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Optimal control of a frictional contact problem with unilateral constraints

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Abstract. We consider a mathematical model that describes a static contact with a nonlinear elastic body and a foundation. The contact boundary is composed of two measurable parts. In one part, the contact is frictionless with Signorini's conditions. In the other part, the normal stress is given and associated with Coulomb's friction law. We state an optimal control problem that consists of leading the stress tensor as close as possible to a given target by acting with a control on the boundary. Then, we study the penalized and regularized control problem for which we establish a convergence result.

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1. Introduction

Contact problems involving deformable bodies are very common in industry and everyday life and play a large role in structural and mechanical systems. Given the significance of these processes, considerable effort has been devoted to modelling and numerical simulation of these problems. The first of frictional contact problems in the context of variational inequalities was carried out in [9]. To get a background in contact mechanics from the mathematical or engineering point of view, the reader can consult for instance [2,12,14,18,21,22,26,23,24,25]. In addition to the numerical study of contact problems at present, we are also interested in studying the optimal control of such problems. Recall that the theory of optimal control of variational inequalities is very elaborate, see for instance [10,18]. In [19], we find the study of the optimal control of linear or nonlinear elliptic problems and variational inequalities. However,

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the optimal control issues for contact models are very significant, but they are not overly developed, see [1,3,4,5,6,7,8,10,13,15,16,18,19,20,26] and the references therein. Recently, in [16,17] two optimal control problems for elastic frictional contact models were studied. In particular, in [17], the authors investigated the optimal control of a frictional contact problem with normal compliance.

In this paper, we consider a nonlinear elastic body which is in static contact with a foundation. The boundary contact is divided into two measurable parts such that their measures must not equal zero at the same time. In one part, the contact is frictionless with unilateral constraints. In the other part, the normal stress is given and the contact is described by Coulomb's friction law. This model of contact was used in [27] to study a viscoelastic contact problem with a long memory. Thus, we contribute by proposing the model from which we derive a variational formulation (Problem P_2) of the mechanical problem and prove the existence and uniqueness of a weak solution. Next, the optimal control problem concerning this model is denoted by C1. It consists of minimizing a cost functional which is convex and continuous. Indeed, we are interested to led the stress tensor field as close as possible to a given target when we act with control on the boundary of the body. We prove that Problem C1 admits at least one solution, and then we introduce a penalized and regularized problem (Problem P_δ) such that the solution converges to the solution of Problem P_2. Also, we introduce a regularized and penalized optimal control problem C2 and obtain a convergence result.

The paper is structured as follows. In section 2, we describe the mechanical model, introduce some notations, establish a variational formulation and prove its weak solvability, Theorem 2.1. In section 3, we state the optimal control problem C1 and prove that it has at least one solution, Theorem 3.2. In section 4, we state and analyze a penalized and regularized optimal control problem, Theorem 4.4.

2. The model and its weak solvability

We denote by $\mathbf{S}_d$ the space of second order symmetric tensors on $\mathbb{R}^d (d = 2, 3)$, while '.' and $|.|$ represent the inner product and the norm on $\mathbf{S}_d$. Thus, for every σ, $\tau \in \mathbf{S}_d$, $\sigma.\tau = \sigma_{ij}\tau_{ij}$, $|\tau| = (\tau.\tau)^{\frac{1}{2}}$. Here and below, the indices i and j lie between 1 and d and the summation convention over repeated indices is adopted. We also use the usual notation for the normal components and the tangential parts of vectors and tensors, respectively, given by $v_\nu = v.\nu = v_i\nu_i$, $v_\tau = v - v_\nu\nu$, $\sigma_\nu = \sigma\nu.\nu$ and $\sigma_\tau = \sigma\nu - \sigma_\nu\nu$.

We consider the following physical setting. Let an elastic body occupy a bounded Lipschitzian domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$. The boundary Γ of Ω is partitioned into three measurable parts such that $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, where Γ_i, $i = 1, 2, 3$, are disjoint and $meas\,(\Gamma_1) > 0$. The body is subjected to volume forces of density φ_0 and tractions φ on Γ_2. On Γ_1, the displacement vanishes and the body is clamped here. Γ_3 is divided into $\Gamma_{3,1}$ and $\Gamma_{3,2}$ such that their measures must not equal zero at the same time. This latter hypothesis allows that where one of the two subsets $\Gamma_{3,1}$ and $\Gamma_{3,2}$ is empty, then the corresponding contact condition below is suppressed from the problem. We assume a frictionless contact with Signorini's conditions on $\Gamma_{3,1}$, and Coulomb's law of dry friction on $\Gamma_{3,2}$.

Under these conditions, the classic formulation for the contact problem is as follows.

Problem P_1. Find a displacement field $u : \Omega \to \mathbb{R}^d$ such that

$$div\sigma\,(u) = -\varphi_0 \text{ in } \Omega, \tag{2.1}$$

$$\sigma\,(u) = \mathcal{F}\varepsilon\,(u) \quad \text{in } \Omega\ , \tag{2.2}$$

$$u = 0 \qquad \text{on } \Gamma_1, \tag{2.3}$$

$$\sigma\nu = \varphi \qquad \text{on } \Gamma_2, \tag{2.4}$$

$$u_\nu \leq 0,\ \sigma_\nu \leq 0,\ \sigma_\nu u_\nu = 0,\ \sigma_\tau = 0 \text{ on } \Gamma_{3,1}, \tag{2.5}$$

$$\left.\begin{array}{l} -\sigma_\nu = S,\ \ |\sigma_\tau| \leq \mu\,|\sigma_\nu|\,, \\ -\sigma_\tau = \mu\,|\sigma_\nu|\,\dfrac{u_\tau}{|u_\tau|} \ \text{ if } u_\tau \neq 0 \end{array}\right\} \text{ on } \Gamma_{3,2}. \tag{2.6}$$

