Stud. Univ. Babeş-Bolyai Math. 70(2025), No. 2, 219–232
DOI: 10.24193/subbmath.2025.2.04

Some classes involving a convolution of analytic functions with some univalency conditions

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Abstract. In this paper, involving a convolution $f * g$, two classes of normalized analytic functions f are defined. Showing an inclusion relation between these classes, various sufficient conditions for functions to be in these classes are established. In particular, varied forms of univalency conditions of the convolution function $f * g$ are given which lead to some univalency conditions of several linear operators.

Mathematics Subject Classification (2010): 30C45, 30C55.

Keywords: Convolution, univalent functions, Dzoik-Srivastava operator, Srivastava-Attiya linear operator, Owa and Srivastava fractional differintegral operator, Jung-Kim-Srivastava integral operator.

1. Introduction

Let $\mathcal{H}$ denote the class of functions analytic in the open unit disk

$$\mathbb{U} = \{z : |z| < 1\},$$

and for $k \in \mathbb{N} = \{1, 2, ...\}$ and $a \in \mathbb{C}$, let

$$\mathcal{H}[a, k] = \left\{f \in \mathcal{H} : f(z) = a + a_k z^k + a_{k+1} z^{k+1} + ...\right\}.$$

Let $\mathcal{A}$ denotes a class of functions in $\mathcal{H}[0, 1]$ of the form

$$f(z) = z + \sum_{k=1}^{\infty} a_{k+1} z^{k+1}. \tag{1.1}$$

Received 19 November 2024; Accepted 11 April 2025.

A subclass of *univalent* functions in $\mathcal{A}$ is denoted by $\mathcal{S}$. Functions $f \in \mathcal{A}$ is said to be in the class $\mathcal{S}^*$, a class of *starlike* functions if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \text{ in } \mathbb{U}.$$

A convolution (Hadamard product) $*$ of $f \in \mathcal{A}$ of the form (1.1) and $g \in \mathcal{A}$ of the form

$$g(z) = z + \sum_{k=1}^{\infty} b_{k+1} z^{k+1}, \tag{1.2}$$

is defined by

$$f(z) * g(z) = z + \sum_{k=1}^{\infty} a_{k+1} b_{k+1} z^{k+1} = g(z) * f(z). \tag{1.3}$$

Note that the convolution preserves the class $\mathcal{A}$.

Several linear operators have been studied in *Geometric Function Theory* so far, which are defined in the form of convolution, differential, integral, and fractional differintegral linear operators. Some of the known linear operators for the class $\mathcal{A}$, are the Dziok-Srivastava convolution operator [5], the Srivastava-Attiya linear operator [19], the Jung-Kim-Srivastava integral operator [7], a multiplier operator [16] and a fractional differintegral operator introduced by Owa and Srivastava [10]. The convolution representation of these operators may be given as follows:

The Dzoik-Srivastava operator [5]: ${}_pH_q\left([\alpha_1]\right) : \mathcal{A} \to \mathcal{A}$ is defined by

$${}_pH_q\left([\alpha_1]\right) f(z) = z\ {}_pF_q(\alpha_1, ..., \alpha_p; \beta_1, ..., \beta_q; z) * f(z) \tag{1.4}$$

where

$${}_pF_q(\alpha_1, ..., \alpha_p; \beta_1, ..., \beta_q; z) = \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p} (\alpha_i)_k}{\prod_{i=1}^{q} (\beta_i)_k} \frac{z^k}{k!}$$

$$\left(p \leq q+1, p, q \in \mathbb{N}_0 = \{0, 1, 2, ...\}, \alpha_i, \beta_i \in \mathbb{C}\ (\beta_i \neq 0, -1, -2, ...); z \in \mathbb{U}\right)$$

is the generalized hypergeometric function ([12, p. 19]). The symbol $(\lambda)_k$ is the Pochhammer symbol defined by

$$(\lambda)_k = \frac{\Gamma(\lambda + k)}{\Gamma(\lambda)} = \lambda(\lambda+1)(\lambda+2)...(\lambda+k-1), k \in \mathbb{N}; (\lambda)_0 = 1.$$

The Srivastava-Attiya linear operator [19]: $J_{a,b} : \mathcal{A} \to \mathcal{A}$ is defined in terms of generalized Hurwitz-Lerch Zeta function $\phi(b, a, z)$ [20] by

$$J_{a,b} f(z) = G_{a,b}(z) * f(z), \tag{1.5}$$

where

$$G_{a,b}(z) = (b+1)^a \left(\phi(b, a, z) - b^{-a}\right) = z + \sum_{k=1}^{\infty} \left(\frac{b+1}{b+n}\right)^a z^{k+1}$$

$$\left(b \in \mathbb{C}\,(b \neq 0, -1, -2, ...), a \in \mathbb{C}; z \in \mathbb{U}\right).$$

The fractional integral operator $D_z^{-\mu}$ of order μ $(\mu > 0)$ for the function $f \in \mathcal{A}$ is defined by (see [9])

$$D_z^{-\mu} f(z) = \frac{1}{\Gamma(\mu)} \int_0^z \frac{f(t)}{(z-t)^{1-\mu}} \mathrm{d}t \quad (z \in \mathbb{U}),$$

where the multiplicity of $(z-t)^{\mu-1}$ is removed by requiring $\log(z-t)$ to be real when $z - t > 0$. Also, the fractional derivative operator D_z^{λ} of order λ $(\lambda \geq 0)$ for the function $f \in \mathcal{A}$ is defined by

$$D_z^{\lambda} f(z) = \begin{cases} \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(t)}{(z-t)^{\lambda}} \mathrm{d}t \ \ (0 \leq \lambda < 1), \\ \frac{d^n}{dz^n} D_z^{\lambda-n} f(z) \ \ (n \leq \lambda < n+1, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}), \end{cases}$$

where the multiplicity of $(z-t)^{-\lambda}$ is understood similarly.

