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Some saturation classes for deferred Riesz and deferred Nörlund means

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Abstract. One of main problem in approximation theory is determination a saturation class for given method. The problem of determining a saturation class has been considered by Zamanski, Sunouchi and Watari and others. Mohaparta and Russel have considered some direct and inverse theorems in approximation of functions. Sunouchi and Watari have studied the Riesz means of type n. In [5], Goel et al. have extended these results by considering Nörlund means. In this paper, we examine some direct and inverse theorems in approximation of functions under weaker conditions by considering Deferred Riesz means and Deferred Nörlund means. Also, we extent above mentioned results.

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1. Introduction

Let f be a 2π-periodic function and $f \in L_p := L_p[-\pi, \pi]$ for $p \geq 1$, where L_p consists of all measurable functions for which denote the L_p-norm with respect to x and defined by

$$\|f\|_p := \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^p dx \right\}^{\frac{1}{p}} .$$

$C_{2\pi}$ denote the set of all continuous functions defined on $[-\pi, \pi]$. For $p = \infty$, $L_p[-\pi, \pi]$ space replace by the space $C_{2\pi}$.

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Each $f \in L^1$ has the Fourier series

$$f(x) \sim \frac{1}{2}a_0 + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx) \equiv \sum_{k=0}^{\infty} A_k(x) \ . \tag{1.1}$$

The partial sum of the first $(n+1)$ terms of the Fourier series of f at a point x is defined by

$$S_n(f;x) = \frac{1}{2}a_0 + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx) \equiv \sum_{k=0}^{n} A_k(x) \ .$$

The conjugate series of the series (1.1) is

$$\sum_{k=1}^{\infty} B_k(x) = \sum_{k=1}^{\infty}(b_k \cos kx - a_k \sin kx) \ .$$

and also the conjugate function $\tilde{f}$ of f is given by

$$\tilde{f}(x) = \frac{1}{2\pi}\int_0^{\pi}\{f(x+t) - f(x-t)\}\cot\frac{t}{2}dt \ .$$

The integral is known as a Cauchy integral. Also, $\tilde{f}$ exists almost everywhere whenever f is integrable.

Moreover, if $\omega_p(\delta, f) = O(\delta^{\alpha})$, then $f \in Lip(\alpha, p)$, $(p \geq 1)$, where

$$\omega_p(\delta, f) = \sup_{|h|\leq\delta} \|f(x+h) - f(x)\|_p$$

is the integral modulus of continuity of $f \in L_p$. Clearly, if $f \in Lip(\alpha, p)$ for some $\alpha > 1$, then f must be constant. So it is interesting only in case of $0 < \alpha \leq 1$. Also for $p \geq 1$, the generalized Minkowski's inequality is given in [7] as follow

$$\left\|\int f(x,t)dt\right\|_p \leq \int \|f(x,t)\|_p dt \ .$$

Throughout the paper, we consider $K_p = \{f \in L_p : \tilde{f} \in Lip(1,p)\}$ for $1 \leq p < \infty$ and $K_\infty = \{f \in C_{2\pi} : \tilde{f} \in Lip1\}$ for $p = \infty$.

In 1932, Agnew [1] defined the Deferred Cesàro mean of the sequence $\{s_k\}$ by

$$(D_{a,b}, s)_n := \frac{s_{a_n+1} + s_{a_n+2} + ... + s_{b_n}}{b_n - a_n} = \frac{1}{b_n - a_n}\sum_{k=a_n+1}^{b_n} s_k$$

where $a = \{a_n\}$ and $b = \{b_n\}$ are sequences of non-negative integers satisfying

$$a_n < b_n \ , \quad n = 1, 2, 3, ... \quad \text{and} \quad \lim_{n\to\infty} b_n = \infty \ .$$

We note here that $D_{a,b}$ is clearly regular for any choice of $\{a_n\}$ and $\{b_n\}$.

Let $\{p_n\}$ be a sequence of non-negative real numbers. Deferred Riesz and Deferred Nörlund means of (1.1) are defined as follows

$$D_a^b R_n(f;x) := \frac{1}{P_{a_n+1}^{b_n}}\sum_{k=a_n+1}^{b_n} p_k S_k(f;x)$$

and

$$D_a^b N_n(f;x) := \frac{1}{P_0^{b_n-a_n-1}} \sum_{k=a_n+1}^{b_n} p_{b_n-k} S_k(f;x)$$

where

$$P_{a_n+1}^{b_n} = \sum_{k=a_n+1}^{b_n} p_k \neq 0 \ , \quad P_0^{b_n-a_n-1} = \sum_{k=0}^{b_n-a_n-1} p_k \neq 0$$

(see [11], [2] and [12]).

Taking $b_n = n$ and $a_n = 0$, Deferred Riesz and Deferred Nörlund means give us classically known Riesz and Nörlund means of the series (1.1), respectively. Also, in case $p_k = 1$ for all k, both of them yield Deferred Cesàro means of $S_k(f;x)$ as follows

$$D_a^b(f;x) := \frac{1}{b_n - a_n} \sum_{k=a_n+1}^{b_n} S_k(f;x) \ .$$

Let $g_k(n)$ $k = 1, 2, \ldots$ be the summating function and consider a family of transform of (1.1) of a summability method G,

$$P_n(x) = \frac{1}{2}a_0 + \sum_{k=1}^{\infty} g_k(n)(a_k \cos kx + b_k \sin kx) \tag{1.2}$$

where the parameter n needs not be discrete.

