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PC-Asymptotically almost automorphic mild solutions for impulsive integro-differential equations with nonlocal conditions

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Abstract. In this article, we study the existence of PC-asymptotically almost automorphic mild solutions of integro-differential equations with nonlocal conditions via resolvent operators in Banach space. Further, we give sufficient conditions for the solutions to depend continuously on the initial condition. Finally, an example is given to validate the theory part.

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1. Introduction

In one of his most influential papers in 1964, S. Bochner introduced almost automorphic functions [14]. In comparison to almost periodic functions, almost automorphic functions are more general. Many authors had established the almost automorphic solution of differential equations in abstract spaces, totically almost automorphic coefficients. For more on asymptotically almost automorphic functions and related issues, we refer the reader to [25] and the references therein.

N'Guérékata [31] is credited with introducing the concept of asymptotically almost automorphy, which serves as the main topic of discussion in this paper. The study of the existence of almost automorphic and asymptotically almost automorphic

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solutions to differential equations is a highly intriguing subject within mathematical analysis. This topic holds significant appeal due to its potential applications in various fields, including physics, economics, mathematical biology, engineering, etc. We recommend reading [16, 15, 22, 32, 33] and its references for more details on the fundamental theory of almost automorphic functions and its applications.

Grimmer's work on utilizing resolvent operators to demonstrate the existence of integro-differential systems. If one is interested in learning more about resolvent operators and integro-differential systems, one can refer to the following sources: [12, 13, 20, 23, 26, 27]. By consulting these references, one can gain a deeper understanding of the subject and explore further studies cited within them for more in-depth information.

Shocks, harvesting, and natural disasters are a few examples of abrupt changes that frequently affect the dynamics of evolution processes. These brief perturbations are frequently treated as having occurred instantly or as impulses. It is crucial to investigate dynamical systems with impulsive effects. Impulsive differential equations can be used to define a variety of mathematical models in the study of population dynamics, biology, ecology, and epidemics, among other topics. For the theory of impulsive differential equations, and impulsive delay differential equations we refer to [5, 6, 28, 10, 11, 7], and the references therein.

On the other hand, evolution equations with nonlocal initial conditions generalize evolution equations with classical initial conditions. Because more information is considered, this notion is more thorough in explaining natural occurrences than the classical one. See [9, 17, 35, 1, 7], and the references therein for further information on the significance of nonlocal conditions in several branches of applied sciences.

Benchohra *et al.* in [8] have established the existence of asymptotically almost automorphic mild solution to some classes of second order semilinear evolution equation. Moreover, in [18] Cao *et al.* discussed the existence of asymptotically almost automorphic mild solutions for a class of nonautonomous semilinear evolution equations.

Motivated by the last two recent works, we will investigate the existence of PC-asymptotically almost automorphic mild solutions for the following impulsive integro-differential equation with nonlocal conditions:

$$\begin{cases} \phi'(\vartheta) = \mathfrak{Z}\phi(\vartheta) + \int_0^\vartheta \Lambda(\vartheta - \upsilon)\phi(\upsilon)d\upsilon + \Psi\left(\vartheta, \phi(\vartheta)\right); \text{ if } \vartheta \in \widetilde{J}, \\ \phi(\vartheta_\imath^+) - \phi(\vartheta_\imath^-) = I_\imath(\phi(\vartheta_\imath^-)), \ \imath \in \mathbb{N}, \\ \phi(0) = \phi_0 + \Xi(\phi), \end{cases} \tag{1.1}$$

where $J = [0; +\infty)$, $\widetilde{J} = J \setminus \{\vartheta_\imath, \ \imath \in \mathbb{N}\}$, $0 = \vartheta_0 < \vartheta_1 < \vartheta_2 < \ldots < \vartheta_\imath \to +\infty$, and $\mathfrak{Z} : D(\mathfrak{Z}) \subset \mho \to \mho$ is the infinitesimal generator of a strongly continuous semigroup $\{T(\vartheta)\}_{\vartheta \geq 0}$, $\Lambda(\vartheta)$ is a closed linear operator with domain $D(\mathfrak{Z}) \subset D(\Lambda(\vartheta))$, $\phi_0 \in \mho$. The nonlinear term Ψ, Ξ and $I_\imath$ are a given functions. $\phi(\vartheta_\imath^+)$ and $\phi(\vartheta_\imath^-)$ denote the left and right limit of ϕ at $\vartheta = \vartheta_\imath$, respectively. $(\mho, \|\cdot\|)$ is a Banach space.

This paper is organized in five sections. Section 2 is reserved for some preliminary results and definitions which will be utilized throughout this manuscript. In Section 3 we study the existence of PC-asymptotically almost automorphic solutions to the system (1.1). And in section 4 we study the continuous dependence of the mild solutions. In section 5, An example is presented to illustrate the efficiency of the result obtained.

2. Preliminaries

In this section, we will go over some of the notations, definitions, and theorems that will be used throughout the work.

Let $J_0 = [0, \vartheta_1]$, $J_\imath = (\vartheta_\imath, \vartheta_{\imath+1}]$, for $\imath \in \mathbb{N}$, $\xi(\vartheta^+) = \lim_{\vartheta \to \vartheta^+} \xi(\vartheta)$, and define the space of piecewise continuous functions:

$$PC(J, \mho) = \Big\{ \xi : J \to \mho : \xi|_{J_\imath} \text{ is continuous for } \imath \in \mathbb{N} \text{ , such that } \xi(\vartheta_\imath^-) \text{ and } \xi(\vartheta_\imath^+) \text{ exist and satisfy } \xi(\vartheta_\imath^-) = \xi(\vartheta_\imath), \text{ for } \imath \in \mathbb{N} \Big\}.$$

Let

$$BPC(J, \mho) = \{\xi \in PC(J, \mho) \ : \ \xi \ \text{ is bounded on } \mathbb{R}^+\},$$

be a Banach space with

$$\|\xi\|_{BPC} = \sup_{\vartheta \in J} \{\|\xi(\vartheta)\|\}.$$

Let $L^1(J, \mho)$ be the Banach space of measurable functions $\aleph : J \to \mho$ which are Bochner integrable, with the norm

$$\|\aleph\|_{L^1} = \int_0^{+\infty} \|\aleph(\vartheta)\| d\vartheta,$$

We consider the following Cauchy problem

$$\begin{cases} \phi'(\vartheta) = \mathfrak{Z}\phi(\vartheta) + \int_0^\vartheta \Lambda(\vartheta - \upsilon)\phi(\upsilon)d\upsilon; & \text{for } \vartheta \geq 0, \\ \phi(0) = \phi_0 \in \mho. \end{cases} \tag{2.1}$$

The existence and properties of a resolvent operator has been discussed in [26]. In what follows, we suppose the following assumptions:

($R1$) $\mathfrak{Z}$ is the infinitesimal generator of a uniformly continuous semigroup $\{T(\vartheta)\}_{\vartheta > 0}$,

($R2$) For all $\vartheta \geq 0, \Lambda(\vartheta)$ is closed linear operator from $D(\mathfrak{Z})$ to $\mho$ and $\Lambda(\vartheta) \in \Lambda(D(\mathfrak{Z}), \mho)$. For any $\phi \in D(\mathfrak{Z})$, the map $\vartheta \to \Lambda(\vartheta)\phi$ is bounded, differentiable and the derivative $\vartheta \to \Lambda'(\vartheta)\phi$ is bounded uniformly continuous on $\mathbb{R}^+$.

