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# Ma-Minda starlikeness of certain analytic functions

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**Abstract.** A normalized analytic function defined on the open unit disc $\mathbb{D}$ is called Ma-Minda starlike if $zf'(z)/f(z)$ is subordinate to the function $\varphi$. For a normalized convex function $f$ defined on $\mathbb{D}$ and $\alpha > 0$, we determine the radius of Ma-Minda starlikeness of the function $g$ defined as $g(z) = (zf'(z)/f(z))^{\alpha} f(z)$ for certain choices of $\varphi$. In particular, we investigate the radius of Janowski starlikeness of the function $g$.

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## 1. Introduction and preliminaries

Let $\mathbb{C}$ denote the complex plane, $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ represent the open unit disc, and $\mathcal{A}$ denote the class of analytic functions defined on $\mathbb{D}$, normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. Additionally, let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of univalent (one-to-one) functions. A function $f \in \mathcal{A}$ is considered starlike if it maps $\mathbb{D}$ onto a domain that is starlike with respect to the origin. Similarly, a function $f \in \mathcal{A}$ is said to be convex if $f(\mathbb{D})$ is a convex set. Let $\mathcal{ST}$ and $\mathcal{CV}$ denote the subclasses of $\mathcal{A}$ respectively consisting of starlike and convex functions. Analytically, we have: $\mathcal{ST} := \{f \in \mathcal{A} : \operatorname{Re}(zf'(z)/f(z)) > 0\}$ and $\mathcal{CV} := \{f \in \mathcal{A} : 1+\operatorname{Re}(zf''(z)/f'(z)) > 0\}$. Alexander's theorem [4] establishes a relationship between these two classes, stating that $f \in \mathcal{CV}$ if and only if $zf' \in \mathcal{ST}$. For two analytic functions $f$ and $g$, we say that $f$ is subordinate to $g$, written $f \prec g$, if there exists an analytic function $w$ satisfying the conditions $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$. This relationship

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implies that $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$. Moreover, if the function $g(z)$ is univalent (one-to-one), then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$. The function $w$ is commonly known as the Schwarz function. Using subordination, Ma and Minda [12] investigated growth, distortion and covering theorems for the class $\mathcal{ST}(\varphi)$ consisting of starlike functions that satisfy the subordination

$$\frac{zf'(z)}{f(z)} \prec \varphi(z),$$

where $\varphi : \mathbb{D} \to \mathbb{C}$ is an analytic function that is univalent with a positive real part, $\varphi(\mathbb{D})$ is starlike with respect to $\varphi(0) = 1$, symmetric about the real axis, and $\varphi'(0) > 0$. Different subclasses of starlike and convex functions are obtained for various choices of $\varphi$. For instance, when $\varphi(z) = (1 + Az)/(1 + Bz)$, where $-1 \leqslant B < A \leqslant 1$, the class $\mathcal{ST}(\varphi)$ is the class $\mathcal{ST}[A, B]$ of Janowski starlike functions [8]. An analytic function $p : \mathbb{D} \to \mathbb{C}$ is known as a Carathéodory function if $p(0) = 1$ and $\operatorname{Re}(p(z)) > 0$ for every $z \in \mathbb{D}$. The class of all Carathéodory functions is denoted as $\mathcal{P}$. For $-1 \leqslant B < A \leqslant 1$ and $p(z) = 1 + c_1 z + \cdots$ with positive real part, we say that $p \in \mathcal{P}[A, B]$ if $p(z) \prec (1 + Az)/(1 + Bz), z \in \mathbb{D}$.

**Lemma 1.1.** [16] *If* $p \in \mathcal{P}[A, B]$, *then*

$$\left| p(z) - \frac{1 - ABr^2}{1 - B^2r^2} \right| \leqslant \frac{(A - B)r}{1 - B^2r^2} \quad (|z| \leqslant r < 1).$$

The class of functions $f \in \mathcal{A}$ with the property that $zf'(z)/f(z)/ \in \mathcal{P}[A, B]$ is denoted by $\mathcal{ST}[A, B]$. In this manuscript, we are interested in the class $\mathcal{J}_1^\alpha$ defined as follows:

$$\mathcal{J}_1^\alpha := \left\{ g \in \mathcal{A} : g(z) = \left( \frac{zf'(z)}{f(z)} \right)^\alpha f(z), \quad f \in \mathcal{CV}, \alpha > 0 \right\}.$$

We determine $\mathcal{ST}(\varphi)$ radius of the class $\mathcal{J}_1^\alpha$ for various choices of $\varphi$. In particular, we consider the following classes of starlike functions:

1. Mendiratta et al. [14] introduced the class consisting of all functions $f \in \mathcal{A}$ such that $zf'(z)/f(z) \prec e^z$ or equivalently $|\log(zf'(z)/f(z))| < 1$.
2. Sharma et al. [18] studied the class $\mathcal{ST}_C = \mathcal{ST}(\varphi_C)$, where $\varphi_C(z) = 1 + (4/3)z + (2/3)z^2$. The boundary of $\varphi_C(\mathbb{D})$ is a cardiod.
3. Raina and Sokól [15] considered the class $\mathcal{ST}_m = \mathcal{ST}(\varphi_m)$, where $\varphi_m(z) = z + \sqrt{1 + z^2}$ and proved that $f \in \mathcal{ST}_m$ if and only if $zf'(z)/f(z) \in \Omega_m := \{w \in \mathbb{C} : |w^2 - 1| < 2|w|\}$ which is the interior of a lune.
4. Kumar and Kamaljeet [20] introduced the class $\mathcal{ST}_\wp = \mathcal{ST}(\varphi_\wp)$, where $\varphi_\wp(z) = 1 + ze^z$. The boundary of $\varphi_\wp(\mathbb{D})$ is a cardiod.
5. The class of starlike functions associated with a nephroid domain, given by $\mathcal{ST}_{Ne} = \mathcal{ST}(\varphi_{Ne})$ where $\varphi_{Ne}(z) = 1 + z - (z^3/3)$ was studied by Wani and Swaminathan [22]. The function $\varphi_{Ne}$ maps the unit circle onto a 2-cusped curve, $\left((u-1)^2 + v^2 - \frac{4}{9}\right)^3 - \frac{4v^2}{3} = 0$.
6. The class $\mathcal{ST}_{SG} = \mathcal{ST}(\varphi_{SG})$ where $\varphi_{SG}(z) = 2/(1 + e^{-z})$ was introduced by Goel and Kumar [7]. The boundary of $\varphi_{SG}(\mathbb{D})$ is a modified sigmoid.
7. Cho et al. [3] introduced the class $\mathcal{ST}_{\sin} = \mathcal{ST}(\varphi_{\sin})$, where $\varphi_{\sin}(z) = 1 + \sin z$.

8. Kumar and Arora [2] defined the class $\mathcal{ST}_h = \mathcal{ST}(\varphi_h)$ where $\varphi_h(z) = 1 + \sinh^{-1}(z)$. The boundary of $\varphi_h(\mathbb{D})$ is petal shaped.

These functions behave like the identity function for small values of $\alpha$ and hence belong to the classes of our interest. However, for $B = -1$, the range of $zg'(z)/g(z)$ is unbounded, and therefore these classes are not contained in various subclasses obtained for special choices of the function $\varphi$. When the inclusion fails, we are interested in the corresponding radius problem. For two subclasses $\mathcal{F}$ and $\mathcal{G}$ of $\mathcal{A}$, the largest number $\mathcal{R} \in (0, 1]$ such that for $0 < r < \mathcal{R}$, $f(rz)/r \in \mathcal{F}$ for every $f \in \mathcal{G}$ is called the $\mathcal{F}$-radius of the class $\mathcal{G}$ and is denoted by $\mathcal{R}_{\mathcal{F}}(\mathcal{G})$. Many radius problems have been extensively explored in recent times [1, 9, 10, 11, 13, 17]. In Theorem 2.1, we obtain the Janowski starlikeness of the class $\mathcal{J}_1^\alpha$ and, in particular, the radius of starlikeness of order $\beta$. Theorem 2.2 gives $\mathcal{ST}(\varphi)$ radius of the class $\mathcal{J}_1^\alpha$ for various choices of $\varphi$ discussed above. To obtain the radii, we find the largest positive number $\mathcal{R}$ less than 1 such that the image of the disc $\mathbb{D}_{\mathcal{R}} := \{z \in \mathbb{C} : |z| < \mathcal{R}\}$ under the mapping $zg'(z)/g(z)$, for $g$ in the classes defined, lie inside the image of the corresponding superordinate functions and the radii obtained are sharp.

