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A new member of the Pell sequences: The pseudo-Pell sequence

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Abstract. In this study, we define a new family of the Pell numbers and establish some properties of the relation to the ordinary Pell numbers. We give some identities the pseudo-Pell numbers. Moreover, we obtain the Binet's formula, generating function formula and some formulas for this new type numbers. Morever, we give the matrix representation of the pseudo-Pell numbers.

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1. Introduction and preliminaries

Almost all branches of contemporary research, including computer sciences, physics, economics, architecture, geostatistics, art, color image processing, and music, employ a large number of integer sequences. One of mathematics' most well-known and intriguing number sequences, the Fibonacci sequence has been the subject of extensive research in the literature. The fascinating features of the Fibonacci sequence have delighted scientific enthusiasts for years. The Fibonacci sequence is generated by a recursive formula

$$F_n = F_{n-1} + F_{n-2}$$

for $n \geq 2$ with $F_0 = 0$ and $F_1 = 1$. The Fibonacci sequence has many interesting properties. For example, the ratio $\dfrac{F_{n+1}}{F_n}$ converges to the golden ratio $\left(\dfrac{1+\sqrt{5}}{2}\right)$ as n tends to infinity. The Fibonacci sequence has been generalized in many ways, some

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by preserving the initial conditions and others by preserving the recurrence relation [1, 4, 6, 7, 10, 11, 14, 16, 19, 18, 21, 22, 27, 29].

The pseudo-Fibonacci and pseudo Lucas sequences was introduced by Ferns [8] as novel generalizations of the Fibonacci and Lucas sequences as follows

$$\Xi_{n+1} = \Xi_n + \Theta_n,$$

$$\Theta_{n+1} = \Xi_{n+1} + \xi\Xi_n$$

with initial conditions $\Xi_1 = 1$ and $\Theta_1 = 1$ in which ξ is a positive integer. We derive the recursion formula by elimination

$$\Xi_{n+2} = 2\Xi_{n+1} + \xi\Xi_n,$$

$$\Theta_{n+2} = 2\Theta_{n+1} + \xi\Theta_n$$

with initial conditions conditions $\Xi_0 = 0$, $\Xi_1 = 1$ and $\Theta_0 = \Theta_1 = 1$ respectively.
The Binet formula for each of the pseudo-Fibonacci and pseudo Lucas sequences is

$$\Xi_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

$$\Theta_n = \frac{\alpha^n + \beta^n}{2}$$

where $\alpha = 1 + \sqrt{1+\xi}$, $\beta = 1 - \sqrt{1+\xi}$.

The Pell sequence is one of the most famous and interesting numerical sequences in mathematics and has been widely studied in the literature. The Pell sequence is generated by a recursive formula

$$P_n = 2P_{n-1} + P_{n-2},$$

for ≥ 2 with $P_0 = 0$ and $P_1 = 1$. Similarly, the Pell sequence has many interesting properties. For example, the ratio $\dfrac{P_{n+1}}{P_n}$ converges to the silver ratio $(1+\sqrt{2})$ as n tends to infinity.

The two basic ways that the Pell sequence has been generalized are either by keeping the recurrence relation constant while changing the starting conditions or by changing the recurrence relation while keeping the beginning circumstances constant. A closed form for the nth term of the sequence, the sum of the first n terms of the sequence, the sum of the first n terms with odd (or even) indices of the sequence, an explicit sum formula, Catalan's identity, Cassini's identity, d'Ocagne's identity, Tagiuri's identity, and generating function are just a few of the properties that have been looked into by various researchers, among many others [2, 3, 12, 13, 15, 17, 20, 23, 24, 25, 26, 28].

In this work, a variety of algebraic properties of the pseudo-Pell and Pell-Lucas numbers will be presented. Some identities will be given for the pseudo-Pell and Pell-Lucas numbers sequences, such as Binet's formula, the generating function formula, and some sum formulas. A matrix representation of these sequences will also be given.

2. The pseudo-Pell sequence

Several integer sequences exist, many of them with charming shapes and several charming characteristics. For instance, two of the most well-known and attractive number sequences are the Pell and Pell-Lucas sequences. The mathematical community is still in awe of their universality and beauty. Additionally, there are countless possibilities to explore, locate, and estimate thanks to the Pell and Pell-Lucas numbers. Mathematical friends, the Pell and Pell-Lucas numbers share many similar characteristics. Our main aim here is to find alternatives to these two series of families.