Theorem 2.1. [26] *Assume that* $(R1) - (R2)$ *hold, then there exists a unique resolvent operator for the Cauchy problem (2.1).*

The concept of "PC-almost automorphic operator" was defined by G.M. N'Guérékata and A. Pankov in [34]. So now, we recall some basic definitions and results on almost automorphic functions and asymptotically almost automorphic functions.

Definition 2.2. A function $\aleph \in PC(\mathbb{R}, \mho)$ is said to be $PC-$almost automorphic if

1. The sequence of impulsive moments $\{\vartheta_\imath\}_{\imath\in\mathbb{N}}$ is a almost automorphic sequence
2. For every sequence of real numbers $\{\tau'_n\}$, there exists a subsequence $\{\tau_{n_\imath}\}$ such that
$$\widehat{\aleph}(\vartheta) = \lim_{\imath\to\infty} \aleph\left(\vartheta + \tau_{n_\imath}\right),$$
is well defined for each $\vartheta \in \mathbb{R}$ and
$$\lim_{\imath\to\infty} \widehat{\aleph}\left(\vartheta - \tau_{n_\imath}\right) = \aleph(\vartheta) \quad \text{for each } \vartheta \in \mathbb{R}.$$
Denote by $AA_{PC}(\mathbb{R},\mho)$ the set of all such functions.

Lemma 2.3. [32] *$AA_{PC}(\mathbb{R},\mho)$ is a Banach space with*
$$\|\aleph\|_{PC^*} = \sup_{\vartheta\in\mathbb{R}} \|\aleph(\vartheta)\|.$$

Definition 2.4. A function $\aleph \in PC(\mathbb{R}\times\mho,\mho)$ is said to be $PC-$almost automorphic if

1. The sequence of impulsive moments $\{\vartheta_\imath\}_{\imath\in\mathbb{N}}$ is a almost automorphic sequence.
2. For every sequence of real numbers $\{\tau'_n\}$, there exists a subsequence $\{\tau_{n_\imath}\}$ such that
$$\lim_{\imath\to\infty} \aleph\left(\vartheta + \tau_{n_\imath}, \phi\right) = \widehat{\aleph}(\vartheta,\phi),$$
is well defined for each $\vartheta \in \mathbb{R}$ and
$$\lim_{\imath\to\infty} \widehat{\aleph}\left(\vartheta - \tau_{n_\imath}, \phi\right) = \aleph(\vartheta,\phi),$$
for each $\vartheta \in \mathbb{R}$ and each $\phi \in \mho$.

The collection of those functions is denoted by $AA_{PC}(\mathbb{R}\times\mho,\mho)$.

The space of all piecewise continuous functions $\widetilde{\aleph} : \mathbb{R}^+ \to \mho$ such that $\lim_{\vartheta\to\infty} \widetilde{\aleph}(\vartheta) = 0$ is denoted by $PC_0\left(\mathbb{R}^+,\mho\right)$. Moreover, we denote $PC_0\left(\mathbb{R}^+\times\mho,\mho\right)$; the space of all piecewise continuous functions from $\mathbb{R}^+\times\mho$ to $\mho$ satisfying $\lim_{\vartheta\to\infty} \widetilde{\aleph}(\vartheta,\phi) = 0$ in ϑ and uniformly in $\phi \in \mho$.

Definition 2.5. A function $\aleph : \mathbb{R}^+ \to \mho$ is said to be $PC-$asymptotically almost automorphic if it can be decomposed as
$$\aleph(\vartheta) = \widehat{\aleph}(\vartheta) + \widetilde{\aleph}(\vartheta),$$
where
$$\widehat{\aleph} \in AA_{PC}(\mathbb{R},\mho),\ \widetilde{\aleph} \in PC_0\left(\mathbb{R}^+,\mho\right).$$

Denote by $\mathfrak{G} = AAA_{PC}\left(\mathbb{R}^+,\mho\right)$ the set of all such functions with the norm
$$\|\phi\|_{\mathfrak{G}} = \sup_{\vartheta\in J}\left\{\|\phi(\vartheta)\|\right\}.$$

Definition 2.6. A function $\aleph : \mathbb{R}^+\times\mho \to \mho$ is said to be $PC-$asymptotically almost automorphic if it can be decomposed as
$$\aleph(\vartheta,\phi) = \widehat{\aleph}(\vartheta,\phi) + \widetilde{\aleph}(\vartheta,\phi),$$
where
$$\widehat{\aleph} \in AA_{PC}(\mathbb{R}\times\mho,\mho), \quad \widetilde{\aleph} \in PC_0\left(\mathbb{R}^+\times\mho,\mho\right).$$

Denote by $AAA_{PC}(\mathbb{R}^+ \times \mho, \mho)$ the set of all such functions.

Lemma 2.7. [21] *Let $\aleph \in AA_{PC}(\mathbb{R}^+ \times \mho, \mho)$ and write $\aleph = \widehat{\aleph} + \widetilde{\aleph}$ where $\widehat{\aleph} \in AA_{PC}(\mathbb{R} \times \mho, \mho), \widetilde{\aleph} \in PC_0(\mathbb{R}^+ \times \mho, \mho)$. Suppose that both $\widehat{\aleph}$ and $\widetilde{\aleph}$ are Lipschitz in $x \in \mho$ uniformly in ϑ, i.e., there exists $L_1, L_2 > 0$ such that*

$$\|\widehat{\aleph}(\vartheta, \phi) - \widehat{\aleph}(\vartheta, \widehat{\phi})\| \leq L_1 \|\phi - \widehat{\phi}\| \text{ for a.e } \vartheta \in \mathbb{R} \text{and each } \phi, \widehat{\phi} \in \mho.$$

and

$$\|\widetilde{\aleph}(\vartheta, \phi) - \widetilde{\aleph}(\vartheta, \widehat{\phi})\| \leq L_2 \|\phi - \widehat{\phi}\| \text{ for a.e } \vartheta \in \mathbb{R}^+ \text{and each } \phi, \widehat{\phi} \in \mho.$$

Then $\phi \in AAA_{PC}(\mathbb{R}^+, \mho)$ implies that $\aleph(\cdot, \phi(\cdot)) \in AAA_{PC}(\mathbb{R}^+, \mho)$.

Lemma 2.8. [29] *$\aleph : \mathbb{R} \times \mho \to \mho$ is PC-almost automorphic, and assume that $\aleph(\vartheta, \cdot)$ is uniformly continuous on each bounded subset $\daleth \subset \mho$ uniformly for $\vartheta \in \mathbb{R}$, that is for any $\varepsilon > 0$, there exists $\varrho > 0$ such that $\phi, \widehat{\phi} \in \daleth$ and $\|\phi(\vartheta) - \widehat{\phi}(\vartheta)\| < \varrho$ imply that $\|\aleph(\vartheta, \phi) - \aleph(\vartheta, \widehat{\phi})\| < \varepsilon$ for all $\vartheta \in \mathbb{R}$. Let $\varphi : \mathbb{R} \to \mho$ be PC-almost automorphic. Then the function $\widehat{\daleth} : \mathbb{R} \to \mho$ defined by $\widehat{\daleth}(\vartheta) = \aleph(\vartheta, \varphi(\vartheta))$ is PC-almost automorphic.*

