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Existence results for some anisotropic possible singular problems via the sub-supersolution method

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Abstract. Using the sub-super solution method, we prove the existence of the solutions for the following anisotropic problem with singularity:

$$\begin{cases} -\sum_{i=1}^{N} \partial_i \left(|\partial_i u|^{p_i-2} \partial_i u \right) = f(x,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary and a given singular nonlinearity $f : \Omega \times (0, \infty) \longrightarrow [0, \infty)$.

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1. Introduction

Partial differential equations with anisotropic operators appear in several scientific domains, in physics for example, such kind of operators models the dynamics of liquids with different conductivities in different directions. Furthermore, in biology for example, such type of operators are related to model describing the spread of epidemics in heterogeneous environments. Regarding the mentioned examples, we point

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out the references [14, 18, 23, 24].
Problems involving anisotropic operators $\vec{p}$-Laplacian

$$-\Delta_{\vec{p}}\, u = -\sum_{i=1}^{N} \partial_i \left(|\partial_i u|^{p_i - 2}\, \partial_i u \right), \tag{1.1}$$

are extensively studied in the literature and we cite them as examples [1, 3, 6, 7, 11]. We note that the operator (1.1) becomes the Laplacian operator in the case of $p_i = 2$ and the p-Laplacian operator that is $\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right)$ in the case of $p_i = p$ for all i. There are many studies on Laplacian and p-Laplacian problems with singularity in the second member, we refer to [19, 4, 22, 16, 25]. There is now a substantial body of work and growing interest in singular problems involving anisotropic operators, some recent results can be found in [2, 20, 17, 14].

In this paper, we study the following anisotropic problem with singularity:

$$\begin{cases} -\sum\limits_{i=1}^{N} \partial_i \left(|\partial_i u|^{p_i - 2} \partial_i u \right) = f(x, u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N (N \geq 3)$ be a bounded domain with smooth boundary and $f : \Omega \times (0, \infty) \to [0, \infty)$ is a continuous function such that $f(., t)$ is in $C^\theta(\Omega)$ with $0 < \theta < 1$. Without loss of generality, we assume that $p_1 \leq ... \leq p_N$.

Against several works that used the approximation methods, we focuse in this work on singular problems which have applications in anisotropic operator using the sub and supersolution method. More precisely, we generalize the existence results existing in [21] through replacing the p-Laplacian operator by the anisotropic one. Moreover, we have weakened conditions given on f. In other part, this work generalise the second member existing in [20, 17] with keeping the same anisotropic operator.

The natural functional space relevant to the problem (1.2) is the anisotropic Sobolev spaces

$$W^{1,\vec{p}}(\Omega) = \left\{ v \in W^{1,1}(\Omega); \partial_i v \in L^{p_i}(\Omega) \right\},$$

and

$$W_0^{1,\vec{p}}(\Omega) = W^{1,\vec{p}}(\Omega) \cap W_0^{1,1}(\Omega),$$

endowed by the usual norm

$$\|v\|_{W_0^{1,\vec{p}}(\Omega)} = \sum_{i=1}^{N} \|\partial_i v\|_{L^{p_i}(\Omega)}.$$

Where ∂u_i denotes the $i-$ th weak partial derivative of u.
In the following, we assume that $\overline{p} < N$, with

$$\frac{1}{\overline{p}} = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{p_i} \quad , \qquad \sum_{i=1}^{N} \frac{1}{p_i} > 1,$$

$$\overline{p}^* = \frac{\overline{p}N}{N-\overline{p}} \quad \text{and} \quad p_\infty = \max\{\overline{p}^*, p_N\}.$$

Then for every $r \in [1, p_\infty]$ the embedding

$$W_0^{1,\vec{p}}(\Omega) \subset L^r(\Omega),$$

is continuous, and compact if $r < p_\infty$. We refer to see [13].
Owing to the absence of a strong maximum principle, we will usually assume that $p_i \geq 2$ for all i.

Definition 1.1. We will say that $u \in W_0^{1,\vec{p}}(\Omega)$ is a solution to (1.2) if and only if, the following equality holds:

$$\sum_{i=1}^{N} \int_\Omega |\partial_i u|^{p_i-2}\, \partial_i u \partial_i \varphi \, dx \; = \; \int_\Omega f(x,u)\varphi \, dx \, , \tag{1.3}$$

for all $\varphi \in W_0^{1,\vec{p}}(\Omega)$.

Now, we are in a position to present our first results. For this, let g be a continuous positive function on $(0,\infty)$. Assume that f and g satisfy the following conditions

(G) $g(0^+) = \lim_{t\to 0^+} g(t) = +\infty$.

(H_0) $\varsigma_\mu(x) = \sup_{t\geq\mu} f(x,t) \in L^r(\Omega)$ for each $\mu > 0$ with $r > \dfrac{N}{\overline{p}}$.

