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Harmonic close-to-convex mappings associated with Sălăgean q-differential operator

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Abstract. In this paper, we define a new subclass $\mathcal{W}(n,\alpha,q)$ of analytic functions and a new subclass $\mathcal{W}^0_{\mathcal{H}}(n,\alpha,q)$ of harmonic functions $f=h+\overline{g}\in\mathcal{H}^0$ associated with Sălăgean q-differential operator. We prove that a harmonic function $f=h+\bar{g}$ belongs to the class $\mathcal{W}^0_{\mathcal{H}}(n,\alpha,q)$ if and only if the analytic functions $h+\epsilon g$ belong to $\mathcal{W}(n,\alpha,q)$ for each ϵ $(|\epsilon|=1)$, and using a method by Clunie and Sheil-Small, we determine a sufficient condition for the class $\mathcal{W}^0_{\mathcal{H}}(n,\alpha,q)$ to be close-to-convex. We provide sharp coefficient estimates, sufficient coefficient condition, and convolution properties for such functions classes. We also determine several conditions of partial sums of $f\in\mathcal{W}^0_{\mathcal{H}}(n,\alpha,q)$.

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1. Introduction

Quantum calculus is the calculus without use of the limits. The history of quantum calculus dates back to the studies of Leonhard Euler (1707-1783) and Carl Gustav Jacobi (1804-1851). Later, geometrical interpretation of the q-calculus has been applied in studies of quantum groups. The great interest to quantum calculus is due to its applications in various branches of mathematics and physics; as for example, in quantum mechanics, analytic number theory, sobolev spaces, group representation theory, theta functions, gamma functions, operator theory and several other areas. For the definitions and properties of q-calculus, one may refer to the books [5] and

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[14]. Jackson [10, 11] was the first who gave some applications of q-calculus by introducing the q-analogues of derivative and integral. The $q-$derivative (or $q-$difference operator) of a function h, defined on a subset of $\mathbb{C}$, is given by

$$(D_q h)(z) = \begin{cases} \frac{h(z)-h(qz)}{(1-q)z}, & z \neq 0 \\ \\ h'(0), & z = 0, \end{cases}$$

where $q \in (0,1)$. Note that $\lim_{q \to 1^-}(D_q h)(z) = h'(z)$ if h is differentiable at z ([10]).

For a function $h(z) = z^k$ $(k \in \mathbb{N})$, we observe that

$$D_q z^k = [k]_q z^{k-1},$$

where

$$[k]_q = \frac{1-q^k}{1-q} = 1 + q + q^2 + ...q^{k-1}$$

is the $q-$number of k. Clearly, $\lim_{q \to 1^-}[k]_q = k$. For more details, one may refer to [14] and references therein.

Connection of q-calculus with geometric function theory was first introduced by Ismail *et al.* [9]. Recently, q-calculus is involved in the theory of analytic functions [7, 8, 21]. But research on q-calculus in connection with harmonic functions is fairly new and not much published (see [12, 23, 22, 28]).

Let $\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\}$ denote an open disk with $r > 0$. The open unit disk will be denoted by $\mathbb{D}_1 = \mathbb{D}$. Let $\mathcal{H}$ denote the class of complex-valued functions $f = u + iv$ which are harmonic in the open unit disk $\mathbb{D}$, where u and v are real-valued harmonic functions in $\mathbb{D}$. Functions $f \in \mathcal{H}$ can also be expressed as $f = h + \overline{g}$, where h the analytic and g the co-analytic parts of f, respectively. A subclass of functions $f = h + \overline{g} \in \mathcal{H}$ with the additional condition $g'(0) = 0$ is denoted by $\mathcal{H}^0$. According to the Lewy's Theorem [15], every harmonic function $f = h + \overline{g} \in \mathcal{H}$ is locally univalent and sense preserving in $\mathbb{D}$ if and only if the Jacobian of f, given by $J_f(z) = |h'(z)|^2 - |\, g'(z)|^2$, is positive in $\mathbb{D}$. This case is equivalent to the existence of an analytic function $\omega(z) = g'(z)/h'(z)$ in $\mathbb{D}$, which is called as the dilatation of f such that

$$|\omega(z)| < 1 \quad \text{for all} \quad z \in \mathbb{D}.$$

Clunie and Sheil-Small [3] introduced the class of all univalent, sense preserving harmonic functions $f = h + \overline{g}$, denoted by $\mathcal{S}_{\mathcal{H}}$, with the normalized conditions $h(0) = 0 = g(0)$ and $h'(0) = 1$. If the function $f = h + \overline{g} \in \mathcal{S}_{\mathcal{H}}$, then

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k \text{ and } g(z) = \sum_{k=1}^{\infty} b_k z^k, \quad (z \in \mathbb{D})\,. \tag{1.1}$$

A subclass of functions $f = h + \overline{g} \in \mathcal{S}_{\mathcal{H}}$ with the condition $g'(0) = 0$ is denoted by $\mathcal{S}^0_{\mathcal{H}}$. Further, the subclass of functions f in $\mathcal{S}_{\mathcal{H}}$ $\left(\mathcal{S}^0_{\mathcal{H}}\right)$, denoted by $\mathcal{K}_{\mathcal{H}}$ $\left(\mathcal{K}^0_{\mathcal{H}}\right)$ consists of functions f that map the unit disk $\mathbb{D}$ onto a convex region, the subclass $\mathcal{S}^*_{\mathcal{H}}$ $\left(\mathcal{S}^{*0}_{\mathcal{H}}\right)$ consists of functions f that are starlike, and the subclass $\mathcal{C}^*_{\mathcal{H}}$ $\left(\mathcal{C}^{*0}_{\mathcal{H}}\right)$ consists of functions f which are close-to-convex. Also, if $g(z) \equiv 0$, the class $\mathcal{S}_{\mathcal{H}}$ reduces to the class $\mathcal{S}$ of univalent functions in the class $\mathcal{A}$. Here, $\mathcal{A}$ is the class of all analytic functions of the form $h(z) = z + \sum_{k=2}^{\infty} a_k z^k$. For more details, we refer [4].

Let $f \in \mathcal{S}$ and be given by $f(z) = \sum_{k=0}^{\infty} a_k z^k$. Then the l^{th} section (partial sum) of f is defined by

$$s_l(f)(z) = \sum_{k=0}^{l} a_k z^k, \quad (l \in \mathbb{N})$$

where $a_0 = 0$ and $a_1 = 1$. For a harmonic function $f = h + \overline{g} \in \mathcal{H}$, where h and g of the form (1.1), the sequences of sections (partial sums) of f is defined by

$$s_{i,j}(f)(z) = s_i(h)(z) + \overline{s_j(g)(z)},$$

where $s_i(h)(z) = \sum_{k=1}^{i} a_k z^k$ and $s_j(g)(z) = \sum_{k=1}^{j} b_k z^k$, $i, j \geq 1$ with $a_1 = 1$.

In [32], it is noted that the partial sums of univalent functions is univalent in the disk $\mathbb{D}_{1/4}$. Starlikeness and convexity of the partial sums of univalent functions was discussed in [29, 30].

