} = \mathbf{0} \;\text{ on}\, \Gamma_1\}.$$

We consider on V the inner product given by

$$(\mathbf{u}, \mathbf{v})_V = (\boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}} \qquad \forall \mathbf{u}, \mathbf{v} \in V,$$

and the associated norm

$$\|\mathbf{v}\|_V = \|\boldsymbol{\varepsilon}(\mathbf{v})\|_{\mathcal{H}} \;\; \forall \mathbf{v} \in V. \tag{2.1}$$

Completeness of the space $(V, \|.\|_V)$ follows from the assumption *meas* $(\Gamma_1) > 0$, since the use of Korn's inequality is allowed.

Moreover, for an element $\mathbf{v} \in V$, we still write $\mathbf{v}$ for the trace of $\mathbf{v}$ on the boundary.

In addition, v_ν and $\mathbf{v}_\tau$ denote the normal and the tangential components of $\mathbf{v}$ on the boundary Γ gave by

$$v_\nu = \mathbf{v}.\boldsymbol{\nu},\ \ \mathbf{v}_\tau = \mathbf{v} - v_\nu \boldsymbol{\nu}.$$

Let Γ_3 be a measurable part of Γ. We can see from the Sobolev trace theorem that there exists a positive constant c_0 which depends on Ω, Γ_1 and Γ_3 such that

$$\|\mathbf{v}\|_{L^2(\Gamma_3)^d} \leq c_0\ \ \|\mathbf{v}\|_V \qquad \forall \mathbf{v} \in V. \tag{2.2}$$

For a regular function $\sigma \in \mathcal{H}$, σ_ν and σ_τ denote the normal and the tangential components of the vector $\boldsymbol{\sigma\nu}$ on Γ, respectively, and we recall that

$$\sigma_\nu = (\boldsymbol{\sigma\nu}).\boldsymbol{\nu},\ \ \boldsymbol{\sigma}_\tau = \boldsymbol{\sigma\nu} - \sigma_\nu \boldsymbol{\nu}.$$

Moreover, we recall the following Green's formula,

$$\int_\Omega \boldsymbol{\sigma}.\boldsymbol{\varepsilon}(\mathbf{v})\ d\mathbf{x} + \int_\Omega Div\ \boldsymbol{\sigma}.\mathbf{v}\ d\mathbf{x} = \int_\Gamma \boldsymbol{\sigma\nu}.\mathbf{v}\ da \qquad \forall\, \mathbf{v} \in V. \tag{2.3}$$

Furthermore, for the internal state variable, we introduce the notation

$$Y = L^2(\Omega)^m \quad m \in \mathbb{N}. \tag{2.4}$$

Finally, for a given Banach space X we use the notation $C\,(0,T;X)$ and $C^1\,(0,T;X)$ for the space of continuous and continuously differentiable functions defined on $[0,T]$ with values in X, respectively. The spaces $C\,(0,T;X)$ and $C^1\,(0,T;X)$ are Banach spaces endowed with the following norms

$$\|\mathfrak{v}\|_{C(0,T;X)} = \max_{t\in[0,T]} \|\mathfrak{v}\,(t)\|_X\,,$$

$$\|\mathfrak{v}\|_{C^1(0,T;X)} = \max_{t\in[0,T]} \|\mathfrak{v}\,(t)\|_X + \max_{t\in[0,T]} \|\dot{\mathfrak{v}}\,(t)\|_X\,.$$

The following fixed point result will be used in section 4 of the paper.

Theorem 2.1. *Let $(X, \|.\|_X)$ be a Hilbert space and let K be a nonempty closed subset of X. Let $\Lambda : C(0,T;K) \longrightarrow C(0,T;K)$ be a nonlinear operator. Assume that there exists $h \in \mathbb{N}$ with the following property: there exists $b \in [0,1)$ and $c \geq 0$ such that*

$$\|\Lambda\eta_1(t) - \Lambda\eta_2(t)\|_X^h \leq b\|\eta_1(t) - \eta_2(t)\|_X^h + c\int_0^t \|\eta_1(s) - \eta_2(s)\|_X^h\, ds,$$

$\forall \eta_1, \eta_2 \in C(0,T;K)$, $\forall t \in [0,T]$. Then, there exists a unique element $\eta^* \in C(0,T;K)$ such that $\Lambda\eta^* = \eta^*$.

Note that here and below, the notation $\Lambda\eta\,(t)$ means the value of the function $\Lambda\eta$, i.e. $\Lambda\eta\,(t) = (\Lambda\eta)\,(t)$.

Next, we recall a second result proved in [15] which will be used in section 4. To this end, we introduce the following setting. Let X be a real Hilbert space with the inner product $(.,.)_X$ and the associated norm $\|.\|_X$ and let K be a subset of X. Let consider the operator $A : K \to X$ and the functionals $j : K \times K \to \mathbb{R}$ and $f : [0,T] \to X$ such that

$$K \text{ is a nonempty closed convex subset of } X. \tag{2.5}$$

$$\begin{cases} \text{(a) There exists } M_A > 0 \text{ such that} \\ (A\mathbf{u}_1 - A\mathbf{u}_2, \mathbf{u}_1 - \mathbf{u}_2)_X \geq M_A \ \|\mathbf{u}_1 - \mathbf{u}_2\|_X^2 \quad \forall \mathbf{u}_1, \mathbf{u}_2 \in K. \\ \text{(b) There exists } L_A > 0 \text{ such that} \\ \|A\mathbf{u}_1 - A\mathbf{u}_2\|_X \leq L_A \ \|\mathbf{u}_1 - \mathbf{u}_2\|_X \quad \forall \mathbf{u}_1, \mathbf{u}_2 \in K. \end{cases} \tag{2.6}$$

$$\begin{cases} \text{(a) The function } j(\mathbf{u}, .) \text{ is convex and lower} \\ \text{semicontinuous on } K, \text{ for all } \mathbf{u} \in X. \\ \text{(b) There exists } \alpha \geq 0 \text{ such that} \\ j(\mathbf{u}_1, \mathbf{v}_2) - j(\mathbf{u}_1, \mathbf{v}_1) + j(\mathbf{u}_2, \mathbf{v}_1) - j(\mathbf{u}_2, \mathbf{v}_2) \\ \quad \leq \alpha \|\mathbf{u}_1 - \mathbf{u}_2\|_X \|\mathbf{v}_1 - \mathbf{v}_2\|_X, \\ \quad \forall \mathbf{u}_1, \mathbf{u}_2 \in X, \ \ \forall \mathbf{v}_1, \mathbf{v}_2 \in K. \end{cases} \tag{2.7}$$

$$f \in C(0, T; X). \tag{2.8}$$

Moreover, we assume that

$$M_A > \alpha, \tag{2.9}$$

where M_A and α are the constants in (2.6) and (2.7) respectively.

We have the following result.

Theorem 2.2. *Assume that* (2.5)-(2.9) *hold. Then there exists a unique function* $\mathbf{u} \in C(0, T; K)$ *such that*

$$\begin{aligned} &(A\mathbf{u}(t), \mathbf{v} - \mathbf{u}(t))_X + j(\mathbf{u}(t), \mathbf{v}) - j(\mathbf{u}(t), \mathbf{u}(t)) \\ &\quad \geq (f(t), \mathbf{v} - \mathbf{u}(t))_X \qquad \forall \mathbf{v} \in K. \end{aligned} \tag{2.10}$$

We can see that (2.10) is a time-dependent quasivariational inequality governed by the functional j which depends on the solution.

3. Problem statement and variational formulation

We consider an elastic-viscoplastic body that occupies a bounded domain $\Omega \subset \mathbb{R}^d$, $(d = 1, 2, 3)$ with a Lipschitz continuous boundary Γ, divided into three measurable parts Γ_1, Γ_2 and Γ_3, such that *meas* $(\Gamma_1) > 0$. The body is acted upon by body forces of density $\mathbf{f}_0$ and surface tractions of density $\mathbf{f}_2$ act on Γ_2. We assume that the body is clamped on Γ_1, and therefore, the displacement field vanishes there. The body may come in contact over Γ_3 with an obstacle, the so-called foundation. Moreover, on Γ_3 we describe the contact with:

a) A unilateral constraint in velocity given by

$$\dot{u}_\nu \leq g,$$

where $g > 0$ is a given bound. Here we assume the nonhomogeneous case and, therefore, g is a function that depends on the spatial variable $\mathbf{x} \in \Gamma_3$.

b) A normal damped response condition associated to Coulomb's law of dry friction, as far as the normal velocity does not reach the bound g. When the normal velocity reaches the limit g, friction follows the Tresca law. Also, We assume a given compatibility condition to accommodate conditions in b) and to ensure the continuity of the friction bound when the normal velocity reaches its maximum value g. Therefore, we can see a natural transition from the Coulomb law (which is valid as far as $0 \leq \dot{u_\nu} \leq g$) to the Tresca friction law (which is valid when $\dot{u_\nu} = g$). Consequently, we obtain the following frictional contact conditions with normal damped response and unilateral constraint

$$\begin{cases} \dot{u}_\nu(t) \le g, \;\; \sigma_\nu(t) + p(\dot{u}_\nu(t)) \le 0, \\ (\dot{u}_\nu(t) - g)(\sigma_\nu(t) + p(\dot{u}_\nu(t))) = 0, \end{cases} \qquad \text{on } \Gamma_3 \times [\,0, T\,]\,,$$

$$\begin{cases} \|\boldsymbol{\sigma}_\tau(t)\| \le \mu\, p(\dot{u}_\nu(t)) \\ -\,\boldsymbol{\sigma}_\tau(t) = \mu\, p(\dot{u}_\nu(t)) \dfrac{\dot{\mathbf{u}}_\tau(t)}{\|\dot{\mathbf{u}}_\tau(t)\|} \qquad \text{if } \dot{\mathbf{u}}_\tau(t) \neq 0, \end{cases} \qquad \text{on } \Gamma_3 \times [\,0, T\,]\,,$$

where p is a positive function such that $p(r) = 0$ for $r \le 0$ and μ denotes the coefficient of friction. More details on these contact conditions can be found in [1].
Furthermore, we assume that the process is quasistatic since the forces and tractions vary slowly in time and, therefore, we neglect the acceleration of the system. Hence, the classical formulation of the contact problem is as follows.