(H_1) There exist two measurable nontrivial functions β, γ and a positive constant λ such that

$$\beta(x) \leq f(x,s) \leqslant \gamma(x) g(s) \text{ for every } \; 0 < s < \lambda, \;\; \text{a.e. } \; x \in \Omega,$$

with $0 \leq \beta(x) \leq \gamma(x)$ a.e. $x \in \Omega$, $\gamma \in L^r(\Omega)$, $r > \dfrac{N}{\overline{p}}$.

Theorem 1.2. *If* $(H_0) - (H_1)$, (G) *hold and* g *is non-increasing, then problem* (1.2) *has a solution in* $W_0^{1,\vec{p}}(\Omega)$.

Theorem 1.3. *If* $(H_0) - (H_1)$, (G) *hold and* g *satisfies the following condition*

$$\limsup_{t\longrightarrow 0^+} tg(t) < +\infty,$$

then problem (1.2) *has a solution in* $W_0^{1,\vec{p}}(\Omega)$.

Remark 1.4. Consider $g(s) = \frac{1}{s^\alpha ln^\beta(s+1)}$, with $0 < \alpha < 1$ and $\beta \geq 1-\alpha$. The function g satisfies the conditions of Theorem 1.2, however g doesn't verify the condition (3) of $(G2)$ of Theorem3.1 in [21].
Also, the function g given by $g(t) = \frac{1}{t^\theta}$ satisfies the conditions of Theorem 1.2 for each $\theta > 0$, but the same function g verifies the condition (3) of $(G2)$ of Theorem [21] for only $\theta > 1$.

This paper is organized as follows: in section 2, we recall some necessary definitions of the classical anisotropic operator, also we mention a technical Lemma and we prove it. In section 3, by using comparison principle and sub-supersolution method, we give the proofs of our results.

2. Preliminaries

Consider the following anisotropic problem:

$$\begin{cases} -\sum_{i=1}^{N} \partial_i \left(|\partial_i u|^{p_i - 2} \partial_i u \right) = f(x,u) & \text{in } \Omega, \\ u = \tau & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where τ in $W^{1,\vec{p}}(\Omega)$.

Definition 2.1. Let $u \in W^{1,\vec{p}}(\Omega)$ such that $u - \tau \in W_0^{1,\vec{p}}(\Omega)$, u is a solution of (2.1) if and only if for every $\varphi \in W_0^{1,\vec{p}}(\Omega)$

$$\int_\Omega \left(\sum_{i=1}^{N} |\partial_i u|^{p_i - 2} \partial_i u \partial_i \varphi \; - \; f(x,u)\varphi \right) dx \; = \; 0\,. \tag{2.2}$$

Definition 2.2. Let $(\underline{u}, \overline{u}) \in W^{1,\vec{p}}(\Omega) \times W^{1,\vec{p}}(\Omega)$,
$\underline{u}$ is called a subsolution of the problem (2.1), if

$$\int_\Omega \sum_{i=1}^{N} |\partial_i \underline{u}|^{p_i - 2} \partial_i \underline{u} \partial_i \varphi \, dx \leq \int_\Omega f(x, \underline{u})\varphi \, dx \quad \text{and} \quad (\underline{u} - \tau)^+ \in W_0^{1,\vec{p}}(\Omega),$$

$\overline{u}$ is said a supersolution of the problem (2.1), if

$$\int_\Omega \sum_{i=1}^{N} |\partial_i \overline{u}|^{p_i - 2} \partial_i \overline{u} \partial_i \varphi \, dx \geq \int_\Omega f(x, \overline{u})\varphi \, dx \quad \text{and} \quad (\overline{u} - \tau)^- \in W_0^{1,\vec{p}}(\Omega),$$

for all functions $0 \leq \varphi \in W_0^{1,\vec{p}}(\Omega)$.

Now, we need to proved the following lemma.

Lemma 2.3. *Let f satisfies (H_0) and $\tau \in W^{1,\overrightarrow{p}}(\Omega)$ with $\tau > 0$ in Ω. Let ϕ_{sub} and ϕ_{super} be sub-solution and super-solution of* (2.1) *respectively with $\phi_{super} > \phi_{sub}$ a.e. in Ω.*
If $0 < \mu < \phi_{sub}$ a.e. in Ω, where μ is a constant, then the problem (2.1) *has at least one positive solution $u \in W^{1,\overrightarrow{p}}(\Omega)$ such that $\phi_{sub} < u < \phi_{super}$ a.e. in Ω.*

Proof. Let $T : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ be defined by

$$T(x,t) := \begin{cases} f(x,\mu) & \text{if } t < \mu, \\ f(x,t) & \text{if } t \geqslant \mu. \end{cases}$$

We will consider the following problem

$$\begin{cases} -\sum\limits_{i=1}^{N} \partial_i \left(|\partial_i u|^{p_i - 2} \partial_i u \right) = T(x,u) & \text{in} \quad \Omega, \\ u = \tau & \text{on} \quad \partial\Omega. \end{cases} \tag{2.3}$$