The convolution or Hadamard product of two analytic functions

$$f_1(z) = \sum_{k=0}^{\infty} a_k z^k \quad \text{and} \quad f_2(z) = \sum_{k=0}^{\infty} b_k z^k$$

is defined by

$$(f_1 * f_2)(z) = \sum_{k=0}^{\infty} a_k b_k z^k, \quad (z \in \mathbb{D}).$$

The convolution of two harmonic functions $f = h + \overline{g}$ and $F = H + \overline{G}$ is defined by

$$(f * F)(z) = (h * H)(z) + \overline{(g * G)(z)}, \quad (z \in \mathbb{D}).$$

In 2013, Li and Ponnusamy [16] investigated properties of functions given by

$$\mathcal{P}_{\mathcal{H}}^0 = \{f = h + \overline{g} \in \mathcal{H}^0 : \Re(h'(z)) > |g'(z)|, \quad z \in \mathbb{D}\}$$

The class $\mathcal{P}_{\mathcal{H}}^0$ is harmonic analogue of the class $\mathcal{R} = \{f \in \mathcal{S} : \Re(f'(z)) > 0, \ z \in \mathbb{D}\}$ introduced by MacGregor [20]. It is known that a harmonic function $f = h + \overline{g}$ belongs to the class $\mathcal{P}_{\mathcal{H}}^0$ if and only if the analytic function $h + \epsilon g$ belongs to $\mathcal{R}$ for each ϵ ($|\epsilon| = 1$).

In 1977, Chichra [2] studied the class $\mathcal{W}(\alpha)$ consisting of functions $f \in \mathcal{A}$ such that $\Re(f'(z) + \alpha z f''(z)) > 0$ for $\alpha \geq 0$ and $z \in \mathbb{D}$. Later, Nagpal and Ravichandran [24] studied the following class

$$\mathcal{W}_{\mathcal{H}}^0 = \{f = h + \overline{g} \in \mathcal{H}^0 : \Re(h'(z) + zh''(z)) > |g'(z) + zg''(z)|, \ z \in \mathbb{D}\},$$

which is harmonic analogue of $\mathcal{W}(1)$. Recently, Ghosh and Vasudevarao [6] defined the class $\mathcal{W}_{\mathcal{H}}^0(\alpha)$ for $\alpha \geq 0$ by

$$\mathcal{W}_{\mathcal{H}}^0(\alpha) = \{f = h + \overline{g} \in \mathcal{H}^0 : \Re(h'(z) + \alpha z h''(z)) > |g'(z) + \alpha z g''(z)|, \ z \in \mathbb{D}\}.$$

In [2], Chichra also studied the class $\mathcal{G}(\alpha)$ of an analytic function f for $\alpha \geq 0$ such that

$$\Re\left[(1-\alpha)\frac{f(z)}{z} + \alpha f'(z)\right] > 0$$

for $|z| < r$ with $r \in (0, 1]$. In 2018, Liu ang Yang [19] defined the class

$$\mathcal{G}^k_{\mathcal{H}}(\alpha) = \left\{ f = h + \bar{g} \in \mathcal{H}^0 : \Re\big((1-\alpha)\frac{h(z)}{z} + \alpha h'(z)\big) > \big|(1-\alpha)\frac{g(z)}{z} + \alpha g'(z)\big| \right\},$$

where $\alpha \geq 0$, $k \geq 1$ and $|z| < r$ with $r \in (0, 1]$.

For an analytic function $h \in \mathcal{A}$, let the Sălăgean q-differential operator be defined by ([7]);

$$\mathcal{D}^0_q h(z) = h(z), \quad \mathcal{D}^1_q h(z) = zD_q h(z), ..., \quad \mathcal{D}^n_q h(z) = zD_q(\mathcal{D}^{n-1}_q h(z)),$$

where $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Making use of h given by (1.1), and simple calculations yield

$$\mathcal{D}^n_q h(z) = h(z) * \mathcal{F}_{q,n}(z) = z + \sum_{k=2}^{\infty} [k]^n_q a_k z^k, \quad (z \in \mathbb{D}) \tag{1.2}$$

where

$$\mathcal{F}_{q,n}(z) = z + \sum_{k=2}^{\infty} [k]^n_q z^k,$$

and $[k]^n_q = \left(\frac{1-q^k}{1-q}\right)^n$, $q \in (0, 1)$. The operator (1.2) easily reduces to the well-known Sălăgean differential operator as $q \to 1^-$ (see [27]).

For a harmonic function $f = h + \overline{g}$ given by (1.1) and the operator $\mathcal{D}^n_q$ defined by (1.2), the harmonic Sălăgean q-differential operator is defined by ([12]);

$$\begin{aligned} \mathcal{D}^n_q f(z) &= \mathcal{D}^n_q h(z) + (-1)^n \overline{\mathcal{D}^n_q g(z)} \\ &= z + \sum_{k=2}^{\infty} [k]^n_q a_k z^k + (-1)^n \overline{\sum_{k=1}^{\infty} [k]^n_q b_k z^k}. \end{aligned}$$

As $q \to 1^-$, the operator $\mathcal{D}^n_q f$ reduces to the Sălăgean differential operator $\mathcal{D}^n f$ for a harmonic function $f = h + \bar{g}$ ([13]).

Motivated by the Sălăgean q-differential operator, we define a new subclass $\mathcal{W}(n, \alpha, q)$ of analytic functions as follows:

Definition 1.1. An analytic function $f \in \mathcal{A}$ is in the class $\mathcal{W}(n, \alpha, q)$ if it satisfies the condition

$$\Re\left(\frac{(1-\alpha)\, \mathcal{D}^n_q f(z) + \alpha \mathcal{D}^{n+1}_q f(z)}{z} \right) > 0, \tag{1.3}$$

where $\mathcal{D}^n_q f(z)$ is the Sălăgean q-differential operator defined by (1.2), and where $\alpha \geq 0$, $n \in \mathbb{N}_0$, $q \in (0, 1)$ and $|z| < r$ with $0 < r \leq 1$.

Remark 1.2. i) Letting $q \to 1^-$, $n = 0$ we get the class $\mathcal{W}(0, \alpha, q) := \mathcal{G}(\alpha)$ introduced by Chichra [2].

ii) Letting $q \to 1^-$, $n = 1$ we get the class $\mathcal{W}(1, \alpha, q) := \mathcal{W}(\alpha)$ introduced by Chichra [2].

iii) Letting $q \to 1^-$, $n = 1$, $\alpha = 0$ we get the class $\mathcal{W}(1, 0, q) := \mathcal{R}$ introduced by MacGregor [20].

Making use of the harmonic Sălăgean q-differential operator, we also define the class $\mathcal{W}^0_{\mathcal{H}}(n, \alpha, q)$ of harmonic functions as follows:

Definition 1.3. A harmonic function $f = h + \overline{g} \in \mathcal{H}^0$ with $h(0) = g(0) = g'(0) = h'(0) - 1 = 0$ is in the class $\mathcal{W}_{\mathcal{H}}^0(n, \alpha, q)$ if it satisfies the condition

$$\Re\left(\frac{(1-\alpha)\,\mathcal{D}_q^n h(z) + \alpha\mathcal{D}_q^{n+1}h(z)}{z}\right) > \left|\frac{(1-\alpha)\,\mathcal{D}_q^n g(z) + \alpha\mathcal{D}_q^{n+1}g(z)}{z}\right|, \tag{1.4}$$

where $\mathcal{D}_q^n f(z)$ is the harmonic Sălăgean q-differential operator, and where $\alpha \geq 0$, $n \in \mathbb{N}_0$, $q \in (0,1)$ and $|z| < r$ with $0 < r \leq 1$.

