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Nonnegative solutions for a class of fourth order singular eigenvalue problems

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Abstract. In this paper, we discuss the existence of nonnegative solutions to a fourth order singular boundary value problem at two points. Our result is based on a recent Birkhoff-Kellogg type fixed point theorem developed on translates of a cone on a Banach space.

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1. Introduction

In the present paper, we investigate the following fourth order singular differential equation with parameter

$$v^{(4)} = \lambda g(t) f(v(t)), \quad 0 < t < 1, \tag{1.1}$$

subject to the boundary conditions

$$v(0) = a_1, \quad v(1) = a_2, \quad v''(0) = a_3, \quad v''(1) = a_4, \tag{1.2}$$

where $a_j \geq 0$, $j \in \{1, 2, 3, 4\}$, are given constants,

(H1). $f \in \mathcal{C}([0, \infty))$,

$$0 < A_1 \leq f(x) \leq A_2 + \sum_{j=0}^{k} B_j x^j, \quad x \in [0, \infty),$$

$A_2 \geq A_1 > 0$ and $B_j \geq 0$, $j \in \{0, \dots, k\}$, $k \in \mathbb{N}_0$, are given constants.

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(H2). $g : (0, 1) \to \mathbb{R}^+$ is continuous and may be singular at $t = 0$ or/and $t = 1$, $g \not\equiv 0$ on $(0, 1)$ and $\int_0^1 s(1-s)g(s)ds < \infty$.

Fourth order two-point boundary value problems (BVPs for short) have been received much attention by many authors due to their importance in physics. Usually, they are essential in describing a vast class of elastic deflections with several types of boundary conditions such as whose ends are simply-supported at 0 and 1 ($v(0) = v(1) = v''(0) = v''(1) = 0$). A great number of research has been devoted to investigate the existence of positive solutions to this class of problems, see [2, 1, 3, 4, 8, 9, 10, 11, 12, 13] and the references therein. The authors in [2] discussed the existence, uniqueness and multiplicity of positive solutions to the following eigenvalue BVP by means of fixed point theorem and degree theory

$$v^{(4)} = \lambda f(t, (v(t)), \quad 0 < t < 1, \tag{1.3}$$

$$v(0) = v(1) = v''(0) = v''(1) = 0, \tag{1.4}$$

where $\lambda > 0$ is a constant and $f : [0, 1] \times [0, \infty) \to [0, \infty)$ is continuous. In [12] by applying a Krasnosel'skii fixed point theorem of cone expansion and compression the author obtained the existence and multiplicity results of equation (1.3) with boundary conditions $v(0) = v(1) = v'(0) = v'(1) = 0$. In the literature, there are few papers devoted to study fourth order singular eigenvalue problems. In the case when $a_j = 0$, $j \in \{1, 2, 3, 4\}$, the BVP (1.1)-(1.2) is investigated in [7] when $f \in \mathcal{C}([0, \infty))$, $f > 0$ on $[0, \infty)$, f is nondecreasing on $[0, \infty)$ and there exist $\delta > 0$, $m \geq 2$ such that $f(u) > \delta u^m$, $u \in [0, \infty)$, and $g \in \mathcal{C}(0, 1)$, $g > 0$ on $(0, 1)$ and $0 < \int_0^1 s(1-s)g(s)ds < \infty$. In [7], Feng and Ge used the method of upper and lower solutions and the fixed point index to discuss the existence of positive solutions.

Our main result is as follows where we do not require any monotonicity assumptions on f, and we do not assume that f is either superlinear or sublinear.

Theorem 1.1. *Suppose that* $(H1)$ *and* $(H2)$ *hold. Then there is a* $\lambda^* > 0$ *such that the BVP* (1.1)-(1.2) *has at least one nonnegative solution for* $\lambda = \lambda^*$.