It is easy to see that ϕ_{sub} and ϕ_{super} are sub and super-solution respectively of this problem. Since $T(x,.)$ is Hölder continuous in $\mathbb{R}$ for each $x \in \Omega$, $|T(x,t)| \leq \varsigma_\mu(x)$ in $\Omega \times \mathbb{R}$ and $\varsigma_\mu \in L^r(\Omega)$ with $r > \frac{N}{\overline{p}}$, then by [[5], Theorem 4.14] the problem (2.3) has a solution $u \in W^{1,\overrightarrow{p}}(\Omega)$ such that $\phi_{sub} \leq u \leq \phi_{super}$, a.e. in Ω. Since $\mu < \phi_{sub}$ a.e. in Ω, then $T(x,u) = f(x,u)$ a.e. in Ω. Finally, we note that u is a solution of (2.1) as claimed. □

3. Proof of the main results

Proof of Theorem 1.2. Let ϕ be a solution of the following problem

$$\begin{cases} -\sum_{i=1}^{N} \partial_i \left(|\partial_i u|^{p_i-2} \partial_i u \right) = \gamma(x) & \text{in} \quad \Omega, \\ u = 1 & \text{on} \quad \partial\Omega. \end{cases} \tag{3.1}$$

As $\gamma \in L^r(\Omega)$ with $r \geq \frac{N}{\overline{p}}$, then according to [[6], Theorem 2.1], we have $\phi \in W^{1,\vec{p}}(\Omega) \cap L^\infty(\Omega)$. Using comparison lemma in [[10], Lemma 2.5], we get $\phi \geq 1$ a.e. in Ω. We can assume without loss of generality that $\phi < \lambda$ a.e. in Ω. If not, we replace λ by $\lambda^* = \max\{\lambda\ ,\ \|\phi\|_{L^\infty(\Omega)} + 1\}$.
From (H_1) and as $\phi \geq 1$ a.e. in Ω, then

$$\begin{aligned} \int_\Omega f(x,\phi)\varphi &\leq \int_\Omega \gamma(x) g(\phi)\varphi \\ &= \int_{\{\phi\geq 1\}} \gamma(x) g(\phi)\varphi \\ &\leq \int_{\{\phi\geq 1\}} \gamma(x) g(1)\varphi. \end{aligned}$$

Without lost of generality, by replacing γ by $g(1)\gamma$ and g by $\frac{g}{g(1)}$, we deduce that

$$\int_\Omega f(x,\phi)\varphi \leq \int_\Omega \gamma(x)\varphi. \tag{3.2}$$

Let $k \in \mathbb{N}^*$, we consider the following problem

$$(P_k) \qquad \begin{cases} -\sum_{i=1}^{N} \partial_i \left(|\partial_i u|^{p_i-2} \partial_i u \right) = f(x,u) & \text{in} \quad \Omega, \\ u = \frac{1}{k} & \text{on} \quad \partial\Omega. \end{cases}$$

From the inequality (3.2) and the condition (H_0), we obtain

$$\begin{aligned} &\int_\Omega \sum_{i=1}^{N} |\partial_i \phi|^{p_i-2}\, \partial_i\phi \partial_i\varphi\, dx - \int_\Omega f(x,\phi)\varphi dx \\ &\qquad \geq \int_\Omega \sum_{i=1}^{N} |\partial_i \phi|^{p_i-2}\, \partial_i\phi \partial_i\varphi\, dx - \int_\Omega \gamma\varphi dx = 0, \end{aligned}$$

for all positive function $\varphi \in W_0^{1,\vec{p}}(\Omega)$ and $(\phi - \frac{1}{k})^- \in W_0^{1,\vec{p}}(\Omega)$. Thus, ϕ is a super-solution of the problem (P_k) in Ω for all $k = 1, 2, ...$

Take ϕ_k be the solution of

$$\begin{cases} -\sum\limits_{i=1}^{N} \partial_i \left(|\partial_i u|^{p_i - 2} \partial_i u \right) = \beta_k(x) & \text{in } \Omega, \\ u = 1/k & \text{on } \partial\Omega, \end{cases} \tag{3.3}$$

for $k = 1, 2, ...,$ where $\beta_k(x) = \min\{\beta(x)\frac{k+1}{k}\}$, for $x \in \Omega$.
Let ϕ_∞ the solution of (3.3) when $k = \infty$ and $\beta_\infty(x) = \min\{\beta(x)\}$. As $\beta_k \in L^r(\Omega)$ with $r > \frac{N}{P}$, it follows that $\phi_k \in L^\infty(\Omega)$ (see [[6], Theorem 2.1]). By the comparison lemma in [[10], Lemma 2.5], we have

$$0 \le \phi_\infty \le \phi_k \le \phi_1 \ \text{ a.e. in } \ \Omega, \ \text{ for all } \ k = 1, 2, ...$$