Remark 1.4. i) Letting $q \to 1^-$, $n = 0$ we get the class $\mathcal{W}_{\mathcal{H}}^0(0, \alpha, q) := \mathcal{G}_{\mathcal{H}}^1(\alpha)$ introduced by Liu ang Yang [19].

ii) Letting $q \to 1^-$, $n = 1$ we get the class $\mathcal{W}_{\mathcal{H}}^0(1, \alpha, q) := \mathcal{W}_{\mathcal{H}}^0(\alpha)$ introduced by Ghosh and Vasudevarao [6].

iii) Letting $q \to 1^-$, $n = 1$, $\alpha = 1$ we get the class $\mathcal{W}_{\mathcal{H}}^0(1, 1, q) := \mathcal{W}_{\mathcal{H}}^0$ introduced by Nagpal and Ravichandran in [24].

iv) Letting $q \to 1^-$, $n = 1$, $\alpha = 0$ we get the class $\mathcal{W}_{\mathcal{H}}^0(1, 0, q) := \mathcal{P}_{\mathcal{H}}^0$ introduced by Li and Ponnusamy [16].

In this paper, we define a new subclass $\mathcal{W}(n, \alpha, q)$ of analytic functions and a new subclass $\mathcal{W}_{\mathcal{H}}^0(n, \alpha, q)$ of harmonic functions $f = h + \overline{g} \in \mathcal{H}^0$ associated with Sălăgean q-differential operator. In Section 2, we prove that a harmonic function $f \in \mathcal{H}^0$ belongs to the class $\mathcal{W}_{\mathcal{H}}^0(n, \alpha, q)$ if and only if the analytic functions $h + \epsilon g$ belong to $\mathcal{W}(n, \alpha, q)$ for each ϵ with $|\epsilon| = 1$, and by a method of Clunie and Sheil-Small, we obtain a sufficient condition for the class $\mathcal{W}_{\mathcal{H}}^0(n, \alpha, q)$ to be close-to-convex. We also provide sharp coefficient estimates and sufficient coefficient condition for such functions classes. In Section 3, we examine that the class $\mathcal{W}_{\mathcal{H}}^0(n, \alpha, q)$ is closed under convex combinations and convolutions of its members. In Section 4, we determine several conditions of partial sums of $f \in \mathcal{W}_{\mathcal{H}}^0(n, \alpha, q)$.

2. Coefficient bounds

Clunie and Sheil-Small proved the following result, which gives a sufficient condition for a harmonic function f to be close-to-convex.

Lemma 2.1. [3] *If h and g are analytic in $\mathbb{D}$ satisfies $|g'(0)| < |h'(0)|$ and the function $f_\epsilon = h + \epsilon g$ is close-to-convex for all complex number ϵ with $|\epsilon| = 1$, then $f = h + \overline{g}$ is close-to-convex..*

Theorem 2.2. *A harmonic mapping $f = h + \overline{g}$ is in $\mathcal{W}_{\mathcal{H}}^0(n, \alpha, q)$ if and only if the analytic function $f_\epsilon = h + \epsilon g$ belongs to $\mathcal{W}(n, \alpha, q)$ for each complex number ϵ with $|\epsilon| = 1$.*

Proof. If $f = h + \overline{g} \in \mathcal{W}^0_{\mathcal{H}}(n,\alpha,q)$, then for each complex number ϵ with $|\epsilon| = 1$

$$\begin{aligned}
&\Re\left(\frac{(1-\alpha)\,\mathcal{D}_q^n f_\epsilon(z) + \alpha\mathcal{D}_q^{n+1} f_\epsilon(z)}{z}\right)\\
&= \Re\left(\frac{(1-\alpha)\,\mathcal{D}_q^n (h(z)+\epsilon g(z)) + \alpha\mathcal{D}_q^{n+1} (h(z)+\epsilon g(z))}{z}\right)\\
&= \Re\left(\frac{(1-\alpha)\,\mathcal{D}_q^n h(z) + \alpha\mathcal{D}_q^{n+1} h(z) + \epsilon\big((1-\alpha)\,\mathcal{D}_q^n g(z) + \alpha\mathcal{D}_q^{n+1} g(z)\big)}{z}\right)\\
&> \Re\left(\frac{(1-\alpha)\,\mathcal{D}_q^n h(z) + \alpha\mathcal{D}_q^{n+1} h(z)}{z}\right) - \left|\frac{(1-\alpha)\,\mathcal{D}_q^n g(z) + \alpha\mathcal{D}_q^{n+1} g(z)}{z}\right| > 0,
\end{aligned}$$

thus $f_\epsilon = h + \epsilon g \in \mathcal{W}(n,\alpha,q)$ for each ϵ with $|\epsilon| = 1$.

Conversely, if $f_\epsilon = h + \epsilon g \in \mathcal{W}(n,\alpha,q)$, then

$$\Re\left(\frac{(1-\alpha)\,\mathcal{D}_q^n h(z) + \alpha\mathcal{D}_q^{n+1} h(z) + \epsilon\big((1-\alpha)\,\mathcal{D}_q^n g(z) + \alpha\mathcal{D}_q^{n+1} g(z)\big)}{z}\right) > 0,\ \ (z \in \mathbb{D}_r)$$

or equivalently

$$\Re\left(\frac{(1-\alpha)\,\mathcal{D}_q^n h(z) + \alpha\mathcal{D}_q^{n+1} h(z)}{z}\right) > -\Re\left(\frac{\epsilon\big((1-\alpha)\,\mathcal{D}_q^n g(z) + \alpha\mathcal{D}_q^{n+1} g(z)\big)}{z}\right), (z \in \mathbb{D}_r).$$

Since $|\epsilon| = 1$ is arbitrary, for an appropriate choice of ϵ we obtain

$$\Re\left(\frac{(1-\alpha)\,\mathcal{D}_q^n h(z) + \alpha\mathcal{D}_q^{n+1} h(z)}{z}\right) > \left|\frac{(1-\alpha)\,\mathcal{D}_q^n g(z) + \alpha\mathcal{D}_q^{n+1} g(z)}{z}\right|,\ \ (z \in \mathbb{D}_r)$$

Hence, $f = h + \overline{g} \in \mathcal{W}^0_{\mathcal{H}}(n,\alpha,q)$. □

Theorem 2.3. *The functions in the class* $\mathcal{W}^0_{\mathcal{H}}(n,\alpha,q)$ *are close-to-convex in* $\mathbb{D}$.

Proof. Let $f = h + \overline{g} \in \mathcal{W}^0_{\mathcal{H}}(n,\alpha,q)$, and let $f_\epsilon = h + \epsilon g \in \mathcal{W}(n,\alpha,q)$ where $|\epsilon| = 1$. By the method used by Ponnusammy *et al.* [25, Theorem 1.3], if $f_\epsilon \in \mathcal{W}(n,\alpha,q)$, then q-derivative of f_ϵ is positive; that is, $\Re\{\mathcal{D}_q^n f_\epsilon\} > 0$, and hence f_ϵ is analytic and close-to-convex function. Therefore,

$$\begin{aligned}
&\Re\{\mathcal{D}_q^n f_\epsilon\} =\\
&\Re\left(\frac{(1-\alpha)\,\mathcal{D}_q^n h(z) + \alpha\mathcal{D}_q^{n+1} h(z) + \epsilon\big((1-\alpha)\,\mathcal{D}_q^n g(z) + \alpha\mathcal{D}_q^{n+1} g(z)\big)}{z}\right)\\
&> \left|\frac{(1-\alpha)\,\mathcal{D}_q^n g(z) + \alpha\mathcal{D}_q^{n+1} g(z)}{z}\right| + \Re\left(\frac{\epsilon\big((1-\alpha)\,\mathcal{D}_q^n g(z) + \alpha\mathcal{D}_q^{n+1} g(z)\big)}{z}\right)\\
&\geq \left|\frac{(1-\alpha)\,\mathcal{D}_q^n g(z) + \alpha\mathcal{D}_q^{n+1} g(z)}{z}\right| - \left|\frac{\epsilon\big((1-\alpha)\,\mathcal{D}_q^n g(z) + \alpha\mathcal{D}_q^{n+1} g(z)\big)}{z}\right| = 0,
\end{aligned}$$

showing that f_ϵ is analytic and close-to-convex function. Thus according to Lemma 2.1 and Theorem 2.2, it follows that the harmonic function $f \in \mathcal{W}^0_{\mathcal{H}}(n,\alpha,q)$ is also close-to-convex in $\mathbb{D}$. □

We now establish the sharp coefficient bounds for functions in the class $\mathcal{W}^0_{\mathcal{H}}(n,\alpha,q)$.

