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On reachability and controllability for a Volterra integro-dynamic system on time scales

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Abstract. The paper studies and relates the notions of reachability and controllability for the Volterra integro-dynamic system on time scales. More specifically, we obtain necessary and sufficient conditions for reachability and controllability. In addition, we obtain an equivalence between the concepts of reachability and controllability studied.

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1. Introduction

The Volterra integro-dynamic systems on time scales have been considered in several articles in the literature, which can be witnessed by [1], [2], [8], [10], and [11]. In [1], [2] and [10], the authors have studied the linear Volterra integro-dynamic system on time scales of the type

$$\begin{cases} x^{\Delta}(t) = A(t)x(t) + \displaystyle\int_{t_0}^{t} K(t,s)x(s)\Delta s + B(t)u(t), \ t \in [0,\infty)_{\mathbb{T}^{\kappa}} \\ x(t_0) = x_0. \end{cases} \tag{1.1}$$

Adıvar [1] introduced the variation of parameters for Eq. (1.1) and then Adıvar and Raffoul [2] used it to obtain the necessary and sufficient conditions for the uniform stability of the zero solutions of Eq. (1.1) employing the resolvent equation. Lupulescu et al. [10] studied asymptotic behaviour of solutions for (1.1). Karpuz and Koyuncuoğlu [8] obtained the necessary and sufficient conditions for the positivity and uniform exponential stability for the Volterra integro-dynamical systems means of Metzler

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matrices. Further, Younus and ur Rahaman [11] studied controllability, observability, and asymptotic behaviour for the Volterra integro-dynamic system on time scales.

Inspired by [3], [9], and [11], the present study investigates the reachability and controllability for system (1.1). Here $\mathbb{T}$ is a time scale $\mathbb{T}_0 = [0, \infty)_{\mathbb{T}^\kappa}$ and $t_0 \in \mathbb{T}_0$ is fixed, $u\colon \mathbb{T}_0 \to \mathbb{R}^m$ is control function, the functions $A\colon \mathbb{T}_0 \to \mathbb{R}^{n\times n}$ and $B\colon \mathbb{T}_0 \to \mathbb{R}^{n\times m}$ are continuous on $\mathbb{T}_0$, and $K\colon \mathbb{T}_0 \times \mathbb{T}_0 \to \mathbb{R}^{n\times n}$ is continuous on $\Omega := \{(t, s) \in \mathbb{T}^\kappa \times \mathbb{T}^\kappa \colon 0 \le s \le t < \infty\}$. Also, the control functions u can admit a finite number of discontinuities at $t_{u_1}, \ldots, t_{u_p}$ in $\mathbb{T}_0 \setminus \{0, \sup \mathbb{T}\}$ with $p \in I(u) \subset \mathbb{N}$ and $t_{u_i} > t_0$ for every $i \in \{1, \ldots, p\}$, such way that for $1 \le i \le p$, there exist the left and right limit of $u(t)$ at $t = t_{u_i}$ in time scale context, i.e., $u(t_{u_i}^-) = \lim_{h\to 0^+} u(t_{u_i} - h)$ and $u(t_{u_i}^+) = \lim_{h\to 0^+} u(t_{u_i} + h)$, respectively. Further, we have $u(t_{u_i}^-) \ne u(t_{u_i}^+) = u(t_{u_i})$. We emphasize that throughout the work $\mathbb{R}^n$ denotes the space of n-dimensional column vectors, equivalently, $\mathbb{R}^n$ also denotes the space of real matrices $n \times 1$.

For system (1.1), we use the notions of reachability and controllability analogous to those given in [3] and establish necessary and sufficient conditions similar to [3, Theorem 1] and [3, Proposition 5]. We also establish an equivalence between the reachability and controllability of (1.1) analogous to [3, Proposition 6]. To do this, we first state and prove the existence result to system (1.1). The novelty of the results obtained here on controllability in relation to [11] are the new necessary and sufficient conditions to controllability. On the other hand, to the best of our knowledge, there is no studies in the time scales literature related to the reachability for Volterra integro-dynamic system on time scales.

The paper is organized as follows. The next section provides useful background concepts of time scales theory, such as the Δ-derivative in addition to the Δ-integral for reading the paper. In Section 3, we define and obtain the existence of solution to system (1.1). Section 4 contains the results concerning the reachability and controllability to system (1.1). Finally, Section 5 brings the conclusions of the work.

2. Preliminaries

In this section, we include basic concepts of time scales theory that will be used throughout the work.

