Stud. Univ. Babeş-Bolyai Math. 70(2025), No. 2, 189–209
DOI: 10.24193/subbmath.2025.2.02

On a unification of Mittag-Leffler function and Wright function

Meera H. Chudasama

Abstract. We introduce here a function that unifies Mittag-Leffler function and Wright function which is referred to here as an UMLW-function. This function turns out to be a solution of an infinite order differential equation. With the aid of this UMLW-function, an integral operator is constructed and shown that it is bounded in Lebesgue measurable space. Further an eigen function property is established for a particular UMLW-function with the help of hyper-Bessel operator and Caputo fractional derivative operator. Some well known functions occur in the illustrations of these properties. At the end, the graphs of this UMLW-function are plotted by suitably specializing the parameters and also compared with the graph of exponential as well as Mittag-Leffler function.

Mathematics Subject Classification (2010): 26A33, 33E12, 40A05.

Keywords: Mittag-Leffler function, Wright function, θ-form differential equation, fractional derivatives and integrals.

1. Introduction and main results

Gosta Mittag-Leffler [15], introduced the function given by

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)},$$

where z is a complex variable and $\alpha \in \mathbb{C}$ with $Re(\alpha) > 0$, which reduces to e^z when $\alpha = 1$.

After some decades, its importance was realized due to its occurrence in many problems of Physics, Chemistry, Biology, and Engineering as a solution of fractional order

Received 26 July 2024; Accepted 22 October 2024.

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differential or integral equations. During the course of time, this function was generalized and studied by many researchers among them Wiman [22], Prabhakar [16], Kiryakova [11], Shukla and Prajapati [19], Srivastava and Tomovski [21], Garra and Polito [10] are worth mentioning.
With the aid of L-exponential function

$$e_k(z) = \sum_{n=0}^{\infty} \frac{z^n}{(n!)^{k+1}} = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma^{k+1}(n+1)} \tag{1.1}$$

(of order k) due to Ricci and Tavkhelidze [18], Garra and Polito [10] defined and studied a generalization of (1.1) in the form:

$$E_{\alpha;\nu,\gamma}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma^{\alpha+1}(\nu n+\gamma)}, \tag{1.2}$$

wherein $x \in \mathbb{R}, \alpha > -1, \nu > 0$, and $\gamma \in \mathbb{R}$, which they called $\alpha-$Mittag-Leffler function. Noticing the *rapid* convergence of the series due to Sikkema [20]

$$\sum_{n=1}^{\infty} \frac{z^n}{n!^n} = \sum_{n=1}^{\infty} \frac{z^n}{\Gamma^n(n+1)}, \tag{1.3}$$

we propose a more general series structure which would also encompass another function, namely the Wright function [12]

$$W_{\lambda,\mu}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\lambda n+\mu) n!}, \ \lambda > -1, \mu \in \mathbb{C}. \tag{1.4}$$

This was introduced and investigated by the eminent British mathematician E. Maitland Wright (in a series of notes starting from 1933) in the framework of the asymptotic theory of partitions [12].
Aiming at the unification of the series given by (1.2), (1.3) and (1.4), we introduce the function defined by the power series as follows.

Definition 1.1. For $Re(\alpha\delta) \geq 0, Re(\beta\delta + \sigma\gamma - \frac{\delta}{2} - r + 1) > 0, \ \alpha, \sigma \neq 0,$ and $\mu, z \in \mathbb{C}$,

$$\mathbb{E}_{\alpha,\beta,\delta}^{\sigma,\nu,\gamma}(\mu, r; z) = \sum_{n=0}^{\infty} \frac{(\mu)_{rn}}{\Gamma^{\delta n}(\alpha n+\beta)\ \Gamma^{\gamma}(\sigma n+\nu)} \frac{z^n}{n!}, \tag{1.5}$$

where $(\mu)_{rn} = \frac{\Gamma(\mu+rn)}{\Gamma(\mu)}$ is generalized Pochhammer symbol.

We shall henceforth referred to this function as UMLW-function.

Remark 1.2. Chudasama M. H. and Dave B. I. studied ℓ-Hypergeometric function, its particular cases and their q-analogues in [4, 3, 7, 5, 6, 2]. In context of the study of these, when $r, \sigma, \gamma, \alpha \in \mathbb{N} \cup \{0\}$ with $z^* = \frac{r^r}{\sigma^{\sigma\gamma}} \frac{z}{\alpha^{\alpha}\ \Gamma^{\delta}(\beta)}$, we have

$$\mathbb{E}_{\alpha,\beta,\delta}^{\sigma,\nu,\gamma}(\mu, r; z) = \frac{1}{\Gamma^{\gamma}(\nu)}$$

$$\times\ {}_rH_{\sigma\gamma}^{\alpha} \left[\begin{matrix} \frac{\mu}{r}, & \frac{\mu+1}{r}, & \ldots, & \frac{\mu+r-1}{r}; & & z^* \\ \left(\frac{\nu}{\sigma}\right)^{\gamma}, & \left(\frac{\nu+1}{\sigma}\right)^{\gamma}, & \ldots, & \left(\frac{\nu+\sigma-1}{\sigma}\right)^{\gamma}; & \left(\frac{\beta}{\alpha}, \ \frac{\beta+1}{\alpha}, \ \ldots, \ \frac{\beta+\alpha-1}{\alpha} : \delta\right); & \end{matrix} \right].$$

The particular cases of (1.5) are worth mentioning; all of them are corresponding to the common choice $\delta = 0$. The substitutions for the other parameters involved are indicated in each special case below.

1. Exponential function : $(\gamma = \mu = r = \nu = \sigma = 1)$

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n+1)} = \mathbb{E}_0^{1,1,1}(1,1;z).$$

2. Mittag-Leffler function [15] : $(\gamma = \mu = r = \nu = 1)$

$$E_\sigma(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\sigma n+1)} = \mathbb{E}_0^{\sigma,1,1}(1,1;z).$$

3. Wiman function [22] : $(\gamma = \mu = r = 1)$

$$E_{\sigma,\nu}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\sigma n+\nu)} = \mathbb{E}_0^{\sigma,\nu,1}(1,1;z).$$

4. Wright function [12] : $(r = 0, \gamma = 1, \sigma \neq 0,\ \nu$ arbitrary$)$

$$W_{\sigma,\nu}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\sigma n+\nu)n!} = \mathbb{E}_0^{\sigma,\nu,1}(\mu,0;z).$$

5. Prabhakar's function [16] : $(r = \gamma = 1)$

$$E_{\sigma,\nu}^{\mu}(z) = \sum_{n=0}^{\infty} \frac{(\mu)_n\ z^n}{\Gamma(\sigma n+\nu)n!} = \mathbb{E}_0^{\sigma,\nu,1}(\mu,1;z).$$

6. Cosine function : $(\gamma = \nu = \mu = r = 1, \sigma = 2,\ z$ is replaced by $-z^2)$

$$\cos(\sqrt{z}) = \sum_{n=0}^{\infty} \frac{(-1)^n z^n}{\Gamma(2n+1)} = \mathbb{E}_0^{2,1,1}(1,1;-z).$$

7. Bessel Maitland function [14] : $(\gamma = 1, r = 0,\ \nu$ is replaced by $\nu+1$, z is replaced by $-z)$

$$J_\nu^\mu(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{\Gamma(\sigma n+\nu+1)\ n!} = \mathbb{E}_0^{\sigma,\nu+1,1}(\mu,0;-z).$$

8. Mainardi's functions [12] : $(\gamma = 1, r = 0, \sigma$ is replaced by $-\sigma$ *with* $0 < \sigma < 1)$ $(\nu = 0$ in $F_\sigma(z))$ $(\nu = 1-\sigma$ in $M_\sigma(z))$

$$F_\sigma(z) = \sum_{n=1}^{\infty} \frac{(-z)^n}{\Gamma(-\sigma n)n!} = \mathbb{E}_0^{-\sigma,0,1}(\mu,0;z);$$

$$M_\sigma(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{\Gamma(-\sigma n+1-\sigma)n!} = \mathbb{E}_0^{-\sigma,1-\sigma,1}(\mu,0;z).$$

9. Kiryakova's function [11] : ($\mu = r = 1, \gamma = m \in \mathbb{N}$, in (1.5); and putting $\mu_1 = \cdots = \mu_m = \nu, \frac{1}{\rho_1} = \cdots = \frac{1}{\rho_m} = \sigma$ in [11, Eq.(13)])

$$E_{\sigma,\nu}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma^m(\sigma n + \nu)} = \mathbb{E}_0^{\sigma,\nu,m}(1, 1; z).$$

10. Garra and Polito's function [10] : ($\mu = r = 1, z = x, \nu = \gamma, \gamma = \alpha + 1, \sigma = \nu$)

$$E_{\alpha;\nu,\gamma}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma^{\alpha+1}(\nu n + \gamma)} = \mathbb{E}_0^{\nu,\gamma,\alpha+1}(1, 1; x).$$

11. Shukla and Prajapati's function [19] : ($\gamma = 1, r \in \mathbb{N} \cup (0, 1)$)

$$E_{\sigma,\nu}^{\mu,r}(z) = \sum_{n=0}^{\infty} \frac{(\mu)_{rn}\ z^n}{\Gamma(\sigma n + \nu)\ n!} = \mathbb{E}_0^{\sigma,\nu,1}(\mu, r; z).$$

12. Srivastava and Tomovski's function [21] : ($\gamma = 1, Re(r) > 0$)

$$E_{\sigma,\nu}^{\mu,r}(z) = \sum_{n=0}^{\infty} \frac{(\mu)_{rn}\ z^n}{\Gamma(\sigma n + \nu)\ n!} = \mathbb{E}_0^{\sigma,\nu,1}(\mu, r; z).$$

Now as a main results, the domain of convergence, differential equation and the integral operator of the UMLW-function are discussed.