Owa and Srivastava [10] introduced a fractional differintegral operator

$$\Omega_z^{\lambda} : \mathcal{A} \to \mathcal{A} \ (-\infty < \lambda < 2)$$

by

$$\Omega_z^{\lambda} f(z) = \Gamma(2-\lambda) z^{\lambda} D_z^{\lambda} f(z) \ \ (z \in \mathbb{U}),$$

where $D_z^{\lambda} f(z)$ is, respectively, the fractional integral of order $\lambda(-\infty < \lambda < 0)$ and a fractional derivative of order λ $(0 \leq \lambda < 2)$. The operator Ω_z^{λ} for the function $f \in \mathcal{A}$ is given in the form of convolution by

$$\Omega_z^{\lambda} f(z) = \ z \ {}_2F_1 \ (2, 1; 2-\lambda; z) * f(z) \ \ (-\infty < \lambda < 2; z \in \mathbb{U}). \tag{1.6}$$

The Jung-Kim-Srivastava integral operator [7] $Q_{\gamma}^{\alpha} : \mathcal{A} \to \mathcal{A}$ ($\alpha > 0, \gamma > -1$) is defined by

$$Q_{\gamma}^{\alpha} f(z) = \begin{pmatrix} \alpha+\gamma \\ \gamma \end{pmatrix} \frac{\alpha}{z^{\gamma}} \int_0^z \left(1 - \frac{t}{z}\right)^{\alpha-1} t^{\gamma-1} f(t) \mathrm{d}t \quad (z \in \mathbb{U})$$

which can also be expressed as follows:

$$Q_{\gamma}^{\alpha} f(z) = \ z \ {}_2F_1 \ (\gamma+1, 1; \alpha+\gamma+1; z) * f(z). \tag{1.7}$$

The multiplier operator $\Im_{\lambda,\mu}^{m} : \mathcal{A} \to \mathcal{A}$, recently studied in [16] (see also [15, 18]) is defined for $m \in \mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}, \mu > -1, \ \lambda > 0$, by

$$\mathcal{J}_{\lambda,\mu}^{m} f(z) = \begin{cases} f(z), & m = 0, \\ \frac{\mu+1}{\lambda} z^{1-\frac{\mu+1}{\lambda}} \int_0^z t^{\frac{\mu+1}{\lambda}-2} \mathcal{J}_{\lambda,\mu}^{m+1} f(t) \mathrm{d}t, & m \in \mathbb{Z}^- = \{-1, -2, \ldots\}, \\ \frac{\lambda}{\mu+1} z^{2-\frac{\mu+1}{\lambda}} \frac{\mathrm{d}}{\mathrm{d}t} \left(z^{\frac{\mu+1}{\lambda}-1} \mathcal{J}_{\lambda,\mu}^{m-1} \ f(z) \right), & m \in \mathbb{Z}^+ = \{1, 2, \ldots\} \end{cases} \tag{1.8}$$

which may be given by

$$\mathcal{J}_{\lambda,\mu}^{m} f(z) = \Phi_{\lambda,\mu}^{m}(z) * f(z), \tag{1.9}$$

where

$$\Phi_{\lambda,\mu}^{m}(z)=\sum_{k=1}^{\infty}\left(1+\frac{\lambda(k-1)}{\mu+1}\right)^{m}z^{k}.$$

Let f and g be analytic functions in the unit disc $\mathbb{U}$. Then we say that f is subordinate to g, and we write $f\prec g$ if there exists a function w analytic in unit disc $\mathbb{U}$, such that

$$w(0)=0,\quad |w(z)|<1\ \ (z\in\mathbb{U})$$

and

$$f(z)=g(w(z)),\forall\ z\in\mathbb{U}.$$

In particular, if g is univalent in $\mathbb{U}$, then we have the following equivalence:

$$f\prec g\Leftrightarrow f(0)=g(0)\text{ and }f(\mathbb{U})\subset g(\mathbb{U}).$$

In [6], Janowski introduced the class $\mathcal{S}^{*}[A,B]$ of functions $f\in\mathcal{A}$ satisfying the condition:

$$\frac{zf'(z)}{f(z)}\prec\frac{1+Az}{1+Bz}\quad(-1\le B<A\le 1;z\in\mathbb{U}).$$

Geometrically, the above subordination condition means that the image of the unit disc $\mathbb{U}$ by the function $\frac{zf'(z)}{f(z)}$ is in the open disc whose endpoints of the diameter are $\frac{1-A}{1-B}$ and $\frac{1+A}{1+B}$ (in case $B\neq-1$) and in the positive half plane $\Re\left(\frac{zf'(z)}{f(z)}\right)>\frac{1-A}{2}$ (in case $B=-1$).

