Stud. Univ. Babeş-Bolyai Math. 69(2024), No. 4, 813–824
DOI: 10.24193/subbmath.2024.4.08

Nonlocal conditions for fractional differential equations

Abdelkader Boucherif, Fatima Dib and Naima Daoudi-Merzagui

Abstract. In this work we use the method of lower and upper solutions to develop an iterative technique, which is not necessarily monotone, and combined with a fixed point theorem to prove the existence of at least one solution of nonlinear fractional differential equations with nonlocal boundary conditions of integral type.

Mathematics Subject Classification (2010): 34A08, 34B10, 34B15.

Keywords: Fractional differential equations, nonlocal conditions, upper and lower solutions, iterative technique, fixed point.

1. Introduction

Numerous phenomena in the applied sciences can be described by fractional differential equations. In fact, several monographs and research papers have been devoted to the study of fractional differential equations and related boundary value problems. We can mention the following books [12], [16], [17], [23], [25], [32], the research papers [1, 4, 11, 13, 14] and the references therein. Following [21], Picone, in 1908, was the first to introduce nonlocal boundary conditions for linear systems of ordinary differential equations. The following survey [19] contains a great number of references on nonlocal boundary problems for second order differential equations. The analysis of elliptic boundary value problems with nonlocal conditions were initiated in the paper [5]. The nonlocal condition has been proven more appropriate and more precise in many physical problems than the classical initial condition. We refer the reader to [3, 6, 9] and the references therein for a motivation regarding nonlocal conditions. The lower and upper solutions method has been proven instrumental for

Received 07 February 2023; Accepted 31 July 2023.

proving the existence and location of solutions of boundary value problems for ordinary differential equation and partial differential equation problems of integer orders. See for example [7, 10, 15, 26]. Many people have been interested in the study of the existence of solutions to boundary value problems for fractional differential equations with nonlocal conditions, see [8], [29], [28] and the references therein.To our modest knowledge only few research articles using the lower and upper solutions method for fractional differential equations are available. See [13, 18, 24].

In this paper, we consider the following class of fractional differential equations

$$D^{\alpha}u(t) + \ f(t,u) = 0, \quad t \in (0,1)\,, \ 1 < \alpha \leq 2, \tag{1.1}$$

with a Neumann condition at the initial point and a nonlocal boundary condition of integral type at the terminal point

$$u'(0) = 0, \quad u(1) = \int_0^1 g\left(u(t)\right) dt. \tag{1.2}$$

Here D^{α} is the Caputo fractional derivative of order $\alpha \in (1,2]$, $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ satisfy conditions that will be specified later. We use the lower and upper solutions method to develop an iterative method, which is not necessarily monotone (see [22], [30]) and combined with the Schauder fixed theorem to prove the existence of at least one solution for problem (1.1) - (1.2).

The rest of this paper is organized as follows. In Section 2 we recall some basic definitions and results that are needed in the rest of the paper. In Section 3, we develop the iterative technique in order to prove our main result concerning the existence of a solution of the problem (1.1) - (1.2). Finally, we give an example to illustrate our main result.

2. Preliminaries

In this section, we recall some basic definitions, notations and few results from fractional calculus that we shall use in the remainder of the paper. Let I denote the compact real interval $[0,1]$ and let $C(I)$ denote the space of continuous functions $\omega : I \to \mathbb{R}$, equipped with the norm

$$\|\omega\|_0 = \max_{t \in I} |\omega(t)|\,.$$

$C^n(I)$, $n \in \mathbb{N}$, is the space of continuous functions $\omega : I \to \mathbb{R}$, such that $\omega^{(k)} \in C(I)$ $k = 0, 1, 2, ..., n$, equipped with the norm

$$\|\omega\|_{C^n} = \sum_{k=0}^{n} \max_{0 \leq t \leq 1} \left|\omega^{(k)}(t)\right|.$$

Definition 2.1. (see [16]) The Riemann-Liouville fractional primitive of order $\alpha > 0$ of a function $f : (0, \infty) \to \mathbb{R}$ is given by

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{0}^{t} (t-s)^{\alpha-1} f(s) ds, \tag{2.1}$$

provided that the right-hand side is pointwise defined on $(0, +\infty)$, and where Γ is the gamma function.

For instance, $I^{\alpha} f$ exists for all $\alpha > 0$, when $f \in L^1(I)$. Notice, also, that when $f \in C(I)$, then $I^{\alpha} f \in C(I)$ and moreover $I^{\alpha} f(0) = 0$. The law of composition $I^{\alpha} I^{\beta} = I^{\alpha+\beta}$ holds for all $\alpha, \beta > 0$.