Moreover $\phi_k \ge k^{-1}$ a.e. in Ω for all $k = 1, 2, ...$
Since $\beta_\infty \in L^\infty(\Omega)$, $\beta_\infty \ne 0$ in Ω and $p_1 \ge 2$, using the Strong Maximum Principle see ([8], Corollary 4.4.) and ([7], Theorem 1.1), we easily see that $\phi_\infty > 0$ for all compact K in Ω.
By comparison lemma in [[10], Lemma 2.5], since $0 \le \beta \le \gamma$ a.e. x in Ω, we deduce that $\phi_k \le \phi$ for a.e. x in Ω and every $k = 1, 2, ...$
Then from the condition (H_0) and since $\phi_k \le \phi < \lambda$ a.e. in Ω for all $k = 1, 2, ...,$ we get

$$\int_\Omega \sum_{i=1}^{N} |\partial_i \phi_k|^{p_i - 2} \partial_i \phi_k \partial_i \varphi \, dx - \int_\Omega f(x, \phi_k)\varphi dx$$
$$\le \int_\Omega \sum_{i=1}^{N} |\partial_i \phi_k|^{p_i - 2} \partial_i \phi_k \partial_i \varphi \, dx - \int_\Omega \gamma \varphi dx = 0,$$

for all positive function φ in $W_0^{1,\vec{p}}(\Omega)$ and $(\phi_k - \frac{1}{k})^+ \in W_0^{1,\vec{p}}(\Omega)$. Hence ϕ_k is a sub-solution of (P_k) for all $k = 1, 2, ...$
Now let $j \in \mathbb{N}^*$, by Lemma 2.3 there exist a solution u_j of the problem (P_j) such that $\phi_j \le u_j \le \phi$ a.e. in Ω. Moreover u_j is a super-solution of (P_{j+1}), using again Lemma 2.3, there is a solution u_{j+1} of the problem (P_{j+1}) where $\phi_{j+1} \le u_{j+1} \le u_j$ a.e. in Ω. By continuing to do so, we build a sequence (u_k) of solutions of the problem (P_k) such that for every $k \ge j$ we have

$$\phi_\infty \le u_{k+1} \le u_k \le ... \le u_j \le \phi \qquad \text{a.e. in } \ \Omega.$$

We should also note that $u_k \ge k^{-1}$ a.e. in Ω. We define $u(x) = \lim\limits_{k\to\infty} u_k(x)$ a.e in Ω.
Now, as ϕ_∞ is locally Hölder continuous in Ω (see [7]) and $\phi_\infty > 0$ for all compact K in Ω, hence $\inf\limits_{supp(\phi)} \phi_\infty > 0$. Take

$$\zeta_k = \frac{u_k - k^{-1}}{g\left(\inf\limits_{supp(\phi)} \phi_\infty\right)}$$

as a test function, then in view of (H_0) and [[12], Theorem 1.3.], we distinguish two cases:

If $g\left(\inf_{supp(\phi)} \phi_\infty\right) \geq 1$, we get the following inequality

$$\begin{aligned}
\frac{\|\zeta_k\|^{p_0}_{W_0^{1,\vec{p}}(\Omega)}}{N^{p_N-1}} - N &\leq \sum_{i=1}^{N} \int_\Omega |\partial_i \zeta_k|^{p_i} dx \\
&\leq \frac{1}{g\left(\inf_{supp(\phi)} \phi_\infty\right)} \sum_{i=1}^{N} \int_\Omega |\partial_i u_k|^{p_i} dx \\
&= \int_\Omega f(x, u_k) \frac{u_k - k^{-1}}{g\left(\inf_{supp(\phi)} \phi_\infty\right)} dx \\
&\leq \int_\Omega f(x, u_k) \frac{u_k}{g\left(\inf_{supp(\phi)} \phi_\infty\right)} dx ,
\end{aligned}$$

where $p_0 = p_1$ if $\|\zeta_k\|_{W_0^{1,\vec{p}}(\Omega)} \geq 1$ and $p_0 = p_N$ if $\|\zeta_k\|_{W_0^{1,\vec{p}}(\Omega)} < 1$.
From (H_1) and since $u_k \leq \phi < \lambda$ for all $k = 1, 2, ...$, a.e. in Ω, we obtain

$$\begin{aligned}
\frac{\|\zeta_k\|^{p_0}_{W_0^{1,\vec{p}}(\Omega)}}{N^{p_N-1}} - N &\leq \int_\Omega \gamma(x) g(u_k) \frac{\phi}{g\left(\inf_{supp(\phi)} \phi_\infty\right)} dx \\
&= \int_{supp(\phi)} \gamma(x) g(u_k) \frac{\phi}{g\left(\inf_{supp(\phi)} \phi_\infty\right)} dx.
\end{aligned}$$

On the other hand as g is non-increasing, $g\left(u_k\right) \leq g(\phi_\infty)$ a.e. in Ω and $g\left(\phi_\infty\right) \leq g\left(\inf_{supp(\phi)} \phi_\infty\right)$ a.e. in $supp(\phi)$. Then according to the above equality, we find

$$\|\zeta_k\|^{p_0}_{W_0^{1,\vec{p}}(\Omega)} \leq \lambda N^{p_N-1} \|\gamma\|_{L^1(\Omega)} + N^{p_N}.$$