Theorem 2.4. *Let $f = h + \overline{g} \in \mathcal{W}^0_{\mathcal{H}}(n, \alpha, q)$ be of the form (1.1) with $b_1 = 0$. Then for any $k \geq 2$*

$$|b_k| \leq \frac{1}{[k]_q^n(1 + \alpha([k]_q - 1))}. \tag{2.1}$$

The result is sharp when f is given by $f(z) = z + \frac{1}{[k]_q^n(1+\alpha([k]_q-1))}\overline{z}^k$.

Proof. Let $f \in \mathcal{W}^0_{\mathcal{H}}(n, \alpha, q)$. Then

$$\Re\left(\frac{(1-\alpha)\,\mathcal{D}_q^n h(z) + \alpha\mathcal{D}_q^{n+1} h(z)}{z}\right) > \left|\frac{(1-\alpha)\,\mathcal{D}_q^n g(z) + \alpha\mathcal{D}_q^{n+1} g(z)}{z}\right|$$

and

$$\frac{(1-\alpha)\,\mathcal{D}_q^n g(z) + \alpha\mathcal{D}_q^{n+1} g(z)}{z} = \sum_{k=2}^{\infty}[k]_q^n(1 + \alpha([k]_q - 1))b_k z^{k-1}.$$

Using the series expansion of g, we derive

$$\begin{aligned}
r^{k-1}[k]_q^n(1+\alpha([k]_q-1))|b_k| &\leq \frac{1}{2\pi}\int_0^{2\pi}\left|\frac{(1-\alpha)\,\mathcal{D}_q^n g(re^{i\theta}) + \alpha\mathcal{D}_q^{n+1} g(re^{i\theta})}{re^{i\theta}}\right| d\theta \\
&\leq \frac{1}{2\pi}\int_0^{2\pi}\Re\left(\frac{(1-\alpha)\,\mathcal{D}_q^n h(re^{i\theta}) + \alpha\mathcal{D}_q^{n+1} h(re^{i\theta})}{re^{i\theta}}\right) d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi}\Re\Big(1 + [k]_q^n\big(1+\alpha([k]_q-1)\big)a_k r^{k-1}\Big) d\theta \\
&= 1.
\end{aligned}$$

Letting $r \to 1^-$ gives the desired bound. □

Remark 2.5. (i) When $q \to 1^-$, $n = 0$ we get the result by Liu ang Yang [19, Corollary 3.2].

(ii) When $q \to 1^-$, $n = 1$ we get the result by Ghosh and Vasudevarao [6, Theorem 4.2].

Theorem 2.6. *Let $f = h + \overline{g} \in \mathcal{W}^0_{\mathcal{H}}(n, \alpha, q)$ be of the form (1.1) with $b_1 = 0$. Then for any $k \geq 2$*

(i) $|a_k| + |b_k| \leq \frac{2}{[k]_q^n(1+\alpha([k]_q-1))}$

(ii) $||a_k| - |b_k|| \leq \frac{2}{[k]_q^n(1+\alpha([k]_q-1))}$

(iii) $|a_k| \leq \frac{2}{[k]_q^n(1+\alpha([k]_q-1))}$

The results are sharp and the equality is held for the function

$$f(z) = z + \sum_{k=2}^{\infty}\frac{2}{[k]_q^n(1 + \alpha([k]_q - 1))}z^k.$$

Proof. Suppose that $f = h + \overline{g} \in \mathcal{W}^0_{\mathcal{H}}(n, \alpha, q)$, then from Theorem 2.2 $f_\epsilon = h + \epsilon g \in \mathcal{W}(n, \alpha, q)$ for ϵ with $|\epsilon| = 1$. Thus for any $|\epsilon| = 1$, we have

$$\Re\left(\frac{(1-\alpha)\,\mathcal{D}_q^n(h(z) + \epsilon g(z)) + \alpha\mathcal{D}_q^{n+1}(h(z) + \epsilon g(z))}{z}\right) > 0, \quad |z| < r.$$

Then there exists an analytic function p of the form $p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k$ with $\Re(p(z)) > 0$ in $\mathbb{D}$ such that

$$\frac{(1-\alpha)\,\mathcal{D}_q^n(h(z)+\epsilon g(z)) + \alpha\mathcal{D}_q^{n+1}(h(z)+\epsilon g(z))}{z} = p(z). \tag{2.2}$$

Comparing coefficients on both sides of (2.2), we have

$$[k]_q^n(1+\alpha([k]_q-1))(a_k+\epsilon b_k) = p_{k-1}, \quad k \geq 2. \tag{2.3}$$

Since $|p_k| \leq 2$ for $k \geq 1$ and ϵ ($|\epsilon| = 1$) is arbitrary, from (2.3) we get

$$[k]_q^n(1+\alpha([k]_q-1))(|a_k|+|b_k|) \leq 2,$$

which proves (i). The last two inequalities are consequences of the first inequality. □

Remark 2.7. (i) When $q \to 1^-$, $n = 0$ we get the result by Liu ang Yang [19, Corollary 3.4].

(ii) When $q \to 1^-$, $n = 1$ we get the result by Ghosh and Vasudevarao [6, Theorem 4.3].

The following result gives a sufficient condition for a function to be belong to $\mathcal{W}_{\mathcal{H}}^0(n,\alpha,q)$.

Theorem 2.8. *Let $f = h + \overline{g} \in \mathcal{H}^0$ be of the form (1.1) with $b_1 = 0$. If*

$$\sum_{k=2}^{\infty}[k]_q^n(1+\alpha([k]_q-1))(|a_k|+|b_k|) \leq 1, \tag{2.4}$$

then $f \in \mathcal{W}_{\mathcal{H}}^0(n,\alpha,q)$.

Proof. Let $f = h+\overline{g} \in \mathcal{H}^0$. Using the series representation of h given by (1.1), we get

$$\begin{aligned}
\Re\left(\frac{(1-\alpha)\,\mathcal{D}_q^n h(z) + \alpha\mathcal{D}_q^{n+1}h(z)}{z}\right) &= \Re\left(1 + \sum_{k=2}^{\infty}[k]_q^n(1+\alpha([k]_q-1)a_k z^{k-1}\right) \\
&> 1 - \sum_{k=2}^{\infty}[k]_q^n(1+\alpha([k]_q-1)|a_k| \\
&\geq \sum_{k=2}^{\infty}[k]_q^n(1+\alpha([k]_q-1)|b_k| \\
&> \left|\sum_{k=2}^{\infty}[k]_q^n(1+\alpha([k]_q-1)b_k z^{k-1}\right| \\
&= \left|\frac{(1-\alpha)\,\mathcal{D}_q^n g(z) + \alpha\mathcal{D}_q^{n+1}g(z)}{z}\right|,
\end{aligned}$$

therefore $f \in \mathcal{W}_{\mathcal{H}}^0(n,\alpha,q)$. □

Remark 2.9. When $q \to 1^-$, $n = 1$ we get the result by Ghosh and Vasudevarao [6, Theorem 4.5].