If there are a positive non-increasing function $\phi(n)$ and a class K of functions in such a way that

$$\| f(x) - P_n(x) \| = o(\phi(n)) \text{ implies } f(x) = \text{constant}; \tag{1.3}$$

$$\| f(x) - P_n(x) \| = O(\phi(n)) \text{ implies } f(x) \in K; \tag{1.4}$$

$$\text{for every } f \in K, \quad \text{one has} \quad \| f(x) - P_n(x) \| = O(\phi(n)), \tag{1.5}$$

then it is said that the method of summation G is saturated with order $\phi(n)$ and its class of saturation in K ([3]).

Ever since the definition of saturation of summability methods was given by Favard [4] many authors have studied the saturation property of operators which are obtained as transforms of the n-th partial sum of the Fourier series by summability methods. Zamanski [14] have studied the notion of determining a saturation class by considering $(C, 1)$. Sunouchi and Watari [13] have obtained the saturation order and class for Cesàro, Abel and the (R, λ, k) method. Goel et al. [5] have examined order and class of saturation of Nörlund means with supremum norms. Mohapatra and Russell [10] have analyzed order and class of (N,c,d)-methods in the L_p spaces. Kuttner, Mohapatra and Sahney [8] have obtained results on saturation for a general class of summability methods in the supremum norm.

In this paper, our object is to extent some of these results under weaker conditions by considering Deferred Riesz means and Deferred Nörlund means.

We shall give some well-known results that we will use them to prove our theorems.

Lemma 1.1. [6] *If f belongs to $Lip(1,p)$, $1 < p \leq \infty$, then f is equivalent to the indefinite integral of a function belonging to L_p. Also, if $f \in Lip1$, then f is the indefinite integral of a bounded function.*

Lemma 1.2. [15] *If $f \in L^p$, $1 < p < \infty$, then $\tilde{f} \in L^p$. Moreover, $\tilde{S}[f] = S[\tilde{f}]$.*

Lemma 1.3. [5]

$$\left| \int_t^\pi \frac{\sin(k+1)u}{u^2} du \right| \leq \begin{cases} 2(k+1)\log\left(\frac{1}{(k+1)t}\right), & 0 < (k+1)t < \frac{1}{e} \\ \frac{2}{(k+1)t^2}\ , & k \geq 0,\ t > 0\ . \end{cases}$$

Lemma 1.4. [9] *Suppose that $d_{nk} \geq 0$ $(\forall n,k)$, $\sum_{k=0}^{\infty} d_{nk} = 1$ and*

$$\sum_{k=1}^{\infty} d_{nk} \log k < \infty\ .$$

Let $\phi(n)$ be a positive function. In order that D should be saturated with order $\phi(n)$ and some class, it is necessary and sufficient that

$$0 < \liminf_{n\to\infty} \frac{\phi(n)}{d_{n0}} < \infty\ .$$

2. Main results

If there are a positive non-increasing function $\phi(n)$ and a class of functions K with the following properties

$$\| f(x) - D_a^b R_n(f;x) \| = o(\phi(n)) \ \Rightarrow\ f \text{ is constant} \tag{2.1}$$

$$\| f(x) - D_a^b R_n(f;x) \| = O(\phi(n)) \ \Rightarrow\ f \in K \tag{2.2}$$

and

$$f \in K \ \Rightarrow \| f(x) - D_a^b R_n(f;x) \| = O(\phi(n)) \tag{2.3}$$

then we say that $D_a^b R_n(f;x)$ is saturated with the order $\phi(n)$ and class K.

Now, we give interesting results for Deferred Riesz means.

Lemma 2.1. *Let $1 \leq p \leq \infty$ and*

$$\frac{1}{P_{a_n+1}^{b_n}} \sum_{k=a_n+1}^{b_n} p_k \log k < \infty.$$

If

$$\| f(x) - D_a^b R_n(f;x) \|_p = o\left(\frac{p_{b_n}}{P_{a_n+1}^{b_n}}\right)$$

then f is constant.

To proof the following lemma we use the same technique in [8].