Note that our main result, in the particular case $a_j = 0$, $j \in \{1, 2, 3, 4\}$, is valid in the case when f is decreasing on $[0, \infty)$, while the corresponding result in [7] is not valid. For instance, $f(x) = 1 + \frac{1}{1+x^2}$, $x \in [0, \infty)$, satisfies $(H1)$ for $A_1 = 1$, $A_2 = 2$, $B_j = 0$, $j \in \{0, \ldots, k\}$, and f is decreasing on $[0, \infty)$, whereupon it does not satisfy the conditions in [7]. Also, the conditions for g in [7] are more restrictive than $(H2)$. For instance, $g(t) = \frac{\left(\frac{1}{2}-t\right)^2}{t(1-t)}$, $t \in (0, 1)$, satisfies $(H2)$ and does not satisfy the conditions in [7] because $g\left(\frac{1}{2}\right) = 0$. Thus, we can consider the particular case of our main result, $a_j = 0$, $j \in \{1, 2, 3, 4\}$, as a complementary result to the result in [7]. The approach used in this paper is to rewrite the (BVP) (1.1)-(1.2) into a perturbed integral equation of which we search for solutions in a suitable subset of a Banach space by means of recent fixed point theorem of Birkhoff-Kellogg type developed by Calamai and Infante in [5]. Note that this fixed point theorem has been applied very recently to discuss the solvability of fourth order retarded equations in [6].

The paper is organized as follows. In Section 2, we give some auxiliary results needed for the proof of our main result. In Section 3, we prove our main result. In Section 4, we give an example.

2. Auxiliary results

Let X be a real Banach space.

Definition 2.1. A mapping $F : \Omega \subset X \to X$ is said to be completely continuous if it is continuous and maps bounded sets into relatively compact sets.

Definition 2.2. A closed, convex set $\mathcal{K}$ of X is said to be cone if

1. $\alpha x \in \mathcal{K}$ for any $\alpha \geq 0$ and for any $x \in \mathcal{K}$,
2. $x, -x \in \mathcal{K}$ implies $x = 0$.

For a given $y \in X$, we consider the translate of a cone $\mathcal{K}$, namely

$$\mathcal{K}_y = \mathcal{K} + y = \{x + y \ : x \in \mathcal{K}\}.$$

Given an open bounded subset D of X we denote $D_{\mathcal{K}_y} = D \cap \mathcal{K}_y$, an open subset of $\mathcal{K}_y$.

Theorem 2.3. [5, Corollary 2.4] *Let $(X, \| \ \|)$ be a real Banach space, $\mathcal{K} \subset X$ be a cone, and $D \subset X$ be an open bounded set with $y \in D_{\mathcal{K}_y}$ and $\overline{D}_{\mathcal{K}_y} \neq \mathcal{K}_y$. Assume that $F : \overline{D}_{\mathcal{K}_y} \to \mathcal{K}$ is a completely continuous map and assume that*

$$\inf_{x \in \partial D_{\mathcal{K}_y}} \|Fx\| > 0.$$

Then there exists $x^* \in \partial D_{\mathcal{K}_y}$ and $\lambda^* \in (0, \infty)$ such that

$$x^* = y + \lambda^* F(x^*).$$

Let

$$y_1(t) = \left(a_1 + \frac{a_4}{6}\right)(1-t) + a_2 t + \frac{a_3}{6}(1-t)^3 + \frac{a_4}{6}(t^3 - 1) + \frac{a_3}{6}(t-1), \quad t \in [0,1].$$

We have

$$0 \leq y_1(t) \leq a_1 + a_2 + a_3 + a_4, \quad t \in [0,1],$$

and

$$\begin{aligned} y_1'(t) &= -a_1 - \frac{a_4}{6} + a_2 - \frac{1}{2}a_3(1-t)^2 + \frac{1}{2}a_4 t^2 + \frac{a_3}{6}, \quad t \in [0,1]. \\ y_1''(t) &= a_3(1-t) + a_4 t, \quad t \in [0,1]. \end{aligned}$$

Hence,

$$y_1(0) = a_1, \quad y_1(1) = a_2, \quad y_1''(0) = a_3, \quad y''(1) = a_4.$$

Set

$$y(t) = -y_1(t), \quad t \in [0,1].$$

Now, consider the BVP

$$\begin{aligned} u^{(4)} &= \lambda g(t) f(u(t) - y(t)), \quad 0 < t < 1, \\ u(0) &= u(1) = u''(0) = u''(1) = 0, \end{aligned} \tag{2.1}$$

where f and g satisfy $(H1)$ and $(H2)$, respectively.