Here (2.1) represents the equilibrium equation where $\sigma = \sigma\,(u)$ denotes the stress tensor and $div\sigma = \sigma_{ij,j}$ is the divergence of σ. Next, equation (2.2) is the elastic constitutive law in which $\varepsilon\,(u)$ is the strain tensor defined by $\varepsilon\,(u) = (\varepsilon_{ij}\,(u))\,,\ \varepsilon_{ij}\,(u) = \frac{1}{2}\,(\partial_j u_i + \partial_i u_j)$ and $\mathcal{F}$ is a given nonlinear function. Equations (2.3) and (2.4) are the displacement and traction boundary conditions, respectively, in which ν denotes the unit outward normal vector on Γ and $\sigma\nu$ represents the Cauchy stress vector. Over $\Gamma_{3,1}$, (2.5) describes the frictionless contact with Signorini's conditions. On $\Gamma_{3,2}$, Coulomb's law of dry friction with the hypothesis that the normal stress is given. In (2.6) S is a nonnegative function, μ is a coefficient of friction and μS a friction bound.

To proceed with the variational formulation, Problem P_1, we need additional notations and need to recall some assumptions in the sequel.

$$\begin{array}{l} H = L^2\,(\Omega)^d\,,\ Q = \left\{\tau = (\tau_{ij})\,;\ \tau_{ij} = \tau_{ji} \in L^2\,(\Omega)\right\} \\ H_1 = \left\{u = (u_i)\,|u_i \in H^1(\Omega),\ i = \overline{1,d}\right\},\ Q_1 = \{\sigma \in Q\,|\text{div } \sigma \in H\} \end{array}$$

H, Q, H_1, H_d are real Hilbert spaces endowed with the respective inner products:

$$\begin{array}{l} (u,v)_H = \displaystyle\int_\Omega u_i v_i dx, \quad \langle\sigma,\tau\rangle_Q = \int_\Omega \sigma_{ij}\tau_{ij}dx, \\ (u,v)_{H_1} = \langle u,v\rangle_H + (\varepsilon(u),\varepsilon(v))_Q\,,\ \ (\sigma,\tau)_{H_d} = \langle\sigma,\tau\rangle_Q + (div\ \sigma, div\tau)_H\,. \end{array}$$

We denote respectively the norms associated with $\|.\|_H$, $\|.\|_Q$, $\|.\|_{H_1}$ and $\|.\|_{H_d}$.

Recall that the following *Green's* formula holds:

For every element $v \in H_1$, we also write v for the trace of v on Γ. Recall that if σ is a regular function, then the following Green's formula holds:

$$(\sigma, \varepsilon\,(v))_Q + (div\sigma, v)_H = \int_\Gamma \sigma\nu.vda \quad \forall v \in H_1,$$

where da is the measure surface element.

Next, let V be the closed subspace of H_1 defined by

$$V = \{v \in H_1;\ v = 0 \text{ on } \Gamma_1\}\,.$$

Since $meas\,(\Gamma_1) > 0$, the following Korn's inequality holds [9],

$$\|\varepsilon\,(v)\|_Q \geq c_\Omega \,\|v\|_{H_1} \quad \forall v \in V, \tag{2.7}$$

where $c_\Omega > 0$ is a constant which depends only on Ω and Γ_1. We equip V with the inner product given by

$$(u, v)_V = (\varepsilon\,(u)\,, \varepsilon(v))_Q,$$

and let $\|.\|_V$ be the associated norm. It follows from (2.7) that the norms $\|.\|_{H_1}$ and $\|.\|_V$ are equivalent and $(V, \|.\|_V)$ is a real Hilbert space. Moreover, by Sobolev's trace theorem, there exists a constant $d_\Omega > 0$ depending only on the domain Ω, Γ_1 and Γ_3 such that

$$\|v\|_{(L^2(\Gamma_3))^d} \leq d_\Omega \,\|v\|_V \quad \forall v \in V. \tag{2.8}$$

We introduce the closed convex set of admissible displacements defined as

$$K = \{v \in V;\, v_\nu \leq 0 \ a.e. \text{ on } \Gamma_{3,1}\}\,.$$

For the study of Problem (P) we adopt the following assumptions on the data:

The operator of elasticity $\mathcal{F}$ satisfies

$$\left\{\begin{array}{l}
(a)\ \ \mathcal{F} : \Omega \times S_d \to S_d; \\
(b)\ \ \text{there exists } M > 0 \text{ such that} \\
\qquad |\mathcal{F}\,(x, \varepsilon_1) - \mathcal{F}\,(x, \varepsilon_2)| \leq M\,|\varepsilon_1 - \varepsilon_2| \\
\qquad\quad \forall \varepsilon_1, \varepsilon_2 \in S_d,\ a.e.\ x \in \Omega; \\
(c)\ \ \text{there exists } m > 0 \text{ such that} \\
\qquad (\mathcal{F}\,(x, \varepsilon_1) - \mathcal{F}\,(x, \varepsilon_2))\,.\,(\varepsilon_1 - \varepsilon_2) \geq m\,|\varepsilon_1 - \varepsilon_2|^2 \\
\qquad\quad \forall \varepsilon_1, \varepsilon_2 \in S_d,\ a.e.\ x \in \Omega; \\
(d)\ \ \text{the mapping } x \to \mathcal{F}\,(x, \varepsilon) \text{ is Lebesgue measurable on } \Omega, \\
\qquad \text{for all } \varepsilon \in S_d; \\
(e)\ \ \mathcal{F}\,(x, 0_{S_d}) \ = 0 \text{ for } a.e.\ x \in \Omega.
\end{array}\right. \tag{2.9}$$

Examples of nonlinear elasticity operators can be found in $[11, 28]$.