3. Convex combinations and convolutions

In this section, we prove that the class $\mathcal{W}_{\mathcal{H}}^{0}(n,\alpha,q)$ is closed under convex combinations and convolutions of its members.

Theorem 3.1. *The class $\mathcal{W}_{\mathcal{H}}^{0}(n,\alpha,q)$ is closed under convex combinations.*

Proof. Suppose $\mathcal{D}_q^n f_i = \mathcal{D}_q^n h_i + (-1)^n \overline{\mathcal{D}_q^n g_i} \in \mathcal{W}_{\mathcal{H}}^{0}(n,\alpha,q)$ for $i = 1, 2, ..., k$ and $\sum_{i=1}^{k} t_i = 1$ $(0 \le t_i \le 1)$. The convex combination of functions $\mathcal{D}_q^n f_i$ can be written as

$$\mathcal{D}_q^n f(z) = \sum_{i=1}^{k} t_i \mathcal{D}_q^n f_i(z) = \mathcal{D}_q^n h(z) + (-1)^n \mathcal{D}_q^n \overline{g(z)}$$

where $\mathcal{D}_q^n h(z) = \sum_{i=1}^{k} t_i \mathcal{D}_q^n h_i(z)$ and $\mathcal{D}_q^n g(z) = \sum_{i=1}^{k} t_i \mathcal{D}_q^n g_i(z)$. Then h and g both are analytic in $\mathbb{D}$ with $h(0) = g(0) = h'(0) - 1 = g'(0) = 0$. A simple computation yields

$$\begin{aligned}
\Re\left(\frac{(1-\alpha)\,\mathcal{D}_q^n h(z) + \alpha \mathcal{D}_q^{n+1} h(z)}{z}\right) &= \Re\left(\sum_{i=1}^{k} t_i \frac{(1-\alpha)\,\mathcal{D}_q^n h_i(z) + \alpha \mathcal{D}_q^{n+1} h_i(z)}{z}\right) \\
&> \left|\sum_{i=1}^{k} t_i \frac{(-1)^n\,(1-\alpha)\,\mathcal{D}_q^n g_i(z) + (-1)^{n+1} \alpha \mathcal{D}_q^{n+1} g_i(z)}{z}\right| \\
&\ge \left|\frac{(1-\alpha)\,\mathcal{D}_q^n g(z) + \alpha \mathcal{D}_q^{n+1} g(z)}{z}\right|.
\end{aligned}$$

This shows that $f \in \mathcal{W}_{\mathcal{H}}^{0}(n,\alpha,q)$ □

A sequence $\{c_k\}_{k=0}^{\infty}$ of non-negative real numbers is said to be a convex null sequence if $c_k \to 0$ as $k \to \infty$, and

$$c_0 - c_1 \ge c_1 - c_2 \ge c_2 - c_3 \ge ... \ge c_{k-1} - c_k \ge ... \ge 0.$$

To prove the convolution results, we need the following lemmas.

Lemma 3.2. [31] *Let $\{c_k\}_{k=0}^{\infty}$ be a convex null sequence. Then the function*

$$s(z) = \frac{c_0}{2} + \sum_{k=1}^{\infty} c_k z^k$$

is analytic, and $\Re(s(z)) > 0$ in $\mathbb{D}$.

Lemma 3.3. [31] *Let the function p be analytic in $\mathbb{D}$ with $p(0) = 1$ and $\Re(p(z)) > 1/2$ in $\mathbb{D}$. Then for any analytic function F in $\mathbb{D}$, the function $p * F$ takes values in the convex hull of the image of $\mathbb{D}$ under F.*

Using Lemmas 3.2 and 3.3, we prove the following lemma.

Lemma 3.4. *Let $F \in \mathcal{W}(n,\alpha,q)$, then $\Re\left(\frac{F(z)}{z}\right) > \frac{1}{2}$.*

Proof. Suppose $F \in \mathcal{W}(n,\alpha,q)$ be given by $F(z) = z + \sum_{k=2}^{\infty} A_k z^k$, then

$$\Re\left(1 + \sum_{k=2}^{\infty} [k]_q^n (1 + \alpha([k]_q - 1)) A_k z^{k-1}\right) > 0,$$

which is equivalent to $\Re(p(z)) > 1/2$ in $\mathbb{D}$, where

$$p(z) = 1 + \frac{1}{2}\sum_{k=2}^{\infty} [k]_q^n (1 + \alpha([k]_q - 1)) A_k z^{k-1}.$$

Now consider a sequence $\{c_k\}_{k=0}^{\infty}$ defined by

$$c_0 = 1 \quad \text{and} \quad c_{k-1} = \frac{2}{[k]_q^n (1 + \alpha([k]_q - 1))} \quad \text{for} \quad k \geq 2.$$

It can be easily seen that the sequence $\{c_k\}_{k=0}^{\infty}$ is convex null sequence and using Lemma 3.2, the function

$$s(z) = 1 + \sum_{k=2}^{\infty} \frac{2}{[k]_q^n (1 + \alpha([k]_q - 1)} z^{k-1}$$

is analytic with $\Re(s(z)) > \frac{1}{2}$ in $\mathbb{D}$. Hence

$$\frac{F(z)}{z} = p(z) * \left(1 + \sum_{k=2}^{\infty} \frac{2}{[k]_q^n (1 + \alpha([k]_q - 1)} z^{k-1}\right)$$

$$= \left(1 + \frac{1}{2}\sum_{k=2}^{\infty} [k]_q^n (1 + \alpha([k]_q - 1)) A_k z^{k-1}\right) * \left(1 + \sum_{k=2}^{\infty} \frac{2}{[k]_q^n (1 + \alpha([k]_q - 1)} z^{k-1}\right)$$

and making use of Lemma 3.3 we observe that $\Re\left(\frac{F(z)}{z}\right) > \frac{1}{2}$ for $z \in \mathbb{D}$. □

Lemma 3.5. *Let F_1 and F_2 belong to $\mathcal{W}(n,\alpha,q)$. Then $F = F_1 * F_2 \in \mathcal{W}(n,\alpha,q)$.*

Proof. Let $F_1(z) = z + \sum_{k=2}^{\infty} A_k z^k$ and $F_2(z) = z + \sum_{k=2}^{\infty} B_k z^k$. Then the convolution of F_1 and F_2 is given by

$$F(z) = (F_1 * F_2)(z) = z + \sum_{k=2}^{\infty} A_k B_k z^k.$$

To prove that $F \in \mathcal{W}(n,\alpha,q)$, we have to show that

$$\Re\left(\frac{(1-\alpha)\,\mathcal{D}_q^n F(z) + \alpha \mathcal{D}_q^{n+1} F(z)}{z}\right) > 0,$$

which is equivalent to

$$\Re\left(1 + \sum_{k=2}^{\infty} [k]_q^n (1 + \alpha([k]_q - 1)) A_k B_k z^{k-1}\right) > 0$$

or

$$\Re\left(1 + \frac{1}{2}\sum_{k=2}^{\infty} [k]_q^n (1 + \alpha([k]_q - 1)) A_k B_k z^{k-1}\right) > \frac{1}{2}. \tag{3.1}$$

Since $F_1 \in \mathcal{W}(n, \alpha, q)$ we have

$$\Re\left(1 + \frac{1}{2}\sum_{k=2}^{\infty}[k]_q^n(1 + \alpha([k]_q - 1))A_k z^{k-1}\right) > \frac{1}{2}$$

and by Lemma 3.4, $F_2 \in \mathcal{W}(n, \alpha, q)$ implies $\Re\left(\frac{F_2(z)}{z}\right) > \frac{1}{2}$ in $\mathbb{D}$ or

$$\Re\left(1 + \frac{1}{2}\sum_{k=2}^{\infty}[k]_q^n(1 + \alpha([k]_q - 1))B_k z^{k-1}\right) > \frac{1}{2}.$$

By applying Lemma 3.3, we conclude we have (3.1). Hence, $F = F_1 * F_2 \in \mathcal{W}(n, \alpha, q)$. □

Now using Lemma 3.5, we prove that the class $\mathcal{W}_{\mathcal{H}}^0(n, \alpha, q)$ is closed under convolutions of its members.