Proof. Let us write $D_a^bR_n(x)$ instead of $D_a^bR_n(f;x)$. By definition of $D_a^bR_n(x)$, we get

$$\begin{aligned}\frac{1}{\pi}\int_{-\pi}^{\pi} D_a^bR_n(x+t)\cos ktdt &= \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{1}{P_{a_n+1}^{b_n}}\sum_{r=a_n+1}^{b_n} p_rS_r(x+t)\cos ktdt \\ &= \frac{1}{P_{a_n+1}^{b_n}}\sum_{r=a_n+1}^{b_n} p_r\frac{1}{\pi}\int_{-\pi}^{\pi} S_r(x+t)\cos ktdt \\ &= \frac{1}{P_{a_n+1}^{b_n}}\sum_{r=k}^{b_n} p_rA_k(x),\end{aligned}$$

since hypothesis and

$$\frac{1}{\pi}\int_{-\pi}^{\pi} S_r(x+t)\cos ktdt = \begin{cases} A_k(x), & r\geq k \\ 0, & r<k\ .\end{cases}$$

Hence, we obtain

$$A_k(x)-\frac{1}{P_{a_n+1}^{b_n}}\sum_{r=k}^{b_n} p_rA_k(x) = A_k(x)\left(\frac{1}{P_{a_n+1}^{b_n}}\sum_{r=k+1}^{b_n} p_r\right).$$

$S_n(f)$ converges to f uniformly whenever f is continuous [15]. So, from hypothesis and generalized Minkowski's inequality we get

$$\begin{aligned} &\left\|\frac{1}{\pi}\int_{-\pi}^{\pi} S_r(x+t)\cos ktdt-\frac{1}{\pi}\int_{-\pi}^{\pi} D_a^bR_n(x+t)\cos ktdt\right\|_p \\ \leq\ &\left\|\frac{1}{\pi}\int_{-\pi}^{\pi} S_r(x+t)\cos ktdt-\frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\cos ktdt\right\|_p \\ &+\left\|\frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\cos ktdt-\frac{1}{\pi}\int_{-\pi}^{\pi} D_a^bR_n(x+t)\cos ktdt\right\|_p \\ \leq\ &\frac{1}{\pi}\int_{-\pi}^{\pi}\|S_r-f\|_pdt+\frac{1}{\pi}\int_{-\pi}^{\pi}\|f-D_a^bR_n\|_pdt \\ =\ &o\left(\frac{p_{b_n}}{P_{a_n+1}^{b_n}}\right).\end{aligned}$$

Therefore for all $k\geq 1$ we have

$$A_k(x)\left(\frac{1}{P_{a_n+1}^{b_n}}\sum_{r=k+1}^{b_n} p_r\right) = o\left(\frac{p_{b_n}}{P_{a_n+1}^{b_n}}\right)$$

i.e.,

$$A_k(x)\left(\frac{p_{k+1}+p_{k+2}\ldots+p_{b_n}}{p_{b_n}}\right) = o(1).$$

Because of $\left(\frac{p_{k+1}+p_{k+2}\ldots+p_{b_n}}{p_{b_n}}\right)\geq 1$ for each $r\geq 1$, we get $A_k(x)=0$. Consequently $f(x)=\frac{1}{2}a_0$ which is a constant. □

Lemma 2.2. *Let the limit*

$$\lim_{n\to\infty}\frac{p_r}{p_{b_n}}=1$$

hold for a fixed $a_n+1\leq r<b_n$. *If the equation*

$$\| f(x)-D_a^bR_n(f;x)\|_p=0\left(\frac{p_{b_n}}{P_{a_n+1}^{b_n}}\right)$$

is hold, then

$$\left\|\sum_{k=a_n+2}^{N}(k-a_n-1)\left(1-\frac{k-(a_n+2)}{N-a_n-1}\right)A_k(x)\right\|_p=O(1).$$

Proof. Suppose that $\Delta_n(x):=f(x)-D_a^bR_n(f;x)$. In this case,

$$\Delta_n(x)\sim\sum_{k=a_n+2}^{b_n}\left(1-\frac{P_k^{b_n}}{P_{a_n+1}^{b_n}}\right)A_k(x).$$

Let $N<b_n$, taking N-th arithmetic mean of $\Delta_n(x)$ we get

$$\sigma_N[x;\Delta_n]=\sum_{k=a_n+2}^{N}\left(1-\frac{P_k^{b_n}}{P_{a_n+1}^{b_n}}\right)\left(1-\frac{k-a_n-2}{N-a_n-1}\right)A_k(x).$$

On account of $\|\Delta_n\|\geq\|\sigma_N[x;\Delta_n]\|$, we obtain

$$\left\|\sum_{k=a_n+2}^{N}\left(1-\frac{P_k^{b_n}}{P_{a_n+1}^{b_n}}\right)\left(1-\frac{k-a_n-2}{N-a_n-1}\right)A_k(x)\right\|_p=O\left(\frac{p_{b_n}}{P_{a_n+1}^{b_n}}\right)$$

$$\Rightarrow\left\|\sum_{k=a_n+2}^{N}\left(\frac{P_{a_n+1}^{b_n}-P_k^{b_n}}{p_{b_n}}\right)\left(1-\frac{k-a_n-2}{N-a_n-1}\right)A_k(x)\right\|_p=O(1)$$

$$\Rightarrow\left\|\sum_{k=a_n+2}^{N}\lim_{n\to\infty}\left(\frac{p_{a_n+1}+p_{a_n+2}+\ldots+p_{k-1}}{p_{b_n}}\right)\left(1-\frac{k}{N+1}\right)A_k(x)\right\|_p=O(1)$$

$$\Rightarrow\left\|\sum_{k=a_n+2}^{N}(k-a_n-1)\left(1-\frac{k}{N+1}\right)A_k(x)\right\|_p=O(1).$$