## 2. Radius estimates of various starlikeness for the class $\mathcal{J}_1^\alpha$

Our first theorem gives the radius of Janowski starlikeness of functions in the class $\mathcal{J}_1^\alpha$ and, in particular, the radius of starlikeness of order $\beta$ (see (2.3)). It follows that the class $\mathcal{J}_1^\alpha$ is a subclass of starlike functions.

**Theorem 2.1.** *The $\mathcal{ST}[A, B]$ radius of the class $\mathcal{J}_1^\alpha$, $\alpha > 0$, is given by*

$$\mathcal{R}_{\mathcal{ST}[A,B]} = \frac{A - B}{1 + \alpha + |A + \alpha B|}.$$

*Proof.* Let $g \in \mathcal{J}_1^\alpha$. Then there is a function $f \in \mathcal{CV}$ satisfying

$$g(z) = \Big(\frac{zf'(z)}{f(z)}\Big)^\alpha f(z).$$

A computation shows that

$$\frac{zg'(z)}{g(z)} = \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) + (1 - \alpha)\left(\frac{zf'(z)}{f(z)}\right). \tag{2.1}$$

Since $f$ is convex, it is starlike of order $1/2$ and therefore we have $1 + zf''(z)/f'(z) \in \mathcal{P} = \mathcal{P}_1[1, -1]$ and $zf'(z)/f(z) \in \mathcal{P}(1/2) := \mathcal{P}_1[0, -1]$. Using the Lemma 1.1, we get

$$\left|1 + \frac{zf''(z)}{f'(z)} - \frac{1 + r^2}{1 - r^2}\right| \leqslant \frac{2r}{1 - r^2} \quad (|z| \leqslant r < 1)$$

and

$$\left|\frac{zf'(z)}{f(z)} - \frac{1}{1 - r^2}\right| \leqslant \frac{r}{1 - r^2} \quad (|z| \leqslant r < 1).$$

These inequalities together with (2.1) immediately yield

$$\left|\frac{zg'(z)}{g(z)} - \frac{1 + \alpha r^2}{1 - r^2}\right| \leqslant \frac{(1 + \alpha)r}{1 - r^2} \quad (|z| \leqslant r < 1). \tag{2.2}$$

1. We first prove the result in the case when $B = -1$. In this case, we write $A$ as $A = 1 - 2\beta$, where $0 \leqslant \beta < 1$ so that $\mathcal{ST}[A, B]$ radius is the same as $\mathcal{ST}(\beta)$ radius. A simple calculation shows that the result in this becomes

$$\mathcal{R}_{\mathcal{ST}(\beta)} = \min\left(1, \frac{1-\beta}{\beta+\alpha}\right) = \begin{cases} 1 & \beta \leqslant \frac{1-\alpha}{2}, \\ \dfrac{1-\beta}{\beta+\alpha} & \beta \geqslant \frac{1-\alpha}{2}. \end{cases} \tag{2.3}$$

With $R = \mathcal{R}_{\mathcal{ST}(\beta)}$, our aim is to show that $\operatorname{Re}(zg'(z)/g(z)) > \beta$ for $|z| = r \leqslant R$ for every $g \in \mathcal{J}_1^\alpha$. The inequality (2.2) shows that

$$\operatorname{Re}\left(\frac{zg'(z)}{g(z)}\right) \geqslant \frac{1+\alpha r^2}{1-r^2} - \frac{(1+\alpha)r}{1-r^2} = \frac{1-\alpha r}{1+r} := \phi(r). \tag{2.4}$$

Since $\phi'(r) = -(1+\alpha)/(1+r)^2$, the function $\phi$ is decreasing for $0 \leqslant r < 1$. For $\beta \leqslant (1-\alpha)/2$, the inequality (2.4) gives

$$\operatorname{Re}\left(\frac{zg'(z)}{g(z)}\right) \geqslant \phi(r) \geqslant \phi(1) = \frac{1-\alpha}{2} \geqslant \beta$$

and so $g \in \mathcal{ST}(\beta)$. For $\beta > (1-\alpha)/2$, the inequality (2.4) gives

$$\operatorname{Re}\left(\frac{zg'(z)}{g(z)}\right) \geqslant \phi(r) \geqslant \phi(R) = \beta$$

for $r \leqslant R$. This shows that $\mathcal{ST}(\beta)$ radius of $\mathcal{J}_1^\alpha$ is at least $R$. To show that the result is sharp, we consider the function $\tilde{g} : \mathbb{D} \to \mathbb{C}$ is given by $\tilde{g}(z) = z/(1-z)^{1+\alpha}$. This function corresponds to the function $\tilde{f} \in \mathcal{CV}$ given by

$$\tilde{f}(z) = \frac{z}{1-z}. \tag{2.5}$$

The function $\tilde{g}$ is clearly starlike of order $(1-\alpha)/2$. The result is therefore sharp for $\beta \leqslant (1-\alpha)/2$. Note that

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{1+\alpha z}{1-z}. \tag{2.6}$$

For $\beta > (1-\alpha)/2$ and $z = R$, using (2.6), we see that

$$\operatorname{Re}\left(\frac{z\tilde{g}'(z)}{\tilde{g}(z)}\right) = \frac{1-\alpha R}{1+R} = \beta,$$

which proves the sharpness of $R$.

2. Now we assume that $B \neq -1$. Let $f \in \mathcal{J}_1^\alpha$. Then, by (2.2), we see that

$$g(\mathbb{D}_r) \subset \{w : |w - c_1(\alpha, r)| \leqslant d_1(\alpha, r)\}$$

where

$$c_1(\alpha, r) := \frac{1+\alpha r^2}{1-r^2} \quad \text{and} \quad d_1(\alpha, r) := \frac{(1+\alpha)r}{1-r^2}.$$

We show that, for $r \leqslant R = \mathcal{R}_{\mathcal{ST}[A,B]}$, the inclusion

$$\{w : |w - c_1(\alpha, r)| \leqslant d_1(\alpha, r)\} \subseteq \{w : |w - a| \leqslant b\}$$

holds where

$$a = \frac{1-AB}{1-B^2} \quad \text{and} \quad b = \frac{A-B}{1-B^2}.$$

Since $\{w : |w-c| \leqslant d\} \subseteq \{w : |w-a| \leqslant b\}$ if and only if $|a-c| \leqslant b-d$ (see [19] and [6]), it is enough to show that, for $r \leqslant R$, the inequality $|a-c_1(\alpha,r)| \leqslant b-d_1(\alpha,r)$ holds. The inequality $|a-c_1(\alpha,r)| \leqslant b-d_1(\alpha,r)$ is equivalent to the inequalities

$$c_1(\alpha,r) + d_1(\alpha,r) \leqslant a+b \tag{2.7}$$

and

$$a-b \leqslant c_1(\alpha,r) - d_1(\alpha,r). \tag{2.8}$$

The inequality (2.7) becomes

$$\frac{1+A}{1+B} \geqslant \frac{1+\alpha r^2+(1+\alpha)r}{1-r^2} = \frac{1+\alpha r}{1-r}.$$

This inequality holds for

$$0 \leqslant r \leqslant \frac{A-B}{1+\alpha+A+\alpha B} := \rho_2.$$

Similarly, the inequality (2.8) becomes

$$\frac{1-A}{1-B} \leqslant \frac{1+\alpha r^2-(1+\alpha)r}{1-r^2} = \frac{1-\alpha r}{1+r}$$

or

$$0 \leqslant r \leqslant \frac{A-B}{1+\alpha-A-\alpha B} := \rho_3.$$

Since

$$\min[\rho_2,\rho_3] = \frac{A-B}{1+\alpha+|A+\alpha B|} = R,$$

it follows that the inequalities (2.7) and (2.8) holds for $0 \leqslant r \leqslant R$. This shows that $\mathcal{ST}[A,B]$ radius of $\mathcal{J}_1^\alpha$ is at least $R$.