Theorem 3.6. *If f_1 and f_2 belong to $\mathcal{W}_{\mathcal{H}}^0(n, \alpha, q)$, then $f_1 * f_2 \in \mathcal{W}_{\mathcal{H}}^0(n, \alpha, q)$.*

Proof. Let $f_1 = h_1 + \overline{g}_1$ and $f_2 = h_2 + \overline{g}_2$ be two functions in $\mathcal{W}_{\mathcal{H}}^0(n, \alpha, q)$. Then the convolution of f_1 and f_2 is defined as $f_1 * f_2 = h_1 * h_2 + \overline{g_1 * g_2}$. In order to prove that $f_1 * f_2 \in \mathcal{W}_{\mathcal{H}}^0(n, \alpha, q)$, we need to prove that $F = h_1 * h_2 + \epsilon(g_1 * g_2) \in \mathcal{W}(n, \alpha, q)$ for each ϵ ($|\epsilon| = 1$). By Lemma 3.5, the class $\mathcal{W}(n, \alpha, q)$ is closed under convolutions for each ϵ ($|\epsilon| = 1$), $h_i + \epsilon g_i \in \mathcal{W}(n, \alpha, q)$ for $i = 1, 2$. Then both F_1 and F_2 given by

$$F_1 = (h_1 - g_1) * (h_2 - \epsilon g_2)$$

and

$$F_2 = (h_1 + g_1) * (h_2 + \epsilon g_2)$$

belong to $\mathcal{W}(n, \alpha, q)$. Since $\mathcal{W}(n, \alpha, q)$ is closed under convex combinations, then the function

$$F = \frac{1}{2}(F_1 + F_2) = h_1 * h_2 + \epsilon(g_1 * g_2)$$

belongs to $\mathcal{W}(n, \alpha, q)$. Thus $\mathcal{W}_{\mathcal{H}}^0(n, \alpha, q)$ is closed under convolution. □

4. Partial sums

In this section, we examine sections (partial sums) of functions in the class $\mathcal{W}_{\mathcal{H}}^0(n, \alpha, q)$.

Theorem 4.1. *Let $f = h + \overline{g} \in \mathcal{W}_{\mathcal{H}}^0(n, \alpha, q)$ with $\alpha \geq 0$. Then for each ϵ ($|\epsilon| = 1$) and $|z| < 1/2$, we have*

$$\Re\left(\frac{(1-\alpha)\,\mathcal{D}_q^n(s_3(h) + \epsilon s_3(g)) + \alpha\mathcal{D}_q^{n+1}(s_3(h) + \epsilon s_3(g))}{z}\right) > \frac{1}{4}.$$

Proof. Let $f = h + \overline{g} \in \mathcal{W}_{\mathcal{H}}^0(n, \alpha, q)$. Then by Theorem 2.2, $h + \epsilon g \in \mathcal{W}(n, \alpha, q)$ for ϵ ($|\epsilon| = 1$), so $\Re f_\epsilon(z) > 0$, where

$$f_\epsilon(z) = \frac{(1-\alpha)\,\mathcal{D}_q^n(h(z) + \epsilon g(z)) + \alpha\mathcal{D}_q^{n+1}(h(z) + \epsilon g(z))}{z} = 1 + \sum_{k=1}^{\infty} p_k z^k.$$

Moreover

$$\begin{aligned}&\frac{(1-\alpha)\,\mathcal{D}_q^n(s_3(h)+\epsilon s_3(g))+\alpha\mathcal{D}_q^{n+1}(s_3(h)+\epsilon s_3(g))}{z}\\&=1+[2]_q^n(1+\alpha([2]_q-1)(a_2+\epsilon b_2)z+[3]_q^n(1+\alpha([3]_q-1)(a_3+\epsilon b_3)z^2\\&=1+p_1z+p_2z^2.\end{aligned}$$

It is easy to see that

$$|2p_2-p_1^2|\le 4-|p_1|^2$$

Let $2p_2-p_1^2=p$. Then $p_2=p/2+p_1^2/2$ and $|p|\le 4-|p_1|^2$. Also, let $p_1z=\gamma+i\beta$ and $\sqrt{pz}=\eta+i\delta$ where β,γ,δ,η are real numbers. Then for $|z|<1/2$

$$\gamma^2+\beta^2=|p_1|^2|z|^2\le\frac{|p_1|^2}{4}$$

and

$$\delta^2=|p||z|^2-\eta^2\le\frac{|p|}{4}-\eta^2\le\frac{4-|p_1|^2}{4}-\eta^2\le 1-(\gamma^2+\beta^2)-\eta^2$$

so that

$$\begin{aligned}&\Re\left(\frac{(1-\alpha)\,\mathcal{D}_q^n(s_3(h)+\epsilon s_3(g))+\alpha\mathcal{D}_q^{n+1}(s_3(h)+\epsilon s_3(g))}{z}\right)\\&=\Re(1+p_1z+p_2z^2)\\&=\Re(1+p_1z+\frac{p}{2}z^2+\frac{p_1^2}{2}z^2)\\&=1+\gamma+\left(\frac{\eta^2}{2}-\frac{\delta^2}{2}\right)+\left(\frac{\gamma^2}{2}-\frac{\beta^2}{2}\right)\\&=1+\gamma+\frac{\eta^2}{2}-\frac{1-\gamma^2-\beta^2-\eta^2}{2}+\frac{\gamma^2}{2}-\frac{\beta^2}{2}\\&=\frac{1}{4}+\left(\gamma+\frac{1}{2}\right)^2+\eta^2\ge\frac{1}{4},\end{aligned}$$

which gives the result. □

Theorem 4.2. *Let $f=h+\overline{g}\in\mathcal{W}_{\mathcal{H}}^0(n,\alpha,q)$, where h and g given by (1.1) with $b_1=0$. Then for each $j\ge 2$, $s_{1,j}(f)\in\mathcal{W}_{\mathcal{H}}^0(n,\alpha,q)$ for $|z|<1/2$.*

Proof. Let $f=h+\overline{g}\in\mathcal{W}_{\mathcal{H}}^0(n,\alpha,q)$. It is clear that

$$s_{1,j}(f)(z)=s_1(h)(z)+\overline{s_j(g)(z)}=z+\sum_{k=2}^{j}\overline{b_kz^k}$$

It follows from Theorem 2.4 that for all $|z| < 1/2$,

$$\begin{aligned}
&\left|\frac{(1-\alpha)\,\mathcal{D}_q^n s_j(g)(z) + \alpha \mathcal{D}_q^{n+1} s_j(g)(z)}{z}\right| \\
&= \left|\sum_{k=2}^{j} [k]_q^n (1+\alpha([k]_q - 1)) b_k z^{k-1}\right| \\
&\le \sum_{k=2}^{j} [k]_q^n (1+\alpha([k]_q - 1)) |b_k| |z^{k-1}| \\
&\le \sum_{k=2}^{j} |z|^{k-1} = \frac{|z|(1-|z|^{j-1})}{1-|z|} < \frac{|z|}{1-|z|} \\
&< 1 = \Re\left(\frac{(1-\alpha)\,\mathcal{D}_q^n s_1(h)(z) + \alpha \mathcal{D}_q^{n+1} s_1(h)(z)}{z}\right).
\end{aligned}$$

This implies that $s_{1,j}(f) \in \mathcal{W}_{\mathcal{H}}^0(n, \alpha, q)$ in $|z| < 1/2$. □

Theorem 4.3. *Let $f = h + \overline{g} \in \mathcal{W}_{\mathcal{H}}^0(n, \alpha, q)$, where h and g given by (1.1) with $b_1 = 0$, and let i and j satisfy of the following conditions:*

(i) $3 \le i < j$,
(ii) $i = j \ge 2$,
(iii) $i = 3$ *and* $j = 2$.