For particular values of A,B, we get $S^{*}[1,-1]=S^{*}$, a class of starlike functions, $S^{*}[1-2\alpha,-1]=S^{*}(\alpha)\ (0\le\alpha<1)$, a class of starlike functions of order α; $S^{*}[1-\alpha,0]=S_{\alpha}^{*}$ and $S^{*}[\alpha,-\alpha]=S^{*}[\alpha]$ (see [2]).

On using convolution, we define following subclasses of the class $\mathcal{A}$:

Definition 1.1. A function $f\in\mathcal{A}$ is said to be in the class $\mathcal{S}^{*}[\mu,g;A,B]$ if for $-1\le B<A\le 1$, $\mu\ge-1$ and for some $g\in\mathcal{A}$ with $0\neq\frac{(f*g)(z)}{z}\in\mathbb{C}$, it satisfies

$$\left(\frac{z}{(f*g)(z)}\right)^{\mu+1}(f*g)'(z)\prec\frac{1+Az}{1+Bz}\ \ (z\in\mathbb{U}),\tag{1.10}$$

where only principle values of the exponent function are considered.

Remark 1.2. Let $\mu=0$ and $g(z)=\frac{z}{1-z}$ $(z\in\mathbb{U})$, we get $\mathcal{S}^{*}[\mu,g;A,B]=\mathcal{S}^{*}[A,B]$.

Remark 1.3. If we put $\mu=0$, $g(z)=\frac{z}{(1-z)^{2}}$ $(z\in\mathbb{U})$ and $A=1-2\alpha,B=-1$ in $\mathcal{S}^{*}[\mu,g;A,B]$ then we obtain the class $\mathcal{K}(\alpha)$ convex functions of order α studied by Robertson [17].

Definition 1.4. A function $f\in\mathcal{A}$ is said to be in the class $\mathcal{B}(g,\mu;\beta)$ if for $\frac{1}{2}<\beta\le 1$, $\mu\ge-1$ and for some $g\in\mathcal{A}$ with $0\neq\frac{(f*g)(z)}{z}\in\mathbb{C}$, it satisfies

$$\left|\left(\frac{z}{(f*g)(z)}\right)^{\mu+1}(f*g)'(z)-\beta\right|<\beta\ (z\in\mathbb{U}),\tag{1.11}$$

where only principle values of the exponent function are considered.

Example 1.5. The following example $\mu = 0$, $f(z) = \frac{z}{(1-z)^2}$ and $g(z) = \frac{z}{1-z}$ satisfies the conditions of Definitions 1.1 and 1.4.

In particular, $\mathcal{S}^*[\mu, g; 1, 0] \equiv \mathcal{B}(g, \mu; 1)$.

Remark 1.6. If $\beta = 1$ and $\mu = 1$, the class condition (1.11) for the class $\mathcal{B}(g, \mu; \beta)$ provides a univalency criterion for the functions $f*g$ according to Ozaki and Nunokawa [11], see also [1, 4].

In this paper, for a certain function $g \in \mathcal{A}$, involving a convolution $f * g$, two classes $\mathcal{S}^*[\mu, g; A, B]$ and $\mathcal{B}(g, \mu; \beta)$ of $f \in \mathcal{A}$, are defined. Showing an inclusion relation between these classes, various sufficient conditions for functions to be in these classes are established. In particular, varied sufficient conditions for univalency of the convolution function $f * g$ are given which lead to the univalency conditions of various known linear operators.

2. Main results

We first prove an inclusion result for the classes $\mathcal{S}^*[\mu, g; A, B]$ and $\mathcal{B}(g, \mu; \beta)$ which is as follows:

Theorem 2.1. *Let $f \in \mathcal{A}$ and $0 \le B < A \le 1$, $\frac{1}{2} < \beta \le 1$ be such that*

$$A \le 2B(1-\beta) + 2\beta - 1. \tag{2.1}$$

Let the classes $\mathcal{S}^*[\mu, g; A, B]$ and $\mathcal{B}(g, \mu; \beta)$ be defined, respectively, by Definitions 1.1 and 1.4. Then

$$\mathcal{S}^*[\mu, g; A, B] \subset \mathcal{B}(g, \mu; \beta).$$

Proof. If $f \in \mathcal{S}^*[\mu, g; A, B]$, then there is a Schwarz function w analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ $(z \in \mathbb{U})$, such that

$$\left(\frac{z}{(f*g)(z)}\right)^{\mu+1} (f*g)'(z) = \frac{1 + Aw(z)}{1 + Bw(z)} \quad (z \in \mathbb{U}). \tag{2.2}$$

Hence, for the given hypotheses (2.1) and for this Schwarz function w given by (2.2), we get

$$\left|\left(\frac{z}{(f*g)(z)}\right)^{\mu+1} (f*g)'(z) - \beta\right| = \left|1 + \frac{(A-B)\,w(z)}{1 + Bw(z)} - \beta\right|$$
$$< \quad 1 + \frac{A-B}{1-B} - \beta \le \beta$$

which implies that $f \in \mathcal{B}(g, \mu; \beta)$. This proves Theorem 2.1. □

Example 2.2. The following example $\mu = 0$, $f(z) = z + \frac{z^2}{2}$ and $g(z) = \frac{z}{1-z}$ satisfies the condition of Theorem 2.1.