To prove the sharpness of $R$ , we again consider the function $\tilde{f} \in \mathcal{CV}$ defined by (2.5). When $A+\alpha B > 0$, then $R = \rho_2$. For $z = \rho_2$, the equation (2.6) gives

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{1+A}{1+B},$$

which proves the sharpness for $\rho_2$. When $A+\alpha B < 0$, then $R = \rho_3$. For $z = -\rho_3$, the equation (2.6) gives

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{1-A}{1-B},$$

which proves the sharpness for $\rho_3$. □

**Theorem 2.2.** *Let $\alpha > 0$. For the class $\mathcal{J}_1^\alpha$, the following radius results hold:*

1. *The $\mathcal{ST}_e$ radius is given by*

$$\mathcal{R}_{\mathcal{ST}_e} = \begin{cases} \frac{e-1}{e\alpha+1} & \text{if} \quad \alpha \geqslant 1 \\ \frac{e-1}{e+\alpha} & \text{if} \quad \alpha \leqslant 1. \end{cases}$$

*2. The $\mathcal{ST}_c$ radius is given by*

$$\mathcal{R}_{\mathcal{ST}_c} = \begin{cases} \frac{2}{3\alpha+1} & \text{if } \alpha \geqslant 1 \\ \frac{2}{\alpha+3} & \text{if } \alpha \leqslant 1. \end{cases}$$

*3. The $\mathcal{ST}_m$ radius is given by*

$$\mathcal{R}_{\mathcal{ST}_m} = \begin{cases} \frac{2-\sqrt{2}}{\alpha-(1-\sqrt{2})} & \text{if } \alpha \geqslant 1 \\ \frac{\sqrt{2}}{\alpha+(1+\sqrt{2})} & \text{if } \alpha \leqslant 1. \end{cases}$$

*4. The $\mathcal{ST}_\wp$ radius is given by*

$$\mathcal{R}_{\mathcal{ST}_\wp} = \begin{cases} \frac{1}{e(1+\alpha)-1} & \text{if } \alpha \geqslant \frac{2}{e-e^{-1}} - 1 \\ \frac{e}{\alpha+e+1} & \text{if } \alpha \leqslant \frac{2}{e-e^{-1}} - 1. \end{cases}$$

*5. The $\mathcal{ST}_{Ne}$ radius is given by*

$$\mathcal{R}_{\mathcal{ST}_{Ne}} = \frac{2}{3\alpha+5}.$$

*6. $\mathcal{ST}_{SG}$ radius is given by*

$$\mathcal{R}_{\mathcal{ST}_{SG}} = \frac{e-1}{(e+1)\alpha+2e}.$$

*7. The $\mathcal{ST}_{\sin}$ radius is given by*

$$\mathcal{R}_{\mathcal{ST}_{\sin}} = \frac{\sin 1}{(1+\alpha)+\sin 1}.$$

*8. The $\mathcal{ST}_h$ radius is given by*

$$\mathcal{R}_{\mathcal{ST}_g} = \frac{\sinh^{-1}(1)}{(1+\alpha)+\sinh^{-1}(1)}.$$

*Proof.* Let $g \in \mathcal{J}_1^\alpha$. For various choices of $\varphi$, we are interested in computing $\mathcal{ST}(\varphi)$ radius of the function $g$. To do this, we first note that, by (2.2), we have $g(\mathbb{D}_r) \subset \{w : |w - c_1(\alpha, r)| \leqslant d_1(\alpha, r)\}$, where

$$c_1(\alpha, r) := \frac{1+\alpha r^2}{1-r^2} \quad \text{and} \quad d_1(\alpha, r) := \frac{(1+\alpha)r}{1-r^2}. \tag{2.9}$$

We compute the largest $R$, such that, for $0 \leqslant r \leqslant R$, the disc $\{w : |w - c_1(\alpha, r)| \leqslant d_1(\alpha, r)\}$ is contained in $\varphi(\mathbb{D})$. For this purpose, we use the formula for the radius $r_a$ of the largest disc centered at $a$ contained in $\varphi(\mathbb{D})$ obtained by various authors. We also need the fact that the center $c_1(\alpha,r)$ is an increasing function of $r$ which follows easily from the equation

$$c_1'(\alpha,r)=\frac{2(1+\alpha)r}{(1-r^2)^2}.$$

One immediate consequence is that $c_1(\alpha,r)\geqslant c_1(\alpha,0)=1$.

1. Let $\Omega_e$ be the image of the unit disc $\mathbb{D}$ under the exponential function $\varphi(z)=e^z$. Mendiratta et al. [14] proved that the inclusion $\{w:|w-a|<r_a\}\subseteq\Omega_e:=\{w:|\log w|<1\}$ holds when

$$r_a=\begin{cases} a-\frac{1}{e} & \text{if } \frac{1}{e}<a\leqslant\frac{e+e^{-1}}{2}\\ e-a & \text{if } \frac{e+e^{-1}}{2}\leqslant a<e.\end{cases}$$

Using this inclusion result, we now show that, for $0\leqslant r\leqslant R:=\mathcal{R}_{\mathcal{ST}_e}$, the disc $\{w:|w-c_1(\alpha,r)|\leqslant d_1(\alpha,r)\}$ is contained in $\Omega_e$ where $c_1(\alpha,r)$ and $d_1(\alpha,r)$ given by (2.9).

First, we consider the case $\alpha\geqslant1$. Let the number

$$\rho_1:=\sqrt{\frac{e+e^{-1}-2}{2\alpha+e+e^{-1}}}<1$$

be the unique root of the equation $c_1(\alpha,r)=(e+e^{-1})/2$. Let the number

$$\rho_2:=\frac{e-1}{\alpha e+1}<1$$

be the positive root of the equation $d_1(\alpha,r)=c_1(\alpha,r)-1/e$ or

$$\frac{1+\alpha r^2}{1-r^2}-\frac{(1+\alpha)r}{1-r^2}=\frac{1-\alpha r}{1+r}=\frac{1}{e}. \tag{2.10}$$

A computation shows that $\rho_2\leqslant\rho_1$ for $\alpha\geqslant1$. We shall show that $R=\mathcal{R}_{\mathcal{ST}_e}=\rho_2$.

Since $c_1(\alpha,r)\geqslant1$, it follows that $c_1(\alpha,r)>1/e$ for $0\leqslant r\leqslant\rho_2<1$. Since $c_1(\alpha,r)$ is an increasing function, for $r\leqslant\rho_1$, we have $c_1(\alpha,r)\leqslant c_1(\alpha,\rho_1)=(e+e^{-1})/2$. Since $c_1(\alpha,r)-d_1(\alpha,r)$ is a decreasing function of $r$, it follows, for $0\leqslant r\leqslant\rho_2$, that

$$c_1(\alpha,r)-d_1(\alpha,r)\geqslant c_1(\alpha,\rho_2)-d_1(\alpha,\rho_2)=1/e$$

and hence

$$d_1(\alpha,r)\leqslant c_1(\alpha,r)-\frac{1}{e}. \tag{2.11}$$

Therefore, for $0\leqslant r\leqslant R=\rho_2$, we have, using (2.2) and (2.11)

$$\left|\frac{zg'(z)}{g(z)}-c_1(\alpha,r)\right|\leqslant c_1(\alpha,r)-\frac{1}{e}.$$

Therefore, the inclusion $\{w:=zg'(z)/g(z):|w-a|<r_a\}\subseteq\Omega_e:=\{w:|\log w|<1\}$ holds which proves that $\mathcal{ST}_e$ radius of $\mathcal{J}_1^\alpha$ is at least $R=\rho_2$.