Then $s_{i,j}(f) \in \mathcal{W}_{\mathcal{H}}^0(n, \alpha, q)$ in $|z| < 1/2$.

Proof. Let $\vartheta_i(h)(z) = \sum_{k=i+1}^{\infty} a_k z^k$ and $\vartheta_j(g)(z) = \sum_{k=j+1}^{\infty} b_k z^k$. Then

$$h = s_i(h) + \vartheta_i(h) \quad \text{and} \quad g = s_j(g) + \vartheta_j(g).$$

To prove $s_{i,j}(f) \in \mathcal{W}_{\mathcal{H}}^0(n, \alpha, q)$, it suffices to prove that $s_i(h) + \epsilon s_j(g) \in \mathcal{W}(n, \alpha, q)$ for ϵ ($|\epsilon| = 1$). If $f \in \mathcal{W}_{\mathcal{H}}^0(n, \alpha, q)$, then

$$\begin{aligned}
&\Re\left(\frac{(1-\alpha)\,\mathcal{D}_q^n (s_i(h) + \epsilon s_j(g)) + \alpha \mathcal{D}_q^{n+1} (s_i(h) + \epsilon s_j(g))}{z}\right) \\
&= \Re\left(\frac{(1-\alpha)\,\mathcal{D}_q^n (h + \epsilon g) + \alpha \mathcal{D}_q^{n+1} (h + \epsilon g)}{z}\right. \\
&\qquad \left. - \frac{(1-\alpha)\,\mathcal{D}_q^n (\vartheta_i(h) + \epsilon \vartheta_j(g)) + \alpha \mathcal{D}_q^{n+1} (\vartheta_i(h) + \epsilon \vartheta_j(g))}{z}\right) \\
&\ge \Re\left(\frac{(1-\alpha)\,\mathcal{D}_q^n (h + \epsilon g) + \alpha \mathcal{D}_q^{n+1} (h + \epsilon g)}{z}\right) \\
&\qquad - \left|\frac{(1-\alpha)\,\mathcal{D}_q^n (\vartheta_i(h) + \epsilon \vartheta_j(g)) + \alpha \mathcal{D}_q^{n+1} (\vartheta_i(h) + \epsilon \vartheta_j(g))}{z}\right|.
\end{aligned} \tag{4.1}$$

By assumption, we see that

$$\frac{(1-\alpha)\,\mathcal{D}_q^n(h+\epsilon g)+\alpha\mathcal{D}_q^{n+1}(h+\epsilon g)}{z} \prec \frac{1+z}{1-z},$$

where $\prec$ is the subordination symbol. From the last relation, we conclude that

$$\Re\left(\frac{(1-\alpha)\,\mathcal{D}_q^n(h+\epsilon g)+\alpha\mathcal{D}_q^{n+1}(h+\epsilon g)}{z}\right) \geq \frac{1-|z|}{1+|z|}. \tag{4.2}$$

Case (i): $3 \leq i < j$

Applying Theorems 2.4 and 2.6, we observe that

$$\begin{aligned}
&\left|\frac{(1-\alpha)\,\mathcal{D}_q^n(\vartheta_i(h)+\epsilon\vartheta_j(g))+\alpha\mathcal{D}_q^{n+1}(\vartheta_i(h)+\epsilon\vartheta_j(g))}{z}\right| \\
&= \left|\sum_{k=i+1}^{j}[k]_q^n(1+\alpha([k]_q-1))a_k z^{k-1}+\sum_{k=j+1}^{\infty}[k]_q^n(1+\alpha([k]_q-1))(a_k+\epsilon b_k)z^{k-1}\right| \\
&\leq \sum_{k=i+1}^{j}2|z|^{k-1}+\sum_{k=j+1}^{\infty}2|z|^{k-1}=2\frac{|z|^i}{1-|z|}
\end{aligned} \tag{4.3}$$

Using (4.1), (4.2) and (4.3), we obtain

$$\Re\left(\frac{(1-\alpha)\,\mathcal{D}_q^n(s_i(h)+\epsilon s_j(g))+\alpha\mathcal{D}_q^{n+1}(s_i(h)+\epsilon s_j(g))}{z}\right) \geq \frac{1-|z|}{1+|z|}-2\frac{|z|^i}{1-|z|}. \tag{4.4}$$

For $4 \leq i < j$ and $|z| = 1/2$, the inequality (4.4) gives

$$\Re\left(\frac{(1-\alpha)\,\mathcal{D}_q^n(s_i(h)+\epsilon s_j(g))+\alpha\mathcal{D}_q^{n+1}(s_i(h)+\epsilon s_j(g))}{z}\right) \geq \frac{1}{3}-\frac{1}{4} > 0.$$

Since $\Re\left(\frac{(1-\alpha)\mathcal{D}_q^n(s_i(h)+\epsilon s_j(g))+\alpha\mathcal{D}_q^{n+1}(s_i(h)+\epsilon s_j(g))}{z}\right)$ is harmonic, it assumes its minimum value on the circle $|z| = 1/2$. Hence, if $4 \leq i < j$ then $s_{i,j}(f) \in \mathcal{W}_{\mathcal{H}}^0(n,\alpha,q)$ in $|z| < 1/2$.

If $i = 3 < j$, then in view of Theorem 2.4 and Theorem 4.1, we attain

$$\begin{aligned}
&\Re\left(\frac{(1-\alpha)\,\mathcal{D}_q^n(s_3(h)+\epsilon s_j(g))+\alpha\mathcal{D}_q^{n+1}(s_3(h)+\epsilon s_j(g))}{z}\right) \\
&= \Re\left(\frac{(1-\alpha)\,\mathcal{D}_q^n(s_3(h)+\epsilon s_3(g))+\alpha\mathcal{D}_q^{n+1}(s_3(h)+\epsilon s_3(g))}{z}\right. \\
&\qquad \left.+\epsilon\sum_{k=4}^{j}[k]_q^n(1+\alpha([k]_q-1))b_k z^{k-1}\right) \\
&\geq \frac{1}{4}-\sum_{k=4}^{j}[k]_q^n(1+\alpha([k]_q-1))|b_k z^{k-1}| \\
&\geq \frac{1}{4}-\frac{|z|^3}{1-|z|}
\end{aligned}$$

so that

$$\Re\left(\frac{(1-\alpha)\,\mathcal{D}_q^n(s_3(h)+\epsilon s_j(g))+\alpha\mathcal{D}_q^{n+1}(s_3(h)+\epsilon s_j(g))}{z}\right)>0$$

for $|z|<1/2$, and thus $s_{3,j}(f)\in\mathcal{W}_{\mathcal{H}}^0(n,\alpha,q)$ in $|z|<1/2$.