We assume that the densities of the body force and the surface traction satisfies

$$\varphi_0 \in H, \qquad \varphi \in \left(L^2\,(\Gamma_2)\right)^d. \tag{2.10}$$

Finally, the coefficient of friction μ and the normal stress S are assumed to satisfy

$$\mu \in L^\infty\,(\Gamma_{3,2}) \text{ and } \mu \geq 0 \ a.e. \text{ on } \Gamma_{3,2}, \tag{2.11}$$

$$S \in L^2\,(\Gamma_{3,2}) \text{ and } S \geq 0 \ a.e. \text{ on } \Gamma_{3,2}. \tag{2.12}$$

Next, we define the functional $j : V \to \mathbb{R}$ by

$$j\,(v) = \int_{\Gamma_{3,2}} (Sv_\nu + \mu S\,|v_\tau|)da,\ \forall v \in V.$$

Using Riesz representation theorem, there exists $f \in V$ such that

$$(f, v)_V = (\varphi_0, v)_H + (\varphi, v)_{(L^2(\Gamma_2))^d}\ \forall v \in V.$$

A standard procedure allows us to derive the following variational formulation from the mechanical P_1.

Problem P_2. *Find* $u \in K$ *such that*

$$(Au, v-u)_V + j(v) - j(u) \geq (f, v-u)_V\,, \ \forall v \in K. \tag{2.13}$$

Here, the operator A is defined by

$$(Au, v)_V = (\mathcal{F}\varepsilon(u), \varepsilon(v))_Q, \ \forall u, v \in V.$$

The main result of this section is on the existence and uniqueness of the weak formulation P_2. One has the following theorem.

Theorem 2.1. *Let* (2.9)*,* (2.10)*,* (2.11) *and* (2.12) *hold. Then, there exists a unique solution of Problem* P_2*.*

Proof. We use (2.9) (b), (2.9) (c) to show that the operator A is Lipschitz continuous and strongly monotone. Using (2.11) and (2.12), we see that the functional $j : V \to \mathbb{R}$ is proper, convex and lower semicontinuous; K is a non empty closed convex of V. Then, it follows from the theory of elliptic variational inequalities (see [24]) that the inequality (2.13) has a unique solution. □

3. The optimal control problem

For a fixed $\varphi_0 \in H$, we consider the state problem below.

Problem Q1. For a given $\varphi \in (L^2(\Gamma_2))^d$ (called control), find $u \in K$ such that

$$\begin{cases} (Au, v-u)_V + \ j(v) - j(u) \\ \geq (\varphi_0, v-u)_H + (\varphi, v-u)_{(L^2(\Gamma_2))^d}, \quad \forall v \in K. \end{cases} \tag{3.1}$$

Theorem 3.1. *Let* (2.9)*,* (2.10)*,* (2.11) *and* (2.12) *hold. Then Problem Q1 has a unique solution.*

By the same arguments used in the proof of Theorem 2.1, this problem has a unique solution $u = u(\varphi)$.

Now, by acting the control on the boundary Γ_2, we focus that the resulting stress be as close to a given target σ_d. We assume that $\sigma_d = \mathcal{F}\varepsilon(u_d)$ where $u_d \in V$ and recall that $\sigma = \mathcal{F}\varepsilon(u)$. Then we have $\|\sigma - \sigma_d\|_Q \leq M\|u - u_d\|_V$ and we see that if $\|u - u_d\|_V$ is sufficiently small, it follows that σ approach σ_d in the sense of $Q-$norm. Thus, we consider the cost functional $\mathcal{L} : V \times (L^2(\Gamma_2))^d \to \mathbb{R}_+$ defined as

$$\mathcal{L}(u, \varphi) = \alpha\|u - u_d\|_V^2 + \beta\|\varphi\|_{(L^2(\Gamma_2))^d}^2\,, \tag{3.2}$$

where $\alpha, \beta > 0$. We define the set U_{ad} of admissible pairs by

$$U_{ad} = \left\{(u, \varphi) \in (K \times (L^2(\Gamma_2))^d), \text{ such that (3.1) is satisfied}\right\}.$$

Then we consider the following optimal control problem.

Problem C1. Find $(u^*, \varphi^*) \in U_{ad}$ such that

$$\mathcal{L}(u^*, \varphi^*) = \min_{(u,\varphi)\in U_{ad}} \mathcal{L}(u, \varphi).$$

Theorem 3.2. *Assume* (2.9)*,* (2.10)*,* (2.11) *and* (2.12)*. Then Problem C1 has at least one solution.*

Proof. We put $v = 0_V$ in (3.1), then, using (2.7), (2.8) and (2.9) (c), we deduce that the solution u of Problem Q1 is bounded in V as

$$\|u\|_V \leq \frac{c_0}{m} \left(\|\varphi_0\|_H + d_\Omega \|\varphi\|_{(L^2(\Gamma_2))^d} + d_\Omega \|S\|_{L^2(\Gamma_{3,2})} \right),$$

where $c_0 > 0$. This estimate below implies that

$$\inf_{(u,\varphi) \in U_{ad}} \{\mathcal{L}(u, \varphi)\} < \infty.$$

Now, let us denote

$$\inf_{(u,\varphi) \in U_{ad}} \{\mathcal{L}(u, \varphi)\} = \theta. \tag{3.3}$$