Problem P. Find a displacement field $\mathbf{u} : \Omega \times [0,T] \to \mathbb{R}^d$, a stress field $\boldsymbol{\sigma} : \Omega \times [0,T] \to \mathbb{S}^d$ and an internal state variable $\mathbf{k} : \Omega \times [0,T] \to \mathbb{R}^m$ such that

$$\begin{aligned} \boldsymbol{\sigma}(t) &= \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)) + \mathcal{B}\boldsymbol{\varepsilon}(\mathbf{u}(t)) \\ &\quad + \int_0^t \mathcal{G}(\boldsymbol{\sigma}(s) - \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(s)), \boldsymbol{\varepsilon}(\mathbf{u}(s)), \mathbf{k}(s))\, ds \quad \text{in } \Omega \times [0,T]\,, \end{aligned} \tag{3.1}$$

$$\dot{\mathbf{k}}(t) = \varphi\left(\boldsymbol{\sigma}(t) - \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)), \boldsymbol{\varepsilon}(\mathbf{u}(t)), \mathbf{k}(t)\right) \quad \text{in } \Omega \times [0,T], \tag{3.2}$$

$$Div\; \boldsymbol{\sigma}(t) + \mathbf{f}_0(t) = \mathbf{0} \quad \text{in } \Omega \times [0,T], \tag{3.3}$$

$$\mathbf{u}(t) = \mathbf{0} \quad \text{on } \Gamma_1 \times [0,T], \tag{3.4}$$

$$\boldsymbol{\sigma}(t)\,.\boldsymbol{\nu} = \mathbf{f}_2(t) \quad \text{on } \Gamma_2 \times [0,T], \tag{3.5}$$

$$\begin{cases} \dot{u}_\nu(t) \le g, \;\; \sigma_\nu(t) + p(\dot{u}_\nu(t)) \le 0, \\ (\dot{u}_\nu(t) - g)(\sigma_\nu(t) + p(\dot{u}_\nu(t))) = 0, \end{cases} \qquad \text{on } \Gamma_3 \times [\,0, T\,]\,, \tag{3.6}$$

$$\begin{cases} \|\boldsymbol{\sigma}_\tau(t)\| \le \mu\, p(\dot{u}_\nu(t)) \\ -\,\boldsymbol{\sigma}_\tau(t) = \mu\, p(\dot{u}_\nu(t)) \dfrac{\dot{\mathbf{u}}_\tau(t)}{\|\dot{\mathbf{u}}_\tau(t)\|} \qquad \text{if } \dot{\mathbf{u}}_\tau(t) \neq 0, \end{cases} \qquad \text{on } \Gamma_3 \times [\,0, T\,]\,, \tag{3.7}$$

$$\mathbf{u}(0) = \mathbf{u}_0,\; \mathbf{k}(0) = \mathbf{k}_0 \quad \text{in } \Omega. \tag{3.8}$$

Now, we describe the problem (3.1)-(3.8). First, equations (3.1) and (3.2) represent the elastic-viscoplastic constitutive law with internal state variable as well as the evolution equation of the latter. Equation (3.3) is the equilibrium equation while conditions (3.4)-(3.5) are the displacement-traction boundary conditions respectively.

The boundary conditions (3.6)-(3.7) describe the mechanical conditions on the contact surface Γ_3 that represents the frictional contact conditions with normal damped response and unilateral constraint in velocity. Finally, (3.8) represents the initial conditions in which $\mathbf{u}_0$ and $\mathbf{k}_0$ are the initial displacement and the initial state variable respectively.

We turn now to the variational formulation of the Problem P. To this end, we assume that the viscosity operator $\mathcal{A}$, the elasticity operator $\mathcal{B}$ and the nonlinear constitutive function $\mathcal{G}$ satisfy

$$\left\{\begin{array}{l}\text{(a) } \mathcal{A}:\Omega\times\mathbb{S}^d\to\mathbb{S}^d.\\ \text{(b) There exists } L_{\mathcal{A}}>0 \text{ such that}\\ \quad \|\mathcal{A}(\mathbf{x},\varepsilon_1)-\mathcal{A}(\mathbf{x},\varepsilon_2)\|\le L_{\mathcal{A}}\ \|\varepsilon_1-\varepsilon_2\|\\ \quad \forall\ \varepsilon_1,\ \varepsilon_2\in\mathbb{S}^d,\text{ a.e. } \mathbf{x}\in\Omega.\\ \text{(c) There exists } M_{\mathcal{A}}>0 \text{ such that}\\ \quad (\mathcal{A}(\mathbf{x},\varepsilon_1)-\mathcal{A}(\mathbf{x},\varepsilon_2)).(\varepsilon_1-\varepsilon_2)\ge m_{\mathcal{A}}\ \|\varepsilon_1-\varepsilon_2\|^2\\ \quad \forall\ \varepsilon_1,\ \varepsilon_2\in\mathbb{S}^d,\text{ a.e. } \mathbf{x}\in\Omega.\\ \text{(d) The mapping } \mathbf{x}\mapsto\mathcal{A}(\mathbf{x},\boldsymbol{\varepsilon}) \text{ is measurable on } \Omega,\\ \quad \text{for any } \boldsymbol{\varepsilon}\in\mathbb{S}^d.\\ \text{(e) The mapping } \mathbf{x}\mapsto\mathcal{A}(\mathbf{x},\mathbf{0}) \text{ belongs to } \mathcal{H}.\end{array}\right. \tag{3.9}$$

$$\left\{\begin{array}{l}\text{(a) } \mathcal{B}:\Omega\times\mathbb{S}^d\to\mathbb{S}^d.\\ \text{(b) There exists } L_{\mathcal{B}}>0 \text{ such that}\\ \quad \|\mathcal{B}(\mathbf{x},\varepsilon_1)-\mathcal{B}(\mathbf{x},\varepsilon_2)\|\le L_{\mathcal{B}}\ \|\varepsilon_1-\varepsilon_2\|\\ \quad \forall\ \varepsilon_1,\ \varepsilon_2\in\mathbb{S}^d,\text{ a.e. } \mathbf{x}\in\Omega.\\ \text{(d) The mapping } \mathbf{x}\mapsto\mathcal{B}(\mathbf{x},\boldsymbol{\varepsilon}) \text{ is measurable on } \Omega,\\ \quad \text{for any } \boldsymbol{\varepsilon}\in\mathbb{S}^d.\\ \text{(e) The mapping } \mathbf{x}\mapsto\mathcal{B}(\mathbf{x},\mathbf{0}) \text{ belongs to } \mathcal{H}.\end{array}\right. \tag{3.10}$$

$$\left\{\begin{array}{l}\text{(a) } \mathcal{G}:\Omega\times\mathbb{S}^d\times\mathbb{S}^d\times\mathbb{R}^m\to\mathbb{S}^d.\\ \text{(b) There exists } L_{\mathcal{G}}>0 \text{ such that}\\ \quad \|\mathcal{G}(\mathbf{x},\boldsymbol{\sigma}_1,\varepsilon_1,\mathbf{k}_1)-\mathcal{G}(\mathbf{x},\boldsymbol{\sigma}_2,\varepsilon_2,\mathbf{k}_2)\|\\ \quad \le L_{\mathcal{G}}\ (\|\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2\|+\|\varepsilon_1-\varepsilon_2\|+\|\mathbf{k}_1-\mathbf{k}_2\|)\\ \quad \forall\ \boldsymbol{\sigma}_1,\ \boldsymbol{\sigma}_2,\ \varepsilon_1,\ \varepsilon_2\in\mathbb{S}^d,\ \mathbf{k}_1,\ \mathbf{k}_2\in\mathbb{R}^m,\text{ a.e. } \mathbf{x}\in\Omega.\\ \text{(c) The mapping } \mathbf{x}\mapsto\mathcal{G}(\mathbf{x},\boldsymbol{\sigma},\boldsymbol{\varepsilon},\mathbf{k}) \text{ is measurable on } \Omega\ ,\\ \quad \text{for any } \boldsymbol{\sigma},\boldsymbol{\varepsilon}\in\mathbb{S}^d\ ,\ \mathbf{k}\in\mathbb{R}^m.\\ \text{(e) The mapping } \mathbf{x}\mapsto\mathcal{G}(\mathbf{x},\mathbf{0},\mathbf{0},\mathbf{0}) \text{ belongs to } \mathcal{H}.\end{array}\right. \tag{3.11}$$

Also, we assume that the constitutive function $\varphi:\Omega\times\mathbb{S}^d\times\mathbb{S}^d\times\ \mathbb{R}^m\to\mathbb{R}^m$ satisfies

$$\left\{\begin{array}{l}\text{(a) There exists } L_{\varphi}>0 \text{ such that}\\ \quad \|\varphi(\mathbf{x},\boldsymbol{\sigma}_1,\ \varepsilon_1,\mathbf{k}_1)-\varphi(\mathbf{x},\boldsymbol{\sigma}_2,\varepsilon_2,\mathbf{k}_2)\|\\ \quad \le L_{\varphi}\ (\|\boldsymbol{\sigma}_1-\boldsymbol{\sigma}_2\|+\|\varepsilon_1-\varepsilon_2\|+\|\mathbf{k}_1-\mathbf{k}_2\|)\\ \quad \forall\ \boldsymbol{\sigma}_1,\ \boldsymbol{\sigma}_2,\ \varepsilon_1,\ \varepsilon_2\in\mathbb{S}^d,\ \mathbf{k}_1,\ \mathbf{k}_2\in\mathbb{R}^m,\text{ a.e. } \mathbf{x}\in\Omega.\\ \text{(b) The mapping } \mathbf{x}\mapsto\varphi(\mathbf{x},\boldsymbol{\sigma},\boldsymbol{\varepsilon},\boldsymbol{k}) \text{ is measurable on } \Omega,\\ \quad \text{for any } \boldsymbol{\sigma},\boldsymbol{\varepsilon}\in\mathbb{S}^d\ ,\ \mathbf{k}\in\mathbb{R}^m.\\ \text{(c) The mapping } \mathbf{x}\mapsto\varphi(\mathbf{x},\mathbf{0},\mathbf{0},\mathbf{0}) \text{ belongs to } Y.\end{array}\right. \tag{3.12}$$