2.1. Time Scales

Given a time scale $\mathbb{T}$, i.e., a nonempty closed subset of the real numbers, here we assume that there exist $a, b \in \mathbb{T}_0$ such that $a < b$. The forward jump operator $\sigma\colon \mathbb{T} \to \mathbb{T}$ is defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} \colon s > t\}$$

and the backward jump operator $\rho\colon \mathbb{T} \to \mathbb{T}$ by

$$\rho(t) = \sup\{s \in \mathbb{T} \colon s < t\}.$$

In this case, we assume that $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$. Also, the graininess function $\mu\colon \mathbb{T} \to [0, +\infty)$ is defined by

$$\mu(t) = \sigma(t) - t.$$

We say that $t \in \mathbb{T}$ is left-dense, left-scattered, right-dense, and right-scattered whenever $\rho(t) = t$, $\rho(t) < t$, $\sigma(t) = t$, and $\sigma(t) > t$, respectively. For $A \subset \mathbb{R}$, we write $A_{\mathbb{T}} = A \cap \mathbb{T}$. In case $\sup \mathbb{T} < +\infty$, we set $\mathbb{T}^\kappa = \mathbb{T} \setminus (\rho(\sup \mathbb{T}), \sup \mathbb{T}]_{\mathbb{T}}$, otherwise, if $\sup \mathbb{T} = +\infty$ we set $\mathbb{T}^\kappa = \mathbb{T}$.
Take a function $f \colon \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^\kappa$. If $\xi \in \mathbb{R}$ is such that, for all $\varepsilon > 0$ there exists $\delta > 0$ satisfying

$$|f(\sigma(t)) - f(s) - \xi(\sigma(t) - s)| \leq \varepsilon |\sigma(t) - s|$$

for all $s \in (t-\delta, t+\delta)_{\mathbb{T}}$, it is said that ξ is the delta derivative of f at t and we denote it by $f^\Delta(t)$.
Now, consider a function $f \colon \mathbb{T} \to \mathbb{R}^n$, $f = (f_1, f_2, \cdots, f_n)$, and $t \in \mathbb{T}^\kappa$. We say that f is Δ-differentiable at t if each component $f_i \colon \mathbb{T} \to \mathbb{R}$ of f is Δ-differentiable at t. In this case $f^\Delta(t) = (f_1^\Delta(t), \ldots, f_n^\Delta(t))$.

2.2. Δ-Integrability

For fixed $a_1, b_1 \in \mathbb{T}_0$ with $a_1 < b_1$, without loss of generality, we consider the time scale $\mathbb{T}_1 = [a_1, b_1]_{\mathbb{T}}$. We denote the family of Δ-measurable sets of $\mathbb{T}_1$ by Δ. We recall that Δ is a σ-algebra of $\mathbb{T}_1$ (see, for instance, [7]).
Suppose that $f \colon \mathbb{T}_1 \to \mathbb{R}$ is a Δ-measurable function, that is, for any $r \in \mathbb{R}$ the set $\{t \in \mathbb{T}_1 \colon f(t) < r\}$ is Δ-measurable. If $E \in \Delta$, we indicate by

$$\int_E f(s)\Delta s$$

the Lebesgue Δ-integral of f over E. Now, if $f \colon \mathbb{T}_1 \to \mathbb{R}^m$ and $E \in \Delta$, then f is Lebesgue Δ-integrable over E if each component $f_i \colon \mathbb{T}_1 \to \mathbb{R}$ of f is Lebesgue Δ-integrable over E. In this case, we have

$$\int_E f(s)\Delta s = \left(\int_E f_1(s)\Delta s, \ldots, \int_E f_m(s)\Delta s \right).$$

Also, if $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^n$, we will indicate by $\mathrm{L}^2(E; \mathbb{R}^m)$ the set of functions $f : \mathbb{T}_1 \to \mathbb{R}^m$ such that the function $\|f\|^2$ is Lebesgue Δ-integrable over E.
In the vector space $\mathrm{L}^2([a_1, b_1]_{\mathbb{T}}; \mathbb{R}^m)$, we can define the inner product

$$\langle f, g \rangle_{L^2} := \int_{[a_1, b_1)_{\mathbb{T}}} g^T(s) f(s) \Delta s,$$

where $f, g \in \mathrm{L}^2([a_1, b_1]_{\mathbb{T}}; \mathbb{R}^m)$ and $g^T(s)$ denotes the transpose of the column vector $g(s) \in \mathbb{R}^m$.

Similarly to [5, Théorème IV.8.], we have the following remark.

Remark 2.1. The vector space $\mathrm{L}^2([a_1, b_1]_{\mathbb{T}}; \mathbb{R}^m)$ is a Banach Space when equipped with the norm induced by the inner product $\langle \cdot, \cdot \rangle_{L^2}$.

We recall that a function $f \colon \mathbb{T}_1 \to \mathbb{R}^n$ is said to be right-dense continuous (rd-continuous) if f is continuous at each right-dense point $t \in \mathbb{T}_1$, and if $\lim_{h \to 0^+} f(t-h)$ exists and finite at each left-dense point $t \in \mathbb{T}_1$.