This completes the proof. □

Lemma 2.3. *Let*

$$M_n(t)=\frac{1}{P_{a_n+1}^{b_n}}\sum_{k=a_n+1}^{b_n}p_k\frac{\cos(k+1/2)t}{\sin(t/2)}$$

and

$$G_n(t)=\int_t^\pi M_n(u)du. \tag{2.4}$$

If

$$\int_0^\pi |G_n(t)| = O\left(\frac{p_{b_n}}{P_{a_n+1}^{b_n}}\right) \tag{2.5}$$

and $f \in K_p$ $(1 < p \leq \infty)$ *then* $\| f(x) - D_a^b R_n(f;x) \|_p = O\left(\frac{p_{b_n}}{P_{a_n+1}^{b_n}}\right)$.

Proof. Let $\tilde{S}_k(\tilde{f};x)$ denote the partial sums of the conjugate series related to $\tilde{f}(x)$. So,

$$\tilde{S}_k(\tilde{f};x) = \frac{1}{2\pi}\int_0^\pi \{\tilde{f}(x+t) - \tilde{f}(x-t)\}\frac{\cos(t/2) - \cos(k+1/2)t}{\sin(t/2)}dt.$$

With a simple analysis, we get

$$D_a^b R_n(\tilde{S}_k(\tilde{f};x)) = \frac{1}{P_{a_n+1}^{b_n}} \sum_{k=a_n+1}^{b_n} p_k \tilde{S}_k(\tilde{f};x)$$

$$= \frac{1}{P_{a_n+1}^{b_n}} \sum_{k=a_n+1}^{b_n} p_k \frac{1}{2\pi}\int_0^\pi \{\tilde{f}(x+t) - \tilde{f}(x-t)\}\cot(t/2)dt$$

$$-\frac{1}{P_{a_n+1}^{b_n}} \sum_{k=a_n+1}^{b_n} p_k \frac{1}{2\pi}\int_0^\pi \{\tilde{f}(x+t) - \tilde{f}(x-t)\}\frac{\cos(k+1/2)t}{\sin(t/2)}dt.$$

By Lemma 1.2, $f \in L_p$ $(1 < p < \infty)$ implies $\tilde{f} \in L_p$. So $\tilde{\tilde{f}} \in L_p$, and thus we obtain $\tilde{S}(\tilde{f}) = S(\tilde{\tilde{f}})$. If $p = \infty$ then it means that $\tilde{f} \in Lip1$. Therefore, we say that $-f + \frac{1}{2}a_0$ is equal to $\tilde{\tilde{f}}$. As a result $\tilde{\tilde{f}} - D_a^b R_n(S_k(\tilde{\tilde{f}};x))$ is identical to $f(x) - D_a^b R_n(S_k(f;x))$. From hypothesis we get

$$f(x) - D_a^b R_n(f;x) = \frac{1}{P_{a_n+1}^{b_n}} \sum_{k=a_n+1}^{b_n} p_k \tilde{\tilde{f}}(x)$$

$$-\frac{1}{P_{a_n+1}^{b_n}} \sum_{k=a_n+1}^{b_n} p_k \frac{1}{2\pi}\int_0^\pi \{\tilde{f}(x+t) - \tilde{f}(x-t)\}\cot(t/2)dt$$

$$+\frac{1}{P_{a_n+1}^{b_n}} \sum_{k=a_n+1}^{b_n} p_k \frac{1}{2\pi}\int_0^\pi \{\tilde{f}(x+t) - \tilde{f}(x-t)\}\frac{\cos(k+1/2)t}{\sin(t/2)}$$

$$= \frac{1}{P_{a_n+1}^{b_n}} \sum_{k=a_n+1}^{b_n} p_k \frac{1}{2\pi}\int_0^\pi \{\tilde{f}(x+t) - \tilde{f}(x-t)\}\cot(t/2)dt$$

$$-\frac{1}{P_{a_n+1}^{b_n}} \sum_{k=a_n+1}^{b_n} p_k \frac{1}{2\pi}\int_0^\pi \{\tilde{f}(x+t) - \tilde{f}(x-t)\}\cot(t/2)dt$$

$$+\frac{1}{P_{a_n+1}^{b_n}} \sum_{k=a_n+1}^{b_n} p_k \frac{1}{2\pi}\int_0^\pi \{\tilde{f}(x+t) - \tilde{f}(x-t)\}\frac{\cos(k+1/2)t}{\sin(t/2)}$$

$$= \frac{1}{2\pi}\int_0^\pi \{\tilde{f}(x+t) - \tilde{f}(x-t)\}M_n(t). \tag{2.6}$$

As $f \in K_p$ by Lemma 1.1, we get $\tilde{f}' \in L_p$, $p > 1$. By integrating in (2.6) we have

$$f(x) - D_a^b R_n(f;x) = -\frac{1}{2\pi}\int_0^\pi \{\tilde{f}'(x+t) + \tilde{f}'(x-t)\}G_n(t)dt.$$

