Stud. Univ. Babeş-Bolyai Math. 70(2025), No. 2, 233–249
DOI: 10.24193/subbmath.2025.2.05

Existence results for a coupled system of higher-order nonlinear differential equations with integral-multipoint boundary conditions

Bashir Ahmad, Sotiris K. Ntouyas,
Ahmed Alsaedi and Ammar B. Wali

Abstract. In this paper, we establish the existence and uniqueness criteria for solutions of an integral-multipoint coupled boundary value problem involving a system of nonlinear higher-order ordinary differential equations. We apply the Leray-Schauder's alternative to prove an existence result for the given problem, while the uniqueness of its solutions is accomplished with the aid of Banach's fixed point theorem. Examples are constructed for illustrating the obtained results.

Mathematics Subject Classification (2010): 34A34, 34B10, 34B15.

Keywords: Ordinary differential equations, system, integral-multipoint boundary conditions, nonlocal, fixed point.

1. Introduction

The topic of boundary value problems is an important area of investigation in view of extensive occurrence of such problems in several diverse disciplines. Examples include conservation laws [8], nano boundary layer fluid flow [4], magnetohydrodynamic flow [18], magneto Maxwell nano-material [19], fluid flow problems [28], cellular systems and aging models [1], etc.

Much of the literature on boundary value problems includes classical boundary conditions. However, these conditions cannot model the physical and chemical processes taking place within the given domain. In order to cope with this situation, the concept of nonlocal conditions representing the changes happening at some interior

Received 13 October 2024; Accepted 10 January 2025.

points or sub-segments of the given domain was introduced. One can find the details and applications of nonlocal boundary conditions in the articles [14, 11, 22, 16, 24, 15] and the references cited therein.

Integral boundary conditions serve as an effective tool in the mathematical modeling of the problems arising in the flow and drag phenomena in arteries [27], heat conduction [9, 20, 10], biomedical CFD [23], etc. In fact, these conditions provide a practical approach to fluid flow problems with arbitrary shaped blood vessels, for instance, see [25]. For the boundary value problems involving integral boundary conditions, for instance, see the papers [26, 21, 3, 7, 2, 6, 5, 13].

In [3], the authors obtained some existence results for nth-order ordinary differential equations and inclusions supplemented with nonlocal multi-point integral boundary conditions:

$$\begin{cases} u^{(n)}(t) = f(t, u(t)),\ u^{(n)}(t) \in F(t, u(t))), t \in [0, 1], \\ u(0) = \delta \displaystyle\int_0^{\xi} u(s)ds,\ u'(0) = 0,\ u''(0) = 0, \dots, u^{(n-2)}(0) = 0, \\ \alpha u(1) + \beta u'(1) = \displaystyle\sum_{i=1}^{m} \gamma_i \int_0^{\beta_i} u(s)ds,\ 0 < \xi < \beta_1 < \beta_2 < \dots < \beta_m < 1, \end{cases}$$

where $f : [0, 1] \times \mathbb{R} \longrightarrow \mathbb{R}$ is a given continuous function, $F : [0, 1] \times \mathbb{R} \longrightarrow \mathcal{P}(\mathbb{R})$, $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$, and $\alpha,\ \beta,\ \gamma_i,\ \delta,\ \xi,\ \beta_i\ (i = 1, 2, \dots, m)$ are appropriately chosen real constants.

In this paper, motivated by [3], we formulate and investigate a boundary value problem for a coupled system of higher-order nonlinear differential equations complemented with coupled integral-multipoint boundary conditions given by

$$\begin{cases} u^{(n)}(t) = f(t, u, v), \qquad v^{(m)}(t) = g(t, u, v), \qquad t \in [0, 1], \\ u(0) = \delta_1 \displaystyle\int_0^{\xi} v(s)\, ds, \qquad u'(0) = 0, \quad u''(0) = 0, \dots, \quad u^{(n-2)}(0) = 0, \\ v(0) = \delta_2 \displaystyle\int_0^{\xi} u(s)\, ds, \qquad v'(0) = 0, \quad v''(0) = 0, \dots, \quad v^{(m-2)}(0) = 0, \\ \epsilon_1 u(1) + \zeta_1 u'(1) = \displaystyle\sum_{i=1}^{p} \gamma_i \int_0^{\beta_i} v(s)ds + \sum_{j=1}^{q} \omega_j v(\eta_j), \\ \epsilon_2 v(1) + \zeta_2 v'(1) = \displaystyle\sum_{i=1}^{p} \widehat{\gamma_i} \int_0^{\beta_i} u(s)ds + \sum_{j=1}^{q} \widehat{\omega_j} u(\eta_j), \end{cases} \tag{1.1}$$

where $0 < \xi < \beta_1 < \beta_2 < \dots < \beta_p < \eta_1 < \eta_2 < \dots < \eta_q < 1$, $\delta_1, \delta_2, \epsilon_1, \epsilon_2, \zeta_1, \zeta_2, \gamma_i, \widehat{\gamma_i}, \omega_j, \widehat{\omega_j} \in \mathbb{R}$, $i = 1, 2, \dots, p$, $j = 1, 2, \dots, q$ and $f, g : [0, 1] \times \mathbb{R}^2 \to \mathbb{R}$ are given functions.

The objective of the present work is to develop the existence theory for the problem (1.1) by applying the standard fixed point theorems. The outcome of the proposed work will be a useful contribution to the existing literature on nonlinear differential systems supplemented with coupled nonlocal integral boundary conditions.

The rest of the paper is arranged as follows. In Section 2, we prove an auxiliary lemma related to the linear variant of the problem (1.1). The main results for the given problem are proved in Section 3. Section 4 contains examples illustrating the main results.

2. An auxiliary lemma

In the following lemma, we solve a linear variant of the system (1.1) and use it to convert the problem (1.1) into a fixed point problem.

Lemma 2.1. *Let* $(J_1K_2 - J_2K_1) \neq 0$, $(1-\delta_1\delta_2\xi^2) \neq 0$ *and* $y_1, y_2 \in C([0,1],\mathbb{R})$. *Then, the linear boundary value problem*

$$\begin{cases} u^{(n)}(t) = y_1(t), \qquad v^{(m)}(t) = y_2(t), \quad t \in [0,1], \\ u(0) = \delta_1 \int_0^{\xi} v(s)\,ds, \qquad u'(0) = 0, \quad u''(0) = 0, \ldots, \quad u^{(n-2)}(0) = 0, \\ v(0) = \delta_2 \int_0^{\xi} u(s)\,ds, \qquad v'(0) = 0, \quad v''(0) = 0, \ldots, \quad v^{(m-2)}(0) = 0, \\ \epsilon_1 u(1) + \zeta_1 u'(1) = \sum_{i=1}^{p} \gamma_i \int_0^{\beta_i} v(s)ds + \sum_{j=1}^{q} \omega_j v(\eta_j), \\ \epsilon_2 v(1) + \zeta_2 v'(1) = \sum_{i=1}^{p} \widehat{\gamma_i} \int_0^{\beta_i} u(s)ds + \sum_{j=1}^{q} \widehat{\omega_j} u(\eta_j), \end{cases} \tag{2.1}$$