The function $p : \Gamma_3 \times \mathbb{R} \longrightarrow \mathbb{R}^+$ satisfies

$$\begin{cases} \text{(a)} \;\; \exists\, L_p > 0 \text{ such that} \\ \qquad |p(\mathbf{x}, r_1) - p(\mathbf{x}, r_2)| \leq L_p\, |r_1 - r_2| \quad \forall r_1, r_2 \in \mathbb{R}, \text{ a.e. } \mathbf{x} \in \Gamma_3. \\ \text{(b)} \;\; (p(\mathbf{x}, r_1) - p(\mathbf{x}, r_2))(r_1 - r_2) \geq 0 \quad \forall r_1, r_2 \in \mathbb{R}, \text{ a.e .} \mathbf{x} \in \Gamma_3. \\ \text{(c)} \;\; \text{The mapping } \mathbf{x} \longmapsto p(\mathbf{x}, r) \text{ is measurable on } \Gamma_3, \\ \;\; \text{for all } r \in \mathbb{R}. \\ \text{(d)} \;\; p(\mathbf{x}, r) = 0 \qquad \forall r \leq 0, \text{ a.e. } \mathbf{x} \in \Gamma_3. \end{cases} \tag{3.13}$$

The friction coefficient μ satisfies

$$\mu \in L^\infty(\Gamma_3), \;\; \mu \geq 0 \;\text{ a.e. } \mathbf{x} \in \Gamma_3. \tag{3.14}$$

The densities of body forces and surface tractions are such that

$$\mathbf{f}_0 \in C\,(0, T; H), \qquad\qquad \mathbf{f}_2 \in C\,(0, T; L^2(\Gamma_2)^d). \tag{3.15}$$

Finally, the initial data verify

$$\mathbf{u}_0 \in U. \tag{3.16}$$

$$\mathbf{k}_0 \in Y. \tag{3.17}$$

After that, we introduce the set of admissible velocities U defined by

$$U = \{\mathbf{v} \in V \;:\; v_\nu \leq g \text{ a.e. on}\, \Gamma_3\}. \tag{3.18}$$

We note that U is a nonempty, closed, convex subset of the space V and, on U, we use the inner product of V.
Next, we use the Green formula (2.3) to find that

$$(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{v} - \dot{\mathbf{u}}(t))_{\mathcal{H}} + (Div\; \boldsymbol{\sigma}(t), \mathbf{v} - \dot{\mathbf{u}}(t))_H = \int_\Gamma \boldsymbol{\sigma\nu}\,(t)\,(\mathbf{v} - \dot{\mathbf{u}}(t))\,da$$
$$= \int_{\Gamma_1} \boldsymbol{\sigma\nu}\,(t)\,(\mathbf{v} - \dot{\mathbf{u}}(t))\,da + \int_{\Gamma_2} \boldsymbol{\sigma\nu}\,(t)\,(\mathbf{v} - \dot{\mathbf{u}}(t))\,da + \int_{\Gamma_3} \boldsymbol{\sigma\nu}\,(t)\,(\mathbf{v} - \dot{\mathbf{u}}(t))\,da,$$

for all $\mathbf{v} \in U$. Since $\mathbf{v} - \dot{\mathbf{u}}(t) = 0$ on Γ_1, $\boldsymbol{\sigma\nu}(t) = \mathbf{f}_2(t)$ on Γ_2 and $Div\; \boldsymbol{\sigma}(t) = -\mathbf{f}_0(t)$ in Ω , we obtain

$$\begin{aligned} (\boldsymbol{\sigma}(t)\,,\, \boldsymbol{\varepsilon}(\mathbf{v} - \dot{\mathbf{u}}(t)))_{\mathcal{H}} &= (\mathbf{f}_0(t), \mathbf{v} - \dot{\mathbf{u}}(t))_H \\ &+ \int_{\Gamma_2} \mathbf{f}_2(t).(\mathbf{v} - \dot{\mathbf{u}}(t))\; da \;+\; \int_{\Gamma_3} \boldsymbol{\sigma\nu}(t).(\mathbf{v} - \dot{\mathbf{u}}(t))\,da. \end{aligned} \tag{3.19}$$

On the other hand, we use Riesz's theorem to define the element $\mathbf{f}(t) \in V$ by

$$(\mathbf{f}(t)\,,\, \mathbf{v})_V \;=\; \int_\Omega \mathbf{f}_0(t).\mathbf{v}\; d\mathbf{x} \;+\; \int_{\Gamma_2} \mathbf{f}_2(t).\mathbf{v}\; da \;\; \forall \mathbf{v} \in V, \tag{3.20}$$

where $\mathbf{f} : [0, T] \to V$. It follows from hypotheses (3.15) that the integral (3.20) is well-defined and we have

$$\mathbf{f} \in C\,(0, T; V)\,. \tag{3.21}$$

Now, we note that

$$\boldsymbol{\sigma\nu}(t)(\mathbf{v} - \dot{\mathbf{u}}(t)) \;= \sigma_\nu(t)(v_\nu - \dot{u}_\nu(t)) \;+\; \boldsymbol{\sigma}_\tau(t)(\mathbf{v}_\tau - \dot{\mathbf{u}}_\tau(t)) \;\; \text{on}\, \Gamma_3 \times [0, T]\,.$$

We combine (3.19), (3.20) and the last equality to obtain

$$\begin{aligned} (\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{v} - \dot{\mathbf{u}}(t)))_{\mathcal{H}} &= (\mathbf{f}(t), \mathbf{v})_V \\ &+ \int_{\Gamma_3} \sigma_\nu(t)(v_\nu - \dot{u}_\nu(t))\,da + \int_{\Gamma_3} \boldsymbol{\sigma_\tau}(t)(\mathbf{v}_\tau - \dot{\mathbf{u}}_\tau(t))\,da \;\forall \mathbf{v} \in U. \end{aligned} \tag{3.22}$$

Next, we write

$$\sigma_\nu(t)(v_\nu - \dot{u}_\nu(t)) = [\sigma_\nu(t) + p(\dot{u}_\nu(t))]\,(v_\nu - g) \\ +[\sigma_\nu(t) + p(\dot{u}_\nu(t))]\,(g - \dot{u}_\nu(t)) - p(\dot{u}_\nu(t))\,(v_\nu - \dot{u}_\nu(t)),$$

for all $\mathbf{v} \in U$. Moreover, (3.4), (3.6) and (3.18) show that

$$\dot{\mathbf{u}}(t) \in U\,,\ \mathbf{u}(t) \in V. \tag{3.23}$$

Thus, we deduce that $v_\nu - g \le 0$ and $\dot{u}_\nu - g \le 0$; in addition, we use the contact conditions (3.6) to obtain

$$\sigma_\nu(t)(v_\nu - \dot{u}_\nu(t)) \ge -p(\dot{u}_\nu(t))\,(v_\nu - \dot{u}_\nu(t)) \qquad \text{on } \Gamma_3.$$

We integrate the last inequality on Γ_3 to find that

$$\int_{\Gamma_3} \sigma_\nu(t)(v_\nu - \dot{u}_\nu(t))\,da \ge -\int_{\Gamma_3} p(\dot{u}_\nu(t))(v_\nu - \dot{u}_\nu(t))\,da. \tag{3.24}$$

Also, we use (3.7) to see that, if $\dot{\mathbf{u}}_\tau \neq \mathbf{0}$, we have

$$\boldsymbol{\sigma}_{\boldsymbol{\tau}}(\mathbf{v}_\tau - \dot{\mathbf{u}}_\tau) = -\mu\, p(\dot{u}_\nu)\frac{\dot{\mathbf{u}}_\tau \mathbf{v}_\tau}{\|\dot{\mathbf{u}}_\tau\|} + \mu\, p(\dot{u}_\nu)\,\|\dot{\mathbf{u}}_\tau\|\,. \tag{3.25}$$

Using the Cauchy-Schwartz inequality, we obtain

$$-\mu\, p(\dot{u}_\nu)\frac{\dot{\mathbf{u}}_\tau \mathbf{v}_\tau}{\|\dot{\mathbf{u}}_\tau\|} + \mu\, p(\dot{u}_\nu)\,\|\dot{\mathbf{u}}_\tau\| \ \ge\ -\mu\, p(\dot{u}_\nu)\,\|\mathbf{v}_\tau\| + \mu\, p(\dot{u}_\nu)\,\|\dot{\mathbf{u}}_\tau\|\,.$$

Now, from (3.25) and the last inequality we find that

$$\boldsymbol{\sigma}_\tau(\mathbf{v}_\tau - \dot{\mathbf{u}}_\tau) \ \ge\ \mu\, p(\dot{u}_\nu)(\|\dot{\mathbf{u}}_\tau\| \ -\ \|\mathbf{v}_\tau\|)\ \text{if}\,\dot{\mathbf{u}}_\tau \neq \mathbf{0}. \tag{3.26}$$

On the other hand, if $\dot{\mathbf{u}}_\tau = \mathbf{0}$, then

$$\boldsymbol{\sigma}_\tau(\mathbf{v}_\tau - \dot{\mathbf{u}}_\tau) = \boldsymbol{\sigma}_\tau \mathbf{v}_\tau.$$