In view of Remark 1.6, for $\beta = 1$ and $\mu = 1$, Theorem 2.1 provides following univalency condition for the convolution $f * g$:

Corollary 2.3. *Let $f \in \mathcal{A}$ and let for some $g \in \mathcal{A}$ with $0 \neq \frac{(f*g)(z)}{z} \in \mathbb{C}$,*

$$\left(\frac{z}{(f*g)(z)}\right)^2 (f*g)'(z) \prec \frac{1+Az}{1+Bz} \quad (0 \leq B < A \leq 1; z \in \mathbb{U}).$$

Then $f * g$ is univalent in $\mathbb{U}$.

Now, we prove certain sufficient conditions for functions to be in the class $\mathcal{S}^*[\mu, g; A, B]$, for this, we apply the method of admissible function used in the following lemma which is the special case of the result [8, (ii) Theorem 2.3h, p. 34].

Lemma 2.4. [8, (ii) Theorem 2.3h, p. 34] *Let Ω be a subset of the complex plane $\mathbb{C}$ and let an admissible function $\psi : \mathbb{C}^2 \times \mathbb{U} \to \mathbb{C}$ satisfies the condition*

$$\psi(Me^{i\theta}, mMe^{i\theta}; z) \notin \Omega$$

for real $M > 0$ and $m \geq k \geq 1$ and $z \in \mathbb{U}$. If the function $w \in \mathcal{H}[a, k]$, then

$$\psi(w(z), zw'(z); z) \in \Omega \Rightarrow |w(z)| < M \quad (z \in \mathbb{U}).$$

Theorem 2.5. *Let $f \in \mathcal{A}$ and let for some $\theta \in \mathbb{R}, m \geq 1$, $-1 \leq B < A \leq 1$,*

$$\left|1 + \frac{(A-B)\, me^{i\theta}}{(1+Ae^{i\theta})(1+Be^{i\theta})}\right| \geq 1. \tag{2.3}$$

If for some $g \in \mathcal{A}$ with $0 \neq (f*g)'(z) \cdot \frac{(f*g)(z)}{z} \in \mathbb{C}$ in $\mathbb{U}$,

$$\left|1 + \frac{z(f*g)''(z)}{(f*g)'(z)} + (\mu+1)\left\{1 - \frac{z(f*g)'(z)}{(f*g)(z)}\right\}\right| < 1, \tag{2.4}$$

then $f \in \mathcal{S}^*[\mu, g; A, B]$.

Proof. Let

$$p(z) = \left(\frac{z}{(f*g)(z)}\right)^{\mu+1} (f*g)'(z) \tag{2.5}$$

and $w \in \mathcal{H}[0, 1]$ be defined by

$$p(z) = \frac{1+Aw(z)}{1+Bw(z)} \quad (z \in \mathbb{U}), \tag{2.6}$$

then w is analytic in $\mathbb{U}$. To prove the theorem we only need to prove $|w(z)| < 1$. For this purpose, we define an admissible function $\Psi : \mathbb{C}^2 \times \mathbb{U} \to \mathbb{C}$ by

$$\Psi(r, s; z) = 1 + \frac{(A-B)\, s}{(1+Ar)(1+Br)} \quad (-1 \leq B < A \leq 1), \tag{2.7}$$

where $r \neq -\frac{1}{A}, -\frac{1}{B}$ (in case $A, B \neq 0$). Then, from (2.3), we have

$$\left|\Psi\left(e^{i\theta}, me^{i\theta}; z\right)\right| \geq 1. \tag{2.8}$$

Differentiating equations (2.6) and (2.5) logarithmically, we obtain

$$\begin{aligned} 1 + \frac{zp'(z)}{p(z)} &= 1 + \frac{(A-B)\, zw'(z)}{(1+Aw(z))(1+Bw(z))} \\ &= 1 + \frac{z(f*g)''(z)}{(f*g)'(z)} + (\mu+1)\left\{1 - \frac{z(f*g)'(z)}{(f*g)(z)}\right\}. \end{aligned} \tag{2.9}$$

Let Ω be a subset of the complex plane $\mathbb{C}$ such that in $\mathbb{C}\setminus\Omega$, the admissible function Ψ satisfies (2.8). Hence, Lemma 2.4 for the case $M=1$ reveals in view of (2.7), (2.9) and (2.4), that

$$|\Psi(w(z), zw'(z); z)| < 1 \Rightarrow |w(z)| < 1 \ \ (z \in \mathbb{U}),$$

which proves that

$$p(z) \prec \frac{1+Az}{1+Bz},$$

and hence $f \in \mathcal{S}^*[\mu, g; A, B]$. □

Theorem 2.6. *Let $f \in \mathcal{A}$ and $-1 \le B < A \le 1$, let*

$$\lambda = \begin{cases} \frac{2\sqrt{A|B|}}{1-AB}, & \text{if } AB < 0 \text{ with } \left|(A+B)\left(1+\frac{1}{AB}\right)\right| \le 4, \\ \frac{A-B}{(1+|A|)(1+|B|)}, & \text{if } AB \ge 0. \end{cases} \tag{2.10}$$

If for some $g \in \mathcal{A}$ with $0 \ne (f*g)'(z) \cdot \frac{(f*g)(z)}{z} \in \mathbb{C}$ in $\mathbb{U}$,

$$\left|\frac{z(f*g)''(z)}{(f*g)'(z)} + (\mu+1)\left\{1 - \frac{z(f*g)'(z)}{(f*g)(z)}\right\}\right| < \lambda, \tag{2.11}$$

then $f \in \mathcal{S}^*[\mu, g; A, B]$.

