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Capacity solution for an elliptic coupled system with lower term in Orlicz spaces

Yassine Ahakkoud, Jaouad Bennouna and Mhamed El Massoudi

Abstract. In this paper, we will deal with the capacity solution for a nonlinear elliptic coupled system with a Leray-Lions operator $Au = -\text{div } \sigma(x, u, \nabla u)$ acting from Orlicz-Sobolev spaces $W_0^1 L_M(\Omega)$ into its dual, where M is an N-function.

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1. Introduction

Let Ω be an open bounded in $\mathbb{R}^N$, $N \geq 1$, and consider the coupled nonlinear elliptic system

$$-\text{div } \sigma(x, u, \nabla u) + \Phi(x, u) = \kappa(u) \mid \nabla u \mid^2 \qquad \text{in} \quad \Omega, \tag{1.1}$$

$$\text{div}(\kappa(u)\nabla\varphi) = 0 \qquad \text{in} \quad \Omega, \tag{1.2}$$

$$\varphi = \varphi_0,\ u = 0 \qquad \text{on} \quad \partial\Omega. \tag{1.3}$$

We assume that the following assumptions hold. Let M and P be two N-functions such that $P \ll M$ (P grows essentially less rapidly than Q) and $\overline{M}$ the N-function conjugate to M (see preliminaries).
$\sigma : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function such that for almost every $x \in \Omega$ and for every $s, s_1, s_2 \in \mathbb{R}$, $\xi, \xi^* \in \mathbb{R}^N$,

$$\mid \sigma(x, s, \xi) \mid \leq \nu[a_0(x) + \overline{M}^{-1} P(k_1 \mid s \mid) + \overline{M}^{-1} M(k_2 \mid \xi \mid)], \tag{1.4}$$

$$\mid \sigma(x, s_1, \xi) - \sigma(x, s_2, \xi) \mid \leq \nu[a_1(x) + \mid s_1 \mid + \mid s_2 \mid + \overline{P}^{-1}(k_3 M(\mid \xi \mid))], \tag{1.5}$$

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$$(\sigma(x,s,\xi)-\sigma(x,s,\xi^*))(\xi-\xi^*)\geq \alpha M(\mid \xi-\xi^*\mid), \tag{1.6}$$

$$\sigma(x,s,0)=0, \tag{1.7}$$

where $a_0(.)\in E_{\overline{M}}(\Omega)$, $a_1(.)\in E_P(\Omega)$ ($E_{\overline{M}}(\Omega)$ and $E_P(\Omega)$ are specific Orlicz spaces) and α, ν, $k_i>0$ (i=1, 2, 3), are given real numbers.
Furthermore, let $\Phi:\Omega\times\mathbb{R}\to\mathbb{R}$ be a Carathéodory function such that for a.e. $x\in\Omega$ and for all $s\in\mathbb{R}$, the growth condition

$$\mid \Phi(x,s)\mid\leq h(x)\overline{M}^{-1}M\left(\frac{\mid s\mid}{\lambda C_0}\right), \tag{1.8}$$

where $\lambda=diam(\Omega)$ and $\|h\|_{L^\infty(\Omega)}<\dfrac{\alpha}{2\lambda}$ and C_0 is a constant large enough.

$$\kappa\in C(\mathbb{R}) \text{ and there exists } \overline{\kappa}\in\mathbb{R} \text{ such that } 0<\kappa(s)\leq\overline{\kappa}, \text{ for all } s\in\mathbb{R}, \tag{1.9}$$

$$\varphi_0\in H^1(\Omega)\cap L^\infty(\Omega). \tag{1.10}$$

In this paper, we will introduce a solution of the coupled system (1.1)-(1.3) called the capacity solution. This type of solution will deal with the phenomena caused by the possible degeneration of the (1.1)-(1.3). Indeed, one cannot use the weak solution of (1.1) since κ can tend towards 0 when $|u|$ tends to infinity and consequently the equation becomes degenerate, no a priori estimates for $\nabla\varphi$ will be available and then φ may not belong to a Sobolev space. To overcome this obstacle, we use the entire function $\Phi=\kappa(u)\nabla\varphi$ instead of φ to show that $\Phi\in(L^2(\Omega))^N$.

The idea of capacity solution is inspired from the weak and renormalized solutions and X. Xu is the first author who introduced the concept in [13] where $\sigma:\mathbb{R}^N\to\mathbb{R}^N$ is a continuous function satisfying the conditions: $\exists\mu>0$, $\forall\mid\xi\mid\gg 1$ (i.e. $\mid\xi\mid$ is large enough), $\mid a(\xi)\mid\leq\mu\mid\xi\mid$, and $\exists\alpha>0$, $\forall\xi,\xi^*\in\mathbb{R}^N$, $\xi\neq\xi^*$, $(\sigma(\xi)-\sigma(\xi^*))(\xi-\xi^*)\geq\alpha\mid\xi-\xi^*\mid^2$. Also, he used this concept in other papers with various conditions (See [14]). Later, from other authors in [7], showed the existence of a capacity solution to the problem (1.1)-(1.3) where $\sigma=\sigma(x,\nabla u)$ is a Leray-Lions operator from $L^p(W^{1,p})$ into $L^{p'}(W^{-1,p'})$, $p\geq 2$, $\frac{1}{p}+\frac{1}{p'}=1$ and $\Phi=\Phi(x,s)$ satisfies the sign condition, and $\mid\Phi(x,s)\mid\leq h_r(x)$ with $h_r\in L^1(\Omega)$, for all $\mid s\mid\leq r$, $\forall r\geq 0$. For the parabolic, we refer the reader to [10]. Recently, the existence of a capacity solution in the context of Orlicz-Sobolev spaces with $\sigma=\sigma(x,u,\nabla u)$ and $H=0$ has been established in [12].

The motivation behind the study of differential equations comes from applications of non Newtonian mechanics turbulence modelling to as an example of an operator for which the present result can be applied, we give

$$\begin{cases} -\Delta_M u+h(x)M^{-1}M(\alpha u)=ru^\zeta e^{\frac{-s}{k_B u}}\mid\nabla u\mid^2 & \text{in} \quad \Omega,\\ \operatorname{div}(\kappa(u)\nabla\varphi)=0 & \text{in} \quad \Omega,\end{cases} \tag{1.11}$$

where $\Delta_M u=-\operatorname{div}\left((1+\mid u\mid)^2 Du\frac{\log(e+Du)}{|Du|}\right)$, $h(.)\in\left(L^\infty(Q_T)\right)^N$ and $M(t)=t\log(e+t)$ is an N -function, φ represent the electric motive force, u the temperature inside the electrical conductor, and $\kappa(u)=ru^\zeta e^{\frac{-s}{k_B u}}$, the electrical conductivity where it means the ability of electrical material to pass charges, where $u>0$, $r,s\in\mathbb{R}^+$, $\zeta\in[-1,1)$ and k_B is the Boltzmann constant. Other applications of the stationary case of the thermostat problem can be found in [8, 15].

Our novelty in the present paper is to give the existence of a capacity solution of (1.1)-(1.3) in the framework of Orlicz spaces with the presence of a perturbation $\Phi(x, u)$. The difficulties encountered during the proof are that the term H satisfies neither the coercivity condition nor the monotony nor the sign condition and the nonlinearity described by N-functions M. The Δ_2-condition is not imposed on the N-functions M, we will lose the reflexivity of the space $L_M(\Omega)$ and $W_0^1 L_M(\Omega)$. To overcome this difficulty, we will first introduce and prove the existence of the solution for the auxiliary elliptic problem (2.8) and by Schauder's fixed point theorem, we show the existence of the uniqueness of the weak solutions for two equations (1.2) and (1.1). Secondly, with adequate approximate problems we establish some a priori estimates for the approximate solution sequence. Finally, we draw a subsequence to obtain a limit function and prove this function is a capacity solution in the sense of Definition 4.1 by virtue of the convergence results of approximate solutions. Note that the second lower order term H is controlled by a non-polynomial growth (see (1.8)). It is similar to those in [2, 3]. Finally, it should be noted that this work is an extension of the results of [12].