In this section, a new generalization of the Pell and Pell-Lucas numbers is introduced. We give some properties of the pseudo-Pell and Pell-Lucas numbers.

Definition 2.1. Let $\xi > 0$ be integer number. The pseudo-Pell and Pell-Lucas numbers be recursively defined by

$$\mathbf{P}_{n+1} = 2\mathbf{P}_n + \mathbf{R}_n \tag{2.1}$$

$$\mathbf{R}_{n+1} = 2\mathbf{P}_{n+1} + \xi\mathbf{P}_n \tag{2.2}$$

with initial conditions $\mathbf{P}_1 = 1$ and $\mathbf{R}_1 = 2$. Actually, by eliminating first the $\mathbf{P}_n$'s and then the $\mathbf{R}_n$'s, from equations (2.1) and (2.2), following the pseudo-Pell and Pell-Lucas numbers are obtained

$$\mathbf{P}_{n+1} = 4\mathbf{P}_n + \xi\mathbf{P}_{n-1} \tag{2.3}$$

$$\mathbf{R}_{n+1} = 4\mathbf{R}_n + \xi\mathbf{R}_{n-1} \tag{2.4}$$

with initial conditions conditions $\mathbf{P}_0 = 0$, $\mathbf{P}_1 = 1$ and $\mathbf{R}_0 = 1$, $\mathbf{R}_1 = 2$ respectively.

From equations (2.3) and (2.4), the associated characteristic polynomial

$$p(x) = x^2 - 4x - \xi.$$

$p(x)$ has the roots $x_1 = 2 + \sqrt{4+\xi}$ and $x_2 = 2 - \sqrt{4+\xi}$. Thus, it is apparent that

$$\begin{aligned} x_1x_2 &= -\xi, \\ x_1 + x_2 &= 4, \\ x_1 - x_2 &= 2\sqrt{4+\xi}, \\ x_1^2 + x_2^2 &= 16 + 2\xi, \\ x_1^2 - x_2^2 &= 8\sqrt{4+\xi}, \\ x_1^2 &= 4x_1 + \xi, \\ x_2^2 &= 4x_2 + \xi. \end{aligned}$$

The first terms of the pseudo-Pell and Pell-Lucas numbers

n	$\mathbf{P}_n$	$\mathbf{R}_n$
0	0	1
1	1	2
2	4	$\xi + 8$
3	$\xi + 16$	$6\xi + 32$
4	$8\xi + 64$	$\xi^2 + 32\xi + 128$
5	$\xi^2 + 48\xi + 256$	$10\xi^2 + 160\xi + 512$
6	$12\xi^2 + 256\xi + 1024$	$\xi^3 + 72\xi^2 + 768\xi + 2048$

2.1. Binet's formula for the pseudo-Pell sequence

The Fibonacci numbers are among the brightest points within a wide range of integer sequences, according to Koshy. We can speculate that this sequence's abundance of intriguing features is one of the reasons it is referenced. Furthermore, Binet's formula can be used to obtain practically all of these features. We will state and prove a general closed formula for the pseudo-Pell sequence.

Theorem 2.2. *The Binet's formula for the pseudo-Pell and Pell-Lucas numbers are*

$$\mathbf{P}_n = \frac{x_1^n - x_2^n}{x_1 - x_2},$$

$$\mathbf{R}_n = \frac{x_1^n + x_2^n}{2}$$

where $x_1 = 2 + \sqrt{4+\xi}$, $x_2 = 2 - \sqrt{4+\xi}$ *and* ξ *is a positive integer.*