Proof. To prove the result, we define an admissible function $\phi : \mathbb{C}^2 \times \mathbb{U} \to \mathbb{C}$ by

$$\phi(r, s; z) = \Psi(r, s; z) - 1, \tag{2.12}$$

where $\Psi(r, s; z)$ is defined by (2.7). Then for some $\theta \in \mathbb{R}$ and for some $m \ge 1$

$$\begin{aligned} \left|\phi\left(e^{i\theta}, me^{i\theta}; z\right)\right| &= \left|\frac{(A-B)\, me^{i\theta}}{(1+Ae^{i\theta})(1+Be^{i\theta})}\right| \\ &= \frac{(A-B)\, m}{|(1+Ae^{i\theta})|\,|(1+Be^{i\theta})|} \\ &= \frac{(A-B)\, m}{\sqrt{1+A^2+2At} \cdot \sqrt{1+B^2+2Bt}} \\ &= \frac{(A-B)\, m}{h(t)}, \end{aligned}$$

where $t = \cos\theta \in [-1, 1]$. Observe that

$$\max_{-1 \le t \le 1} h(t) = \begin{cases} (1+A)(1+B), & \text{if } 0 \le B < A \le 1, \\ (1-A)(1-B), & \text{if } -1 \le B < A \le 0, \end{cases}$$

Hence,

$$\left|\phi\left(e^{i\theta}, me^{i\theta}; z\right)\right| \ge \frac{A-B}{(1+|A|)(1+|B|)}, \ \text{if } AB \ge 0.$$

Further, if $-1 \le B < 0 < A \le 1$, i.e. if $AB < 0$, then the function $h(t)$ attains its maximum value at

$$t^* = -\frac{(A+B)(1+AB)}{4AB} \in [-1, 1].$$

Hence, if $AB < 0$ with the condition: $4AB \leq (A+B)(1+AB) \leq -4AB$ or equivalently, $\left|(A+B)\left(1+\frac{1}{AB}\right)\right| \leq 4$,

$$h(t^*) = \frac{(A-B)(1-AB)}{2\sqrt{A|B|}}.$$

So,

$$\left|\phi\left(e^{i\theta}, me^{i\theta}; z\right)\right| \geq \frac{2\sqrt{A|B|}}{1-AB}, \text{ if } AB < 0 \text{ with } \left|(A+B)\left(1+\frac{1}{AB}\right)\right| \leq 4.$$

Hence,

$$\left|\phi\left(e^{i\theta}, me^{i\theta}; z\right)\right| \geq \lambda, \tag{2.13}$$

where λ is given by (2.10). Thus, in view of (2.12) and for $p(z)$ defined by (2.5), we get from (2.9),

$$\begin{aligned} |\phi(w(z), zw'(z); z)| &= \left|\frac{zp'(z)}{p(z)}\right| \qquad (2.14) \\ &= \left|\frac{z(f*g)''(z)}{(f*g)'(z)} + (\mu+1)\left\{1 - \frac{z(f*g)'(z)}{(f*g)(z)}\right\}\right|. \end{aligned}$$

Let Λ be a subset of the complex plane $\mathbb{C}$ such that in $\mathbb{C}\setminus\Lambda$, the admissible function ϕ satisfies (2.13). Hence, applying Lemma 2.4 (in case $M = 1$), from (2.14) and (2.11)

$$|\phi(w(z), zw'(z); z)| < \lambda \Rightarrow |w(z)| < 1 \ \ (z \in \mathbb{U}),$$

which proves

$$p(z) \prec \frac{1+Az}{1+Bz}.$$

This establishes Theorem 2.6. □

From Theorem 2.1 and Theorem 2.6, we obtain following result.

Corollary 2.7. *Let $f \in \mathcal{A}$ and $0 \leq B < A \leq 1$, $\frac{1}{2} < \beta \leq 1$ be such that*

$$A \leq 2B(1-\beta) + 2\beta - 1. \tag{2.15}$$

If for some $g \in \mathcal{A}$ with $\frac{(f*g)(z)}{z} \neq 0$ in $\mathbb{U}$ and for $\mu \geq -1$,

$$\left|\frac{z(f*g)''(z)}{(f*g)'(z)} + (\mu+1)\left\{1 - \frac{z(f*g)'(z)}{(f*g)(z)}\right\}\right| < \frac{A-B}{(1+A)(1+B)} \ \ (z \in \mathbb{U}), \tag{2.16}$$

then $f \in \mathcal{B}(g, \mu; \beta)$.

Proof. Applying Theorem 2.6 for $0 \leq B < A \leq 1$, we get $f \in \mathcal{S}^*[\mu, g; A, B]$ if and condition (2.16) holds, and from Theorem 2.1, $\mathcal{S}^*[\mu, g; A, B] \subset \mathcal{B}(g, \mu; \beta)$ if (2.15) holds. Hence, this proves the result. □

Example 2.8. The following example $\mu = 0$, $f(z) = z + \frac{z^n}{n}$ and $g(z) = \frac{z}{1-z}$ satisfies the condition of Corollary 2.7.

