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# On the stabilization of a thermoelastic laminated beam system with microtemperature effects

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**Abstract.** The present article investigates a one dimensional thermoelastic laminated beam with microtemperature effects. Using the energy method we prove in the case of zero thermal conductivity that the unique dissipation due to the microtemperatures is strong enough to exponentially stabilize the system if and only if the wave speeds of the system are equal. Our result is new and improves previous results in the literature.

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**Keywords:** Laminated beam, microtemperatures, exponential stability, energy method.

## 1. Introduction

In this paper, we address the following thermoelastic laminated beams with microtemperature effects

$$\begin{cases} \rho\varphi_{tt} + G\left(\psi - \varphi_x\right)_x = 0, \\ I_\rho\left(3s - \psi\right)_{tt} - D\left(3s - \psi\right)_{xx} - G\left(\psi - \varphi_x\right) = 0, \\ 3I_\rho s_{tt} - 3Ds_{xx} + 3G\left(\psi - \varphi_x\right) + 4\gamma s - \delta\theta + m\omega_x = 0, \\ c\theta_t + \kappa_1\omega_x + \delta s_t = 0, \\ \alpha\omega_t - \kappa_2\omega_{xx} + \kappa_3\omega + \kappa_1\theta_x + ms_{tx} = 0, \end{cases} \tag{1.1}$$

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for $(x,t) \in (0,1) \times \mathbb{R}_+$, system (1.1) is complemented with the following boundary conditions

$$\begin{aligned} &\varphi_x(0,t) = \psi(0,t) = s(0,t) = \theta(0,t) = \omega_x(0,t) = 0, && t > 0, \\ &\varphi_x(1,t) = \psi(1,t) = s(1,t) = \theta(1,t) = \omega_x(1,t) = 0, && t > 0, \end{aligned} \tag{1.2}$$

and the initial data

$$\begin{aligned} &\varphi(x,0) = \varphi_0(x), \quad \psi(x,0) = \psi_0(x), \quad s(x,0) = s_0(x), && x \in (0,1), \\ &\varphi_t(x,0) = \varphi_1(x), \quad \psi_t(x,0) = \psi_1(x), \quad s_t(x,0) = s_1(x), && x \in (0,1), \\ &\theta(x,0) = \theta_0(x), \quad \omega(x,0) = \omega_0(x), && x \in (0,1), \end{aligned} \tag{1.3}$$

where the functions $\varphi(x,t)$ is the transversal displacement of the beam, $\psi$ is the volume fraction difference, $(3s(x,t) - \psi(x,t))$ is the effective rotation angle, $\theta$ is the relative temperature and $\omega$ is the microtemperature difference and the coefficients, $\rho$, $I_\rho$, $D$, $G$, and $\gamma$ are positive constant coefficients represent the density, the shear stiffness, the mass moment of inertia, the flexural rigidity, and the adhesive damping weight. And the coefficients $\gamma$, $\kappa_1$, $\kappa_2$, $\kappa_3$, $c$, $m$ and $\alpha$ are positive constants represent the physical parameters describing the coupling between the various constituents of the materials.

The initial data $(\varphi_0, \varphi_1, \psi_0, \psi_1, s_0, s_1, \theta_0, \omega_0)$ are assumed to belong to a suitable functional space.

The laminated beam model describes a vibrating structure of an interfacial slip. It consists of two layered beams of uniform thickness which are attached by an adhesive layer of small thickness in such a way that small amount of slip is possible while they are continuously in contact with each other. And with the increasing demand of advanced performance, the vibration suppression of the laminated beams has been one of the main research topics in smart materials and structures, and these composite laminates usually have superior structural properties such as adaptability.

The laminated beam problem was first introduced by Hansen and Spies in [14]. In that paper, the authors derived the mathematical model for two-layered beams with structural damping due to the interfacial slip, namely

$$\begin{cases} \rho\varphi_{tt} + G(\psi - \varphi_x)_x = 0, \\ I_\rho(3s - \psi)_{tt} - D(3s - \psi)_{xx} - G(\psi - \varphi_x) = 0, \\ 3I_\rho s_{tt} - 3Ds_{xx} + 3G(\psi - \varphi_x) + 4\gamma s + 4\alpha s_t = 0. \end{cases} \tag{1.4}$$

In recent years, researchers have focused on the study of the well-posedness and asymptotic stability properties of (1.4). With additional dampings on the first two equations or some sort of boundary damping mechanism, the authors [4, 5, 20, 21, 22, 27, 28, 32] showed that system (1.4) can be stabilized exponentially.

Regarding thermoelastic laminated-beam models, Apalara [2] analyzed a laminated beam system with thermal effect in the slip instead of the frictional damping

$(4\alpha s_t)$. More precisely, he studied the following laminated beam system

$$\begin{cases} \rho\varphi_{tt} + G(\psi - \varphi_x)_x = 0, \\ I_\rho(3s - \psi)_{tt} - D(3s - \psi)_{xx} - G(\psi - \varphi_x) = 0, \\ 3I_\rho s_{tt} - 3Ds_{xx} + 3G(\psi - \omega_x) + 4\gamma s + \delta\theta_x = 0, \\ c\theta_t - k\theta_{xx} + \delta s_{tx} = 0, \end{cases}$$

and he came to the conclusion that an exponential stability result is achievable in the case of equal wave speeds, that is,

$$\frac{\rho}{G} = \frac{I_\rho}{D}.$$

We refer the reader to [1, 3, 7, 6, 10, 8, 9, 11, 13, 17, 18, 16, 25, 26, 23, 29] and the references cited therein for some other results.