is equivalent to a pair of integral equations

$$\begin{aligned} u(t) \;=\; & \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} y_1(s)ds + N_1(t) \int_0^{\xi} \frac{(\xi-s)^m}{m!} y_2(s)ds \\ & + N_2(t) \int_0^{\xi} \frac{(\xi-s)^n}{n!} y_1(s)ds \\ & + N_3(t)\Big[\sum_{i=1}^{p} \gamma_i \int_0^{\beta_i} \frac{(\beta_i - s)^m}{m!} y_2(s)ds + \sum_{j=1}^{q} \omega_j \int_0^{\eta_j} \frac{(\eta_j - s)^{m-1}}{(m-1)!} y_2(s)ds \\ & - \int_0^1 \frac{(1-s)^{n-2}}{(n-1)!} [\epsilon_1(1-s) + \zeta_1(n-1)] y_1(s)ds \Big] \\ & + N_4(t)\Big[\sum_{i=1}^{p} \widehat{\gamma_i} \int_0^{\beta_i} \frac{(\beta_i - s)^n}{n!} y_1(s)ds + \sum_{j=1}^{q} \widehat{\omega_j} \int_0^{\eta_j} \frac{(\eta_j - s)^{n-1}}{(n-1)!} y_1(s)ds \\ & - \int_0^1 \frac{(1-s)^{m-2}}{(m-1)!} [\epsilon_2(1-s) + \zeta_2(m-1)] y_2(s)ds \Big], \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} v(t) \;=\;& \int_0^t \frac{(t-s)^{m-1}}{(m-1)!}y_2(s)ds + N_5(t)\int_0^\xi \frac{(\xi-s)^m}{m!}y_2(s)ds \\ &+N_6(t)\int_0^\xi \frac{(\xi-s)^n}{n!}y_1(s)ds \\ &+N_7(t)\Big[\sum_{i=1}^p \gamma_i \int_0^{\beta_i}\frac{(\beta_i-s)^m}{m!}y_2(s)ds + \sum_{j=1}^q \omega_j\int_0^{\eta_j}\frac{(\eta_j-s)^{m-1}}{(m-1)!}y_2(s)ds \\ &-\int_0^1 \frac{(1-s)^{n-2}}{(n-1)!}[\epsilon_1(1-s)+\zeta_1(n-1)]y_1(s)ds\Big] \\ &+N_8(t)\Big[\sum_{i=1}^p \widehat{\gamma_i} \int_0^{\beta_i}\frac{(\beta_i-s)^n}{n!}y_1(s)ds + \sum_{j=1}^q \widehat{\omega_j}\int_0^{\eta_j}\frac{(\eta_j-s)^{n-1}}{(n-1)!}y_1(s)ds \\ &-\int_0^1 \frac{(1-s)^{m-2}}{(m-1)!}[\epsilon_2(1-s)+\zeta_2(m-1)]y_2(s)ds\Big], \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} N_1(t) &= \Delta_1+\Delta_5 t^{n-1}, \quad N_2(t)=\Delta_2+\Delta_6 t^{n-1}, \quad N_3(t)=\Delta_3+\Delta_7 t^{n-1}, \\ N_4(t) &= \Delta_4+\Delta_8 t^{n-1}, \quad N_5(t)=\Delta_9+\Delta_{13} t^{m-1}, \quad N_6(t)=\Delta_{10}+\Delta_{14} t^{m-1}, \\ N_7(t) &= \Delta_{11}+\Delta_{15} t^{m-1}, \quad N_8(t)=\Delta_{12}+\Delta_{16} t^{m-1}, \end{aligned}$$

$$\Delta_1 = \rho_1 + \frac{\rho_4(n\delta_1\xi^m K_1 + m\delta_1\delta_2\xi^{n+1}K_2) - \rho_6(n\delta_1\xi^m J_1 + m\delta_1\delta_2\xi^{n+1}J_2)}{M(1-\delta_1\delta_2\xi^2)mn},$$

$$\Delta_2 = \rho_2 + \frac{\rho_5(n\delta_1\xi^m K_1 + m\delta_1\delta_2\xi^{n+1}K_2) - \rho_7(n\delta_1\xi^m J_1 + m\delta_1\delta_2\xi^{n+1}J_2)}{M(1-\delta_1\delta_2\xi^2)n},$$

$$\Delta_3 = \frac{(n\delta_1\xi^m K_1 + m\delta_1\delta_2\xi^{n+1}K_2)}{M(1-\delta_1\delta_2\xi^2)mn}, \quad \Delta_4 = \frac{(n\delta_1\xi^m J_1 + m\delta_1\delta_2\xi^{n+1}J_2)}{M(1-\delta_1\delta_2\xi^2)mn},$$

$$\Delta_5 = \frac{(\rho_4K_2-\rho_6J_2)}{M}, \quad \Delta_6 = \frac{(\rho_5K_2-\rho_7J_2)}{M}, \quad \Delta_7=\frac{K_2}{M}, \quad \Delta_8=\frac{J_2}{M},$$

$$\Delta_9 = \rho_3 + \frac{\rho_4(n\delta_1\delta_2\xi^{m+1} K_1 + m\delta_1\xi^{n}K_2) - \rho_6(n\delta_1\delta_2\xi^{m+1} J_1 + m\delta_1\xi^{n}J_2)}{M(1-\delta_1\delta_2\xi^2)mn},$$

$$\Delta_{10} = \rho_1 + \frac{\rho_5(n\delta_1\delta_2\xi^{m+1} K_1 + m\delta_1\xi^{n}K_2) - \rho_7(n\delta_1\delta_2\xi^{m+1} J_1 + m\delta_1\xi^{n}J_2)}{M(1-\delta_1\delta_2\xi^2)mn},$$

$$\Delta_{11} = \frac{(n\delta_1\delta_2\xi^{m+1}K_1 + m\delta_1\xi^n K_2)}{M(1-\delta_1\delta_2\xi^2)mn}, \quad \Delta_{12} = \frac{(n\delta_1\delta_2\xi^{m+1}J_1 + m\delta_1\xi^n J_2)}{M(1-\delta_1\delta_2\xi^2)mn},$$

$$\Delta_{13} = \frac{(\rho_4 K_1 - \rho_6 J_1)}{M}, \quad \Delta_{14} = \frac{(\rho_5 K_1 - \rho_7 J_1)}{M}, \quad \Delta_{15} = \frac{K_1}{M}, \quad \Delta_{16} = \frac{J_1}{M},$$

$$\rho_1 = \frac{1}{1-\delta_1\delta_2\xi^2}, \ \rho_2 = \frac{A_2}{1-\delta_1\delta_2\xi^2}, \ \rho_3 = \frac{B_2}{1-\delta_1\delta_2\xi^2}, \ \rho_4 = \frac{B_2D_1 - C_1}{1-\delta_1\delta_2\xi^2},$$

$$\rho_5 = \frac{D_1 - A_2C_1}{1-\delta_1\delta_2\xi^2}, \rho_6 = \frac{B_2E_1 - F_1}{1-\delta_1\delta_2\xi^2}, \ \rho_7 = \frac{E_1 - A_2F_1}{1-\delta_1\delta_2\xi^2},$$

$$J_1 = \frac{\epsilon_1\delta_1\delta_2\xi^{n+1} - \delta_2\xi^n D_1}{n(1-\delta_1\delta_2\xi^2)} + \epsilon_1 + \zeta_1(n+1),$$

$$J_2 = \frac{\delta_1\xi^m(\delta_2\xi D_1 - \epsilon_1)}{m(1-\delta_1\delta_2\xi^2)} + D_2, \ K_1 = \frac{\delta_1\delta_2\xi^{n+1}F_1 - \delta_2\xi^{n+1}\epsilon_2}{n(1-\delta_1\delta_2\xi^2)} + F_2,$$

$$K_2 = \frac{\delta_1\xi^m(\delta_2\xi\epsilon_2 - F_1)}{m(1-\delta_1\delta_2\xi^2)} + D_2, \ M = J_1K_2 - J_2K_1,$$

$$D_1 = \sum_{i=1}^{p}\gamma_i\beta_i + \sum_{j=1}^{q}\omega_j\eta_j, D_2 = \sum_{i=1}^{p}\gamma_i\frac{\beta_i^m}{m} + \sum_{j=1}^{q}\omega_j\eta_j^{m-1},$$

$$F_1 = \sum_{i=1}^{p}\widehat{\gamma_i}\beta_i + \sum_{j=1}^{q}\widehat{\omega_j}, \ F_2 = \sum_{i=1}^{p}\widehat{\gamma_i}\frac{\beta_i^n}{n} + \sum_{j=1}^{q}\widehat{\omega_j}\eta_j^{n-1}. \tag{2.4}$$