Again, in view of the Remark 1.6, for $\beta = 1$ and $\mu = 1$, above Corollary 2.7 provides the following univalency condition for the convolution $f * g$:

Corollary 2.9. *Let $f \in \mathcal{A}$ and $0 \leq B < A \leq 1$. If for some $g \in \mathcal{A}$ with*

$$0 \neq (f * g)'(z) \cdot \frac{(f * g)(z)}{z} \in \mathbb{C} \ in \ \mathbb{U},$$

$$\left| \frac{z(f * g)''(z)}{(f * g)'(z)} + 2\left\{1 - \frac{z(f * g)'(z)}{(f * g)(z)}\right\}\right| < \frac{A - B}{(1 + A)(1 + B)} \quad (z \in \mathbb{U}),$$

then $f * g$ is univalent in $\mathbb{U}$.

Also, for the special values: $A = 1 - 2\alpha$ $\left(0 \leq \alpha < \frac{1}{2}\right)$ and $B = -1$, Theorem 2.6 provides following result:

Corollary 2.10. *Let $f \in \mathcal{A}$ and $0 \leq \alpha < \frac{1}{2}$, $\mu \geq -1$. If for some $g \in \mathcal{A}$ with*

$$0 \neq (f * g)'(z) \cdot \frac{(f * g)(z)}{z} \in \mathbb{C} \ in \ \mathbb{U},$$

$$\left| \frac{z(f * g)''(z)}{(f * g)'(z)} + (\mu + 1)\left\{1 - \frac{z(f * g)'(z)}{(f * g)(z)}\right\}\right| < \frac{\sqrt{1 - 2\alpha}}{1 - \alpha} \quad (z \in \mathbb{U}),$$

then

$$\left(\frac{z}{(f * g)(z)}\right)^{\mu+1} (f * g)'(z) \prec \frac{1 + (1 - 2\alpha)z}{1 - z} \quad (z \in \mathbb{U}).$$

In our next result we give some more sufficient conditions for the class $\mathcal{S}^*[\mu, g; A, B]$ in case $B = 0$.

Theorem 2.11. *Let $f \in \mathcal{A}$ and let $0 < A \leq 1$. If for some $g \in \mathcal{A}$ with*

$$0 \neq (f * g)'(z) \cdot \frac{(f * g)(z)}{z} \in \mathbb{C}$$

in $\mathbb{U}$, any one of the following conditions holds

$$\begin{aligned} &\left| \left(\frac{z}{(f * g)(z)}\right)^{\mu+1} (f * g)'(z) \right. \\ &\left. \left[\frac{z(f * g)''(z)}{(f * g)'(z)} + (\mu + 1)\left\{1 - \frac{z(f * g)'(z)}{(f * g)(z)}\right\}\right]\right| \\ < \ & A \ (z \in \mathbb{U}), \end{aligned} \tag{2.17}$$

$$\begin{aligned} &\left| \left(\frac{(f * g)(z)}{z}\right)^{\mu+1} \frac{1}{(f * g)'(z)} \right. \\ &\left. \left[\frac{z(f * g)''(z)}{(f * g)'(z)} + (\mu + 1)\left\{1 - \frac{z(f * g)'(z)}{(f * g)(z)}\right\}\right]\right| \\ < \ & \frac{A}{(1 + A)^2} \ (z \in \mathbb{U}), \end{aligned} \tag{2.18}$$

$$\left| \frac{\left[\frac{z(f*g)''(z)}{(f*g)'(z)} + (\mu + 1)\left\{1 - \frac{z(f*g)'(z)}{(f*g)(z)}\right\}\right]}{\left(\frac{z}{(f*g)(z)}\right)^{\mu+1} (f * g)'(z) - 1} \right| < \frac{1}{1 + A} \quad (z \in \mathbb{U}), \tag{2.19}$$

then $f \in \mathcal{S}^*[\mu, g; A, 0]$.

Proof. Let $p(z)$ be defined by (2.5). Then $p \in \mathcal{H}[1,1]$ and by the hypothesis $0 \neq p(z) \in \mathbb{C}$ in $\mathbb{U}$. Then from (2.9), we obtain

$$zp'(z) = \left(\frac{z}{(f*g)(z)}\right)^{\mu+1} (f*g)'(z) \quad \times$$
$$\left[\frac{z(f*g)''(z)}{(f*g)'(z)} + (\mu+1)\left\{1 - \frac{z(f*g)'(z)}{(f*g)(z)}\right\}\right], \tag{2.20}$$

$$\frac{zp'(z)}{(p(z))^2} = \left(\frac{(f*g)(z)}{z}\right)^{\mu+1} \frac{1}{(f*g)'(z)} \quad \times \tag{2.21}$$
$$\left[\frac{z(f*g)''(z)}{(f*g)'(z)} + (\mu+1)\left\{1 - \frac{z(f*g)'(z)}{(f*g)(z)}\right\}\right],$$

and

$$\frac{zp'(z)}{p(z)(p(z)-1)} = \frac{\left[\frac{z(f*g)''(z)}{(f*g)'(z)} + (\mu+1)\left\{1 - \frac{z(f*g)'(z)}{(f*g)(z)}\right\}\right]}{\left(\frac{z}{(f*g)(z)}\right)^{\mu+1} (f*g)'(z) - 1}, \tag{2.22}$$

where in (2.22) the singularity of the function at $z = 0$, is being removed by the numerator. To prove the result, we use the similar method used in the above proofs of Theorems 2.5 and 2.6 for the case if $B = 0$. Let $u(z)$ be defined by

$$p(z) = 1 + Au(z). \tag{2.23}$$

Then $u(0) = 0$ and now we prove $|u(z)| < 1$ in $\mathbb{U}$. For this, we may define admissible function $\eta_i : \mathbb{C}^2 \times \mathbb{U} \to \mathbb{C}$ for each $i = 1, 2, 3$, by

$$\eta_1(r, s; z) = As, \tag{2.24}$$

$$\eta_2(r, s; z) = \frac{As}{(1+Ar)^2} \quad \left(r \neq -\frac{1}{A}\right),$$

and

$$\eta_3(r, s; z) = \frac{s}{r(1+Ar)} \quad \left(r \neq 0, -\frac{1}{A}\right).$$