In the matter of microtemperature effects, we bring up the study of Djeradi et al. [12] where they examined the joint of microtemperature, nonlinear structure damping, along with nonlinear time-varying delay term, and time- varying coefficient on a thermoelastic laminated beam. They examined the system

$$\begin{cases} \rho\varphi_{tt} + G\left(\psi - \varphi_x\right)_x = 0, \\ I_\rho\left(3s - \psi\right)_{tt} - D\left(3s - \psi\right)_{xx} - G\left(\psi - \varphi_x\right) = 0, \\ 3I_\rho s_{tt} - 3Ds_{xx} + 3G\left(\psi - \varphi_x\right) + 4\gamma s + \delta\theta_x + m\omega_x \\ \qquad\qquad + \beta\mathfrak{b}(t)\mathfrak{h}_1(s_t(x,t)) + \mu\mathfrak{b}(t)\mathfrak{h}_2(s_t(x, t - \varsigma(t))) = 0, \\ c\theta_t - \kappa_0\theta_{xx} + \kappa_1\omega_x + \delta s_{tx} = 0, \\ \alpha\omega_t - \kappa_2\omega_{xx} + \kappa_3\omega + \kappa_1\theta_x + ms_{tx} = 0, \end{cases}$$

and established a general decay result in the case of equal wave speeds and particular assumptions related to nonlinear terms.

The coupled system we've described involves several physical phenomena, including thermoelasticity, laminated beams, and microtemprature effects. For example, a laminated beam consists of multiple layers of different materials bonded together, thermoelasticity refers to the combined behavior of thermal and elastic properties of the materials, and microtemperature refers to the consideration of temperature variations at a very small scale, which can influence the overall behavior of the coupled system.

Taking the above observations into account, we consider the one-dimensional thermoelastic laminated beam problem with microtemperature effects and without thermal conductivity (1.1)-(1.3), and we establish that the dissipation due solely to microtemperature is adequate to stabilize the system exponentially in the case of equal wave speeds. i,e.

$$\chi = \frac{\rho}{G} - \frac{I_\rho}{D} = 0. \tag{1.5}$$

Concerning the stability of some thermoelastic systems with microtemperature effects and without thermal conductivity, we refer the reader to [15, 24, 31].

In order to be able to use Poincaré's inequality for $\varphi$ and $\omega$, we perform the following transformation. From the first equation in (1.1) and boundary conditions, it follows that

$$\frac{d^2}{dt^2}\int_0^1 \varphi(x,t)dx = 0, \quad \forall t \geq 0,$$

and therefore

$$\int_0^1 \varphi(x,t)dx = t\int_0^1 \varphi_1(x,t)dx + \int_0^1 \varphi_0(x,t)dx, \quad \forall t \geq 0.$$

Consequently, if we set

$$\overline{\varphi}(x,t) = \varphi(x,t) - t\int_0^1 \varphi_1(x)dx - \int_0^1 \varphi_0(x)dx, \quad t \geq 0,$$

we get

$$\int_0^1 \overline{\varphi}(x,t)dx = 0, \quad \forall t \geq 0.$$

Now, from the fifth equation of (1.1) and the boundry conditions, we get

$$\frac{d}{dt}\int_0^1 \omega(x,t)dx + \frac{\kappa_3}{\alpha}\int_0^1 \omega(x,t)dx = 0, \quad \forall t \geq 0,$$

thus

$$\int_0^1 \omega(x,t)dx = \left(\int_0^1 \omega_0(x)dx\right) e^{-\frac{\kappa_3}{\alpha}t},$$

so, if we put

$$\overline{\omega}(x,t) = \omega(x,t) - \left(\int_0^1 \omega_0(x)dx\right) e^{-\frac{\kappa_3}{\alpha}t}, \quad t \geq 0,$$

we obtain

$$\int_0^1 \overline{\omega}(x,t)dx = 0, \forall t \geq 0.$$

Clearly, the use of Poincaré's inequality for $\overline{\varphi}$ and $\overline{\omega}$ is justified, and $(\overline{\varphi}, \psi, s, \theta, \overline{\omega})$ satisfies the same equations in (1.1)-(1.3). Subsequently, we work with $\overline{\varphi}$ and $\overline{\omega}$ instead of $\varphi$ and $\omega$ but write $\varphi, \omega$ for simplicity of notation.

For completeness we present a short discussion of the well-posedness and the semigroup formulation of (1.1)-(1.3). For this purpose, we denote by $\xi$ the effective rotation angle, that is, $\xi = 3s - \psi$. Then, system (1.1)-(1.3) is equivalent to

$$
\begin{cases}
\rho\varphi_{tt} + G\left(3s - \xi - \varphi_x\right)_x = 0, \\
I_\rho \xi_{tt} - D\xi_{xx} - G\left(3s - \xi - \varphi_x\right) = 0, \\
3I_\rho s_{tt} - 3Ds_{xx} + 3G\left(3s - \xi - \varphi_x\right) + 4\gamma s - \delta\theta + m\omega_x = 0, \\
c\theta_t + \kappa_1\omega_x + \delta s_t = 0, \\
\alpha\omega_t - \kappa_2\omega_{xx} + \kappa_3\omega + \kappa_1\theta_x + ms_{tx} = 0, \\
\varphi_x(0,t) = \xi(0,t) = s(0,t) = \theta(0,t) = \omega_x(0,t) = 0, \\
\varphi_x(1,t) = \xi(1,t) = s(1,t) = \theta(1,t) = \omega_x(1,t) = 0, \\
\varphi(x,0) = \varphi_0(x),\ \xi(x,0) = \xi_0(x),\ s(x,0) = s_0(x), \\
\varphi_t(x,0) = \varphi_1(x),\ s_t(x,0) = s_1(x),\ \xi_t(x,0) = \xi_1(x), \\
\theta(x,0) = \theta_0(x),\ \omega(x,0) = \omega_0(x).
\end{cases}
\tag{1.6}
$$