Theorem 1.3. *For $Re(\alpha\delta) \geq 0, Re(\beta\delta + \sigma\gamma - \frac{\delta}{2} - r + 1) > 0$, and $\alpha, \sigma \neq 0$, the UMLW-function (1.5) is an entire function.*

Proof. We have

$$\mathbb{E}_{\alpha,\beta,\delta}^{\sigma,\nu,\gamma}(\mu, r; z) = \sum_{n=0}^{\infty} \frac{(\mu)_{rn}}{\Gamma^{\delta n}(\alpha n + \beta)\Gamma^{\gamma}(\sigma n + \nu)} \frac{z^n}{n!} = \sum_{n=0}^{\infty} \varphi_n z^n \text{ (say)}. \tag{1.6}$$

Using Cauchy-Hadamard formula:

$$\frac{1}{R} = \lim_{n\to\infty} \sup \sqrt[n]{|\varphi_n|} = \lim_{\mathrm{n}\to\infty} \sup \left| \frac{(\mu)_{\mathrm{rn}}}{\Gamma^{\delta \mathrm{n}}(\alpha \mathrm{n} + \beta)\Gamma^{\gamma}(\sigma \mathrm{n} + \nu)\ \mathrm{n}!} \right|^{\frac{1}{\mathrm{n}}},$$

and then applying Stirlng's asymptotic Formula [9]

$$\Gamma(an + b) \quad \sim \quad \sqrt{2\pi}\ e^{-(an+b)}\ (an + b)^{an+b-\frac{1}{2}} \tag{1.7}$$

for large n and for $a = r, \alpha, \sigma$ and $b = \mu, \beta, \nu$ respectively we have

$$\begin{aligned} \frac{1}{R} \quad &= \quad \lim_{n\to\infty} \sup \left| \frac{\Gamma(\mu + rn)}{\Gamma(\mu)\ \Gamma^{\delta n}(\alpha n + \beta)\ \Gamma^{\gamma}(\sigma n + \nu)\ \Gamma(n + 1)} \right|^{\frac{1}{n}} \qquad (1.8) \\ &\sim \quad \lim_{n\to\infty} \sup \Bigg\{ \left| \frac{\sqrt{2\pi}\ e^{-(\mu+rn)}(\mu + rn)^{\mu+rn-\frac{1}{2}}}{\sqrt{2\pi}\ e^{-(\mu)}(\mu)^{\mu-\frac{1}{2}}} \right|^{\frac{1}{n}} \\ &\quad \times \left| \frac{1}{\sqrt{2\pi}\ e^{-(\alpha n+\beta)}(\alpha n + \beta)^{\alpha n+\beta-\frac{1}{2}}} \right|^{\delta} \end{aligned}$$

$$
\begin{aligned}
& \times\left|\frac{1}{\sqrt{2\pi}\ e^{-(\sigma n+\nu)}(\sigma n+\nu)^{\sigma n+\nu-\frac{1}{2}}}\right|^{\frac{\gamma}{n}}\left|\frac{1}{\sqrt{2\pi}\ e^{-(n+1)}(n+1)^{n+1-\frac{1}{2}}}\right|^{\frac{1}{n}}\Bigg\} \\
= \ & \lim_{n\to\infty}\sup\Bigg\{|e^{-r}|\left|\frac{(rn)^{r+\frac{\mu}{n}-\frac{1}{2n}}\left(1+\frac{\mu}{rn}\right)^{\frac{\mu}{n}-\frac{1}{2n}}\left(1+\frac{\mu}{rn}\right)^{r}}{\mu^{\frac{\mu}{n}-\frac{1}{2n}}}\right| \\
& \times\left|\frac{e^{\beta\delta}}{(\sqrt{2\pi})^{\delta}}\right|\left|\frac{e^{\alpha\delta n}}{(\alpha n)^{\delta(\alpha n+\beta-\frac{1}{2})}\left(1+\frac{\beta}{\alpha n}\right)^{\alpha\delta n}\left(1+\frac{\beta}{\alpha n}\right)^{\beta\delta-\frac{\delta}{2}}}\right| \\
& \times|e^{\sigma\gamma}|\left|\frac{e^{\frac{\gamma\nu}{n}}}{(\sqrt{2\pi})^{\frac{\gamma}{n}}(\sigma n)^{\sigma\gamma+\frac{\gamma\nu}{n}-\frac{\gamma}{2n}}\left(1+\frac{\nu}{\sigma n}\right)^{\sigma\gamma+\frac{\gamma\nu}{n}-\frac{\gamma}{2n}}}\right| \\
& \times\left|\frac{e^{1+\frac{1}{n}}}{(\sqrt{2\pi})^{\frac{1}{n}}n^{1+\frac{1}{n}-\frac{1}{2n}}\left(1+\frac{1}{n}\right)^{1-\frac{1}{n}}}\right|\Bigg\} \\
= \ & \left|\frac{e^{\beta\delta+\sigma\gamma-r+1}\ r^{r}}{(\sqrt{2\pi})^{\delta}\sigma^{\sigma\gamma}\alpha^{\beta\delta-\frac{\delta}{2}}}\right|\lim_{n\to\infty}\sup\Bigg\{\left|\frac{n^{r}r^{\frac{\mu}{n}-\frac{1}{2n}}(n^{\frac{1}{n}})^{\mu-\frac{1}{2}}\left(1+\frac{\mu}{rn}\right)^{\frac{\mu}{n}-\frac{1}{2n}}\left(1+\frac{\mu}{rn}\right)^{r}}{\mu^{\frac{\mu}{n}-\frac{1}{2n}}}\right| \\
& \times\left|\frac{\left(\frac{e}{\alpha}\right)}{n}\right|^{\alpha\delta n}\left|\frac{1}{n^{\beta\delta-\frac{\delta}{2}}\left(1+\frac{\beta}{\alpha n}\right)^{\alpha\delta n}\left(1+\frac{\beta}{\alpha n}\right)^{\beta\delta-\frac{\delta}{2}}}\right| \\
& \times\left|\frac{e^{\frac{\gamma\nu}{n}}}{(\sqrt{2\pi})^{\frac{\gamma}{n}}\sigma^{\frac{\gamma\nu}{n}-\frac{\gamma}{2n}}\ n^{\sigma\gamma}\ (n^{\frac{1}{n}})^{\gamma\nu-\frac{\gamma}{2n}}}\right| \\
& \times\left|\frac{1}{\left(1+\frac{\nu}{\sigma n}\right)^{\sigma\gamma}\left(1+\frac{\nu}{\sigma n}\right)^{\frac{\sigma\gamma}{n}-\frac{\gamma}{n}}}\right|\left|\frac{e^{\frac{1}{n}}\left(1+\frac{1}{n}\right)^{\frac{1}{n}-1}}{(\sqrt{2\pi})^{\frac{1}{n}}\ n(n^{\frac{1}{n}})^{\frac{1}{2}}}\right|\Bigg\} \qquad (1.9) \\
= \ & \left|\frac{e^{\beta\delta+\sigma\gamma-r+1}r^{r}}{(\sqrt{2\pi})^{\delta}\sigma^{\sigma\gamma}\alpha^{\beta\delta-\frac{\delta}{2}}}\right|\left|\frac{1}{e^{\beta\delta+1}}\right|\lim_{n\to\infty}\left\{\left|\frac{\left(\frac{e}{\alpha}\right)}{n}\right|^{\alpha\delta n}\left|\frac{1}{n^{\beta\delta+\sigma\gamma-\frac{\delta}{2}-r+1}}\right|\right\} \\
= \ & 0,
\end{aligned}
$$

provided that $Re(\alpha\delta)\geq 0, Re(\beta\delta+\gamma\sigma-\frac{\delta}{2}-r+1)>0$, and $\alpha,\sigma\neq 0$.
Thus, $R=\infty$. □

Remark 1.4.

1. We stick to the conditions proved in Theorem 1.3 throughout the article unless it is specified.
2. The series $\sum\varphi_n z^n$ thus, converges uniformly in any compact subset of $\mathbb{C}$.

Next, to obtain the differential equation we define an operator as follows.

Definition 1.5. Let $f(z) = \sum_{n=1}^{\infty} a_n z^n$, $0 \neq z \in \mathbb{C}$, $p \in \mathbb{N} \cup \{0\}$ and $\alpha \in \mathbb{C}$ with $Re(\alpha) > 0$. Define [4, 7]

$$ {}_p\Delta_\alpha^\theta(f(z)) = \begin{cases} \sum_{n=1}^{\infty} a_n (\alpha)_{n-1}^p (\theta+\alpha-1)^{pn} z^n, & phif \quad p \in \mathbb{N} \\ f(z), & phif \quad p = 0 \end{cases}, \tag{1.10} $$

where $\theta = z\frac{d}{dz}$ and $(\theta+c)^n = \underbrace{(\theta+c)(\theta+c)\ldots(\theta+c)}_{\text{n times}}$, c is a constant.

Using this operator, we have now obtained the differential equation of the UMLW-function in the following theorem.

Theorem 1.6. *If* $\alpha = 1, \gamma \in \mathbb{N} \cup \{0\}, \sigma, r, \delta \in \mathbb{N}$ *and* $\beta \neq 0, -1, -2, \ldots$ *then*

$$ w = \mathbb{E}_{1,\beta,\delta}^{\sigma,\nu,\gamma}(\mu, r; z) = \sum_{n=0}^{\infty} \frac{(\mu)_{rn}}{\Gamma^{\delta n}(n+\beta)\Gamma^\gamma(\sigma n+\nu)} \frac{z^n}{n!} $$

satisfies the differential equation :

$$ \left\{ \left[\left\{ {}_\delta\Delta_\beta^\theta \right\} \left(\prod_{i=0}^{\sigma-1} \left(\theta + \frac{\nu+i}{\sigma} - 1 \right)^\gamma \right) \theta \right] - z_* \prod_{j=0}^{r-1} \left[\theta + \frac{\mu+j}{r} \right] \right\} w = 0, \tag{1.11} $$

where $z_* = \frac{r^r}{\sigma^{\sigma\gamma}} \frac{z}{\Gamma^\delta(\beta)}$.