Definition 2.2. (see [16]) The Caputo fractional derivative of order $\alpha > 0$ of a C^n function $f : (0, \infty) \to \mathbb{R}$ is given by

$$D^{\alpha} f(t) = I^{n-\alpha} f^{(n)}(t) = \frac{1}{\Gamma(n-\alpha)} \int_{0}^{t} \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \tag{2.2}$$

where $n = [\alpha] + 1$ and $[\alpha]$ is the integer part of α, provided that the right-hand side is pointwise defined on $(0, +\infty)$.

Notice that $D^{\alpha} c = 0$, where c is a real constant.

Remark 2.3. It is well known (see for instance [16, Lemma 2.22 page 96], [31, Lemma 3.6 page 6]) that for $\alpha > 0$

$$I^{\alpha} D^{\alpha} u(t) = u(t) + c_0 + c_1 t + ... + c_{n-1} t^{n-1}, \text{ for all } t \in I,$$

where $n = [\alpha] + 1$, and $c_0, c_1, ..., c_{n-1}$ are real constants.

Lemma 2.4. *Let $\alpha > 0$. Then the differential equation on I*

$$D^{\alpha} u(t) = 0$$

has solutions $u(t) = c_0 + c_1 t + ... + c_{n-1} t^{n-1}$, $t \in I$, $c_0, c_1, ..., c_{n-1}$ are real constants and $n = [\alpha] + 1$.

Lemma 2.5. *Let $\alpha \in (1, 2)$. Then the homogeneous problem*

$$\begin{cases} D^{\alpha} u(t) = 0, & t \in I \\ u'(0) = 0, \ u(1) = 0 \end{cases}$$

has only the trivial solution $u(t) = 0$ for all $t \in I$.

Lemma 2.6. *Let $f \in C^2(I)$. Then for any $\alpha \in (1, 2)$ $D^{\alpha} f$ exists and is continuous on I.*

Proof. It follows from (2.2) with $\alpha \in (1, 2)$ that

$$D^{\alpha} f(t) = I^{2-\alpha} f''(t) = \frac{1}{\Gamma(2-\alpha)} \int_{0}^{t} \frac{f''(\tau)}{(t-\tau)^{\alpha-1}} d\tau,$$

so that

$$|D^{\alpha} f(t)| \leq \frac{1}{\Gamma(2-\alpha)} \int_0^t \frac{|f''(\tau)|}{(t-\tau)^{\alpha-1}} d\tau \leq \frac{\|f''\|_0}{\Gamma(2-\alpha)} \int_0^t \frac{1}{(t-\tau)^{\alpha-1}} d\tau.$$

Since

$$\int_0^t \frac{1}{(t-\tau)^{\alpha-1}} d\tau = \frac{t^{2-\alpha}}{2-\alpha} \leq \frac{1}{2-\alpha}$$

we obtain

$$\|D^{\alpha} f\|_0 \leq \frac{\|f''\|_0}{(2-\alpha)\,\Gamma(2-\alpha)}.$$

To prove the continuity of $D^{\alpha} f$ on I, let $t \geq t_0 \in I$. Then

$$D^{\alpha} f(t) - D^{\alpha} f(t_0) = \frac{1}{\Gamma(2-\alpha)} \int_0^t \frac{f''(\tau)}{(t-\tau)^{\alpha-1}} d\tau - \frac{1}{\Gamma(2-\alpha)} \int_0^{t_0} \frac{f''(\tau)}{(t_0-\tau)^{\alpha-1}} d\tau$$

$$= \frac{1}{\Gamma(2-\alpha)} \int_0^{t_0} \frac{f''(\tau)}{(t-\tau)^{\alpha-1}} d\tau + \frac{1}{\Gamma(2-\alpha)} \int_{t_0}^{t} \frac{f''(\tau)}{(t-\tau)^{\alpha-1}} d\tau - \frac{1}{\Gamma(2-\alpha)} \int_0^{t_0} \frac{f''(\tau)}{(t_0-\tau)^{\alpha-1}} d\tau$$

$$= \frac{1}{\Gamma(2-\alpha)} \int_0^{t_0} f''(\tau) \left(\frac{1}{(t-\tau)^{\alpha-1}} - \frac{1}{(t_0-\tau)^{\alpha-1}} \right) d\tau + \frac{1}{\Gamma(2-\alpha)} \int_{t_0}^{t} \frac{f''(\tau)}{(t-\tau)^{\alpha-1}} d\tau.$$

Notice that $(t_0-\tau)^{\alpha-1} \leq (t-\tau)^{\alpha-1}$ so that

$$\left| \frac{1}{(t-\tau)^{\alpha-1}} - \frac{1}{(t_0-\tau)^{\alpha-1}} \right| = \frac{1}{(t_0-\tau)^{\alpha-1}} - \frac{1}{(t-\tau)^{\alpha-1}}.$$