Lemma 2.9. [29] *Suppose that $\aleph(\vartheta, \phi) = \widehat{\aleph}(\vartheta, \phi) + \widetilde{\aleph}(\vartheta, \phi)$ is an asymptotically almost automorphic function with $\widehat{\aleph} \in AA_{PC}(\mathbb{R} \times \mho, \mho), \widetilde{\aleph} \in PC_0(\mathbb{R}^+ \times \mho, \mho)$, and $\widehat{\aleph}$ is uniformly continuous on any bounded subset $\daleth \subset X$ uniformly for $\vartheta \in \mathbb{R}$. Then $\phi \in AAA_{PC}(\mathbb{R}, \mho)$ implies $\aleph \in AAA_{PC}(\mathbb{R}, \mho)$*

Now, for $\vartheta \in \mathbb{R}^+$ we define the following functions:

$$\Phi_1(\vartheta) = \int_{-\infty}^{\vartheta} \Re(\vartheta - \upsilon) Y(\upsilon) d\upsilon, \text{ and } \Phi_2(\vartheta) = \int_0^{\vartheta} \Re(\vartheta - \upsilon) Z(\upsilon) d\upsilon.$$

Lemma 2.10. *We assume that*

($R3$) *The resolvent $\Re(\vartheta)$ is exponentially stable i.e, there exist $\mathfrak{X}_{\Re} \geq 1$ and $b \geq 0$, such that*

$$\|\Re(\vartheta)\|_{B(\mho)} \leq \mathfrak{X}_{\Re} e^{-b\vartheta}, \quad \text{for all } \vartheta \in J.$$

Then

(i) *If $Y \in AA_{PC}(\mathbb{R}, \mho)$, then $\Phi_1 \in AA_{PC}(\mathbb{R}, \mho)$.*
(ii) *If $Z \in PC_0(\mathbb{R}^+, \mho)$, then $\Phi_2 \in PC_0(\mathbb{R}^+, \mho)$.*

Proof. For (i), choose a bounded subset $\daleth$ of $\mho$ such that $Y(\vartheta) \in \daleth$ for all $\vartheta \in \mathbb{R}$. Since $Y \in AA_{PC}(\mathbb{R}, \mho)$ and the resolvent $\Re(\vartheta)$ is exponentially stable it follows that for every sequence of real numbers τ'_n, we can extract a subsequence $\tau_{n_\imath}$ such that

(i_1) $\lim_{\imath \to +\infty} Y(\vartheta + \tau_{n_\imath}) = \widetilde{Y}(\vartheta)$,
(i_2) $\lim_{\imath \to +\infty} \widetilde{Y}(\vartheta - \tau_{n_\imath}) = Y(\vartheta)$.

Write

$$\widetilde{\Phi}_1(\vartheta) := \int_{-\infty}^{\vartheta} \Re(\vartheta - \upsilon) \widetilde{Y}(\upsilon) d\upsilon, \quad \vartheta \in \mathbb{R}^+.$$

Then

$$\|\Phi_1(\vartheta + \tau_{n_\imath}) - \widetilde{\Phi}_1(\vartheta)\| = \left\| \int_{-\infty}^{\vartheta + \tau_{n_\imath}} \Re(\vartheta + \tau_{n_\imath} - \upsilon) Y(\upsilon) d\upsilon - \int_{-\infty}^{\vartheta} \Re(\vartheta - \upsilon) \widetilde{Y}(\upsilon) \right\|$$

$$
\begin{aligned}
&= \left\| \int_{-\infty}^{\vartheta} \Re(\vartheta - \upsilon) Y(\upsilon + \tau_{n_\imath}) d\upsilon - \int_{-\infty}^{\vartheta} \Re(\vartheta - \upsilon) \widetilde{Y}(\upsilon) \right\| \\
&\leq \int_{-\infty}^{\vartheta} \|\Re(\vartheta - \upsilon)\| \left\| Y(\upsilon + \tau_{n_\imath}) - \widetilde{Y}(\upsilon) \right\| d\upsilon \\
&\leq \frac{\mathfrak{X}_{\Re}}{b} \sup_{\vartheta \in \mathbb{R}} \left\| Y(\vartheta + \tau_{n_\imath}) - \widetilde{Y}(\vartheta) \right\|.
\end{aligned}
$$

Since the resolvent $\Re(\vartheta)$ is exponentially stable together with the Lebesgue dominated convergence theorem and (i_1) it follows that

$$\lim_{\imath \to +\infty} \Phi_1(\vartheta + \tau_{n_\imath}) = \widetilde{\Phi}_1(\vartheta), \quad \vartheta \in \mathbb{R}.$$

Similarly by (i_2) we can prove that

$$\lim_{\imath \to +\infty} \widetilde{\Phi}_1(\vartheta - \tau_{n_\imath}) = \Phi_1(\vartheta), \quad \vartheta \in \mathbb{R}.$$

Hence, $\Phi_1 \in AA_{PC}(\mathbb{R}, \mho)$.
Now for (ii), one can choose $\varkappa > 0$ such that

$$\|Z(\vartheta)\| < \varepsilon, \qquad \forall \vartheta > \varkappa.$$

This enables us to conclude that for all $\vartheta > \varkappa$,

$$
\begin{aligned}
\|\Phi_2(\vartheta)\| &= \left\| \int_0^{\vartheta} \Re(\vartheta - \upsilon) Z(\upsilon) d\upsilon \right\| \\
&= \left\| \int_0^{\varkappa} \Re(\vartheta - \upsilon) Z(\upsilon) d\upsilon + \int_{\varkappa}^{\vartheta} \Re(\vartheta - \upsilon) Z(\upsilon) d\upsilon \right\| \\
&\leq \left\| \int_0^{\varkappa} \Re(\vartheta - \upsilon) Z(\upsilon) d\upsilon \right\| + \left\| \int_{\varkappa}^{\vartheta} \Re(\vartheta - \upsilon) Z(\upsilon) d\upsilon \right\| \\
&\leq \frac{\mathfrak{X}_{\Re} e^{-b(\vartheta - \varkappa)}}{b} \|Z\| + \frac{\mathfrak{X}_{\Re} \varepsilon}{b}.
\end{aligned}
$$

Consequently $\lim_{\vartheta \to +\infty} \|\Phi_2(\vartheta)\| = 0$.
Now, we define the Kuratowski measure of noncompactness.

Definition 2.11. [4] Let $\Bbbk$ be a Banach space and $\nabla_{\Bbbk}$ the bounded subsets of $\Bbbk$. The Kuratowski measure of noncompactness is the map $\alpha : \nabla_{\Bbbk} \to [0, \infty)$ defined by

$$\alpha(\Im) = \inf\{\epsilon > 0 : \Im \subseteq \cup_{i=1}^{n} \Im_i \text{ and } \operatorname{diam}(\Im_i) \leq \epsilon\}; \text{ here } \Im \in \nabla_{\Bbbk},$$

where

$$\operatorname{diam}(\Im_i) = sup\{\|\xi - \widehat{\xi}\| : \xi, \widehat{\xi} \in \Im_i\}.$$

Lemma 2.12. [24] *If Y is a bounded subset of a Banach space $\Bbbk$, then for each $\epsilon > 0$, there is a sequence $\{\phi_\imath\}_{\imath=1}^{\infty} \subset Y$ such that*

$$\alpha(Y) \leq 2\alpha\left(\{\phi_\imath\}_{\imath=1}^{\infty}\right) + \epsilon.$$