Let $X = \mathcal{C}([0,1])$ be endowed with the norm $\|u\| = \max_{t\in[0,1]} |u(t)|$. Define

$$\mathcal{K} = \{u \in X : u(t) \geq 0, \quad t \in [0,1]\}.$$

Since $0 \leq \int_0^1 s(1-s)g(s)ds < \infty$, there exists a nonnegative constant C_0 such that

$$\int_0^1 s(1-s)g(s)ds = C_0.$$

Because $g \not\equiv 0$ on $(0,1)$, there are $C_1 > 0$, $s_0 \in (0,1)$ and $\epsilon > 0$ such that $s_0-\epsilon, s_0+\epsilon \in (0,1)$ and

$$g(s) \geq C_1, \;\; s \in (s_0-\epsilon, s_0+\epsilon).$$

Define

$$G(t,s) = \begin{cases} t(1-s)\frac{2s-s^2-t^2}{6}, & 0 \leq t \leq s \leq 1, \\ s(1-t)\frac{2t-t^2-s^2}{6}, & 0 \leq s \leq t \leq 1. \end{cases}$$

We have

$$0 \leq G(t,s) \leq \frac{1}{6}s(1-s) \leq \frac{1}{6}, \quad 0 \leq t, s \leq 1,$$

Note that

$$\begin{aligned} \int_0^1 G(s_0+\epsilon, s)g(s)ds &\geq \int_{s_0-\epsilon}^{s_0+\epsilon} G(s_0+\epsilon, s)g(s)ds \\ &\geq C_1 \int_{s_0-\epsilon}^{s_0+\epsilon} G(s_0+\epsilon, s)ds \\ &= C_1 \int_{s_0-\epsilon}^{s_0+\epsilon} s(1-s_0-\epsilon)\frac{2(s_0+\epsilon)-(s_0+\epsilon)^2-s^2}{6}ds \\ &\geq \frac{2}{3}C_1\epsilon(s_0-\epsilon)^2(1-s_0-\epsilon)^2 \\ &> 0. \end{aligned}$$

For $u \in X$, define the operator

$$Tu(t) = \int_0^1 G(t,s)g(s)f(u(s)-y(s))ds, \quad t \in [0,1].$$

In [7], it is proved that any fixed point $u \in X$ of the operator λT is a solution to the BVP (2.1). Fix $C_2 > a_1 + a_2 + a_3 + a_4$ arbitrarily. Define

$$D = \{u \in X : \|u\| < C_2\}.$$

We have that D is an open bounded set in X, $y \in D$ and $D_{\mathcal{K}_y} = D \cap \mathcal{K}_y \neq \mathcal{K}_y$. Note that for any $u \in \overline{D}_{\mathcal{K}_y}$, we have

$$u(t) = y(t) + z(t), \;\; t \in [0,1],$$

for some $z \in \mathcal{K}$, and so $u(t) - y(t) = z(t) \geq 0$, $t \in [0,1]$, and

$$\begin{aligned}
f(u(t)-y(t)) &\leq \left(A_2 + \sum_{j=0}^{k} B_j (u(t)-y(t))^j\right) \\
&\leq \left(A_2 + \sum_{j=0}^{k} B_j 2^j \left(|u(t)|^j + |y_1(t)|^j\right)\right) \\
&\leq \left(A_2 + \sum_{j=0}^{k} B_j 2^j \left(C_2^j + (a_1+a_2+a_3+a_4)^j\right)\right), \quad t \in [0,1].
\end{aligned}$$

2.1. Proof of the main result

Since $f \in \mathcal{C}([0,\infty))$ and $g \in \mathcal{C}(0,1)$, we have that $T : D_{\mathcal{K}_y} \to \mathcal{K}$ is a continuous operator. Next, for $u \in \overline{D}_{\mathcal{K}_y}$, we have

$$\begin{aligned}
Tu(t) &= \int_0^1 G(t,s)g(s)f(u(s)-y(s))ds \\
&\leq \frac{1}{6}\left(A_2 + \sum_{j=0}^{k} B_j 2^j \left(C_2^j + (a_1+a_2+a_3+a_4)^j\right)\right) \int_0^1 s(1-s)g(s)\,ds \\
&= \frac{1}{6}C_0 \left(A_2 + \sum_{j=0}^{k} B_j 2^j \left(C_2^j + (a_1+a_2+a_3+a_4)^j\right)\right), \quad t \in [0,1],
\end{aligned}$$

whereupon

$$\|Tu\| \leq C_0 \left(A_2 + \sum_{j=0}^{k} B_j 2^j \left(C_2^j + (a_1+a_2+a_3+a_4)^j\right)\right).$$