Clearly, by introducing the vector function $U = \left(\varphi, \phi, \xi, u, s, v, \theta, \omega\right)^T$, where $\phi = \varphi_t$, $u = \xi_t$, and $v = s_t$, system (1.6) can be written as

$$
\begin{cases}
\frac{d}{dt}U(t) = \mathcal{A}U(t), \qquad t > 0, \\
U(0) = U_0 = \left(\varphi_0, \varphi_1, \xi_0, \xi_1, s_0, s_1, \theta_0, \omega_0\right)^T,
\end{cases}
\tag{1.7}
$$

where $\mathcal{A}$ is a differential operator defined by

$$
\mathcal{A}U = \begin{pmatrix}
\phi \\
-\frac{G}{\rho}\left(3s - \xi - \varphi_x\right)_x \\
u \\
\frac{1}{I_\rho}\Big(D\xi_{xx} + G\left(3s - \xi - \varphi_x\right)\Big) \\
v \\
\frac{1}{3I_\rho}\Big(3Ds_{xx} - 3G\left(3s - \xi - \varphi_x\right) - 4\gamma s + \delta\theta - m\omega_x\Big) \\
-\frac{1}{c}\Big(\kappa_1\omega_x + \delta v\Big) \\
\frac{1}{\alpha}\Big(\kappa_2\omega_{xx} - \kappa_3\omega - \kappa_1\theta_x - mv_x\Big)
\end{pmatrix}.
$$

We consider the following spaces

$$
\begin{aligned}
&L^2_*(0,1) = \left\{\Psi \in L^2(0,1) :\ \int_0^1 \Psi(x)\,dx = 0\right\}, \\
&H^1_*(0,1) = H^1(0,1) \cap L^2_*(0,1), \\
&H^2_*(0,1) = \left\{\Psi \in H^2(0,1) :\ \Psi_x(0) = \Psi_x(1) = 0\right\}.
\end{aligned}
$$

The energy space

$$
\begin{aligned}
\mathcal{H} = H^1_*(0,1) \times L^2_*(0,1) \times H^1_0(0,1) \times L^2(0,1) \times H^1_0(0,1) \times L^2(0,1) \\
\times L^2(0,1) \times L^2_*(0,1)
\end{aligned}
$$

is a Hilbert space with respect to the inner product

$$\begin{aligned} \langle U, \tilde{U} \rangle_{\mathcal{H}} = & \rho \int_0^1 \phi\tilde{\phi} dx + I_\rho \int_0^1 u\tilde{u} dx + 3I_\rho \int_0^1 v\tilde{v} dx + c \int_0^1 \theta\tilde{\theta} dx \\ & + \alpha \int_0^1 \omega\tilde{\omega} dx + G \int_0^1 (3s - \xi - \varphi_x) \left(3\tilde{s} - \tilde{\xi} - \tilde{\varphi}_x\right) dx \\ & + D \int_0^1 \xi_x \tilde{\xi}_x dx + 4\gamma \int_0^1 s\tilde{s} dx + 3D \int_0^1 s_x \tilde{s}_x dx, \end{aligned} \tag{1.8}$$

for $U = (\varphi, \phi, \xi, u, s, v, \theta, \omega)^T \in \mathcal{H}$ and $\tilde{U} = \left(\tilde{\varphi}, \tilde{\phi}, \tilde{\xi}, \tilde{u}, \tilde{s}, \tilde{v}, \tilde{\theta}, \tilde{\omega}\right)^T \in \mathcal{H}$.

The domain of $\mathcal{A}$ is then

$$\mathcal{D}(\mathcal{A}) = \left\{ U \in \mathcal{H} \,\middle|\, \begin{array}{ll} \varphi, \omega \in H_*^2(0,1) \cap H_*^1(0,1); & \xi, s \in H^2(0,1) \cap H_0^1(0,1); \\ \phi \in H_*^1(0,1); & u, v, \theta \in H_0^1(0,1) \end{array} \right\}.$$

Using the standard semigroup method (see, for instance [19, 30]), one easily establishes the following well-posedness result:

**Theorem 1.1.** *Let $U_0 \in \mathcal{H}$, then there exists a unique solution $U \in C(\mathbb{R}^+, \mathbb{H})$ of problem* (1.1)-(1.3)*. Moreover, if $U_0 \in \mathcal{D}(\mathcal{A})$. Then $U \in C(\mathbb{R}^+, \mathcal{D}(\mathcal{A})) \cap C^1(\mathbb{R}^+, \mathcal{H})$.*

This paper is organized as follows. In section 2, we state and prove some technical lemmas needed in the proof of our main results. In section 3, we show that the system is exponentially stable under condition (1.5). In what follows, we use $c_1$ to denote a generic positive constant.

## 2. Technical lemmas

This section is devoted to the statements and proofs of some technical lemmas needed for the proof of our stability result.