By generalized Minkowski's inequality, we obtain

$$\begin{aligned} \left\|f(x) - D_a^b R_n(f;x)\right\|_p &= \left\|-\frac{1}{2\pi}\int_0^\pi \{\tilde{f}'(x+t) + \tilde{f}'(x-t)\}G_n(t)dt\right\|_p \\ &\le \int_0^\pi \left\|\{\tilde{f}'(x+t) + \tilde{f}'(x-t)\}\right\|_p |G_n(t)|dt \\ &\le M\int_0^\pi |G_n(t)|dt = O\left(\frac{p_{b_n}}{P_{a_n+1}^{b_n}}\right). \end{aligned}$$

This completes the proof of Lemma. □

Theorem 2.4. *Let* $1 < p \le \infty$, (a_n) *and* (b_n) *be sequences of non-negative integers satisfying*

$$a_n < b_n, \quad \lim_{n\to\infty} b_n = \infty$$

and $\{p_n\}$ *be a sequence of non-negative real numbers such that*

$$\sum_{k=a_n+1}^{b_n} |p_k - p_{k+1}| = O(p_{b_n}) \tag{2.7}$$

$p_{a_n+1} = 0$, $p_{b_n+1} = 0$. *If* $f \in K_p$, $1 < p \le \infty$, *then*

$$\| f(x) - D_a^b R_n(f;x) \|_p = O\left(\frac{p_{b_n}}{P_{a_n+1}^{b_n}}\right). \tag{2.8}$$

Proof. Due to Lemma 2.3, it is enough to show (2.5). By Abel's transform, we get

$$M_n(t) = \frac{1}{2\sin^2(t/2)}\frac{1}{P_{a_n+1}^{b_n}}\left\{\sum_{k=a_n+1}^{b_n}(\sin(k+1)t)(p_k - p_{k+1})\right\}$$

Since

$$\frac{1}{2\sin^2(t/2)} = \frac{2}{t^2} + O(1),$$

we have

$$\begin{aligned} M_n(t) &= \frac{2}{t^2}\frac{1}{P_{a_n+1}^{b_n}}\left\{\sum_{k=a_n+1}^{b_n}(\sin(k+1)t)(p_k - p_{k+1})\right\} \\ &\quad + O\left(\frac{1}{P_{a_n+1}^{b_n}}\left\{\sum_{k=a_n+1}^{b_n}(\sin(k+1)t)(p_k - p_{k+1})\right\}\right) \end{aligned}$$

$$= \quad \frac{2}{t^2}\frac{1}{P^{b_n}_{a_n+1}}\left\{\sum_{k=a_n+1}^{b_n}(\sin(k+1)t)(p_k-p_{k+1})\right\}+O\left(\frac{p_{b_n}}{P^{b_n}_{a_n+1}}\right).$$

From the last equation we obtain

$$\int_0^\pi |G_n(t)|dt \le \frac{1}{P^{b_n}_{a_n+1}}\sum_{k=a_n+1}^{b_n}|p_k-p_{k+1}|\int_0^\pi\left|\int_t^\pi \frac{\sin(k+1)u}{u^2}du\right|dt.$$

To complete the proof, we shall show the following equation

$$I := \int_0^\pi\left|\int_t^\pi \frac{\sin(k+1)u}{u^2}du\right|dt = O(1)\ . \tag{2.9}$$

By Lemma 1.3 we get

$$\begin{aligned} I := \int_0^\pi\left|\int_t^\pi \frac{\sin(k+1)u}{u^2}du\right|dt &= \int_0^{1/e(k+1)}\left|\int_t^\pi \frac{\sin(k+1)u}{u^2}du\right|dt \\ &\quad + \int_{1/e(k+1)}^\pi\left|\int_t^\pi \frac{\sin(k+1)u}{u^2}du\right|dt \\ &\le \int_0^{1/e(b_k+1)} 2(k+1)\log\left(\frac{1}{(k+1)t}\right)dt \\ &\quad + \int_{1/e(b_k+1)}^\pi \frac{2}{(k+1)t^2}dt = O(1). \end{aligned}$$

This completes the proof. □

Now, we can give our results for Deferred Nörlund means.

If there are a positive non-increasing function $\phi(n)$ and a class of functions K with the following properties

$$\|\ f(x) - D^b_aN_n(f;x)\ \| = \ o(\phi(n)) \ \Rightarrow\ f \text{ is constant} \tag{2.10}$$

$$\|\ f(x) - D^b_aN_n(f;x)\ \| = \ O(\phi(n)) \ \Rightarrow\ f \in K \tag{2.11}$$

and

$$f\in K \ \Rightarrow \|\ f(x) - D^b_aN_n(f;x)\ \| = \ O(\phi(n)) \tag{2.12}$$

then we say that $D^b_aN_n(f;x)$ is saturated with the order $\phi(n)$ and class K.