Lemma 2.13. [30] *If* $\{\phi_\imath\}_{\imath=0}^{\infty} \subset L^1$ *is uniformly integrable, then the function* $\vartheta \to \alpha(\{\phi_\imath(\vartheta)\}_{\imath=0}^{\infty})$ *is measurable and*

$$\alpha\left(\left\{\int_0^{\vartheta} \phi_\imath(\upsilon)d\upsilon\right\}_{\imath=0}^{\infty}\right) \leq 2\int_0^{\vartheta} \alpha\left(\{\phi_\imath(\upsilon)\}_{\imath=0}^{\infty}\right)d\upsilon.$$

Theorem 2.14. (*Darbo's fixed point theorem,* [19]). *Let* $\Im$ *be a nonempty, bounded, closed and convex subset of a Banach space* $\Bbbk$ *and let* $T : \Im \to \Im$ *be a continuous mapping. Assume that there exists a constant* $\imath \in [0, 1)$*, such that*

$$\alpha(TM) \leq \imath\alpha(M),$$

for any nonempty subset M *of* $\Im$*. Then* T *has a fixed point in set* $\Im$*.*

3. The main result

In this section we discuss existence of PC-asymptotically almost automorphic mild solutions via resolvent operators for problem (1.1). In order to establish a measure of noncompactness in the space $\mathfrak{G}$, let us first recall the specific measure of noncompactness that results from [8]. This measure will be used in our main results. Let us fix a nonempty bounded subset $\Im$ in the space $\mathfrak{G}$, for $\widehat{\xi} \in \Im$, $\varkappa > 0$, $\epsilon > 0$ *and* $\kappa, \tau \in [0, \varkappa]$, such that $|\kappa - \tau| \leq \epsilon$. We denote $\omega^{\varkappa}(\widehat{\xi}, \epsilon)$ the modulus of continuity of the function $\widehat{\xi}$ on the interval $[0, \varkappa]$, namely,

$$\begin{aligned} \omega^{\varkappa}(\widehat{\xi}, \epsilon) &= \sup\{\|\widehat{\xi}(\kappa) - \widehat{\xi}(\tau)\| \ ; \ \kappa, \tau \in [0, \varkappa] \cap \widetilde{J}\}, \\ \omega_0(\Im) &= \lim_{\varkappa\to+\infty} \lim_{\epsilon\to 0} \sup\{\omega^{\varkappa}(\widehat{\xi}, \epsilon) \ ; \ \widehat{\xi} \in \Im\}. \end{aligned}$$

Finally, consider the function χ_* defined on the family of subset of $\mathfrak{G}$ by the formula

$$\chi_*(\Im) = \omega_0(\Im) + \sup_{\vartheta\in J} \alpha(\Im(\vartheta)),$$

and notice that if the set $\Im$ is equicontinuous and equiconvergent, then $\omega_0(\Im) = 0$.

Definition 3.1. A function $\phi \in \mathfrak{G}$ is called a PC-asymptotically almost automorphic mild solution of problem (1.1), if it satisfies the following integral equation

$$\begin{aligned} \phi(\vartheta) = \Re(\vartheta)(\phi_0 + \Xi(\phi)) &+ \int_0^{\vartheta} \Re(\vartheta - \upsilon)\Psi(\upsilon, \phi(\upsilon))d\upsilon \\ &+ \sum_{0<\vartheta_\imath<\vartheta} \Re(\vartheta - \vartheta_\imath)I_\imath(\phi(\vartheta_\imath^-)), \quad \vartheta \in J. \end{aligned}$$

The following hypotheses will be used in the sequel.

($A1$) Assume that ($R1$) – ($R3$) hold.

($A2$) *i*) The sequence of impulsive moments $\vartheta_\imath$ is asymptotically almost automorphic.

ii) $\Psi : J \times \mho \longrightarrow \mho$ is a Carathéodory function and PC-asymptotically almost automorphic ie., $\Psi(\vartheta, \phi) = \widehat{\aleph}(\vartheta, \phi) + \widetilde{\aleph}(\vartheta, \phi)$ with

$$\widehat{\aleph} \in AA_{PC}(\mathbb{R} \times \mho, \mho), \quad \widetilde{\aleph} \in PC_0(\mathbb{R}^+ \times \mho, \mho).$$

iii) There exists a function $\beta \in L^1(J, \mathbb{R}^+)$, such that :

$$\|\Psi(\vartheta, x) - \Psi(\vartheta, \phi)\| \ \leq \ \beta(\vartheta)\|x - \phi\|, \text{ for all } x, \phi \in \mho.$$

Also we assume that $\Psi(\vartheta, 0) = 0$.

$(A3)$ $\Xi : \mathfrak{G} \to \mho$ is continuous and there exists $L_\Xi > 0$, such that,

$$\|\Xi(\xi) - \Xi(\widehat{\xi})\| \leq L_\Xi \|\xi - \widehat{\xi}\|_{\mathfrak{G}}, \text{ for all } \xi, \widehat{\xi} \in \mathfrak{G}.$$

Also we assume that $\Xi(0) = 0$.

$(A4)$ $I_\imath : \mho \to \mho$ is Lipschitz continuous with Lipschitz constants $m_\imath$, $\imath \in \mathbb{N}$, such that

$$\|I_\imath(\kappa_3) - I_\imath(\kappa_4)\| \leq m_\imath \|\kappa_3 - \kappa_4\|, \text{for all } \kappa_3, \ \kappa_4 \in \mho, \ \imath \in \mathbb{N}.$$

And $I_\imath(0) = 0$.

Theorem 3.2. *Assume that the conditions* $(A1) - (A4)$ *are satisfied. If*

$$\mathfrak{X}_{\Re} \left(L_\Xi + 4\|\beta\|_{L^1} + \sum_{\imath=0}^{\infty} m_\imath \right) < 1,$$

then the problem (1.1) has a PC-asymptotically almost automorphic mild solution.

Proof. Consider the operator $\Theta : \mathfrak{G} \longrightarrow \mathfrak{G}$ defined by

$$(\Theta\phi)(\vartheta) = \Re(\vartheta)(\phi_0 + \Xi(\phi)) + \int_0^\vartheta \Re(\vartheta - \upsilon)\Psi(\upsilon, \phi(\upsilon))d\upsilon$$
$$+ \sum_{0<\vartheta_\imath<\vartheta} \Re(\vartheta - \vartheta_\imath) I_\imath(\phi(\vartheta_\imath^-)), \quad \vartheta \in J,$$

where $\phi \in \mathfrak{G}$ with $\phi = \phi_1 + \phi_2$, ϕ_1 is the principal term and ϕ_2 the corrective term of ϕ_1.

Step 1 : Θ is well-defined, i.e $\Theta(\mathfrak{G}) \subset \mathfrak{G}$.

We have $\Theta(\mathfrak{G}) \subset PC(J, \mho)$. Now, let

$$\zeta(\vartheta) = \Re(\vartheta) \left(\phi_0 + \Xi(\phi)\right),$$

then

$$\|\zeta(\vartheta)\| \leq \mathfrak{X}_{\Re} e^{-bt} \left(\|\phi_0\| + L_\Xi \|\phi\|\right).$$

Since $b > 0$, we get $\lim_{\vartheta \longrightarrow +\infty} |\zeta(\vartheta)| = 0$. Thus $\zeta \in PC_0(\mathbb{R}^+, \mho)$.