In order to prove this theorem, we first prove the following lemma which allows us to actually apply the operator ${}_\delta\Delta_\beta^\theta$ onto the operand w.
For the sake of brevity, we put

$$ \left\{ {}_\delta\Delta_\beta^\theta \right\} \prod_{i=0}^{\sigma-1} \left(\theta + \frac{\nu+i}{\sigma} - 1 \right)^\gamma \theta = {}_{\beta,\delta}\Theta_{\sigma,\nu,\gamma}. $$

In this notation, we have

Lemma 1.7. *If* $\alpha = 1, \gamma \in \mathbb{N} \cup \{0\}, \sigma, r, \delta \in \mathbb{N}$ *and* $\beta \neq 0, -1, -2, \ldots$ *with*

$$ w = \mathbb{E}_{1,\beta,\delta}^{\sigma,\nu,\gamma}(\mu, r; z) = \sum_{n=0}^{\infty} \frac{(\mu)_{rn}}{\Gamma^{\delta n}(n+\beta)\Gamma^\gamma(\sigma n+\nu)} \frac{z^n}{n!} $$

and

$$ {}_{\beta,\delta}\Theta_{\sigma,\nu,\gamma}(w) = \sum_{n=0}^{\infty} f_n(\mu, r, \beta, \delta, \sigma, \nu, \gamma; z) \ \textit{(say)}, $$

then the operator ${}_{\beta,\delta}\Theta_{\sigma,\nu,\gamma}$ *is applicable to* w *provided that the series*

$$ \sum_{n=0}^{\infty} \varphi_n \ f_n(\mu, r, \beta, \delta, \sigma, \nu, \gamma; z) $$

converges (cf. [20, Definition 11, p.20]*).*

Proof. We first write

$$\frac{1}{\Gamma^{\gamma}(\sigma n+\nu)}=\frac{1}{\Gamma^{\gamma}(\nu)}\frac{\Gamma^{\gamma}(\nu)}{\Gamma^{\gamma}(\sigma n+\nu)}=\frac{1}{\Gamma^{\gamma}(\nu)}\left[\frac{1}{(\nu)_{\sigma n}}\right]^{\gamma},$$

then applying the formula [17, Lemma 6, p. 22]

$$(a)_{kn}=k^{kn}\left(\frac{a}{k}\right)_n\cdots\left(\frac{a+k-1}{k}\right)_n,$$

for $a=\mu,\nu$ and $k=r,\sigma$ respectively, we have

$$\begin{aligned} w &= \sum_{n=0}^{\infty}\frac{(\mu)_{rn}\ z^n}{\Gamma^{\delta n}(n+\beta)\Gamma^{\gamma}(\sigma n+\nu)\ n!}\frac{\Gamma^{\delta n}(\beta)}{\Gamma^{\delta n}(\beta)}\\ &= \sum_{n=0}^{\infty}\frac{(\mu)_{rn}}{(\beta)_n^{\delta n}\Gamma^{\gamma}(\sigma n+\nu)\ n!}\frac{z^n}{\Gamma^{\delta n}(\beta)}\\ &= \frac{1}{\Gamma^{\gamma}(\nu)}\sum_{n=0}^{\infty}\frac{\left(\frac{\mu}{r}\right)_n\cdots\left(\frac{\mu+r-1}{r}\right)_n}{\left(\frac{\nu}{\sigma}\right)_n^{\gamma}\cdots\left(\frac{\nu+\sigma-1}{\sigma}\right)_n^{\gamma}\ n!}\frac{r^{rn}\ z^n}{\Gamma^{\delta n}(\beta)\sigma^{\sigma\gamma n}(\beta)_n^{\delta n}}. \end{aligned} \tag{1.12}$$

Now take

$$\frac{r^r\ z}{\sigma^{\sigma^{\gamma}}\ \Gamma^{\delta}(\beta)}=z_*,$$

then

$$w=\frac{1}{\Gamma^{\gamma}(\nu)}\sum_{n=0}^{\infty}\frac{\left(\frac{\mu}{r}\right)_n\cdots\left(\frac{\mu+r-1}{r}\right)_n}{\left(\frac{\nu}{\sigma}\right)_n^{\gamma}\cdots\left(\frac{\nu+\sigma-1}{\sigma}\right)_n^{\gamma}}\frac{z_*^n}{(\beta)_n^{\delta n}\ n!}.$$

Now consider

$$\begin{aligned} &{}_{\beta,\delta}\Theta_{\sigma,\nu,\gamma}(w)\\ &= \frac{1}{\Gamma^{\gamma}(\nu)}\Bigg\{\left[\left\{{}_{\delta}\Delta_{\beta}^{\theta}\right\}\left(\prod_{i=0}^{\sigma-1}\left(\theta+\frac{\nu+i}{\sigma}-1\right)^{\gamma}\right)\right]\\ &\quad\times\sum_{n=1}^{\infty}\frac{\left(\frac{\mu}{r}\right)_n\cdots\left(\frac{\mu+r-1}{r}\right)_n}{\left(\frac{\nu}{\sigma}\right)_n^{\gamma}\cdots\left(\frac{\nu+\sigma-1}{\sigma}\right)_n^{\gamma}}\frac{z_*^n}{(\beta)_n^{\delta n}\ (n-1)!}\Bigg\}\\ &= \frac{1}{\Gamma^{\gamma}(\nu)}\left\{\left\{{}_{\delta}\Delta_{\beta}^{\theta}\right\}\left(\theta+\frac{\nu-\sigma}{\sigma}\right)^{\gamma}\left(\theta+\frac{\nu-\sigma+1}{\sigma}\right)^{\gamma}\cdots\left(\theta+\frac{\nu-1}{\sigma}\right)^{\gamma}\right\}\\ &\quad\times\sum_{n=1}^{\infty}\frac{\left(\frac{\mu}{r}\right)_n\cdots\left(\frac{\mu+r-1}{r}\right)_n}{\left(\frac{\nu}{\sigma}\right)_n^{\gamma}\cdots\left(\frac{\nu+\sigma-1}{\sigma}\right)_n^{\gamma}}\frac{z_*^n}{(\beta)_n^{\delta n}\ (n-1)!}\\ &= \frac{1}{\Gamma^{\gamma}(\nu)}\left\{\left\{{}_{\delta}\Delta_{\beta}^{\theta}\right\}\left(\theta+\frac{\nu-\sigma}{\sigma}\right)^{\gamma}\left(\theta+\frac{\nu-\sigma+1}{\sigma}\right)^{\gamma}\cdots\left(\theta+\frac{\nu-2}{\sigma}\right)^{\gamma}\right\}\\ &\quad\times\sum_{n=1}^{\infty}\frac{\left(\frac{\mu}{r}\right)_n\cdots\left(\frac{\mu+r-1}{r}\right)_n}{\left(\frac{\nu}{\sigma}\right)_n^{\gamma}\cdots\left(\frac{\nu+\sigma-1}{\sigma}\right)_n^{\gamma}}\frac{z_*^n}{(\beta)_n^{\delta n}\ (n-1)!}\left[n+\frac{\nu-1}{\sigma}\right]^{\gamma}\\ &= \frac{1}{\Gamma^{\gamma}(\nu)}\left\{\left\{{}_{\delta}\Delta_{\beta}^{\theta}\right\}\left(\theta+\frac{\nu-\sigma}{\sigma}\right)^{\gamma}\left(\theta+\frac{\nu-\sigma+1}{\sigma}\right)^{\gamma}\cdots\left(\theta+\frac{\nu-2}{\sigma}\right)^{\gamma}\right\} \end{aligned}$$

$$\times \sum_{n=1}^{\infty} \frac{\left(\frac{\mu}{r}\right)_n \cdots \left(\frac{\mu+r-1}{r}\right)_n}{\left(\frac{\nu}{\sigma}\right)_n^{\gamma} \cdots \left(\frac{\nu+\sigma-1}{\sigma}\right)_n^{\gamma}} \frac{z_*^n}{(\beta)_n^{\delta n}\ (n-1)!} \left[\frac{\nu+\sigma n-1}{\sigma}\right]^{\gamma}.$$

Continuing in this manner, we finally arrive at

$$\begin{aligned}
&{}_{\beta,\delta}\Theta_{\sigma,\nu,\gamma}(w) \\
&= \frac{1}{\Gamma^{\gamma}(\nu)}\ \left\{{}_{\delta}\Delta_{\beta}^{\theta}\right\} \left\{\sum_{n=0}^{\infty} \frac{\left(\frac{\mu}{r}\right)_n \cdots \left(\frac{\mu+r-1}{r}\right)_n}{\left(\frac{\nu}{\sigma}\right)_n^{\gamma} \cdots \left(\frac{\nu+\sigma-1}{\sigma}\right)_n^{\gamma}} \frac{z_*^n}{(\beta)_n^{\delta n}\ ((n-1)!}\right\} \\
&\quad \times \left(\frac{\nu+\sigma n-1}{\sigma}\right)^{\gamma} \left(\frac{\nu+\sigma n-2}{\sigma}\right)^{\gamma} \cdots \left(\frac{\nu+\sigma n-\sigma}{\sigma}\right)^{\gamma} \\
&= \frac{1}{\Gamma^{\gamma}(\nu)}\ \left\{{}_{\delta}\Delta_{\beta}^{\theta}\right\} \left\{\sum_{n=1}^{\infty} \frac{\left(\frac{\mu}{r}\right)_n \cdots \left(\frac{\mu+r-1}{r}\right)_n}{\left(\frac{\nu}{\sigma}\right)_{n-1}^{\gamma} \cdots \left(\frac{\nu+\sigma-1}{\sigma}\right)_{n-1}^{\gamma}} \frac{z_*^n}{(\beta)_n^{\delta n}\ (n-1)!}\right\} \\
&= \frac{1}{\Gamma^{\gamma}(\nu)} \sum_{n=1}^{\infty} \frac{\left(\frac{\mu}{r}\right)_n \cdots \left(\frac{\mu+r-1}{r}\right)_n}{\left(\frac{\nu}{\sigma}\right)_{n-1}^{\gamma} \cdots \left(\frac{\nu+\sigma-1}{\sigma}\right)_{n-1}^{\gamma}} \frac{(\beta)_{n-1}^{\delta}\ (\theta+\beta-1)^{\delta n}\ z_*^n}{(\beta)_n^{\delta n}\ (n-1)!}. \qquad (1.13)
\end{aligned}$$