If $g\left(\inf_{supp(\phi)} \phi_\infty\right) < 1$, we have

$$\begin{aligned}
\frac{\|u_k - k^{-1}\|^{p_0}_{W_0^{1,\vec{p}}(\Omega)}}{N^{p_N-1}} - N &\leq \sum_{i=1}^{N} \int_\Omega \left|\partial_i \left(u_k - k^{-1}\right)\right|^{p_i} dx \\
&= \int_\Omega f(x, u_k) \left(u_k - k^{-1}\right) dx \\
&\leq \int_{supp(\phi)} \gamma(x) g(u_k) \phi dx ,
\end{aligned}$$

where $p_0 = p_1$ if $\|u_k - k^{-1}\|_{W_0^{1,\vec{p}}(\Omega)} \geq 1$ and $p_0 = p_N$ if $\|u_k - k^{-1}\|_{W_0^{1,\vec{p}}(\Omega)} < 1$.
Since $g\left(u_k\right) \leq g\left(\inf_{supp(\phi)} \phi_\infty\right) < 1$ a.e. in $supp(\phi)$ and $\phi < \lambda$ for a.e. in Ω, then we

obtain

$$\|u_k - k^{-1}\|^{p_0}_{W_0^{1,\overrightarrow{p}}(\Omega)} \leq \lambda N^{p_N-1}\|\gamma\|_{L^1(\Omega)} + N^{p_N},$$

which implies the inequality

$$\begin{aligned}\|\zeta_k\|^{p_0}_{W_0^{1,\overrightarrow{p}}(\Omega)} &= \frac{1}{g\left(\inf\limits_{supp(\phi)} \phi_\infty\right)^{p_0}}\|u_k - k^{-1}\|^{p_0}_{W_0^{1,\overrightarrow{p}}(\Omega)} \\ &\leq \frac{1}{g\left(\inf\limits_{supp(\phi)} \phi_\infty\right)^{p_0}}\left(\lambda N^{p_N-1}\|\gamma\|_{L^1(\Omega)} + N^{p_N}\right)\end{aligned}$$

and thus

$$\|\zeta_k\|^{p_0}_{W_0^{1,\overrightarrow{p}}(\Omega)} \leq \frac{\lambda N^{p_N-1}\|\gamma\|_{L^1(\Omega)} + N^{p_N}}{g\left(\inf\limits_{supp(\phi)} \phi_\infty\right)^{p_0}}.$$

Finally, we conclude that $\zeta_k \in W_0^{1,\overrightarrow{p}}(\Omega) \cap L^\infty(\Omega)$ for every k.
Since (ζ_k) is bounded in $W_0^{1,\overrightarrow{p}}(\Omega)$, it follows that $\zeta_k \rightharpoonup v$ in $W_0^{1,\overrightarrow{p}}(\Omega)$ and (ζ_k) converge weakly to the same limit in $W^{1,\overrightarrow{p}}(\Omega)$. As (u_k) is bounded in $W^{1,\overrightarrow{p}}(\Omega)$, we have $u_k \rightharpoonup u$ in $W^{1,\overrightarrow{p}}(\Omega)$, strongly in $L^p(\Omega)$ and almost everywhere in Ω.
In other part, we have $u_k = g\left(\inf\limits_{supp(\phi)} \phi_\infty\right)\zeta_k + k^{-1} \rightharpoonup g\left(\inf\limits_{supp(\phi)} \phi_\infty\right) v$ in $W^{1,\overrightarrow{p}}(\Omega)$, strongly in $L^p(\Omega)$ and almost everywhere in Ω. Therefore, we can conclude that $u = g\left(\inf\limits_{supp(\phi)} \phi_\infty\right) v$ almost everywhere in Ω, we easily see that $v \in W_0^{1,\overrightarrow{p}}(\Omega)$ which implies that $u \in W_0^{1,\overrightarrow{p}}(\Omega)$.

Let Ω_0 be a compact domain in Ω. We define $\mu = \min\limits_{\Omega_0} \phi_\infty$, from ([7], Theorem 1.1), $\phi_\infty > 0$ a.e. in Ω, we have $\mu > 0$. Hence

$$|(f(x,u_k) - f(x,u_j))(u_k - u_j)| \leqslant 4\varsigma_\mu(x)\phi,$$

which implies that

$$\sum_{i=1}^N \int_{\Omega_0} \left(|\partial_i u_k|^{p_i-2}\partial_i u_k - |\partial_i u_j|^{p_i-2}\partial_i u_j\right)\partial_i(u_k - u_j)\,dx \to 0 \tag{3.4}$$

as $k, j \to \infty$. From ([15], Proposition 1.) and (3.4), we get

$$\sum_{i=1}^N \int_{\Omega_0} |\partial_i u_k - \partial_i u_j|^{p_i} dx \to 0, \quad k, j \to \infty. \tag{3.5}$$