**Lemma 2.1.** *Let $(\varphi, \psi, s, \theta, \omega)$ be the solution of* (1.1)-(1.3)*, then the energy functional defined by*

$$\begin{aligned} E(t) = \frac{1}{2} \int_0^1 \Big[ & \rho\varphi_t^2 + I_\rho (3s_t - \psi_t)^2 + 3I_\rho s_t^2 + D (3s_x - \psi_x)^2 \\ & + 3Ds_x^2 + 4\gamma s^2 + G (\psi - \varphi_x)^2 + c\theta^2 + \alpha\omega^2 \Big] dx, \ \forall t \geq 0, \end{aligned} \tag{2.1}$$

*satisfies, along a strong solution of* (1.1)-(1.3),

$$E'(t) = -\kappa_2 \int_0^1 \omega_x^2 dx - \kappa_3 \int_0^1 \omega^2 dx \leq 0, \qquad \forall t \geq 0. \tag{2.2}$$

*Proof.* Equation (2.2) follows by multiplying the five equations of system (1.1) by $\varphi_t, (3s_t - \psi_t), s_t, \theta$ and $\omega$ respectively, integrating by parts over $(0,1)$, boundary conditions (1.2) and summing up. $\square$

**Lemma 2.2.** *The functional $F_1(t)$ defined by*

$$F_1(t) = \frac{3\alpha I_\rho}{m}\int_0^1 s_t\left(\int_0^x \omega(y)dy\right)dx + \frac{3\kappa_1 I_\rho}{m}\int_0^1 s\theta dx + \frac{3\kappa_1 I_\rho \delta}{2mc}\int_0^1 s^2 dx, \tag{2.3}$$

*satisfies, for any $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$, the estimate*

$$\begin{aligned} F_1'(t) \leq & - I_\rho \int_0^1 s_t^2 dx + \varepsilon_1 \int_0^1 s_x^2 dx + \varepsilon_2 \int_0^1 (\psi - \varphi_x)^2 dx \\ & + \varepsilon_3 \int_0^1 \theta^2 dx + c_1\left(1 + \frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2} + \frac{1}{\varepsilon_3}\right)\int_0^1 \omega^2 dx \\ & + c_1\left(1 + \frac{1}{\varepsilon_1}\right)\int_0^1 \omega_x^2 dx. \end{aligned} \tag{2.4}$$

*Proof.* By taking the derivative of $F_1$, using (1.1), integrating by parts and the fact that $\int_0^1 \omega(x)\,dx = 0$, we get,

$$\begin{aligned} F_1'(t) = & - \frac{3\alpha D}{m}\int_0^1 s_x \omega dx - \frac{3\alpha G}{m}\int_0^1 (\psi - \varphi_x)\int_0^x \omega(y)dydx \\ & - \frac{4\alpha\gamma}{m}\int_0^1 s\int_0^x \omega(y)dydx + \frac{\alpha\delta}{m}\int_0^1 \theta\int_0^x \omega(y)dydx \\ & + \alpha\int_0^1 \omega^2 dx + \frac{3I_\rho\kappa_2}{m}\int_0^1 s_t\omega_x dx - 3I_\rho\int_0^1 s_t^2 dx \\ & - \frac{3I_\rho\kappa_3}{m}\int_0^1 s_t\int_0^x \omega(y)dydx - \frac{3I_\rho\kappa_1^2}{mc}\int_0^1 s\omega_x dx. \end{aligned} \tag{2.5}$$

Using Young's, Poincaré's and Cauchy-Schwarz inequalities, we have, for any $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$

$$-\frac{3\alpha D}{m}\int_0^1 s_x\omega dx \leq \frac{\varepsilon_1}{4}\int_0^1 s_x^2 dx + \frac{c_1}{\varepsilon_1}\int_0^1 \omega^2 dx, \tag{2.6}$$

$$\begin{aligned} & -\frac{3\alpha G}{m}\int_0^1 (\psi - \varphi_x)\int_0^x \omega(y)dydx \\ & \qquad \leq \varepsilon_2 \int_0^1 (\psi - \varphi_x)^2 dx + \frac{c_1}{\varepsilon_2}\int_0^1 \left(\int_0^x \omega(y)dy\right)^2 dx \\ & \qquad \leq \varepsilon_2 \int_0^1 (\psi - \varphi_x)^2 dx + \frac{c_1}{\varepsilon_2}\int_0^1 \omega^2 dx, \end{aligned} \tag{2.7}$$

similarly,

$$-\frac{4\alpha\gamma}{m}\int_0^1 s\int_0^x \omega(y)dydx \leq \frac{\varepsilon_1}{4}\int_0^1 s_x^2 dx + \frac{c_1}{\varepsilon_1}\int_0^1 \omega^2 dx, \tag{2.8}$$

$$\frac{\alpha\delta}{m}\int_0^1 \theta \int_0^x \omega(y)dydx \leq \varepsilon_3 \int_0^1 \theta^2 dx + \frac{c_1}{\varepsilon_3}\int_0^1 \omega^2 dx, \tag{2.9}$$

$$\frac{3I_\rho\kappa_2}{m}\int_0^1 s_t\omega_x dx \leq I_\rho \int_0^1 s_t^2 dx + c_1 \int_0^1 \omega_x^2 dx, \tag{2.10}$$

$$-\frac{3I_\rho\kappa_3}{m}\int_0^1 s_t \int_0^x \omega(y)dydx \leq I_\rho \int_0^1 s_t^2 dx + c_1 \int_0^1 \omega^2 dx, \tag{2.11}$$

$$-\frac{3I_\rho\kappa_1^2}{mc}\int_0^1 s\omega_x dx \leq \frac{\varepsilon_1}{2}\int_0^1 s_x^2 dx + \frac{c_1}{\varepsilon_1}\int_0^1 \omega_x^2 dx. \tag{2.12}$$