Now, observe that for $\theta = z\frac{d}{dz}$,

$$\begin{aligned}
(\theta+\beta-1)\ z_*^n &= (\theta+\beta-1)\left(\frac{r^r\ z}{\sigma^{\sigma^{\gamma}}\ \Gamma^{\delta}(\beta)}\right)^n \\
&= \left(\frac{r^r}{\sigma^{\sigma^{\gamma}}\ \Gamma^{\delta}(\beta)}\right)^n\ (\theta+\beta-1)z^n \\
&= \left(\frac{r^r}{\sigma^{\sigma^{\gamma}}\ \Gamma^{\delta}(\beta)}\right)^n\ (z\ nz^{n-1}+\beta z^n - z^n) \\
&= \left(\frac{r^r}{\sigma^{\sigma^{\gamma}}\ \Gamma^{\delta}(\beta)}\right)^n\ (n+\beta-1)\ z^n \\
&= (n+\beta-1)\ z_*^n.
\end{aligned}$$

Similarly $(\theta+\beta-1)^2\ z_*^n = (n+\beta-1)^2\ z_*^n$ and in general, for $\delta, n \in \mathbb{N}\cup\{0\}$, $(\theta+\beta-1)^{\delta n}\ z_*^n = (n+\beta-1)^{\delta n}\ z_*^n$. Using this in (1.13), we have

$$\begin{aligned}
{}_{\beta,\delta}\Theta_{\sigma,\nu,\gamma}(w) &= \frac{1}{\Gamma^{\gamma}(\nu)} \sum_{n=1}^{\infty} \frac{\left(\frac{\mu}{r}\right)_n \cdots \left(\frac{\mu+r-1}{r}\right)_n}{\left(\frac{\nu}{\sigma}\right)_{n-1}^{\gamma} \cdots \left(\frac{\nu+\sigma-1}{\sigma}\right)_{n-1}^{\gamma}} \frac{(\beta)_{n-1}^{\delta}(n+\beta-1)^{\delta n}\ z_*^n}{(\beta)_n^{\delta n}\ (n-1)!} \\
&= \frac{1}{\Gamma^{\gamma}(\nu)} \sum_{n=1}^{\infty} \frac{\left(\frac{\mu}{r}\right)_n \cdots \left(\frac{\mu+r-1}{r}\right)_n}{\left(\frac{\nu}{\sigma}\right)_{n-1}^{\gamma} \cdots \left(\frac{\nu+\sigma-1}{\sigma}\right)_{n-1}^{\gamma}} \frac{z_*^n}{(\beta)_{n-1}^{\delta n-\delta}\ (n-1)!} \\
&= \frac{z_*}{\Gamma^{\gamma}(\nu)} \sum_{n=0}^{\infty} \frac{\left(\frac{\mu}{r}\right)_n \cdots \left(\frac{\mu+r-1}{r}\right)_n}{\left(\frac{\nu}{\sigma}\right)_n^{\gamma} \cdots \left(\frac{\nu+\sigma-1}{\sigma}\right)_n^{\gamma}}\ \frac{(\mu+rn)_r}{r^r} \frac{z_*^n}{(\beta)_n^{\delta n}\ n!} \qquad (1.14) \\
&= \sum_{n=0}^{\infty}\ f_n(\mu, r, \beta, \delta, \sigma, \nu, \gamma; z)\ \text{(say)}.
\end{aligned}$$

To complete the proof of lemma, it remains to show that

$$\sum_{n=0}^{\infty} \varphi_n \; f_n(\mu, r, \beta, \delta, \sigma, \nu, \gamma; z) = \sum_{n=0}^{\infty} \frac{(\mu)_{rn}^2 \; (\mu + rn)_r}{\Gamma^{2\delta n}(n+\beta) \; \Gamma^{2\gamma}(\sigma n + \nu)} \; \frac{z_*^{n+1}}{(n!)^2}$$

is convergent.
For that take

$$\xi_n \quad = \quad \frac{(\mu)_{rn}^2 \; (\mu + rn)_r}{\Gamma^{2\delta n}(n+\beta) \; \Gamma^{2\gamma}(\sigma n + \nu) \; (n!)^2}.$$

Using Cauchy Hadamard formula:

$$\frac{1}{R} = \lim_{n\to\infty} \sup \sqrt[n]{|\xi_n|} = \lim_{\mathrm{n}\to\infty} \sup \left| \frac{(\mu)_{\mathrm{rn}}^2 \; (\mu + \mathrm{rn})_{\mathrm{r}}}{\Gamma^{2\delta \mathrm{n}}(\mathrm{n}+\beta) \; \Gamma^{2\gamma}(\sigma \mathrm{n} + \nu) \; (\mathrm{n}!)^2} \right|^{\frac{1}{\mathrm{n}}},$$

and then applying Stirlng's asymptotic formula (1.7), we have

$$\begin{aligned} \frac{1}{R} \quad &= \quad \lim_{n\to\infty} \sup \left| \frac{\Gamma^2(\mu + \mathrm{rn}) \; \Gamma(\mu + \mathrm{rn} + \mathrm{r})}{\Gamma^2(\mu) \; \Gamma(\mathrm{r}) \; \Gamma^{2\delta\mathrm{n}}(\mathrm{n}+\beta) \; \Gamma^{2\gamma}(\sigma\mathrm{n}+\nu) \; \Gamma^2(\mathrm{n}+1)} \right|^{\frac{1}{\mathrm{n}}} \\ &= \quad \lim_{n\to\infty} \sup \left| \frac{\Gamma(\mu + \mathrm{rn}) \; \Gamma^{\frac{1}{2}}(\mu + \mathrm{rn} + \mathrm{r})}{\Gamma(\mu) \; \Gamma^{\frac{1}{2}}(\mathrm{r}) \; \Gamma^{\delta\mathrm{n}}(\mathrm{n}+\beta) \; \Gamma^{\gamma}(\sigma\mathrm{n}+\nu) \; \Gamma(\mathrm{n}+1)} \right|^{\frac{2}{\mathrm{n}}}. \end{aligned}$$

Proceeding in the similar manner from (1.8) to (1.9), we get

$$\begin{aligned} \frac{1}{R} \quad &\sim \quad \left| \frac{e^{2(\beta\delta + \sigma\gamma - r + 1)} r^{2r}}{\Gamma^2(\mu)(\sqrt{2\pi})^{2\delta} \sigma^{2\sigma\gamma}} \right| \left| \frac{1}{e^{2\beta\delta+2}} \right| \lim_{n\to\infty} \left| \frac{e}{n} \right|^{2\delta n} \left| \frac{1}{n^{2(\beta\delta + \sigma\gamma - \frac{\delta}{2} - r + 1)}} \right| \\ &= \quad 0, \end{aligned}$$

provided that $Re(\beta\delta + \gamma\sigma - \frac{\delta}{2} - r + 1) > 0$, and $\alpha, \sigma \neq 0$.
This completes the proof of Lemma. □

Proof. (of Theorem 1.6) From (1.14), we have

$$_{\beta,\delta}\Theta_{\sigma,\nu,\gamma}(w) \quad = \quad \frac{z_*}{\Gamma^{\gamma}(\nu)} \sum_{n=0}^{\infty} \frac{\left(\frac{\mu}{r}\right)_n \cdots \left(\frac{\mu+r-1}{r}\right)_n}{\left(\frac{\nu}{\sigma}\right)_n^{\gamma} \cdots \left(\frac{\nu+\sigma-1}{\sigma}\right)_n^{\gamma}} \; \frac{(\mu + rn)_r}{r^r} \; \frac{z_*^n}{(\beta)_n^{\delta n} \; n!}. \qquad (1.15)$$

On the other hand,

$$\begin{aligned} &z_* \left\{ \prod_{j=0}^{r-1} \left[\theta + \frac{\mu + j}{r} \right] \right\} w \\ &= \quad \frac{z_*}{\Gamma^{\gamma}(\nu)} \left(\theta + \frac{\mu}{r}\right) \ldots \left(\theta + \frac{\mu + r - 2}{r}\right) \sum_{n=0}^{\infty} \frac{\left(\frac{\mu}{r}\right)_n \cdots \left(\frac{\mu+r-1}{r}\right)_n}{\left(\frac{\nu}{\sigma}\right)_n^{\gamma} \cdots \left(\frac{\nu+\sigma-1}{\sigma}\right)_n^{\gamma} \; (\beta)_n^{\delta n} \; n!} \\ &\quad \times \left(\theta + \frac{\mu + r - 1}{r}\right) z_*^n \\ &= \quad \frac{z_*}{\Gamma^{\gamma}(\nu)} \left(\theta + \frac{\mu}{r}\right) \ldots \left(\theta + \frac{\mu + r - 2}{r}\right) \end{aligned}$$

$$
\begin{aligned}
&\times \sum_{n=0}^{\infty} \frac{\left(\frac{\mu}{r}\right)_n \cdots \left(\frac{\mu+r-1}{r}\right)_n}{\left(\frac{\nu}{\sigma}\right)_n^{\gamma} \cdots \left(\frac{\nu+\sigma-1}{\sigma}\right)_n^{\gamma}} \frac{z_*^n}{(\beta)_n^{\delta n}\ n!} \left(\frac{\mu+rn+r-1}{r}\right) \\
= &\ \frac{z_*}{\Gamma^{\gamma}(\nu)} \left(\theta+\frac{\mu}{r}\right) \cdots \left(\theta+\frac{\mu+r-3}{r}\right) \sum_{n=0}^{\infty} \frac{\left(\frac{\mu}{r}\right)_n \cdots \left(\frac{\mu+r-1}{r}\right)_n}{\left(\frac{\nu}{\sigma}\right)_n^{\gamma} \cdots \left(\frac{\nu+\sigma-2}{\sigma}\right)_n^{\gamma}\ (\beta)_n^{\delta n}\ n!} \\
&\times \left(\frac{\mu+rn+r-1}{r}\right)\left(\theta+\frac{\mu+r-2}{r}\right) z_*^n \\
= &\ \frac{z_*}{\Gamma^{\gamma}(\nu)} \left(\theta+\frac{\mu}{r}\right) \cdots \left(\theta+\frac{\mu+r-3}{r}\right) \\
&\times \sum_{n=0}^{\infty} \frac{\left(\frac{\mu}{r}\right)_n \cdots \left(\frac{\mu+r-1}{r}\right)_n}{\left(\frac{\nu}{\sigma}\right)_n^{\gamma} \cdots \left(\frac{\nu+\sigma-1}{\sigma}\right)_n^{\gamma}} \frac{z_*^n}{(\beta)_n^{\delta n}\ n!} \left(\frac{\mu+rn+r-1}{r}\right)\left(\frac{\mu+rn+r-2}{r}\right).
\end{aligned}
$$