Then, $T(\overline{D}_{\mathcal{K}_y})$ is uniformly bounded. Moreover, for $u \in \overline{D}_{\mathcal{K}_y}$ and $t_1, t_2 \in [0,1], t_1 < t_2$, the Lebesgue dominated convergence theorem guarantees that

$$\begin{aligned}
&|Tu(t_1) - Tu(t_2)| \\
&\leq \int_0^1 |G(t_1,s) - G(t_2,s)| g(s) f(u(s)-y(s)) ds\, ds \\
&\leq \left(A_2 + \sum_{j=0}^{k} B_j 2^j \left(C_2^j + (a_1+a_2+a_3+a_4)^j\right)\right) \int_0^1 g(s)|G(t_1,s) - G(t_2,s)| ds \\
&\to 0, \quad t_1 \to t_2,
\end{aligned}$$

Therefore, $T(\overline{D}_{\mathcal{K}_y})$ is equicontinuous. According to the Arzelà-Ascoli compactness criterion, we conclude that the operator $T : \overline{D}_{\mathcal{K}_y} \to \mathcal{K}$ is completely continuous.

Observe that, for $u \in \partial D_{\mathcal{K}_y}$,

$$\begin{aligned}\max_{t\in[0,1]} |Tu(t)| \geq Tu(s_0+\epsilon) &= \int_0^1 G(s_0+\epsilon, s)g(s)f(u(s)-y(s))ds \\ &\geq A_1 \int_0^1 G(s_0+\epsilon, s)g(s)ds \\ &\geq \frac{2}{3}A_1C_1\epsilon(s_0-\epsilon)^2(1-s_0-\epsilon)^2 \\ &> 0.\end{aligned}$$

Consequently

$$\inf_{u\in\partial D_{\mathcal{K}_y}} \|Tu\| \geq \frac{2}{3}A_1C_1\epsilon(s_0-\epsilon)^2(1-s_0-\epsilon)^2 > 0.$$

Now, applying Theorem 2.3, we conclude that there are $\lambda^* \in (0,\infty)$ and $u^* \in \partial D_{\mathcal{K}_y}$ such that

$$u^*(t) = y(t) + \lambda^* \int_0^1 G(t,s)g(s)f(u^*(s)-y(s))ds, \quad t \in [0,1].$$

Let

$$v^*(t) = u^*(t) - y(t), \quad t \in [0,1].$$

Then

$$\begin{aligned} v^*(0) &= u^*(0) - y(0) = a_1, \\ v^*(1) &= u^*(1) - y(1) = a_2, \\ v^{*\prime\prime}(0) &= u^{*\prime\prime}(0) - y''(0) = a_3, \\ v^{*\prime\prime}(1) &= u^{*\prime\prime}(1) - y''(1) = a_4 \end{aligned}$$

and

$$v^*(t) = \lambda \int_0^1 G(t,s)g(s)f(v^*(s))ds, \quad t \in [0,1],$$

whereupon

$$v^{*(4)}(t) = \lambda g(t) f(v^*(t)), \quad 0 < t < 1.$$

Since $u^* \in \partial D_{\mathcal{K}_y}$, we have that $u^*(t) = y(t) + z^*(t)$, $t \in [0,1]$, for some $z^* \in \mathcal{K}$, and then

$$v^*(t) = u^*(t) - y(t) = z^*(t) + y(t) - y(t) = z^*(t) \geq 0, \quad t \in [0,1].$$

3. An example

Consider the BVP

$$\begin{aligned} u^{(4)} &= \lambda \tfrac{\left(\frac{1}{2}-t\right)^2}{t(1-t)}\left(1+\tfrac{1}{1+(u(t))^2}\right), \quad t\in(0,1), \\ u(0) &= 0, \quad u(1)=1, \quad u''(0)=\tfrac{1}{2}, \quad u''(1)=1. \end{aligned} \tag{3.1}$$

Here

$$f(x) = 1+\frac{1}{1+x^2}, \quad x\in[0,\infty), \quad g(t) = \frac{\left(\frac{1}{2}-t\right)^2}{t\,(1-t)}, \quad t\in(0,1),$$

and

$$a_0 = 0, \quad a_1 = 1, \quad a_2 = \frac{1}{2}, \quad a_3 = 1.$$

By our main result, it follows that the BVP (3.1) has at least one nonnegative solution.

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