Case (ii): $i=j\geq 2$

If $i=j\geq 4$, then the inequality (4.4) gives $s_{i,j}(f)\in\mathcal{W}_{\mathcal{H}}^0(n,\alpha,q)$ in $|z|<1/2$. For $i=j=2$, $s_{2,2}(f)(z)=z+a_2z^2+\overline{b_2z^2}$. Using Theorem 2.6, we get

$$\begin{aligned}&\Re\big(1+[2]_q^n(1+\alpha([2]_q-1))(a_2+\epsilon b_2)z\big)\\&\geq 1-[2]_q^n(1+\alpha([2]_q-1))|a_2+\epsilon b_2||z|\\&\geq 1-2|z|>0\end{aligned}$$

in $|z|<1/2$.

If $i=j=3$, then Theorem 4.1 shows that $s_{3,3}(f)\in\mathcal{W}_{\mathcal{H}}^0(n,\alpha,q)$ in $|z|<1/2$. Therefore, we prove that for $i=j\geq 2$, $s_{i,j}(f)\in\mathcal{W}_{\mathcal{H}}^0(n,\alpha,q)$ in $|z|<1/2$.

Case (iii): $i=3$ and $j=2$.

In view of Theorems 2.4 and 4.1, we have

$$\Re\left(\frac{(1-\alpha)\,\mathcal{D}_q^n(s_3(h)+\epsilon s_2(g))+\alpha\mathcal{D}_q^{n+1}(s_3(h)+\epsilon s_2(g))}{z}\right)$$

$$=\Re\left(\frac{(1-\alpha)\,\mathcal{D}_q^n(s_3(h)+\epsilon s_3(g))+\alpha\mathcal{D}_q^{n+1}(s_3(h)+\epsilon s_3(g))}{z}-\epsilon[3]_q^n(1+\alpha([3]_q-1))b_3z^2\right)$$

$$\geq\frac{1}{4}-|z|^2=\frac{1}{4}-\frac{1}{2^2}=0$$

for $|z|<1/2$. Thus $s_{3,2}(f)\in\mathcal{W}_{\mathcal{H}}^0(n,\alpha,q)$ in $|z|<1/2$. □

References

[1] Beig, S., Ravichandran, V., *Convolution and convex combination of harmonic mappings*, Bull. Iranian Math. Soc., **45**(2019), no. 5, 1467-1486.

[2] Chichra, P.N., *New subclass of the class of close-to-convex function*, Proc. Amer. Math. Soc., **62**(1977), 37–43.

[3] Clunie, J., Sheil-Small, T., *Harmonic univalent functions*, Ann. Acad. Sci. Fenn. Ser. A.I., **9**(1984), 3–25.

[4] Dorff, M., Rolf, J., *Anamorphosis, mapping problems, and harmonic univalent functions*, Explorations in Complex Analysis, Math. Assoc. of America, Inc., Washington, DC, (2012), 197–269.

[5] Gasper, G., Rahman, M., *Basic Hypergeometric Series*, Cambridge University Press, Cambridge, 2004.

[6] Ghosh, N., Vasudevarao, A., *On a subclass of harmonic close-to-convex mappings*, Monatsh. Math., **188**(2019), 247–267.

[7] Govindaraj, M., Sivasubramanian, S., *On a class of analytic functions related to conic domains involving q-calculus*, Analysis Math., **43**(2017), 475–487.

[8] Hussain, S., Khan, S., Zaighum, M.A., Darus, M., *Certain subclass of analytic functions related with conic domains and associated with Sălăgean q-differential operator*, AIMS Mathematics, **2**(2017), 622–634.

[9] Ismail, M.E.H., Merkes, E., Styer, D., *A generalizations of starlike functions*, Complex Variables Theory Appl., **14**(1990), 77–84.

[10] Jackson, F.H., *On q-functions and a certain difference operator*, Trans. Royal Soc. Edinburgh, **46**(1909), 253–281.

[11] Jackson, F.H., *On q-definite integrals*, Quart. J. Pure Appl. Math., **41**(1910), 193–203.

[12] Jahangiri, J.M., *Harmonic univalent functions defined by q-calculus operator*, Int. J. Math. Anal. Appl., **5**(2018), 39–43.

[13] Jahangiri, J.M., Murugusundaramoorthy, G., Vijaya, K., *Sălăgean type harmonic univalent functions*, South J. Pure. Appl. Math., **2**(2002), 77–82.

[14] Kac, V., Cheung, P., *Quantum Calculus*, Springer-Verlag, New York, 2002.

[15] Lewy, H., *On the non vanishing of the Jacobian in certain one-to-one mappings*, Bull. Amer. Math. Soc., **42**(1936), 689–692.

[16] Li, L., Ponnusamy, S., *Disk of convexity of sections of univalent harmonic functions*, J. Math. Anal. Appl., **408**(2013), 589–596.

[17] Li, L., Ponnusamy, S., *Solution to an open problem on convolutions of harmonic mappings*, Complex Var. Elliptic Equ., **58**(2013), no. 12, 1647–1653.

[18] Liu, Z., Ponnusamy, S., *Univalency of convolutions of univalent harmonic right half-plane mappings*, Comput. Methods Funct. Theory, **17**(2017), no. 2, 289–302.

[19] Liu, M.S., Yang, L.M., *Geometric properties and sections for certain subclasses of harmonic mappings*, Monatsh. Math., **190**(2019), 353–387.

[20] MacGregor, T.H., *Functions whose derivative has a positive real part*, Trans. Amer. Math. Soc., **104**(1962), 532–537.

[21] Mahmood, S., Sokół, J., *New subclass of analytic functions in conical domain associated with Ruscheweyh q-differential operator*, Results Math., **71**(2017), 1345–1357.

[22] Mishra, O., Sokol, J., *Generalized q-starlike harmonic functions*, Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A-Mat., **115**(2021).

[23] Murugusundaramoorthy, G., Jahangiri, J.M., *Ruscheweyh-type harmonic functions defined by q-differential operators*, Khayyam J. Math., **5**(2019), 79–88.

[24] Nagpal, S., Ravichandran, V., *Construction of subclasses of univalent harmonic mappings*, J. Korean Math. Soc., **53**(2014), 567–592.

[25] Ponnusamy, S., Yamamoto, H., Yanagihara, H., *Variability regions for certain families of harmonic univalent mappings*, Complex Variables and Elliptic Equations, **58**(2013), no. 1, 23–34.

[26] Ruscheweyh, S., Sheil-Small, T., *Hadamard products of Schlicht functions and the Pólya-Schoenberg conjecture*, Comment. Math. Helv., **48**(1973), 119–135.

[27] Sălăgean, G.S., *Subclasses of univalent functions*, Lecture notes in Math., Springer-Verlage, **1013**(1983), 362–372.

[28] Sharma, P., Mishra, O., *A class of harmonic functions associated with a $q-$Sălăgean operator*, U.P.B. Sci. Bull., Series A, **82**(2020), no. 3, 10 pages.

[29] Silverman, H., *Partial sums of starlike and convex functions*, J. Math. Anal. Appl., **209**(1997), 221–227.

[30] Silvia, E.M., *On partial sums of convex functions of order alpha*, Houston J. Math., **11**(1985), 397–404.

[31] Singh, R., Singh, S., *Convolution properties of a class of starlike functions*, Proc. Amer. Math. Soc., **106**(1989), 145–152.

[32] Szegö, G., *Zur theorie der schlichten Abbildungen*, Math. Ann., **100** (1928), 188–211.

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