Hence

$$|D^{\alpha} f(t) - D^{\alpha} f(t_0)| \leq \frac{\|f''\|_0}{\Gamma(2-\alpha)} \int_0^{t_0} \left(\frac{1}{(t_0-\tau)^{\alpha-1}} - \frac{1}{(t-\tau)^{\alpha-1}} \right) d\tau$$

$$+ \frac{\|f''\|_0}{\Gamma(2-\alpha)} \int_{t_0}^{t} \frac{1}{(t-\tau)^{\alpha-1}} d\tau.$$

Simple integrations give

$$\int_0^{t_0} \left(\frac{1}{(t_0-\tau)^{\alpha-1}} - \frac{1}{(t-\tau)^{\alpha-1}} \right) d\tau = \frac{1}{2-\alpha} \left((t-t_0)^{2-\alpha} + t_0^{2-\alpha} - t^{2-\alpha} \right), \tag{2.3}$$

and

$$\int_{t_0}^{t} \frac{1}{(t-\tau)^{\alpha-1}} d\tau = \frac{(t-t_0)^{2-\alpha}}{2-\alpha}.$$

Using lemma 2 in [14] we have for $t \geq t_0$ and $1 > 2-\alpha \geq 0$

$$\left|t_0^{2-\alpha} - t^{2-\alpha}\right| \leq (t-t_0)^{2-\alpha}.$$

Combining the above computations we see that

$$|D^\alpha f(t) - D^\alpha f(t_0)| \leq \frac{3\,(t-t_0)^{2-\alpha}}{(2-\alpha)\,\Gamma(2-\alpha)}.$$

If $t_0 > t$, we interchange the role of t and t_0 in the preceding computations and we arrive at the same result. Therefore

$$\lim_{t\to t_0} |D^\alpha f(t) - D^\alpha f(t_0)| = 0.$$

□

The following results play an important role in the proof of our main result.

Theorem 2.7. [2, Corollary 2.1 page 3] *Let $f \in C^2(I)$ attains its minimum over the interval I at the point $t_0 \in (0,1)$ and $f'(0) \leq 0$. Then $D^\alpha f(t_0) \geq 0$ for any $\alpha \in (1,2)$.*

Changing f to $-f$ we obtain

Theorem 2.8. *Let $f \in C^2(I)$ attains its maximum over the interval I at the point $t_0 \in (0,1)$ and $f'(0) \geq 0$. Then $D^\alpha f(t_0) \leq 0$ for any $\alpha \in (1,2)$.*

Definition 2.9. [20]. Let E and F be Banach spaces. An operator $T : E \to F$ is called a completely continuous operator if T is continuous and maps any bounded subset of E into relatively compact subset of F.

Theorem 2.10. *(Schauder fixed point theorem,* [27]*) If Ω is a closed bounded convex subset of a Banach space E and $T : \Omega \to \Omega$ is completely continuous, then T has at least one fixed point in Ω.*

We shall use the following notation. For $U, V \in C^2(I)$, $U \leq V$ means $U(t) \leq V(t)$ for all $t \in I$. Also, $[U, V] := \{v \in C^2(I);\ U \leq v \leq V\}$.

3. Main result

In this section, we shall apply the lower and upper solutions method to develop an iterative technique to prove the existence of solutions to problem (1.1) - (1.2).

Definition 3.1. We call a function $\underline{u}$ a lower solution for problem (1.1) - (1.2), if $\underline{u} \in C^2(I)$ and

$$\begin{cases} D^\alpha \underline{u}(t) + f(t, \underline{u}(t)) \geq 0, & t \in (0,1) \\ \underline{u}'(0) = 0, \quad \underline{u}(1) \leq \int\limits_0^1 g\left(\underline{u}(t)\right) dt. \end{cases}$$

Definition 3.2. We call a function $\overline{u}$ an upper solution for problem (1.1) - (1.2), if $\overline{u} \in C^2(I)$ and

$$\begin{cases} D^\alpha \overline{u}(t) + f(t, \overline{u}(t)) \leq 0, & t \in (0,1) \\ \overline{u}'(0) = 0, \quad \overline{u}(1) \geq \int\limits_0^1 g\left(\overline{u}(t)\right) dt. \end{cases}$$

Definition 3.3. A solution of (1.1) - (1.2) is a function $u \in C^2(I)$ that is both a lower solution and an upper solution of the problem.

Define a truncation operator $\tau : C^2(I) \to [\underline{u}, \overline{u}]$ by

$$\tau(y) = \max\{\underline{u}, \min(y, \overline{u})\}.$$

Then $\tau(y) = \underline{u}$ if $y \leq \underline{u}$, $\tau(y) = y$ if $y \in [\underline{u}, \overline{u}]$ and $\tau(y) = \overline{u}$ if $y \geq \overline{u}$. Moreover τ is a continuous and bounded operator. In fact, we have

$$\|\tau(u)\|_0 \leq \max\left(\|\underline{u}\|_0, \|\overline{u}\|_0\right).$$

We now provide sufficient conditions on the nonlinearities f, g that will allow us to investigate problem (1.1) - (1.2).
(**H1**) $f : I \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies $(f(t, v_1) - f(t, v_2))(v_1 - v_2) < 0$, for all $t \in I$, $v_1 > v_2$.
(**H2**) $g : \mathbb{R} \to \mathbb{R}$ is continuous and nondecreasing.