From assumption $(A2)$, we can write

$$\begin{aligned} \Psi(\vartheta, \phi(\vartheta)) &= \widehat{\aleph}(\vartheta, \phi_2(\vartheta)) + \Psi(\vartheta, \phi(\vartheta)) - \Psi(\vartheta, \phi_2(\vartheta)) + \widetilde{\aleph}(\vartheta, \phi_2(\vartheta)) \\ &= \widehat{\aleph}(\vartheta, \phi_2(\vartheta)) + \mathfrak{U}(\vartheta, \phi(\vartheta)). \end{aligned}$$

Then, we get

$$\int_0^\vartheta \Re(\vartheta - \upsilon)\Psi(\upsilon, \phi(\upsilon))d\upsilon$$
$$= \int_0^\vartheta \Re(\vartheta - \upsilon)\widehat{\aleph}(\upsilon, \phi_2(\upsilon))d\upsilon + \int_0^\vartheta \Re(\vartheta - \upsilon)\mathfrak{U}(\upsilon, \phi(\upsilon))d\upsilon$$

$$= \int_{-\infty}^{\vartheta} \Re(\vartheta - \upsilon)\widehat{\aleph}(\upsilon, \phi_2(\upsilon))d\upsilon + \int_{-\infty}^{0} \Re(\vartheta - \upsilon)\widehat{\aleph}(\upsilon, \phi(\upsilon))d\upsilon$$
$$+ \int_{0}^{\vartheta} \Re(\vartheta - \upsilon)\mathfrak{U}(\upsilon, \phi(\upsilon))d\upsilon$$
$$= \Upsilon_1\phi(\vartheta) + \Upsilon_2\phi(\vartheta),$$

where

$$(\Upsilon_1\phi)(\vartheta) = \int_{-\infty}^{\vartheta} \Re(\vartheta - \upsilon)\widehat{\aleph}(\upsilon, \phi_2(\upsilon))d\upsilon,$$
$$(\Upsilon_2\phi)(\vartheta) = \int_{-\infty}^{0} \Re(\vartheta - \upsilon)\widehat{\aleph}(\upsilon, \phi(\upsilon))d\upsilon + \int_{0}^{\vartheta} \Re(\vartheta - \upsilon)\mathfrak{U}(\upsilon, \phi(\upsilon))d\upsilon,$$

and

$$(\Delta_1\phi)(\vartheta) = \int_{0}^{\vartheta} \Re(\vartheta - \upsilon)\mathfrak{U}(\upsilon, \phi(\upsilon))d\upsilon,$$
$$(\Delta_2\phi)(\vartheta) = \int_{-\infty}^{0} \Re(\vartheta - \upsilon)\widehat{\aleph}(\upsilon, \phi_2(\upsilon))d\upsilon.$$

Using $(A2)$ and Lemma 2.8, We deduce that $\upsilon \longrightarrow \widehat{\aleph}(\upsilon, \phi_2(\upsilon))$ is in $AA_{PC}(\mathbb{R} \times \mho, \mho)$. Thus, by Lemma 2.8, we obtain

$$\Upsilon_1\phi \in AA_{PC}(\mathbb{R}^+ \times \mho, \mho).$$

Let us prove that $\Delta_1\phi \in PC_0(\mathbb{R}^+ \times \mho, \mho)$, indeed by definition $\mathfrak{U} \in PC_0(\mathbb{R}^+ \times \mho, \mho)$, that means given $\varepsilon > 0$, there exists $\varkappa > 0$ such that for $\vartheta \geq \varkappa$, we have $\|\mathfrak{U}(\vartheta, \phi)\| \leq \varepsilon$. Therefore if $\vartheta \geq \varkappa$, we get

$$\int_{\varkappa}^{\vartheta} \|\Re(\vartheta - \upsilon)\|\|\mathfrak{U}(\upsilon, \phi(\upsilon))\|d\upsilon \leq \mathfrak{X}_{\Re}\varepsilon \int_{\varkappa}^{\vartheta} e^{-b(\vartheta-\upsilon)}d\upsilon \leq \frac{\mathfrak{X}_{\Re}}{b}\varepsilon,$$

then

$$\|(\Delta_1\phi)(\vartheta)\| \leq \frac{\mathfrak{X}_{\Re}}{b}\varepsilon.$$

Thus

$$\Delta_1 \in PC_0(\mathbb{R}^+ \times \mho, \mho).$$

Next, let us show that $\Delta_2\phi \in PC_0(\mathbb{R}^+ \times \mho, \mho)$, we have

$$\begin{aligned} \|(\Delta_2\phi)(\vartheta)\| &= \left\|\int_{-\infty}^{0} \Re(\vartheta - \upsilon)\widehat{\aleph}(\upsilon, \phi_2(\upsilon))d\upsilon\right\| \\ &\leq \mathfrak{X}_{\Re} \sup_{\vartheta\in\mathbb{R}} \|\widehat{\aleph}(\vartheta, \phi_2(\vartheta))\| \int_0^{\varkappa} e^{-b(\vartheta-\upsilon)}d\upsilon \\ &\quad +\mathfrak{X}_{\Re}\|\widehat{\aleph}\|_{PC^*} \frac{e^{-b(\vartheta-\varkappa)}}{b} \longrightarrow 0, \ as\ \vartheta \longrightarrow \infty. \end{aligned}$$

Therefore, $\Delta_2\phi \in PC_0(\mathbb{R}^+ \times \mho, \mho)$.
Also we have

$$\| \sum_{0<\vartheta_\imath<\vartheta} \Re(\vartheta - \vartheta_\imath)I_\imath(\phi(\vartheta_\imath)\| \leq \mathfrak{X}_{\Re}\|\phi\|_{\mathfrak{S}} \sum_{\imath=0}^{+\infty} e^{-b(\vartheta-\vartheta_\imath)}m_\imath \longrightarrow 0, \ as\ \vartheta \longrightarrow \infty.$$

Consequently, from the previous estimates we deduce that $\Theta(\mathfrak{G}) \subset \mathfrak{G}$.
Next, we shall check that operator Θ satisfies all conditions of Darbo's theorem.
Let $\Im_\theta = \{\phi \in \mathfrak{G} \ ; \ \|\phi\|_{\mathfrak{G}} \leq \ \theta\}$, the set $\Im_\theta$ is bounded, closed and convex.
Step 2 : $\Theta(\Im_\theta) \subset \Im_\theta$.
For each $\phi \in \Im_\theta$ and by $(A1)$, $(A2)$ and $(A3)$, we have

$$\|\Xi(\phi)\| \leq L_\Xi \|\phi\|_{\mathfrak{G}}.$$

Then,

$$\begin{aligned}
\|\Theta\phi(\vartheta)\| &\leq \mathfrak{X}_{\Re}(\|\phi_0\| + \|\Xi(\phi)\|) + \mathfrak{X}_{\Re}\int_0^\vartheta \|\Psi(\upsilon, \phi(\upsilon))\| d\upsilon \\
&\quad + \mathfrak{X}_{\Re} \sum_{0<\vartheta_\imath<\vartheta} \|I_\imath(\phi(\vartheta_\imath^-))\| \\
&\leq \mathfrak{X}_{\Re}(\|\phi_0\| + L_\Xi\theta) + \mathfrak{X}_{\Re}\theta\|\beta\|_{L^1} + \mathfrak{X}_{\Re}\theta\sum_{\imath=0}^{\infty} m_\imath.
\end{aligned}$$

Hence $\Theta(\Im_\theta) \subset \Im_\theta$, provided that

$$\theta > \frac{\mathfrak{X}_{\Re}\|\phi_0\|}{1 - \mathfrak{X}_{\Re}(L_\Xi + \|\beta\|_{L^1} + \sum_{\imath=0}^{\infty} m_\imath)}.$$