Estimate (2.4) follows by substituting (2.6)(2.12) into (2.5). □

**Lemma 2.3.** *The functional $F_2(t)$ defined by*

$$F_2(t) = \frac{\alpha c}{\kappa_1}\int_0^1 \theta \left(\int_0^x \omega(y)dy\right) dx, \tag{2.13}$$

*satisfies, the following estimate*

$$F_2'(t) \leq -\frac{c}{2}\int_0^1 \theta^2 dx + c_1 \int_0^1 s_t^2 dx + c_1 \int_0^1 \omega^2 dx + c_1 \int_0^1 \omega_x^2 dx. \tag{2.14}$$

*Proof.* Direct computations, using (1.1), integrating by parts and the fact that $\int_0^1 \omega(x)\, dx = 0$, yield

$$\begin{aligned} F_2'(t) = &-c\int_0^1 \theta^2 dx + \alpha \int_0^1 \omega^2 dx - \frac{\alpha\delta}{\kappa_1}\int_0^1 s_t \int_0^x \omega(y)dydx \\ &+ \frac{c\kappa_2}{\kappa_1}\int_0^1 \theta\omega_x dx - \frac{c\kappa_3}{\kappa_1}\int_0^1 \theta \int_0^x \omega(y)dydx - \frac{mc}{\kappa_1}\int_0^1 \theta s_t dx. \end{aligned} \tag{2.15}$$

By virtue of Young's and Cauchy-Schwarz inequalities, we find

$$-\frac{\alpha\delta}{\kappa_1}\int_0^1 s_t \int_0^x \omega(y)dydx \leq c_1 \int_0^1 s_t^2 dx + c_1 \int_0^1 \omega^2 dx, \tag{2.16}$$

$$\frac{c\kappa_2}{\kappa_1}\int_0^1 \theta\omega_x dx \leq \frac{c}{8}\int_0^1 \theta^2 dx + c_1 \int_0^1 \omega_x^2 dx, \tag{2.17}$$

$$-\frac{c\kappa_3}{\kappa_1}\int_0^1 \theta \int_0^x \omega(y)dydx \leq \frac{c}{8}\int_0^1 \theta^2 dx + c_1 \int_0^1 \omega^2 dx, \tag{2.18}$$

$$-\frac{mc}{\kappa_1}\int_0^1 \theta s_t dx \leq \frac{c}{4}\int_0^1 \theta^2 dx + c_1 \int_0^1 s_t^2 dx, \tag{2.19}$$

which yields the desired result (2.14), by inserting (2.16)(2.19) into (2.15). □

**Lemma 2.4.** *The functional $F_3(t)$ defined by*

$$F_3(t) = -\frac{D\rho}{G}\int_0^1 \varphi_t s_x dx + I_\rho \int_0^1 s_t\left(\psi - \varphi_x\right)dx, \tag{2.20}$$

*satisfies, for any $\varepsilon_4 > 0$, the estimate*

$$\begin{aligned} F_3'(t) \leq & -\frac{G}{2}\int_0^1 \left(\psi - \varphi_x\right)^2 dx + \varepsilon_4 \int_0^1 \left(3s_t - \psi_t\right)^2 dx \\ & + c_1\left(1 + \frac{1}{\varepsilon_4}\right)\int_0^1 s_t^2 dx + c_1 \int_0^1 s^2 dx + c_1 \int_0^1 \theta^2 dx \\ & + c_1 \int_0^1 \omega_x^2 dx + D\chi \int_0^1 \varphi_{tx} s_t dx. \end{aligned} \tag{2.21}$$

*Proof.* Differentiating $F_3$, using (1.1) and intgrating by parts, we obtain

$$\begin{aligned} F_3'(t) = & -\frac{D\rho}{G}\int_0^1 \varphi_t s_{tx} dx - G\int_0^1 \left(\psi - \varphi_x\right)^2 dx - \frac{4}{3}\gamma \int_0^1 s\left(\psi - \varphi_x\right)dx \\ & + \frac{\delta}{3}\int_0^1 \theta\left(\psi - \varphi_x\right)dx - \frac{m}{3}\int_0^1 \omega_x\left(\psi - \varphi_x\right)dx \\ & + I_\rho \int_0^1 s_t \psi_t dx - I_\rho \int_0^1 s_t \varphi_{tx} dx. \end{aligned}$$

Using the simple equality $\psi_t = -\left(3s_t - \psi_t\right) + 3s_t$, we arrive at

$$\begin{aligned} F_3'(t) = & -G\int_0^1 \left(\psi - \varphi_x\right)^2 dx - \frac{4}{3}\gamma \int_0^1 s\left(\psi - \varphi_x\right)dx + 3I_\rho \int_0^1 s_t^2 dx \\ & + \frac{\delta}{3}\int_0^1 \theta\left(\psi - \varphi_x\right)dx - \frac{m}{3}\int_0^1 \omega_x\left(\psi - \varphi_x\right)dx \\ & - I_\rho \int_0^1 s_t\left(3s_t - \psi_t\right)dx + D\chi \int_0^1 \varphi_{tx} s_t dx. \end{aligned} \tag{2.22}$$