The contents of this article are summarized as follows: Section 2 presents the mathematical preliminaries. In Section 3, we make precise all the basic assumptions on σ, H, κ, φ and some technical results. Finally, in Section 4, we give the definition of a capacity solution of (1.1)-(1.3) and we prove the main result (Theorem 4.2).

2. Preliminaries

Let $M : \mathbb{R}^+ \to \mathbb{R}^+$ be an N-function, that is, M is continuous, convex, with $M(t) > 0$ for $t > 0$, $\frac{M(t)}{t} \to 0$ as $t \to 0$, and $\frac{M(t)}{t} \to +\infty$ as $t \to +\infty$. Equivalently, M admits the representation $M(t) = \int_0^t a(s)ds$, where $a : \mathbb{R}^+ \to \mathbb{R}^+$ is nondecreasing, right continuous, with $a(0) = 0, a(t) > 0$ for $t > 0$, and $a(t) \to +\infty$ as $t \to +\infty$. The N-function $\overline{M}$conjugate to M is defined by $\overline{M}(t) = \int_0^t \overline{a}(s)ds$, where $\overline{a} : \mathbb{R}^+ \to \mathbb{R}^+$, is given by $\overline{a}(t) = \sup_{s \geq 0}\{s : a(s) \leq t\}$.

The N-function M is said to satisfy the Δ_2-condition if, for some k, $M(2t) \leq kM(t)$ for all $t \in IR^+$.

We will extend these N-functions into even functions on all $\mathbb{R}$. Let P and Q be two N-functions. $P \ll Q$ means that P grows essentially less rapidly than Q, that is, for each $\epsilon > 0$, $\frac{P(t)}{Q(\epsilon t)} \to 0$ as $t \to +\infty$. This is the case if and only if $\lim_{t\to+\infty} \frac{Q^{-1}(t)}{P^{-1}(t)} = 0$.

The Orlicz class $K_M(\Omega)$ (resp. the Orlicz space $L_M(\Omega)$), is defined as the set of (equivalence classes of) real valued measurable functions u on Ω such that

$$\int_\Omega M(|u(x)|)dx < +\infty \left(\text{resp.} \int_\Omega M\left(\frac{|u(x)|}{\lambda}\right) dx < +\infty \quad \text{for some} \quad \lambda > 0\right).$$

The set $L_M(\Omega)$ is Banach space under the norm

$$\|u\|_M = \inf\left\{\lambda > 0 : \int_\Omega M\left(\frac{|u(x)|}{\lambda}\right) dx \leq 1\right\},$$

and $K_M(\Omega)$ is a convex subset of $L_M(\Omega)$. The closure in $L_M(\Omega)$ of the set of bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_M(\Omega)$. The dual of $E_M(\Omega)$ can be identified with $L_{\overline{M}}(\Omega)$ by means of the pairing $\int_\Omega uvdx$ and the dual norm of $L_{\overline{M}}(\Omega)$ is equivalent to $\|u\|_{\overline{M},\Omega}$. We now turn to the Orlicz-Sobolev space, $W^1L_M(\Omega)$) [resp. $W^1E_M(\Omega)$] is the space of all functions u such that u and its distributional derivatives up to order 1 lie in $L_M(\Omega)$ [resp. $E_M(\Omega)$]. It is a Banach space under the norm

$$\|u\|_{1,M} = \sum_{|\alpha|\leq 1} \|D^\alpha u\|_M.$$

Let $W^{-1}L_{\overline{M}}(\Omega)$ [resp. $W^{-1}E_{\overline{M}}(\Omega)$] denote the space of distributions on Ω which can be written as sums of derivatives of order ≤ 1 of functions in $L_{\overline{M}}(\Omega)$ [resp. $E_{\overline{M}}(\Omega)$]. It is a Banach space under the usual quotient norm (for more details see [4]).

Lemma 2.1. ([11]) *For all $u \in W_0^1L_M(\Omega)$ with meas$(\Omega) < +\infty$ one has*

$$\int_\Omega M\left(\frac{|u|}{\lambda}\right) dx \leq \int_\Omega M(|\nabla u|)dx. \tag{2.1}$$

where $\lambda = diam(\Omega)$, is the diameter of Ω.

Statement of useful results.

We assume that there exists four positive constants γ_0 and γ_1 such that

$$|u|^2 \leq \gamma_0 M(u), \quad \text{and } |u|^2 \leq \gamma_1 P(u) \quad \text{for all} \quad u \geq 0, \tag{2.2}$$

Hence, the following continuous inclusions hold true:

$$L_M(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow L_{\overline{M}}(\Omega), \quad \text{and } L_P(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow L_{\overline{P}}(\Omega). \tag{2.3}$$

And we also deduce that

$$W_0^1L_M(\Omega) \hookrightarrow H_0^1(\Omega), \quad \text{and } H^{-1}(\Omega) \hookrightarrow W^{-1}L_{\overline{M}}(\Omega). \tag{2.4}$$

Example 2.2. The N-function $M(t) = tlog(e+t)$ verifies the previous results.

Consider the following set $\mathbf{W} = \left\{\omega \in E_M(\Omega) : \int_\Omega M(\frac{|\omega|}{\lambda C_0})dx \leq 1\right\}$.

It is closed and convex. Indeed let $\omega_n \in \mathbf{W}$ such that $\omega_n \to \omega$ strongly in $E_M(\Omega)$, then for any $\epsilon > 0$, there exists n_0 such that for all $n \geq n_0$ we have $\|\omega_n - \omega\|_M \leq \epsilon$ and $\|\frac{\omega}{\lambda C_0}\|_M \leq \|\frac{\omega_n-\omega}{\lambda C_0}\|_M + \|\frac{\omega_n}{\lambda C_0}\|_M \leq \frac{\epsilon}{\lambda C_0} + 1$. Let tends as $\epsilon \to 0$ we have $\omega \in \mathbf{W}$; Thus $\mathbf{W}$ is closed. And since M is convex function, we deduce that $\mathbf{W}$ is also convex. Now, let is start by this first result that we will use later. Suppose that σ verifies the following strong hypothesis: There exists $a_3(.) \in E_{\overline{M}}(\Omega)$, and $\nu > 0$ and $k_4 \geq 0$, such that for almost every $x \in \Omega$ and for all $s \in \mathbb{R}$, $\xi \in \mathbb{R}^N$,

$$|\sigma(x,s,\xi)| \leq \nu[a_3(x) + \overline{M}^{-1}(M(k_4 |\xi|))], \tag{2.5}$$

$\kappa \in C(\mathbb{R})$ and there exist κ_1 and $\kappa_2 \in \mathbb{R}$ such that

$$0 < \kappa_1 \leq \kappa(s) \leq \kappa_2, \text{ for all } s \in \mathbb{R}. \tag{2.6}$$

Then

$$\operatorname{div}(\kappa(\omega)\varphi\nabla\varphi) \in H^{-1}(\Omega). \tag{2.7}$$

Proof. Let $\omega \in \mathbf{W}$, we consider the elliptic problem

$$\begin{cases} \operatorname{div}(\kappa(\omega)\nabla\varphi) = 0 & \text{in}\,\Omega, \\ \varphi = \varphi_0 & \text{on}\,\partial\Omega. \end{cases} \tag{2.8}$$

By applying Lax-Milgram's theorem, we prove that there exists a unique solution $\varphi \in H^1(\Omega)$ to (2.8) and, by (1.10) and the maximum principle, we get

$$\|\varphi\|_{L^\infty(\Omega)} \leq \|\varphi_0\|_{L^\infty(\Omega)}. \tag{2.9}$$

Multiplying the first equation of (2.8) by $\varphi - \varphi_0 \in H_0^1(\Omega)$ we get

$$\int_\Omega \kappa(\omega)\nabla(\varphi - \varphi_0) = 0,$$

therefore

$$\kappa_1 \int_\Omega |\nabla\varphi|^2\, dx \leq \int_\Omega \kappa(\omega) |\nabla\varphi||\nabla\varphi_0|\, dx \leq \kappa_2 \int_\Omega |\nabla\varphi||\nabla\varphi_0|\, dx.$$