The Cauchy integral and the Lebesgue Δ-integral of an rd-continuous function can be related as follows. For this, let $f\colon \mathbb{T}_1 \to \mathbb{R}$ be an rd-continuous function. From [4], we can see that function f has an antiderivative $F : \mathbb{T} \to \mathbb{R}$, if $F^{\Delta}(t) = f(t)$ for each $t \in \mathbb{T}_1^{\kappa}$. Thus, the Cauchy integral of f is defined as

$$\int_c^d f(s)\Delta s = F(d) - F(c)$$

for all $c, d \in \mathbb{T}_1$. Hence, the Cauchy integral and the Lebesgue Δ-integral of f relate as

$$\int_c^d f(s)\Delta s = \int_{[c,d)_{\mathbb{T}_1}} f(s)\Delta s$$

with $c, d \in \mathbb{T}_1$ and $c \leq d$.
More about the integration on time scales, can be found in [4], [6], and [7].

3. Existence to Eq. (1.1)

Here we define the solution to system (1.1) and then establish the existence of solution in Theorem 3.1. For this, we first consider the principal matrix solution $Z(t,s)$ of the integro-dynamic equation

$$\begin{cases} x^{\Delta}(t) = A(t)x(t) + \displaystyle\int_s^t K(t,\tau)x(\tau)\Delta\tau, \ t \in [s,\infty)_{\mathbb{T}^{\kappa}} \\ x(s) = x_0, \end{cases} \tag{3.1}$$

where $s \in \mathbb{T}^{\kappa}$. The principal matrix solution of Eq. (3.1) is the $n \times n$ matrix function $Z(t,s)$ defined as

$$Z(t,s) = [x^1(t,s), \ldots, x^n(t,s)]$$

where $x^i(t,s)$, $i = 1, \ldots, n$, are the linearly independent solutions of Eq. (3.1). Given the control function $u(t) \in \mathbb{R}^m$, we define the solution x of system (1.1) as follows. If u is continuous on $\mathbb{T}_0$, as can be seen in [1, Theorem 19], the solution x of Eq. (1.1) on $\mathbb{T}_0$ is given by

$$x(t) = Z(t,t_0)x_0 + \int_{t_0}^t Z(t,\sigma(s))B(s)u(s)\Delta s, \tag{3.2}$$

where $Z(t,s)$ is the principal matrix solution of Eq. (3.1).
Now, if the control function $u(t) \in \mathbb{R}^m$ admits the discontinuities $t_{u_1}, \ldots, t_{u_p}$ in $\mathbb{T}_0 \setminus \{0, \sup \mathbb{T}\}$, with $p \in I(u) \subset \mathbb{N}$, for $i = 1$, we consider the continuous function $w_1 : [0,\infty)_{\mathbb{T}^{\kappa}} \to \mathbb{R}^m$ defined by

$$w_i(t) = \begin{cases} u(t) & \text{if } t \in [0, t_{u_i})_{\mathbb{T}^{\kappa}}, \\ u(t_{u_i}^-) & \text{if } t \in [t_{u_i}, \infty)_{\mathbb{T}^{\kappa}}, \end{cases} \tag{3.3}$$

and for $1 < i \leq p$, we take the continuous function $w_i : [0,\infty)_{\mathbb{T}^{\kappa}} \to \mathbb{R}^m$ given by

$$w_i(t) = \begin{cases} u(t_{u_{i-1}}^+) & \text{if } t \in [0, t_{u_{i-1}})_{\mathbb{T}^{\kappa}}, \\ u(t) & \text{if } t \in [t_{u_{i-1}}, t_{u_i})_{\mathbb{T}^{\kappa}}, \\ u(t_{u_i}^-) & \text{if } t \in [t_{u_i}, \infty)_{\mathbb{T}^{\kappa}}. \end{cases} \tag{3.4}$$

We also consider the continuous function $w_{p+1} : [0, \infty)_{\mathbb{T}^\kappa} \to \mathbb{R}^m$ given by

$$w_{p+1}(t) = \begin{cases} u(t_{u_p}^+) & \text{if } t \in [0, t_{u_p})_{\mathbb{T}^\kappa}, \\ u(t) & \text{if } t \in [t_{u_p}, \infty)_{\mathbb{T}^\kappa}. \end{cases} \tag{3.5}$$