Then, there exists a minimizing sequence $(u^n, \varphi^n) \subset U_{ad}$ such that

$$\lim_{n \to \infty} \mathcal{L}(u^n, \varphi^n) = \theta. \tag{3.4}$$

The sequence (u^n, φ^n) is bounded in $V \times (L^2(\Gamma_2))^d$, so there exists an element

$$(u^*, \varphi^*) \in V \times (L^2(\Gamma_2))^d$$

such that passing to a subsequence still denoted by (u^n, φ^n), we deduce that as $n \to \infty$,

$$u^n \to u^* \text{ weakly in } V. \tag{3.5}$$

We note that K is a closed convex subset of the space V and $(u^n) \subset K$. Then the convergence (3.5) implies that $u^* \in K$.

$$\varphi^n \to \varphi^* \text{ weakly in } (L^2(\Gamma_2))^d. \tag{3.6}$$

Now, we need to prove that

$$u^n \to u^* \text{ strongly in } V \text{ as } n \to \infty. \tag{3.7}$$

Indeed, as $(u^n, \varphi^n) \in U_{ad}$, then u^n is the solution of the inequality below.

$$\begin{cases} (Au^n, v - u^n)_V + j(v) - j(u^n) \\ \geq (\varphi_0, v - u^n)_H + (\varphi^n, v - u^n)_{(L^2(\Gamma_2))^d}, \quad \forall v \in K. \end{cases} \tag{3.8}$$

Using (2.9) (c) and (3.8), we deduce that

$$\begin{aligned} m \|u^n - u^*\|_V^2 &\leq (Au^n - Au^*, u^n - u^*)_V \\ &\leq -(Au^*, u^n - u^*)_V + j(u^*) - j(u^n) \\ &\quad + (\varphi_0, u^n - u^*)_H + (\varphi^n, u^n - u^*)_{(L^2(\Gamma_2))^d}. \end{aligned} \tag{3.9}$$

Using (3.5), we have that

$$\lim_{n \to \infty} (Au^*, u^n - u^*)_V = 0.$$

On the other hand, since $u^n \to u^*$ weakly in V implies $u^n \to u^*$ strongly in H, then $\lim\limits_{n \to +\infty} (\varphi_0, u^n - u^*)_H = 0$. Also, as (φ^n) is bounded in $(L^2(\Gamma_2))^d$, then using that (3.5) implies $u^n \to u^*$ strongly in $(L^2(\Gamma_2))^d$. It follows that

$$\lim_{n \to \infty} (\varphi^n, u^n - u^*)_{(L^2(\Gamma_2))^d} = 0 \text{ and } \lim_{n \to \infty} j(u^n) = j(u^*).$$

Thus, the right hand side of inequality (3.9) tends to zero as $n \to +\infty$ and then we get (3.7) . Moreover, using (3.6) and (3.7), we pass to the limit as $n \to +\infty$ in (3.8) to obtain that u^* satisfies the inequality (3.1) with $\varphi = \varphi^*$. Hence, from Theorem 3.1, we deduce that

$$(u^*, \varphi^*) \in U_{ad}. \tag{3.10}$$

On the other hand, the functional $\mathcal{L}$ is convex and lower semicontinuous, then it is weakly lower semicontinuous. So we deduce that

$$\liminf_{n\to+\infty} \mathcal{L}\left(u^n, \varphi^n\right) \geq \mathcal{L}\left(u^*, \varphi^*\right). \tag{3.11}$$

It follows now from (3.4) and (3.11) that

$$\theta \geq \mathcal{L}\left(u^*, \varphi^*\right). \tag{3.12}$$

In addition, (3.3) yields

$$\mathcal{L}\left(u^*, \varphi^*\right) \geq \theta. \tag{3.13}$$

Then, to end the proof, it suffices to combine the inequalities (3.12) and (3.13) .□ □

4. The penalized and regularized optimal control problem

Let $\delta > 0$, we replace the contact condition (2.5) by the condition

$$\sigma_{\delta\nu}\left(u\right) = -\frac{1}{\delta}(u_\nu)_+$$

where we recall that for $r \in \mathbb{R}$, $r_+ = \max\left(r, 0\right)$, and consider the smooth function

$$\psi\left(x\right) = \sqrt{x^2 + \delta^2}.$$

Now, we introduce the following penalized and regularized problem.

Problem P_δ**.** Find $u^\delta \in V$ such that

$$\begin{aligned}
&(Au^\delta, v - u^\delta)_V + \frac{1}{\delta}\left((u_\nu^\delta)_+, v_\nu - u_\nu^\delta\right)_{L^2(\Gamma_{3,1})} + \int_{\Gamma_{3,2}} \mu S\left(\psi\left(v_\tau\right) - \psi\left(u_\tau^\delta\right)\right) da \\
&+ \int_{\Gamma_{3,2}} S\left(v_\nu - u_\nu^\delta\right) da \geq (f, v - u^\delta)_V \quad \forall v \in V.
\end{aligned} \tag{4.1}$$

Theorem 4.1. *Assume that* (2.9), (2.10), (2.11) *and* (2.12) *hold.Then, there exists a unique solution of Problem* P_δ*.*

Proof. We define the operator $B : V \to V$ by

$(Bu, v)_V = (Au, v)_V + \frac{1}{\delta}\left((u_\nu)_+, v_\nu\right)_{L^2(\Gamma_{3,1})} \forall u, v \in V.$