Proof. Solving the system of ordinary differential equations in (2.1), we get

$$\begin{cases} u(t) = \int_0^t \frac{(t-s)^{n-1}}{(n-1)!}y_1(s)ds + c_0 + c_1t + \ldots + c_{n-1}t^{n-1}, \\ v(t) = \int_0^t \frac{(t-s)^{m-1}}{(m-1)!}y_2(s)ds + b_0 + b_1t + \ldots + b_{m-1}t^{m-1}, \end{cases} \tag{2.5}$$

where $c_i, b_i \in \mathbb{R}$, $i = 0, 1, \ldots, n-1, m-1$, are arbitrary constants. Making use of the conditions $u'(0) = 0$, $u''(0) = 0, \ldots, u^{(n-2)}(0) = 0$ and $v'(0) = 0$, $v''(0) = 0, \ldots,$ $v^{(m-2)}(0) = 0$ in (2.5), we get $c_1 = c_2 = \ldots, c_{n-2} = 0, b_0 = b_1 = \ldots, b_{m-1} = 0$. In consequence, (2.5) takes the form

$$\begin{cases} u(t) = \int_0^t \frac{(t-s)^{n-1}}{(n-1)!}y_1(s)ds + c_0 + c_{n-1}t^{n-1}, \\ v(t) = \int_0^t \frac{(t-s)^{m-1}}{(m-1)!}y_2(s)ds + b_0 + b_{m-1}t^{m-1}. \end{cases} \tag{2.6}$$

Using (2.6) in the conditions $u(0) = \delta_1 \int\limits_0^\xi v(s)ds$ and $v(0) = \delta_2 \int\limits_0^\xi u(s)ds$, we get

$$c_0 = \delta_1 \int_0^\xi \frac{(\xi - r)^m}{m!}y_2(r)dr + \delta_1 b_0\xi + \delta_1 b_{m-1}\frac{\xi^m}{m}, \tag{2.7}$$

and

$$b_0 = \delta_2 \int_0^{\xi} \frac{(\xi - r)^n}{n!} y_1(r)dr + c_0\delta_2\xi + c_{n-1}\delta_2\frac{\xi^n}{n}. \tag{2.8}$$

Now, inserting (2.6) in the conditions:

$$\epsilon_1 u(1) + \zeta_1 u'(1) = \sum_{i=1}^{p} \gamma_i \int_0^{\beta_i} v(s)ds + \sum_{j=1}^{q} \omega_j v(\eta_j),$$

$$\epsilon_2 v(1) + \zeta_2 v'(1) = \sum_{i=1}^{p} \widehat{\gamma_i} \int_0^{\beta_i} u(s)ds + \sum_{j=1}^{q} \widehat{\omega_j} u(\eta_j),$$

we obtain

$$\begin{aligned} &c_0\epsilon_1 + c_{n-1}[\epsilon_1 + \zeta_1(n-1)] + \int_0^1 \frac{(1-s)^{n-2}[\epsilon_1(1-s) + \zeta_1(n-1)]}{(n-1)!} y_1(s)ds \\ &= b_0\left[\sum_{i=1}^{p} \gamma_i\beta_i + \sum_{j=1}^{q} \omega_j\eta_j\right] + b_{m-1}\left[\sum_{i=1}^{p} \gamma_i \frac{\beta_i^m}{m} + \sum_{j=1}^{q} \omega_j \eta_j^{m-1}\right] \\ &\quad + \sum_{i=1}^{p} \gamma_i \int_0^{\beta_i} \frac{(\beta_i - s)^m}{m!} y_2(s)ds + \sum_{j=1}^{q} \omega_j \int_0^{\eta_j} \frac{(\eta_j - s)^{m-1}}{(m-1)!} y_2(s)ds, \end{aligned} \tag{2.9}$$

$$\begin{aligned} &b_0\epsilon_2 + b_{m-1}[\epsilon_2 + \zeta_2(m-1)] + \int_0^1 \frac{(1-s)^{m-2}[\epsilon_2(1-s) + \zeta_2(m-1)]}{(m-1)!} y_2(s)ds \\ &= c_0\left[\sum_{i=1}^{p} \widehat{\gamma_i}\beta_i + \sum_{j=1}^{q} \widehat{\omega_j}\right] + c_{n-1}\left[\sum_{i=1}^{p} \widehat{\gamma_i} \frac{\beta_i^n}{n} + \sum_{j=1}^{q} \widehat{\omega_j} \eta_j^{n-1}\right] \\ &\quad + \sum_{i=1}^{p} \widehat{\gamma_i} \int_0^{\beta_i} \frac{(\beta_i - s)^n}{n!} y_1(s)ds + \sum_{j=1}^{q} \widehat{\omega_j} \int_0^{\eta_j} \frac{(\eta_j - s)^{n-1}}{(n-1)!} y_1(s)ds. \end{aligned} \tag{2.10}$$

We can express equations (2.7)-(2.10) in the form

$$\begin{cases} c_0 - A_2 b_0 - A_3 b_{m-1} = A_1, \\ -B_2 c_0 + b_0 - B_3 c_{n-1} = B_1, \\ C_1 c_0 - D_1 b_0 + C_2 c_{n-1} - D_2 b_{m-1} = D_3 - C_3, \\ -F_1 c_0 + E_1 b_0 - F_2 c_{n-1} + E_2 b_{m-1} = F_3 - E_3, \end{cases} \tag{2.11}$$

where D_1, D_2, F_1 and F_2 are given in (2.4) and

$$A_1 = \delta_1\Big[\int_0^{\xi} \frac{(\xi - r)^m}{m!} y_2(r)dr\Big], \qquad A_2 = \delta_1\xi, \qquad A_3 = \delta_1\frac{\xi^m}{m},$$

$$B_1 = \delta_2\Big[\int_0^{\xi} \frac{(\xi - r)^n}{n!} y_1(r)dr\Big], \qquad B_2 = \delta_2\xi, \qquad B_3 = \delta_2\frac{\xi^n}{n},$$

$$
\begin{aligned}
C_1 &= \epsilon_1, \qquad C_2 = \epsilon_1 + \zeta_1(n-1), \\
C_3 &= \int_0^1 \frac{(1-s)^{n-2}[\epsilon_1(1-s)+\zeta_1(n-1)]}{(n-1)!} y_1(s)ds, \\
D_3 &= \sum_{i=1}^{p} \gamma_i \int_0^{\beta_i} \frac{(\beta_i - s)^m}{m!} y_2(s)ds + \sum_{j=1}^{q} \omega_j \int_0^{\eta_j} \frac{(\eta_j - s)^{m-1}}{(m-1)!} y_2(s)ds, \\
E_1 &= \epsilon_2, \qquad E_2 = \epsilon_2 + \zeta_2(n-1), \\
E_3 &= \int_0^1 \frac{(1-s)^{m-2}[\epsilon_2(1-s)+\zeta_2(m-1)]}{(m-1)!} y_2(s)ds, \\
F_3 &= \sum_{i=1}^{p} \widehat{\gamma_i} \int_0^{\beta_i} \frac{(\beta_i - s)^n}{n!} y_1(s)ds + \sum_{j=1}^{q} \widehat{\omega_j} \int_0^{\eta_j} \frac{(\eta_j - s)^{n-1}}{(n-1)!} y_1(s)ds.
\end{aligned} \tag{2.12}
$$

Solving the first two equations in (2.11) for c_0 and b_0 in term of c_{n-1} and $b_{m-1}$and using the notation in (2.12), we obtain

$$
\begin{cases}
c_0 = G_1 + G_2 b_{m-1} + G_3 c_{n-1}, \\
\\
b_0 = H_1 + H_2 b_{m-1} + H_3 c_{n-1},
\end{cases} \tag{2.13}
$$

where

$$
\begin{aligned}
G_1 &= \frac{A_1 + A_2 B_1}{r_1}, \; G_2 = \frac{A_3}{r_1}, \; G_3 = \frac{A_2 B_3}{r_1}, \; H_1 = \frac{A_1 B_2 + B_1}{r_1}, \\
H_2 &= \frac{A_3 B_2}{r_1}, \qquad H_3 = \frac{B_3}{r_1}, \qquad r_1 = 1 - \delta_1 \delta_2 \xi^2.
\end{aligned} \tag{2.14}
$$