Applying Young's and Poincaré's inequalities, for $\varepsilon_4 > 0$, we get

$$-\frac{4}{3}\gamma \int_0^1 s\left(\psi - \varphi_x\right)dx \leq \frac{G}{8}\int_0^1 \left(\psi - \varphi_x\right)^2 dx + c_1 \int_0^1 s^2 dx, \tag{2.23}$$

$$\frac{\delta}{3}\int_0^1 \theta\left(\psi - \varphi_x\right)dx \leq \frac{G}{8}\int_0^1 \left(\psi - \varphi_x\right)^2 dx + c_1 \int_0^1 \theta^2 dx, \tag{2.24}$$

$$-\frac{m}{3}\int_0^1 \omega_x\left(\psi - \varphi_x\right)dx \leq \frac{G}{4}\int_0^1 \left(\psi - \varphi_x\right)^2 dx + c_1 \int_0^1 \omega_x^2 dx, \tag{2.25}$$

$$-I_\rho \int_0^1 s_t\left(3s_t - \psi_t\right)dx \leq \varepsilon_4 \int_0^1 \left(3s_t - \psi_t\right)^2 dx + \frac{c_1}{\varepsilon_4}\int_0^1 s_t^2 dx. \tag{2.26}$$

By substituting (2.23)-(2.26) into (2.22), we obtain (2.21). □

**Lemma 2.5.** *The functional $F_4(t)$ defined by*

$$F_4(t) = -\frac{D\rho}{G}\int_0^1 \varphi_t\left(3s_x - \psi_x\right)dx + I_\rho \int_0^1 \left(\psi - \varphi_x\right)\left(3s_t - \psi_t\right)dx, \tag{2.27}$$

*satisfies, the estimate*

$$\begin{aligned} F_4'(t) \leq &- \frac{I_\rho}{2}\int_0^1 \left(3s_t - \psi_t\right)^2 dx + G\int_0^1 \left(\psi - \varphi_x\right)^2 dx \\ &+ c_1 \int_0^1 s_t^2 dx + D\chi \int_0^1 \left(3s_t - \psi_t\right)\varphi_{tx} dx. \end{aligned} \tag{2.28}$$

*Proof.* Direct differentiation of $F_4$, using (1.1) and then integrating by parts, gives

$$\begin{aligned} F_4'(t) = &- \frac{D\rho}{G}\int_0^1 \varphi_t\left(3s_x - \psi_x\right)_t dx + G\int_0^1 \left(\psi - \varphi_x\right)^2 dx \\ &+ I_\rho \int_0^1 \left(3s_t - \psi_t\right)\psi_t dx - I_\rho \int_0^1 \left(3s_t - \psi_t\right)\varphi_{tx} dx. \end{aligned}$$

By using the equality $\psi_t = -\left(3s_t - \psi_t\right) + 3s_t$, we obtain

$$\begin{aligned} F_4'(t) = &- I_\rho \int_0^1 \left(3s_t - \psi_t\right)^2 dx + 3I_\rho \int_0^1 s_t\left(3s_t - \psi_t\right)dx \\ &+ G\int_0^1 \left(\psi - \varphi_x\right)^2 dx + D\chi \int_0^1 \left(3s_t - \psi_t\right)\varphi_{tx} dx. \end{aligned}$$

Estimate (2.28) follows thanks Youngs inequality. □

**Lemma 2.6.** *The functional $F_5(t)$ defined by*

$$F_5(t) = -\rho\int_0^1 \left(\int_0^x \varphi_t(y)dy\right) s dx + I_\rho \int_0^1 s_t s dx, \tag{2.29}$$

*satisfies, for $\varepsilon_5 > 0$, the estimate*

$$\begin{aligned} F_5'(t) \leq &- \frac{D}{2}\int_0^1 s_x^2 dx - \gamma\int_0^1 s^2 dx + \varepsilon_5 \int_0^1 \varphi_t^2 dx + c_1\int_0^1 \theta^2 dx \\ &+ c_1 \int_0^1 \omega_x^2 dx + c_1\left(1 + \frac{1}{\varepsilon_5}\right)\int_0^1 s_t^2 dx. \end{aligned} \tag{2.30}$$

*Proof.* The derivative of $F_5$, using (1.1), integration by parts and the boundary conditions, give

$$\begin{aligned} F_5'(t) = &- D\int_0^1 s_x^2 dx - \frac{4}{3}\gamma\int_0^1 s^2 dx + \frac{\delta}{3}\int_0^1 \theta s dx - \frac{m}{3}\int_0^1 \omega_x s dx \\ &+ I_\rho \int_0^1 s_t^2 dx - \rho\int_0^1 s_t\left(\int_0^x \varphi_t(y)dy\right)dx. \end{aligned} \tag{2.31}$$

By using Young's, Poincaré's and Cauchy-Schwarz inequalities, for $\varepsilon_5 > 0$, we have

$$\frac{\delta}{3}\int_0^1 \theta s dx \leq \frac{\gamma}{3}\int_0^1 s^2 dx + c_1\int_0^1 \theta^2 dx, \tag{2.32}$$

$$-\frac{m}{3}\int_0^1 \omega_x s dx \leq \frac{D}{2}\int_0^1 s_x^2 dx + c_1\int_0^1 \omega_x^2 dx, \tag{2.33}$$

$$-\rho\int_0^1 s_t\left(\int_0^x \varphi_t(y)dy\right)dx \leq \varepsilon_5\int_0^1 \varphi_t^2 dx + \frac{c_1}{\varepsilon_5}\int_0^1 s_t^2 dx. \tag{2.34}$$