To prove the sharpness, consider the function $\tilde g\in\mathcal{J}_1^\alpha$ defined by $\tilde g(z)=z/(1-z)^{1+\alpha}$. Since

$$\frac{z\tilde g'(z)}{\tilde g(z)}=\frac{1+\alpha z}{1-z},$$

we have, for $z = -\rho_2$,

$$\left|\log\left(\frac{z\tilde{g}'(z)}{\tilde{g}(z)}\right)\right| = \left|\log\left(\frac{1-\alpha\rho_2}{1+\rho_2}\right)\right| = \left|\log\left(\frac{1}{e}\right)\right| = 1,$$

which proves the sharpness for $\rho_2$.

We now consider the case when $\alpha \leqslant 1$. Let the number

$$\rho_3 := \frac{e-1}{e+\alpha} < 1$$

be the positive root of the equation $d_1(\alpha, r) = e - c_1(\alpha, r)$ or

$$\frac{1+\alpha r^2}{1-r^2} + \frac{(1+\alpha)r}{1-r^2} = \frac{1+\alpha r}{1-r} = e. \tag{2.12}$$

A computation shows that $\rho_3 \geqslant \rho_1$ for $\alpha \leqslant 1$. We shall show that $R = \mathcal{R}_{\mathcal{ST}_e} = \rho_3$. For $0 \leqslant r \leqslant \rho_3 < 1$ it follows that $c_1(\alpha, R) < e$ . Since $c_1(\alpha, r)$ is an increasing function, for $r \leqslant \rho_1$, we have $c_1(\alpha, r) \leqslant c_1(\alpha, \rho_1) = (e + e^{-1})/2$. Since $c_1(\alpha, r) + d_1(\alpha, r)$ is an increasing function of $r$, it follows, for $0 \leqslant r \leqslant \rho_3$, that

$$c_1(\alpha, r) + d_1(\alpha, r) \leqslant c_1(\alpha, \rho_3) + d_1(\alpha, \rho_3) = e$$

and hence

$$d_1(\alpha, r) \leqslant e - c_1(\alpha, r). \tag{2.13}$$

Therefore, for $0 \leqslant r \leqslant R = \rho_3$, we have, using (2.2) and (2.13)

$$\left|\frac{zg'(z)}{g(z)} - c_1(\alpha, r)\right| \leqslant e - c_1(\alpha, r).$$

Therefore, the inclusion $\{w := zg'(z)/g(z) : |w-a| < r_a\} \subseteq \Omega_e := \{w : |\log w| < 1\}$ holds which proves that $\mathcal{ST}_e$ radius of $\mathcal{J}_1^\alpha$ is at least $R = \rho_3$.

To prove the sharpness, consider the function $\tilde{g} \in \mathcal{J}_1^\alpha$ defined by $\tilde{g}(z) = z/(1-z)^{1+\alpha}$. Since

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{1+\alpha z}{1-z},$$

we have, for $z = \rho_3$,

$$\left|\log\left(\frac{z\tilde{g}'(z)}{\tilde{g}(z)}\right)\right| = \left|\log\left(\frac{1+\alpha\rho_3}{1-\rho_3}\right)\right| = |\log e| = 1,$$

proving the sharpness for $\rho_3$.

2. Let $\Omega_C$ be the image of the unit disc $\mathbb{D}$ under the function $\varphi_C(z) = 1 + (4/3)z + (2/3)z^2$. Sharma et al. [18] proved that the inclusion $\{w : |w-a| < r_a\} \subset \varphi_C(\mathbb{D}) = \Omega_C$ holds when

$$r_a = \begin{cases} a - \frac{1}{3} & \text{if } \frac{1}{3} < a \leqslant \frac{5}{3} \\ 3 - a & \text{if } \frac{5}{3} \leqslant a < 3. \end{cases}$$

Using the inclusion result, we now show that, for $0 \leqslant r \leqslant R$ the disc $\{w : |w - c_1(\alpha, r)| \leqslant d_1(\alpha, r)\}$ is contained in $\Omega_C$ where $c_1(\alpha, r)$ and $d_1(\alpha, r)$ given by (2.9).

We first consider the case $\alpha \geqslant 1$. Let the number

$$\rho_1 := \sqrt{\frac{2}{3\alpha+5}} < 1$$

be the unique root of the equation $c_1(\alpha, r) = 5/3$. Let the number

$$\rho_2 := \frac{2}{3\alpha+1} < 1$$

be the positive root of the equation $d_1(\alpha, r) = c_1(\alpha, r) - 1/3$ or

$$\frac{1+\alpha r^2}{1-r^2} - \frac{(1+\alpha)r}{1-r^2} = \frac{1-\alpha r}{1+r} = \frac{1}{3}. \tag{2.14}$$

A computation shows that $\rho_2 \leqslant \rho_1$ for $\alpha \geqslant 1$. We shall show that $R = \mathcal{R}_{\mathcal{ST}_C} = \rho_2$.

Since $c_1(\alpha, r) \geqslant 1$, it follows that $c_1(\alpha, r) > 1/3$ for $0 \leqslant r \leqslant \rho_2 < 1$. Since $c_1(\alpha, r)$ is an increasing function, for $r \leqslant \rho_1$, we have $c_1(\alpha, r) \leqslant c_1(\alpha, \rho_1) = 5/3$. Since $c_1(\alpha, r) - d_1(\alpha, r)$ is a decreasing function of $r$, it follows, for $0 \leqslant r \leqslant \rho_2$, that

$$c_1(\alpha, r) - d_1(\alpha, r) \geqslant c_1(\alpha, \rho_2) - d_1(\alpha, \rho_2) = 1/3$$

and hence

$$d_1(\alpha, r) \leqslant c_1(\alpha, r) - \frac{1}{3}. \tag{2.15}$$

Therefore, for $0 \leqslant r \leqslant R = \rho_2$, we have, using (2.2) and (2.15)

$$\left|\frac{zg'(z)}{g(z)} - c_1(\alpha, r)\right| \leqslant c_1(\alpha, r) - \frac{1}{3}.$$

Therefore, the inclusion $\{w := zg'(z)/g(z) : |w-a| < r_a\} \subseteq \varphi_C(\mathbb{D}) = \Omega_C$ holds which proves that $\mathcal{ST}_C$ radius of $\mathcal{J}_1^\alpha$ is at least $R = \rho_2$.

To prove the sharpness, consider the function $\tilde{g} \in \mathcal{J}_1^\alpha$ defined by $\tilde{g}(z) = z/(1-z)^{1+\alpha}$. Since

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{1+\alpha z}{1-z},$$

we have, for $z = -\rho_2$,

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{1-\alpha\rho_2}{1+\rho_2} = \frac{1}{3} = \varphi_C(-1),$$

which proves the sharpness for $\rho_2$.

