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Subclass of analytic functions on q-analogue connected with a new linear extended multiplier operator

Ekram E. Ali

Abstract. Using a new linear extended multiplier q-Choi-Saigo-Srivastava operator $D_{\alpha,\beta}^{m,q}(\mu,\tau)$ we define a subclass $\Theta_{\alpha,\beta}^{m,q}(\mu,\tau,N,M)$ subordination and the newly defined q-analogue of the Choi-Saigo-Srivastava operator to the class of analytic functions. For this class, conclusions are drawn that include coefficient estimates, integral representation, linear combination, weighted and arithmetic means, and radius of starlikeness.

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1. Introduction and preliminaries

Let A denote the normalized analytical function family f of the form:

$$f(\varsigma)=\varsigma+\sum_{\vartheta=2}^{\infty}a_{\vartheta}\varsigma^{\vartheta},\ \varsigma\in\mathbb{D}, \tag{1.1}$$

in the open unit disc $\mathbb{D}:=\{\varsigma\in\mathbb{C}:|\varsigma|<1\}$. Let $S\subset$ A be a class of functions which are univalent in $\mathbb{D}$. If f and $\hbar$ are analytic in $\mathbb{D}$ we say that f is *subordinate* to $\hbar$, denoted $f(\varsigma)\prec\hbar(\varsigma)$, if there exists an analytic function ϖ, with $\varpi(0)=0$ and $|\varpi(\varsigma)|<1$ for all $\varsigma\in\mathbb{D}$, such that $f(\varsigma)=\hbar(\varpi(\varsigma))$, $\varsigma\in\mathbb{D}$. In addition, if $\hbar$ is univalent in $\mathbb{D}$, then the next equivalent ([8, 22] and [23]) holds:

$$f(\varsigma)\prec\hbar(\varsigma)\Leftrightarrow f(0)=\hbar(0)\quad\text{and}\quad f(\mathbb{D})\subset\hbar(\mathbb{D}).$$

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For f given by (1.1) and $\hbar$ of the form

$$\hbar(\varsigma) = \varsigma + \sum_{\vartheta=2}^{\infty} b_{\vartheta}\varsigma^{\vartheta},\ \varsigma \in \mathbb{D},$$

the well-known *convolution product* is

$$(f * \hbar)(\varsigma) := \varsigma + \sum_{\vartheta=2}^{\infty} a_{\vartheta}b_{\vartheta}\varsigma^{\vartheta},\ \varsigma \in \mathbb{D}.$$

The class $S^*(\delta)$ of *starlike functions of order* δ, is said to include a function $f \in \mathrm{A}$ if

$$\mathrm{Re}\left(\frac{\varsigma f'(\varsigma)}{f(\varsigma)}\right) > \delta,\ (0 \le \delta < 1).$$

We observe that the class of *starlike functions*, $S^*(0) = S^*$, holds true. An analytic function $\hbar$ with $\hbar(0) = 1$ is definitely in the Ĵanowski class $P[N, M]$, iff

$$\hbar(\varsigma) \prec \frac{1 + N\varsigma}{1 + M\varsigma} \quad (-1 \le M < N \le 1).$$

The class $P[N, M]$ of Ĵanowski functions was investigated by Ĵanowski [16].

Scholars have recently been inspired by the study of the q-derivative, it is useful in mathematics and related fields. Jackson [13, 14], presented the q-analogue of the derivative and integral operator and also suggested some of its applications. Kanas and Raducanu [17] provided the q-analogue of the Ruscheweyh differential operator and looked into some of its features by using the concept of convolution. Aldweby and Darus [1], Mahmood and Sokol [20], and others looked into various sorts of analytical functions defined by the q-analogue of the Ruscheweyh differential operator see [2, 3, 7, 12, 15, 18, 21, 24, 29, 30] for further details.