We deduce from the Cauchy-Schwarz inequality that

$$\int_\Omega |\nabla\varphi|^2\, dx \leq C = C(\kappa_1, \kappa_2, \varphi_0). \tag{2.10}$$

Notice that $\kappa(\omega) \mid \nabla\varphi \mid^2 \in L^1(\Omega)$, this term is also belongs to the space $H^{-1}(\Omega)$. Indeed, let $\psi \in \mathcal{D}(\Omega)$ and taking $\psi\varphi$ as a test function in (2.8), we have

$$\int_\Omega \kappa(\omega)\nabla\varphi\nabla(\psi\varphi)dx = 0,$$

then

$$\int_\Omega \kappa(\omega) |\nabla\varphi|^2 \psi dx = -\int_\Omega \kappa(\omega)\varphi\nabla\varphi\nabla\psi dx = \langle \operatorname{div}(\kappa(\omega)\varphi\nabla\varphi), \psi\rangle_{\mathcal{D}'(\Omega),\mathcal{D}(\Omega)}.$$

Thus

$$\kappa(\omega) |\nabla\varphi|^2 = \operatorname{div}(\kappa(\omega)\varphi\nabla\varphi) \quad \text{in } \mathcal{D}'(\Omega). \tag{2.11}$$

Since $\kappa(\omega)\varphi\nabla\varphi \in L^2(\Omega)^N$, we deduce (2.7). □

3. Main result

Theorem 3.1. *Assume (1.4)-(2.3), with (2.5) and (2.6) instead of (1.4) and (1.9), respectively. Then there exists a weak solution (u, φ) to problem (1.1)-(1.3), that is,*

$$\begin{cases} u \in W_0^1 L_M(\Omega),\ \sigma(x, u, \nabla u) \in L_{\overline{M}}(\Omega)^N, \\ \varphi - \varphi_0 \in H_0^1(\Omega) \cap L^\infty(\Omega), \\ \displaystyle\int_\Omega \sigma(x, u, \nabla u)\nabla\psi + \int_\Omega \Phi(x, u)\psi = -\int_\Omega \kappa(u)\varphi\nabla\varphi\nabla\psi, \quad \textit{for all} \quad \psi \in W_0^1 L_M(\Omega), \\ \displaystyle\int_\Omega \kappa(u)\nabla\varphi\nabla\psi = 0, \quad \textit{for all} \quad \psi \in H_0^1(\Omega). \end{cases}$$

Proof. Let consider the following variational formulation problem

$$\begin{cases} u \in W_0^1 L_M(\Omega),\ \sigma(x,\omega,\nabla u) \in L_{\overline{M}}(\Omega),\ \Phi(x,\omega) \in L_{\overline{M}}(\Omega), \\ \displaystyle\int_\Omega \sigma(x,\omega,\nabla u)\nabla\phi + \int_\Omega \Phi(x,\omega)\phi = -\int_\Omega \kappa(\omega)\varphi\nabla\varphi\nabla\phi, \text{for all } \phi \in W_0^1 L_M(\Omega), \\ u = 0 \text{ on } \partial\Omega. \end{cases} \tag{3.1}$$

Notice that $\operatorname{div}(\kappa(\omega)\varphi\nabla\varphi) \in H^{-1}(\Omega) \hookrightarrow W^{-1}L_{\overline{M}}(\Omega)$, and the existence of solution to (3.1) is derived by an application of the result obtained in [11]. Also we can check that the solution of (3.1) is unique [5].

Lemma 3.2. *Let u be a weak solution of problem (3.1). Then we have $\mid \nabla u \mid \in K_M(\Omega)$, and the estimates*

$$\int_\Omega M(\mid \nabla u \mid)dx \leq C_1, \tag{3.2}$$

$$\|\sigma(x,\omega,\nabla u)\|_{\overline{M},\Omega} \leq C_2, \tag{3.3}$$

Where C_1 and C_2 are two positive constants that do not depend on ω.

Proof. Let $\eta > 0$ such that $\dfrac{\mid \nabla u \mid}{\eta} \in K_M(\Omega)$. Since $\varphi \in H^1(\Omega) \subset W^1 L_M(\Omega)$, there exist $\beta > 0$ such that $\dfrac{\kappa_2\|\varphi_0\|_{L^\infty(\Omega)} \mid \nabla\varphi \mid}{\beta} \in K_{\overline{M}}(\Omega)$, we take $\psi = u$ as a test function in (3.1). In view of (1.6), (1.7), (1.8), (2.6), (2.9) and Young's inequality, and Lemma 2.1, we get

$$\begin{aligned}
\frac{\alpha}{\eta\beta}\int_\Omega M(\mid \nabla u \mid)dx &\leq \frac{1}{\eta\beta}\int_\Omega \sigma(x,\omega,\nabla u)\nabla u dx \\
&\leq \frac{\lambda}{\beta}\int_\Omega \mid \Phi(x,\omega) \mid \frac{\mid u \mid}{\eta\lambda}dx + \int_\Omega \frac{\kappa_2\|\varphi_0\|_{L^\infty(\Omega)} \mid \nabla\varphi \mid}{\beta}\frac{\mid \nabla u \mid}{\eta}dx \\
&\leq \frac{\lambda}{\beta}\int_\Omega h(x)\overline{M}^{-1}M\left(\frac{\mid \omega \mid}{\lambda C_0}\right)\frac{\mid u \mid}{\eta\lambda}dx + \int_\Omega \frac{\kappa_2\|\varphi_0\|_{L^\infty(\Omega)} \mid \nabla\varphi \mid}{\beta}\frac{\mid \nabla u \mid}{\eta}dx \\
&\leq \frac{\lambda\|h\|_{L^\infty(\Omega)}}{\beta}\int_\Omega M\left(\frac{\mid \omega \mid}{\lambda C_0}\right)dx + \frac{\lambda\|h\|_{L^\infty(\Omega)}}{\beta}\int_\Omega M\left(\frac{\mid u \mid}{\eta\lambda}\right)dx \\
&+ \int_\Omega \overline{M}\left(\frac{\kappa_2\|\varphi_0\|_{L^\infty(\Omega)} \mid \nabla\varphi \mid}{\beta}\right)dx + \int_\Omega M\left(\frac{\mid \nabla u \mid}{\eta}\right)dx \\
&\leq \frac{\lambda\|h\|_{L^\infty(\Omega)}}{\beta} + \frac{\lambda\|h\|_{L^\infty(\Omega)}}{\beta}\int_\Omega M\left(\frac{\mid \nabla u \mid}{\eta}\right)dx \\
&+ \int_\Omega \overline{M}\left(\frac{\kappa_2\|\varphi_0\|_{L^\infty(\Omega)} \mid \nabla\varphi \mid}{\beta}\right)dx + \int_\Omega M\left(\frac{\mid \nabla u \mid}{\eta}\right)dx < \infty.
\end{aligned}$$