Hence for $i = 1$, let $x_i = x_1$ be the solution of integro-dynamic equation

$$\begin{cases} x_1^\Delta(t) = A(t)x_1(t) + \displaystyle\int_{t_0}^{t} K(t,s)x_1(s)\Delta s + B(t)w_1(t), \\ x_1(t_0) = x_0 \end{cases} \tag{3.6}$$

on $[0, \infty)_{\mathbb{T}^\kappa}$ and for $1 < i \le p$, let x_i be the solution of integro-dynamic equation

$$\begin{cases} x_i^\Delta(t) = A(t)x_i(t) + \displaystyle\int_{t_{u_{i-1}}}^{t} K(t,s)x_i(s)\Delta s + B(t)w_i(t), \\ x_i(t_{u_{i-1}}) = x_{i-1}(t_{u_{i-1}}) \end{cases} \tag{3.7}$$

on $[t_{u_{i-1}}, \infty)_{\mathbb{T}^\kappa}$. Futhermore, let x_{p+1} be the solution of integro-dynamic equation

$$\begin{cases} x_{p+1}^\Delta(t) = A(t)x_{p+1}(t) + \displaystyle\int_{t_{u_p}}^{t} K(t,s)x_{p+1}(s)\Delta s + B(t)w_{p+1}(t), \\ x_{p+1}(t_{u_p}) = x_p(t_{u_p}) \end{cases} \tag{3.8}$$

on $[t_{u_p}, \infty)_{\mathbb{T}^\kappa}$. Thus, we define the solution x of system (1.1) as

$$x(t) = \begin{cases} x_1(t) & \text{if } t \in [0, t_{u_1})_{\mathbb{T}^\kappa}, \\ x_i(t) & \text{if } t \in [t_{u_{i-1}}, t_{u_i})_{\mathbb{T}^\kappa},\ 1 < i \le p, \\ x_{p+1}(t) & \text{if } t \in [t_{u_p}, \infty)_{\mathbb{T}^\kappa}. \end{cases}$$

The principal matrix $Z(t, s)$ is said to be transition matrix if $Z(s, s) = \mathrm{Id}$. According to [11, Lemma 2.2], the transition matrix $Z(t, s)$ of Eq. (3.1) admits, among others, the following properties:

(i) $Z(t, s) = Z(t, \tau)Z^{-1}(s, \tau)$;
(ii) $Z(t, s) = Z^{-1}(s, t)$;
(iii) $Z(t, r)Z(r, s) = Z(s, t)$.

Theorem 3.1. *Suppose that the control function $u\colon \mathbb{T}_0 \to \mathbb{R}^m$ in Eq.* (1.1) *admits the discontinuities $t_{u_1}, \ldots, t_{u_p}$ in $\mathbb{T}_0 \setminus \{0, \sup \mathbb{T}\}$. Then the solution x of system* (1.1) *is given by*

$$x(t) = Z(t, t_0)x_0 + \int_{[t_0,t)_{\mathbb{T}}} Z(t, \sigma(s))B(s)u(s)\Delta s$$

on $[0, \infty)_{\mathbb{T}^\kappa}$.

Proof. For the control function $u(t) \in \mathbb{R}^m$ we consider continuous functions w_i $(1 \le i \le p)$ and w_{p+1} as defined in Eqs. (3.3), (3.4), and (3.5). Let x_1 be the solution of Eq. (3.6) on $[0, \infty)_{\mathbb{T}^\kappa}$, and for $1 < i \le p$, x_i be the solution of Eq. (3.7) on $[t_{u_{i-1}}, \infty)_{\mathbb{T}^\kappa}$. Also, let x_{p+1} be the solution of Eq. (3.8) on $[t_{u_p}, \infty)_{\mathbb{T}^\kappa}$. Hence, the solution x of system (1.1) is given by

$$x(t) = \begin{cases} x_1(t) & \text{if } t \in [0, t_{u_1})_{\mathbb{T}^\kappa}, \\ x_i(t) & \text{if } t \in [t_{u_{i-1}}, t_{u_i})_{\mathbb{T}^\kappa},\ 1 < i \le p, \\ x_{p+1}(t) & \text{if } t \in [t_{u_p}, \infty)_{\mathbb{T}^\kappa}. \end{cases}$$

Using [1, Theorem 19] repeatedly, we have

$$x_1(t) = Z(t,t_0)x_0 + \int_{t_0}^{t} Z(t,\sigma(s))B(s)w_1(s)\Delta s$$

for $t \in [0, t_{u_1}]_{\mathbb{T}^\kappa}$,

$$x_i(t) = Z(t,t_{u_{i-1}})x_{i-1}(t_{u_{i-1}}) + \int_{t_{u_{i-1}}}^{t} Z(t,\sigma(s))B(s)w_i(s)\Delta s$$

for $t \in [t_{u_{i-1}}, t_{u_i}]_{\mathbb{T}^\kappa}$, $1 < i \leq p$, and

$$x_{p+1}(t) = Z(t,t_{u_p})x_p(t_{u_p}) + \int_{t_{u_p}}^{t} Z(t,\sigma(s))B(s)w_{p+1}(s)\Delta s$$