The primary goal of the current study is to express a Choi-Saigo-Srivastava operator q-analogue based on convolutions. It also offers a few intriguing applications for this operator at the outset.We will now discuss the essential concept of the q-calculus, which was created by Ĵackson [14] and is pertinent to our ongoing research.

Ĵackson [13, 14] defined the q-derivative operator D_q of a function f:

$$D_q f(\varsigma) := \partial_q f(\varsigma) = \frac{f(q\varsigma) - f(\varsigma)}{(q-1)\varsigma},\ q \in (0, 1),\ \varsigma \neq 0.$$

Remark that if the function f is in the type (1.1), thus, it implies

$$D_q f(\varsigma) = D_q\left(\varsigma + \sum_{\vartheta=2}^{\infty} a_{\vartheta}\varsigma^{\vartheta}\right) = 1 + \sum_{\vartheta=2}^{\infty} [\vartheta]_q a_{\vartheta}\varsigma^{\vartheta-1}, \tag{1.2}$$

where $[\vartheta]_q$ is

$$[\vartheta]_q := \frac{1 - q^{\vartheta}}{1 - q} = 1 + \sum_{\kappa=1}^{\vartheta-1} q^{\kappa},\ [0]_q := 0,$$

and

$$\lim_{q \to 1^-} [\vartheta]_q = \vartheta.$$

The definition of the q-number shift factorial for every non-negative integer ϑ is

$$[\vartheta, q]! := \begin{cases} 1, & \text{if} \quad \vartheta = 0, \\ [1, q]\,[2, q]\,[3, q]\ldots.\,[\vartheta, q]\,, & \text{if} \quad \vartheta \in \mathbb{N}. \end{cases}$$

By combining the notion of convolution with a definition of the q-derivative,Wang et al. introduced in [30] the *q-analogue Choi-Saigo-Srivastava operator* $I^q_{\alpha,\beta} : \mathrm{A} \to \mathrm{A}$,

$$I^q_{\alpha,\beta} f(\varsigma) := f(\varsigma) * \mathcal{F}_{q,\alpha+1,\beta}(\varsigma), \ \varsigma \in \mathbb{D} \quad (\alpha > -1, \ \beta > 0), \tag{1.3}$$

where

$$\begin{aligned} \mathcal{F}_{q,\alpha+1,\beta}(\varsigma) &= \varsigma + \sum_{\vartheta=2}^{\infty} \frac{\Gamma_q(\beta+\vartheta-1)\Gamma_q(\alpha+1)}{\Gamma_q(\beta)\Gamma_q(\alpha+\vartheta)} \varsigma^\vartheta \\ &= \varsigma + \sum_{\vartheta=2}^{\infty} \frac{[\beta, q]_{\vartheta-1}}{[\alpha+1, q]_{\vartheta-1}} \varsigma^\vartheta, \ \varsigma \in \mathbb{D}, \end{aligned} \tag{1.4}$$

where $[\beta, q]_\vartheta$ is the *q-generalized Pochhammer symbol* for $\beta > 0$ defined by

$$[\beta, q]_\vartheta := \begin{cases} 1, & \text{if} \quad \vartheta = 0, \\ [\beta]_q \ [\beta+1]_q \ldots [\beta+\vartheta-1]_q\,, & \text{if} \quad \vartheta \in \mathbb{N}. \end{cases} \tag{1.5}$$

Thus,

$$I^q_{\alpha,\beta} f(\varsigma) = \varsigma + \sum_{\vartheta=2}^{\infty} \frac{[\beta, q]_{\vartheta-1}}{[\alpha+1, q]_{\vartheta-1}} a_\vartheta \varsigma^\vartheta, \ \varsigma \in \mathbb{D}, \tag{1.6}$$

while

$$I^q_{0,2} f(\varsigma) = \varsigma D_q f(\varsigma) \quad \text{and} \quad I^q_{1,2} f(\varsigma) = f(\varsigma).$$