Using that for $a, b \in \mathbb{R}$, $(a - b)\left(a_+ - b_+\right) \geq \left(a_+ - b_+\right)^2$ and $|a_+ - b_+| \leq |a - b|$, we deduce by (2.8) and (2.9) that the operator B is Lipschitz continuous and strongly monotone as for all $u, v \in V$:

$$\begin{aligned}
&\|Bu - Bv\|_V \leq (M + \frac{d_\Omega^2}{\delta}) \|u - v\|_V , \\
&(Bu - Bv, u - v)_V \geq m \|u - v\|_V^2 .
\end{aligned}$$

So, there exists a unique solution u^δ of (4.1). In addition, take $v = 0$ in (4.1) and use (2.7), (2.8) and (2.9) (c) implies that

$$\left\|u^\delta\right\|_V \leq \frac{c_0}{m}(\|\varphi_0\|_H + d_\Omega \|\varphi\|_{(L^2(\Gamma_2))^d} + d_\Omega \|S\|_{L^2(\Gamma_{3,2})}). \tag{4.2}$$

□

Now for a fixed $\varphi_0 \in H$, we define the penalized and regularized state problem as follows.

Problem Q2. For a given $\varphi \in (L^2(\Gamma_2))^d$ (called control), find $u^\delta \in V$ such that

$$(Au^\delta, v - u^\delta)_V + \frac{1}{\delta}\left((u^\delta_\nu)_+, v_\nu - u^\delta_\nu\right)_{L^2(\Gamma_{3,1})} + \int_{\Gamma_{3,2}} \mu S\left(\psi(v_\tau) - \psi\left(u^\delta_\tau\right)\right) da$$

$$+ \int_{\Gamma_{3,2}} S(v_\nu - u_\nu)\, da \geq (\varphi_0, v - u^\delta)_H + (\varphi, v - u^\delta)_{(L^2(\Gamma_2))^d} \quad \forall v \in V.$$

With the same arguments used in Theorem 4.1 this problem has a unique solution. Moreover, we define the set of admissible pairs as

$$U^\delta_{ad} = \left\{(u, \varphi) \in V \times (L^2(\Gamma_2))^d, \text{ such that } (4.1) \text{ is satisfied}\right\}.$$

Then using the functional $\mathcal{L}$, given by (3.2), we formulate below the regularized and penalized optimal control problem.

Problem C2. Find $(\bar{u}^\delta, \bar{\varphi}^\delta) \in U^\delta_{ad}$ such that

$$\mathcal{L}\left(\bar{u}^\delta, \bar{\varphi}^\delta\right) = \min_{(u,\varphi)\in U^\delta_{ad}} \{\mathcal{L}(u, \varphi)\}.$$

With arguments similar to those used in Theorem 3.1, the following result can be proved.

Theorem 4.2. *Assume* (2.9), (2.10), (2.11) *and* (2.12) *hold.Then, Problem C2 has at least one solution.*

In the first part of this section, we prove that the unique solution of the penalized and regularized state problem Q2 converges to the unique solution of the state problem Q1. More precisely, the following theorem takes place.

Theorem 4.3. *Assume that* (2.9), (2.10), (2.11) *and* (2.12) *hold. Then, the following strong convergence holds:*

$$u^\delta \to u \textit{ strongly in } V \quad \textit{as } \delta \to 0. \tag{4.3}$$

Proof. Taking into account (4.2), it follows that there exists an element $\tilde{u} \in V$ such that passing to a subsequence still denoted in the same way, we have the convergence:

$$u^\delta \to \tilde{u} \text{ weakly in } V \quad \text{as } \delta \to 0. \tag{4.4}$$

Now take $v \in K$ in (4.1) and taking account that for $a, b \in \mathbb{R}$,

$$(a_+ - b_+)(a - b) \geq (a_+ - b_+)^2,$$

we deduce that

$$
\begin{aligned}
&(Au^\delta, v-u^\delta)_V+\int_{\Gamma_{3,2}}\mu S\left(\psi\left(v_\tau\right)-\psi\left(u_\tau^\delta\right)\right)da+\int_{\Gamma_{3,2}}S(v_\nu-u_\nu^\delta)da\\
&\geq(\varphi_0,v-u^\delta)_H+(\varphi,v-u^\delta)_{(L^2(\Gamma_2))^d}\quad\forall v\in K.
\end{aligned}
\tag{4.5}
$$

Using (2.11) and (2.12), we have that $\int_{\Gamma_{3,2}}\mu S(\psi\left(v_\tau\right)-|v_\tau|)da=O\left(\delta\right)$, then

$$
\int_{\Gamma_{3,2}}\mu S(\psi\left(v_\tau\right))\rightarrow\int_{\Gamma_{3,2}}\mu S\left|v_\tau\right|da\text{ as }\delta\rightarrow 0. \tag{4.6}
$$

On the other hand, we have

$$
\int_{\Gamma_{3,2}}\mu S\psi\left(u_\tau^\delta\right)da=\int_{\Gamma_{3,2}}\mu S(\psi\left(u_\tau^\delta\right)-\left|u_\tau^\delta\right|)da+\int_{\Gamma_{3,2}}\mu S\left|u_\tau^\delta\right|da.
$$

By (2.11) and (2.12), we have that

$$
\int_{\Gamma_{3,2}}\mu S(\psi\left(u_\tau^\delta\right)-\left|u_\tau^\delta\right|)da=O\left(\delta\right).
$$

Then,

$$
\int_{\Gamma_{3,2}}\mu S\psi\left(u_\tau^\delta\right)da=O\left(\delta\right)+\int_{\Gamma_{3,2}}\mu S\left|u_\tau^\delta\right|da. \tag{4.7}
$$