We observe that

$$u_k \longrightarrow u \quad \text{in} \quad L^{p_i}(\Omega_0). \tag{3.6}$$

From (3.5), (3.6), we obtain that (u_k) is Cauchy sequence in $W^{1,\overrightarrow{p}}(\Omega_0)$ which is a Banach space, therefore $u_k \longrightarrow u$ in $W^{1,\overrightarrow{p}}(\Omega_0)$. We conclude that for any compact

set Ω_0 in Ω, there exist a subsequence (u_k) such that $u_k \longrightarrow u$ in $W^{1,\overrightarrow{p}}(\Omega_0)$.
We mention the following estimates. We have for all $p_i \geq 2$ with $i \in \{1, 2, ..., N\}$

$$\begin{aligned} \| (|\partial_i u_k| + |\partial_i u|)^{\frac{(p_i-2)p_i}{p_i-1}} \|_{L^{p_i-1/(p_i-2)}(\Omega_0)} &= \left(\int_{\Omega_0} (|\partial_i u_k| + |\partial_i u|)^{p_i} \, dx \right)^{p_i-2/(p_i-1)} \\ &\leq 2^{p_i-2} \left(\int_{\Omega_0} |\partial_i u_k|^{p_i} + |\partial_i u|^{p_i} dx \right)^{p_i-2/(p_i-1)} \\ &\leq 2^{p_i-2} M, \end{aligned} \tag{3.7}$$

where M is a positive constant independent of x. Using Hölders inequality, we get

$$\int_{\Omega_0} (|\partial_i u_k| + |\partial_i u|)^{(p_i-2)p_i'} \, dx \leq \| (|\partial_i u_k| + |\partial_i u|)^{\frac{(p_i-2)p_i}{p_i-1}} \|_{L^{p_i-1/(p_i-2)}(\Omega_0)} (|\Omega_0|^{p_i-1}). \tag{3.8}$$

By the inequality (3.7), we have

$$\int_{\Omega_0} (|\partial_i u_k| + |\partial_i u|)^{(p_i-2)p_i'} \, dx \leq 2^{p_i-2} M |\Omega_0|^{p_i-1}. \tag{3.9}$$

Using again Hölders inequality, we obtain

$$\begin{aligned} &\sum_{i=1}^{N} \int_{\Omega_0} |\partial_i u_k - \partial_i u| \, (|\partial_i u_k| + |\partial_i u|)^{p_i-2} \, dx \\ &\qquad \leq \sum_{i=1}^{N} \|\partial_i u_k - \partial_i u\|_{L^{p_i}(\Omega_0)} \| \, (|\partial_i u_k| + |\partial_i u|)^{p_i-2} \, \|_{L^{p_i'}(\Omega_0)}, \end{aligned}$$

from the inequality (3.9), we deduce that

$$\begin{aligned} &\sum_{i=1}^{N} \int_{\Omega_0} |\partial_i u_k - \partial_i u| \, (|\partial_i u_k| + |\partial_i u|)^{p_i-2} \, dx \\ &\qquad \leq \; M 2^{p_N-2} \, (|\Omega_0| + 1)^{p_N-1} \sum_{i=1}^{N} \|\partial_i u_k - \partial_i u\|_{L^{p_i}(\Omega_0)} \\ &\qquad \leq \; M 2^{p_N-2} \, (|\Omega_0| + 1)^{p_N-1} \, \|u_k - u\|_{W^{1,\overrightarrow{p}}(\Omega_0)}. \end{aligned} \tag{3.10}$$

Now, we recall the fallowing useful inequality (see [9]) that hold for all a, b in $\mathbb{R}^N$ and $p_i \geq 2$ for all $i = 1, 2, ..., N$

$$||a|^{p_i-2}a - |b|^{p_i-2}b| \leq c(|a| + |b|)^{p_i-2}|a - b|, \tag{3.11}$$

where c is a positive constant independent of a and b. By estimation (3.10) and inequality (3.11), it follows that

$$\lim_{k \to +\infty} \sum_{i=1}^{N} \int_{\Omega_0} ||\partial_i u_k|^{p_i-2} \partial_i u_k - |\partial_i u|^{p_i-2} \partial_i u| dx = 0 \, . \tag{3.12}$$

Let $\xi \in C_0^\infty(\Omega)$ such that supp $(\xi) \subseteq \Omega_0 \subset \Omega$. From the limite (3.12), we conclude that

$$\sum_{i=1}^{N} \int_\Omega |\partial_i u_k|^{p_i-2}\partial_i u_k \partial_i \xi \; dx \longrightarrow \sum_{i=1}^{N} \int_\Omega |\partial_i u|^{p_i-2}\partial_i u \partial_i \xi \; dx \qquad \text{as } k \longrightarrow +\infty. \quad (3.13)$$

On the other hand, since $|f(x,u_k)\xi| \leq C\varsigma_\mu(x)$ a.e. in Ω_0, where C is a positive constant independent of x and $\varsigma_\mu \in L^1(\Omega)$, we obtain

$$\int_\Omega f\left(x,u_k\right)\xi \; dx \to \int_\Omega f(x,u)\xi \; dx. \quad (3.14)$$

Hence by (3.13) and (3.14), we conclude that for all $\xi \in C_0^\infty(\Omega)$

$$\sum_{i=1}^{N} \int_\Omega |\partial_i u|^{p_i-2}\partial_i u \partial_i \xi \; dx \;=\; \int_\Omega f(x,u)\xi \; dx.$$