Then for some $\theta \in \mathbb{R}$ and for some $m \geq 1$,

$$\left|\eta_1\left(e^{i\theta}, me^{i\theta}; z\right)\right| = Am \geq A, \tag{2.25}$$

$$\left|\eta_2\left(e^{i\theta}, me^{i\theta}; z\right)\right| = \frac{Am}{|1+Ae^{i\theta}|^2} \geq \frac{A}{(1+A)^2}, \tag{2.26}$$

and

$$|\eta_3(r, s; z)| = \frac{m}{|1+Ae^{i\theta}|} \geq \frac{1}{1+A}. \tag{2.27}$$

Then from (2.23)

$$\begin{aligned} zp'(z) &= zAu'(z), & (2.28)\\ \frac{zp'(z)}{(p(z))^2} &= \frac{zAu'(z)}{(1+Au(z))^2}, & (2.29)\\ \frac{zp'(z)}{p(z)(p(z)-1)} &= \frac{zu'(z)}{(1+Au(z))\,u(z)}. & (2.30)\end{aligned}$$

Let for each $i = 1, 2, 3$, Ω_i be a subset of the complex plane $\mathbb{C}$ such that in $\mathbb{C}\setminus\Omega_i$, the admissible function η_i satisfies for each $i = 1, 2, 3$, the conditions, (2.25), (2.26) and (2.27). Hence, by Lemma 2.4 for $M = 1$, in view of (2.20), (2.21) and (2.22), from the conditions (2.17), (2.18) and (2.19) and using the values (2.28), (2.29) and (2.30), we get

$$\begin{aligned} |\eta_1(u(z), zu'(z); z)| &= |zp'(z)| < A \Rightarrow |u(z)| < 1,\\ |\eta_2(u(z), zu'(z); z)| &= \left|\frac{zp'(z)}{(p(z))^2}\right| < \frac{A}{(1+A)^2} \Rightarrow |u(z)| < 1,\\ |\eta_3(u(z), zu'(z); z)| &= \left|\frac{zp'(z)}{p(z)(p(z)-1)}\right| < \frac{1}{1+A} \Rightarrow |u(z)| < 1.\end{aligned}$$

This proves the Theorem 2.11. □

Using Theorem 2.1 for the case $B = 0$, we obtain following result from Theorem 2.11.

Corollary 2.12. *Let $f \in \mathcal{A}$ and let $\frac{1}{2} < \beta \le 1, 0 < A \le 2\beta - 1$. If for some $g \in \mathcal{A}$ with $0 \ne (f*g)'(z) \cdot \frac{(f*g)(z)}{z} \in \mathbb{C}$ in $\mathbb{U}$, any one of the conditions (2.17), (2.18) and (2.19) in Theorem 2.11 holds, then $f \in \mathcal{B}(g, \mu; \beta)$.*

In addition to the Corollaries 2.3 and 2.9, Corollary 2.12 provides for $\beta = 1$ and $\mu = 1$, the following univalency condition for the convolution function $f * g$.

Corollary 2.13. *Let $f \in \mathcal{A}$ and let $0 < A \le 1$. If for some $g \in \mathcal{A}$ with*

$$0 \ne (f*g)'(z) \cdot \frac{(f*g)(z)}{z} \in \mathbb{C} \text{ in } \mathbb{U},$$

$f * g$ satisfies any one of the following conditions:

$$\left|\frac{z^2(f*g)'(z)}{((f*g)(z))^2}\left(\frac{z(f*g)''(z)}{(f*g)'(z)} + 2\left\{1 - \frac{z(f*g)'(z)}{(f*g)(z)}\right\}\right)\right| < A \ \ (z \in \mathbb{U}),$$

$$\left|\frac{((f*g)(z))^2}{z^2(f*g)'(z)}\left(\frac{z(f*g)''(z)}{(f*g)'(z)} + 2\left\{1 - \frac{z(f*g)'(z)}{(f*g)(z)}\right\}\right)\right| < \frac{A}{(1+A)^2} \ \ (z \in \mathbb{U}),$$

$$\left|\frac{\frac{z(f*g)''(z)}{(f*g)'(z)} + 2\left\{1 - \frac{z(f*g)'(z)}{(f*g)(z)}\right\}}{\frac{z^2(f*g)'(z)}{((f*g)(z))^2} - 1}\right| < \frac{1}{1+A} \quad (z \in \mathbb{U}),$$

then $f * g$ is univalent in $\mathbb{U}$.