Proof. The pseudo-Pell sequence's characteristic equation $x^2 - 4x - \xi = 0$, and its real roots are $x_1 = 2 + \sqrt{4+\xi}$ and $x_2 = 2 - \sqrt{4+\xi}$. Then the sequences $\mathbf{P}_n = \eta(x_1)^n + \mu(x_2)^n$, for $n \geq 0$, and with η, μ real numbers are solutions of equation. Let us determine the constants η and μ, considering that $\mathbf{P}_0 = 0$ and $\mathbf{P}_1 = 1$, and we obtain the linear system,

$$\eta + \mu = 0$$

$$\eta x_1 + \mu x_2 = 1.$$

We find $\mu = -\frac{1}{x_1 - x_2}$ and $\eta = \frac{1}{x_1 - x_2}$. So we have that

$$\mathbf{P}_n = \frac{x_1^n - x_2^n}{x_1 - x_2}.$$

Similarly,

$$\mathbf{R}_n = \frac{x_1^n + x_2^n}{2}.$$ □

Corollary 2.3. *We let* $n = -m$ *where* m *is a positive integer. From Binet's formula, we find the negative subscript terms of the pseudo-Pell and Pell-Lucas numbers.*

$$\mathbf{P}_{-n} = -\frac{\mathbf{P}_n}{(-\xi)^n},$$

$$\mathbf{R}_{-n} = \frac{\mathbf{R}_n}{(-\xi)^n}.$$

Theorem 2.4. *For* $n \geq 0$, *the following identity holds*

$$\mathbf{R}_{n+1} - 2\mathbf{R}_n = (4+\xi)\mathbf{P}_n$$

where $\mathbf{P}_n$ *and* $\mathbf{R}_n$ *are the nth pseudo-Pell and Pell-Lucas numbers, respectively.*

Proof.

$$\mathbf{R}_{n+1} - 2\mathbf{R}_n = \frac{x_1^{n+1} + x_2^{n+1}}{2} - 2\frac{x_1^n + x_2^n}{2} = \frac{x_1^n\left[x_1 - 2\right] + x_2^n\left[x_2 - 2\right]}{2}$$

$$= \frac{\left[x_1^n - x_2^n\right]\sqrt{4+\xi}}{2} = (4+\xi)\mathbf{P}_n.$$ □

Corollary 2.5. *In (2.3) and (2.4) let $\xi = 4$. We get $\mathbf{P}_n = 2^{n-1}P_n$ and $\mathbf{R}_n = 2^{n-1}R_n$ where P_n and R_n are the nth Pell and Pell-Lucas numbers, respectively.*

2.2. Generating function for the pseudo-Pell sequence

A strong method for resolving linear homogeneous recurrence relations is offered by generating functions. We shall systematically apply generating functions for linear recurrence relations with nonconstant coefficients, even though they are usually employed in conjunction with linear recurrence relations with constant coefficients. Now, we consider the generating functions for the pseudo-Pell sequence.

Theorem 2.6. *The generating formula for the pseudo-Pell and Pell-Lucas numbers are*

$$\sum_{n=0}^{\infty} \mathbf{P}_n t^n = \frac{t}{1-4t-\xi t^2},$$

$$\sum_{n=0}^{\infty} \mathbf{R}_n t^n = \frac{1-2t}{1-4t-\xi t^2}.$$

Proof. Let $h(t)$ be the generating function for the pseudo-Pell numbers as $\sum_{n=0}^{\infty} \mathbf{P}_n t^n$. We get the following equations

$$4th(t) = 4\sum_{n=0}^{\infty} \mathbf{P}_n t^{n+1} \text{ and } \xi t^2 h(t) = \xi \sum_{n=0}^{\infty} \mathbf{P}_n t^{n+2}.$$

After the needed calculations, the generating function for the pseudo-Pell numbers is obtained as

$$\sum_{n=0}^{\infty} \mathbf{P}_n t^n = \frac{t}{1-4t-\xi t^2}.$$

Similarly,

$$\sum_{n=0}^{\infty} \mathbf{R}_n t^n = \frac{1-2t}{1-4t-\xi t^2}.$$

□

Theorem 2.7. *The following identities holds*

$$(4+\xi)\sum_{i=0}^{n} \mathbf{P}_i + \sum_{i=0}^{n} \mathbf{R}_i = \mathbf{R}_{n+1} - 1, \tag{2.5}$$

$$(2+\xi)\sum_{i=0}^{n+1} \mathbf{P}_i - \sum_{i=0}^{n+1} \mathbf{R}_i = \xi \mathbf{P}_{n+1} \tag{2.6}$$

where $\mathbf{P}_n$ and $\mathbf{R}_n$ are the nth pseudo-Pell and Pell-Lucas numbers, respectively.