Relation (2.30) follows by substituting (2.32)-(2.34) into (2.31). □

**Lemma 2.7.** *The functional $F_6$ defined by*

$$F_6(t) = -\rho\int_0^1 \varphi_t\varphi dx, \tag{2.35}$$

*satisfies, the estimate*

$$F_6'(t) \leq -\rho\int_0^1 \varphi_t^2 dx + \frac{D}{4}\int_0^1 (3s_x - \psi_x)^2 dx$$
$$+ c_1\int_0^1 s_x^2 dx + c_1\int_0^1 (\psi - \varphi_x)^2 dx. \tag{2.36}$$

*Proof.* Direct differentiation of $F_6$, using (1.1) and then integrating by parts, gives

$$F_6'(t) = -G\int_0^1 \varphi_x(\psi - \varphi_x)dx - \rho\int_0^1 \varphi_t^2 dx.$$

Using the simple relation $\varphi_x = -(\psi - \varphi_x) - (3s - \psi) + 3s$, we get

$$F_6'(t) = G\int_0^1 (\psi - \varphi_x)^2 dx + G\int_0^1 (\psi - \varphi_x)(3s - \psi)dx$$
$$- 3G\int_0^1 (\psi - \varphi_x)s\,dx - \rho\int_0^1 \varphi_t^2 dx.$$

Using Young's and Poincaré's inequalities, lead to the desired estimation. □

**Lemma 2.8.** *The functional $F_7$ defined by*

$$F_7(t) = I_\rho\int_0^1 (3s - \psi)(3s_t - \psi_t)dx, \tag{2.37}$$

*satisfies, the estimate*

$$F_7'(t) \leq -\frac{D}{2}\int_0^1 (3s_x - \psi_x)^2 dx + I_\rho\int_0^1 (3s_t - \psi_t)^2 dx$$
$$+ c_1\int_0^1 (\psi - \varphi_x)^2 dx. \tag{2.38}$$

*Proof.* A simple differentiation of $F_7$, using (1.1) together with integration by parts, yield

$$F_7'(t) = I_\rho\int_0^1 (3s_t - \psi_t)^2 dx - D\int_0^1 (3s_x - \psi_x)^2 dx$$
$$+ G\int_0^1 (\psi - \varphi_x)(3s - \psi)dx.$$

The use of Young's and Poincaré's inequalities lead to (2.38). □

## 3. Stability result

In this section, we prove under the condition of equal wave-speed propagation (1.5) that the energy associated with (1.1)(1.3) is exponentially stable. To achieve this goal, we define a Lyapunov functional $\mathcal{L}$ and show that it is equivalent to the energy functional $E$.

**Lemma 3.1.** *Let* $(\varphi, \psi, s, \theta, \omega)$ *be the solution of* (1.1)-(1.3) *and assume* $\chi = 0$. *Then, for* $N$, $N_1, N_2, N_3, N_4, N_5 > 0$ *to be chosen appropriately later, the functional defined by*

$$\begin{aligned} \mathcal{L}(t) = NE(t) + N_1F_1(t) + N_2F_2(t) + N_3F_3(t) + N_4F_4(t) \\ + N_5F_5(t) + F_6(t) + F_7(t), \end{aligned} \tag{3.1}$$

*satisfies, for* $N$ *sufficiently large,*

$$\tau_1\, E(t) \leq \mathcal{L}(t) \leq \tau_2\, E(t), \qquad \forall t \geq 0, \tag{3.2}$$

*and the estimate*

$$\mathcal{L}'(t) \leq -\tau_3 E(t), \tag{3.3}$$

*where* $\tau_1, \tau_2$ *and* $\tau_3$ *are positive constants.*

*Proof.* From (3.1) and the Lemmas in Section 2, it follows that

$$\begin{aligned} \Big|\mathcal{L}(t) - NE(t)\Big| \leq & \frac{3\alpha I_\rho}{m} N_1 \int_0^1 \Big| s_t \int_0^x \omega(y)dy \Big| dx + \frac{3\kappa_1 I_\rho}{m} N_1 \int_0^1 \Big| s\theta \Big| dx \\ & + \frac{3\kappa_1 I_\rho \delta}{2mc} N_1 \int_0^1 s^2 dx + \frac{\alpha c}{\kappa_1} N_2 \int_0^1 \Big| \theta \int_0^x \omega(y)dy \Big| dx \\ & + \frac{D\rho}{G} N_3 \int_0^1 \Big| \varphi_t s_x \Big| dx + I_\rho N_3 \int_0^1 \Big| s_t\left(\psi - \varphi_x\right) \Big| dx \\ & + \frac{D\rho}{G} N_4 \int_0^1 \Big| \varphi_t\left(3s_x - \psi_x\right) \Big| dx \\ & + I_\rho N_4 \int_0^1 \Big| \left(\psi - \varphi_x\right)\left(3s_t - \psi_t\right) \Big| dx \\ & + \rho N_5 \int_0^1 \Big| s \int_0^x \varphi_t(y)dy \Big| dx + I_\rho N_5 \int_0^1 \Big| s_t s \Big| dx \\ & + \rho \int_0^1 \Big| \varphi_t \varphi \Big| dx + I_\rho \int_0^1 \Big| \left(3s_t - \psi_t\right)\left(3s - \psi\right) \Big| dx. \end{aligned}$$

Exploiting Young's, Cauchy-Schwarz and Poincaré's inequalities, we get

$$\Big|\mathcal{L}(t) - NE(t)\Big| \leq c_1 \int_0^1 \Big[\varphi_t^2 + (3s_t - \psi_t)^2 + s_t^2 + (3s_x - \psi_x)^2 \\ + s_x^2 + s^2 + (\psi - \varphi_x)^2 + \theta^2 + \omega^2\Big] dx.$$