From the Cauchy-Schwartz inequality and (3.7), we obtain

$$\begin{aligned} \boldsymbol{\sigma}_\tau.\mathbf{v}_\tau &\ge -\|\boldsymbol{\sigma}_\tau\|\,.\,\|\mathbf{v}_\tau\| \\ &\ge -\mu\, p(\dot{u}_\nu).\,\|\mathbf{v}_\tau\|\,. \end{aligned}$$

Since $\dot{\mathbf{u}}_\tau = \mathbf{0}$, the last inequality can be written as follows

$$\boldsymbol{\sigma}_\tau.\mathbf{v}_\tau - \boldsymbol{\sigma}_\tau.\dot{\mathbf{u}}_\tau \quad \ge \quad -\mu\, p(\dot{u}_\nu)\,\|\mathbf{v}_\tau\| + \mu\, p(\dot{u}_\nu)\,\|\dot{\mathbf{u}}_\tau\|\ \ ,$$

which yields

$$\boldsymbol{\sigma}_\tau(\mathbf{v}_\tau - \dot{\mathbf{u}}_\tau) \quad \ge \quad \mu\, p(\dot{u}_\nu)(\|\dot{\mathbf{u}}_\tau\| - \|\mathbf{v}_\tau\|)\ \text{if}\,\dot{\mathbf{u}}_\tau = \mathbf{0}. \tag{3.27}$$

We conclude from (3.26) and (3.27) that

$$\int_{\Gamma_3} \boldsymbol{\sigma}_\tau(t)(\mathbf{v}_\tau - \dot{\mathbf{u}}_\tau(t))\,da \quad \ge \quad \int_{\Gamma_3} \mu\, p(\dot{u}_\nu(t))(\|\dot{\mathbf{u}}_\tau(t)\| - \|\mathbf{v}_\tau\|)\,da. \tag{3.28}$$

We gather (3.22),(3.24) and (3.28) to find that

$$\begin{aligned} (\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{v} - \dot{\mathbf{u}}(t)))_{\mathcal{H}} \ge (\mathbf{f}(t), \mathbf{v} - \dot{\mathbf{u}}(t))_V + \textstyle\int_{\Gamma_3} p\,(\dot{u}_\nu(t))\,(\dot{u}_\nu(t) - v_\nu)\ da \\ + \textstyle\int_{\Gamma_3} \mu\, p\,(\dot{u}_\nu(t))\,(\|\dot{\mathbf{u}}_\tau(t)\| - \|\mathbf{v}_\tau\|)\ da. \end{aligned} \tag{3.29}$$

To finalize the variational formulation of Problem P, we use again the Riesz's theorem to define the operator $P : V \to V$ by

$$(P\mathbf{u}, \mathbf{v})_V = \int_{\Gamma_3} p(u_\nu)\, v_\nu \, da \ \ \forall\, \mathbf{u},\, \mathbf{v} \in V. \tag{3.30}$$

It follows from (2.2) and hypotheses (3.13) that

$$(P\mathbf{u} - P\mathbf{v}, \mathbf{u} - \mathbf{v})_V \geq 0, \ \|P\mathbf{u} - P\mathbf{v}\|_V \leq c_0^2 L_p \|\mathbf{u} - \mathbf{v}\|_V \quad \forall \mathbf{u}, \mathbf{v} \in V, \tag{3.31}$$

which means that $P : V \to V$ is monotone and Lipschitz continuous.
Finally, we define the function $j : U \times U \to \mathbb{R}^+$ by

$$j(\mathbf{u}, \mathbf{v}) = \int_{\Gamma_3} \mu\, p\,(u_\nu)\, \|\mathbf{v}_\tau\| \; da \ \ \forall \mathbf{u}, \mathbf{v} \in U. \tag{3.32}$$

We use now (3.30) and (3.32) to see that (3.29) becomes

$$\begin{aligned} &(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{v} - \dot{\mathbf{u}}(t)))_{\mathcal{H}} + (P\dot{\mathbf{u}}(t), \mathbf{v} - \dot{\mathbf{u}}(t))_V \\ &\quad + j(\dot{\mathbf{u}}(t), \mathbf{v}) - j(\dot{\mathbf{u}}(t), \dot{\mathbf{u}}(t)) \geq (\mathbf{f}(t), \mathbf{v} - \dot{\mathbf{u}}(t))_V, \ \forall \mathbf{v} \in U. \end{aligned} \tag{3.33}$$

Lastly, we integrate (3.2) from 0 to t by using initial conditions (3.8) and we use (3.33) and (3.1) to obtain the following variational formulation of Problem P.
Problem PV. Find a displacement field $\mathbf{u} : [0, T] \to U$, a stress field $\boldsymbol{\sigma} : [0, T] \to \mathcal{H}$ and an internal state variable $\mathbf{k} : [0, T] \to Y$ such that

$$\boldsymbol{\sigma}(t) = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)) + \mathcal{B}\boldsymbol{\varepsilon}(\mathbf{u}(t)) + \int_0^t \mathcal{G}(\boldsymbol{\sigma}(s) - \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(s)), \boldsymbol{\varepsilon}(\mathbf{u}(s)), \mathbf{k}(s)) \, ds, \tag{3.34}$$

$$\mathbf{k}(t) = \int_0^t \varphi(\boldsymbol{\sigma}(s) - \mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}(s)), \boldsymbol{\varepsilon}(\mathbf{u}(s)), \mathbf{k}(s)) \, ds + \mathbf{k}_0, \tag{3.35}$$

$$\begin{aligned} &(\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\dot{\mathbf{u}}(t)))_{\mathcal{H}} + (P\dot{\mathbf{u}}(t), \mathbf{v} - \dot{\mathbf{u}}(t))_V \\ &\quad + j(\dot{\mathbf{u}}(t), \mathbf{v}) - j(\dot{\mathbf{u}}(t), \dot{\mathbf{u}}(t)) \geq (\mathbf{f}(t), \mathbf{v} - \dot{\mathbf{u}}(t))_V \ \forall \mathbf{v} \in U, \end{aligned} \tag{3.36}$$

$$\mathbf{u}(0) = \mathbf{u}_0. \tag{3.37}$$

4. Existence and uniqueness result

In this section, we study the existence and the uniqueness of the solution of the variational problem PV introduced in section 3. We summarize this study in the following result.
Theorem 4.1. *Assume that hypotheses* (3.9) - (3.17) *are satisfied. Then, there exists a constant* $L_0 > 0$ *such that, if* $L_p < L_0$, *then the problem PV has a unique solution* $\{\mathbf{u}, \boldsymbol{\sigma}, \mathbf{k}\}$. *Moreover , the solution satisfies*

$$\begin{aligned} &\mathbf{u} \in C^1(0, T; V), \\ &\boldsymbol{\sigma} \in C(0, T; \mathcal{H}_1), \\ &\mathbf{k} \in C^1(0, T; Y). \end{aligned} \tag{4.1}$$

Now let's move on to the proof of Theorem 4.1 which will be carried out in several steps. We assume in what follows that hypotheses (3.9)-(3.17) are satisfied. We use the product space $\mathcal{H} \times Y$ endowed with the norm

$$\|\boldsymbol{\eta}\|_{\mathcal{H}\times Y} = \|\boldsymbol{\eta}^{(1)}\|_{\mathcal{H}} + \|\boldsymbol{\eta}^{(2)}\|_{Y} \quad \forall \boldsymbol{\eta} = (\boldsymbol{\eta}^{(1)}, \boldsymbol{\eta}^{(2)}) \in \mathcal{H} \times Y. \tag{4.2}$$

Step 1. For all $\boldsymbol{\eta} = (\boldsymbol{\eta}^{(1)}, \boldsymbol{\eta}^{(2)}) \in C(0,T;\mathcal{H} \times Y)$, we consider the following intermediate variational problem.

Problem PV_η. Find a displacement field $\mathbf{u}_\eta : [0,T] \to U$ such that

$$\begin{aligned}&(\mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}_\eta(t)), \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\dot{\mathbf{u}}_\eta(t)))_{\mathcal{H}} + (\boldsymbol{\eta}^{(1)}(t), \boldsymbol{\varepsilon}(\mathbf{v}) - \boldsymbol{\varepsilon}(\dot{\mathbf{u}}_\eta(t)))_{\mathcal{H}}\\ &+(P\dot{\mathbf{u}}_\eta(t), \mathbf{v} - \dot{\mathbf{u}}_\eta(t))_V + j(\dot{\mathbf{u}}_\eta(t), \mathbf{v}) - j(\dot{\mathbf{u}}_\eta(t), \dot{\mathbf{u}}_\eta(t))\\ &\quad \geq (\mathbf{f}(t), \mathbf{v} - \dot{\mathbf{u}}_\eta(t))_V \quad \forall \mathbf{v} \in U.\end{aligned} \tag{4.3}$$

$$\mathbf{u}_\eta(0) = \mathbf{u}_0. \tag{4.4}$$

We have the following existence and uniqueness result.