Lemma 2.5. *Let* $1\le p\le\infty$ *and*

$$\frac{1}{P_0^{b_n-a_n-1}}\sum_{k=a_n+1}^{b_n}p_{b_n-k}\log k<\infty.$$

If

$$\|\ f(x) - D^b_aN_n(f;x)\ \|_p = o\left(\frac{p_0}{P_0^{b_n-a_n-1}}\right)$$

then f *is constant.*

Proof. Let us write $D_a^b N_n(x)$ instead of $D_a^b N_n(f;x)$. Now we get

$$\begin{aligned} & \frac{1}{\pi}\int_{-\pi}^{\pi} D_a^b N_n(x+t)\cos ktdt \\ = & \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1}{P_0^{b_n-a_n-1}} \sum_{r=a_n+1}^{b_n} p_{b_n-r} S_r(x+t)\cos ktdt \\ = & \frac{1}{P_0^{b_n-a_n-1}} \sum_{r=a_n+1}^{b_n} p_{b_n-r} \frac{1}{\pi}\int_{-\pi}^{\pi} S_r(x+t)\cos ktdt \\ = & \frac{1}{P_0^{b_n-a_n-1}} \sum_{r=k}^{b_n} p_{b_n-r} A_k(x) \ , \end{aligned}$$

since summation and integration can be replace by hypothesis and since

$$\frac{1}{\pi}\int_{-\pi}^{\pi} S_r(x+t)\cos ktdt = \begin{cases} A_k(x), & r \geq k \\ 0, & r < k. \end{cases}$$

Hence, we obtain

$$A_k(x) - \frac{1}{P_0^{b_n-a_n-1}} \sum_{r=k}^{b_n} p_{b_n-r} A_k(x) = A_k(x)\left(\frac{1}{P_0^{b_n-a_n-1}} \sum_{r=k+1}^{b_n} p_{b_n-r}\right)$$

$S_n(f)$ converges to f uniformly whenever f is continuous [15]. So, from hypothesis and generalized Minkowski's inequality we get

$$\begin{aligned} & \left\| \frac{1}{\pi}\int_{-\pi}^{\pi} S_r(x+t)\cos ktdt - \frac{1}{\pi}\int_{-\pi}^{\pi} D_a^b N_n(x+t)\cos ktdt \right\|_p \\ \leq & \left\| \frac{1}{\pi}\int_{-\pi}^{\pi} S_r(x+t)\cos ktdt - \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\cos ktdt \right\|_p \\ & + \left\| \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+t)\cos ktdt - \frac{1}{\pi}\int_{-\pi}^{\pi} D_a^b N_n(x+t)\cos ktdt \right\|_p \\ \leq & \frac{1}{\pi}\int_{-\pi}^{\pi} \|S_r - f\|_p dt + \frac{1}{\pi}\int_{-\pi}^{\pi} \|f - D_a^b N_n\|_p dt \\ = & o\left(\frac{p_0}{P_0^{b_n-a_n-1}}\right). \end{aligned}$$

Hence we have

$$A_k(x)\left(\frac{1}{P_{a_n+1}^{b_n}} \sum_{r=k+1}^{b_n} p_{b_n-r}\right) = o\left(\frac{p_0}{P_0^{b_n-a_n-1}}\right)$$

for all $k \geq 1$, i.e.,

$$A_k(x)\left(\frac{p_{b_n-k-1} + p_{b_n-k-2}\cdots + p_0}{p_0}\right) = o(1).$$

Because of $\left(\frac{p_{b_n-k-1}+p_{b_n-k-2}\dots+p_0}{p_0}\right)\geq 1$, $A_k(x)=0$ for each $r\geq 1$. Consequently $f(x)=\frac{1}{2}a_0$ which is a constant. □

Lemma 2.6. *Let the limit*

$$\lim_{n\to\infty}\frac{p_{b_n-k+1}}{p_0}=1$$

hold for a fixed $a_n+2\leq k<b_n$. If the equation

$$\| f(x)-D_a^bN_n(f;x)\|_p=O\left(\frac{p_0}{P_0^{b_n-a_n-1}}\right)$$

is hold, then

$$\left\|\sum_{k=a_n+2}^{N}(k-a_n-1)\left(1-\frac{k-(a_n+2)}{N-a_n-1}\right)A_k(x)\right\|_p=O(1).$$

Proof. Suppose that $\Delta_n(x):=f(x)-D_a^bN_n(f;x)$. In this case,

$$\Delta_n(x)\sim\sum_{k=a_n+2}^{b_n}\left(1-\frac{P_0^{b_n-k}}{P_0^{b_n-a_n-1}}\right)A_k(x).$$

Let $N<b_n$. Taking N-th arithmetic mean of $\Delta_n(x)$ we have

$$\sigma_N[x;\Delta_n]=\sum_{k=a_n+2}^{N}\left(1-\frac{P_0^{b_n-k}}{P_0^{b_n-a_n-1}}\right)\left(1-\frac{k-a_n-2}{N-a_n-1}\right)A_k(x).$$