Substituting the values of c_0 and b_0 from (2.13) in the last two equations of (2.11), we get

$$
\begin{cases}
c_{n-1} J_1 - b_{m-1} J_2 = J_3, \\
c_{n-1} K_1 - b_{m-1} K_2 = K_3,
\end{cases} \tag{2.15}
$$

where J_1, J_2, K_1, K_2 are given in (2.4) and

$$
\begin{aligned}
J_3 &= \frac{A_1(B_2 D_1 - C_1) + B_1(D_1 - A_2 C_1)}{r_1} + D_3 - C_3, \\
K_3 &= \frac{A_1(B_2 E_1 - F_1) + B_1(E_1 - A_2 F_1)}{r_1} + E_3 - F_3.
\end{aligned} \tag{2.16}
$$

Solving the system (2.15) for b_{m-1} and c_{n-1}, we find that

$$
\begin{cases}
c_{n-1} = \dfrac{J_3 K_2 - J_2 K_3}{J_1 K_2 - J_2 K_1}, \\
\\
b_{m-1} = \dfrac{J_3 K_1 - J_1 K_3}{J_1 K_2 - J_2 K_1}.
\end{cases} \tag{2.17}
$$

Inserting (2.17) in (2.13), we obtain

$$\begin{cases} c_0 = \dfrac{G_1(J_1K_2 - J_2K_1) + G_2(J_3K_1 - J_1K_3) + G_3(J_3K_2 - J_2K_3)}{J_1K_2 - J_2K_1}, \\ b_0 = \dfrac{H_1(J_1K_2 - J_2K_1) + H_2(J_3K_1 - J_1K_3) + H_3(J_3K_2 - J_2K_3)}{J_1K_2 - J_2K_1}. \end{cases} \tag{2.18}$$

Substituting the above values of c_{n-1}, b_{m-1}, c_0 and b_0 into (2.6) together with the notation (2.4), we obtain the solution (2.2) and (2.3). One can obtain the converse of the lemma by direct computation. □

In the sequel, we set

$$\begin{aligned} &\bar{N}_i = \max_{t\in[0,1]} |N_i(t)|,\ i = 1,2,\dots,8, \\ &\sigma_1 = \frac{\xi^{m+1}}{(m+1)!},\ \sigma_2 = \frac{\xi^{n+1}}{(n+1)!}, \\ &\sigma_3 = \sum_{i=1}^{p} |\gamma_i| \frac{\beta_i^{m+1}}{(m+1)!} + \sum_{i=1}^{p} |\omega_j| \frac{\eta_j^m}{m!}, \quad \sigma_4 = \left(\frac{|\epsilon_1|}{n!} + \frac{|\zeta_1|}{(n-1)!}\right), \\ &\sigma_5 = \sum_{i=1}^{p} |\widehat{\gamma}_i| \frac{\beta_i^{n+1}}{(n+1)!} + \sum_{i=1}^{p} |\widehat{\omega}_j| \frac{\eta_j^n}{n!}, \quad \sigma_6 = \left(\frac{|\epsilon_2|}{m!} + \frac{|\zeta_2|}{(m-1)!}\right), \end{aligned} \tag{2.19}$$

where $N_i(t), i = 1,2,\dots,8$, are given in (2.4).

3. Main results

In the forthcoming analysis, we need the assumptions:

(H_1) There exist real constants $\widehat{m}_i, \widehat{n}_i \geq 0, i = 1,2$ and $\widehat{m}_0 > 0, \widehat{n}_0 > 0$ such that $\forall u, v \in \mathbb{R}$,

$$|f(t,u,v)| \leq \widehat{m}_0 + \widehat{m}_1|u| + \widehat{m}_2|v|, \quad |g(t,u,v)| \leq \widehat{n}_0 + \widehat{n}_1|u| + \widehat{n}_2|v|;$$

(H_2) There exist positive constants ℓ_1 and ℓ_2 such that, $\forall t \in [0,1]$ and $u_i, v_i \in \mathbb{R}$, $i = 1,2$,

$$|f(t,u_1,v_1) - f(t,u_2,v_2)| \leq \ell_1(|u_1 - u_2| + |v_1 - v_2|),$$
$$|g(t,u_1,v_1) - g(t,u_2,v_2)| \leq \ell_2(|u_1 - u_2| + |v_1 - v_2|).$$

For the sake of convenience in the mathematical computations, we set

$$\begin{aligned} &Q_0 = \min\{1 - (Q_1\widehat{m}_1 + Q_2\widehat{n}_1), 1 - (Q_1\widehat{m}_2 + Q_2\widehat{n}_2)\}, \\ &Q_1 = q_1 + q_2,\ Q_2 = \bar{q_1} + \bar{q_2}, \\ &q_1 = \tfrac{1}{n!} + \bar{N}_2\sigma_2 + \bar{N}_3\sigma_4 + \bar{N}_4\sigma_5,\ \bar{q_1} = \bar{N}_1\sigma_1 + \bar{N}_3\sigma_3 + \bar{N}_4\sigma_6, \\ &q_2 = \bar{N}_6\sigma_2 + \bar{N}_7\sigma_4 + \bar{N}_8\sigma_5,\ \bar{q_2} = \frac{1}{m!} + \bar{N}_5\sigma_1 + \bar{N}_7\sigma_3 + \bar{N}_8\sigma_6. \end{aligned} \tag{3.1}$$

Let $\mathcal{X} = \{u(t) \mid u(t) \in C([a,b])\}$ be the space equipped with norm

$$\|u\| = \sup\{|u(t)|,\ t \in [a,b]\}.$$

Then, $(\mathcal{X}, \|\cdot\|)$ is a Banach space and consequently, the product space $(\mathcal{X}\times\mathcal{X}, \|(u,v)\|)$ is also a Banach space endowed with the norm $\|(u,v)\| = \|u\|+\|v\|$ for $(u,v) \in \mathcal{X}\times\mathcal{X}$. By Lemma 1, we define an operator $\mathcal{T} : \mathcal{X}\times\mathcal{X} \to \mathcal{X}\times\mathcal{X}$ associated with the problem (1.1) as

$$\mathcal{T}(u,v)(t) := (\mathcal{T}_1(u,v)(t), \mathcal{T}_2(u,v)(t)),$$

where

$$\begin{aligned}
&\mathcal{T}_1(u,v)(t)\\
=\ & \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} f(s,u,v)ds + N_1(t)\int_0^\xi \frac{(\xi-s)^m}{m!} g(s,u,v)ds\\
&+N_2(t)\int_0^\xi \frac{(\xi-s)^n}{n!} f(s,u,v)ds + N_3(t)\Big[\sum_{i=1}^p \gamma_i \int_0^{\beta_i} \frac{(\beta_i-s)^m}{m!} g(s,u,v)ds\\
&+\sum_{j=1}^q \omega_j \int_0^{\eta_j} \frac{(\eta_j-s)^{m-1}}{(m-1)!} g(s,u,v)ds\\
&-\int_0^1 \frac{(1-s)^{n-2}}{(n-1)!}(\epsilon_1(1-s)+\zeta_1(n-1))f(s,u,v)ds\Big]\\
&+N_4(t)\Big[\sum_{i=1}^p \widehat{\gamma_i}\int_0^{\beta_i} \frac{(\beta_i-s)^n}{n!} f(s,u,v)ds + \sum_{j=1}^q \widehat{\omega_j}\int_0^{\eta_j} \frac{(\eta_j-s)^{n-1}}{(n-1)!} f(s,u,v)ds\\
&-\int_0^1 \frac{(1-s)^{m-2}}{(m-1)!}(\epsilon_2(1-s)+\zeta_2(m-1))g(s,u,v)ds\Big],
\end{aligned} \tag{3.2}$$