Consequently, the identity (1.3) holds for every ξ in $C_0^\infty(\Omega)$. Now it remains to shows that identity (1.3) is satisfied for every $\xi \in W_0^{1,\overrightarrow{p}}(\Omega)$. Let $\nu \in W_0^{1,\overrightarrow{p}}(\Omega)$, choose a sequence (η_k) of non-negative functions in $C_0^\infty(\Omega)$ such that

$$\eta_k \to |\nu| \;\text{ in } W_0^{1,\overrightarrow{p}}(\Omega).$$

For subsequence if necessary, we can suppose that $\eta_k \to |\nu|$ a.e. in Ω, then through the Fatou's lemma and Hölder's inequality, we have

$$\begin{aligned}
\left|\int_\Omega f(x,u)\nu\right| \leq \int_\Omega f(x,u)|\nu| &\leq \liminf_{k\to\infty} \int_\Omega f(x,u)\eta_k \\
&= \liminf_{k\to\infty} \sum_{i=1}^{N} \int_\Omega |\partial_i u|^{p_i-2}\partial_i u \partial_i \eta_k \\
&\leq \liminf_{k\to\infty} \sum_{i=1}^{N} \left\| |\partial_i u|^{p_i-2}\partial_i u \right\|_{L^{p_i'}(\Omega)} \|\partial_i \eta_k\|_{L^{p_i}(\Omega)} \\
&\leq \liminf_{k\to\infty} \sum_{i=1}^{N} \|\partial_i u\|_{L^{p_i}(\Omega)}^{p_i-1} \|\partial_i \eta_k\|_{L^{p_i}(\Omega)} \\
&\leq \|u\|_{W_0^{1,\overrightarrow{p}}(\Omega)}^{q-1} \liminf_{k\to\infty} \sum_{i=1}^{N} \|\partial_i \eta_k\|_{L^{p_i}(\Omega)} \\
&\leq \|u\|_{W_0^{1,\overrightarrow{p}}(\Omega)}^{q-1} \liminf_{k\to\infty} \|\eta_k\|_{W_0^{1,\overrightarrow{p}}(\Omega)} \\
&\leq \|u\|_{W_0^{1,\overrightarrow{p}}(\Omega)}^{q-1} \|\nu\|_{W_0^{1,\overrightarrow{p}}(\Omega)},
\end{aligned}$$

with $q = p_1$ if $\|u\|_{W_0^{1,\overrightarrow{p}}(\Omega)} < 1$ and $q = p_N$ if $\|u\|_{W_0^{1,\overrightarrow{p}}(\Omega)} \geq 1$. Now for $\xi \in W_0^{1,\overrightarrow{p}}(\Omega)$, choosing again a sequence (ξ_k) of function in $C_0^\infty(\Omega)$ such that $\xi_k \to \xi$. By taking $\nu = \xi_k - \xi$ in the previous inequality, we get

$$\lim_{k\to\infty} \int_\Omega f(x,u)\xi_k \; dx = \int_\Omega f(x,u)\xi \; dx \; .$$

Furthermore

$$\lim_{k\to+\infty}\sum_{i=1}^{N}\int_{\Omega}|\partial_i u|^{p_i-2}\partial_i u\partial_i\xi_k\,dx \;=\; \sum_{i=1}^{N}\int_{\Omega}|\partial_i u|^{p_i-2}\partial_i u\partial_i\xi\,dx.$$

Hence (1.3) holds for every ξ in $W_0^{1,\overrightarrow{p}}(\Omega)$. Consequently $u\in W_0^{1,\overrightarrow{p}}(\Omega)$ is a solution of (1.2) such that $\phi_\infty\le u\le\phi$ a.e. in Ω. □

Proof of Theorem 1.3. From Lemma 2.3 and comparison lemma in [[10], Lemma 2.5], and by following the same steps of the proof of Theorem 1.2, we can build a sequence (u_k) of solutions of the problem (P_k) such that

$$\phi_\infty\le u_{k+1}\le u_k\le ...\le u_j\le\phi \qquad \text{a.e. in}\quad \Omega,\ \text{for } k\ge j,$$

where (P_k) is defined in the proof of Theorem 1.2. We also note that $u_k\ge k^{-1}$ a.e. in Ω. We define $u(x)=\lim_{k\to\infty}u_k(x)$ a.e in Ω.
We take $\zeta_k=u_k-k^{-1}$ as a test function. From the condition (H_0) and [[12], Theorem 1.3.], we have

$$\begin{aligned}\frac{\|\zeta_k\|^{p_0}_{W_0^{1,\overrightarrow{p}}(\Omega)}}{N^{p_N-1}}-N &\le \sum_{i=1}^{N}\int_{\Omega}|\partial_i u_k|^{p_i}dx\\ &= \int_{\Omega}f(x,u_k)\left(u_k-k^{-1}\right)dx\\ &\le \int_{\Omega}f(x,u_k)u_k\,dx\\ &\le \int_{supp(u_k)}\gamma(x)g(u_k)u_k\,dx\,,\end{aligned}\tag{3.15}$$

where $p_0=p_1$ if $\|\zeta_k\|_{W_0^{1,\overrightarrow{p}}(\Omega)}\ge 1$ and $p_0=p_N$ if $\|\zeta_k\|_{W_0^{1,\overrightarrow{p}}(\Omega)}<1$.
Since $\limsup_{t\longrightarrow 0^+}tg(t)<+\infty$, then there exist tow positive constants C and ϵ such that

$$tg(t)\leqslant C\ \text{ for all }\ 0<t<\epsilon.$$

If $0<u_k<\epsilon$, we obtain

$$\gamma(x)g(u_k)u_k\;\le\;C\gamma(x)\qquad\text{a.e. in}\quad supp(u_k).\tag{3.16}$$