We now consider the case when $\alpha \leqslant 1$. Let the number

$$\rho_3 := \frac{2}{\alpha+3} < 1$$

be the positive root of the equation $d_1(\alpha, r) = 3 - c_1(\alpha, r)$ or

$$\frac{1+\alpha r^2}{1-r^2} + \frac{(1+\alpha)r}{1-r^2} = \frac{1+\alpha r}{1-r} = 3. \tag{2.16}$$

A computation shows that $\rho_3 \geqslant \rho_1$ for $\alpha \leqslant 1$. We shall show that $R = \mathcal{R}_{\mathcal{ST}_C} = \rho_3$. For $0 \leqslant r \leqslant \rho_3 < 1$ it follows that $c_1(\alpha, R) < 3$ . Since $c_1(\alpha, r)$ is an increasing function, for $r \leqslant \rho_1$, we have $c_1(\alpha,r) \leqslant c_1(\alpha,\rho_1) = 5/3$. Since $c_1(\alpha,r)+d_1(\alpha,r)$ is an increasing function of $r$, it follows, for $0 \leqslant r \leqslant \rho_3$, that

$$c_1(\alpha,r) + d_1(\alpha,r) \leqslant c_1(\alpha,\rho_3) + d_1(\alpha,\rho_3) = 3$$

and hence

$$d_1(\alpha,r) \leqslant 3 - c_1(\alpha,r). \tag{2.17}$$

Therefore, for $0 \leqslant r \leqslant R = \rho_3$, we have, using (2.2) and (2.17)

$$\left|\frac{zg'(z)}{g(z)} - c_1(\alpha,r)\right| \leqslant 3 - c_1(\alpha,r).$$

Therefore, the inclusion $\{w := zg'(z)/g(z) : |w-a| < r_a\} \subseteq \varphi_C(\mathbb{D}) = \Omega_C$ holds which proves that $\mathcal{ST}_C$ radius of $\mathcal{J}_1^\alpha$ is at least $R = \rho_3$.

To prove the sharpness, consider the function $\tilde{g} \in \mathcal{J}_1^\alpha$ defined by $\tilde{g}(z) = z/(1-z)^{1+\alpha}$. Since

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{1+\alpha z}{1-z},$$

we have, for $z = \rho_3$,

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{1+\alpha\rho_3}{1-\rho_3} = 3 = \varphi_C(1),$$

proving the sharpness for $\rho_3$.

3. Let $\Omega_m$ be the image of the unit disc $\mathbb{D}$ under the function $\varphi_m(z) = z + \sqrt{1+z^2}$. Gandhi and Ravichandran [5] proved that the inclusion $\{w : |w-a| < r_a\} \subset \varphi_m(\mathbb{D}) = \Omega_m := \{w : \left|w^2-1\right| < 2|w|\}$ holds when

$$r_a = 1 - |\sqrt{2} - a|$$

for $\sqrt{2}-1 < a \leqslant \sqrt{2}+1$. Using the inclusion result, we now show that, for $0 \leqslant r \leqslant R$ the disc $\{w : |w - c_1(\alpha,r)| \leqslant d_1(\alpha,r)\}$ is contained in $\Omega_m$ where $c_1(\alpha,r)$ and $d_1(\alpha,r)$ given by (2.9).

First, we consider the case $\alpha \geqslant 1$. Let the number

$$\rho_1 := \sqrt{\frac{\sqrt{2}-1}{\alpha+\sqrt{2}}} < 1$$

be the unique root of the equation $c_1(\alpha,r) = \sqrt{2}$. Let the number

$$\rho_2 := \frac{2-\sqrt{2}}{\alpha-(1-\sqrt{2})} < 1$$

be the positive root of the equation $d_1(\alpha,r) = c_1(\alpha,r) - (\sqrt{2}-1)$ or

$$\frac{1+\alpha r^2}{1-r^2} - \frac{(1+\alpha)r}{1-r^2} = \frac{1-\alpha r}{1+r} = \sqrt{2}-1. \tag{2.18}$$

A computation shows that $\rho_2 \leqslant \rho_1$ for $\alpha \geqslant 1$. We shall show that $R = \mathcal{R}_{\mathcal{ST}_m} = \rho_2$.

Since $c_1(\alpha,r) \geqslant 1$, it follows that $c_1(\alpha,r) > \sqrt{2}-1$ for $0 \leqslant r \leqslant \rho_2 < 1$. Since $c_1(\alpha,r)$ is an increasing function, for $r \leqslant \rho_1$, we have $c_1(\alpha,r) \leqslant c_1(\alpha,\rho_1) = \sqrt{2}$.

Since $c_1(\alpha,r) - d_1(\alpha,r)$ is a decreasing function of $r$, it follows, for $0 \leqslant r \leqslant \rho_2$, that

$$c_1(\alpha,r) - d_1(\alpha,r) \geqslant c_1(\alpha,\rho_2) - d_1(\alpha,\rho_2) = \sqrt{2}-1$$

and hence

$$d_1(\alpha,r) \leqslant c_1(\alpha,r) - (\sqrt{2}-1). \tag{2.19}$$

Therefore, for $0 \leqslant r \leqslant R = \rho_2$, we have, using (2.2) and (2.19)

$$\left|\frac{zg'(z)}{g(z)} - c_1(\alpha,r)\right| \leqslant c_1(\alpha,r) - (\sqrt{2}-1).$$

Therefore, the inclusion $\{w := zg'(z)/g(z) : |w-a| < r_a\} \subseteq \varphi_m(\mathbb{D}) = \Omega_m$ holds which proves that $\mathcal{ST}_m$ radius of $\mathcal{J}_1^\alpha$ is at least $R = \rho_2$.

To prove the sharpness, consider the function $\tilde{g} \in \mathcal{J}_1^\alpha$ defined by $\tilde{g}(z) = z/(1-z)^{1+\alpha}$. Since

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{1+\alpha z}{1-z},$$

we have, for $z = -\rho_2$,

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{1-\alpha\rho_2}{1+\rho_2} = \sqrt{2}-1 = \varphi_m(-1),$$

which proves the sharpness for $\rho_2$.

We now consider the case when $\alpha \leqslant 1$. Let the number

$$\rho_3 := \frac{\sqrt{2}}{\alpha + (1+\sqrt{2})} < 1$$

be the positive root of the equation $d_1(\alpha,r) = \sqrt{2}+1-c_1(\alpha,r)$ or

$$\frac{1+\alpha r^2}{1-r^2} + \frac{(1+\alpha)r}{1-r^2} = \frac{1+\alpha r}{1-r} = \sqrt{2}+1. \tag{2.20}$$

A computation shows that $\rho_3 \geqslant \rho_1$ for $\alpha \leqslant 1$. We shall show that $R = \mathcal{R}_{\mathcal{ST}_m} = \rho_3$. For $0 \leqslant r \leqslant \rho_3 < 1$ it follows that $c_1(\alpha,R) < \sqrt{2}+1$ . Since $c_1(\alpha,r)$ is an increasing function, for $r \leqslant \rho_1$, we have $c_1(\alpha,r) \leqslant c_1(\alpha,\rho_1) = \sqrt{2}$. Since $c_1(\alpha,r) + d_1(\alpha,r)$ is an increasing function of $r$, it follows, for $0 \leqslant r \leqslant \rho_3$, that

$$c_1(\alpha,r) + d_1(\alpha,r) \leqslant c_1(\alpha,\rho_3) + d_1(\alpha,\rho_3) = \sqrt{2}+1$$

and hence

$$d_1(\alpha,r) \leqslant \sqrt{2}+1-c_1(\alpha,r). \tag{2.21}$$

Therefore, for $0 \leqslant r \leqslant R = \rho_3$, we have, using (2.2) and (2.21)

$$\left|\frac{zg'(z)}{g(z)} - c_1(\alpha,r)\right| \leqslant \sqrt{2}+1-c_1(\alpha,r).$$

Therefore, the inclusion $\{w := zg'(z)/g(z) : |w-a| < r_a\} \subseteq \varphi_m(\mathbb{D}) = \Omega_m$ holds which proves that $\mathcal{ST}_m$ radius of $\mathcal{J}_1^\alpha$ is at least $R = \rho_3$.

To prove the sharpness, consider the function $\tilde{g} \in \mathcal{J}_1^\alpha$ defined by $\tilde{g}(z) = z/(1-z)^{1+\alpha}$. Since

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{1+\alpha z}{1-z},$$

we have, for $z = \rho_3$,

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{1+\alpha\rho_3}{1-\rho_3} = \sqrt{2}+1 = \varphi_m(1),$$

proving the sharpness for $\rho_3$.