Proof. The proof is carried out using elimination in the equations of theorem (2.4). □

Corollary 2.8. *From equations (2.5) and (2.6), the following equations are obtained,*

$$\sum_{i=0}^{n+1} \mathbf{P}_i = \frac{(4+\xi)\mathbf{P}_{n+1} + \xi \mathbf{P}_n - \frac{1}{2}}{3+\xi},$$

$$\sum_{i=0}^{n+1} \mathbf{R}_i = \frac{(8+3\xi)\mathbf{P}_{n+1} + \xi(2+\xi)\mathbf{P}_n - (2+\xi)\frac{1}{2}}{3+\xi}.$$

2.3. Matrix representation for the pseudo-Pell sequence

A close connection between matrices and Fibonacci numbers was shown in Charles'[9] work on what he called the Q matrix. An interesting pattern emerges from this work. The power of matrices was exploited to obtain new identities and results involving Fibonacci numbers. We will give the matrix representation of the pseudo-Pell sequence.

Definition 2.9. The basic matrix of the pseudo-Pell and pseudo-Pell-Lucas sequence is

$$Q = \begin{bmatrix} 4 & \xi \\ 1 & 0 \end{bmatrix}$$

where $\xi > 0$ is a integer. Based on the Cayley-Hamilton Theorem, the pseudo-Pell and pseudo-Pell-Lucas's characteristic polynomial is given as

$$p(\lambda) = det(\lambda I - Q).$$

$$p(\lambda) = det(\lambda I - Q) = \begin{vmatrix} \lambda - 4 & -\xi \\ -1 & \lambda \end{vmatrix} = \lambda^2 - 4\lambda - \xi.$$

Then, if $p(\lambda) = \lambda^2 - 4\lambda - \xi$.

Theorem 2.10. *Let $n > 0$ be an integer. The following equality holds*

$$a) \quad \begin{bmatrix} 4 & \xi \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} \mathbf{P}_2 & \mathbf{P}_1 \\ \mathbf{P}_1 & \mathbf{P}_0 \end{bmatrix} = \begin{bmatrix} \mathbf{P}_{n+2} & \mathbf{P}_{n+1} \\ \mathbf{P}_{n+1} & \mathbf{P}_n \end{bmatrix}$$

$$b) \quad \begin{bmatrix} 0 & 1 \\ \frac{1}{\xi} & -\frac{4}{\xi} \end{bmatrix}^n \begin{bmatrix} \mathbf{P}_2 & \mathbf{P}_1 \\ \mathbf{P}_1 & \mathbf{P}_0 \end{bmatrix} = \begin{bmatrix} \mathbf{P}_{-n+2} & \mathbf{P}_{-n+1} \\ \mathbf{P}_{-n+1} & \mathbf{P}_{-n} \end{bmatrix}$$

$$c) \quad \begin{bmatrix} 0 & 1 \\ \xi & 4 \end{bmatrix}^n \begin{bmatrix} \mathbf{P}_0 \\ \mathbf{P}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{P}_n \\ \mathbf{P}_{n+1} \end{bmatrix}$$

$$d) \quad \begin{bmatrix} -\frac{4}{\xi} & \frac{1}{\xi} \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} \mathbf{P}_0 \\ \mathbf{P}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{P}_{-n} \\ \mathbf{P}_{-n+1} \end{bmatrix}$$

$$e) \quad \begin{bmatrix} \mathbf{P}_1 & \mathbf{P}_0 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ \xi & 0 \end{bmatrix}^n = \begin{bmatrix} \mathbf{P}_{n+1} & \mathbf{P}_n \end{bmatrix}$$

$$f) \quad \begin{bmatrix} \mathbf{P}_1 & \mathbf{P}_0 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{\xi} \\ 1 & -\frac{4}{\xi} \end{bmatrix}^n = \begin{bmatrix} \mathbf{P}_{-n+1} & \mathbf{P}_{-n} \end{bmatrix}$$

Proof. For the prove, we utilize induction principle on n. The equality hold for $n = 1$. Now assume that the equality is true for $n > 1$. Then, we can verify for $n + 1$ as follows

$$a) \quad \begin{bmatrix} 4 & \xi \\ 1 & 0 \end{bmatrix}^{n+1} \begin{bmatrix} \mathbf{P}_2 & \mathbf{P}_1 \\ \mathbf{P}_1 & \mathbf{P}_0 \end{bmatrix} = \begin{bmatrix} 4 & \xi \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 4 & \xi \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} \mathbf{P}_2 & \mathbf{P}_1 \\ \mathbf{P}_1 & \mathbf{P}_0 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & \xi \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{P}_{n+2} & \mathbf{P}_{n+1} \\ \mathbf{P}_{n+1} & \mathbf{P}_n \end{bmatrix} = \begin{bmatrix} \mathbf{P}_{n+3} & \mathbf{P}_{n+2} \\ \mathbf{P}_{n+2} & \mathbf{P}_{n+1} \end{bmatrix}.$$