Then we deduce that $|\nabla u| \in K_M(\Omega)$. Let prove the estimate (3.2), by (1.6), (1.7), (1.8), (2.2) and Young's inequality, and Lemma 2.1, we obtain

$$\begin{aligned}
\alpha \int_\Omega M(|\nabla u|)dx &\leq \int_\Omega \sigma(x,\omega,\nabla u)\nabla u dx \\
&\leq \int_\Omega |\Phi(x,\omega)| \frac{|u|}{\lambda}dx + \int_\Omega \kappa_2\|\varphi_0\|_{L^\infty(\Omega)} |\nabla\varphi||\nabla u| \, dx \\
&\leq \lambda \int_\Omega h(x)\overline{M}^{-1}M\left(\frac{|\omega|}{\lambda C_0}\right)\frac{|u|}{\lambda}dx + \frac{(\kappa_2\|\varphi_0\|_{L^\infty(\Omega)})^2}{2\alpha\epsilon}\int_\Omega |\nabla\varphi|^2 + \frac{\alpha\epsilon}{2}\int_\Omega |\nabla u|^2 \, dx \\
&\leq \lambda\|h\|_{L^\infty(\Omega)}\int_\Omega M\left(\frac{|\omega|}{\lambda C_0}\right)dx + \lambda\|h\|_{L^\infty(\Omega)}\int_\Omega M\left(\frac{|u|}{\lambda}\right)dx \\
&+ \frac{(\kappa_2\|\varphi_0\|_{L^\infty(\Omega)})^2}{2\alpha\epsilon}\int_\Omega |\nabla\varphi|^2 \, dx + \frac{\alpha\epsilon}{2}\int_\Omega |\nabla u|^2 \, dx \\
&\leq \lambda\|h\|_{L^\infty(\Omega)} + \lambda\|h\|_{L^\infty(\Omega)}\int_\Omega M(|\nabla u|)dx \\
&+ \frac{(\kappa_2\|\varphi_0\|_{L^\infty(\Omega)})^2}{2\alpha\epsilon}C(\kappa_1,\kappa_2,\varphi_0) + \frac{\alpha\epsilon\gamma_0}{2}\int_\Omega M(|\nabla u|)dx.
\end{aligned}$$

which implies, that

$$(\alpha - \lambda\|h\|_{L^\infty(\Omega)} - \frac{\alpha\epsilon\gamma_0}{2})\int_\Omega M(|\nabla u|)dx \leq \lambda\|h\|_{L^\infty(\Omega)} + \frac{(\kappa_2\|\varphi_0\|_{L^\infty(\Omega)})^2}{2\alpha\epsilon}C(\kappa_1,\kappa_2,\varphi_0).$$

Then by choosing ϵ such that $\alpha - 2\lambda\|h\|_{L^\infty(\Omega)} - \frac{\alpha\epsilon\gamma_0}{2} > 0$, as a consequence, we have the estimate (3.2).

Remark 3.3.

- We take the constant C_0 and C_1 such that

  $$\lambda\|h\|_{L^\infty(\Omega)} + \frac{(\kappa_2\|\varphi_0\|_{L^\infty(\Omega)})^2}{2\alpha\epsilon}C(\kappa_1,\kappa_2,\varphi_0) < C_0\left(\alpha - 2\lambda\|h\|_{L^\infty(\Omega)} - \frac{\alpha\epsilon\gamma_0}{2}\right)$$

  and

  $$C_0 < C_1. \tag{3.4}$$

- It is clear that u belongs also to $\mathbf{W}$ and do not depends on ω.

  On the other hand, from the previous prove and (1.6), we also have

  $$\int_\Omega \sigma(x,\omega,\nabla u)\nabla u dx \leq \frac{C_1}{\alpha}. \tag{3.5}$$

From (1.6), (3.5) and Young's inequality, we get

$$\int_\Omega \sigma(x,\omega,\nabla u)\nabla\psi dx \leq \int_\Omega \sigma(x,\omega,\nabla u)\nabla u dx - \int_\Omega \sigma(x,\omega,\nabla\psi)(\nabla u - \nabla\psi)dx$$

$$\leq \frac{C_1}{\alpha} + \int_\Omega \mid \sigma(x,\omega,\nabla\psi) \mid\mid \nabla u \mid dx + \int_\Omega \mid \sigma(x,\omega,\nabla\psi) \mid\mid \nabla\psi \mid dx$$

$$\leq \frac{C_1}{\alpha} + 4\nu\int_\Omega \overline{M}\left(\frac{\sigma(x,\omega,\nabla\psi)}{2\nu}\right) + 2\nu\int_\Omega \left[M(\mid \nabla u \mid) + M(\mid \nabla\psi \mid)\right] dx.$$

by (2.5), we get

$$\int_\Omega \overline{M}\left(\frac{\sigma(x,\omega,\nabla\psi)}{2\nu}\right) dx \leq \int_\Omega \frac{1}{2}(\overline{M}(a_3(x))dx + M(k_4 \mid \nabla\psi \mid))dx.$$

Choosing $\psi \in W_0^1E_M(\Omega)$ such that $\|\nabla\psi\|_{M,\Omega} = \dfrac{1}{k_4+1}$, then

$$\int_\Omega \sigma(x,\omega,\nabla u)\nabla\psi dx \leq C,$$

finally to deduce the estimate (3.3), we use the dual norm on $L_{\overline{M}}(\Omega)$. □

Now, let define the operator $T : \omega \in \mathbf{W} \longrightarrow u \in W_0^1L_M(\Omega) \hookrightarrow E_M(\Omega)$, where u is the unique solution to (3.1), then due to the estimate (3.2), T is a compact operator. Moreover, from (3.2), (3.4) and Lemma 2.1, we have $T(\mathbf{W}) \subset \mathbf{W}$. And to satisfy the hypotheses of Schauder's fixed point theorem for T, it remains to be shown that T is a continuous operator. Indeed, taking a sequence $(\omega_n) \subset \mathbf{W}$ such that $\omega_n \to \omega$ strongly in $E_M(\Omega)$ and let $u_n = T(\omega_n)$, φ_n, $F_n = \kappa(\omega_n)\varphi_n\nabla\varphi_n$ and $F = \kappa(\omega)\varphi\nabla\varphi$. We have to show that

$$u_n \to u = T(\omega) \text{ strongly in } E_M(\Omega).$$

Owing to (3.2), we have $\nabla u \in L_M(\Omega)^N$. We also have $\omega_n \to \omega$ strongly in $L^2(\Omega)$ and thus, we may extract a subsequence, still denoted in the same way, such that $\omega_n \to \omega$ a.e. in Ω. Then it is easy task to show that $\varphi_n \to \varphi$ strongly in $H^1(\Omega)$ and, consequently, also for another subsequence denoted in the same way, $F_n \to F$ strongly in $L^2(\Omega)$.
Since $(\omega_n) \subset L_M(\Omega)$ is bounded, we deduce for a subsequence,

$$u_n \to U \text{ in } E_M(\Omega), \text{ for some } U \in E_M(\Omega), \tag{3.6}$$

$$\nabla u_n \to \nabla U \text{ weakly in } L^2(\Omega)^N. \tag{3.7}$$

By subtracting the respective equations of (3.1) for u_n and u, and taking $\phi = u_n - u$ as a test function, we obtain

$$\int_\Omega(\sigma(x,\omega_n,\nabla u_n) - \sigma(x,\omega,\nabla u))(\nabla u_n - \nabla u)dx + \int_\Omega(\Phi(x,\omega_n) - \Phi(x,\omega))(u_n - u)dx$$
$$= -\int_\Omega(F_n - F)(\nabla u_n - \nabla u)dx. \tag{3.8}$$

For the first term of the right hand-side of (3.8):
Using (1.6), we get

$$(\sigma(x,\omega_n,\nabla u_n)-\sigma(x,\omega,\nabla u))(\nabla u_n-\nabla u)\geq \alpha M(\mid\nabla(u_n-u)\mid)$$
$$+(\sigma(x,\omega_n,\nabla u)-\sigma(x,\omega,\nabla u))(\nabla u_n-\nabla u).$$

Let $B_n=\sigma(x,\omega_n,\nabla u)-\sigma(x,\omega,\nabla u)$, then $\mid B_n\mid\to 0$ a.e. in Ω. For a given positive number δ_0, to be chosen later, we have

$$\int_\Omega \mid B_n\nabla(u_n-u)\mid=\int_{\{|\nabla(u_n-u)|\leq\delta_0\}}\mid B_n\nabla(u_n-u)\mid$$
$$+\int_{\{|\nabla(u_n-u)|>\delta_0\}}\mid B_n\nabla(u_n-u)\mid \tag{3.9}$$

For the first term of the right-hand side of (3.9), we have

$$\int_{\{|\nabla(u_n-u)|\leq\delta_0\}}\mid B_n\nabla(u_n-u)\mid\leq\delta_0\int_\Omega\mid B_n\mid$$
$$=\delta_0\int_{\{|B_n|\leq 4\nu\}}\mid B_n\mid+\delta_0\int_{\{|B_n|>4\nu\}}\mid B_n\mid.$$