Remark 2.14. For $A = 1$ and $g(z) = \frac{z}{1-z}$ $(z \in \mathbb{U})$, above Corollary 229 coincides with the result [13, Corollary 3.2, p.361] for a function $f \in \mathcal{A}$, and with the result [14, Theorem 1, p. 2135] for the function $f(z) = K_{\nu,c}(z)$, where $K_{\nu,c}(z)$ is a normalized form of generalized Bessel function defined in [3, 21] by

$$K_{\nu,c}(z) = z + \sum_{k=2}^{\infty} \frac{\left(\frac{-c}{4}\right)^{k-1}}{(\nu+1)_{k-1}} \frac{z^k}{(k-1)!} \, (c \in \mathbb{C},\ \nu \in \mathbb{R},\ \nu \neq -1, -2, ...; z \in \mathbb{U}) .$$

3. Concluding remark

By considering special form of the function g, from our main results, we may obtain results involving several linear operators of the class $\mathcal{A}$, some of the known linear operators are mentioned in the *Introduction* section. We give here only the results giving varied univalency conditions of the Dzoik-Srivastava operator by taking $g(z) = z\ {}_pF_q(\alpha_1, ..., \alpha_p; \beta_1, ..., \beta_q; z)$. Results giving univalency conditions of other linear operators mentioned in the *Introduction* section may similarly be obtained by taking $g(z) = G_{a,b}(z)$, $z\ {}_2F_1\ (2, 1; 2-\lambda; z)$, $z\ {}_2F_1\ (\gamma+1, 1; \alpha+\gamma+1; z)$ and $\Phi^m_{\lambda,\mu}(z)$, respectively, in the Corollaries 2.3, 2.9 and 2.13.

Corollary 3.1. *Let $f \in \mathcal{A}$ and ${}_pH_q([\alpha_1]) f$ be defined by (1.4) with*

$$0 \neq \frac{{}_pH_q([\alpha_1]) f(z)}{z} \in \mathbb{C} \text{ in } \mathbb{U}.$$

If

$$\left(\frac{z}{{}_pH_q([\alpha_1]) f(z)}\right)^2 ({}_pH_q([\alpha_1]) f)'(z) \prec \frac{1+Az}{1+Bz} \quad (0 \leq B < A \leq 1; z \in \mathbb{U}),$$

then ${}_pH_q([\alpha_1]) f$ is univalent in $\mathbb{U}$.

Corollary 3.2. *Let $f \in \mathcal{A}$ and ${}_pH_q([\alpha_1]) f$ be defined by (1.4) with*

$$0 \neq ({}_pH_q([\alpha_1]) f)'(z) \cdot \frac{{}_pH_q([\alpha_1]) f(z)}{z} \in \mathbb{C} \text{ in } \mathbb{U}.$$

If

$$\left| \frac{z ({}_pH_q([\alpha_1]) f)''(z)}{({}_pH_q([\alpha_1]) f)'(z)} + 2\left\{1 - \frac{z ({}_pH_q([\alpha_1]) f)'(z)}{{}_pH_q([\alpha_1]) f(z)}\right\} \right| < \frac{A-B}{(1+A)(1+B)}$$
$$(0 \leq B < A \leq 1; z \in \mathbb{U}),$$

then ${}_pH_q([\alpha_1]) f$ is univalent in $\mathbb{U}$.

Corollary 3.3. *Let $f \in \mathcal{A}$ and ${}_pH_q([\alpha_1]) f$ be defined by (1.4) with*

$$0 \neq ({}_pH_q([\alpha_1]) f)'(z) \cdot \frac{{}_pH_q([\alpha_1]) f(z)}{z} \in \mathbb{C} \text{ in } \mathbb{U}.$$

If for $0 < A \leq 1; z \in \mathbb{U}$, any one of the following conditions:

$$\left| \frac{z^2 ({}_pH_q([\alpha_1]) f)'(z)}{({}_pH_q([\alpha_1]) f(z))^2} \left(\frac{z ({}_pH_q([\alpha_1]) f)''(z)}{({}_pH_q([\alpha_1]) f)'(z)} + 2\left\{1 - \frac{z ({}_pH_q([\alpha_1]) f)'(z)}{{}_pH_q([\alpha_1]) f(z)}\right\}\right) \right| < A,$$

$$\left|\frac{\left({}_pH_q\left([\alpha_1]\right)f(z)\right)^2}{z^2\left({}_pH_q\left([\alpha_1]\right)f\right)'(z)}\left(\frac{z\left({}_pH_q\left([\alpha_1]\right)f\right)''(z)}{\left({}_pH_q\left([\alpha_1]\right)f\right)'(z)}+2\left\{1-\frac{z\left({}_pH_q\left([\alpha_1]\right)f\right)'(z)}{{}_pH_q\left([\alpha_1]\right)f(z)}\right\}\right)\right|$$
$$< \frac{A}{(1+A)^2},$$

$$\left|\frac{\frac{z({}_pH_q([\alpha_1])f)''(z)}{({}_pH_q([\alpha_1])f)'(z)}+2\left\{1-\frac{z({}_pH_q([\alpha_1])f)'(z)}{{}_pH_q([\alpha_1])f(z)}\right\}}{\frac{z^2({}_pH_q([\alpha_1])f)'(z)}{({}_pH_q([\alpha_1])f(z))^2}-1}\right|<\frac{1}{1+A},$$

holds, then ${}_pH_q\left([\alpha_1]\right)f$ *is univalent in* $\mathbb{U}$.

Acknowledgment. The authors are grateful to the reviewers for their valuable comments and suggestions which have significantly improved the quality and presentation of the paper in the present enriched form.

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