Theorem 3.4. *Assume that Problem (1.1) - (1.2) has a lower solution* $\underline{u}$*, an upper solution* $\overline{u}$ *such that* $\underline{u}(t) \leq \overline{u}(t)$*, for all* $t \in I$*, and (H1), (H2) hold. Then Problem (1.1) - (1.2) has at least one solution* $u^* \in C^2(I)$ *such that* $\underline{u}(t) \leq u^*(t) \leq \overline{u}(t), t \in I$.

Proof. The proof will be given in several steps.
Step1: Modification of the problem. Let $\phi : I \times [\underline{u}, \overline{u}] \to \mathbb{R}$ and $\psi : [\underline{u}, \overline{u}] \to \mathbb{R}$ be defined, respectively, by

$$\phi(t, u) = f(t, \tau(u)), \ \psi(u) = g(\tau(u)).$$

It is clear that ϕ, ψ are continuous and bounded. Moreover, for $v_1 > v_2$ in $[\underline{u}, \overline{u}]$, we have $\tau(v_1) = v_1$ and $\tau(v_2) = v_2$, so that $\phi(t, v_1) = f(t, v_1)$ and $\phi(t, v_2) = f(t, v_2)$ hence $(\phi(t, v_1) - \phi(t, v_2))(v_1 - v_2) < 0$ for all $t \in I$ and $\underline{u} \leq v_2 < v_1 \leq \overline{u}$. Similarly, $\psi(u) = g(u)$, for all $u \in [\underline{u}, \overline{u}]$, so that ψ is nondecreasing in $[\underline{u}, \overline{u}]$.

We consider the following modified boundary value problem

$$\begin{cases} D^\alpha u(t) + \phi(t, u(t)) = 0, & t \in (0,1), 1 < \alpha \leq 2 \\ u'(0) = 0, \quad u(1) = \int\limits_0^1 \psi(u(t))\,dt \end{cases} \tag{3.1}$$

We will show that the modified problem (3.1) has at least one solution $u^* \in [\underline{u}, \overline{u}]$. It follows that $\tau(u^*) = u^*$ so that $\phi(t, u^*) = f(t, u^*)$, $\psi(u^*) = g(u^*)$. This implies that u^* is a solution of our original problem (1.1) - (1.2).
Step2. Let $b \in \mathbb{R}$. Consider the auxiliary problem

$$\begin{cases} D^\alpha u(t) + \phi(t, u(t)) = 0, & t \in (0,1), 1 < \alpha \leq 2 \\ u'(0) = 0, \quad u(1) = b \end{cases} \tag{3.2}$$

Claim. If (H1) is satisfied then (3.2) has a unique solution.
Proof. Uniqueness. Assume (3.2) has two solutions y, z. We show that $y(t) = z(t)$ for all $t \in I$. Suppose, on the contrary, that there is $\xi \in I$ such that $y(\xi) \neq z(\xi)$. Assume, for definiteness that $y(\xi) > z(\xi)$. Let $w(t) = y(t) - z(t)$ for all $t \in I$. Then $w(\xi) > 0$. By the continuity of w on I it follows that there exists $\xi_0 \in I$ such that

$w(\xi_0) := \max_{t\in I} w(t) > 0$. Then $\xi_0 \in [0,1)$ because $w'(0) = w(1) = 0$. Theorem 2.8 implies that $D^\alpha w(\xi_0) \le 0$. Then

$$0 \ge D^\alpha w(\xi_0) w(\xi_0) = \left(D^\alpha y(\xi_0) - D^\alpha z(\xi_0)\right)\left(y(\xi_0) - z(\xi_0)\right).$$

It follows from the first equation in (3.2) and (H1) that

$$0 \ge -(\phi(\xi_0, y(\xi_0) - \phi(\xi_0, z(\xi_0))\left(y(\xi_0) - z(\xi_0)\right) > 0.$$

This contradiction shows that $y(t) \le z(t)$ for all $t \in I$. Similarly we show that $z(t) \le y(t)$ for all $t \in I$. Therefore $y(t) = z(t)$ for all $t \in I$.
Existence. It follows from Remark 2.3 that for $u \in C^2(I)$ and any $\alpha \in (1,2)$ we have

$$I^\alpha D^\alpha u(t) = u(t) - c_0 - c_1 t, \text{ for all } t \in I,$$

where c_0, c_1 are real constants. The first equation in (3.2) implies that

$$u(t) = -I^\alpha \phi(t, u(t)) + c_0 + c_1 t, \text{ for all } t \in I.$$

Simple computations lead to

$$u(t) = \int_0^1 G(t,s)\phi(s,u(s))ds + b, \tag{3.3}$$

where $G(t,s)$ is Green's function corresponding to the linear homogeneous problem. This function exists because the homogeneous problem has only the trivial solution, see Lemma 2.5. It is given by