Since $\|\Delta_n\|\geq\|\sigma_N[x;\Delta_n]\|$ we obtain

$$\left\|\sum_{k=a_n+2}^{N}\left(1-\frac{P_0^{b_n-k}}{P_0^{b_n-a_n-1}}\right)\left(1-\frac{k-a_n-2}{N-a_n-1}\right)A_k(x)\right\|_p=O\left(\frac{p_0}{P_0^{b_n-a_n-1}}\right)$$

$$\left\|\sum_{k=a_n+2}^{N}\left(\frac{P_0^{b_n-a_n-1}-P_0^{b_n-k}}{p_0}\right)\left(1-\frac{k-a_n-2}{N-a_n-1}\right)A_k(x)\right\|_p=O(1)$$

$$\left\|\sum_{k=a_n+2}^{N}\lim_{n\to\infty}\left(\frac{p_{b_n-k+1}+\dots+p_{b_n-a_n-1}}{p_0}\right)\left(1-\frac{k}{N+1}\right)A_k(x)\right\|_p=O(1)$$

$$\left\|\sum_{k=a_n+2}^{N}(k-a_n-1)\left(1-\frac{k}{N+1}\right)A_k(x)\right\|_p=O(1).$$

This completes the proof. □

Lemma 2.7. *Let*

$$M_n(t)=\frac{1}{P_0^{b_n-a_n-1}}\sum_{k=a_n+1}^{b_n}p_{b_n-k}\frac{\cos(k+1/2)t}{\sin(t/2)}$$

and

$$G_n(t) = \int_t^{\pi} M_n(u)du. \tag{2.13}$$

If

$$\int_0^{\pi} |G_n(t)| = O\left(\frac{p_o}{P_0^{b_n - a_n - 1}}\right) \tag{2.14}$$

and $f \in K_p (1 < p \leq \infty)$ *then* $\| f(x) - D_a^b N_n(f;x) \|_p = O\left(\frac{p_o}{P_0^{b_n - a_n - 1}}\right)$.

Proof. Let $\tilde{S}_k(\tilde{f};x)$ denote the partial sums of the conjugate series related to $\tilde{f}(x)$. So,

$$\tilde{S}_k(\tilde{f};x) = \frac{1}{2\pi}\int_0^{\pi}\{\tilde{f}(x+t) - \tilde{f}(x-t)\}\frac{\cos(t/2) - \cos(k+1/2)t}{\sin(t/2)}dt.$$

With a simple analysis, we get

$$\begin{aligned}
D_a^b N_n(\tilde{S}_k(\tilde{f};x)) &= \frac{1}{P_0^{b_a - n - 1}} \sum_{a_n+1}^{b_n} p_{b_n - k} \tilde{S}_k(\tilde{f};x) \\
= & \frac{1}{P_0^{b_a - n - 1}} \sum_{a_n+1}^{b_n} p_{b_n - k} \frac{1}{2\pi} \int_0^{\pi} \{\tilde{f}(x+t) - \tilde{f}(x-t)\}\cot(t/2)dt \\
& - \frac{1}{P_{a_n+1}^{b_n}} \sum_{a_n+1}^{b_n} p_{b_n - k} \frac{1}{2\pi} \int_0^{\pi} \{\tilde{f}(x+t) - \tilde{f}(x-t)\}\frac{\cos(k+1/2)t}{\sin(t/2)}dt.
\end{aligned}$$

By Lemma 1.2, $f \in L_p$ $(1 < p < \infty)$ implies $\tilde{f} \in L_p$. So $\tilde{\tilde{f}} \in L_p$, and we get $\tilde{S}(\tilde{f}) = S(\tilde{\tilde{f}})$. If $p = \infty$ then it means that $\tilde{f} \in Lip1$. Therefore, $-f + \frac{1}{2}a_0$ is equal to $\tilde{\tilde{f}}$. Thus $\tilde{\tilde{f}} - D_a^b N_n(S_k(\tilde{\tilde{f}};x))$ is identical to $f(x) - D_a^b N_n(S_k(f;x))$. From hypothesis we get

$$\begin{aligned}
& f(x) - D_a^b N_n(f;x) \\
= & \frac{1}{P_0^{b_a - n - 1}} \sum_{a_n+1}^{b_n} p_{b_n - k} \frac{1}{2\pi} \int_0^{\pi} \{\tilde{f}(x+t) - \tilde{f}(x-t)\}\cot(t/2)dt \\
& - \frac{1}{P_0^{b_a - n - 1}} \sum_{a_n+1}^{b_n} p_{b_n - k} \frac{1}{2\pi} \int_0^{\pi} \{\tilde{f}(x+t) - \tilde{f}(x-t)\}\cot(t/2)dt \\
& + \frac{1}{P_0^{b_a - n - 1}} \sum_{a_n+1}^{b_n} p_{b_n - k} \frac{1}{2\pi} \int_0^{\pi} \{\tilde{f}(x+t) - \tilde{f}(x-t)\}\frac{\cos(k+1/2)t}{\sin(t/2)}dt \\
= & \frac{1}{2\pi}\int_0^{\pi}\{\tilde{f}(x+t) - \tilde{f}(x-t)\}M_n(t).
\end{aligned} \tag{2.15}$$

As $f \in K_p$ by Lemma 1.1, we have $\tilde{f}' \in L_p$, $p > 1$. By integrating in (2.15), we get

$$f(x) - D_a^b N_n(f;x) = -\frac{1}{2\pi}\int_0^{\pi}\{\tilde{f}'(x+t) + \tilde{f}'(x-t)\}G_n(t)dt.$$