Step 3: Θ is continuous.
Let x_m be a sequence such that $\phi_m \to \phi_*$ in $\mathfrak{G}$, then we have,

$$\begin{aligned}
&\|(\Theta\phi_m)(\vartheta) - (\Theta\phi_*)(\vartheta)\| \\
&\quad \leq \quad \mathfrak{X}_{\Re}\|\Xi(\phi_m) - \Xi(\phi_*)\| + \mathfrak{X}_{\Re}\int_0^\vartheta \|\Psi(\upsilon, \phi_m(\upsilon)) - \Psi(\upsilon, \chi_*(\upsilon))\| d\upsilon \\
&\quad\quad + \mathfrak{X}_{\Re}\sum_{0<\vartheta_\imath<\vartheta} m_\imath \|\phi_m(\vartheta_\imath^-) - \phi_*(\vartheta_\imath)\|.
\end{aligned}$$

Since the function Ψ is Carathédory and Ξ is continuous, the Lebesgue dominated converge theorem implies that :

$$\|(\Theta\phi_m) - (\Theta\phi_*)\|_{\mathfrak{G}} \to 0, \ as \ m \to +\infty.$$

Thus, Θ is continuous.
Step 4: $\Theta(\Im_\theta)$ is equicontinuous. Let $\vartheta_1, \vartheta_2 \in J$ with $\vartheta_2 > \vartheta_1$. For all $\phi \in \Im_\theta$, we have

$$\begin{aligned}
&\|(\Theta\phi)(\vartheta_2) - (\Theta\phi)(\vartheta_1)\| \\
&\quad = \quad \|\int_0^{\vartheta_2} \Re(\vartheta_2 - \upsilon)\Psi(\upsilon, \phi(\upsilon)) d\upsilon - \int_0^{\vartheta_1} \Re(\vartheta_1 - \upsilon)\Psi(\upsilon, \phi(\upsilon)) d\upsilon \\
&\quad\quad + \sum_{0<\vartheta_\imath<\vartheta_2} \Re(\vartheta_2 - \vartheta_\imath) I_\imath(\phi(\vartheta_\imath)) - \sum_{0<\vartheta_\imath<\vartheta_1} \Re(\vartheta_1 - \vartheta_\imath) I_\imath(\phi(\vartheta_\imath))\| \\
&\quad \leq \quad \int_0^{\vartheta_1} \|\Re(\vartheta_2 - \upsilon) - \Re(\vartheta_1 - \upsilon)\| \, \|\Psi(\upsilon, \phi(\upsilon))\| \, d\upsilon
\end{aligned}$$

$$
\begin{aligned}
& + \int_{\vartheta_1}^{\vartheta_2} \|\Re(\vartheta_2 - \upsilon)\| \, \|\Psi(\upsilon, \phi(\upsilon))d\upsilon\| \\
& + \sum_{\vartheta_1 < \vartheta_\imath < \vartheta_2} \|\Re(\vartheta_2 - \vartheta_\imath)\| \|I_\imath(\phi(\vartheta_\imath))\| \\
& + \sum_{0 < \vartheta_\imath < \vartheta_1} \|(\Re(\vartheta_2 - \vartheta_\imath) - \Re(\vartheta_1 - \vartheta_\imath)\| \|I_\imath(\phi(\vartheta_\imath))\| \\
\leq \;& \theta \int_0^{\vartheta_1} \|(\Re(\vartheta_2 - \upsilon) - \Re(\vartheta_1 - \upsilon))\| \beta(\upsilon)d\upsilon + \mathfrak{X}_\Re \theta \int_{\vartheta_1}^{\vartheta_2} \beta(\upsilon)d\upsilon \\
& + \theta \sum_{0 < \vartheta_\imath < \vartheta_1} m_\imath \|\Re(\vartheta_2 - \vartheta_\imath) - \Re(\vartheta_1 - \vartheta_\imath)\| + \mathfrak{X}_\Re \theta \sum_{\vartheta_1 < \vartheta_\imath < \vartheta_2} m_\imath e^{-b(\vartheta_2 - \vartheta_\imath)}.
\end{aligned}
$$

Since $\Re(\vartheta)$ is strongly continuous and $\beta \in L^1$, we get

$$
\|(\Theta\phi)(\vartheta_2) - (\Theta\phi)(\vartheta_1)\| \longrightarrow 0 \quad as \quad \vartheta_2 \longrightarrow \vartheta_1,
$$

which implies that $\Theta(\Im_\theta)$ is equicontinuous.
Step 5: $\Theta(\Im_\theta)$ is equiconvergent.
For $\phi \in \Im_\theta$ and $\vartheta \in J$, we have

$$
\begin{aligned}
\|(\Theta\phi)(\vartheta)\| \;\leq\;& \|\Re(\vartheta)\|_{B(\mho)}[\|\phi_0\| + \|\Xi(\phi)\|] + \int_0^{\vartheta} \|\Re(\vartheta - \upsilon)\| \beta(\upsilon) \|\phi(\upsilon)\| d\upsilon \\
& + \sum_{0 < \vartheta_\imath < \vartheta} \|\Re(\vartheta - \vartheta_\imath)\| \|I_\imath(\phi(\vartheta_\imath))\| \\
\leq\;& \mathfrak{X}_\Re e^{-b\vartheta} (\|\phi_0\| + L_\Xi \theta) + \mathfrak{X}_\Re \theta \int_0^{\vartheta} e^{-b(\vartheta - \upsilon)} \beta(\upsilon) d\upsilon \\
& + \mathfrak{X}_\Re \theta \sum_{\imath=0}^{p} e^{-b(\vartheta - \vartheta_\imath)} m_\imath \\
& \longrightarrow \mathfrak{X}_\Re (\|\phi_0\| + L_\Xi \theta) + \mathfrak{X}_\Re \theta \|\beta\|_{L^1} \; as \quad \vartheta \longrightarrow +\infty.
\end{aligned}
$$

Then

$$
\|(\Theta\phi)(\vartheta) - (\Theta\phi)(+\infty)\| \longrightarrow 0 \quad as \quad \vartheta \longrightarrow +\infty.
$$

Step 6: Let ∇ be a bounded equicontinuous subset of $\Im_\theta$, we have $\{\Theta(\nabla)\}$ is equicontinuous and in addition to the estimate given in step 1 and step 5 we have, $\omega_0 (\Theta(\nabla)) = 0$.
From Lemma 2.12 and 2.13 it follow that for any $\varrho > 0$, there exists a sequence $\{\phi_m\}_{\imath=0}^{\infty} \subset \nabla$ such that

$$
\begin{aligned}
& \alpha \left(\int_0^{\vartheta} \Re(\vartheta - \upsilon) \Psi(\upsilon, \phi(\upsilon)) d\upsilon \; ; \phi \in \nabla \right) \\
& \leq 2\alpha \left(\int_0^{\vartheta} \Re(\vartheta - \upsilon) \Psi(\upsilon, \phi_m(\upsilon)) d\upsilon \; ; \phi \in \nabla \right) + \varrho \\
& \leq 4 \int_0^{\vartheta} \alpha \Big(\Re(\vartheta - \upsilon) \Psi(\upsilon, \phi_m(\upsilon)) d\upsilon \quad ; \phi \in \nabla \Big) + \varrho.
\end{aligned}
$$