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} (1-s)^{\alpha-1}, & 0 \le t < s \le 1 \\ (1-s)^{\alpha-1} - (t-s)^{\alpha-1}, & 0 \le s < t \le 1 \end{cases}.$$

Conversely, if $u \in C(I)$ is a solution of (3.3) then $u \in C^2(I)$ and is a solution of (3.2). Indeed, let $v(t) = \int_0^1 G(t,s)\phi(s,u(s))ds + b$. Then

$$v(t) = b - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \phi(s,u(s))ds + \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} \phi(s,u(s))ds.$$

$$v(t) = b - I^\alpha \phi(\cdot, u(\cdot))(t) + I^\alpha \phi(\cdot, u(\cdot))(1)$$

Obviously $v(1) = b$. Also

$$v'(t) = -\frac{\alpha-1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-2} \phi(s,u(s))ds,$$

so that $v'(0) = 0$. Moreover, it ias clear that $v' \in C^1(I)$, i.e. $v \in C^2(I)$. By Lemma 2.6 $D^\alpha v(t)$ exists and

$$\begin{aligned} D^\alpha v(t) &= D^\alpha\left(b - I^\alpha \phi(\cdot, u(\cdot))(t) + I^\alpha \phi(\cdot, u(\cdot))(1)\right) \\ &= -D^\alpha I^\alpha \phi(\cdot, u(\cdot))(t) = -\phi(t, u(t)). \end{aligned}$$

Hence

$$D^\alpha u(t) = -\phi(t, u(t)),$$

i.e.

$$D^\alpha u(t) + \phi(t, u(t)) = 0.$$

Now, define an operator $T : C(I) \to C(I)$ by the right-hand side of (3.3), i.e.

$$(Tu)(t) = \int_0^1 G(t,s)\phi(s,u(s))ds + b, \text{ for all } t \in I. \tag{3.4}$$

We show that T is continuous and uniformly bounded. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $C(I)$ which converges uniformly to u. Then $u \in C(I)$. It follows from the uniform continuity of $G(.,.)$ on the compact rectangle $I\times I$ there is $G_0 > 0$ such that $|G(t,s)| \leq G_0$ for all $(t,s) \in I \times I$. Also, $\phi : I \times [\underline{u}, \overline{u}] \to \mathbb{R}$ is continuous and bounded. It follows that the exists $M_\phi > 0$ such that $|\phi(t,u(t))| \leq M_\phi$ for all $t \in I$. The equation (3.4) implies that

$$\|Tu_n - Tu\|_0 \leq G_0 \int_0^1 |\phi(s,u_n(s)) - \phi(s,u(s))|\, ds \to 0 \text{ as } n \to \infty.$$

Moreover

$$\|T(u)\|_0 \leq G_0 M_\phi + |b| := \rho.$$

Let $\Omega := \{u \in C(I);\ \|u\|_0 \leq \rho\}$. Then Ω is a closed, bounded and convex subset of $C(I)$. Also, $T(\Omega) \subset \Omega$. Now, we show that $\overline{T(\Omega)}$ is a compact subset of $C(I)$. First, $T(\Omega)$ is equicontinuous. Let $(t,s) \in I \times I$ with $s \leq t$. Then for all $u \in \Omega$

$$(Tu)(t) - (Tu)(s) = \int_0^1 (G(t,\sigma) - G(s,\sigma))\, \phi(\sigma, u(\sigma))d\sigma.$$

It follows that

$$|(Tu)(t) - (Tu)(s)| \leq \int_0^1 |G(t,\sigma) - G(s,\sigma)|\, |\phi(\sigma,u(\sigma))|\, d\sigma.$$

$$|(Tu)(t) - (Tu)(s)| \leq M_\phi \int_0^1 |G(t,\sigma) - G(s,\sigma)|\, d\sigma. \tag{3.5}$$

The uniform continuity of Green's function implies that for every $\epsilon > 0$ there is $\delta_1 > 0$ such that for all $(t,s) \in I \times I$ with $|t-s| < \delta_1$ we have

$$|G(t,\sigma) - G(s,\sigma)| \leq \frac{\epsilon}{M_\phi}.$$

It follows from (3.5) that for all $(t,s) \in I \times I$ with $|t-s| < \delta_1$ we have for all $u \in \Omega$

$$|(Tu)(t) - (Tu)(s)| < \epsilon.$$

Next, we show that $\overline{T(\Omega)}$ is equicontinuous. Given $\epsilon > 0$ there is $\delta > 0$ such that for any $u \in \Omega$ and $|t-s| < \delta$ we have

$$|(Tu)(t) - (Tu)(s)| < \epsilon/3. \tag{3.6}$$

Now, let $v \in \overline{T(\Omega)}$. Then there is $u \in \Omega$ such that

$$\|v - T(u)\|_0 \leq \epsilon/3, \tag{3.7}$$

i.e.