Definition 1.1. [4] For $\mu \geq 0$, and $\tau > -1$, with the aid of the operator $I^q_{\alpha,\beta}$ we will define a new linear extended multiplier q-Choi-Saigo-Srivastava operator $D^{m,q}_{\alpha,\beta}(\mu, \tau) :$ $\mathrm{A} \to \mathrm{A}$ as follows:

$$\begin{aligned} D^{0,q}_{\alpha,\beta}(\mu, \tau) f(\varsigma) =&: D^q_{\alpha,\beta}(\mu, \tau) f(\varsigma) = f(\varsigma), \\ D^{1,q}_{\alpha,\beta}(\mu, \tau) f(\varsigma) =& \left(1 - \frac{\mu}{\tau+1}\right) I^q_{\alpha,\beta} f(\varsigma) + \frac{\mu}{\tau+1} \varsigma D_q \left(I^q_{\alpha,\beta} f(\varsigma)\right) \\ =& \varsigma + \sum_{\vartheta=2}^{\infty} \left(\frac{[\beta, q]_{\vartheta-1}}{[\alpha+1, q]_{\vartheta-1}} \cdot \frac{\tau+1+\mu\left([\vartheta]_q - 1\right)}{\tau+1} \right) a_\vartheta \varsigma^\vartheta, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ D^{m,q}_{\alpha,\beta}(\mu, \tau) f(\varsigma) =& D^q_{\alpha,\beta}(\mu, \tau) \left(D^{m-1,q}_{\alpha,\beta}(\mu, \tau) f(\varsigma) \right), \ m \geq 1, \end{aligned}$$

where $\mu \geq 0$, $\tau > -1$, $m \in \mathbb{N}_0$, $\alpha > -1$, $\beta > 0$ and $0 < q < 1$.

If $f \in \mathrm{A}$ given by (1.1), from (1.6) and the above definition Thus, it implies

$$D^{m,q}_{\alpha,\beta}(\mu, \tau) f(\varsigma) = \varsigma + \sum_{\vartheta=2}^{\infty} \aleph^{m,q}_{\alpha,\beta}(\vartheta, \mu, \tau) a_\vartheta \varsigma^\vartheta, \ \varsigma \in \mathbb{D}, \tag{1.7}$$

where

$$\aleph_{\alpha,\beta}^{m,q}(\vartheta,\mu,\tau) := \left(\frac{[\beta,q]_{\vartheta-1}}{[\alpha+1,q]_{\vartheta-1}} \cdot \frac{\tau+1+\mu\left([\vartheta]_q-1\right)}{\tau+1}\right)^m. \tag{1.8}$$

From (1.3) and (1.8), then

$$D_{\alpha,\beta}^{m,q}(\mu,\tau)f(\varsigma) = \underbrace{\left[\left(I_{\alpha,\beta}^{q}f(\varsigma)*\wp_{\mu,\tau}^{q}(\varsigma)\right)*\ldots*\left(I_{\alpha,\beta}^{q}f(\varsigma)*\wp_{\mu,\tau}^{q}(\varsigma)\right)\right]}_{n-\text{times}}*f(\varsigma),$$

where

$$\wp_{\mu,\tau}^{q}(\varsigma) := \frac{\varsigma-\left(1-\frac{\mu}{\tau+1}\right)q\varsigma^2}{(1-\varsigma)(1-q\varsigma)}.$$

Remark 1.2. The operator $D_{\alpha,\beta}^{m,q}(\mu,\tau)f(\varsigma)$ should be noticed that the following generalizes a number of other operators previously covered, for instance:

(i) For $q\to 1^-$, $\alpha=1$, $\beta=2$, and $\tau=0$, we obtain the operator D_μ^m investigated by Al-Oboudi [5];

(ii) If $q\to 1^-$, $\alpha=1$, $\beta=2$, $\mu=1$ and $\tau=0$, we obtain the operator D^m introduced by Sălăgean [27];

(iii) Taking $q\to 1^-$, $\alpha=1$ and $\beta=2$, we obtain the operator $I^m(\lambda,\kappa)$ studied Cătaş [9];