The first of these integrals converges to zero. As for the second one, using the fact that $\frac{\mid B_n\mid}{4\nu}>1$ on the set $\{\mid B_n\mid>4\nu\}$ and (2.2), it yields

$$\delta_0\int_{\{|B_n|>4\nu\}}\mid B_n\mid\leq 4\nu\delta_0\int_{\{|B_n|>4\nu\}}\left(\frac{\mid B_n\mid}{4\nu}\right)^2\leq 4\nu\gamma_1\delta_0\int_\Omega P\left(\frac{\mid B_n\mid}{4\nu}\right).$$

In virtue of (1.5) and while $P\ll M$ for $\epsilon k_3\leq 1$, we deduce

$$P\left(\frac{\mid B_n\mid}{4\nu}\right)\leq\frac{1}{4}(P(a_1)+P(\omega)+P(\omega_n)+k_3M(\mid\nabla u\mid)),$$

and since $P(\omega_n)\to P(\omega)$ strongly in $L^1(\Omega)$, by Lebesgue's dominated theorem it yields that

$$\lim_{n\to\infty}\int_\Omega P\left(\frac{\mid B_n\mid}{4\nu}\right)=0,$$

consequently,

$$\lim_{n\to\infty}\int_{\{|\nabla(u_n-u)|\leq\delta_0\}}\mid B_n\nabla(u_n-u)\mid=0.$$

For the second term of the right-hand side of (3.9), we use Young's inequality and (2.2). It yields

$$\int_{\{|\nabla(u_n-u)|>\delta_0\}}\mid B_n\nabla(u_n-u)\mid\leq\frac{1}{\alpha\varepsilon_0}\int_E\mid B_n\mid^2+\frac{\alpha\varepsilon_0}{4}\int_{\{|\nabla(u_n-u)|>\delta_0\}}\mid\nabla(u_n-u)\mid^2$$
$$\leq\frac{16\gamma_1\nu^2}{\alpha\varepsilon_0}\int_\Omega P\left(\frac{\mid B_n\mid}{4\nu}\right)+\frac{\alpha\gamma_1\varepsilon_0}{4}\int_{\{|\nabla(u_n-u)|>\delta_0\}}P(\mid\nabla(u_n-u)\mid).$$

It has been already shown that the first of these terms converges to zero. As for the second one, since $P \ll M$, we fix $\delta_0 > 0$ such $P(s) \leq M(s)$ for all $s > \delta_0$. Then

$$\frac{\alpha\gamma_1\varepsilon_0}{4}\int_{\{|\nabla(u_n-u)|>\delta_0\}} P(|\nabla(u_n-u)|)dx \leq \frac{\alpha\gamma_1\varepsilon_0}{4}\int_{\{|\nabla(u_n-u)|>\delta_0\}} M(|\nabla(u_n-u)|)dx.$$

By taking $\varepsilon_0 = \dfrac{2}{\lambda_0}$, we obtain

$$\frac{\alpha}{2}\int_{\{|\nabla(u_n-u)|>\delta_0\}} P(|\nabla(u_n-u)|)dx \leq \frac{\alpha}{2}\int_{\{|\nabla(u_n-u)|>\delta_0\}} M(|\nabla(u_n-u)|)dx.$$

For the second term of the right hand-side of (3.8):

$$\begin{aligned}\int_\Omega (\Phi(x,\omega_n)-\Phi(x,\omega))(u_n-u)dx &\leq \int_\Omega |\Phi(x,\omega_n)-\Phi(x,\omega)||u_n-u|\,dx\\ &\leq \int_\Omega \overline{M}\left(\frac{\lambda\,|\Phi(x,\omega_n)-\Phi(x,\omega)|}{\alpha_0}\right)dx + \alpha_0\int_\Omega M\left(\frac{|u_n-u|}{\lambda}\right)dx\\ &\leq \int_\Omega \overline{M}\left(\frac{\lambda\,|\Phi(x,\omega_n)-\Phi(x,\omega)|}{\alpha_0}\right)dx + \alpha_0\int_\Omega M(|\nabla(u_n-u)|)dx.\end{aligned} \tag{3.10}$$

From above, we deduce the following estimate, for some sequence (ϵ_n) such that $\epsilon_n \to 0$,

$$\begin{aligned}&(\frac{\alpha}{2}-\alpha_0)\int_\Omega M(|\nabla(u_n-u)|)dx\\ &\leq \int_\Omega |(F_n-F)\nabla(u_n-u)|\,dx + \alpha_0\int_\Omega \overline{M}\left(\frac{\lambda\,|\Phi(x,\omega_n)-\Phi(x,\omega)|}{\alpha_0}\right)dx + \epsilon_n.\end{aligned} \tag{3.11}$$

Choosing $\alpha_0 = \dfrac{\alpha}{4}$ and by (2.3), we obtain

$$\begin{aligned}\frac{\alpha}{4\gamma_0}\|\nabla(u_n-u)\|^2_{L^2(\Omega)} &\leq \int_\Omega |(F_n-F)\nabla(u_n-u)|\,dx\\ &\quad + \alpha_0\int_\Omega \overline{M}\left(\frac{\lambda\,|\Phi(x,\omega_n)-\Phi(x,\omega)|}{\alpha_0}\right)dx + \epsilon_n.\end{aligned}$$

Using Poincare's inequality, we get

$$\begin{aligned}\|u_n-u\|^2_{L^2(\Omega)} &\leq C\int_\Omega |(F_n-F)\nabla(u_n-u)|\,dx\\ &\quad + C\alpha_0\int_\Omega \overline{M}\left(\frac{\lambda\,|\Phi(x,\omega_n)-\Phi(x,\omega)|}{\alpha_0}\right)dx + \epsilon_n.\end{aligned} \tag{3.12}$$

We have $F_n \to F$ strongly in $L^2(\Omega)^N$ and $\nabla(u_n-u)$ is bounded in $L^2(\Omega)^N$. On the other hand $\omega_n \to \omega$ strongly in B_R, we may extract a subsequence, still denoted the same way, such that $\omega_n \to \omega$ a.e. in Ω. In addition, the function H is continuous with respect to its second argument, then from (1.8) and dominate convergence's theorem $\int_\Omega \overline{M}\left(\frac{\lambda\,|\Phi(x,\omega_n)-\Phi(x,\omega)|}{\alpha_0}\right)dx$ converges to 0.

Then the right-hand side in (3.12) converges to zero. In conclusion, $u_n \to u$ strongly in $L^2(\Omega)$. Since this limit does not depend upon the subsequence one may extract, it is in fact the whole sequence (u_n) which converges to u strongly in $L^2(\Omega)$. On other hand, in virtue of (3.6), we also have $u_n \to U$ strongly in $L^2(\Omega)$, so that $u = U$ and we can rewrite (3.6) to give $u_n \to u$ strongly in $E_M(\Omega)$. This shows that T is continuous and this ends the proof of theorem 3.1. □

4. An existence result

The definition of a capacity solution of (1.1)-(1.3) can be stated as follows.

Definition 4.1. A triplet (u, φ, Φ) is called a capacity solution of (1.1)-(1.3) if the following conditions are fulfilled:
(R_1) $u \in W_0^1 L_M(\Omega)$, $\sigma(x, u, \nabla u) \in L_{\overline{M}}(\Omega)^N$, $\Phi(x, u) \in L^1(\Omega)$, $\Phi(x, u).u \in L^1(\Omega)$, $\varphi \in L^\infty(\Omega)$, $\Phi \in L^2(\Omega)^N$.
(R_2) (u, φ, Φ) verifies the system of elliptic equations

$$\begin{cases} -\text{div}\ \sigma(x, u, \nabla u) + \Phi(x, u) = \text{div}(\varphi\Phi) & \text{in} \quad \Omega, \\ \text{div}(\Phi) = 0 & \text{in} \quad \Omega. \end{cases}$$

(R_3) For every $S \in C_0^1(\Omega) = \{\phi \in C^1(\Omega)/\text{supp}(\phi) \text{ is compact}\}$, one has

$$S(u)\varphi - S(0)\varphi_0 \in H_0^1(\Omega), \quad \text{and} \quad S(u)\Phi = \kappa(u)[\nabla(S(u)\varphi) - \varphi\nabla S(u)].$$

Our most general result reads as follows.