For any bounded set $\nabla \subset \mho$ and $\vartheta \in J$, and by [2] the Lipschitz conditions on the functions $\Psi,\ \Xi$ and $I_\imath$, we get

$$\begin{aligned} \alpha(\Psi(\vartheta,\nabla(\vartheta))) &\leq \beta(\vartheta)\alpha(\nabla(\vartheta)), \\ \alpha(I_\imath(\vartheta,\nabla(\vartheta))) &\leq m_\imath\alpha(\nabla(\vartheta)), \\ \alpha(\Xi(\nabla(\vartheta))) &\leq L_\Xi\chi_*(\nabla). \end{aligned}$$

Then

$$\begin{aligned} \alpha(\Theta\nabla(\vartheta)) &\leq \mathfrak{X}_\Re L_\Xi \chi_*(\nabla) + 4\int_0^\vartheta \mathfrak{X}_\Re\beta(\upsilon)\alpha(\nabla(\upsilon))d\upsilon \\ &\quad + \mathfrak{X}_\Re \sum_{0<\vartheta_\imath<\vartheta} m_\imath\alpha(\nabla(\vartheta_\imath)) + \varrho \\ &\leq \mathfrak{X}_\Re L_\Xi\chi_*(\nabla) + 4\mathfrak{X}_\Re\|\beta\|_{L^1}\sup_{\vartheta\in J}\alpha(\nabla(\vartheta)) \\ &\quad + \mathfrak{X}_\Re \sum_{0<\vartheta_\imath<\vartheta} m_\imath\alpha(\nabla(\vartheta_\imath)) + \varrho. \end{aligned}$$

Since ϱ is arbitrary, we obtain

$$\alpha(\Theta\nabla(\vartheta)) \leq \mathfrak{X}_\Re L_\Xi\chi_*(\nabla) + \mathfrak{X}_\Re\left(4\|\beta\|_{L^1} + \sum_{0<\vartheta_\imath<\vartheta} m_\imath\right)\sup_{\vartheta\in J}\alpha(\nabla(\vartheta)).$$

Therefore

$$\chi_*(\Theta\nabla) \leq \mathfrak{X}_\Re\left(L_\Xi + 4\|\beta\|_{L^1} + \sum_{0<\vartheta_\imath<\vartheta} m_\imath\right)\chi_*(\nabla).$$

Thus Θ is χ_*-contraction. By Theorem 2.13 we conclude that Θ has at least one fixed point $\phi \in \Im_\theta$, which is a PC-asymptotically almost automorphic mild solution of problem (1.1) .

4. Continuous dependence on the initial condition

In this section we need the following lemma:

Lemma 4.1. [3] *Let the following inequality holds:*

$$\xi(\vartheta) \leq a(\vartheta) + \int_0^\vartheta b(\upsilon)d\upsilon + \sum_{0\leq\vartheta_\imath<\vartheta} \varsigma_\imath\xi\left(\vartheta_\imath^-\right), \quad \vartheta \geq 0,$$

where $\xi, a, b \in PC\left(\mathbb{R}^+, \mathbb{R}^+\right)$, *and* a *is nondecreasing,* $b(\vartheta) > 0,\ \varsigma_\imath > 0,\ \imath \in \mathbb{N}$. *Then, for* $\vartheta \in \mathbb{R}^+$, *the following inequality is valid:*

$$\xi(\vartheta) \leq a(\vartheta)(1+\varsigma)^\imath \exp\left(\int_0^\vartheta b(\upsilon)d\upsilon\right) \quad \vartheta \in [\vartheta_\imath, \vartheta_{\imath+1}], \imath \in \mathbb{N},$$

where $\varsigma = \max\{\varsigma_\imath : \imath \in \mathbb{N}\}$.

Theorem 4.2. *If the assumption of Theorems 3.2 are fulfilled, then the solution of the problem (1.1) depends continuously on the initial condition.*

Proof. Let $\phi_0, \phi_0^* \in \mho$. From Theorem 3.2, there exist $\phi(\cdot, \phi_0)$, $\phi^*(\cdot, \phi_0^*) \in \mathfrak{G}$ such that

$$\phi(\vartheta) = \Re(\vartheta)(\phi_0 + \Xi(\phi)) + \int_0^\vartheta \Re(\vartheta - \upsilon)\Psi(\upsilon, \phi(\upsilon))d\upsilon + \sum_{0<\vartheta_\imath<\vartheta} \Re(\vartheta - \vartheta_\imath)I_\imath(\phi(\vartheta_\imath^-)),\ \vartheta \in J,$$

and

$$\phi^*(\vartheta) = \Re(\vartheta)[\phi_0^* + \Xi(\phi)] + \int_0^\vartheta \Re(\vartheta - \upsilon)\Psi(\upsilon, \phi^*(\upsilon))d\upsilon + \sum_{0<\vartheta_\imath<\vartheta} \Re(\vartheta - \vartheta_\imath)I_\imath((\vartheta_\imath^-)),\ \vartheta \in J.$$

Then for $\varpi(\vartheta) = \|\phi(\vartheta) - \phi^*(\vartheta)\|$, we have

$$\begin{aligned} \sup_{\vartheta \in J} \varpi(\vartheta) &\le \mathfrak{X}_\Re \|\phi_0 - \phi_0^*\| + \mathfrak{X}_\Re \|\Xi(\phi) - \Xi(\phi^*)\| + \mathfrak{X}_\Re \int_0^\vartheta \beta(\upsilon)\varpi(\upsilon)d\upsilon \\ &\quad + \mathfrak{X}_\Re \sum_{0<\vartheta_\imath<\vartheta} m_\imath \varpi(\vartheta_\imath^-) \\ &\le \mathfrak{X}^* \|\phi_0 - \phi^*\| + \mathfrak{X}^* \int_0^\vartheta \beta(\upsilon)\varpi(\upsilon)d\upsilon + \sum_{0<\vartheta_\imath<\vartheta} \mathfrak{X}^* m_\imath \varpi(\vartheta_\imath^-), \end{aligned}$$

where $\mathfrak{X}^* = \frac{\mathfrak{X}_\Re}{1 - \mathfrak{X}_\Re L_\Xi}$.
Now, applying Lemma 4.1, we get

$$\|\phi - \phi^*\|_{\mathfrak{G}} \le \mathfrak{X}^* \delta (1 + m^*)^\imath \exp(\mathfrak{X}^* \|\beta\|_{L^1}),$$

where $m^* = \mathfrak{X}^* \max_{\imath \in N} m_\imath$.
Therefore if δ is small enough, we obtain

$$\|\phi - \phi^*\|_{\mathfrak{G}} \le \epsilon.$$

It follows that the PC-Asymptotically almost automorphic mild solutions of the problem (1.1) depends continuously on the initial condition.