Proceeding in this way, we finally arrive at

$$
\begin{aligned}
&z_* \left\{ \prod_{j=0}^{r-1} \left[\theta+\frac{\mu+j}{r}\right] \right\} w \\
&= \frac{z_*}{\Gamma^{\gamma}(\nu)\ r^r} \sum_{n=0}^{\infty} \frac{\left(\frac{\mu}{r}\right)_n \cdots \left(\frac{\mu+r-1}{r}\right)_n}{\left(\frac{\nu}{\sigma}\right)_n^{\gamma} \cdots \left(\frac{\nu+\sigma-1}{\sigma}\right)_n^{\gamma}} \frac{z_*^n}{(\beta)_n^{\delta n}\ n!} \\
&\quad \times(\mu+rn+r-1)(\mu+rn+r-2)\ldots(\mu+rn+1)(\mu+rn) \\
&= \frac{z_*}{\Gamma^{\gamma}(\nu)} \sum_{n=0}^{\infty} \frac{\left(\frac{\mu}{r}\right)_n \cdots \left(\frac{\mu+r-1}{r}\right)_n}{\left(\frac{\nu}{\sigma}\right)_n^{\gamma} \cdots \left(\frac{\nu+\sigma-1}{\sigma}\right)_n^{\gamma}} \frac{(\mu+rn)_r}{r^r} \frac{z_*^n}{(\beta)_n^{\delta n}\ n!}. \qquad (1.16)
\end{aligned}
$$

The differential equation (1.11) now follows from (1.15) and (1.16). □

We next define an integral operator of $\mathbb{E}^{\sigma,\nu,\gamma}_{\alpha,\beta,\delta}(\mu, r; x)$ as follows.

Definition 1.8. For $Re(\nu) > 0$,

$$
\mathcal{I}_{a+}\varphi(x) = \int_a^x (x-y)^{\nu-1} \mathbb{E}^{\sigma,\nu,\gamma}_{\alpha,\beta,\delta}(\mu, r; \lambda(x-y)^{\sigma})\varphi(y)dy. \qquad (1.17)
$$

For this operator, we prove

Theorem 1.9. *The operator $\mathcal{I}_{a+}$ defined in (1.17) is bounded in $L(a,b)$, the space of all Lebesgue measurable functions on finite interval (a,b) and*

$$
\| \mathcal{I}_{a+}\ \varphi \|_1 \le\ M\ \ \| \varphi \|_1,
$$

where

$$
M = \sum_{n=0}^{\infty} \frac{|(\mu)_{rn}|\ |\lambda|^n\ (b-a)^{Re(\nu)+Re(\sigma)n}}{|\Gamma^{\delta n}(\alpha n+\beta)|\ |\Gamma^{\gamma}(\sigma n+\nu)|\ n!\ (Re(\nu)+Re(\sigma)n)}.
$$

We need the following lemma for proving this theorem.

Lemma 1.10. *The series*

$$\sum_{n=0}^{\infty} u_n = \sum_{n=0}^{\infty} \frac{|(\mu)_{rn}|\ |\lambda|^n\ (b-a)^{Re(\nu)+Re(\sigma)n}}{|\Gamma^{\delta n}(\alpha n+\beta)|\ |\Gamma^{\gamma}(\sigma n+\nu)|\ n!\ (Re(\nu)+Re(\sigma)n)}$$

converges absolutely under the convergence conditions as stated in Theorem 1.3.

The proof runs almost parallel to that of Theorem 1.3. Hence we omit the proof.

Proof. (of Theorem 1.9)
From the definition of integral operator (1.17), we have

$$\begin{aligned}\| \mathcal{I}_{a+}\varphi \|_1 &= \int_a^b \left| \int_a^x (x-y)^{\nu-1} \mathbb{E}_{\alpha,\beta,\delta}^{\sigma,\nu,\gamma}(\mu, r; \lambda(x-y)^{\sigma})\varphi(y)dy \right| dx \\ &\leq \int_a^b \int_a^x (x-y)^{Re(\nu)-1} \left| \mathbb{E}_{\alpha,\beta,\delta}^{\sigma,\nu,\gamma}(\mu, r; \lambda(x-y)^{\sigma}) \right| |\varphi(y)|\, dy\ dx.\end{aligned}$$

Changing the order of integration, gives

$$\| \mathcal{I}_{a+}\varphi \|_1 \leq \int_a^b \int_y^b (x-y)^{Re(\nu)-1} \left| \mathbb{E}_{\alpha,\beta,\delta}^{\sigma,\nu,\gamma}(\mu, r; \lambda(x-y)^{\sigma}) \right| dx\ \ |\varphi(y)|\, dy.$$

Now taking $x - y = u$, we get

$$\begin{aligned}\| \mathcal{I}_{a+}\varphi \|_1 &\leq \int_a^b \int_0^{b-y} u^{Re(\nu)-1} \left| \mathbb{E}_{\alpha,\beta,\delta}^{\sigma,\nu,\gamma}(\mu, r; \lambda u^{\sigma}) \right| du\ \ |\varphi(y)|\, dy \\ &\leq \int_a^b \left[\int_0^{b-a} u^{Re(\nu)-1} \left| \mathbb{E}_{\alpha,\beta,\delta}^{\sigma,\nu,\gamma}(\mu, r; \lambda u^{\sigma}) \right| du \right] |\varphi(y)|\, dy.\end{aligned}$$

Using the Definition 1.1 of $\mathbb{E}_{\alpha,\beta,\delta}^{\sigma,\nu,\gamma}(\mu, r; x)$, we obtain

$$\begin{aligned}\| \mathcal{I}_{a+}\varphi \|_1 &\leq \sum_{n=0}^{\infty} \frac{|(\mu)_{rn}|\ |\lambda|^n}{|\Gamma^{\delta n}(\alpha n+\beta)|\ |\Gamma^{\gamma}(\sigma n+\nu)|\ n!} \\ &\quad \times \int_0^{b-a} u^{Re(\nu)+Re(\sigma)n-1} du \int_a^b |\varphi(y)|\, dy \\ &= \sum_{n=0}^{\infty} \frac{|(\mu)_{rn}|\ |\lambda|^n\ (b-a)^{Re(\nu)+Re(\sigma)n}}{|\Gamma^{\delta n}(\alpha n+\beta)|\ |\Gamma^{\gamma}(\sigma n+\nu)|\ n!\ (Re(\nu)+Re(\sigma)n)} \| \varphi \|_1 .\end{aligned}$$

The series on the r.h.s. is of real constants which converges absolutely by Lemma 1.10. Hence denoting its sum by M, the theorem follows. □

2. Other results

In this section, we derive some results involving certain fractional order derivatives and obtain the eigen function property of the UMLW-function. At last, some special cases and graphs of the UMLW-function are compared.

Definition 2.1. The Riemann-Liouville fractional integral (RL-integral) operator of order $\alpha \in \mathbb{C}, \Re(\alpha) > 0$ is defined as [13]

$$I_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\ dt, x > a. \tag{2.1}$$

Definition 2.2. The Riemann-Liouville fractional derivative (RL-derivative) of order $\alpha \in \mathbb{C}, m-1 < Re(\alpha) \leq m, m \in \mathbb{N}$ is defined as [13]

$$D_a^\alpha f(x) = D_a^m I_a^{m-\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} D^m \left\{ \int_a^x (x-t)^{m-\alpha-1} f(t)\ dt \right\}, x > a, \tag{2.2}$$

where $D^m = \frac{d^m}{dx^m}$.

Definition 2.3. The Caputo derivative of order $\alpha \in \mathbb{C}, m-1 < Re(\alpha) \leq m, m \in \mathbb{N}$ is [13]

$$_cD_a^\alpha f(x) = I_a^{m-\alpha} D_a^m f(x) = \frac{1}{\Gamma(m-\alpha)} \int_a^x D^m(f(t))(x-t)^{\alpha-1}\ dt, x > a, \tag{2.3}$$

where $D^m = \frac{d^m}{dx^m}$.

Then following hold true.

$$(1) \qquad I_a^\alpha (x-a)^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} (x-a)^{\beta+\alpha}, \beta > -1, \alpha \geq 0. \tag{2.4}$$

$$(2) \qquad D_a^\alpha (x-a)^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\alpha-\beta+1)} (x-a)^{\beta-\alpha}, \beta > -1, \alpha \geq 0. \tag{2.5}$$

We take $a = 0$ now onwards. We define below hyper-Bessel type operators.