Thus, the first step of the theorem can be proved easily. Similarly, the other steps of the proof are seen by induction on n. Matrix representations of the pseudo-Pell-Lucas numbers are proved similarly. □

2.4. Sums of the pseudo-Pell sequence

Finite sums of sequences have always been important for people working in this field. Some researchers examined the sum of product two consecutive terms, some researchers examined the sum of squares. In a new sequence found, the study of the sums of sequence terms became important. We present some results concerning sums of terms of the pseudo-Pell and Pell-Lucas sequence.

Theorem 2.11. *The following equalities hold*

$$a)\quad \sum_{i=0}^{n}\mathbf{P}_{2i}=\frac{\xi^2\mathbf{P}_{2n}-\mathbf{P}_{2n+2}+\mathbf{P}_2}{(\xi-5)(\xi+3)},$$

$$b)\quad \sum_{i=0}^{n}\mathbf{P}_{2i+1}=\frac{\xi^2\mathbf{P}_{2n+1}-\mathbf{P}_{2n+3}+(1-\xi)}{(\xi-5)(\xi+3)},$$

$$c)\quad \sum_{i=0}^{n}\mathbf{R}_{2i}=\frac{\xi^2\mathbf{R}_{2n}-\mathbf{R}_{2n+2}-\mathbf{R}_2+2}{(\xi-5)(\xi+3)},$$

$$d)\quad \sum_{i=0}^{n}\mathbf{R}_{2i+1}=\frac{\xi^2\mathbf{R}_{2n+1}-\mathbf{R}_{2n+3}+2(1-\xi)}{(\xi-5)(\xi+3)}.$$

where $\mathbf{P}_n$ and $\mathbf{R}_n$ are the nth pseudo-Pell and Pell-Lucas numbers, respectively.

Proof.

$$a)\quad \sum_{i=0}^{n}\mathbf{P}_{2i}=\sum_{i=0}^{n}\frac{x_1^{2i}-x_2^{2i}}{x_1-x_2}=\frac{1}{x_1-x_2}\left[\sum_{i=0}^{n}(x_1^2)^i-\sum_{i=0}^{n}(x_2^2)^i\right]$$

$$=\frac{1}{x_1-x_2}\left[\frac{x_1^{2n+2}-1}{x_1^2-1}-\frac{x_2^{2n+2}-1}{x_2^2-1}\right]$$

$$=\frac{1}{(x_1^2-1)(x_2^2-1)}\left[\frac{\xi^2(x_1^{2n}-x_2^{2n})-(x_1^{2n+2}-x_2^{2n+2})+(x_1^2-x_2^2)}{x_1-x_2}\right]$$

$$=\frac{\xi^2\mathbf{P}_{2n}-\mathbf{P}_{2n+2}+\mathbf{P}_2}{(\xi-5)(\xi+3)}$$

Other sums are shown in a similar way. □

Theorem 2.12. *The sum of squares of the first n terms and the sum of products of consecutive terms of the pseudo-Pell and Pell-Lucas sequence are*

$$a)\quad \sum_{i=0}^{n}\mathbf{P}_i^2=\frac{\xi^2\mathbf{R}_{2n}-\mathbf{R}_{2n+2}-\xi-7}{2(\xi-5)(\xi+3)(\xi+4)}+\frac{(-\xi)^{n+1}-1}{2(\xi+1)(\xi+4)},$$

$$b)\quad \sum_{i=0}^{n}\mathbf{R}_i^2=\frac{\xi^2\mathbf{R}_{2n}-\mathbf{R}_{2n+2}-\xi-7}{2(\xi-5)(\xi+3)}-\frac{(-\xi)^{n+1}-1}{2(\xi+1)},$$