$$|(Tu)(t) - v(t)| < \epsilon/3.$$

Hence for $|t-s|<\delta$ we have for all $v \in \overline{T(\Omega)}$

$$|v(t)-v(s)| \leq |v(t)-(Tu)(t)| + |(Tu)(t)-(Tu)(s)| + |(Tu)(s)-v(s)| < \epsilon.$$

By Ascoli-Arzela Theorem we conclude that $\overline{T(\Omega)}$ is a compact subset of $C(I)$. Hence the operator T is completely continuous. Schauder fixed point theorem (see Theorem 2.10) implies that T has a fixed point v^* in Ω, which is unique as shown earlier. So that

$$v^*(t) = T(v^*(t)) = \int_0^1 G(t,s)\phi(s,v^*(s))ds + b, \text{ for all } t \in I. \tag{3.8}$$

It follows from (3.8) that v^* is the (unique) solution of the auxiliary problem (3.2). **Step3**. We develop an iterative method to show that the modified problem has at least one solution. Define a sequence $(u_k)_{k\in\mathbb{N}}$ in the following way. Let $u_0 = \underline{u}$ and for $k \geq 1$

$$\begin{cases} D^\alpha u_k(t) + \phi(t,u_k(t)) = 0, & t \in (0,1), 1<\alpha\leq 2 \\ u_k'(0) = 0, \quad u_k(1) = \int\limits_0^1 \psi(u_{k-1}(t))\,dt \end{cases} \tag{3.9}$$

Notice that $u_k'(0)$ and $u_k(1)$ do not depend on the unknown function u_k. We see that problem (3.9) is similar to the previous auxiliary problem (3.2). Therefore, for each $k \in \mathbb{N}$, (3.9) has a unique solution $u_k \in \Omega$. This implies that the sequence $(u_k)_{k\in\mathbb{N}}$ is uniformly bounded. Hence it has convergent subsequence $(u_{kj})_{j\in\mathbb{N}}$. Observe that the subsequence $\left(u_{k_j-1}\right)_{j\in\mathbb{N}}$ may not converge to the same limit as the subsequence $(u_{kj})_{j\in\mathbb{N}}$. We use a diagonalization process to have $\lim\limits_{kj\to\infty} u_{kj} = \lim\limits_{k_j-1} u_{k_j-1} = u^*$. It follows from (3.3) that (3.9) is equivalent to

$$u_k(t) = \int_0^1 G(t,s)\phi(s,u_k(s))ds + \int\limits_0^1 \psi(u_{k-1}(t))\,dt, \text{ for all } t \in I.$$

Take limit as $kj \to \infty$, using the continuity of ϕ and ψ, we obtain

$$u^*(t) = \int_0^1 G(t,s)\phi(s,u^*(s))ds + \int\limits_0^1 \psi(u^*(t))\,dt, \text{ for all } t \in I.$$

Therefore

$$\begin{cases} D^\alpha u^*(t) + \phi(t,u^*(t)) = 0, & t \in (0,1), 1<\alpha\leq 2 \\ u^{*'}(0) = 0, \quad u^*(1) = \int\limits_0^1 \psi(u^*(t))\,dt \end{cases} \tag{3.10}$$

Step4. To complete the proof of our main result we need to prove that $\underline{u} \leq u^* \leq \overline{u}$, i.e. for all $t \in I$

$$\underline{u}(t) \leq u^*(t) \leq \overline{u}(t).$$

We only show that $\underline{u}(t) \leq u^*(t)$ for all $t \in I$. We proceed by contradiction. Assume, on the contrary that there is $t_1 \in (0,1)$ such that $\underline{u}(t_1) > u^*(t_1)$. Let $w(t) = \underline{u}(t) - u^*(t)$ for all $t \in I$. Then $w \in C(I) \cap C^2(0,1)$ and $w(t_1) > 0$. It follows that there is $t_0 \in I$ such that $w(t_0) = \max_{t\in I} w(t) > 0$. It follows from Theorem 2.8 that $D^\alpha w(t_0) \le 0$. The first equation in (3.1) and assumption (H1) imply that

$$\begin{aligned} 0 \ge D^\alpha w(t_0)\ \ w(t_0) &= \left(D^\alpha \underline{u}(t_0) - D^\alpha u^*(t_0)\right)\left(\underline{u}(t_0) - u^*(t_0)\right) \\ &= -\left(\phi(t_0, \underline{u}(t_0)) - \phi(t_0, u^*(t_0))\right)\left(\underline{u}(t_0) - u^*(t_0)\right) > 0. \end{aligned}$$