With compactness arguments, as $u_\tau^\delta\rightarrow\tilde{u}_\tau$ strongly in $\left(L^2\left(\Gamma_2\right)\right)^d$, we have that

$$
\int_{\Gamma_{3,2}}\mu S\left|u_\tau^\delta\right|da\rightarrow\int_{\Gamma_{3,2}}\mu S\left|\tilde{u}_\tau\right|da\text{ as }\delta\rightarrow 0.
$$

Then from (4.7) we deduce that

$$
\int_{\Gamma_{3,2}}\mu S\psi\left(u_\tau^\delta\right)da\rightarrow\int_{\Gamma_{3,2}}\mu S\left|\tilde{u}_\tau\right|da\text{ as }\delta\rightarrow 0. \tag{4.8}
$$

Then using (2.11), (2.12), (4.4), (4.5), (4.6), (4.8) and the compact imbedding $H^{\frac{1}{2}}\left(\Gamma\right)\hookrightarrow L^2\left(\Gamma\right)$, yields

$$
\begin{aligned}
\limsup_{\delta\rightarrow 0}\left(Au^\delta,u^\delta-v\right)_V&\leq(\varphi_0,\tilde{u}-v)_H+(\varphi,\tilde{u}-v)_{(L^2(\Gamma_2))^d}\\
&\quad+j(\tilde{u})-j\left(v\right)\quad\forall v\in K.
\end{aligned}
\tag{4.9}
$$

Using now the pseudo-monotonicity of A, we deduce that

$$
\liminf_{\delta\rightarrow 0}\left(Au^\delta,u^\delta-v\right)_V\geq(A\tilde{u},\tilde{u}-v)_V\quad\forall v\in V. \tag{4.10}
$$

Then, we combine (4.9) and (4.10) to get that

$$
\left\{\begin{array}{l}(A\tilde{u}\ ,v-\tilde{u})_V+j\left(v\right)-j(\tilde{u})\\ \geq(\varphi_0,v-\tilde{u})_H+(\varphi,v-\tilde{u})_{(L^2(\Gamma_2))^d}\quad\forall v\in K.\end{array}\right. \tag{4.11}
$$

On the other hand, take $v=0$ in (4.1) implies that

$$
\left((\tilde{u}_\nu^\delta)_+,\tilde{u}_\nu^\delta\right)_{L^2(\Gamma_{3,1})}\leq\delta\left((\varphi_0,u^\delta)_H+(\varphi,u^\delta)_{(L^2(\Gamma_2))^d}-\left(S,u_\nu^\delta\right)_{L^2(\Gamma_{3,2})}\right).
$$

Then, from this inequality, we deduce that

$$\|(\tilde{u}_\nu)_+\|_{L^2(\Gamma_{3,1})} \leq \liminf_{\delta\to 0} \left\|(u_\nu^\delta)_+\right\|_{L^2(\Gamma_{3,1})} \leq$$
$$\lim_{\delta\to 0}\sqrt{\frac{c_0\delta}{m}}\left(\|\varphi_0\|_H + d_\Omega\|\varphi\|_{(L^2(\Gamma_2))^d} + d_\Omega\|S\|_{L^2(\Gamma_{3,2})}\right).$$

This inequality above implies that $\|(\tilde{u}_\nu)_+\|_{L^2(\Gamma_{3,1})} = 0$, then $(\tilde{u}_\nu)_+ = 0$ *a.e.* on $\Gamma_{3,1}$. Hence, it follows that $\tilde{u} \in K$. Then, we deduce that $\tilde{u}$ is a solution of Problem P_1, so that $u = \tilde{u}$ from the uniqueness part of Theorem 2.1. Now, we have all ingredients to end the proof of Theorem 4.2. Indeed, by the arguments used above, it follows that any weakly convergent subsequence of the sequence $(u_\delta) \subset V$ converges weakly to the unique solution u of Problem P_2. Estimate (4.2) implies that the sequence (u_δ) is bounded in V. Thus, by a standard compactness argument, we conclude that the whole sequence (u_δ) converges weakly to u. Then we use (2.9) (c) to have

$$\begin{aligned} m\left\|u^\delta - u\right\|_V^2 &\leq (Au^\delta - Au, u^\delta - u)_V \\ &= (Au^\delta, u^\delta - u)_V - (Au, u^\delta - u)_V \end{aligned} \tag{4.12}$$

Now take $v = u$ in (4.5) and (4.9), then as $u = \tilde{u}$, we get

$$0 \leq \liminf_{\delta\to 0}\left(Au^\delta, u^\delta - u\right)_V \leq \limsup_{\delta\to 0}\left(Au^\delta, u^\delta - u\right)_V \leq 0.$$

Hence,

$$\lim_{\delta\to 0}\left(Au^\delta, u^\delta - u\right)_V = 0.$$

Moreover, from (4.12) since $\lim_{\delta\to 0}\left(Au, u^\delta - u\right)_V = 0$, we deduce that

$$\lim_{\delta\to 0}\left\|u^\delta - u\right\|_V = 0.$$

Then, we obtain (4.3). $\square$

Next, we prove the convergence result below.