Definition 2.4. For $x \in \mathbb{R} \setminus \{0\}$ and $\ell \in \mathbb{N} \cup \{0\}$, the hyper-Bessel type operators denoted and defined by

$$({}^x\mathbf{I}^\alpha)^n = \underbrace{\mathbf{I}^\alpha x^{-\alpha} \mathbf{I}^\alpha \ldots \mathbf{I}^\alpha x^{-\alpha} \mathbf{I}^\alpha}_{(n+1) \text{ integrals}}, \text{ for n} = 0, 1, 2, \ldots, \tag{2.6}$$

and

$$({}^x\mathbf{D}^\alpha)^{\ell n} = \underbrace{\mathbf{D}^\alpha x^\alpha \mathbf{D}^\alpha \ldots \mathbf{D}^\alpha x^\alpha \mathbf{D}^\alpha}_{(\ell n+1) \text{ derivatives}}, \text{ for n} = 0, 1, 2, \ldots, \tag{2.7}$$

where $\mathbf{I}^\alpha$ denotes the RL-integral and $\mathbf{D}^\alpha$ will be either RL-derivative D^α or Caputo derivative $_cD^\alpha$ defined in (2.1), (2.2) and (2.3) respectively for $a = 0$.

Theorem 2.5. *For UMLW-function (1.5) with* $\alpha,\beta,\delta,\sigma,\nu,\mu \in \mathbb{C}, \eta > 0, x \neq 0$ *and* $\gamma \in \mathbb{N}$*, the hyper-Bessel type operators furnish*

$$({}^{x}\mathbf{I}^{\eta})^{\gamma}\left(x^{\nu-1}\mathbb{E}^{\sigma,\nu,\gamma+1}_{\alpha,\beta,\delta}(\mu,r;x^{\sigma})\right) = x^{\nu+\eta-1}\mathbb{E}^{\sigma,\nu+\eta,\gamma+1}_{\alpha,\beta,\delta}(\mu,r;x^{\sigma}) \tag{2.8}$$

and

$$({}^{x}\mathbf{D}^{\eta})^{\gamma}\left(x^{\nu-1}\mathbb{E}^{\sigma,\nu,\gamma+1}_{\alpha,\beta,\delta}(\mu,r;x^{\sigma})\right) = x^{\nu-\eta-1}\mathbb{E}^{\sigma,\nu-\eta,\gamma+1}_{\alpha,\beta,\delta}(\mu,r;x^{\sigma}), \tag{2.9}$$

where ${}^{x}I^{\eta}$ *and* ${}^{x}\mathbf{D}^{\eta}$ *are as defined in the Definition 2.4.*
That is, a fractional integration or differentiation transforms the function (1.5) with the ν*-parameter is increased or decreased by the order of integration or differentiation respectively.*

Proof. The equation (2.8) is proved below which uses (2.4). In fact,

$$\begin{aligned}
&({}^{x}\mathbf{I}^{\eta})^{\gamma}\left(x^{\nu-1}\mathbb{E}^{\sigma,\nu,\gamma+1}_{\alpha,\beta,\delta}(\mu,r;x^{\sigma})\right)\\
&= \underbrace{I^{\eta}x^{-\eta}I^{\eta}\dots I^{\eta}x^{-\eta}I^{\eta}}_{\gamma+1 \text{ integrals}}\sum_{n=0}^{\infty}\frac{(\mu)_{rn}\ x^{\sigma n+\nu-1}}{\Gamma^{\delta n}(\alpha n+\beta)\Gamma^{\gamma+1}(\sigma n+\nu)n!}\\
&= \underbrace{I^{\eta}x^{-\eta}I^{\eta}\dots I^{\eta}x^{-\eta}I^{\eta}}_{\gamma \text{ integrals}}\sum_{n=0}^{\infty}\frac{(\mu)_{rn}\ x^{\sigma n+\nu-1}}{\Gamma^{\delta n}(\alpha n+\beta)\Gamma^{\gamma+1}(\sigma n+\nu)n!}\frac{\Gamma(\sigma n+\nu)}{\Gamma(\sigma n+\nu+\eta)}\\
&= \dots = I^{\eta}\sum_{n=0}^{\infty}\frac{(\mu)_{rn}\ x^{\sigma n+\nu-1}}{\Gamma^{\delta n}(\alpha n+\beta)\Gamma^{\gamma+1}(\sigma n+\nu)n!}\frac{\Gamma^{\gamma}(\sigma n+\nu)}{\Gamma^{\gamma}(\sigma n+\nu+\eta)}\\
&= x^{\nu+\eta-1}\sum_{n=0}^{\infty}\frac{(\mu)_{rn}\ x^{\sigma n}}{\Gamma^{\delta n}(\alpha n+\beta)\Gamma^{\gamma+1}(\sigma n+\nu+\eta)n!}\\
&= x^{\nu+\eta-1}\mathbb{E}^{\sigma,\nu+\eta,\gamma}_{\alpha,\beta,\delta}(\mu,r;x^{\sigma}).
\end{aligned}$$

We next prove (2.9).
Observing that

$$\begin{aligned}
&({}^{x}\mathbf{D}^{\eta})^{\gamma}\left(x^{\nu-1}\mathbb{E}^{\sigma,\nu,\gamma+1}_{\alpha,\beta,\delta}(\mu,r;x^{\sigma})\right)\\
&= \underbrace{D^{\eta}x^{\eta}D^{\eta}\dots D^{\eta}x^{\eta}D^{\eta}}_{\gamma+1 \text{ derivatives}}\sum_{n=0}^{\infty}\frac{(\mu)_{rn}\ x^{\sigma n+\nu-1}}{\Gamma^{\delta n}(\alpha n+\beta)\Gamma^{\gamma+1}(\sigma n+\nu)n!}\\
&= \underbrace{D^{\eta}x^{\eta}D^{\eta}\dots D^{\eta}x^{\eta}D^{\eta}}_{\gamma \text{ derivatives}}\sum_{n=0}^{\infty}\frac{(\mu)_{rn}\ x^{\sigma n+\nu-1}}{\Gamma^{\delta n}(\alpha n+\beta)\Gamma^{\gamma+1}(\sigma n+\nu)n!}\frac{\Gamma(\sigma n+\nu)}{\Gamma(\sigma n+\nu-\eta)}\\
&= \dots = D^{\eta}\sum_{n=0}^{\infty}\frac{(\mu)_{rn}\ x^{\sigma n+\nu-1}}{\Gamma^{\delta n}(\alpha n+\beta)\Gamma^{\gamma+1}(\sigma n+\nu)\ n!}\frac{\Gamma^{\gamma}(\sigma n+\nu)}{\Gamma^{\gamma}(\sigma n+\nu-\eta)}\\
&= x^{\nu-\eta-1}\sum_{n=0}^{\infty}\frac{(\mu)_{rn}\ x^{\sigma n}}{\Gamma^{\delta n}(\alpha n+\beta)\Gamma^{\gamma+1}(\sigma n+\nu-\eta)n!}\\
&= x^{\nu-\eta-1}\mathbb{E}^{\sigma,\nu-\eta,\gamma}_{\alpha,\beta,\delta}(\mu,r;x^{\sigma}).
\end{aligned}$$

Hence the required result. □

For deriving the eigen function property, we first define the following operator.

Definition 2.6. Let $f(x) = \sum\limits_{n=1}^{\infty} a_n x^n$, $|x| < R,\ R > 0$. Define an operator for $x \neq 0$ as

$$\mathbf{I}_x f = \int\limits_0^{\infty} e^{-\frac{t}{x}} x^{-1} f(t)\ dt. \tag{2.10}$$

With the aid of this and the Caputo fractional derivative, we next define the eigen function operator below.

Definition 2.7. Let $f(x) = \sum\limits_{n=0}^{\infty} a_n x^n$, $|x| < R,\ R > 0$ and $\ell, k \in \mathbb{N} \cup \{0\}$. Define an operator for $x \neq 0$ as

$$ {}^{\ell}_{D_x}\Omega^k_I \left(f(x^\eta)\right) = \underbrace{\mathbf{I}_x \mathbf{I}_x \cdots \mathbf{I}_x}_{\ell \text{ integrals}} \left({}^x\mathbf{D}^\eta\right)^k\ f\left(\left({}^x\mathbf{D}^\eta\right)^\ell x^\eta\right), \tag{2.11}$$

where ${}^x\mathbf{D}^\eta$ represents an operator defined in (2.7) with $\mathbf{D}^\alpha$ as the Caputo derivative and $\mathbf{I}_x$ is as defined in (2.10).

Theorem 2.8. *For $\beta = \sigma = \mu = r = 1$ and $\nu = \alpha, n-1 < Re(\alpha) < n, n \in \mathbb{N}$, the UMLW-function $\mathbb{E}^{\alpha,1,\gamma+1}_{\alpha,1,\delta}(1,1;\lambda x^\alpha) = \sum\limits_{n=0}^{\infty} \frac{\lambda^n\ x^{\alpha n}}{\Gamma^{\delta n+\gamma+1}(\alpha n+1)} := \mathbb{E}^{\gamma}_{\alpha,\delta}(\lambda x^\alpha)$, say, $\lambda \in \mathbb{C}, \alpha, x > 0$, is an eigen function of the operator ${}^{\ell}_{D_x}\Omega^{\gamma}_I$ in (2.11).*
That is,

$$ {}^{\delta}_{D_x}\Omega^{\gamma}_I \left(\mathbb{E}^{\gamma}_{\alpha,\delta}(\lambda x^\alpha)\right) = \lambda\ \mathbb{E}^{\gamma}_{\alpha,\delta}(\lambda x^\alpha),\ \lambda \in \mathbb{R} - \{0\}.$$

Proof. Note that

$$\begin{aligned}
&{}^{\delta}_{D_x}\Omega^{\gamma}_I \left(\mathbb{E}^{\gamma}_{\alpha,\delta}(\lambda x^\alpha)\right)\\
&= \underbrace{\mathbf{I}_x \mathbf{I}_x \cdots \mathbf{I}_x}_{\delta \text{ integrals}} \left({}^x\mathbf{D}^\alpha\right)^\gamma\ \mathbb{E}^{\gamma}_{\alpha,\delta}\left(\lambda\left({}^x\mathbf{D}^\alpha\right)^\delta x^\alpha\right)\\
&= \underbrace{\mathbf{I}_x \mathbf{I}_x \cdots \mathbf{I}_x}_{\delta \text{ integrals}} \left({}^x\mathbf{D}^\alpha\right)^\gamma\ \sum_{n=0}^{\infty} \frac{\lambda^n}{\Gamma^{\delta n+\gamma+1}(\alpha n+1)} \left({}^x\mathbf{D}^\alpha\right)^{\delta n}\ x^{\alpha n}\\
&= \underbrace{\mathbf{I}_x \mathbf{I}_x \cdots \mathbf{I}_x}_{\delta \text{ integrals}} \sum_{n=0}^{\infty} \frac{\lambda^n}{\Gamma^{\delta n+\gamma+1}(\alpha n+1)} \left({}^x\mathbf{D}^\alpha\right)^{\delta n+\gamma}\ x^{\alpha n}.
\end{aligned} \tag{2.12}$$