We arrive at a contradiction. Thus, $w(t) = \underline{u}(t) - u^*(t) \le 0$ for all $t \in (0,1)$. Now, if $t_0 = 0$ we have $w'(0) = 0$. If $w(0) > 0$, it follows from the continuity of w that there exists a small interval $[0,a] \subset I$ such that $w(t) > 0$ for all $t \in [0,a]$. This is not possible from the previous argument. Hence $w(0) = \underline{u}(0) - u^*(0) \le 0$. Also, if $t_0 = 1$ we have from the definition of $\underline{u}$ and u^* and the monotonicity of ψ

$$\underline{u}(1) - u^*(1) \le \int_0^1 \psi(\underline{u}(t))\,dt - \int_0^1 \psi(u^*(t))\,dt = \int_0^1 \left(\psi(\underline{u}(t)) - \psi(u^*(t))\right)dt \le 0.$$

Therefore $\underline{u}(t) \le u^*(t)$ for all $t \in I$. Similarly we show that $u^*(t) \le \overline{u}(t)$ for all $t \in I$. We infer that $\underline{u}(t) \le u^*(t) \le \overline{u}(t)$ for all $t \in I$, i.e. $u^* \in [\underline{u}, \overline{u}]$. We deduce that for all $t \in I$

$$\phi(t, u^*(t)) = f(t, u^*(t)), \text{ and } \psi(u^*(t)) = g(u^*(t)).$$

Consequently,

$$\begin{cases} D^\alpha u^*(t) + f(t, u^*(t)) = 0, \quad t \in (0,1), 1 < \alpha \le 2 \\ u'^*(0) = 0, \quad u^*(1) = \int_0^1 g(u^*(t))\,dt \end{cases}.$$

Finally, we see that u^* is the desired solution to our original problem. This completes the proof of our main result. □

Example 3.5. We consider the following boundary value problem

$$\begin{cases} D^{\frac{3}{2}} u(t) - 1 + e^{-u(t)} = 0, \ t \in (0,1) \\ u'(0) = 0, \quad u(1) = \int_0^1 \left(1 - e^{-u(t)}\right) dt. \end{cases} \tag{3.11}$$

We have $\alpha = 3/2$, $f(t,u) = -1 + e^{-u}$, and $g(u) = 1 - e^{-u}$. We see that f, g are continuous. For $u > v$ the mean value theorem implies

$$f(t,u) - f(t,v) = \left(-1 + e^{-u}\right) - \left(-1 + e^{-v}\right) = \left(e^{-u} - e^{-v}\right) = -e^{-z}(u - v),$$

where z is in the segment $[v,u]$. Hence $(f(t,u) - f(t,v))(u - v) = -e^{-z}(u - v)^2 < 0$. Hence f satisfies (H1). Also, $g'(u) = e^{-u} > 0$, so that g satisfies (H2). We see that $\underline{u}(t) = 0$ is a lower solution for problem (3.11) and $\overline{u}(t) = 1$ is an upper solution for problem (3.11). Applying Theorem 3.4, we see that the problem (3.11) has at least one solution $u^* \in C^2(I)$ with $0 \le u^*(t) \le 1$, for all $t \in I$. Notice that we have obtained the existence of a nonnegative solution.

Acknowledgments. The authors are grateful to an anonymous referee and Professor Radu Precup for comments and suggestions that led to the improvement of the presentation of the manuscript.

References

[1] Agarwal, R. P., Benchohra, M., Hamani, S., *Boundary value problems for differential inclusions with fractional order*, Adv. Stud. Contemp. Math.,**16**(2008), no. 2, 181-196.

[2] Al-Refai, M., *On the fractional derivatives at extreme points*, Electron. J. Qual. Theory Differ. Equ., (2012), no. 55, 1-5.

[3] Balachandran, K., Uchiyama, K., *Existence of solutions of nonlinear integrodifferential equations of Sobolev type with nonlocal conditions in Banach spaces*, Proc. Indian Acad. Sci. Math. Sci., **110**(2000), 225-232.

[4] Benchohra, M., Hamani, S., *Nonlinear boundary value problems for differential inclusions with Caputo fractional derivative*, Topol. Methods Nonlinear Anal., **32**(2008), 115-130.

[5] Bitsadze, A.V., Samarskii, A.A., *Some elementary generalizations of linear elliptic boundary value problems*, Dokl. Akad. Nauk SSSR, **185**(1969), no. 4, 739-740.

[6] Boucherif, A., *Nonlocal conditions for two-endpoint problems*, Int. J. Difference Eq., **15**(2020), 321-334.