Theorem 4.4. *Assume that* (2.9), (2.10), (2.11), (2.12) *hold and let* $(\bar{u}^\delta, \bar{\varphi}^\delta)$ *be a solution of Problem C2. Then, there exists a solution* $(\bar{u}, \bar{\varphi})$ *of Problem C1 such that after passing to a subsequence still denoted in the same way, the following convergences as* $\delta \to 0$ *hold :*

$$\begin{aligned} &(a)\ \bar{u}^\delta \to \bar{u} \text{ strongly in } V, \\ &(b)\ \bar{\varphi}^\delta \to \bar{\varphi} \text{ weakly in } (L^2(\Gamma_2))^d. \end{aligned} \tag{4.13}$$

Proof. Let $u_0^\delta \in V$ be the unique solution of Problem Q2 with $\varphi = 0_{(L^2(\Gamma_2))^d}$. We have

$$\mathcal{L}\left(u_0^\delta, 0_{(L^2(\Gamma_2))^d}\right) = \alpha\left\|u_0^\delta - u_d\right\|_V^2 \leq 2\alpha\left(\left\|u_0^\delta\right\|_V^2 + \|u_d\|_V^2\right).$$

On the other hand, by (2.7), (2.8) and (2.9) (c), we have

$$\left\|u_0^\delta\right\|_V \leq \frac{c_1}{m}\left(\|\varphi_0\|_H + d_\Omega\|S\|_{L^2(\Gamma_{3,2})}\right),$$

where $c_1 > 0$. Then, denote $\frac{c_1}{m}\left(\|\varphi_0\|_H + d_\Omega \|S\|_{L^2(\Gamma_{3,2})}\right) = C$, we deduce that

$$\mathcal{L}\left(\bar{u}^\delta, \bar{\varphi}^\delta\right) \leq \mathcal{L}\left(u_0^\delta, 0_{(L^2(\Gamma_2))^d}\right) \leq 2\alpha(C^2 + \|u_d\|_V^2).$$

Therefore, $\left(\bar{u}^\delta, \bar{\varphi}^\delta\right)$ is a bounded sequence in $V \times (L^2(\Gamma_2))^d$. Consequently, there exists $(\bar{u}, \bar{\varphi}) \in V \times (L^2(\Gamma_2))^d$ such that passing to a subsequence still denoted in the same way, we have the convergences as $\delta \to 0$:

$$\begin{array}{rcl} \bar{u}^\delta & \rightharpoonup & \bar{u} \text{ weakly in } V, \\ \bar{\varphi}^\delta & \rightharpoonup & \bar{\varphi} \text{ weakly in } (L^2(\Gamma_2))^d. \end{array}$$

Moreover, denote $j_\delta(v) = \int_{\Gamma_{3,2}} (S\mu\psi(v_\tau) + Sv_\nu)da$, we see that

$$\begin{array}{rcl} m\left\|\bar{u}^\delta - \bar{u}\right\|_V^2 & \leq & \left(A\bar{u} - A\bar{u}^\delta, \bar{u} - \bar{u}^\delta\right)_V \\ & \leq & \left(A\bar{u}, \bar{u} - \bar{u}^\delta\right)_V + j_\delta(\bar{u}) - j_\delta\left(\bar{u}^\delta\right) \\ & & +(\varphi_0, \bar{u} - u^\delta)_H + (\varphi, \bar{u} - u^\delta)_{(L^2(\Gamma_2))^d}. \end{array}$$

Then, taking in mind that $\bar{u}^\delta \rightharpoonup \bar{u}$ weakly in V implies that $\bar{u}^\delta \to \bar{u}$ strongly in $(L^2(\Gamma_2))^d$, it follows that $j_\delta(\bar{u}) - j_\delta\left(\bar{u}^\delta\right) \to 0$ as $\delta \to 0$. Hence we deduce that the right hand side of the above inequality tends to zero, thus we obtain (4.13) (a). Also, we must prove that $(\bar{u}, \bar{\varphi}) \in U_{ad}$. Indeed, using (4.3), it follows that as $\delta \to 0$, the following convergences hold:

$$(A\bar{u}^\delta, v - \bar{u}^\delta)_V \to (A\bar{u}, v - \bar{u})_V,$$

$$\lim_{\delta \to 0}(j_\delta(v) - j_\delta(\bar{u})) = j(v) - j(\bar{u}),$$

$$(\varphi_0, v - u^\delta)_H + (\varphi, v - u^\delta)_{(L^2(\Gamma_2))^d} \to (\varphi_0, v - \bar{u})_H + (\varphi, v - \bar{u})_{(L^2(\Gamma_2))^d}.$$

Therefore, passing to the limit as $\delta \to 0$ in (4.5), we deduce that $(\bar{u}, \bar{\varphi})$ satisfies (3.1) and $(\bar{u}, \bar{\varphi}) \in U_{ad}$.Let now (u^*, φ^*) be a solution of Problem C1 and let us consider the sequence $\left(u^\delta\right)_\delta$ such that, for each $\delta > 0$, u^δ is the unique solution of Problem Q2 with $\varphi^* \in (L^2(\Gamma_2))^d$. Obviously, for every $\delta > 0$, $\left(u^\delta, \varphi^*\right) \in U_{ad}^\delta$. Using Theorem 4.3 we deduce that

$$\left(u^\delta, \varphi^*\right) \to (u^*, \varphi^*) \text{ in } V \times (L^2(\Gamma_2))^d \text{ as } \delta \to 0. \tag{4.14}$$

Since the functional $\mathcal{L}$ is convex and continuous, we have

$$\mathcal{L}(u^*, \varphi^*) \leq \liminf_{\delta \to 0} \mathcal{L}\left(\bar{\varphi}^\delta, \bar{u}^\delta\right). \tag{4.15}$$

Also, as $\left(\bar{u}^\delta, \bar{\varphi}^\delta\right)$ is a solution of Problem C2, we have

$$\limsup_{\delta \to 0} \mathcal{L}\left(\bar{u}^\delta, \bar{\varphi}^\delta\right) \leq \limsup_{\delta \to 0} \mathcal{L}\left(u^\delta, \bar{\varphi}\right). \tag{4.16}$$