For $n = 0$, $\left({}^x\mathbf{D}^\alpha\right)^{0+\gamma}\ x^0 = \underbrace{{}_cD^\alpha x^\alpha\, {}_cD^\alpha \ldots {}_cD^\alpha x^\alpha\, {}_cD^\alpha}_{\gamma+1 \text{ derivatives}}(1) = 0.$

For $n = 1$,

$$\left({}^x\mathbf{D}^\alpha\right)^{\delta+\gamma} x^\alpha = \underbrace{{}_cD^\alpha x^\alpha\, {}_cD^\alpha \ldots {}_cD^\alpha x^\alpha\, {}_cD^\alpha}_{\delta+\gamma+1 \text{ derivatives}}\ (x^\alpha). \tag{2.13}$$

Observing that

$$
\begin{aligned}
{}_cD^{\alpha}x^{\alpha} &= \frac{1}{\Gamma(m-\alpha)}\int\limits_0^x \alpha(\alpha-1)\ldots(\alpha-m+1)\; t^{\alpha-m}\;(x-t)^{m-\alpha-1}dt \\
&= \frac{\alpha(\alpha-1)\ldots(\alpha-m+1)}{\Gamma(m-\alpha)}x^{m-\alpha-1}\int\limits_0^x t^{\alpha-m}\;\left(1-\frac{t}{x}\right)^{m-\alpha-1} dt
\end{aligned}
$$

and substituting $t = xu$, we further have

$$
\begin{aligned}
{}_cD^{\alpha}x^{\alpha} &= \frac{\alpha(\alpha-1)\ldots(\alpha-m+1)}{\Gamma(m-\alpha)}\int\limits_0^1 u^{\alpha-m}\;(1-u)^{m-\alpha-1}\,du \\
&= \frac{\alpha(\alpha-1)\ldots(\alpha-m+1)}{\Gamma(m-\alpha)}\;\Gamma(m-\alpha)\Gamma(\alpha-m+1) \\
&= \Gamma(\alpha+1).
\end{aligned}
$$

Using this repeatedly in (2.13), we finally arrive at

$$({}^x\mathbf{D}^{\alpha})^{\delta+\gamma}\; x^{\alpha} \;=\; \Gamma^{\delta+\gamma+1}(\alpha+1).$$

Now for $n = 2$,

$$({}^x\mathbf{D}^{\alpha})^{2\delta+\gamma}\; x^{2\alpha} = \underbrace{{}_cD^{\alpha}x^{\alpha}{}_cD^{\alpha}\ldots{}_cD^{\alpha}x^{\alpha}{}_cD^{\alpha}}_{2\delta+\gamma+1\text{ derivatives}}\; x^{2\alpha}.$$

We begin with

$$
\begin{aligned}
&{}_cD^{\alpha}(x^{2\alpha}) \\
&= \frac{1}{\Gamma(m-\alpha)}\int\limits_0^x (2\alpha)(2\alpha-1)\ldots(2\alpha-m+1)\; t^{2\alpha-m}\;(x-t)^{m-\alpha-1}dt \\
&= \frac{(2\alpha)(2\alpha-1)\ldots(2\alpha-m+1)}{\Gamma(m-\alpha)}x^{m-\alpha-1}\int\limits_0^x t^{2\alpha-m}\;\left(1-\frac{t}{x}\right)^{m-\alpha-1} dt \\
&= \frac{(2\alpha)(2\alpha-1)\ldots(2\alpha-m+1)}{\Gamma(m-\alpha)}x^{2\alpha-\alpha}\frac{\Gamma(m-\alpha)\;\Gamma(2\alpha-m+1)}{\Gamma(2\alpha-\alpha+1)} \\
&= \frac{\Gamma(2\alpha+1)}{\Gamma(2\alpha-\alpha+1)}x^{2\alpha-\alpha}.
\end{aligned}
$$

Therefore,

$$({}^x\mathbf{D}^{\alpha})^{2\delta+\gamma}\; x^{2\alpha} = \frac{\Gamma^{2\delta+\gamma+1}(2\alpha+1)}{\Gamma^{2\delta+\gamma+1}(2\alpha-\alpha+1)}x^{2\alpha-\alpha}.$$

In general,

$$({}^x\mathbf{D}^{\alpha})^{\delta n+\gamma}x^{\alpha n} = \frac{\Gamma^{\delta n+\gamma+1}(\alpha n+1)}{\Gamma^{\delta n+\gamma+1}(\alpha n-\alpha+1)}\; x^{\alpha n-\alpha}. \tag{2.14}$$

Now substituting (2.14) in (2.12) and then applying the operator defined in (2.10), we find that

$$
\begin{aligned}
&{}^{\delta}_{D_x}\Omega^{\gamma}_{I}\left(\mathbb{E}^{\gamma}_{\alpha,\delta}(\lambda x^{\alpha})\right)\\
&= \underbrace{\mathbf{I}_x\mathbf{I}_x\cdots\mathbf{I}_x}_{\delta\text{ fold integrals}} \sum_{n=1}^{\infty} \frac{\lambda^n}{\Gamma^{\delta n+\gamma+1}(\alpha n+1)} \frac{\Gamma^{\delta n+\gamma+1}(\alpha n+1)}{\Gamma^{\delta n+\gamma+1}(\alpha n-\alpha+1)}\, x^{\alpha n-\alpha}\\
&= \underbrace{\mathbf{I}_x\mathbf{I}_x\cdots\mathbf{I}_x}_{\delta-1\text{ fold integrals}} \sum_{n=1}^{\infty} \frac{\lambda^n}{\Gamma^{\delta n+\gamma+1}(\alpha n-\alpha+1)} \int_0^{\infty} e^{-\frac{t}{x}}\, x^{-1}\, t^{\alpha n-\alpha} dt\\
&= \underbrace{\mathbf{I}_x\mathbf{I}_x\cdots\mathbf{I}_x}_{\delta-1\text{ fold integrals}} \sum_{n=1}^{\infty} \frac{\lambda^n}{\Gamma^{\delta n+\gamma+1}(\alpha n-\alpha+1)}\, x^{\alpha n-\alpha-1} \int_0^{\infty} e^{-\frac{t}{x}} \left(\frac{t}{x}\right)^{\alpha n-\alpha} dt\\
&= \underbrace{\mathbf{I}_x\mathbf{I}_x\cdots\mathbf{I}_x}_{\delta-1\text{ fold integrals}} \sum_{n=1}^{\infty} \frac{\lambda^n}{\Gamma^{\delta n+\gamma+1}(\alpha n-\alpha+1)}\, x^{\alpha n-\alpha}\, \Gamma(\alpha n-\alpha+1).
\end{aligned}
$$

Continuing in this way by applying the operator $\mathbf{I}_x$, $\delta-1$ times, we finally arrive at

$$
\begin{aligned}
{}^{\delta}_{D_x}\Omega^{\gamma}_{I}\left(\mathbb{E}^{\gamma}_{\alpha,\delta}(\lambda x^{\alpha})\right) &= \sum_{n=1}^{\infty} \frac{\lambda^n}{\Gamma^{\delta n+\gamma+1}(\alpha n-\alpha+1)}\, x^{\alpha n-\alpha}\, \Gamma^{\delta}(\alpha n-\alpha+1)\\
&= \sum_{n=1}^{\infty} \frac{\lambda^n\, x^{\alpha n-\alpha}}{\Gamma^{\delta n-\delta+\gamma+1}(\alpha n-\alpha+1)}.
\end{aligned}
$$

Hence,

$$
{}^{\delta}_{D_x}\Omega^{\gamma}_{I}\left(\mathbb{E}^{\gamma}_{\alpha,\delta}(\lambda x^{\alpha})\right) = \lambda\, \mathbb{E}^{\gamma}_{\alpha,\delta}(\lambda x^{\alpha}). \tag{2.15}
$$

□

Remark 2.9. From the definition of $\mathbb{E}^{\gamma}_{\alpha,\delta}(x)$ in the Theorem 2.8, observe that $\mathbb{E}^{\gamma}_{\alpha,\delta}(0)=1$ and from (2.15), ${}^{\delta}_{D_x}\Omega^{\gamma}_{I}\left(\mathbb{E}^{\gamma}_{\alpha,\delta}(\lambda x^{\alpha})\right) = \lambda\, \mathbb{E}^{\gamma}_{\alpha,\delta}(\lambda x^{\alpha}),\ \ \lambda \in \mathbb{R}\setminus\{0\}$.

3. Application

In the view of [8], we now discuss the application of particular UMLW-function discussed in the Theorem 2.8. Let $\mathcal{D}$ be the bounded domain in $\mathbb{R}^d$ with sufficiently smooth boundary $\partial\mathcal{D}$. We consider the infinite order fractional evolution type problem

$$
{}^{\ell}_{D_x}\Omega^{\gamma}_{I}\, u(x,t) = u_t(x,t),\ \ t\in[0,T],\quad T>0; \tag{3.1}
$$

$$
u(0,t) = f(t), \tag{3.2}
$$

where the operator ${}^{\ell}_{D_x}\Omega^{\gamma}_{I}$ is as defined in (2.11) in L^{∞}-space and is operating only on the variable x and $f(t)\in\mathcal{C}[0,T]$.