[7] Boucherif, A., Bouguima, S. M., Benbouziane, Z., Al-Malki, N., *Third order problems with nonlocal conditions of integral type*, Bound. Value Probl., (2014), no. 137.

[8] Boucherif, A., Ntouyas, S.K., *Nonlocal initial value problems for first order fractional differential equations*, Dynam. Systems Appl., **20**(2011), 247-260.

[9] Byszewski, L., *Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem*, J. Math. Anal. Appl., **162**(1991), 494-505.

[10] Cabada, A., *The method of lower and upper solutions for second, third, fourth and higher order boundary value problems*, J. Math. Anal. Appl., **185**(1994), no. 2, 302-320.

[11] Chang, Y.-K. and Nieto, J.J., *Some new existence results for fractional differential inclusions with boundary conditions*, Math. Comput. Model., **49**(2009), 605-609.

[12] Diethelm, K., *The Analysis of Fractional Differential Equations*, Springer-Verlag, Berlin, 2010.

[13] Ding, Y., Wei, Z., *On the extremal solution for a nonlinear boundary value problems of fractional p-Laplacian differential equation*, Filomat, **30**(2016), no. 14, 3771-3778.

[14] Furati, K.M., Tatar, N.-E., *An existence result for nonlocal fractional differential problem*, J. Fract. Calc., **26**(2004), 43-51.

[15] Henderson, J., Thompson, H.B., *Existence of multiple solutions for second order boundary value problems*, J. Differential Equations, **166**(2000), no. 2, 443-454.

[16] Kilbas, A.A., Srivastava, H.M., Trujillo, J.J., *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, 2006.

[17] Lakshmikanthan, V., Leela, S., Devi, J.V., *Theory of Fractional Dynamic Systems*, Cambridge Scientific Publishers, 2009.

[18] Lin, L., Liu, X., Fang, H., *Method of upper and lower solutions for fractional differential equations*, Electron. J. Differential Equations, (2012), no. 100, 1-13.

[19] Ma, R., *A survey on nonlocal boundary value problems*, Appl. Math. E-Notes, **7**(2007), 257-279.

[20] Mawhin, J., *Topological degree methods in nonlinear boundary value problems*, in: NSFCBMS Regional Conference Series in Math., Amer. Math. Soc. Colloq. Publ., 1979.

[21] Mawhin, J., Szymanska-Debowska, K., *Convexity, topology and nonlinear differential systems with nonlocal boundary conditions: A survey*, Rend. Istit. Mat. Univ. Trieste, **51**(2019), 125-166.

[22] McRae, F.A., *Monotone iterative technique for periodic boundary value problems of Caputo fractional differential equations*, Commun. Appl. Anal., **14**(2010), 73-80.

[23] Miller, K.S., Ross, B., *An Introduction to the Fractional Calculus and Differential Equations*, John Wiley, 1993.

[24] Pawar, G.U., Salunke, J.N., *Upper and Lower Solution Method for Nonlinear Fractional Differential Equations Boundary Value Problem*, J. Comput. Math., **9**(2018), 164-173.

[25] Podlubny, I., *Fractional Differential Equations*, Academic Press, 1999.

[26] Rachankov, I., *Upper and lower solutions and topological degree*, J. Math. Anal. Appl., **234**(1999), no. 1, 311-327.

[27] Smart, D.R., *Fixed Point Theorems*, Cambridge University Press 1974.

[28] Wang, Y., Liang, S., Wang, Q., *Existence results for fractional differential equations with integral and multi-point boundary conditions*, Bound. Value Probl., (2018), no. 4.

[29] Wang, X., Wang, L., Zeng, Q., *Fractional differential equations with integral boundary conditions*, J. Nonlinear Sci. Appl., **8**(2015), 309-314.

[30] Xie, W., Xiao, J., Luo, Z., *Existence of extremal solutions for nonlinear fractional differential equation with nonlinear boundary conditionals*, Appl. Math. Lett., **41**(2015), 46-51.

[31] Zhang, S., *Positive solutions for boundary value problems of nonlinear fractional differential*, Electron. J. Differential Equations, (2006), no. 36, 1-12.

[32] Zhou, Y., *Basic Theory of Fractional Differential Equations*, World Scientific, Singapore, 2014.

Abdelkader Boucherif
University of Tlemcen, Department of Mathematics,
B.P. 119, 13000 Tlemcen, Algeria
e-mail: boucherif@yahoo.com

Fatima Dib
University Center of Maghnia, B.P. 600, 13300 Maghnia, Algeria,
Laboratory of Dynamic Systems and Applications,
University of Tlemcen, B.P. 119, 13000 Tlemcen, Algeria
e-mail: fatimadib1967@yahoo.fr

Naima Daoudi-Merzagui
University of Tlemcen, Department of Mathematics,
B.P. 119, 13000 Tlemcen, Algeria
e-mail: nmerzagui@yahoo.fr