5. An example

Consider the following partial differential equation :

$$\begin{cases} \frac{\partial}{\partial \vartheta}(\phi(\vartheta, x)) = \frac{\partial^2 \phi(\vartheta, x)}{\partial x^2} + \int_0^\vartheta \Gamma(\vartheta - \upsilon)\frac{\partial^2 \phi(\upsilon, x)}{\partial x^2} d\upsilon + \frac{\cos^2(\vartheta)\sin(\pi\phi(\vartheta, x))}{30\sqrt{1+\vartheta^2}(1+|\phi(\vartheta, x)|)e^\vartheta} \\ \qquad + \frac{e^{-\vartheta}\cos^2(\vartheta)}{6\sqrt{1+\vartheta^2}} \sin\left(\frac{1}{\cos(\vartheta) + \cos\sqrt{2}\vartheta + 2}\right)|(\phi(\vartheta, x))|,\ \vartheta \in \widehat{J}, \quad x \in [0, \pi], \\ I_\imath \phi(\vartheta_\imath, x) = \frac{7^{-\imath}\phi(\vartheta_\imath^-, x)}{9\sqrt{1 + |\phi(\vartheta_\imath^-, x)|}},\ \text{for } \imath \in \mathbb{N},\ \textit{and } x \in (0, \pi). \\ \phi(\vartheta, 0) = \phi(\vartheta, \pi) = 0, \quad \vartheta \in \mathbb{R}^+, \\ \phi(0, x) + \frac{9}{28}\sum_{i=1}^{2} \frac{1}{3^i}\phi\left(\frac{1}{i}, x\right) = e^x, \quad x \in [0, \pi], \end{cases} \tag{5.1}$$

where $\widehat{J} = \mathbb{R}^+ - \{\vartheta_\imath\}_{\imath\in\mathbb{N}}$, and $\{\vartheta_\imath\}$ is an almost automorphic sequence of positive real numbers.
Let $\mho = L^2(0,\pi)$ be the space of 2-integrable functions from $[0,\pi]$ into $\mathbb{R}^+$.
Define

$$\phi(\vartheta)(x) = \phi(\vartheta, x), \;\; \Xi(\phi) = \frac{9}{28}\sum_{i=1}^{2}\frac{1}{3^i}\phi\left(\frac{1}{i}, x\right),$$

and

$$\Psi(\vartheta, \phi(\vartheta)) = \frac{e^{-\vartheta}\cos^2(\vartheta)}{6\sqrt{1+\vartheta^2}}\sin\left(\frac{1}{\cos\vartheta + \cos\sqrt{2}\vartheta + 2}\right)\|\phi(\vartheta)\| + \frac{e^{-\vartheta}\cos^2(\vartheta)\sin\pi\phi(\vartheta)}{30\sqrt{1+\vartheta^2}(1+|\phi(\vartheta)|)},$$

$$I_\imath(\phi(\vartheta_\imath^-)) = \frac{7^{-\imath}\phi(\vartheta_\imath^-)}{9\sqrt{1+|\phi(\vartheta_\imath^-)|}}.$$

Consider the operator $\Lambda(\vartheta) : \mho \mapsto \mho$ as follows:

$$\Lambda(\vartheta)z = \Gamma(\vartheta)\mathfrak{Z}z, \;\; for\ \vartheta \geq 0, \;\; z \in D(\mathfrak{Z}),$$

where $\mathfrak{Z}$ is defined by

$$\begin{cases} D(\mathfrak{Z}) = \{\varphi \in \mho \;/\; \varphi,\; \varphi' \text{ are AC},\; \varphi'' \in L^2(0,\pi)\;,\; \varphi(0) = \varphi(\pi) = 0\}, \\ (\mathfrak{Z}\varphi)(x) = \frac{\partial^2\varphi(\vartheta,x)}{\partial x^2}. \end{cases}$$

It is well known that $\mathfrak{Z}$ generates a strongly continuous semigroup $(T(\vartheta))_{\vartheta\geq 0}$, which is dissipative and compact with $\|T(\vartheta)\| \leq e^{-\phi^2\vartheta}$, and for some $\sigma > \frac{1}{\phi^2}$. We assume that

$$\|\Gamma(\vartheta)\| \leq \frac{e^{-\phi^2\vartheta}}{\sigma}, \;\; \text{and } \|\Gamma'(\vartheta)\| \leq \frac{e^{-\phi^2\vartheta}}{\sigma^2}.$$

It follows from [26], that $\|\Re(\vartheta)\| \leq e^{-\jmath\vartheta}$, where $\jmath = 1 - \sigma^{-1}$.
Then $(A1)$ hold with $\mathfrak{X}_\Re = 1$ and $b = 1 - \sigma^{-1}$.
Consequently, the problem can be written in the abstract form (1.1) with $\mathfrak{Z}$, Λ, Ξ and Ψ as defined above.
Now, let

$$\Psi(\vartheta, \phi(\vartheta)) = \widehat{\aleph}(\vartheta, \phi(\vartheta)) + \widetilde{\aleph}(\vartheta, \phi(\vartheta)),$$

where

$$\widehat{\aleph}(\vartheta, \phi(\vartheta)) = \frac{e^{-\vartheta}\cos^2(\vartheta)}{6\sqrt{1+\vartheta^2}}\sin\left(\frac{1}{\cos\vartheta + \cos\sqrt{2}\vartheta + 2}\right)|\phi(\vartheta)|,$$

$$\widetilde{\aleph}(\vartheta, \phi(\vartheta)) = \frac{e^{-\vartheta}\cos^2(\vartheta)\sin\pi\phi(\vartheta)}{30\sqrt{1+\vartheta^2}(1+|\phi(\vartheta)|)}.$$

Then it is easy to verify that the function $\widehat{\aleph}, \widetilde{\aleph} : \mathbb{R}^+ \times \mho \longrightarrow \mho$ are continuous and $\widehat{\aleph} \in AA(\mathbb{R}^+ \times \mho; \mho)$, with

$$\|\widehat{\aleph}(\vartheta, z_1(\vartheta)) - \widehat{\aleph}(\vartheta, z_2(\vartheta))\| \leq \frac{1}{6}\|z_1(\vartheta) - z_2(\vartheta)\|, \text{ for all } \vartheta \in J, z_1, z_2 \in \mho,$$

$$\|\widetilde{\aleph}(\vartheta, z(\vartheta))\| \leq \frac{1}{30\sqrt{1+\vartheta^2}}, \text{ for all } \vartheta \in J, \;\; z \in \mho,$$

which implies that $\widetilde{\aleph} \in PC_0(\mathbb{R}^+ \times \mho, \mho)$.
For the function Ψ, we can make the following estimates:

$$\|\Psi(\vartheta, \phi_1(\vartheta)) - \Psi(\vartheta, \phi_2(\vartheta))\| \quad \leq \quad \frac{e^{-\vartheta}\cos^2\vartheta}{6\sqrt{1+\vartheta^2}}\|\phi_1(\vartheta) - \phi_2(\vartheta)\|.$$

For every $\vartheta \in J$ and $\Im \subset \mho$, we have

$$\alpha(\Psi(\vartheta, \Im(\vartheta))) \leq \frac{e^{-\vartheta}\cos^2\vartheta}{6\sqrt{1+\vartheta^2}}\alpha(\Im(\vartheta)),$$

Then, $\beta(\vartheta) = \frac{e^{-\vartheta}\cos^2\vartheta}{6\sqrt{1+\vartheta^2}}$, which belongs to $L^1(J, \mathbb{R}^+)$. We have also the following estimates,

$$\|\Xi(\phi_1) - \Xi(\phi_1)\| \leq \frac{1}{7}\|\phi_1 - \phi_2\|_{\mathfrak{S}},$$

$$\|I_\imath\phi_1(\vartheta_\imath) - I_\imath\phi_2(\vartheta_\imath)\| \leq \frac{7^{-\imath}}{9}\|\phi_1(\vartheta_\imath) - \phi_2(\vartheta_\imath)\|,$$

and

$$\mathfrak{X}_{\Re}\left(L_\Xi + 4\|\beta\|_{L^1} + \sum_{\imath=0}^{\infty} m_\imath\right) \simeq 0,6 < 1.$$

Thus, Theorem 3.2 yields, then the problem (5.1) has a PC-asymptotically almost automorphic mild solution.

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