and

$$\begin{aligned}
&\mathcal{T}_2(u,v)(t)\\
=\ & \int_0^t \frac{(t-s)^{m-1}}{(m-1)!} g(s,u,v)ds + N_5(t)\int_0^\xi \frac{(\xi-s)^m}{m!} g(s,u,v)ds\\
&+N_6(t)\int_0^\xi \frac{(\xi-s)^n}{n!} f(s,u,v)ds + N_7(t)\Big[\sum_{i=1}^p \gamma_i \int_0^{\beta_i} \frac{(\beta_i-s)^m}{m!} g(s,u,v)ds\\
&+\sum_{j=1}^q \omega_j \int_0^{\eta_j} \frac{(\eta_j-s)^{m-1}}{(m-1)!} g(s,u,v)ds\\
&-\int_0^1 \frac{(1-s)^{n-2}}{(n-1)!}(\epsilon_1(1-s)+\zeta_1(n-1))f(s,u,v)ds\Big]
\end{aligned}$$

$$+N_8(t)\Big[\sum_{i=1}^{p}\widehat{\gamma_i}\int_0^{\beta_i}\frac{(\beta_i-s)^n}{n!}f(s,u,v)ds+\sum_{j=1}^{q}\widehat{\omega_j}\int_0^{\eta_j}\frac{(\eta_j-s)^{n-1}}{(n-1)!}f(s,u,v)ds$$
$$-\int_0^1\frac{(1-s)^{m-2}}{(m-1)!}(\epsilon_2(1-s)+\zeta_2(m-1))g(s,u,v)ds\Big]. \tag{3.3}$$

3.1. Existence of solutions

In this subsection, we discuss the existence of solutions for the problem (1.1) by using Leray-Schauder's alternative [17], which is stated below.

Lemma 3.1. *Let $T : K \to K$ be a completely continuous operator (that is, a map restricted to any bounded set in K is compact). Let*

$$\psi(T) = \{x \in K : x = \varphi T(x) \text{ for some } 0 < \varphi < 1\}.$$

Then, either the set $\psi(T)$ is unbounded or T has at least one fixed point.

Theorem 3.2. *Let $f, g : [0,1]\times\mathbb{R}^2 \to \mathbb{R}$ be continuous functions. Assume that condition (H_1) holds, and*

$$Q_1\widehat{m}_1 + Q_2\widehat{n}_1 < 1 \qquad Q_1\widehat{m}_2 + Q_2\widehat{n}_2 < 1,$$

where Q_1 and Q_2 are given by (3.1). Then, there exists at least one solution for the problem (1.1) on $[0,1]$.

Proof. First of all, we show that the operator $\mathcal{T} : \mathcal{X}\times\mathcal{X} \to \mathcal{X}\times\mathcal{X}$ is completely continuous. Notice that the operator $\mathcal{T}$ is continuous as the functions f and g are continuous. Let $\Psi = \{(u,v) \in \mathcal{X}\times\mathcal{X} : \|(u,v)\| \le \rho\}$. For any $u, v \in \Psi$ we have

$$\begin{aligned}|f(t,u,v)| &\le \widehat{m}_0+\widehat{m}_1|u|+\widehat{m}_2|v| \le \widehat{m_0}+(\widehat{m_1}+\widehat{m_2})(\|u\|+\|v\|)\\ &\le \widehat{m_0}+(\widehat{m_1}+\widehat{m_2})\rho := \kappa_f,\end{aligned}$$

and similarly

$$|g(t,u,v)| \le \widehat{n_0}+(\widehat{n_1}+\widehat{n_2})\rho := \kappa_g.$$

Then, for any $(u,v) \in B_\rho$, we obtain

$$\begin{aligned}&|\mathcal{T}_1(u,v)(t)|\\ \le\ & \sup_{t\in[0,1]}\Bigg\{\int_0^t\frac{(t-s)^{n-1}}{(n-1)!}|f(s,u,v)|ds+|N_1(t)|\int_0^\xi\frac{(\xi-s)^m}{m!}|g(s,u,v)|ds\\ &+|N_2(t)|\int_0^\xi\frac{(\xi-s)^n}{n!}|f(s,u,v)|ds\\ &+|N_3(t)|\Bigg[\int_0^1\frac{(1-s)^{n-2}}{(n-1)!}[\epsilon_1(1-s)+\zeta_1(n-1)]|f(s,u,v)|ds\\ &+\sum_{i=1}^p\gamma_i\int_0^{\beta_i}\frac{(\beta_i-s)^m}{m!}|g(s,u,v)|ds+\sum_{j=1}^q\omega_j\int_0^{\eta_j}\frac{(\eta_j-s)^{m-1}}{(m-1)}|g(s,u,v)|ds\Bigg]\\ &+|N_4(t)|\Bigg[\int_0^1\frac{(1-s)^{m-2}}{(m-1)!}[\epsilon_2(1-s)+\zeta_2(m-1)]|g(s,u,v)|ds\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{p}\widehat{\gamma_i}\int_0^{\beta_i}\frac{(\beta_i-s)^n}{n!}|f(s,u,v)|ds+\sum_{j=1}^{q}\widehat{\omega_j}\int_0^{\eta_j}\frac{(\eta_j-s)^{n-1}}{(n-1)}|f(s,u,v)|ds\Bigg]\Bigg\}\\
\le\ &\kappa_f\left[\frac{1}{n!}+\bar{N}_2\sigma_2+\bar{N}_3\sigma_4+\bar{N}_4\sigma_5\right]+\kappa_g\left[\bar{N}_1\sigma_1+\bar{N}_3\sigma_3+\bar{N}_4\sigma_6\right]\\
\le\ &\kappa_f q_1+\kappa_g\bar{q_1},
\end{aligned}
$$

which implies that $\|\mathcal{T}_1(u,v)\|\le\kappa_f q_1+\kappa_g\bar{q_1}$, where q_1 and $\bar{q_1}$ are given in (3.1). Similarly, one can obtain that $\|\mathcal{T}_2(u,v)\|\le\kappa_f q_2+\kappa_g\bar{q_2}$, where q_2 and $\bar{q_2}$ are defined in (3.1). From the forgoing inequalities, we get $\|\mathcal{T}(u,v)\|\le\kappa_f Q_1+\kappa_g Q_2$, where Q_1 and Q_2 are given in (3.1), which shows that the operator $\mathcal{T}$ is uniformly bounded. Next, we establish that $\mathcal{T}$ is equicontinuous. For $t_1,t_2\in[0,1]$ with $t_1<t_2$, we have

$$
\begin{aligned}
&|\mathcal{T}_1(u_1,v_1)(t_2)-\mathcal{T}_1(u_2,v_2)(t_1)|\\
\le\ &\kappa_f\left|\int_0^{t_2}\frac{(t_2-s)^{n-1}}{(n-1)!}f(s,u,v)ds-\int_0^{t_1}\frac{(t_1-s)^{n-1}}{(n-1)!}f(s,u,v)ds\right|\\
&+|N_1(t_2)-N_1(t_1)|\int_0^{\xi}\frac{(\xi-s)^m}{m!}|g(s,u,v)|ds\\
&+|N_2(t_2)-N_2(t_1)|\int_0^{\xi}\frac{(\xi-s)^n}{n!}|f(s,u,v)|ds\\
&+|N_3(t_2)-N_3(t_1)|\Bigg[\int_0^1\frac{(1-s)^{n-2}}{(n-1)!}[\epsilon_1(1-s)+\zeta_1(n-1)]|f(s,u,v)|ds\\
&+\sum_{i=1}^{p}\gamma_i\int_0^{\beta_i}\frac{(\beta_i-s)^m}{m!}|g(s,u,v)|ds+\sum_{j=1}^{q}\omega_j\int_0^{\eta_j}\frac{(\eta_j-s)^{m-1}}{(m-1)}|g(s,u,v)|ds\Bigg]\\
&+|N_4(t_2)-N_4(t_1)|\Bigg[\int_0^1\frac{(1-s)^{m-2}}{(m-1)!}[\epsilon_2(1-s)+\zeta_2(m-1)]|g(s,u,v)|ds\\
&+\sum_{i=1}^{p}\widehat{\gamma_i}\int_0^{\beta_i}\frac{(\beta_i-s)^n}{n!}|f(s,u,v)|ds+\sum_{j=1}^{q}\widehat{\omega_j}\int_0^{\eta_j}\frac{(\eta_j-s)^{n-1}}{(n-1)}|f(s,u,v)|ds\Bigg]\Bigg\}\\
\le\ &\frac{\kappa_f}{n!}\left(2(t_2-t_1)^n+|t_2^n-t_1^n|\right)+|N_1(t_2)-N_1(t_1)|\kappa_g\sigma_1+|N_2(t_2)-N_2(t_1)|\kappa_f\sigma_2\\
&+|N_3(t_2)-N_3(t_1)|(\kappa_f\sigma_4+\kappa_g\sigma_3)+|N_4(t_2)-N_4(t_1)|(\kappa_f\sigma_5+\kappa_g\sigma_6),
\end{aligned}
$$

which tends to zero as $(t_2-t_1)\to 0$ independent of $(u,v)\in\Psi$. In a similar manner, it can be shown that $|\mathcal{T}_2(u_1,v_1)(t_2)-\mathcal{T}_2(u_2,v_2)(t_1)|\to 0$ as $(t_2-t_1)\to 0$ independent of $(u,v)\in\Psi$. Thus, the operator $\mathcal{T}$ is equicontinuous.