Using (4.13), we have

$$\limsup_{\delta \to 0} \mathcal{L}\left(u^\delta, \bar{\varphi}\right) = \mathcal{L}(\bar{u}, \bar{\varphi}), \tag{4.17}$$

and as $(\bar{u}, \bar{\varphi})$ is a solution of Problem C1, then

$$\mathcal{L}(\bar{u}, \bar{\varphi}) \leq \mathcal{L}(u^*, \varphi^*). \tag{4.18}$$

Thus, from (4.15)-(4.18) , we deduce that $\mathcal{L}(\bar{u}, \bar{\varphi}) = \mathcal{L}(u^*, \varphi^*)$. □

References

[1] Amassad, A., Chenais, D., Fabre, C., *Optimal control of an elastic contact problem involving Tresca friction law*, Nonlinear Anal., **48**(2002), 1107-1135.

[2] Barbu, V., *Optimal Control of Variational Inequalities*, Pitman Advanced Publishing, Boston, 1984.

[3] Bartosz, K., Kalita, P., *Optimal control for a class of dynamic viscoelastic contact problems with adhesion*, Dynam. Systems Appl., **21**(2012), 269-292.

[4] Bonnans, J.F., Tiba, D., *Pontryagin's principle in the control of semilinear elliptic variational inequalities*, Appl. Math. Optim., **23**(1991) 299-312.

[5] Capatina, A., *Optimal control of a Signorini contact problem*, Numer. Funct. Anal. and Optimiz., **21**(2000), 817-828.

[6] Capatina, A., Timofte, C., *Boundary optimal control for quasistatic bilateral frictional contact problems*, Nonlinear Anal., **94**(2014), 84–99.

[7] Denkowski, Z., Migorski, S., Ochal, A., *Optimal control for a class of mechanical thermoviscoelastic frictional contact problems*, Control Cybernet., **36(3)**(2007), 611-632.

[8] Denkowski, Z., Migorski, S., Ochal, A., *A class of optimal control problems for piezoelectric frictional contact models*, Nonlinear Anal. Real World Appl., **12**(2011), 1883-1895.

[9] Duvaut, G., Lions, J.-L., *Les Inéquations en Mécanique et en Physique*, Dunod, Paris, 1972.

[10] Friedman, A., *Optimal control for variational inequalities*, SIAM J. Control Optim., **24**(1986), 439-451.

[11] Han, W., Sofonea, M., *Quasistatic Contact Problems in Viscoelasticity and Viscoplasticity*, AMS/IP Studies in Advanced Mathematics, vol. 30, AMS, Rhode Island, 2002.

[12] Kikuchi, N., Oden, J.T., *Contact Problems in Elasticity: A Study of Variational Inequalities and Finite Element Methods*, SIAM, Philadelphia, 1988.

[13] Kimmerle, S.-J., Moritz, R., *Optimal control of an elastic Tyre-Damper system with road contact*, Proc. Appl. Math. Mech., **14(1)**(2014), 875-876.

[14] Laursen, T., *Computational Contact and Impact Mechanics*, Springer Berlin Heidelberg,2002.

[15] Lions, J.-L., *Contrôle Optimal des Systèmes Gouvernés par des Équations aux Dérivées Partielles*, Dunod, Paris, 1968.

[16] Matei, A., Micu, S., *Boundary optimal control for a frictional contact problem with normal compliance*, Appl. Math. Optim., **78(2)**(2018), 379–401.

[17] Matei, A., Micu, S., *Boundary optimal control for nonlinear antiplane problems*, Nonlinear Anal., **74(5)**(2011), 1641-1652.

[18] Mignot, R., *Contrôle dans les inéquations variationnelles elliptiques*, J. Func. Anal., **22**(1976), 130-185.

[19] Mignot, R., Puel, J.-P., *Optimal control in some variational inequalities*, SIAM J. Control Optim, **22**(1984), 466-476.

[20] Neittaanmaki, P., Sprekels, J., Tiba, D., *Optimization of Elliptic Systems: Theory and Applications*, Springer Monographs in Mathematics, Springer-Verlag, New York, 2006.

[21] Oden, J.T., Martins, J.A.C., *Models and computational methods for dynamic friction phenomena*, Comput. Methods Appl. Mech. Engrg., **52**(1985), 527–634.

[22] Popov, V.L., *Contact Mechanics and Friction*, Springer, Heidelberg, 2010.

[23] Shillor, M., Sofonea, M., Telega, J.J., *Models and Variational Analysis of Quasistatic Contact*, Lecture Notes in Physics, Vol. 655, Springer, Berlin Heidelberg, 2004.

[24] Sofonea, M., Matei, A., *Variational Inequalities with Applications. A Study of Antiplane Frictional Contact Problems*, Adv. Mech. Math., 18, Springer, 2009.

[25] Sofonea, M., Matei, A., *Mathematical Models in Contact Mechanics*, London Math. Soc. Lecture Note Ser. 398, Cambridge University Press, 2012.

[26] Touzaline, A., *Optimal control of a frictional contact problem*, Acta Math. Appl. Sin. Engl. Ser., **31(4)**(2015), 991-1000.

[27] Xu, W., Wang, C., et al., *Numerical analysis of doubly-history dependent variational inequalities in contact mechanics*, Fixed Point Theory Algorithms Sci. Eng. 2021, **24**(2021).

[28] Zeidler, E., *Nonlinear Functional Analysis and its Applications. IV: Applications to Mathematical Physics*, Springer, New York, 1988.

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