for $t \in [t_{u_p}, \infty)_{\mathbb{T}^\kappa}$. Thus the solution x is given by

$$x(t) = Z(t,t_0)x_0 + \int_{[t_0,t)_{\mathbb{T}}} Z(t,\sigma(s))B(s)u(s)\Delta s$$

for $t \in [0, t_{u_1}]_{\mathbb{T}^\kappa}$,

$$x(t) = Z(t,t_{u_{i-1}})x(t_{u_{i-1}}) + \int_{[t_{u_{i-1}},t)_{\mathbb{T}}} Z(t,\sigma(s))B(s)u(s)\Delta s$$

for $t \in [t_{u_{i-1}}, t_{u_i}]_{\mathbb{T}^\kappa}$, $1 < i \leq p$, and for $t \in [t_{u_p}, \infty)_{\mathbb{T}^\kappa}$, it is expressed by

$$x(t) = Z(t,t_{u_p})x(t_{u_p}) + \int_{[t_{u_p},t)_{\mathbb{T}}} Z(t,\sigma(s))B(s)u(s)\Delta s.$$

Note that for $p = 1$ and $t \in [t_{u_1}, \infty)_{\mathbb{T}^\kappa}$, the solution is given by

$$x(t) = Z(t,t_{u_1})x(t_{u_1}) + \int_{[t_{u_1},t)_{\mathbb{T}}} Z(t,\sigma(s))B(s)u(s)\Delta s$$

with

$$x(t_{u_1}) = Z(t_{u_1},t_0)x_0 + \int_{[t_0,t_{u_1})_{\mathbb{T}}} Z(t_{u_1},\sigma(s))B(s)u(s)\Delta s,$$

and by using properties of transition matrices and integrals, we can conclude that

$$x(t) = Z(t,t_0)x_0 + \int_{[t_0,t)_{\mathbb{T}}} Z(t,\sigma(s))B(s)u(s)\Delta s$$

for each $t \in [t_{u_1}, \infty)_{\mathbb{T}^\kappa}$. Thus,

$$x(t) = Z(t,t_0)x_0 + \int_{[t_0,t)_{\mathbb{T}}} Z(t,\sigma(s))B(s)u(s)\Delta s$$

for every $t \in [0, \infty)_{\mathbb{T}^\kappa}$. Now, for $p > 1$ and $t \in [t_{u_1}, t_{u_2}]_{\mathbb{T}^\kappa}$, the solution is

$$x(t) = Z(t,t_{u_1})x(t_{u_1}) + \int_{[t_{u_1},t)_{\mathbb{T}}} Z(t,\sigma(s))B(s)u(s)\Delta s,$$

where

$$x(t_{u_1}) = Z(t_{u_1},t_0)x_0 + \int_{[t_0,t_{u_1})_{\mathbb{T}}} Z(t_{u_1},\sigma(s))B(s)u(s)\Delta s.$$

Again, we can use properties of transition matrices and integrals to conclude that

$$x(t) = Z(t,t_0)x_0 + \int_{[t_0,t)_{\mathbb{T}}} Z(t,\sigma(s))B(s)u(s)\Delta s$$

for every $t \in [t_{u_1}, t_{u_2}]_{\mathbb{T}^\kappa}$. Similarly, recursively we get

$$x(t) = Z(t,t_0)x_0 + \int_{[t_0,t)_{\mathbb{T}}} Z(t,\sigma(s))B(s)u(s)\Delta s$$

for $t \in [t_{u_{i-1}}, t_{u_i}]_{\mathbb{T}^\kappa}$ and $1 \leq i \leq p$. In this case, for $t \in [t_{u_p}, \infty)_{\mathbb{T}^\kappa}$, we have

$$x(t) = Z(t,t_{u_p})x(t_{u_p}) + \int_{[t_{u_p},t)_{\mathbb{T}}} Z(t,\sigma(s))B(s)u(s)\Delta s$$

with

$$x(t_{u_p}) = Z(t_{u_p},t_0)x_0 + \int_{[t_0,t_{u_p})_{\mathbb{T}}} Z(t_{u_p},\sigma(s))B(s)u(s)\Delta s.$$

Hence, we can also deduce that

$$x(t) = Z(t,t_0)x_0 + \int_{[t_0,t)_{\mathbb{T}}} Z(t,\sigma(s))B(s)u(s)\Delta s$$

for each $t \in [t_{u_p}, \infty)_{\mathbb{T}^\kappa}$. Therefore

$$x(t) = Z(t,t_0)x_0 + \int_{[t_0,t)_{\mathbb{T}}} Z(t,\sigma(s))B(s)u(s)\Delta s$$

for each $t \in [0, \infty)_{\mathbb{T}^\kappa}$. This completes the proof. □

4. Reachability and controllability

Now, we consider the notions of reachability and controllability for system (1.1) analogous to those given in [3]. Then we establish the results on necessary and sufficient conditions for reachability and controllability (theorems 4.3, 4.4, 4.6, and 4.7). Thus, in theorems 4.3 and 4.4, we establish the results on reachability and in theorems 4.6 and 4.7, we establish results on controllability. Besides, we also establish an equivalence that relates reachability and controllability (Theorem 4.8).