4. Let $\Omega_\wp$ be the image of the unit disc $\mathbb{D}$ under the function $\varphi_\wp(z) = 1+ze^z$. Kumar and Kamaljeet [20] proved that the inclusion $\{w : |w-a| < r_a\} \subset \varphi_\wp(\mathbb{D}) = \Omega_\wp$ holds when

$$r_a = \begin{cases} (a-1)+\frac{1}{e} & \text{if } 1-\frac{1}{e} < a \leqslant 1+\frac{e-e^{-1}}{2} \\ e-(a-1) & \text{if } 1+\frac{e-e^{-1}}{2} \leqslant a < 1+e. \end{cases}$$

Using the inclusion result, we now show that, for $0 \leqslant r \leqslant R$ the disc $\{w : |w-c_1(\alpha,r)| \leqslant d_1(\alpha,r)\}$ is contained in $\Omega_\wp$ where $c_1(\alpha,r)$ and $d_1(\alpha,r)$ given by (2.9).

First, we consider the case $\alpha \geqslant 1$. Let the number

$$\rho_1 := \sqrt{\frac{e-e^{-1}}{2(1+\alpha)+e-e^{-1}}} < 1$$

be the unique root of the equation $c_1(\alpha,r) = 1+(e-e^{-1})/2$. Let the number

$$\rho_2 := \frac{1}{e(1+\alpha)-1} < 1$$

be the positive root of the equation $d_1(\alpha,r) = c_1(\alpha,r) - 1 + (1/e)$ or

$$\frac{1+\alpha r^2}{1-r^2} - \frac{(1+\alpha)r}{1-r^2} = \frac{1-\alpha r}{1+r} = \frac{1}{e} - 1. \tag{2.22}$$

A computation shows that $\rho_2 \leqslant \rho_1$ for $\alpha \geqslant 1$. We shall show that $R = \mathcal{R}_{\mathcal{ST}_\wp} = \rho_2$.

Since $c_1(\alpha,r) \geqslant 1$, it follows that $c_1(\alpha,r) > 1-(1/e)$ for $0 \leqslant r \leqslant \rho_2 < 1$. Since $c_1(\alpha,r)$ is an increasing function, for $r \leqslant \rho_1$, we have $c_1(\alpha,r) \leqslant c_1(\alpha,\rho_1) = 1+(e-e^{-1})/2$. Since $c_1(\alpha,r) - d_1(\alpha,r)$ is a decreasing function of $r$, it follows, for $0 \leqslant r \leqslant \rho_2$, that

$$c_1(\alpha,r) - d_1(\alpha,r) \geqslant c_1(\alpha,\rho_2) - d_1(\alpha,\rho_2) = \frac{1}{e} - 1$$

and hence

$$d_1(\alpha,r) \leqslant c_1(\alpha,r) - 1 + \frac{1}{e}. \tag{2.23}$$

Therefore, for $0 \leqslant r \leqslant R = \rho_2$, we have, using (2.2) and (2.23)

$$\left|\frac{zg'(z)}{g(z)} - c_1(\alpha,r)\right| \leqslant c_1(\alpha,r) - 1 + \frac{1}{e}.$$

Therefore, the inclusion $\{w := zg'(z)/g(z) : |w-a| < r_a\} \subseteq \varphi_\wp(\mathbb{D}) = \Omega_\wp$ holds which proves that $\mathcal{ST}_\wp$ radius of $\mathcal{J}_1^\alpha$ is at least $R = \rho_2$.

To prove the sharpness, consider the function $\tilde{g} \in \mathcal{J}_1^\alpha$ defined by $\tilde{g}(z) = z/(1-z)^{1+\alpha}$. Since

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{1+\alpha z}{1-z},$$

we have, for $z = -\rho_2$,

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{1-\alpha\rho_2}{1+\rho_2} = 1 - e^{-1} = \varphi_{\wp}(-1),$$

which proves the sharpness for $\rho_2$.

We now consider the case when $\alpha \leqslant 1$. Let the number

$$\rho_3 := \frac{e}{\alpha+e+1} < 1$$

be the positive root of the equation $d_1(\alpha, r) = e + 1 - c_1(\alpha, r)$ or

$$\frac{1+\alpha r^2}{1-r^2} + \frac{(1+\alpha)r}{1-r^2} = \frac{1+\alpha r}{1-r} = e+1. \tag{2.24}$$

A computation shows that $\rho_3 \geqslant \rho_1$ for $\alpha \leqslant 1$. We shall show that $R = \mathcal{R}_{\mathcal{ST}_{\wp}} = \rho_3$. For $0 \leqslant r \leqslant \rho_3 < 1$ it follows that $c_1(\alpha, R) < e + 1$ . Since $c_1(\alpha, r)$ is an increasing function, for $r \leqslant \rho_1$, we have $c_1(\alpha, r) \leqslant c_1(\alpha, \rho_1) = 1 + (e - e^{-1})/2$. Since $c_1(\alpha, r) + d_1(\alpha, r)$ is an increasing function of $r$, it follows, for $0 \leqslant r \leqslant \rho_3$, that

$$c_1(\alpha, r) + d_1(\alpha, r) \leqslant c_1(\alpha, \rho_3) + d_1(\alpha, \rho_3) = e + 1$$

and hence

$$d_1(\alpha, r) \leqslant e + 1 - c_1(\alpha, r). \tag{2.25}$$

Therefore, for $0 \leqslant r \leqslant R = \rho_3$, we have, using (2.2) and (2.25)

$$\left|\frac{zg'(z)}{g(z)} - c_1(\alpha, r)\right| \leqslant e + 1 - c_1(\alpha, r).$$

Therefore, the inclusion $\{w := zg'(z)/g(z) : |w - a| < r_a\} \subseteq \varphi_{\wp}(\mathbb{D}) = \Omega_{\wp}$ holds which proves that $\mathcal{ST}_{\wp}$ radius of $\mathcal{J}_1^\alpha$ is at least $R = \rho_3$.

To prove the sharpness, consider the function $\tilde{g} \in \mathcal{J}_1^\alpha$ defined by $\tilde{g}(z) = z/(1-z)^{1+\alpha}$. Since

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{1+\alpha z}{1-z},$$

we have, for $z = \rho_3$,

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{1+\alpha\rho_3}{1-\rho_3} = 1 + e = \varphi_{\wp}(1),$$

proving the sharpness for $\rho_3$.

5. Let $\Omega_{Ne}$ be the image of the unit disc $\mathbb{D}$ under the function $\varphi_{Ne}(z) = 1 + z - (z^3/3)$. Wani and Swaminathan [21] proved that the inclusion $\{w : |w - a| < r_a\} \subset \varphi_{Ne}(\mathbb{D}) = \Omega_{Ne}$ holds when

$$r_a = \begin{cases} a - \frac{1}{3} & \text{if } \frac{1}{3} < a \leqslant 1 \\ \frac{5}{3} - a & \text{if } 1 \leqslant a < \frac{5}{3}. \end{cases}$$

Using the inclusion result, we now show that, for $0 \leqslant r \leqslant R$ the disc $\{w : |w - c_1(\alpha, r)| \leqslant d_1(\alpha, r)\}$ is contained in $\Omega_{Ne}$ where $c_1(\alpha, r)$ and $d_1(\alpha, r)$ given by (2.9).