Finally, it will be verified that the set $\psi=\{(u,v)\in\mathcal{X}\times\mathcal{X}|(u,v)=\varphi\mathcal{T}(u,v),\ 0<\varphi<1\}$ is bounded. Let $(u,v)\in\psi$. Then $(u,v)=\varphi\mathcal{T}(u,v)$ for any $t\in[0,1]$. Therefore, we have $u(t)=\varphi\mathcal{T}_1(u,v)(t)$, $v(t)=\varphi\mathcal{T}_2(u,v)(t)$. In consequence, it follows by the assumption (H_1) that

$$|u(t)|=q_1\widehat{m}_0+\bar{q_1}\widehat{n}_0+(q_1\widehat{m}_1+\bar{q_1}\widehat{n}_1)\|u\|+(q_1\widehat{m}_2+\bar{q_1}\widehat{n}_2)\|v\|, \tag{3.4}$$

and

$$|v(t)|=q_2\widehat{m}_0+\bar{q_2}\widehat{n}_0+(q_2\widehat{m}_1+\bar{q_2}\widehat{n}_1)\|u\|+(q_2\widehat{m}_2+\bar{q_2}\widehat{n}_2)\|v\|, \tag{3.5}$$

where $q_1, q_2.\bar{q}_1$, and $\bar{q}_2$ are given in (3.1). From (3.4) and (3.5), we have

$$\begin{aligned}\|u\|+\|v\| &\leq (q_1+q_2)\widehat{m}_0+(\bar{q_1}+\bar{q_2})\widehat{n}_0+[(q_1+q_2)\widehat{m}_1+(\bar{q_1}+\bar{q_2})\widehat{n}_1]\|u\| \\ &\quad +[(q_1+q_2)\widehat{m}_2+(\bar{q_1}+\bar{q_2})\widehat{n}_2]\|v\|,\end{aligned}$$

which, in view of (3.1), can be written as

$$\|(u,v)\| \leq \frac{Q_1\widehat{m}_0+Q_2\widehat{n}_0}{Q_0}.$$

This shows that the set ψ is bounded. Hence, by Lemma 3.1, the operator $\mathcal{T}$ has at least one fixed point. Therefore, the problem (1.1) has at least one solution on $[0,1]$. This completes the proof. □

3.2. Uniqueness of solutions

Here, we establish the uniqueness of solutions for the problem (1.1) by means of Banach's contractions mapping principle [12].

Theorem 3.3. *Suppose that $f, g : [0,1]\times\mathbb{R}^2 \to \mathbb{R}$ are continuous functions, the assumption (H_2) and the following condition*

$$Q_1\ell_1+Q_2\ell_2<1, \tag{3.6}$$

hold, where Q_1 and Q_2 are given in (3.1). Then, the problem (1.1) has a unique solution on $[0,1]$.

Proof. Firstly, we show that $\mathcal{T}B_r \subset B_r$, where $B_r=\{(u,v)\in\mathcal{X}\times\mathcal{X} : \|(u,v)\|\leq r\}$ is a closed ball with

$$r \geq \frac{Q_1N_1+Q_2N_2}{1-(Q_1\ell_1+Q_2\ell_2)}. \tag{3.7}$$

Let us set $\sup_{t\in[0,1]}|f(t,0,0)|=\mu_1$ and $\sup_{t\in[0,1]}|g(t,0,0)|=\mu_2$. Then, by the assumption (H_2), we have

$$\begin{aligned}|f(s,u(s),v(s))| &= |f(s,u(s),v(s))-f(s,0,0)+f(s,0,0)| \\ &\leq |f(s,u(s),v(s))-f(s,0,0)|+|f(s,0,0)| \\ &\leq \ell_1(\|u\|+\|v\|)+\mu_1 \leq \ell_1\|(u,v)\|+\mu_1 \leq \ell_1 r+\mu_1.\end{aligned}$$

Likewise, one can obtain that

$$|g(s,u(s),v(s))| \leq \ell_2 r+\mu_2.$$

For $(u,v)\in B_r$, we have

$$\begin{aligned}&|\mathcal{T}_1(u,v)(t)| \\ \leq\ & \sup_{t\in[0,1]}\left\{\int_0^t \frac{(t-s)^{n-1}}{(n-1)!}|f(s,u,v)|ds+|N_1(t)|\int_0^\xi \frac{(\xi-s)^m}{m!}|g(s,u,v)|ds\right. \\ &+|N_2(t)|\int_0^\xi \frac{(\xi-s)^n}{n!}|f(s,u,v)|ds \\ &+|N_3(t)|\left[\int_0^1 \frac{(1-s)^{n-2}}{(n-1)!}[\epsilon_1(1-s)+\zeta_1(n-1)]|f(s,u,v)|ds\right.\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{p}\gamma_i\int_0^{\beta_i}\frac{(\beta_i-s)^m}{m!}|g(s,u,v)|ds+\sum_{j=1}^{q}\omega_j\int_0^{\eta_j}\frac{(\eta_j-s)^{m-1}}{(m-1)}|g(s,u,v)|ds\Bigg]\\
&+|N_4(t)|\Bigg[\int_0^1\frac{(1-s)^{m-2}}{(m-1)!}[\epsilon_2(1-s)+\zeta_2(m-1)]|g(s,u,v)|ds\\
&+\sum_{i=1}^{p}\widehat{\gamma_i}\int_0^{\beta_i}\frac{(\beta_i-s)^n}{n!}|f(s,u,v)|ds+\sum_{j=1}^{q}\widehat{\omega_j}\int_0^{\eta_j}\frac{(\eta_j-s)^{n-1}}{(n-1)}|f(s,u,v)|ds\Bigg]\Bigg\}\\
\le\;& [\ell_1 r+\mu_1]\Big[\frac{1}{n!}+\bar{N}_2\sigma_2+\bar{N}_3\sigma_4+\bar{N}_4\sigma_5\Big]+[\ell_2 r+\mu_2]\Big[\bar{N}_1\sigma_1+\bar{N}_3\sigma_3+\bar{N}_4\sigma_6\Big]\\
\le\;& q_1(\ell_1 r+\mu_1)+\bar{q_1}(\ell_2 r+\mu_2),
\end{aligned}
$$

which implies that

$$\|\mathcal{T}_1(u,v))\|\le q_1(\ell_1 r+\mu_1)+\bar{q_1}(\ell_2 r+\mu_2).$$

Similarly, we can get

$$\|\mathcal{T}_2(u,v))\|\le q_2(\ell_1 r+\mu_1)+\bar{q_2}(\ell_2 r+\mu_2).$$

From the above estimates together with (3.7), it follows that $\|\mathcal{T}(u,v)\|\le r$. Since $(u,v)\in B_r$ is an arbitrary element, therefore $\mathcal{T}B_r\subset B_r$.
Now, we show that the operator $\mathcal{T}$ is a contraction. For $(u_1,v_1),(u_2,v_2)\in\mathcal{X}\times\mathcal{X}$, we have