Suppose that $\mathcal{U}$ denotes the set of control functions to system (1.1). Hence, if $\tau \in (0,\infty)_{\mathbb{T}^\kappa}$ and $t_0 \in [0,\tau)_{\mathbb{T}}$, $\mathcal{U}(t_0,\tau)$ will denote the functions from the set $\mathcal{U}$ restricted to $[t_0,\tau]_{\mathbb{T}}$. We point out that $\mathcal{U}(t_0,\tau)$ is a subspace of $\mathrm{L}^2([t_0,\tau]_{\mathbb{T}};\mathbb{R}^m)$.

Definition 4.1.

1. A state $x_1 \in \mathbb{R}^n$ is said to be reachable at time $\tau \in [0,\infty)_{\mathbb{T}^\kappa}$ if for some $t_0 \in [0,\tau)_{\mathbb{T}}$ and for every initial state $x_0 \in \mathbb{R}^n$, there is an input $u \in \mathcal{U}(t_0,\tau)$ such that the state x of system (1.1) with $x(t_0) = x_0$ satisfies $x(\tau) = x_1$.
2. The system (1.1) is called reachable at time $\tau \in [0,\infty)_{\mathbb{T}^\kappa}$ if each state $x_1 \in \mathbb{R}^n$ is reachable at time τ.

3. The reachability map $\mathcal{B}_r(t_0,\tau)\colon \mathcal{U}(t_0,\tau)\to\mathbb{R}^n$ is defined as
$$\mathcal{B}_r(t_0,\tau)[u]=\int_{[t_0,\tau)_{\mathbb{T}}} Z(\tau,\sigma(s))B(s)u(s)\Delta s.$$
4. The adjoint of $\mathcal{B}_r(t_0,\tau)$, indicated by $\mathcal{B}_r^*(t_0,\tau)\colon \mathbb{R}^n\to\mathcal{U}(t_0,\tau)$, is defined by
$$\mathcal{B}_r^*(t_0,\tau)[w](t)=B^T(t)Z^T(\tau,\sigma(t))w,$$
for all $w\in\mathbb{R}^n$ and for any $t\in[t_0,\tau]_{\mathbb{T}}$.
5. The Gramian reachability map $\mathcal{G}_B^r(t_0,\tau)\colon\mathbb{R}^n\to\mathbb{R}^n$ is given by
$$\begin{aligned}\mathcal{G}_B^r(t_0,\tau)[w]&=\mathcal{B}_r(t_0,\tau)\circ\mathcal{B}_r^*(t_0,\tau)[w]\\&=\int_{[t_0,\tau)_{\mathbb{T}}} Z(\tau,\sigma(s))B(s)B^T(s)Z^T(\tau,\sigma(s))w\Delta s.\end{aligned}$$

Definition 4.2.

1. A state $x_0\in\mathbb{R}^n$ is said to be controllable at time $t_0\in[0,\infty)_{\mathbb{T}^\kappa}$ if for some $\tau\in(t_0,\infty)_{\mathbb{T}^\kappa}$ there exists an input $u\in\mathcal{U}(t_0,\tau)$ such that the state x of system (1.1) with $x(t_0)=x_0$ satisfies $x(\tau)=0$.
2. The system (1.1) is called controllable at time $t_0\in[0,\infty)_{\mathbb{T}^\kappa}$ if each state $x_0\in\mathbb{R}^n$ is controllable at time t_0.
3. The controllability map $\mathcal{B}_c(t_0,\tau)\colon\mathcal{U}(t_0,\tau)\to\mathbb{R}^n$ is defined as
$$\mathcal{B}_c(t_0,\tau)[u]=\int_{[t_0,\tau)_{\mathbb{T}}} Z(t_0,\sigma(s))B(s)u(s)\Delta s.$$
4. The adjoint of $\mathcal{B}_c(t_0,\tau)$, indicated by $\mathcal{B}_c^*(t_0,\tau)\colon\mathbb{R}^n\to\mathcal{U}(t_0,\tau)$, is given by
$$\mathcal{B}_c^*(t_0,\tau)[w](t)=B^T(t)Z^T(t_0,\sigma(t))w,$$
for all $w\in\mathbb{R}^n$ and all $t\in[t_0,\tau]_{\mathbb{T}}$.
5. The Gramian controllability map $\mathcal{G}_B^c(t_0,\tau)\colon\mathbb{R}^n\to\mathbb{R}^n$ is defined by
$$\begin{aligned}\mathcal{G}_B^c(t_0,\tau)[w]&=\mathcal{B}_c(t_0,\tau)\circ\mathcal{B}_c^*(t_0,\tau)[w]\\&=\int_{[t_0,\tau)_{\mathbb{T}}} Z(t_0,\sigma(s))B(s)B^T(s)Z^T(t_0,\sigma(s))w\Delta s.\end{aligned}$$