Theorem 4.2. *Assume that (1.4)-(2.3) hold true. Then there exists a capacity solution to problem (1.1)-(1.3).*

Proof of the theorem 4.2
Step 1: Approximative problem.
For every $n \in \mathbb{N}^*$, let us define the following approximation of κ, a and g:

$$\kappa_n(s) = \kappa(s) + \frac{1}{n}, \quad \sigma_n(x, s, \xi) = \sigma(x, T_n(s), \xi), \quad \Phi_n(x, s) = \frac{\Phi(x, s)}{1 + \dfrac{1}{n} \mid \Phi(x, s) \mid},$$

for all $x \in \Omega$ and $\xi \in \mathbb{R}^N$.
Let us now consider the approximate system

$$-\text{ div } \sigma_n(x, u_n, \nabla u_n) + \Phi_n(x, u_n) = \kappa_n(u_n) \mid \nabla\varphi_n \mid^2 \text{ in } \Omega, \tag{4.1}$$

$$\text{div}(\kappa_n(u_n)\nabla\varphi_n) = 0 \text{ in } \Omega, \tag{4.2}$$

$$u_n = 0, \text{ on } \partial\Omega, \tag{4.3}$$

$$\varphi_n = \varphi_0, \text{ on } \partial\Omega. \tag{4.4}$$

From (1.4), we deduce

$$\mid \sigma(x, T_n(s), \xi) \mid \leq \nu \left[a_0(x) + \overline{M}^{-1}(P(k_1 \mid T_n(s) \mid)) + \overline{M}^{-1}(M(k_2 \mid \xi \mid)) \right],$$

where $\left(a_0(x)+\overline{M}^{-1}(P(k_1 n))\right)\in E_{\overline{M}}(\Omega)$.
In view of (1.9), we have that

$$n^{-1}\leq \kappa_n(s)\leq \overline{\kappa}+1=\kappa_3,\ \text{for all } s\in\mathbb{R}. \tag{4.5}$$

We have also $|\ \Phi_n(x,s)\ |\leq|\ \Phi(x,s)\ |$ and $|\ \Phi_n(x,s)\ |\leq n$. Thus, we can apply Theorem 3.1 to deduce the existence of a weak solution (u_n,φ_n) to the system (4.1)-(4.4).
From the maximum principle

$$\|\varphi_n\|_{L^\infty(\Omega)}\leq\|\varphi_0\|_{L^\infty(\Omega)}; \tag{4.6}$$

hence, there exists a function $\varphi\in L^\infty(\Omega)$ and a subsequence, still denoted φ_n, such that

$$\varphi_n\rightarrow\varphi,\ \text{weakly-}*\ \text{ in } L^\infty(\Omega). \tag{4.7}$$

Now let multiply (4.2) by $\varphi_n-\varphi_0\in H_0^1(\Omega)$ and integrate over Ω. We get

$$\int_\Omega \kappa_n(u_n)\nabla\varphi_n\nabla(\varphi_n-\varphi_0)dx=0;$$

hence

$$\int_\Omega \kappa_n(u_n)\mid\nabla\varphi_n\mid^2 dx\leq C_3,\ \text{for all } n\in\mathbb{N}^*, \tag{4.8}$$

where $C_3=C(\overline{\kappa},\|\varphi_0\|_{L^\infty(\Omega)})$. Consequently, the sequence $(\kappa_n(u_n)\nabla\varphi_n)$ is bounded in $L^2(\Omega)^N$. Thus, there exists a function $\phi\in L^2(\Omega)^N$ and a subsequence, still denoted in the same way, such that

$$\kappa_n(u_n)\nabla\varphi_n\rightarrow\phi \text{ weakly in } L^2(\Omega)^N. \tag{4.9}$$

This weak limit function $\phi\in L^2(\Omega)^N$ is in fact the third component of the triplet appearing in the defintion (4.1) of a capacity solution.
Taking u_n as a function test in (4.1), we obtain

$$\int_\Omega \sigma(x,T_n(u_n),\nabla u_n)\nabla u_n dx+\int_\Omega \Phi_n(x,u_n)u_n dx=-\int_\Omega \kappa_n(u_n)\varphi_n\nabla\varphi_n\nabla u_n dx. \tag{4.10}$$

Since $u_n\in W_0^1L_M(\Omega)$, and $\varphi_n\in H^1(\Omega)\subset W^1L_{\overline{M}}(\Omega)$, there exist η_n and $\beta_n>0$ such that $\frac{|\nabla u_n|}{\eta_n}\in K_M(\Omega)$ and $\frac{\kappa_3\|\varphi_0\|_{L^\infty(\Omega)}|\nabla\varphi_n|}{\beta_n}\in K_{\overline{M}}(\Omega)$.

From (1.6), (1.7), (4.4) and (4.6) and Young's inequality, and Lemma 2.1, we obtain

$$
\begin{aligned}
\alpha \int_\Omega M(| \nabla u_n |)dx &\le \int_\Omega \sigma(x, T_n(u_n), \nabla u_n)\nabla u_n dx \\
&\le 2\lambda \|h\|_{L^\infty(\Omega)} \int_\Omega M\left(\frac{| u_n |}{\lambda C_0}\right) dx + \eta_n\beta_n \int_\Omega \overline{M}\left(\frac{\kappa_3 \|\varphi_0\|_{L^\infty(\Omega)} | \nabla\varphi_n |}{\beta_n}\right) dx \\
&+ \eta_n\beta_n \int_\Omega M\left(\frac{| \nabla u_n |}{\eta_n}\right) dx \\
&\le 2\lambda \|h\|_{L^\infty(\Omega)} \int_\Omega M(| \nabla u_n |)dx + \eta_n\beta_n \int_\Omega \overline{M}\left(\frac{\kappa_3 \|\varphi_0\|_{L^\infty(\Omega)} | \nabla\varphi_n |}{\beta_n}\right) dx \\
&+ \eta_n\beta_n \int_\Omega M\left(\frac{| \nabla u_n |}{\eta_n}\right) dx.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
(\alpha - 2\lambda \|h\|_{L^\infty(\Omega)}) \int_\Omega M(| \nabla u_n |)dx &\le \eta_n\beta_n \int_\Omega \overline{M}\left(\frac{\kappa_3 \|\varphi_0\|_{L^\infty(\Omega)} | \nabla\varphi_n |}{\beta_n}\right) dx \\
&+ \eta_n\beta_n \int_\Omega M\left(\frac{| \nabla u_n |}{\eta_n}\right) dx < \infty.
\end{aligned}
$$

and thus $| \nabla u_n | \in K_M(\Omega)$. On the other hand

$$
\begin{aligned}
\alpha \int_\Omega M(| \nabla u_n |)dx &\le \int_\Omega \sigma(x, T_n(u_n), \nabla u_n)\nabla u_n dx \\
&\le 2\lambda \|h\|_{L^\infty(\Omega)} \int_\Omega M\left(\frac{| u_n |}{\lambda C_0}\right) dx + \frac{(\kappa_3 \|\varphi_0\|_{L^\infty(\Omega)})^2}{\alpha\epsilon_1} \int_\Omega \kappa(u_n) | \nabla\varphi_n |^2 dx \\
&+ \frac{\alpha\epsilon_1}{4} \int_\Omega | \nabla u_n |^2 dx \\
&\le 2\lambda \|h\|_{L^\infty(\Omega)} \int_\Omega M(| \nabla u_n |)dx + \frac{(\kappa_3 \|\varphi_0\|_{L^\infty(\Omega)})^2}{\alpha\epsilon_1} C_3 + \frac{\alpha\epsilon_1\gamma_0}{4} \int_\Omega M(| \nabla u_n |)dx.
\end{aligned}
$$

Then

$$
\left(\frac{\alpha\epsilon_1\gamma_0}{4} - 2\lambda \|h\|_{L^\infty(\Omega)}\right) \int_\Omega M(| \nabla u_n |)dx \le \frac{(\kappa_3 \|\varphi_0\|_{L^\infty(\Omega)})^2}{\alpha} C_3.
$$