$$c)\quad \sum_{i=0}^{n}\mathbf{P}_i\mathbf{P}_{i+1}=\frac{1}{4(\xi+4)}\left[\frac{\xi^2\mathbf{R}_{2n+1}-\mathbf{R}_{2n+3}+2(1-\xi)}{(\xi-5)(\xi+3)}+\frac{2(-\xi)^{n+1}-1}{(\xi+1)}\right],$$

$$d)\quad \sum_{i=0}^{n}\mathbf{R}_i\mathbf{R}_{i+1}=\frac{1}{2}\left[\frac{\xi^2\mathbf{R}_{2n+1}-\mathbf{R}_{2n+3}+2(1-\xi)}{(\xi-5)(\xi+3)}-\frac{2(-\xi)^{n+1}-1}{(\xi+1)}\right].$$

where $\mathbf{P}_n$ *and* $\mathbf{R}_n$ *are the nth pseudo-Pell and Pell-Lucas numbers, respectively.*

Proof.

$$a)\quad \sum_{i=0}^{n}\mathbf{P}_i^2=\sum_{i=0}^{n}\left(\frac{x_1^i-x_2^i}{x_1-x_2}\right)^2=\frac{1}{(x_1-x_2)^2}\left[\sum_{i=0}^{n}(x_1^2)^i+\sum_{i=0}^{n}(x_2^2)^i-2\sum_{i=0}^{n}(x_1x_2)^i\right]$$

$$=\frac{\xi^2\mathbf{R}_{2n}-\mathbf{R}_{2n+2}-\xi-7}{2(\xi-5)(\xi+3)(\xi+4)}+\frac{(-\xi)^{n+1}-1}{2(\xi+1)(\xi+4)}.$$

Sums are shown in a similar way. □

Some special equalities well-known for the Pell and Pell-Lucas sequences have also been calculated for the pseudo-Pell and Pell-Lucas numbers. The proofs of these equations are omitted. $\mathbf{P}_n$ and $\mathbf{R}_n$ be the *nth* pseudo-Pell and Pell-Lucas numbers such that $\xi > 0$ integer, respectively. Then, the following equalities hold:

a) Tagiuri's Identity:

$$\mathbf{P}_{m+k}\mathbf{P}_{n-k}-\mathbf{P}_m\mathbf{P}_n=-(\xi)^{n-k}\mathbf{P}_k\mathbf{P}_{m-n+k}$$

$$\mathbf{R}_{m+k}\mathbf{R}_{n-k}-\mathbf{R}_m\mathbf{R}_n=(\xi+4)(\xi)^{n-k}\mathbf{R}_k\mathbf{R}_{m-n+k}$$

b) d'Ocagne's Identity:

$$\mathbf{P}_{m+1}\mathbf{P}_{n-1}-\mathbf{P}_m\mathbf{P}_n=-(\xi)^{n-1}\mathbf{P}_{m-n+1}$$

$$\mathbf{R}_{m+1}\mathbf{R}_{n-1}-\mathbf{R}_m\mathbf{R}_n=2(\xi+4)(\xi)^{n-1}\mathbf{R}_{m-n+1}$$

c) Catalan's Identity:

$$\mathbf{P}_{n+k}\mathbf{P}_{n-k}-\mathbf{P}_n\mathbf{P}_n=-(\xi)^{n-k}\mathbf{P}_k^2$$

$$\mathbf{R}_{n+k}\mathbf{R}_{n-k}-\mathbf{R}_n\mathbf{R}_n=(\xi+4)(\xi)^{n-k}\mathbf{R}_k^2$$

d) Cassini's Identity:

$$\mathbf{P}_{n+1}\mathbf{P}_{n-1}-\mathbf{P}_n\mathbf{P}_n=-(\xi)^{n-1}$$

$$\mathbf{R}_{n+1}\mathbf{R}_{n-1}-\mathbf{R}_n\mathbf{R}_n=4(\xi+4)(\xi)^{n-1}$$

3. Some numerical examples

In this section, we show four numerical examples for verify our theoretical results. Let us examine the case $\xi = 3$ in some of the results we obtained. For $\xi = 3$, the pseudo-Pell sequence will be

$$\mathbf{P}_{n+1} = 4\mathbf{P}_n + 3\mathbf{P}_{n-1}$$

and the characteristic equation will be $p(x) = x^2 - 4x - 3$.