The necessary and sufficient condition for the reachability of (1.1) in terms of the Gramian reachability matrix is proved in Theorem 4.3.

Theorem 4.3. *For a fixed $\tau\in[0,\infty)_{\mathbb{T}^\kappa}$, the system* (1.1) *is reachable at time τ if and only if there exists $t_0\in[0,\tau)_{\mathbb{T}}$ such that the $n\times n$ Gramian reachability matrix defined by*
$$\mathcal{G}_r(t_0,\tau):=\int_{[t_0,\tau)_{\mathbb{T}}} Z(\tau,\sigma(s))B(s)B^T(s)Z^T(\tau,\sigma(s))\Delta s$$
is invertible, where $Z(t,s)$ is the transition matrix of Eq. (3.1).

Proof. Suppose the Gramian matrix $\mathcal{G}_r(t_0,\tau)$ is invertible. Define the input $u\in\mathcal{U}(t_0,\tau)$ by
$$u(t)=-B^T(t)Z^T(\tau,\sigma(t))\mathcal{G}_r^{-1}(t_0,\tau)(Z(\tau,t_0)x_0-x_1),\tag{4.1}$$

where $x_0, x_1 \in \mathbb{R}^n$. Then, the state x of system (1.1) with $x(t_0) = x_0$ is such that

$$x(\tau) = Z(\tau, t_0)x_0 + \int_{[t_0,\tau)_{\mathbb{T}}} Z(\tau, \sigma(s))B(s)u(s)\Delta s.$$

Now, substituting the value of $u(s)$ from (4.1), we get

$$\begin{aligned} x(\tau) &= Z(\tau, t_0)x_0 \\ &\quad - \mathcal{G}_r^{-1}(t_0, \tau)(Z(\tau, t_0)x_0 - x_1) \int_{[t_0,\tau)_{\mathbb{T}}} Z(\tau, \sigma(s))B(s)B^T(s) \times Z^T(\tau, \sigma(s))\Delta s \\ &= x_1. \end{aligned}$$

Therefore the system (1.1) is reachable at time τ. On the other hand, assume $\mathcal{G}_r(t_0, \tau)$ not invertible and the system (1.1) reachable at time τ. Hence there exists a nonzero vector $x_a \in \mathbb{R}^n$ such that

$$\begin{aligned} 0 &= x_a^T \mathcal{G}_r(t_0, \tau)x_a \\ &= \int_{[t_0,\tau)_{\mathbb{T}}} x_a^T Z(\tau, \sigma(s))B(s)B^T(s)Z^T(\tau, \sigma(s))x_a \Delta s \\ &= \int_{[t_0,\tau)_{\mathbb{T}}} \left\| B^T(s)Z^T(\tau, \sigma(s))x_a \right\|^2 \Delta s. \end{aligned}$$

This gives

$$B^T(s)Z^T(\tau, \sigma(s))x_a = 0, s \in [t_0, \tau]_{\mathbb{T}},$$

i.e.,

$$x_a^T Z(\tau, \sigma(s))B(s) = 0, s \in [t_0, \tau]_{\mathbb{T}}. \tag{4.2}$$

Since the system (1.1) is reachable at time τ, for $x_0 = Z(t_0, \tau)x_a + Z(t_0, \tau)x_1$, there exists an input $u \in \mathcal{U}(t_0, \tau)$ such that the state x of system (1.1) with $x(t_0) = x_0$ obeys

$$x(\tau) = x_1 = Z(\tau, t_0)x_0 + \int_{[t_0,\tau)_{\mathbb{T}}} Z(\tau, \sigma(s))B(s)u(s)\Delta s.$$

Hence,

$$x_1 = x_a + x_1 + \int_{[t_0,\tau)_{\mathbb{T}}} Z(\tau, \sigma(s))B(s)u(s)\Delta s$$

and we deduce that

$$x_a = -\int_{[t_0,\tau)_{\mathbb{T}}} Z(\tau, \sigma(s))B(s)u(s)\Delta s.$$

Now, multiplying the last equation by x_a^T and using Eq. (4.2) we get

$$\begin{aligned} x_a^T x_a &= -\int_{[t_0,\tau)_{\mathbb{T}}} x_a^T Z(\tau, \sigma(s))B(s)u(s)\Delta s \\ &= 0. \end{aligned}$$

That is, $\|x_a\| = 0$ and hence $x_a = 0$, a contradiction. Thus, the Gramian matrix $\mathcal{G}_r(t_0, \tau)$ is invertible. □

Theorem 4.4 given below establishes necessary and sufficient conditions to reachability for system (1.1).