Theorem 3.1. *If* $\delta, \gamma \in \mathbb{N} \cup \{0\}$, $n-1 < Re(\alpha) < n, n \in \mathbb{N}$ *then the solution of (3.1)-(3.2) is given by*

$$u(x,t) = \mathbb{E}^{\gamma}_{\alpha,\delta}\left(x^{\alpha}\,\frac{\partial}{\partial t}\right) f(t).$$

Proof. To prove the theorem, we prove that $u(x,t) = \mathbb{E}^{\gamma}_{\alpha,\delta}\left(x^{\alpha}\,\frac{\partial}{\partial t}\right) f(t)$ satisfies the problem described by (3.1)-(3.2).
Here, noticing that ${}^{\ell}_{D_x}\Omega^{\gamma}_{I}$ is the operator operating only on the variable x, we have from the Theorem 2.8,

$$\begin{aligned} {}^{\ell}_{D_x}\Omega^{\gamma}_{I}\, u(x,t) &= {}^{\ell}_{D_x}\Omega^{\gamma}_{I}\left\{\mathbb{E}^{\gamma}_{\alpha,\delta}\left(x^{\alpha}\,\frac{\partial}{\partial t}\right) f(t)\right\} = \frac{\partial}{\partial t}\,\mathbb{E}^{\gamma}_{\alpha,\delta}\left(x^{\alpha}\,\frac{\partial}{\partial t}\right) f(t) \\ &= u_t(x,t). \end{aligned}$$

To complete the proof of the theorem it sufficient to prove that

$$\lim_{x\to 0} ||u(x,.) - f||_{\infty} = 0. \tag{3.3}$$

Observe that

$$\begin{aligned} &\lim_{x\to 0} ||u(x,.) - f||_{\infty} \\ &= \lim_{x\to 0}\left|\left|\mathbb{E}^{\gamma}_{\alpha,\delta}\left(x^{\alpha}\,\frac{\partial}{\partial t}\right) f - f\right|\right|_{\infty} \\ &= \lim_{x\to 0}\ ||f||_{\infty}\ \left|\left|\mathbb{E}^{\gamma}_{\alpha,\delta}\left(x^{\alpha}\,\frac{\partial}{\partial t}\right) - 1\right|\right|_{\infty}. \end{aligned}$$

But $\lim\limits_{x\to 0}\left|\left|\mathbb{E}^{\gamma}_{\alpha,\delta}\left(x^{\alpha}\,\frac{\partial}{\partial t}\right) - 1\right|\right|_{\infty} = 0$ by Remark 2.9 and $f \in \mathcal{C}[0,T]$ proves (3.3). □

Now, some of the special cases of the properties proved for UMLW-function are shown.

Differential equation:

We illustrate the reducibility of the differential equation of Theorem 1.6 corresponding to the special cases namely Garra and Polito's function and Srivastava and Tomovski's function as follows.

(i) By taking $\delta = 0, \mu = r = 1$ and replacing γ by $\gamma + 1$, z by x in (1.11) we obtain with $x^* = \frac{x}{\sigma^{\sigma(\gamma+1)}}$, the equation

$$\left\{\prod_{i=0}^{\sigma-1}\left[\theta + \tfrac{\nu+i}{\sigma} - 1\right]^{\gamma+1}\ \theta - x^*\,(\theta+1)\right\} w = 0,$$

where the solution $w = E_{\gamma;\sigma,\nu}(x) = \sum\limits_{n=0}^{\infty} \frac{x^n}{\Gamma^{\gamma+1}(\sigma n+\nu)}$ is Garra and Polito's function.

(ii) The Srivastava-Tomovski's function $w = E^{\mu,r}_{\sigma,\nu}(z) = \sum\limits_{n=0}^{\infty} \frac{(\mu)_{rn}\ (z)^n}{\Gamma(\sigma n+\nu)\ n!}$ satisfies the differential equation

$\left\{\prod\limits_{i=0}^{\sigma-1}\left[\theta + \frac{\nu+i}{\sigma} - 1\right]\ \theta - z^*\prod\limits_{j=0}^{r-1}\left[\theta + \frac{\mu+j}{r}\right]\right\} w = 0$, with substitutions $\delta = 0$ and $\gamma = 1$ in (1.11), and $z^* = \frac{r^r}{\sigma^\sigma} z$.

Integral Operator:
(i) By taking $\delta = 0, \mu = r = 1$ and replacing γ by $\gamma + 1$, z by x in Theorem 1.9, we obtain

$$M = \sum_{n=0}^{\infty} \frac{|\lambda|^n \ (b-a)^{Re(\nu)+Re(\sigma)n}}{|\Gamma^{\gamma+1}(\sigma n + \nu)| \ (Re(\nu) + Re(\sigma)n)},$$

which is a bound of integral operator of Garra and Polito's function in Lebesgue Measurable space.
(ii) In Theorem 1.9, on making substitutions $\delta = 0$ and $\gamma = 1$, we find

$$M = \sum_{n=0}^{\infty} \frac{|(\mu)_{rn}| \ |\lambda|^n \ (b-a)^{Re(\nu)+Re(\sigma)n}}{|\Gamma^{\gamma}(\sigma n + \nu)| \ n! \ (Re(\nu) + Re(\sigma)n)},$$

which is nothing but the Integral operator of Srivastava and Tomovski's function [21, Theorem 2, Eq.(2.15)].
Eigen function property:
It is interesting to note that the substitutions $\delta = 0, \mu = r = 1$ and $z = x$ in Theorem 2.8, yields the eigen function property of the Garra and Polito's function [10, Theorem 3.6, p. 776]

$$\mathbb{E}^{\gamma}_{\alpha,0}(x^{\alpha}) = E_{\gamma;\alpha,1}(x^{\alpha}) = \sum_{n=0}^{\infty} \frac{x^{\alpha n}}{\Gamma^{\gamma+1}(\alpha n + 1)}$$

with respect to the operator $({}^{x}\mathbf{D}^{\alpha})^{\gamma} := {}^{0}_{D_x}\Omega^{\gamma}_{I}$.
Following are the graphs of UMLW-function for the specific values of the parameters involved.

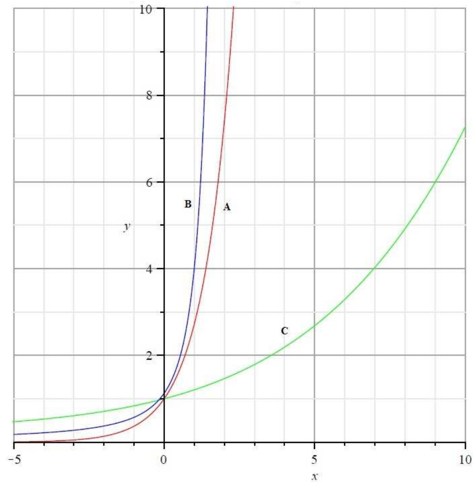

FIGURE 1. Graph A: $\exp(x)$, Graph B: $E_{\frac{1}{2},\frac{1}{3}}(x)$, Graph C : $E^{4,2,\frac{1}{2}}_{\frac{1}{2},\frac{1}{2},\frac{1}{3}}(1,2;x)$

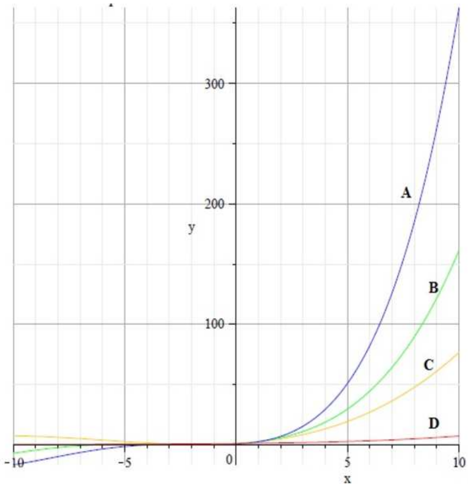

FIGURE 2. Graph A : $E^{\frac{1}{2},\frac{1}{2},1}_{\frac{1}{2},\frac{1}{2},1}(1,0;x)$, Graph B : $E^{\frac{1}{2},1,0}_{\frac{1}{2},1,1}(1,0;x)$, Graph C : $E^{\frac{1}{10},1,4}_{\frac{1}{2},1,2}(\frac{1}{2},\frac{1}{3};x)$, Graph D : $E^{4,2,\frac{1}{2}}_{\frac{1}{2},\frac{1}{2},\frac{1}{3}}(1,2;x)$

In the Figure 1, a graph of particular UMLW-function $\left(E^{4,2,\frac{1}{2}}_{\frac{1}{2},\frac{1}{2},\frac{1}{3}}(1,2;x)\right)$ is compared with that of exponential function and ML-function in two parameters indicated by Graph C, Graph A and Graph B respectively. And in the Figure 2, the graphs of certain specialized UMLW-functions are plotted. These are indicated in Figure 2 as Graph A, Graph B, Graph C and Graph D. Click: https://drive.google.com/file/d/0Bwly1qnYQNxZbEJnc0JQdW5tNE0/view?usp=sharing for more detail.

4. Conclusion

As a specific instance of the hypergeometric function ${}_pF_q$ if $\alpha \in \mathbb{N} \cup \{0\}$, the new function defined in (1.5) may clearly be viewed as an extension of the Mittag-Leffler and Wright functions (${}_1F_\alpha$ and ${}_0F_\alpha$), reduced to the hyper Bessel function ${}_0F_q$. However, in the power series, q in the second index and the summation index n both go to infinity at this point. It's also noteworthy to note that it solves an infinite order differential equation, which may arise in the turbulence field or in a system with an infinite number of degrees of freedom. Also, the integral involves this newly defined UMLW function as a kernel is bouned in $L(a,b)$. Notably, the specific instance of this new function possessing an eigen function characteristic concerning the hyper Bessel type fraction operators via which the infinite order evolution type problem is formulated is also intriguing.

Acknowledgment. Author is indebted to her guide Prof. B. I. Dave, for his valuable guidance. The author is deeply grateful to the referees for their valuable comments and suggestions that enormously benefited the final version of the article.

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Meera H. Chudasama
Department of Mathematics,
Sardar Patel University,
Vallabh Vidyanagar-388120,
Gujarat, India
e-mail: meera.chudsama@yahoo.co.in