Lemma 4.2. *If $L_p < L_0$, then there exists a unique solution $\mathbf{u}_\eta$ to Problem PV_η such that $\mathbf{u}_\eta \in C^1(0,T;V)$. Moreover, if $\mathbf{u}_i = \mathbf{u}_{\eta_i}$ are two solutions to Problem PV_η corresponding to $\boldsymbol{\eta}_i \in C(0,T;\mathcal{H}\times Y)$, $i = 1, 2$, then there exists a constant $c > 0$ such that*

$$||\dot{\mathbf{u}}_1(t) - \dot{\mathbf{u}}_2(t)||_V \leq c\ \|\boldsymbol{\eta}_1(t) - \boldsymbol{\eta}_2(t)\|_{\mathcal{H}\times Y} \quad \forall t \in [0,T]. \tag{4.5}$$

Proof. First, we use Riesz's Theorem to define the operator $A : V \to V$ and the function $\mathbf{f}_\eta : [0,T] \to V$ by equalities

$$(A\mathbf{u}, \mathbf{v})_V = (\mathcal{A}\boldsymbol{\varepsilon}(\mathbf{u}),\ \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}} + (P\mathbf{u}, \mathbf{v})_V, \tag{4.6}$$

$$(\mathbf{f}_\eta(t), \mathbf{v})_V = (\mathbf{f}(t), \mathbf{v})_V - (\boldsymbol{\eta}^{(1)}(t), \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}}, \tag{4.7}$$

for all $\mathbf{u}, \mathbf{v} \in V$ and $t \in [0,T]$. Hence, (4.3) becomes

$$\begin{aligned}&(A\dot{\mathbf{u}}_\eta(t), \mathbf{v} - \dot{\mathbf{u}}_\eta(t))_V + j(\dot{\mathbf{u}}_\eta(t), \mathbf{v}) - j(\dot{\mathbf{u}}_\eta(t), \dot{\mathbf{u}}_\eta(t))\\ &\quad \geq (\mathbf{f}_\eta(t), \mathbf{v} - \dot{\mathbf{u}}_\eta(t))_V \ \forall \mathbf{v} \in U,\end{aligned} \tag{4.8}$$

and using the notation $\mathbf{w}_\eta(t) = \dot{\mathbf{u}}_\eta(t)$, we can see that the last inequality can be written as follows

$$\begin{aligned}&(A\mathbf{w}_\eta(t), \mathbf{v} - \mathbf{w}_\eta(t))_V + j(\mathbf{w}_\eta(t), \mathbf{v}) - j(\mathbf{w}_\eta(t), \mathbf{w}_\eta(t))\\ &\quad \geq (\mathbf{f}_\eta(t), \mathbf{v} - \mathbf{w}_\eta(t))_V \ \forall \mathbf{v} \in U,\end{aligned} \tag{4.9}$$

Next, we apply Theorem 2.2 for $K = U$ and $X = V$. First, we note that the space U defined in (3.18) satisfies conditions (2.5). Next, we consider $\mathbf{w_1}, \mathbf{w_2} \in V$ and we use the monotonicity of the operator P expressed in (3.31) as well as (3.9)(c) and (2.1) to obtain

$$(A\mathbf{w}_1 - A\mathbf{w}_2, \mathbf{w}_1 - \mathbf{w}_2)_V \geq M_{\mathcal{A}}||\mathbf{w}_1 - \mathbf{w}_2||_V^2. \quad \forall \mathbf{w}_1, \mathbf{w}_2 \in V,$$

which shows that A is strongly monotone with constant $M_A = M_{\mathcal{A}}$.

On the other hand, for $\mathbf{w}_1, \mathbf{w}_2, \mathbf{v} \in V$, we use the Lipschitz continuity of P expressed in (3.31) as well as (3.9)(b) and (2.1) to find

$$(A\mathbf{w}_1 - A\mathbf{w}_2, \mathbf{v})_V \leq (L_{\mathcal{A}} + c_0^2\, L_p)\, ||\mathbf{w}_1 - \mathbf{w}_2||_V ||\mathbf{v}||_V.$$

By choosing $\mathbf{v} = A\mathbf{w}_1 - A\mathbf{w}_2$ in the last inequality we obtain

$$||A\mathbf{w}_1 - A\mathbf{w}_2||_V \leq (L_{\mathcal{A}} + c_0^2\, L_p)\, ||\mathbf{w}_1 - \mathbf{w}_2||_V,$$

which means that A is a Lipschitz continuous operator with constant $L_A = L_{\mathcal{A}} + c_0^2 L_p$. We conclude that conditions (2.6) are satisfied.
Now we prove conditions (2.7) on the function j. First, it is easy to see that $j(\mathbf{w}, .)$ is a semi-norm on V, for all $\mathbf{w} \in V$. Moreover, we recall that $\|\mathbf{v}_\tau\| \leq \|\mathbf{v}\|$ and we use (3.13), (3.14) and (2.2) to see that for all $\mathbf{w} \in V$,

$$j(\mathbf{w}, \mathbf{v}) \leq c \, \|\mathbf{v}\|_V.$$

We conclude that $j(\mathbf{w}, .)$ is a continuous semi-norm on V and thus it is convex and lower semi-continuous on V, which means that it satisfies condition (2.7)(a) of Theorem 2.2. Now, for all $\mathbf{w}_1, \mathbf{w}_2, \mathbf{v}_1, \mathbf{v}_2 \in V$, we use assumptions (3.13) (a) and (3.14), after a simple calculation we obtain

$$\begin{aligned} & j(\mathbf{w}_1, \mathbf{v}_2) - j(\mathbf{w}_1, \mathbf{v}_1) + j(\mathbf{w}_2, \mathbf{v}_1) - j(\mathbf{w}_2, \mathbf{v}_2) \\ & \qquad \leq \mu \, L_p \int_{\Gamma_3} |w_{1\nu} - w_{2\nu}| \; \big| \, \|\mathbf{v}_{1\tau}\| - \|\mathbf{v}_{2\tau}\| \, \big| \; da \; . \end{aligned}$$

Next, it is well known that $|w_{1\nu} - w_{2\nu}| \leq \|\mathbf{w}_1 - \mathbf{w}_2\|$, $\big| \, \|\mathbf{v}_{1\tau}\| - \|\mathbf{v}_{2\tau}\| \, \big| \leq \|\mathbf{v}_1 - \mathbf{v}_2\|$. Thus, we obtain

$$\begin{aligned} & j(\mathbf{w}_1, \mathbf{v}_2) - j(\mathbf{w}_1, \mathbf{v}_1) + j(\mathbf{w}_2, \mathbf{v}_1) - j(\mathbf{w}_2, \mathbf{v}_2) \\ & \qquad \leq \mu \, L_p \, \|\mathbf{w}_1 - \mathbf{w}_2\|_{L^2(\Gamma_3)^d} \, \|\mathbf{v}_1 - \mathbf{v}_2\|_{L^2(\Gamma_3)^d} \, . \end{aligned}$$

Hence, inequality (2.2) yields

$$\begin{aligned} & j(\mathbf{w}_1, \mathbf{v}_2) - j(\mathbf{w}_1, \mathbf{v}_1) + j(\mathbf{w}_2, \mathbf{v}_1) - j(\mathbf{w}_2, \mathbf{v}_2) \\ & \leq c_0^2 \, \mu \, L_p \, \|\mathbf{w}_1 - \mathbf{w}_2\|_V \, \|\mathbf{v}_1 - \mathbf{v}_2\|_V \ , \end{aligned} \tag{4.10}$$

for all $\mathbf{w}_1, \mathbf{w}_2, \mathbf{v}_1, \mathbf{v}_2 \in V$. We note that the last inequality shows that the condition (2.7) (b) is satisfied for $\alpha = c_0^2 \, \mu \, L_p$.
Moreover, we use (3.21) and (4.7) and we recall that $\boldsymbol{\eta} \in C(0, T; \mathcal{H} \times Y)$ to deduce that $\mathbf{f}_\eta \in C(0, T; V)$; i.e. $\mathbf{f}_\eta$ satisfies (2.8). Finally, for the condition (2.9) to be satisfied, we choose $L_0 = \dfrac{M_{\mathcal{A}}}{c_0^2 \, \mu}$. As a consequence, if $L_p < L_0$, then $M_{\mathcal{A}} > c_0^2 \, \mu \, L_p$, which means that condition (2.9) of Theorem 2.2 is now satisfied. We conclude that there exists a unique solution $\mathbf{w}_\eta \in C(0, T; U)$ to the quasivariational (4.9). Now, we use (4.4) and we define the displacement $\mathbf{u}_\eta$ by

$$\mathbf{u}_\eta(t) = \int_0^t \mathbf{w}_\eta(s) \, ds + \mathbf{u}_0. \tag{4.11}$$

It results from the last equality and hypothesis (3.16) that $\mathbf{u}_\eta \in C^1(0, T; V)$ is the unique solution of the quasivariational inequality (4.8). Finally, we can see that, by substituting (4.6)-(4.7) in (4.8), we find that $\mathbf{u}_\eta \in C^1(0, T; V)$ is the unique solution of PV_η, which concludes the existence and uniqueness part of Lemma 4.2.
We turn now to the proof of estimate (4.5). To this end, let consider

$$\boldsymbol{\eta}_1 = (\boldsymbol{\eta}_1^{(1)}, \boldsymbol{\eta}_1^{(2)}), \; \boldsymbol{\eta}_2 = (\boldsymbol{\eta}_2^{(1)}, \boldsymbol{\eta}_2^{(2)}) \in C(0, T; \mathcal{H} \times Y)$$

and let use the notations $\mathbf{u}_1 = \mathbf{u}_{\eta_1}$, $\mathbf{u}_2 = \mathbf{u}_{\eta_2}$. We write (4.3) for $\mathbf{u}_\eta(t) = \mathbf{u}_1(t)$ and $\mathbf{v} = \dot{\mathbf{u}}_2\,(t)$ and then for $\mathbf{u}_\eta(t) = \mathbf{u}_2(t)$ with $\mathbf{v} = \dot{\mathbf{u}}_1\,(t)$, after a simple calculation we obtain

$$\begin{aligned}&(\mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}_1(t))-\mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}_2(t)),\boldsymbol{\varepsilon}(\dot{\mathbf{u}}_1(t))-\boldsymbol{\varepsilon}(\dot{\mathbf{u}}_2(t)))_{\mathcal{H}}\\&+(P\dot{\mathbf{u}}_1(t)-P\dot{\mathbf{u}}_2(t),\dot{\mathbf{u}}_1(t)-\dot{\mathbf{u}}_2(t))_V\\&\qquad\le(\boldsymbol{\eta}_1^{(1)}(t)-\boldsymbol{\eta}_2^{(1)}(t),\boldsymbol{\varepsilon}(\dot{\mathbf{u}}_2(t))-\boldsymbol{\varepsilon}(\dot{\mathbf{u}}_1(t)))_{\mathcal{H}}\\&\qquad\ +j(\dot{\mathbf{u}}_1(t),\dot{\mathbf{u}}_2(t))-j(\dot{\mathbf{u}}_1(t),\dot{\mathbf{u}}_1(t))+j(\dot{\mathbf{u}}_2(t),\dot{\mathbf{u}}_1(t))-j(\dot{\mathbf{u}}_2(t),\dot{\mathbf{u}}_2(t)).\end{aligned}$$