$$
\begin{aligned}
&|\mathcal{T}_1(u_1,v_1)(t)-\mathcal{T}_1(u_2,v_2)(t)|\\
\le\;& \sup_{t\in[0,1]}\Bigg\{\int_0^t\frac{(t-s)^{n-1}}{(n-1)!}|f(s,u_1,v_1)-f(s,u_2,v_2)|ds\\
&+|N_1(t)|\int_0^{\xi}\frac{(\xi-s)^m}{m!}|g(s,u_1,v_1)-g(s,u_2,v_2)|ds\\
&+|N_2(t)|\int_0^{\xi}\frac{(\xi-s)^n}{n!}|f(s,u_1,v_1)-f(s,u_2,v_2)|ds\\
&+|N_3(t)|\Bigg[\int_0^1\frac{(1-s)^{n-2}}{(n-1)!}[\epsilon_1(1-s)+\zeta_1(n-1)]|f(s,u_1,v_1)-f(s,u_2,v_2)|ds\\
&+\sum_{i=1}^{p}\gamma_i\int_0^{\beta_i}\frac{(\beta_i-s)^m}{m!}|g(s,u_1,v_1)-g(s,u_2,v_2)|ds\\
&+\sum_{j=1}^{q}\omega_j\int_0^{\eta_j}\frac{(\eta_j-s)^{m-1}}{(m-1)}|g(s,u_1,v_1)-g(s,u_2,v_2)|ds\Bigg]\\
&+|N_4(t)|\Bigg[\int_0^1\frac{(1-s)^{m-2}}{(m-1)!}[\epsilon_2(1-s)+\zeta_2(m-1)]|g(s,u_1,v_1)-g(s,u_2,v_2)|ds\\
&+\sum_{i=1}^{p}\widehat{\gamma_i}\int_0^{\beta_i}\frac{(\beta_i-s)^n}{n!}|f(s,u_1,v_1)-f(s,u_2,v_2)|ds\\
&+\sum_{j=1}^{q}\widehat{\omega_j}\int_0^{\eta_j}\frac{(\eta_j-s)^{n-1}}{(n-1)}|f(s,u_1,v_1)-f(s,u_2,v_2)|ds\Bigg]\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \quad \ell_1\Big[\frac{1}{n!}+\bar{N_2}\sigma_2+\bar{N_3}\sigma_4+\bar{N_4}\sigma_5\Big](|u_1-u_2|+|v_1-v_2|)\\
&\qquad +\ell_2\Big[\bar{N_1}\sigma_1+\bar{N_3}\sigma_3+\bar{N_4}\sigma_6\Big](|u_1-u_2|+|v_1-v_2|)\\
&\leq \quad (\ell_1 q_1+\ell_2\bar{q_1})(|u_1-u_2|+|v_1-v_2|),
\end{aligned}
$$

which implies that

$$\|\mathcal{T}_1(u_1,v_1)-\mathcal{T}_1(u_2,v_2)\|\leq(\ell_1 q_1+\ell_2\bar{q_1})(|u_1-u_2|+|v_1-v_2|). \tag{3.8}$$

In a similar manners, we get

$$\|\mathcal{T}_2(u_1,v_1)-\mathcal{T}_2(u_2,v_2)\|\leq(\ell_1 q_2+\ell_2\bar{q_2})(|u_1-u_2|+|v_1-v_2|). \tag{3.9}$$

From (3.8) and (3.9), we deduce that

$$\|\mathcal{T}(u_1,v_1)-\mathcal{T}(u_2,v_2)\|\leq(Q_1\ell_1+Q_2\ell_2)(\|u_1-u_2\|+\|v_1-v_2\|),$$

where Q_1 and Q_2 are given in (3.1). By the assumption (3.7), it follows from the above inequality that the operator $\mathcal{T}$ is a contraction. Thus, by the Banach's contraction mapping principle, the operator $\mathcal{T}$ has a unique fixed point, which corresponds to a unique solution to the problem (1.1) on $[0,1]$. □

4. Examples

Example 4.1. Consider the integral-multipoint boundary value problem of nonlinear differential equations

$$
\begin{cases}
u^{(3)}(t)=\dfrac{1}{t^2+9}+\dfrac{1}{\sqrt{t^2+4}}\dfrac{|u^2|}{(1+|u|)}+\dfrac{e^{-t}}{4+t^4}\sin v, & t\in[0,1],\\
v^{(4)}(t)=\dfrac{e^{-t}}{16}+\dfrac{u\cos v}{\sqrt{t^2+36}}+\dfrac{v}{(t^2+5)}\dfrac{|u^2|}{(1+|u|)}, & t\in[0,1],\\
u(0)=\delta_1\displaystyle\int_0^{\xi}v(s)ds,\ u'(0)=0,\ v(0)=\delta_2\int_0^{\xi}u(s)ds,\ v'(0)=0,\ v''(0)=0,\\
\epsilon_1 u(1)+\zeta_1 u'(1)=\displaystyle\sum_{i=1}^{4}\gamma_i\int_0^{\beta_i}v(s)ds+\sum_{j=1}^{3}\omega_j v(\eta_j),\\
\epsilon_2 v(1)+\zeta_2 v'(1)=\displaystyle\sum_{i=1}^{4}\widehat{\gamma_i}\int_0^{\beta_i}u(s)ds+\sum_{j=1}^{3}\widehat{\omega_j}u(\eta_j),
\end{cases}
\tag{4.1}
$$

where $n=3$, $m=4$, $\delta_1=1.2$, $\delta_2=1.5$, $\epsilon_1=0.7$, $\epsilon_2=0.4$, $\zeta_1=2.6$, $\zeta_2=2.1$, $\xi=0.1$, $\beta_1=0.2$, $\beta_2=0.3$, $\beta_3=0.4$, $\beta_4=0.5$, $\eta_1=0.6$, $\eta_2=0.7$, $\eta_3=0.8$, $\gamma_1=0.325$, $\gamma_2=0.572$, $\gamma_3=0.811$, $\gamma_4=0.124$, $\omega_1=0.267$, $\omega_2=0.489$, $\omega_3=0.712$, $\widehat{\gamma}_1=0.452$, $\widehat{\gamma}_2=0.695$, $\widehat{\gamma}_3=0.831$, $\widehat{\gamma}_4=0.203$, $\widehat{\omega}_1=0.378$, $\widehat{\omega}_2=0.617$, $\widehat{\omega}_3=0.954$.

Using the given data in (2.4), (2.19) and (3.1), we find that $\bar{N}_1\approx0.978509$, $\bar{N}_2\approx0.3632664$, $\bar{N}_3\approx0.172781$, $\bar{N}_4\approx0.020315$, $\bar{N}_5\approx0.621273$, $\bar{N}_6\approx1.038184$, $\bar{N}_7\approx0.035554$, $\bar{N}_8\approx0.221574$, $\sigma_1\approx0.0000008$, $\sigma_2\approx0.000004$, $\sigma_3\approx0.017157$, $\sigma_4\approx1.416667$, $\sigma_5\approx0.118329$, $q_1\approx0.413846$, $q_2\approx0.076591$, $\bar{q}_1\approx0.010413$, $\bar{q}_2\approx0.123521$, $Q_1\approx0.490437$, $Q_2\approx0.133934$.

Also it is easy to find that $|f(t,u,v)| \leq 1/9 + 1/2|u| + 1/4|v|$, $|g(t,u,v)| \leq 1/16+1/6|u|+1/5|v|$, $Q_1\widehat{m}_1+Q_2\widehat{n}_1 \approx 0.267541 < 1$ and $Q_1\widehat{m}_2+Q_2\widehat{n}_2 \approx 0.149396 < 1$. Clearly all the assumptions of Theorem 3.2 are satisfied. Therefore, there exists at least one solution to the problem (4.1).