Theorem 4.4. *Suppose $\tau \in [0,\infty)_{\mathbb{T}^\kappa}$. The system* (1.1) *is reachable at time τ if and only if there exists $t_0 \in [0,\tau)_{\mathbb{T}}$ such that one of the following statements is satisfied.*

(i). *The operator $\mathcal{B}_r(t_0,\tau)$ is onto (surjective).*
(ii). *The operator $\mathcal{B}_r^*(t_0,\tau)$ is one to one (injective).*
(iii). *The Gramian reachability operator $\mathcal{G}_B^r(t_0,\tau)$ is invertible.*
(iv). *There is a positive constant γ such that*

$$\|w\|^2 \leq \gamma \int_{[t_0,\tau)_{\mathbb{T}}} \|B^T(s)Z^T(\tau,\sigma(s))w\|^2 \Delta s$$

for all $w \in \mathbb{R}^n$.

Remark 4.5. In Theorem 4.4, the equivalence between statement (iii) and reachability of system (1.1) at time τ can be obtained from Theorem 4.3, since the $n \times n$ Gramian reachability matrix $\mathcal{G}_r(t_0,\tau)$ is the matrix representation of operator $\mathcal{G}_B^r(t_0,\tau)$ relative to the canonical basis.

From [11, Theorem 2.4] we have the following result.

Theorem 4.6. *Assume $t_0 \in [0,\infty)_{\mathbb{T}^\kappa}$. Hence, the system* (1.1) *is controllable at time t_0 if, and only if, there exists $\tau \in (t_0,\infty)_{\mathbb{T}^\kappa}$ such that the $n \times n$ controllability Gramian matrix defined by*

$$\mathcal{G}_c(t_0,\tau) = \int_{[t_0,\tau)_{\mathbb{T}}} Z(t_0,\sigma(s))B(s)B^T(s)Z^T(t_0,\sigma(s))\Delta s$$

is invertible, where $Z(t,s)$ is the transition matrix of Eq. (3.1).

The necessary and sufficient conditions to controllability for system (1.1) is given below.

Theorem 4.7. *Assume $t_0 \in [0,\infty)_{\mathbb{T}^\kappa}$. The system* (1.1) *is controllable at time t_0 if and only if there exists $\tau \in (t_0,\infty)_{\mathbb{T}^\kappa}$ such that one of the following statements is satisfied.*

(i). *The operator $\mathcal{B}_c(t_0,\tau)$ is onto (surjective).*
(ii). *The operator $\mathcal{B}_c^*(t_0,\tau)$ is one to one (injective).*
(iii). *The Gramian controllability operator $\mathcal{G}_B^c(t_0,\tau)$ is invertible.*
(iv). *There is a positive constant γ such that*

$$\|w\|^2 \leq \gamma \int_{[t_0,\tau)_{\mathbb{T}}} \|B^T(s)Z^T(t_0,\sigma(s))w\|^2 \Delta s$$

for every $w \in \mathbb{R}^n$.

Proof. The equivalence between controllability of system (1.1) at time τ and statement (iii) follows from Theorem 4.6, since the $n \times n$ Gramian controllability matrix $\mathcal{G}_c(t_0,\tau)$ is the matrix representation of operator $\mathcal{G}_B^c(t_0,\tau)$ relative to the canonical basis. The remaining equivalences can be obtained as in proof of [3, Proposition 5]. □

Finally, we have the following equivalence that relates reachability and controllability.

Theorem 4.8. *If system* (1.1) *is reachable at time* τ *in* $[0,\infty)_{\mathbb{T}^\kappa}$ *if and only if the system* (1.1) *is controllable at some time* $t_0 < \tau$ *in* $[0,\infty)_{\mathbb{T}^\kappa}$.

5. Conclusions

The paper studies the notions of reachability and controllability for linear Volterra integro-dynamic system on time scales. In such a way that the study carried out here on reachability is a pioneer in the time scales literature. In Theorems 4.3, 4.4, 4.6, and 4.7, we establish results on the necessary and sufficient conditions to reachability and controllability. Also, we relate the notions of reachability and controllability in Theorem 4.8. In Theorem 4.7, new necessary and sufficient conditions to controllability are obtained. On the other hand, Theorems 4.3 and 4.4 establish new necessary and sufficient conditions to reachability.

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