On the one hand, we note that $\mathbf{u}_i\in C^1(0,T;V)$, $i=1,2$; this implies $\dot{\mathbf{u}}_i(t)\in V$, $i=1,2$. Then, we use (3.9) (c), the monotonicity of P expressed in (3.31) and (2.1) to find

$$\begin{aligned}&(\mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}_1(t))-\mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}_2(t)),\boldsymbol{\varepsilon}(\dot{\mathbf{u}}_1(t))-\boldsymbol{\varepsilon}(\dot{\mathbf{u}}_2(t)))_{\mathcal{H}}\\&+(P\dot{\mathbf{u}}_1(t)-P\dot{\mathbf{u}}_2(t),\dot{\mathbf{u}}_1(t)-\dot{\mathbf{u}}_2(t))_V\\&\qquad\ge M_{\mathcal{A}}\left\|\dot{\mathbf{u}}_1(t)-\dot{\mathbf{u}}_2(t)\right\|_V^2.\end{aligned}$$

On the othor hand, we use (4.10) to deduce that

$$\begin{aligned}&(\mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}_1(t))-\mathcal{A}\boldsymbol{\varepsilon}(\dot{\mathbf{u}}_2(t)),\boldsymbol{\varepsilon}(\dot{\mathbf{u}}_1(t))-\boldsymbol{\varepsilon}(\dot{\mathbf{u}}_2(t)))_{\mathcal{H}}\\&+(P\dot{\mathbf{u}}_1(t)-P\dot{\mathbf{u}}_2(t),\dot{\mathbf{u}}_1(t)-\dot{\mathbf{u}}_2(t))_V\\&\qquad\le\left\|\boldsymbol{\eta}_1^{(1)}(t)-\boldsymbol{\eta}_2^{(1)}(t)\right\|_{\mathcal{H}}\left\|\boldsymbol{\varepsilon}(\dot{\mathbf{u}}_2(t))-\boldsymbol{\varepsilon}(\dot{\mathbf{u}}_1(t))\right\|_{\mathcal{H}}\\&\qquad\ +c_0^2\,\mu L_p\left\|\dot{\mathbf{u}}_1(t)-\dot{\mathbf{u}}_2(t)\right\|_V\left\|\dot{\mathbf{u}}_1(t)-\dot{\mathbf{u}}_2(t)\right\|_V.\end{aligned}$$

We combine the two last inequalities and we recall (2.1) to find

$$M_{\mathcal{A}}\left\|\dot{\mathbf{u}}_1(t)-\dot{\mathbf{u}}_2(t)\right\|_V\le\left\|\boldsymbol{\eta}_1^{(1)}(t)-\boldsymbol{\eta}_2^{(1)}(t)\right\|_{\mathcal{H}}+c_0^2\,\mu L_p\left\|\dot{\mathbf{u}}_1(t)-\dot{\mathbf{u}}_2(t)\right\|_V.$$

Now we use (4.2) to deduce that

$$\left\|\boldsymbol{\eta}_1^{(1)}(s)-\boldsymbol{\eta}_2^{(1)}(s)\right\|_{\mathcal{H}}\le\left\|\boldsymbol{\eta}_1(s)-\boldsymbol{\eta}_2(s)\right\|_{\mathcal{H}\times Y}.$$

Then, we combine the last two inequalities to obtain

$$(M_{\mathcal{A}}-c_0^2\,\mu\,L_p)\left\|\dot{\mathbf{u}}_1(t)-\dot{\mathbf{u}}_2(t)\right\|_V\le\left\|\boldsymbol{\eta}_1(t)-\boldsymbol{\eta}_2(t)\right\|_{\mathcal{H}\times Y}.$$

Finally, we recall that the condition (2.9) of Theorem 2.2 is satisfied for $M_A=M_{\mathcal{A}}$ and $\alpha=c_0^2\mu\,L_p$, which yields $M_{\mathcal{A}}>c_0^2\,\mu\,L_p$; i.e. $M_{\mathcal{A}}-c_0^2\mu\,L_p>0$. Hence, we conclude that the estimate (4.5) is satisfied. □

Step 2. In the second step of the proof of Theorem 4.1, we denote by $\mathbf{k}_\eta\in C(0,T;Y)$ the function defined by

$$\mathbf{k}_\eta(t)=\int_0^t\boldsymbol{\eta}^{(2)}(s)\,ds+\mathbf{k}_0. \tag{4.12}$$

Step 3. The third step of the proof consists of using the displacement field $\mathbf{u}_\eta$ which was obtained in Lemma 4.2 and the function $\mathbf{k}_\eta$ defined in (4.12) to consider the following problem.

Problem Q_η. Find a stress field $\boldsymbol{\sigma}_\eta:[0,T]\to\mathcal{H}$ such that

$$\boldsymbol{\sigma}_\eta(t)=\mathcal{B}\boldsymbol{\varepsilon}(\mathbf{u}_\eta(t))+\int_0^t\mathcal{G}(\boldsymbol{\sigma}_\eta(s),\boldsymbol{\varepsilon}(\mathbf{u}_\eta(s)),\mathbf{k}_\eta(s))\,ds, \tag{4.13}$$

In the study of Problem Q_η we have the following result.
Lemma 4.3. *There exists a unique solution to Problem* Q_η *which satisfies* $\boldsymbol{\sigma}_\eta \in C(0,T;\mathcal{H})$. *Moreover, for all* $\boldsymbol{\eta}_i \in C(0,T;\mathcal{H}\times Y)$, $i=1,2$, *if* $\boldsymbol{\sigma}_i = \boldsymbol{\sigma}_{\eta_i}$ *and* $\mathbf{u}_i = \mathbf{u}_{\eta_i}$ *represent the solutions of Problems* Q_η *and* PV_η *respectively and* $\mathbf{k}_i = \mathbf{k}_{\eta_i}$, $i=1,2$ *are defined by* (4.12) *then there exists* $c>0$ *such that*

$$\begin{aligned}&\| \boldsymbol{\sigma}_1(t) - \boldsymbol{\sigma}_2(t)\|_{\mathcal{H}} \leq \\ &c\left(\|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_V + \int_0^t \|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_V \, ds + \int_0^t \|\mathbf{k}_1(s) - \mathbf{k}_2(s)\|_Y \, ds\right),\end{aligned} \tag{4.14}$$

$\forall t \in [0,T]$.

Proof. We introduce the operator $\Lambda_\eta : C(0,T;\mathcal{H}) \longrightarrow C(0,T;\mathcal{H})$ defined by

$$\Lambda_\eta \boldsymbol{\sigma}(t) \;=\; \mathcal{B}\varepsilon(\mathbf{u}_\eta(t)) + \int_0^t \mathcal{G}(\boldsymbol{\sigma}(s), \varepsilon(\mathbf{u}_\eta(s)), \mathbf{k}_\eta(s))\, ds. \tag{4.15}$$

First, we can see that assumptions (3.10) and (3.11) on $\mathcal{B}$ and $\mathcal{G}$ show that the operator Λ_η is well-defined. Next, for all $\boldsymbol{\sigma} \in C(0,T;\mathcal{H})$ and $t \in [0,T]$, we consider $\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2 \in C(0,T;\mathcal{H})$ and we use (4.15) and (3.11)(b) to obtain for all $t \in [0,T]$,

$$\|\Lambda_\eta \boldsymbol{\sigma}_1(t) - \Lambda_\eta \boldsymbol{\sigma}_2(t)\|_{\mathcal{H}} \leq \quad L_{\mathcal{G}} \int_0^t \|\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)\|_{\mathcal{H}} \; ds \;.$$

The reiteration of the last inequality p times yields

$$\left\|\Lambda_\eta^p \boldsymbol{\sigma}_1(t) - \Lambda_\eta^p \boldsymbol{\sigma}_2(t)\right\|_{\mathcal{H}} \leq (L_{\mathcal{G}})^p \underbrace{\int_0^t \int_0^s \ldots \int_0^r}_{p \text{ times}} \|\boldsymbol{\sigma}_1(l) - \boldsymbol{\sigma}_2(l)\|_{\mathcal{H}} \; dl,$$

which implies

$$\left\|\Lambda_\eta^p \boldsymbol{\sigma}_1 - \Lambda_\eta^p \boldsymbol{\sigma}_2\right\|_{C(0,T;\mathcal{H})} \leq \frac{c^p \, T^p}{p!} \; \|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2\|_{C(0,T;\mathcal{H})} \,.$$

It results from the last inequality that for p large enough, $\lim_{p\to+\infty} \dfrac{c^p T^p}{p!} = 0$; and therefore the operator Λ_η^p is a contraction on the Banach space $C(0,T;\mathcal{H})$. So we can deduce that there exists a unique function $\boldsymbol{\sigma}_\eta \in C(0,T;\mathcal{H})$ such that

$$\Lambda_\eta \boldsymbol{\sigma}_\eta \;=\; \boldsymbol{\sigma}_\eta.$$

The last equality combined with (4.15) shows that $\boldsymbol{\sigma}_\eta$ is a solution of Q_η. Its uniqueness follows from the uniqueness of the fixed point of the operator Λ_η.
Now, let consider $\boldsymbol{\eta}_1, \boldsymbol{\eta}_2 \in C(0,T;\mathcal{H}\times Y)$ and, for $i=1,2$, we use the notations $\mathbf{u}_{\eta_i} = \mathbf{u}_i, \boldsymbol{\sigma}_{\eta_i} = \boldsymbol{\sigma}_i$ and $\mathbf{k}_{\eta_i} = \mathbf{k}_i$. We use assumptions (3.10)(b) and (3.11)(b) on $\mathcal{B}$ and $\mathcal{G}$ as well as (2.1) to find

$$\begin{aligned}\|\boldsymbol{\sigma}_1(t) - \boldsymbol{\sigma}_2(t)\|_{\mathcal{H}} \leq c\,(&\|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_V + \int_0^t \|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_V \; ds \\ &+ \int_0^t \|\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)\|_{\mathcal{H}} \; ds \; + \int_0^t \|\mathbf{k}_1(s) - \mathbf{k}_2(s)\|_Y \; ds),\end{aligned} \tag{4.16}$$

for all $t \in [0,T]$. We use now (4.16) and a Gronwell argument to deduce the estimate (4.14). □