Example 4.2. Consider the system of ordinary differential equations

$$\begin{cases} u^{(3)}(t) = \dfrac{1}{\sqrt{t^2+100}} \tan^{-1} u + \dfrac{1}{(t^2+10)} \dfrac{|v|}{(1+|v|)} + \dfrac{e^{-t}}{4}, & t \in [0,1], \\ v^{(4)}(t) = \dfrac{e^{-t}}{t^2+2} \sin u + \dfrac{1}{\sqrt{t^2+4}} \cos v + \dfrac{t^2+4}{\sqrt{t^3+4}}, & t \in [0,1], \end{cases} \tag{4.2}$$

subject to the boundary conditions in Example 4.1.

Observe that $\ell_1 = 1/10, \ell_2 = 1/2$ as

$$|f(t,u_1,v_1) - f(t,u_2,v_2)| \leq \frac{1}{10}(|u_1-u_2| + |v_1-v_2|),$$

$$|g(t,u_1,v_1) - g(t,u_2,v_2)| \leq \frac{1}{2}(|u_1-u_2| + |v_1-v_2|).$$

Moreover, $Q_1\ell_1 + Q_2\ell_2 \approx 0.088091 < 1$. Thus, the hypotheses of Theorem 3.3 are satisfied and hence its conclusion applies to the problem (4.2).

5. Conclusions

We have developed the existence and uniqueness results for a new class of coupled systems of two nonlinear ordinary differential equations of order n and m subject to the coupled integral-multipoint boundary conditions. Our results are not only new in the given configuration but also yield some new ones by fixing the parameters involved in the given boundary data. In future, we plan to develop the multivalued version of the problem studied in this paper.

References

[1] Adomian, G., Adomian, G.E., *Cellular systems and aging models*, Comput. Math. Appl., **11**(1985), 283-291.

[2] Ahmad, B., Almalki, A., Ntouyas, S.K., Alsaedi, A., *Existence results for a self-adjoint coupled system of nonlinear second-order ordinary differential inclusions with nonlocal integral boundary conditions*, J. Inequal. Appl., (2022), Paper No. 111, 41 pp.

[3] Ahmad, B., Alsaedi, A., Al-Malki, N., *On higher-order nonlinear boundary value problems with nonlocal multipoint integral boundary conditions*, Lith. Math. J., **56**(2016), 143-163.

[4] Akyildiz, F.T., Bellout, H., Vajravelu, K., Van Gorder, R.A., *Existence results for third order nonlinear boundary value problems arising in nano boundary layer fluid flows over stretching surfaces*, Nonlinear Anal. Real World Appl., **12**(2011), 2919-2930.

[5] Alsaedi, A., Almalki, A., Ntouyas, S.K., Ahmad, B., *Existence theory for a self-adjoint coupled system of nonlinear ordinary differential equations with nonlocal integral multi-point boundary conditions*, Appl. Math. E-Notes, **22**(2022), 496-508.

[6] Alsaedi, A., Alsulami, M., Srivastava, H.M., Ahmad, B., Ntouyas, S.K., *Existence theory for nonlinear third-order ordinary differential equations with nonlocal multi-point and multi-strip boundary conditions*, Symmetry, **11**(2019), 281.

[7] Boukrouche, M., Tarzia, D.A., *A family of singular ordinary differential equations of the third order with an integral boundary condition*, Bound. Value Probl., (2018), 2018:32.

[8] Bressan, A., *Hyperbolic Systems of Conservation Laws. The One-Dimensional Cauchy Problem*, Oxford University Press, Oxford, UK, 2000.

[9] Cannon, J.R., *The solution of the heat equation subject to the specification of energy*, Quart. Appl. Math., *21*(1963), 155-160.

[10] Chegis, R.Y., *Numerical solution of a heat conduction problem with an integral condition*, Litovsk. Mat. Sb., **24** (1984), 209-215.

[11] Clark, S., Henderson, J., *Uniqueness implies existence and uniqueness criterion for non local boundary value problems for third-order differential equations*, Proc. Amer. Math. Soc., **134**(2006), 3363-3372.

[12] Deimling, K., *Nonlinear Functional Analysis*, Springer Verlag, Berlin, Heidelberg, 1985.

[13] Elango, S., Govindarao, L., Vadivel, R., *A comparative study on numerical methods for Fredholm integro-differential equations of convection-diffusion problem with integral boundary conditions*, Appl. Numer. Math., **207** (2025), 323-338.

[14] Eloe, P.W., Ahmad, B., *Positive solutions of a nonlinear nth order boundary value problem with nonlocal conditions*, Appl. Math. Lett., **18**(2005), 521-527.

[15] Feng, M., Zhang, X., Ge, W., *Existence theorems for a second order nonlinear differential equation with nonlocal boundary conditions and their applications*, J. Appl. Math. Comput., **33** (2010), 137-153.

[16] Graef, J.R., Webb, J.R.L., *Third order boundary value problems with nonlocal boundary conditions*, Nonlinear Anal., **71**(2009), 1542-1551.

[17] Granas, A., Dugundji, J., *Fixed Point Theory*, Springer-Verlag, New York, NY, 2005.

[18] Hayat, T., Aziz, A., Muhammad, T., Ahmad, B., *On magnetohydrodynamic flow of second grade nanofluid over a nonlinear stretching sheet*, J. Magn. Magn. Mater., **408**(2016), 99-106.

[19] Hayat, T., Kiyani, M.Z., Ahmad, I., Ahmad, B., *On analysis of magneto Maxwell nanomaterial by surface with variable thickness*, Internat. J. Mech. Sci., **131** (2017), 1016-1025.

[20] Ionkin, N.I., *The solution of a certain boundary value problem of the theory of heat conduction with a nonclassical boundary condition* (Russian), Diff. Uravn., **13**(1977), 294-304.

[21] Karaca, I.Y., Fen, F.T., *Positive solutions of nth-order boundary value problems with integral boundary conditions*, Math. Model. Anal., **20**(2015), 188-204.

[22] Ma, R., *A survey on nonlocal boundary value problems*, Appl. Math. E-Notes, **7**(2007), 257-279.

[23] Nicoud, F., Schfonfeld, T., *Integral boundary conditions for unsteady biomedical CFD applications*, Int. J. Numer. Meth. Fluids, **40**(2002), 457-465.

[24] Sun, Y., Liu, L., Zhang, J., Agarwal, R.P., *Positive solutions of singular three-point boundary value problems for second-order differential equations*, J. Comput. Appl. Math., **230**(2009), 738-750.

[25] Taylor, C., Hughes, T., Zarins, C., *Finite element modeling of blood flow in arteries*, Comput. Methods Appl. Mech. Engrg., **158**(1998), 155-196.

[26] Webb, J.R.L., Infante, G., *Positive solutions of nonlocal boundary value problems involving integral conditions*, NoDEA Nonlinear Differential Equations Appl., **15**(2008), 45-67.

[27] Womersley, J.R., *Method for the calculation of velocity, rate of flow and viscous drag in arteries when the pressure gradient is known*, J. Psychol., **127**(1955), 553-563.

[28] Zheng, L., Zhang, X., *Modeling and analysis of modern fluid problems. Mathematics in Science and Engineering*, Elsevier/Academic Press, London, 2017.

Bashir Ahmad
Nonlinear Analysis and Applied Mathematics (NAAM) Research Group,
Department of Mathematics, Faculty of Science, King Abdulaziz University,
P.O. Box 80203, Jeddah 21589, Saudi Arabia
e-mail: bashirahmad_qau@yahoo.com

Sotiris K. Ntouyas
Department of Mathematics, University of Ioannina, 451 10, Ioannina, Greece
e-mail: sntouyas@uoi.gr

Ahmed Alsaedi
Nonlinear Analysis and Applied Mathematics (NAAM) Research Group,
Department of Mathematics, Faculty of Science, King Abdulaziz University,
P.O. Box 80203, Jeddah 21589, Saudi Arabia
e-mail: aalsaedi@hotmail.com

Ammar B. Wali
Nonlinear Analysis and Applied Mathematics (NAAM) Research Group,
Department of Mathematics, Faculty of Science, King Abdulaziz University,
P.O. Box 80203, Jeddah 21589, Saudi Arabia
e-mail: awali0012@stu.kau.edu.sa