Let the number

$$\rho_3 := \frac{2}{3\alpha + 5} < 1$$

be the positive root of the equation $d_1(\alpha, r) = (5/3) - c_1(\alpha, r)$ or

$$\frac{1 + \alpha r^2}{1 - r^2} + \frac{(1 + \alpha)r}{1 - r^2} = \frac{1 + \alpha r}{1 - r} = \frac{5}{3}. \tag{2.26}$$

We shall show that $R = \mathcal{R}_{\mathcal{ST}_\varphi} = \rho_3$. For $0 \leqslant r \leqslant R < 1$ it follows that $1 \leqslant c_1(\alpha, r) \leqslant c_1(\alpha, R) < 5/3$. Since $c_1(\alpha, r) + d_1(\alpha, r)$ is an increasing function of $r$, it follows, for $0 \leqslant r \leqslant \rho_3$, that

$$c_1(\alpha, r) + d_1(\alpha, r) \leqslant c_1(\alpha, \rho_3) + d_1(\alpha, \rho_3) = \frac{5}{3}$$

and hence

$$d_1(\alpha, r) \leqslant \frac{5}{3} - c_1(\alpha, r). \tag{2.27}$$

Therefore, for $0 \leqslant r \leqslant R = \rho_3$, we have, using (2.2) and (2.27)

$$\left| \frac{zg'(z)}{g(z)} - c_1(\alpha, r) \right| \leqslant \frac{5}{3} - c_1(\alpha, r).$$

Therefore, the inclusion $\{w := zg'(z)/g(z) : |w - a| < r_a\} \subseteq \varphi_{Ne}(\mathbb{D}) = \Omega_{Ne}$ holds which proves that $\mathcal{ST}_{Ne}$ radius of $\mathcal{J}_1^\alpha$ is at least $R = \rho_3$.

To prove the sharpness, consider the function $\tilde{g} \in \mathcal{J}_1^\alpha$ defined by $\tilde{g}(z) = z/(1 - z)^{1+\alpha}$. Since

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{1 + \alpha z}{1 - z},$$

we have, for $z = \rho_3$,

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{1 + \alpha\rho_3}{1 - \rho_3} = \frac{5}{3} = \varphi_{Ne}(1),$$

proving the sharpness for $\rho_3$.

6. Let $\Omega_{SG}$ be the image of the unit disc $\mathbb{D}$ under the function $\varphi_{SG}(z) = 2/(1+e^{-z})$. Goel and Kumar [7] proved that the inclusion $\{w : |w - a| < r_a\} \subset \varphi_{SG}(\mathbb{D}) = \Omega_{SG} := \{w : |\log w/(2 - w)| < 1\}$ holds when

$$r_a = \frac{e - 1}{e + 1} - |a - 1|$$

for $2/(1 + e) < a < 2e/(1 + e)$. Using the inclusion result, we now show that, for $0 \leqslant r \leqslant R$ the disc $\{w : |w - c_1(\alpha, r)| \leqslant d_1(\alpha, r)\}$ is contained in $\Omega_{SG}$ where $c_1(\alpha, r)$ and $d_1(\alpha, r)$ given by (2.9).

Let the number

$$\rho_3 := \frac{e - 1}{(e + 1)\alpha + 2e} < 1$$

be the positive root of the equation $d_1(\alpha, r) = (e-1)/(e+1) + 1 - c_1(\alpha, r)$ or

$$\frac{1+\alpha r^2}{1-r^2} + \frac{(1+\alpha)r}{1-r^2} = \frac{1+\alpha r}{1-r} = \frac{e-1}{e+1} + 1. \tag{2.28}$$

We shall show that $R = \mathcal{R}_{\mathcal{ST}_{SG}} = \rho_3$. For $0 \leqslant r \leqslant R < 1$ it follows that $2/(1+e) < 1 \leqslant c_1(\alpha, r) \leqslant c_1(\alpha, R) \leqslant c_1(\alpha, \rho_3) + d_1(\alpha, \rho_3) = 2e/(1+e)$. Since $c_1(\alpha, r) + d_1(\alpha, r)$ is an increasing function of $r$, it follows for $0 \leqslant r \leqslant \rho_3$, that

$$c_1(\alpha, r) + d_1(\alpha, r) \leqslant c_1(\alpha, \rho_3) + d_1(\alpha, \rho_3) = \frac{e-1}{e+1} + 1$$

and hence

$$d_1(\alpha, r) \leqslant \frac{e-1}{e+1} + 1 - c_1(\alpha, r). \tag{2.29}$$

Therefore, for $0 \leqslant r \leqslant R = \rho_3$, we have, using (2.2) and (2.29)

$$\left| \frac{zg'(z)}{g(z)} - c_1(\alpha, r) \right| \leqslant \frac{e-1}{e+1} + 1 - c_1(\alpha, r).$$

Therefore, the inclusion $\{w := zg'(z)/g(z) : |w - a| < r_a\} \subseteq \varphi_{SG}(\mathbb{D}) = \Omega_{SG} := \{w : |\log w/(2-w)| < 1\}$ holds which proves that $\mathcal{ST}_{SG}$ radius of $\mathcal{J}_1^\alpha$ is at least $R = \rho_3$.

To prove the sharpness, consider the function $\tilde{g} \in \mathcal{J}_1^\alpha$ defined by $\tilde{g}(z) = z/(1-z)^{1+\alpha}$. Since

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{1+\alpha z}{1-z},$$

we have, for $z = \rho_3$,

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{1+\alpha\rho_3}{1-\rho_3} = \frac{2e}{e+1} = \varphi_{SG}(1),$$

proving the sharpness for $\rho_3$.

7. Let $\Omega_{\sin}$ be the image of the unit disc $\mathbb{D}$ under the function $\varphi_{\sin}(z) = 1 + \sin z$. Cho et al. [3] proved that the inclusion $\{w : |w - a| < r_a\} \subset \varphi_{\sin}(\mathbb{D}) = \Omega_{\sin}$ holds when

$$r_a = \sin 1 - |a - 1|$$

for $1 - \sin 1 < a < 1 + \sin 1$. Using the inclusion result, we now show that, for $0 \leqslant r \leqslant R$ the disc $\{w : |w - c_1(\alpha, r)| \leqslant d_1(\alpha, r)\}$ is contained in $\Omega_{\sin}$ where $c_1(\alpha, r)$ and $d_1(\alpha, r)$ given by (2.9).

Let the number

$$\rho_3 := \frac{\sin 1}{(1+\alpha) + \sin 1} < 1$$

be the positive root of the equation $d_1(\alpha, r) = (\sin 1) + 1 - c_1(\alpha, r)$ or

$$\frac{1+\alpha r^2}{1-r^2} + \frac{(1+\alpha)r}{1-r^2} = \frac{1+\alpha r}{1-r} = 1 + \sin 1. \tag{2.30}$$

We shall show that $R = \mathcal{R}_{\mathcal{ST}_{\sin}} = \rho_3$. For $0 \leqslant r \leqslant R < 1$ it follows that $1 - \sin 1 < 1 \leqslant c_1(\alpha, r) \leqslant c_1(\alpha, R) \leqslant c_1(\alpha, \rho_3) + d_1(\alpha, \rho_3) = 1 + \sin 1$. Since $c_1(\alpha, r) + d_1(\alpha, r)$ is an increasing function of $r$, it follows, for $0 \leqslant r \leqslant \rho_3$, that

$$c_1(\alpha, r) + d_1(\alpha, r) \leqslant c_1(\alpha, \rho_3) + d_1(\alpha, \rho_3) = 1 + \sin 1$$

and hence

$$d_1(\alpha, r) \leqslant 1 + \sin 1 - c_1(\alpha, r). \tag{2.31}$$

Therefore, for $0 \leqslant r \leqslant R = \rho_3$, we have, using (2.2) and (2.31)

$$\left| \frac{zg'(z)}{g(z)} - c_1(\alpha, r) \right| \leqslant 1 + \sin 1 - c_1(\alpha, r).$$

Therefore, the inclusion $\{w := zg'(z)/g(z) : |w - a| < r_a\} \subseteq \varphi_{\sin}(\mathbb{D}) = \Omega_{\sin}$ holds which proves that $\mathcal{ST}_{\sin}$ radius of $\mathcal{J}_1^\alpha$ is at least $R = \rho_3$.

To prove the sharpness, consider the function $\tilde{g} \in \mathcal{J}_1^\alpha$ defined by $\tilde{g}(z) = z/(1-z)^{1+\alpha}$. Since

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{1 + \alpha z}{1 - z},$$

we have, for $z = \rho_3$,

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{1 + \alpha\rho_3}{1 - \rho_3} = 1 + \sin 1 = \varphi_{\sin}(1),$$

proving the sharpness for $\rho_3$.