Step 4. In this step, we use the properties of $\mathcal{B}$, $\mathcal{G}$ and φ to define the operator $\Lambda : C(0,T;\mathcal{H}\times Y) \to C(0,T;\mathcal{H}\times Y)$ which maps every element $\boldsymbol{\eta} = (\boldsymbol{\eta}^{(1)}, \boldsymbol{\eta}^{(2)}) \in C(0,T;\mathcal{H}\times Y)$ into the element $\Lambda\boldsymbol{\eta}$ given by

$$\Lambda\boldsymbol{\eta}(t) = \left(\mathcal{B}\boldsymbol{\varepsilon}(\mathbf{u}_\eta(t)) + \int_0^t \mathcal{G}\left(\boldsymbol{\sigma}_\eta(s), \boldsymbol{\varepsilon}(\mathbf{u}_\eta(s)), \mathbf{k}_\eta(s)\right) ds, \varphi\left(\boldsymbol{\sigma}_\eta(t), \boldsymbol{\varepsilon}(\mathbf{u}_\eta(t)), \mathbf{k}_\eta(t)\right)\right). \tag{4.17}$$

Here, for all $\boldsymbol{\eta} \in C(0,T;\mathcal{H}\times Y)$, $\mathbf{u}_\eta$ and $\boldsymbol{\sigma}_\eta$ represent respectively the displacement field and the stress field provided in Lemmas 4.2 and 4.3. Moreover, $\mathbf{k}_\eta$ is the internal state variable given by (4.12). We have the following result.

Lemma 4.4. *The operator Λ has a unique fixed point $\boldsymbol{\eta}^* \in C(0,T;\mathcal{H}\times Y)$.*

Proof. Let consider $\boldsymbol{\eta}_1, \boldsymbol{\eta}_2 \in C(0,T;\mathcal{H}\times Y)$ and let use the notations $\mathbf{u}_{\eta_i} = \mathbf{u}_i$, $\boldsymbol{\sigma}_{\eta_i} = \boldsymbol{\sigma}_i$, $\mathbf{k}_{\eta_i} = \mathbf{k}_i$, for $i = 1,2$. We use (4.17),(4.2), (3.10)(b), (3.11)(b), (3.12)(a) and (2.1) to obtain

$$\begin{aligned} &\|\Lambda\boldsymbol{\eta}_1(t) - \Lambda\boldsymbol{\eta}_2(t)\|_{\mathcal{H}\times Y} \\ &\le c\left(\|\boldsymbol{\sigma}_1(t) - \boldsymbol{\sigma}_2(t)\|_{\mathcal{H}} + \|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_V + \|\mathbf{k}_1(t) - \mathbf{k}_2(t)\|_Y\right) \\ &+c\int_0^t \left(\|\boldsymbol{\sigma}_1(s) - \boldsymbol{\sigma}_2(s)\|_{\mathcal{H}} + \|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_V + \|\mathbf{k}_1(s) - \mathbf{k}_2(s)\|_Y\right) ds\ . \end{aligned} \tag{4.18}$$

On the one hand, definition (4.12) yields

$$\|\mathbf{k}_1(t) - \mathbf{k}_2(t)\|_Y \quad \le \quad \int_0^t \left\|\boldsymbol{\eta}_1^{(2)}(s) - \boldsymbol{\eta}_2^{(2)}(s)\right\|_Y \; ds\ ,$$

and, by using (4.2), we deduce that

$$\|\mathbf{k}_1(t) - \mathbf{k}_2(t)\|_Y \quad \le \quad \int_0^t \|\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s)\|_{\mathcal{H}\times Y} \; ds \qquad \forall t \in [0,T]. \tag{4.19}$$

On the other hand, we use the initial condition (3.8) to write

$$\mathbf{u}_i = \mathbf{u}_0 + \int_0^t \dot{\mathbf{u}}_i(s)\ ds \qquad , i = 1,2.$$

Hence,

$$\|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_V \le \int_0^t \|\dot{\mathbf{u}}_1(s) - \dot{\mathbf{u}}_2(s)\|_V \; ds\ .$$

The last inequality combined with the estimate (4.5) implies

$$\|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_V \le c\int_0^t \|\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s)\|_{\mathcal{H}\times Y} \; ds, \tag{4.20}$$

Now, we combine (4.18)-(4.20) and the estimate (4.14) to deduce that

$$\begin{aligned} &\|\Lambda\boldsymbol{\eta}_1(t) - \Lambda\boldsymbol{\eta}_2(t)\|_{\mathcal{H}\times Y} \\ &\le c\int_0^t \|\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s)\|_{\mathcal{H}\times Y}\, ds + c\int_0^t \left(\int_0^s \|\boldsymbol{\eta}_1(r) - \boldsymbol{\eta}_2(r)\|_{\mathcal{H}\times Y} \; dr\right) ds \\ &+c\int_0^t \left(\int_0^s \left(\int_0^r \|\boldsymbol{\eta}_1(l) - \boldsymbol{\eta}_2(l)\|_{\mathcal{H}\times Y} \; dl\right) dr\right) ds, \end{aligned}$$

which yields

$$\|\Lambda\boldsymbol{\eta}_1(t) - \Lambda\boldsymbol{\eta}_2(t)\|_{\mathcal{H}\times Y} \le c\int_0^t \|\boldsymbol{\eta}_1(s) - \boldsymbol{\eta}_2(s)\|_{\mathcal{H}\times Y} \; ds.$$

Finally, we apply Theorem 2.1 to conclude that there exists a unique fixed point $\boldsymbol{\eta}^{*} \in C(0,T;\mathcal{H}\times Y)$ of the operator Λ . □

Now we have all the ingredients to prove Theorem 4.1.

Proof. **Existence.** Let $\boldsymbol{\eta}^{*}=(\boldsymbol{\eta}^{(1)*},\boldsymbol{\eta}^{(2)*})\in C(0,T;\mathcal{H}\times Y)$ be the fixed point of the operator Λ which is defined by (4.17). We use the notations

$$\mathbf{u}\left(t\right) = \mathbf{u}_{\boldsymbol{\eta}^{*}}\left(t\right) \tag{4.21}$$

$$\mathbf{k}\left(t\right) = \mathbf{k}_{\boldsymbol{\eta}^{*}}\left(t\right) \tag{4.22}$$

$$\boldsymbol{\sigma}\left(t\right) = \mathcal{A}\boldsymbol{\varepsilon}\left(\dot{\mathbf{u}}\left(t\right)\right) + \boldsymbol{\sigma}_{\boldsymbol{\eta}^{*}}\left(t\right) . \tag{4.23}$$

We prove that $(\mathbf{u},\boldsymbol{\sigma},\mathbf{k})$ is a solution of the Problem PV with regularity (4.1). In fact, we write (4.13) for $\boldsymbol{\eta}=\boldsymbol{\eta}^{*}$ and we use the notations (4.21)-(4.23) to obtain (3.34). Next, we write (4.3) for $\boldsymbol{\eta}=\boldsymbol{\eta}^{*}$ and we use (4.21) to find that

$$\begin{aligned}&\left(\mathcal{A}\boldsymbol{\varepsilon}\left(\dot{\mathbf{u}}\left(t\right)\right),\boldsymbol{\varepsilon}\left(\mathbf{v}\right)-\boldsymbol{\varepsilon}\left(\dot{\mathbf{u}}(t)\right)\right)_{\mathcal{H}}+\left(\boldsymbol{\eta}^{(1)*}\left(t\right),\boldsymbol{\varepsilon}\left(\mathbf{v}\right)-\boldsymbol{\varepsilon}\left(\dot{\mathbf{u}}(t)\right)\right)_{\mathcal{H}}\\&+(P\dot{\mathbf{u}}(t),\mathbf{v}-\dot{\mathbf{u}}(t))_{V}+j(\dot{\mathbf{u}}(t),\mathbf{v})-j(\dot{\mathbf{u}}(t),\dot{\mathbf{u}}(t))\geq\left(\mathbf{f}\left(t\right),\mathbf{v}-\dot{\mathbf{u}}(t)\right)_{V} ,\end{aligned} \tag{4.24}$$

for all $\mathbf{v}\in V$ and $t\in[0,T]$. Now, we recall that $\Lambda\boldsymbol{\eta}^{*}=\boldsymbol{\eta}^{*}=(\boldsymbol{\eta}^{(1)*},\boldsymbol{\eta}^{(2)*})$. Hence, definition (4.17) and the notations (4.21)-(4.23) yield

$$\boldsymbol{\eta}^{(1)*}\left(t\right)=\mathcal{B}\boldsymbol{\varepsilon}(\mathbf{u}(t))+\int_{0}^{t}\mathcal{G}\left(\boldsymbol{\sigma}\left(s\right)-\mathcal{A}\boldsymbol{\varepsilon}\left(\dot{\mathbf{u}}\left(s\right)\right),\boldsymbol{\varepsilon}\left(\mathbf{u}\left(s\right)\right),\mathbf{k}\left(s\right)\right)\,ds, \tag{4.25}$$

$$\boldsymbol{\eta}^{(2)*}\left(t\right)=\varphi\left(\boldsymbol{\sigma}\left(t\right)-\mathcal{A}\boldsymbol{\varepsilon}\left(\dot{\mathbf{u}}(t)\right),\boldsymbol{\varepsilon}\left(\mathbf{u}\left(t\right)\right),\mathbf{k}\left(t\right)\right) . \tag{4.26}$$