Taking $\epsilon_1 = \frac{4}{\gamma_0}$, we obtain

$$
\int_\Omega M(| \nabla u_n |)dx \le C_4. \tag{4.11}
$$

It follows that (u_n) is bounded in $W_0^1 L_M(\Omega)$. Consequently, there exists a subsequence, still denoted (u_n), and a function $u \in W_0^1 L_M(\Omega)$ such that

$$
u_n \rightharpoonup u \quad \text{in} \quad W_0^1 L_M(\Omega) \quad \text{for} \quad \sigma(\Pi L_M, \Pi E_{\overline{M}}), \tag{4.12}
$$

and since the embedding $W_0^1 L_M(\Omega) \hookrightarrow E_M(\Omega)$ is compact, we also have

$$
u_n \to u \quad \text{strongly in} \quad E_M(\Omega) \quad \text{and a.e. in} \quad \Omega, \tag{4.13}
$$

On the other hand, let $\psi \in W_0^1 E_M(\Omega)^N$ be arbitrary with $\|\nabla\psi\|_{(M)} = \dfrac{1}{k_2+1}$.
In view of the monotonicity of a_n, one easily has

$$\begin{aligned}\int_\Omega \sigma(x,u_n,\nabla u_n)\nabla\psi dx &\le \int_\Omega \sigma(x,u_n,\nabla u_n)\nabla u_n dx - \int_\Omega \sigma(x,u_n,\nabla\psi)(\nabla u_n - \nabla\psi)dx \\ &\le C + \int_\Omega \mid \sigma(x,u_n,\nabla\psi) \mid\mid \nabla u_n \mid dx + \int_\Omega \mid \sigma(x,u_n,\nabla\psi) \mid\mid \nabla\psi \mid dx.\end{aligned} \tag{4.14}$$

For the first integral in the right side, we use the Young's inequality to have

$$\int_\Omega \mid \sigma(x,u_n,\nabla\psi) \mid\mid \nabla u_n \mid dx \le 3\nu \int_\Omega \left[\overline{M}\left(\frac{\sigma(x,u_n,\nabla\psi)}{3\nu}\right) + M(\mid \nabla u_n \mid)\right] dx,$$

using (1.4) we have

$$3\nu\overline{M}\left(\frac{\sigma(x,T_n(u_n),\nabla\psi)}{3\nu}\right) \le \nu(\overline{M}(a_0(x)) + P(k_1T_n(u_n)) + M(k_2\nabla\psi)),$$

Since (u_n) is bounded in $W_0^1L_M(\Omega)$, and owing to Poincare's inequality, there exist $\lambda > 0$ such that $\displaystyle\int_\Omega M\left(\frac{u_n}{\lambda}\right) dx \le 1$ for all $n \in \mathbb{N}^*$. Also, since $P \ll M$, there exist $s_0 > 0$ such that $P(k_1 s) \le P(k_1 s_0) + M\left(\dfrac{s}{\lambda}\right)$ for all $s \in \mathbb{R}$.
Consequently,

$$\begin{aligned}3\nu \int_\Omega \overline{M}\left(\frac{\sigma(x,T_n(u_n),\nabla\psi)}{3\nu}\right) dx &\le \nu \int_\Omega \left(\overline{M}(a_0(x)) + P(k_1T_n(u_n)) + M(k_2\nabla\psi)\right) dx \\ &\le C,\end{aligned}$$

and thus $\displaystyle\int_\Omega \mid \sigma_n(x,u_n,\nabla\psi) \mid . \mid \nabla u_n\| dx \le C$, for all $n \in \mathbb{N}^*$ and $\psi \in W_0^1E_M(\Omega)^N$ such that $\|\nabla\psi\|_{(M)} = \dfrac{1}{k_2+1}$.
On the other hand, the second integral in (4.14), namely

$$\int_\Omega \mid \sigma_n(x,u_n,\nabla u_n) \mid . \mid \nabla u_n \mid dx \le C$$

can be dealt in the same way so that it is easy to check that it is also bounded. Gathering all these estimates, and using the dual norm, one easily deduce that

$$(\sigma_n(x,u_n,\nabla u_n)) \quad \text{is bounded in} \quad L_{\overline{M}}(\Omega)^N. \tag{4.15}$$

Thus, up to a subsequence, still denoted in the same way, there exists $\varpi \in L_{\overline{M}}(\Omega)^N$ such that

$$(\sigma_n(x,u_n,\nabla u_n)) \rightharpoonup \varpi \quad \text{in} \quad L_{\overline{M}}(\Omega)^N \quad \text{for} \quad \sigma(\Pi L_M, \Pi E_{\overline{M}}). \tag{4.16}$$

Step 2: Almost everywhere convergence of the gradient.
In this step, we may extract a subsequence of (u_n), still denoted the same way, such that

$$\nabla u_n \to \nabla u \quad \text{a.e. in} \quad \Omega, \quad \text{as} \quad n \to +\infty. \tag{4.17}$$

Let $v_j \in \mathcal{D}(\Omega)$ be a sequence that $v_j \to u$ in $W_0^1 L_M(\Omega)$ for the modular convergence see [9]. Setting for $s > 0$, $\Omega_s = \{x \in \Omega : \mid \nabla T_K(u) \mid \leq s\}$ and $\Omega_s^j = \{x \in \Omega : |\nabla T_K(v_j) \mid \leq s\}$ and denoting by χ^s and χ_s^j the characteristic functions of Ω_s and Ω_s^j respectively. And we denote by $\epsilon(i, j, \beta, n)$ the quantities such that

$$\lim_{i\to\infty} \lim_{j\to\infty} \lim_{\beta\to\infty} \lim_{n\to\infty} \epsilon(i, j, \beta, n) = 0.$$

For any $\eta > 0$ and $n, j \geq 1$, we may use the admissible test function

$$\varphi_{n,j}^{\eta} = T_\eta(u_n - T_K(v_j))$$

in (4.1). This leads to

$$\begin{aligned} \int_\Omega \sigma_n(x, u_n, \nabla u_n)\nabla T_\eta(u_n - T_K(v_j))dx + \int_\Omega \Phi_n(x, u_n)T_\eta(u_n - T_K(v_j))dx \\ = \int_\Omega \kappa_n(u_n) \mid \nabla\varphi_n \mid^2 \nabla T_\eta(u_n - T_K(v_j))dx. \end{aligned} \tag{4.18}$$

By Young's inequality and Lemma 2.1, we have

$$\begin{aligned} \int_\Omega |\Phi_n(x, u_n) \mid T_\eta(u_n - T_K(v_j))dx &\leq \int_\Omega \mid \Phi(x, u_n) \mid T_\eta(u_n - T_K(v_j))dx \\ &\leq \int_\Omega h(x)\overline{M}^{-1} M\left(\frac{\mid u_n \mid}{\lambda C_0}\right) T_\eta(u_n - T_K(v_j))dx \\ &\leq \eta\|h\|_{L^\infty(\Omega)} \int_\Omega M(\mid \nabla u_n \mid)dx + \eta\|h\|_{L^\infty(\Omega)} M(1)\text{meas}(\Omega) \\ &\leq C\eta. \end{aligned}$$

Using (4.8) and above result, we get

$$\int_\Omega \sigma_n(x, u_n, \nabla u_n)\nabla T_\eta(u_n - T_K(v_j))dx \leq C\eta. \tag{4.19}$$