8. Let $\Omega_h$ be the image of the unit disc $\mathbb{D}$ under the function $\varphi_h(z) = 1 + \sinh^{-1}(z)$. Kumar and Arora [2] proved that the inclusion $\{w : |w - a| < r_a\} \subset \varphi_h(\mathbb{D}) = \Omega_h$ holds when

$$r_a = \begin{cases} a - (1 - \sinh^{-1}(1)) & \text{if} \quad 1 - \sinh^{-1}(1) < a \leqslant 1 \\ 1 + \sinh^{-1}(1) - a & \text{if} \quad 1 \leqslant a < 1 + \sinh^{-1}(1). \end{cases}$$

Using the inclusion result, we now show that, for $0 \leqslant r \leqslant R$ the disc $\{w : |w - c_1(\alpha, r)| \leqslant d_1(\alpha, r)\}$ is contained in $\Omega_h$ where $c_1(\alpha, r)$ and $d_1(\alpha, r)$ given by (2.9).

Let the number

$$\rho_3 := \frac{\sinh^{-1}(1)}{(1 + \alpha) + \sinh^{-1}(1)} < 1$$

be the positive root of the equation $d_1(\alpha, r) = 1 + \sinh^{-1}(1) - c_1(\alpha, r)$ or

$$\frac{1 + \alpha r^2}{1 - r^2} + \frac{(1 + \alpha)r}{1 - r^2} = \frac{1 + \alpha r}{1 - r} = 1 + \sinh^{-1}(1). \tag{2.32}$$

We shall show that $R = \mathcal{R}_{\mathcal{ST}_{\sin}} = \rho_3$. For $0 \leqslant r \leqslant R < 1$ it follows that $1 - \sinh^{-1}(1) < 1 \leqslant c_1(\alpha, r) \leqslant c_1(\alpha, R) \leqslant c_1(\alpha, \rho_3) + d_1(\alpha, \rho_3) = 1 + \sinh^{-1}(1)$. Since $c_1(\alpha, r) + d_1(\alpha, r)$ is an increasing function of $r$, it follows, for $0 \leqslant r \leqslant \rho_3$, that

$$c_1(\alpha, r) + d_1(\alpha, r) \leqslant c_1(\alpha, \rho_3) + d_1(\alpha, \rho_3) = 1 + \sinh^{-1}(1)$$

and hence

$$d_1(\alpha, r) \leqslant 1 + \sinh^{-1}(1) - c_1(\alpha, r). \tag{2.33}$$

Therefore, for $0 \leqslant r \leqslant R = \rho_3$, we have, using (2.2) and (2.33)

$$\left| \frac{zg'(z)}{g(z)} - c_1(\alpha, r) \right| \leqslant 1 + \sinh^{-1}(1) - c_1(\alpha, r).$$

Therefore, the inclusion $\{w := zg'(z)/g(z) : |w - a| < r_a\} \subseteq \varphi_h(\mathbb{D}) = \Omega_h$ holds which proves that $\mathcal{ST}_h$ radius of $\mathcal{J}_1^\alpha$ is at least $R = \rho_3$.

To prove the sharpness, consider the function $\tilde{g} \in \mathcal{J}_1^\alpha$ defined by $\tilde{g}(z) = z/(1-z)^{1+\alpha}$. Since

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{1 + \alpha z}{1 - z},$$

we have, for $z = \rho_3$,

$$\frac{z\tilde{g}'(z)}{\tilde{g}(z)} = \frac{1 + \alpha\rho_3}{1 - \rho_3} = 1 + \sinh^{-1}(1) = \varphi_h(1),$$

proving the sharpness for $\rho_3$. □

## References


[1] Ali, R.M., Jain, N.K., Ravichandran, V., *Radii of starlikeness associated with the lemniscate of Bernoulli and the left-half plane*, Appl. Math. Comput., **218**(11)(2012), 6557–6565.

[2] Arora, K., Sivaprasad Kumar, S., *Starlike functions associated with a petal shaped domain*, Bull. Korean Math. Soc., **59**(4)(2022), 993–1010.

[3] Cho, N.E., Kumar, V., Kumar, S.S., Ravichandran, V., *Radius problems for starlike functions associated with the sine function*, Bull. Iranian Math. Soc., **45**(1)(2019), 213–232.

[4] Duren, P.L., *Univalent Functions*, vol. **259** of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*, Springer-Verlag, New York, 1983.

[5] Gandhi, S., Ravichandran, V., *Starlike functions associated with a lune*, Asian-Eur. J. Math., **10**(4)(2017), 1750064, 12.

[6] Gangadharan, A., Ravichandran, V., Shanmugam, T.N., *Radii of convexity and strong starlikeness for some classes of analytic functions*, J. Math. Anal. Appl., **211**(1)(1997), 301–313.

[7] Goel, P., Sivaprasad Kumar, S., *Certain class of starlike functions associated with modified sigmoid function*, Bull. Malays. Math. Sci. Soc., **43**(1)(2020), 957–991.

[8] Janowski, W., *Some extremal problems for certain families of analytic functions*, I, Ann. Polon. Math., **28**(1973), 297–326.

[9] Kanaga, R., Ravichandran, V., *Starlikeness for certain close-to-star functions*, Hacet. J. Math. Stat., **50**(2)(2021), 414–432.

[10] Lecko, A., Ravichandran, V., Sebastian, A., *Starlikeness of certain non-univalent functions*, Anal. Math. Phys., **11**(4)(2021), Paper No. 163, 23.

[11] Lee, S.K., Khatter, K., Ravichandran, V., *Radius of starlikeness for classes of analytic functions*, Bull. Malays. Math. Sci. Soc., **43**(6)(2020), 4469–4493.

[12] Ma, W.C., Minda, D., *A unified treatment of some special classes of univalent functions*, In: Proceedings of the Conference on Complex Analysis (Tianjin, 1992), Conf. Proc. Lecture Notes Anal., I, pages 157–169. Int. Press, Cambridge, MA, 1994.

[13] Madhumitha, S., Ravichandran, V., *Radius of starlikeness of certain analytic functions*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM, **115**(4)(2021), Paper No. 184, 18.

[14] Mendiratta, R., Nagpal, S., Ravichandran, V., *On a subclass of strongly starlike functions associated with exponential function*, Bull. Malays. Math. Sci. Soc., **38**(1)(2015), 365–386.

[15] Raina, R.K., Sokół, J., *Some properties related to a certain class of starlike functions*, C.R. Math. Acad. Sci. Paris, **353**(11)(2015), 973–978.

[16] Ravichandran, V., Rø nning, F., Shanmugam, T.N., *Radius of convexity and radius of starlikeness for some classes of analytic functions*, Complex Variables Theory Appl., **33**(1-4)(1997), 265–280.

[17] Sebastian, A., Ravichandran, V., *Radius of starlikeness of certain analytic functions*, Math. Slovaca, **71**(1)(2021), 83–104.

[18] Sharma, K., Jain, N.K., Ravichandran, V., *Starlike functions associated with a cardioid*, Afr. Mat., **27**(5-6)(2016), 923–939.

[19] Silverman, H., Silvia, E.M., *Subclasses of starlike functions subordinate to convex functions*, Canad. J. Math., **37**(1)(1985), 48–61.

[20] Sivaprasad Kumar, S., Kamaljeet, G., *A cardioid domain and starlike functions*, Anal. Math. Phys., **11**(2)(2021), Paper No. 54, 34.

[21] Wani, L.A., Swaminathan, A., *Radius problems for functions associated with a nephroid domain*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM, **114**(4)(2020), Paper No. 178, 20.

[22] Wani, L.A., Swaminathan, A., *Starlike and convex functions associated with a nephroid domain*, Bull. Malays. Math. Sci. Soc., **44**(1)(2021), 79–104.



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