We use (4.26) and (4.12) to see that (3.35) is satisfied. Next, we substitute (4.25) in (4.24) and we use (3.34) to see that (3.36) is also satisfied.
Finally, (3.37) and the regularities of $\mathbf{u}$ and $\mathbf{k}$ which are given in (4.1) follow from the Lemma 4.2 and (4.12), combined with the fact that $\boldsymbol{\eta}^{(2)*}\in C(0,T;Y)$.
Moreover, for the stress tensor $\boldsymbol{\sigma}$, we use (4.23), (3.9) and we recall that from Lemma 4.3 we have $\boldsymbol{\sigma}_{\boldsymbol{\eta}^{*}}(t)\in\mathcal{H}$; hence, we deduce that $\boldsymbol{\sigma}(t)\in\mathcal{H}$. As for the regularity of $\boldsymbol{\sigma}$, we use again (4.23) to find that for all $t_1,t_2\in[0,T]$,

$$\left\|\boldsymbol{\sigma}\left(t_1\right)-\boldsymbol{\sigma}\left(t_2\right)\right\|_{\mathcal{H}}\leq\left\|\mathcal{A}\boldsymbol{\varepsilon}\left(\dot{\mathbf{u}}\left(t_1\right)\right)-\mathcal{A}\boldsymbol{\varepsilon}\left(\dot{\mathbf{u}}\left(t_2\right)\right)\right\|_{\mathcal{H}}+\left\|\boldsymbol{\sigma}_{\eta^{*}}\left(t_1\right)-\boldsymbol{\sigma}_{\eta^{*}}\left(t_2\right)\right\|_{\mathcal{H}} .$$

Thus, hypothesis (3.9)(b) on the operator $\mathcal{A}$ and (2.1) yield

$$\left\|\boldsymbol{\sigma}\left(t_1\right)-\boldsymbol{\sigma}\left(t_2\right)\right\|_{\mathcal{H}}\leq L_{\mathcal{A}}\left\|\dot{\mathbf{u}}\left(t_1\right)-\dot{\mathbf{u}}\left(t_2\right)\right\|_{V}+\left\|\boldsymbol{\sigma}_{\eta^{*}}\left(t_1\right)-\boldsymbol{\sigma}_{\eta^{*}}\left(t_2\right)\right\|_{\mathcal{H}} .$$

The last inequality combined with regularities $\mathbf{u}\in C^1(0,T;V)$ and $\boldsymbol{\sigma}_{\eta^{*}}\in C(0,T;\mathcal{H})$ derived respectively from Lemmas 4.2 and 4.3 shows that $\boldsymbol{\sigma}\in C(0,T;\mathcal{H})$. In order to obtain the regularity $\boldsymbol{\sigma}\in C(0,T;\mathcal{H}_1)$, we test (3.36) with $\mathbf{v}=\dot{\mathbf{u}}+\boldsymbol{\varphi}$ and then with $\mathbf{v}=\dot{\mathbf{u}}-\boldsymbol{\varphi}$, where $\boldsymbol{\varphi}\in C_0^{\infty}\left(\Omega\right)^{d}$ and we recall that j is a positive function; after a simple calculation, we obtain

$$\left(\boldsymbol{\sigma}\left(t\right),\boldsymbol{\varepsilon}\left(\boldsymbol{\varphi}\right)\right)_{\mathcal{H}}=\left(\mathbf{f}\left(t\right),\boldsymbol{\varphi}\right)_{V}\quad\forall t\in\left[0,T\right],\quad\forall\boldsymbol{\varphi}\in C_0^{\infty}\left(\Omega\right)^{d}.$$

Then we use (3.20) to deduce that

$$\left(\boldsymbol{\sigma}\left(t\right),\boldsymbol{\varepsilon}\left(\boldsymbol{\varphi}\right)\right)_{\mathcal{H}}=\left(\mathbf{f}_0\left(t\right),\boldsymbol{\varphi}\right)_{H} .$$

Thus, from the definition of weak divergence we conclude that

$$(-Div\ \boldsymbol{\sigma}(t), \boldsymbol{\varphi})_H = (\mathbf{f}_0(t), \boldsymbol{\varphi})_H \quad \forall \boldsymbol{\varphi} \in C_0^\infty(\Omega)^d.$$

Since the space $C_0^\infty(\Omega)^d$ is dense in $L^2(\Omega)^d$ we deduce that

$$Div\ \boldsymbol{\sigma}(t) = -\ \mathbf{f}_0(t) \quad \forall t \in [0,T]. \tag{4.27}$$

The last equality combined with the hypothesis (3.15) on $\mathbf{f_0}$ implies $Div\ \boldsymbol{\sigma}(t) \in H$ and, therefore, $\boldsymbol{\sigma}(t) \in \mathcal{H}_1$. Finally, we note that the norm on $\mathcal{H}_1$ allows us to write

$$\|\boldsymbol{\sigma}(t_1) - \boldsymbol{\sigma}(t_2)\|^2_{\mathcal{H}_1} \quad = \|\boldsymbol{\sigma}(t_1) - \boldsymbol{\sigma}(t_2)\|^2_{\mathcal{H}} + \|Div\ (\boldsymbol{\sigma}(t_1)) - Div\ (\boldsymbol{\sigma}(t_2))\|^2_H\ .$$

Thus, (4.27), (3.15) and the regularity $\boldsymbol{\sigma} \in C(0,T;\mathcal{H})$ imply $\boldsymbol{\sigma} \in C(0,T;\mathcal{H}_1)$; which completes the proof of the existence of a solution $(\mathbf{u}, \boldsymbol{\sigma}, \mathbf{k})$ to Problem PV with regularity (4.1).

Uniqueness. The uniqueness of the solution $(\mathbf{u}, \boldsymbol{\sigma}, \mathbf{k})$ of Problem PV follows from the uniqueness of the fixed point of the operator Λ combined with the unique solvability of the intermediate problems PV_η and Q_η guaranteed by Lemmas 4.2 and 4.3. □

References

[1] Barboteu, M., Danan, D., Sofonea, M., *Analysis of a contact problem with normal damped response and unilateral constraint*, ZAMM-Z Angew Math. Mech., **96**(2015), no. 4, 408-428.

[2] Chau, O., Awbi, B., *Quasistatic thermoviscoelastic frictional contact with damped response*, Appl. Anal., **86**(2004), no. 6, 635-648.

[3] Chouchane, L., Selmani, L., *On a viscoplastic model coupling adhesion and damage*, An. Univ. Oradea Fasc. Mat., **15**(2008), 63-89.

[4] Chouchane, L., Selmani, L., *A frictionless elastic-viscoplastic contact problem with normal comliance, adhesion and damage*, Stud. Univ. Babeş-Bolyai Math., **54**(2009), no. 1, 43-64.

[5] Chouchane, L., Selmani, L., *A history-dependent frictional contact problem with wear for thermoviscoelastic materials*, Math. Model. Anal., **24**(2019), no. 3, 351-371.

[6] Eck, C., Jarušek, J., Krbeč, M., *Unilateral contact problems: Variational methods and existence theorems*, Pure Applied Mathematics, Chapman/CRC Press, New York, 2005.

[7] Fernández, J.R., Han, W., Viaño, J.M., *Variational and numerical analysis of a frictionless contact problem for elastic-viscoplastic materials with internal state variables*, Quart. J. Mech. Appl. Math., **54**(2001), no. 4, 501-522.

[8] Latreche, S., Selmani, L., *Analysis of a frictionless contact problem with adhesion for piezoelectric materials*, Taiwanese J. Math., **21**(2017), 81-105.

[9] Selmani, M., Selmani, L., *A frictional contact problem with normal damped response for elastic-visco-plastic materials*, Rend. Circ. Mat. Palermo, **59**(2010), 67-85.

[10] Selmani, M., Selmani, L., *Analysis of frictionless contact problem for elastic-viscoplastic materials*, Nonlinear Anal. Model. Control, **17**(2012), no. 1, 99-117.

[11] Sofonea, M., Bollati, J., Tarzia, D.A., *Optimal control of differential quasivariational inequalities with applications in contact mechanics*, J. Math. Anal. Appl., **493**(2021), no. 2, 124567.

[12] Sofonea, M., Chouchane, L., *Quasistatic adhesive contact of piezoelectric cylinders*, Nonlinear Anal. Model. Control, **14**(2009), no. 1, 123-142.

[13] Sofonea, M., Chouchane, L., Selmani, L., *Analysis of an antiplane contact problem with adhesion for electro-viscoelastic materials*, Nonlinear Anal. Model. Control, **13**(2008), no. 3, 379-395.

[14] M Sofonea, M., Han, W., Barboteu, M., *Analysis of a viscoelastic contact problem with multivalued normal compliance and unilateral constraint*, Comput. Methods Appl. Mech. Engrg., **264**(2013), 12-22.

[15] Sofonea, M., Matei, A., *Mathematical Models in Contact Mechanics*, London Mathematical Society Lecture Note Series, vol. 398, Cambridge University Press, Cambridge, 2012.

[16] Sofonea, M., Pătrulescu, F., Farcas, A., *A viscoplastic contact problem with normal compliance, unilateral constraint and memory term*, Appl. Math. Optim., **69**(2014), no. 2, 175-198.

[17] Sofonea, M., Souleimane, Y., *A viscoelastic sliding contact problem with normal compliance, unilateral constraint and memory term*, Mediterr. J. Math., **13**(2016), no. 5, 2863-2886.

Lamia Chouchane
"Sétif 1" University,
Laboratory of Mathematical Analysis and Applications (LAMA),
El Maabouda Campus, 19000 Sétif, Algeria
e-mail: l_chouchane@yahoo.fr

Dounia Bouchelil
"Sétif 1" University, Faculty of Sciences,
El Bez, 19000 Sétif, Algeria
e-mail: douniabouchelil@outlook.fr