Let's study the left-hand side of (4.19). We have

$$\begin{aligned} &\int_\Omega \sigma_n(x, u_n, \nabla u_n)\nabla T_\eta(u_n - T_K(v_j))dx \\ &= \int_{\{|u_n - T_K(v_j)| \leq \eta\}} \sigma_n(x, u_n, \nabla u_n)\nabla(u_n - T_K(v_j))dx \\ &= \int_{\{|u_n| > K\} \cap \{|u_n - T_K(v_j)| \leq \eta\}} \sigma_n(x, u_n, \nabla u_n)\nabla(u_n - T_K(v_j))dx \\ &+ \int_{\{|u_n| \leq K\} \cap \{|u_n - T_K(v_j)| \leq \eta\}} \sigma_n(x, u_n, \nabla u_n)\nabla(u_n - T_K(v_j))dx \\ &= \int_{\{|T_K(u_n) - T_K(v_j)| \leq \eta\}} \sigma_n(x, T_n(u_n), \nabla T_n(u_n))(\nabla T_n(u_n) - \nabla T_K(v_j))dx \\ &+ \int_{\{|u_n| > K\} \cap \{|u_n - T_K(v_j)| \leq \eta\}} \sigma_n(x, u_n, \nabla u_n)\nabla u_n dx \\ &- \int_{\{|u_n| > K\} \cap \{|u_n - T_K(v_j)| \leq \eta\}} \sigma_n(x, u_n, \nabla u_n)\nabla T_K(v_j)dx. \end{aligned}$$

which yields, thanks to (1.6) and (1.7),

$$\begin{cases} \int_\Omega \sigma_n(x,u_n,\nabla u_n)\nabla T_\eta(u_n - T_K(v_j))dx \\ \geq \int_{\{|u_n - T_K(v_j)|\leq \eta\}} \sigma_n(x,T_K(u_n),\nabla T_K(u_n))(\nabla T_n(u_n) - \nabla T_K(v_j))dx \\ - \int_{\{|u_n|>K\}\cap\{|u_n - T_K(v_j)|\leq \eta\}} \sigma_n(x,u_n,\nabla u_n)\nabla T_K(v_j)dx. \end{cases} \tag{4.20}$$

Let $0 < \delta < 1$, we define

$$\Theta_{n,K} = (\sigma(x,T_K(u_n),\nabla T_K(u_n)) - \sigma(x,T_K(u_n),\nabla T_K(u)))(\nabla T_K(u_n) - \nabla T_K(u)).$$

Using the similar technic as in [1], we obtain,

$$\int_{\Omega_r} \Theta^\delta_{n,K} dx \leq C_1 \text{meas}\{x\in\Omega :| T_K(u_n) - T_K(v_j) |> \eta\}^{1-\delta} + C_2(\epsilon(n,j,s,\eta))^\delta.$$

Which yields, by passing to the limit sup over n, j, s and η

$$\limsup_{n\to\infty} \int_{\Omega_r} \Big((\sigma(x,T_K(u_n),\nabla T_K(u_n)) - \sigma(x,T_K(u_n),\nabla T_K(u)))$$
$$\times(\nabla T_K(u_n) - \nabla T_K(u))\Big)^\delta dx = 0.$$

Thus, passing to a subsequence if necessary, $\nabla u_n \to \nabla u$ a.e. in Ω_r, and since r is arbitrary,

$$\nabla u_n \to \nabla u \quad \text{a.e. in} \quad \Omega.$$

Remark 4.3. A consequence of (4.17) is that,

$$\sigma(x,u_n,\nabla u_n) \rightharpoonup \sigma(x,u,\nabla u) \quad \text{in} \quad L_{\overline{M}}(\Omega)^N \quad \text{for} \quad \sigma(\Pi L_{\overline{M}}, \Pi E_M). \tag{4.21}$$

Step 3: Equi-integrability of the nonlinearitie $\Phi_n(x,u_n)$.
We shall now prove that $\Phi_n(x,u_n) \to \Phi(x,u)$ strongly in $L^1(\Omega)$ by using Vitali's theorem. Since $\Phi_n(x,u_n) \to \Phi(x,u)$ a.e in Ω, it is suffices to prove that $\Phi_n(x,u_n)$ are equi-integrable in Ω. Indeed, let ϵ and for any measurable subset $D \subset \Omega$. Using (1.8), Young's inequality and Lemma 2.1, we have

$$\int_D | \Phi_n(x,u_n) | dx \leq \int_D h(x)\overline{M}^{-1} M\left(\frac{| u_n |}{\lambda C_0}\right) dx$$
$$\leq \|h\|_{L^\infty(\Omega)} \int_D M(| \nabla u_n |)dx + \|h\|_{L^\infty(\Omega)} M(1)\text{meas}(D).$$

According to Lemma 3.2 in [6], we have $M(| \nabla u_n |) \to M(| \nabla u |)$ in $L^1(\Omega)$, and there exists $\eta(\epsilon) > 0$ such that

$$\|h\|_{L^\infty(\Omega)} \int_D M(| \nabla u_n |)dx \leq \frac{\epsilon}{2} \quad \text{and} \quad \|h\|_{L^\infty(\Omega)} M(1)\text{meas}(D) \leq \frac{\epsilon}{2},$$

such that $\text{meas}(D) < \eta(\epsilon)$. Then, by Vitali's theorem we conclude that

$$\Phi_n(x,u_n) \to \Phi(x,u) \quad \text{strongly in} \quad L^1(\Omega).$$

Using again (1.8), Young inequality and Lemma 2.1, we obtain

$$\int_D |\Phi_n(x,u_n)| \,.\, |u_n| \, dx \leq \lambda \int_D h(x)\overline{M}^{-1} M\left(\frac{|u_n|}{\lambda C_0}\right)\frac{|u_n|}{\lambda} dx$$
$$\leq 2\lambda \|h\|_{L^\infty(\Omega)} \int_D M(|\nabla u_n|) dx.$$

and in the same way, we show that

$$\Phi_n(x,u_n)u_n \to \Phi(x,u)u \quad \text{strongly in} \quad L^1(\Omega).$$

So that $\Phi(x,u)u \in L^1(\Omega)$.

Step 4: Passage to the limit.
The next result analyze the behavior of certain subsequences of (φ_n). They will allow us, to pass to the limit in the approximate system (4.1)-(4.4) to show the existence of a capacity solution to the system (1.1)-(1.3).

Lemma 4.4. [10] *Let (u_n, φ_n) be a weak solution to the system (4.1)-(4.4), $u \in E_M(\Omega)$ and $\varphi \in L^\infty(\Omega)$ the limits functions appearing, respectively in (4.7) and (4.13). Then for any function $S \in C_0^1(\mathbb{R})$,*

- *there exists a subsequence, still denoted in the same way, such that*
$$S(u_n)\varphi_n \rightharpoonup S(u)\varphi \quad \textit{weakly in} \quad H^1(\Omega). \tag{4.22}$$
- *Moreover, if $0 \leq S \leq 1$, then there exists a constant $C > 0$, independant of S, such that*
$$\limsup_{n\to\infty} \int_\Omega \kappa_n(u_n) \,|\, \nabla(S(u_n)\varphi_n - S(u)\varphi) \,|^2 \leq C\|S'\|_\infty(1 + \|S'\|_\infty). \tag{4.23}$$
- *There exists a subsequence $(\varphi_{n_k}) \subset (\varphi_n)$ such that*
$$\lim_{n\to\infty} \int_\Omega |\varphi_{n_k} - \varphi| = 0. \tag{4.24}$$

Finally, the condition (R_1) and (R_2) of the Definition 4.1 are fulfilled. In order to obtain the condition (R_3), using (4.17), (4.22) and (4.24), it is enough to make $k \to +\infty$ in the expression

$$S(u_{n_k})\kappa_{n_k}(u_{n_k})\nabla\varphi_{n_k} = \kappa_{n_k}(u_{n_k})[\nabla(S(u_{n_k})\varphi_{n_k}) - \varphi_{n_k}\nabla S(u_{n_k})].$$

This completes the proof of theorem 4.2.

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Yassine Ahakkoud
Laboratory L2MASI, Faculty of Sciences Dhar El Mahraz,
University Sidi Mohamed Ben Abdellah, Fez, Morocco
e-mail: `yassine.ahakkoud@usmba.ac.ma`

Jaouad Bennouna
Laboratory L2MASI, Faculty of Sciences Dhar El Mahraz,
University Sidi Mohamed Ben Abdellah, Fez, Morocco
e-mail: `Jbennouna@hotmail.com`

Mhamed El Massoudi
Laboratory L2MASI, Faculty of Sciences Dhar El Mahraz,
University Sidi Mohamed Ben Abdellah, Fez, Morocco
e-mail: `elmassoudi09@gmail.com`