Consequently, we have

$$\Big|\mathcal{L}(t) - NE(t)\Big| \leq c_1 E(t),$$

that is,

$$(N - c_1)\, E(t) \leq \mathcal{L}(t) \leq (N + c_1)\, E(t).$$

By choosing $N$ large enough, (3.2) follows. Next, to prove (3.3), we take the derivative of $\mathcal{L}(t)$, use (2.2), (2.4), (2.14), (2.21), (2.28), (2.30), (2.36), (2.38), and set

$$\varepsilon_1 = \frac{DN_5}{4N_1}, \quad \varepsilon_2 = \frac{GN_3}{4N_1}, \quad \varepsilon_3 = \frac{cN_2}{4N_1}, \quad \varepsilon_4 = \frac{I_\rho N_4}{4N_3}, \quad \varepsilon_5 = \frac{\rho}{2N_5}.$$

So, we arrive at

$$\begin{aligned} \mathcal{L}'(t) \leq &- \frac{\rho}{2}\int_0^1 \varphi_t^2 dx - \left[\frac{I_\rho}{4}N_4 - I_\rho\right]\int_0^1 (3s_t - \psi_t)^2\, dx - \frac{D}{4}\int_0^1 (3s_x - \psi_x)^2\, dx \\ &- \left[I_\rho N_1 - c_1 N_2 - c_1 N_3\left(1 + \frac{N_3}{N_4}\right) - c_1 N_4 - c_1 N_5\,(1 + N_5)\right]\int_0^1 s_t^2 dx \\ &- \left[\frac{G}{4}N_3 - GN_4 - c_1\right]\int_0^1 (\psi - \varphi_x)^2\, dx - \left[\frac{D}{4}N_5 - c_1\right]\int_0^1 s_x^2 dx \\ &- [\gamma N_5 - c_1 N_3]\int_0^1 s^2 dx - \left[\frac{c}{4}N_2 - c_1 N_3 - c_1 N_5\right]\int_0^1 \theta^2 dx \\ &- \left[N\kappa_2 - c_1 N_1\left(1 + \frac{N_1}{N_5}\right) - c_1 N_2 - c_1 N_3 - c_1 N_5\right]\int_0^1 \omega_x^2 dx \\ &- \left[N\kappa_3 - c_1 N_1\left(1 + \frac{N_1}{N_5} + \frac{N_1}{N_3} + \frac{N_1}{N_2}\right) - c_1 N_2\right]\int_0^1 \omega^2 dx. \end{aligned}$$

At this point, we choose the constants carefully. First, let us take $N_4 > 4$. We then choose $N_3$ large enough such that

$$\frac{G}{4}N_3 - GN_4 - c_1 > 0.$$

After that, we select $N_5$ large enough so that

$$\gamma N_5 - c_1 N_3 > 0 \qquad \text{and} \qquad \frac{D}{4}N_5 - c_1 > 0.$$

Next, we choose $N_2$ large enough such that

$$\frac{c}{4}N_2 - c_1 N_3 - c_1 N_5 > 0.$$

Then, we pick $N_1$ so large that

$$I_\rho N_1 - c_1 N_2 - c_1 N_3 \left(1 + \frac{N_3}{N_4}\right) - c_1 N_4 - c_1 N_5 (1 + N_5) > 0.$$

Finallay, we choose $N$ very large enough (even larger so that (3.2) remains valid) such that

$$N\kappa_2 - c_1 N_1 \left(1 + \frac{N_1}{N_5}\right) - c_1 N_2 - c_1 N_3 - c_1 N_5 > 0,$$

and

$$N\kappa_3 - c_1 N_1 \left(1 + \frac{N_1}{N_5} + \frac{N_1}{N_3} + \frac{N_1}{N_2}\right) - c_1 N_2 > 0.$$

Therefore, we arrive at

$$\mathcal{L}'(t) \leq -\tau_4 \int_0^1 \Big[\varphi_t^2 + (3s_t - \psi_t)^2 + s_t^2 + (3s_x - \psi_x)^2 \\ + s_x^2 + s^2 + (\psi - \varphi_x)^2 + \theta^2 + \omega_x^2 + \omega^2\Big] dx, \quad \tau_4 > 0.$$

We finally use Poincaré's inequality to substitute $-\int_0^1 \omega_x^2 dx$ by $-\int_0^1 \omega^2 dx$ and, hence, (3.3) is established. □

We are now ready to state and prove the following exponential stability result.

**Theorem 3.2.** *Let* $(\varphi, \psi, s, \theta, \omega)$ *be the solution of* (1.1)-(1.3) *and assume* (1.5). *Then, there exist two positive constants* $\lambda_1, \lambda_2$ *such that the energy functional satisfies*

$$E(t) \leq \lambda_1 e^{-\lambda_2 t}, \qquad \forall t \geq 0. \tag{3.4}$$

*Proof.* The combination of (3.2) and (3.3) gives

$$\mathcal{L}'(t) \leq -\lambda_2 \mathcal{L}(t), \qquad t \geq 0, \tag{3.5}$$

where $\lambda_2 = \frac{\tau_3}{\tau_2}$. A simple integration of (3.5) over $(0, t)$ yields

$$\mathcal{L}(t) \leq \mathcal{L}(0) e^{-\lambda_2 t}, \qquad t \geq 0.$$

which yields the desired result (3.4) by using the other side of the equivalence relation (3.2) again. □

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