For $\xi = 3$, The basic matrix of the pseudo-Pell sequence is

$$Q = \begin{bmatrix} 4 & 3 \\ 1 & 0 \end{bmatrix}.$$

For $\xi = 3$, some sum formulas will be as follows

$$\sum_{i=0}^{n+1} \mathbf{P}_i = \frac{14\mathbf{P}_{n+1} + 6\mathbf{P}_n - 1}{12},$$

$$\sum_{i=0}^{n} \mathbf{P}_{2i} = -\frac{9\mathbf{P}_{2n} - \mathbf{P}_{2n+2} + \mathbf{P}_2}{12},$$

$$\sum_{i=0}^{n} \mathbf{P}_{2i+1} = -\frac{9\mathbf{P}_{2n+1} - \mathbf{P}_{2n+3} - 2}{12},$$

$$\sum_{i=0}^{n} \mathbf{P}_i^2 = -\frac{9\mathbf{R}_{2n} - \mathbf{R}_{2n+2} - 10}{168} + \frac{(-3)^{n+1} - 1}{56},$$

$$\sum_{i=0}^{n} \mathbf{P}_i\mathbf{P}_{i+1} = -\frac{1}{28}\left[\frac{9\mathbf{R}_{2n+1} - \mathbf{R}_{2n+3} - 4}{12} + \frac{2(-3)^{n+1} - 1}{4}\right].$$

Some well-known special equations for the pseudo-Pell sequence are obtained for $\xi = 3$ as follows

a) Tagiuri's Identity:

$$\mathbf{P}_{m+k}\mathbf{P}_{n-k} - \mathbf{P}_m\mathbf{P}_n = -(3)^{n-k}\mathbf{P}_k\mathbf{P}_{m-n+k}.$$

b) d'Ocagne's Identity:

$$\mathbf{P}_{m+1}\mathbf{P}_{n-1} - \mathbf{P}_m\mathbf{P}_n = -(3)^{n-1}\mathbf{P}_{m-n+1}.$$

c) Catalan's Identity:

$$\mathbf{P}_{n+k}\mathbf{P}_{n-k} - \mathbf{P}_n\mathbf{P}_n = -(3)^{n-k}\mathbf{P}_k^2.$$

d) Cassini's Identity:

$$\mathbf{P}_{n+1}\mathbf{P}_{n-1} - \mathbf{P}_n\mathbf{P}_n = -(3)^{n-1}.$$

4. Discussion and conclusions

Cigler[5] obtained the Fibonacci and Lucas polynomials, which are defined by

$$F_n(x,s) = xF_{n-1}(x,s) + sF_{n-2}(x,s),$$
$$L_n(x,s) = xL_{n-1}(x,s) + sL_{n-2}(x,s),$$

where $F_0(x,s) = 0$, $F_1(x,s) = 1$, $L_0(x,s) = 2$ and $L_1(x,s) = x$.

There is a relationship between the special cases of pseudo-Pell and Pell-Lucas numbers and the special cases of Fibonacci and Lucas polynomials defined by Cigler as follows

$$\mathbf{P}_n = F_n(4,\xi)$$

and

$$\mathbf{R}_n = \frac{L_n(4,\xi)}{2}.$$

This study presents the pseudo-Pell and Pell-Lucas sequences. We obtain this new sequence, which was not defined in the literature before. Some very important properties of sequence, such as characteristic equation, generating functions, and Binet's formula, are investigated. We obtain the matrix representation of the pseudo-Pell and Pell-Lucas numbers. There have been a large number of studies on numerical sequences in the literature lately, and these sequences have been employed extensively in a variety of academic fields, including biology, finance, physics, architecture, nature, and the arts. Since this study includes some new results, it contributes to the literature by providing essential information concerning the number sequences. Research in these fields can benefit from the pseudo-Pell and pseudo-Pell-Lucas sequences as well. Therefore, we hope that this new number system and properties that we have found will offer a new perspective to the researchers. Some further investigations are as follows:

- Studying the properties of the pseudo-Pell and pseudo-Pell-Lucas sequences quaternions (hybrid, octonion, sedenion, etc.) might be intriguing.
- Examining the partial infinite sum obtained from the pseudo-Pell numbers and pseudo-Pell-Lucas sequences' reciprocals would be fascinating.
- Non-positive values of the integer ξ may also be worth examining.

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