By generalized Minkowski's inequality, we obtain

$$\begin{aligned}\left\|f(x)-D_a^bN_n(f;x)\right\|_p &= \left\|-\frac{1}{2\pi}\int_0^\pi\{\tilde{f}'(x+t)+\tilde{f}'(x-t)\}G_n(t)dt\right\|_p\\ &\leq \int_0^\pi\left\|\{\tilde{f}'(x+t)+\tilde{f}'(x-t)\}\right\|_p|G_n(t)|dt\\ &\leq M.\int_0^\pi|G_n(t)|dt=O\left(\frac{p_o}{P_0^{b_n-a_n-1}}\right).\end{aligned}$$

This completes the proof of Lemma. □

Theorem 2.8. *Let* $1<p\leq\infty$, (a_n) *and* (b_n) *be sequences of non-negative integers satisfying*

$$a_n<b_n\ ,\quad \lim_{n\to\infty}b_n=\infty$$

and $\{p_n\}$ *be a sequence of non-negative real numbers such that*

$$\sum_{k=a_n+1}^{b_n}|p_{b_n-k}-p_{b_n-k-1}|=O(p_0), \tag{2.16}$$

$p_{b_n-a_n-1}=0$ *and* $p_{-1}=0$. *If* $f\in K_p(1<p\leq\infty)$ *then*

$$\|\ f(x)-D_a^bN_n(f;x)\ \|_p=O\left(\frac{p_o}{P_0^{b_n-a_n-1}}\right). \tag{2.17}$$

Proof. Due to the Lemma 2.7, it is enough to show (2.14). From Abel's transform, we get

$$M_n(t)=\frac{1}{2\sin^2(t/2)}\frac{1}{P_{a_n+1}^{b_n}}\left\{\sum_{k=a_n+1}^{b_n}(\sin(k+1)t)(p_{b_n-k}-p_{b_n-k-1})\right\}.$$

Since

$$\frac{1}{2\sin^2(t/2)}=\frac{2}{t^2}+O(1),$$

we have

$$\begin{aligned}M_n(t) &= \frac{2}{t^2}\frac{1}{P_0^{b_n-a_n-1}}\left\{\sum_{k=a_n+1}^{b_n}(\sin(k+1)t)(p_{b_n-k}-p_{b_n-k-1})\right\}\\ &\quad+O\left(\frac{1}{P_0^{b_n-a_n-1}}\left\{\sum_{k=a_n+1}^{b_n}(\sin(k+1)t)(p_{b_n-k}-p_{b_n-k-1})\right\}\right)\\ &= \frac{2}{t^2}\frac{1}{P_0^{b_n-a_n-1}}\left\{\sum_{k=a_n+1}^{b_n}(\sin(k+1)t)(p_{b_n-k}-p_{b_n-k-1})\right\}\\ &\quad+O\left(\frac{p_0}{P_0^{b_n-a_n-1}}\right).\end{aligned}$$

From the last equation we get

$$\int_0^\pi |G_n(t)|dt \leq \frac{1}{P_0^{b_n-a_n-1}} \sum_{k=a_n+1}^{b_n} |p_{b_n-k} - p_{b_n-k-1}| \int_0^\pi \left| \int_t^\pi \frac{\sin(k+1)u}{u^2} du \right| dt.$$

By Theorem 2.4 and hypothesis, we have

$$\int_0^\pi |G_n(t)| = O\left(\frac{p_o}{P_0^{b_n-a_n-1}}\right).$$

This completes the proof. □

If we take $p_k = 1$ for all k, both of them yield Deferred Cesàro means of the series (1.1). So we get following corollary.

Corollary 2.9. *Let*

$$M_n(t) = \frac{\cos[((b_n + a_n + 2)/2)t] \sin[((b_n - a_n)/2)t]}{(b_n - a_n) \sin^2(t/2)}$$

and

$$G_n(t) = \int_t^\pi M_n(u)du.$$

If

$$\int_0^\pi |G_n(t)| = O\left(\frac{1}{b_n - a_n}\right) \tag{2.18}$$

and $f \in K_p (1 < p \leq \infty)$ *then* $\| f(x) - D_a^b(f;x) \|_p = O\left(\frac{1}{b_n - a_n}\right)$.

If we take $p_k = 1$ for all k, $a_n = 0$ and $b_n = \lambda(n)$, where $\lambda(n)$ is a strictly increasing sequence of positive integers, both of them yield C_λ-method. So, we immediately get following corollary.

Corollary 2.10. *Let*

$$M_n(t) = \frac{1}{\lambda(n)} \left(\frac{\cos((\lambda(n) + 2)/2)t. \sin((\lambda(n) - 1)/2)t}{\sin^2(t/2)}\right)$$

and

$$G_n(t) = \int_t^\pi M_n(u)du.$$

If

$$\int_0^\pi |G_n(t)| = O\left(\frac{1}{\lambda(n)}\right) \tag{2.19}$$

and $f \in K_p$ $(1 < p \leq \infty)$ *then* $\| f(x) - \sigma_n^\lambda(f;x) \|_p = O\left(\frac{1}{\lambda(n)}\right)$.

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