If $\epsilon\le u_k\le\lambda$, as g is continuous on $(0,\infty)$, we get

$$\gamma(x)g(u_k)u_k\;\le\;\lambda M\gamma(x)\qquad\text{a.e. in}\quad supp(u_k),\tag{3.17}$$

with M is a constant positive such that $g(s)<M$ for all $\epsilon\le s\le\lambda$. By the inequality (3.16) and (3.17), we deduce

$$\gamma(x)g(u_k)u_k\;\le\;\max\{\lambda M,C\}\gamma(x)\qquad\text{a.e. in}\quad supp(u_k).\tag{3.18}$$

From the inequality (3.15), (3.18) and as $\gamma\in L^r(\Omega)$ with $r>\frac{N}{\overline{p}}$, we obtain

$$\|\zeta_k\|^{p_0}_{W_0^{1,\overrightarrow{p}}(\Omega)}<\max\{\lambda M,C\}N^{p_N-1}\|\gamma\|_{L^1(\Omega)}+N^{p_N}.$$

Thus the sequence (ζ_k) is bounded in $W_0^{1,\overrightarrow{p}}(\Omega)$.
Following the same techniques of the proof of Theorem 1.2. We prove the existence of solution
$u \in W_0^{1,\overrightarrow{p}}(\Omega)$ of the problem (1.2) such that $\phi_\infty \le u \le \phi$ a.e. in Ω. □

Remark 3.1. Note that if the conditions $(H_0)-(H_1)$, (G) are satisfied and we replace the condition of g in the Theorem 1.2 by $h(s) = sg(s)$ where $s > 0$ is nondecreasing. Then the problem (1.2) has a solution.
It suffices to show that

$$\int_\Omega f\,(x, u_k)\, u_k\, dx \; < \infty.$$

In fact

$$\int_\Omega f\,(x, u_k)\, u_k\, dx \le \int_\Omega \gamma(x) g(u_k) u_k\, dx.$$

As h is nondecreasing for all $s > 0$, it follows that

$$\begin{aligned}\int_\Omega f\,(x, u_k)\, u_k\, dx &\le \int_{supp(\phi)} \gamma(x) g(\phi)\phi\, dx \\ &\le \int_{supp(\phi)} \gamma(x) g(\|\phi\|_{L^\infty(\Omega)}) \|\phi\|_{L^\infty(\Omega)}\, dx \\ &\le g(\|\phi\|_{L^\infty(\Omega)}) \|\phi\|_{L^\infty(\Omega)} \|\gamma\|_{L^1(\Omega)} < \infty\,.\end{aligned}$$

Corollary 3.2. *Let g be a nonincreasing function from $(0,\infty)$ to $(0,\infty)$, satisfies (G). Suppose that*

$$\int_0^\lambda g(x)\, dx \; < +\infty$$

for same $\lambda > 0$. If $f(x,t) = \gamma(x)g(t)$ for some non-trivial and non-negative $\gamma \in L^r(\Omega)$ with $r > \frac{N}{\overline{p}}$, then (1.2) has a weak solution in $W_0^{1,\overrightarrow{p}}(\Omega)$.

Proof. Using the fact that $f(x,t) = \gamma(x)g(t)$ and $\gamma \in L^r(\Omega)$ with $r > \frac{N}{\overline{p}}$, then conditions $(H_0) - (H_1)$ are satisfied. Hence, similar to the proof of Theorem 1.3, we can build a sequence (u_k) of solutions of the problem (P_k) such that

$$\phi_\infty \le u_{k+1} \le u_k \le ... \le u_j \le \phi \qquad \text{a.e. in} \quad \Omega, \text{ for } k \ge j.$$

In addition, since $\int_0^\lambda g(x)\, dx \; < +\infty$, then $tg(t) \le M$ for all $0 < t < \lambda$ and some positive constant M, thus

$$\gamma(x) g(u_k) u_k \; \le \; M\gamma(x) \qquad \text{a.e. in} \quad supp(u_k).$$

As in the proof of Theorem 1.3, we combine the above inequality with (3.15), we get

$$\|\zeta_k\|^{p_0}_{W_0^{1,\overrightarrow{p}}(\Omega)} < MN^{p_N-1} \|\gamma\|_{L^1(\Omega)} + N^{p_N},$$

where $\zeta_k = u_k - k^{-1}$. Thus ζ_k is bounded in $W_0^{1,\overrightarrow{p}}(\Omega)$. The proof is completed. □

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