(iv) Considering $\alpha=1$, $\beta=2$ and $\tau=0$, we get $D_{\mu,q}^m$ presented and analysed by Aouf et al. [7];

(v) For $\alpha=1$, $\beta=2$, $\mu=1$ and $\tau=0$, we obtain the operator S_q^m investigated by Govindaraj and Sivasubramanian [12];

(vi) If $q\to 1^-$ we obtain $D_{\mu,\tau,\beta}^{m,\alpha}$ presented and investigated by El-Ashwah et al. [11] for $q=2$, $s=1$, $\alpha_1=\beta$, $\alpha_2=1$, $\beta_1=\alpha+1$;

(vii) If $q\to 1^-$, $\alpha=1$, $\beta=2$ and $\mu=1$, we obtain the operator I_τ^m, $\tau\geq 0$, investigated by Cho and Srivastava [10];

(viii) Given $q\to 1^-$, $\mu=\tau=0$ and $m=1$, we get $I_{\alpha,\beta}^q$ presented and analysed by Wang et al. [30];

(ix) Given $q\to 1^-$, $\alpha:=1-\alpha$, $\beta=2$, and $\tau=0$, we obtain the operator $D_\mu^{m,\alpha}$ investigate by Al-Oboudi and Al-Amoudi [6];

(x) If $\alpha:=1-\varrho$ and $\beta=2$, we get $D_{q,\varrho}^{m,\lambda,\kappa}$ investigated by Kota and El-Ashwah [18];

(xi) Given $\beta=2$, $\mu=0$ and $\tau=0$, we obtain the q-analogue integral operator of Noor $I_{\alpha,2}^q$ presented and investigated by [29];

(xii) If $q\to 1^-$, $\beta=2$, $\mu=0$ and $\tau=0$, we get the differential operator I^ϑ studied in [25, 26];

(xiii) For $q\to 1^-$, $\beta=2$, $\alpha:=1-\alpha$, $\mu=0$ and $\tau=0$, we obtain the Owa-Srivastava operator $I_{1-\alpha,2}$ presented and analysed in [28].

Definition 1.3. Let $-1\leq M<N\leq 1$ and $0<q<1$. $f\in\Theta_{\alpha,\beta}^{m,q}(\mu,\tau,N,M)$ if it satisfies

$$\frac{\varsigma\partial_q(D_{\alpha,\beta}^{m,q}(\mu,\tau)f(\varsigma))}{D_{\alpha,\beta}^{m,q}(\mu,\tau)f(\varsigma)} \prec \frac{1+N\varsigma}{1+M\varsigma}.$$

Equivalently, $f \in \Theta^{m,q}_{\alpha,\beta}(\mu, \tau, N, M)$ iff

$$\left| \frac{\frac{\varsigma \partial_q (D^{m,q}_{\alpha,\beta}(\mu,\tau) f(\varsigma))}{D^{m,q}_{\alpha,\beta}(\mu,\tau) f(\varsigma)} - 1}{N - M \left(\frac{\varsigma \partial_q (D^{m,q}_{\alpha,\beta}(\mu,\tau) f(\varsigma))}{D^{m,q}_{\alpha,\beta}(\mu,\tau) f(\varsigma)} \right)} \right| < 1. \tag{1.9}$$

We must apply the following lemma in order to validate one of our findings.

Lemma 1.4. [19] *Let* $-1 \le M_2 \le M_1 < N_1 \le N_2 \le 1$. *Then*

$$\frac{1 + N_1 \varsigma}{1 + M_1 \varsigma} \prec \frac{1 + N_2 \varsigma}{1 + M_2 \varsigma}.$$

Throughout this paper, we suppose that $\mu \ge 0$, $\tau > -1$, $m \in \mathbb{N}_0$, $\alpha > -1$, $\beta > 0$, $0 < q < 1$ and $-1 \le M < N \le 1$, We furthermore assume that all coefficients a_n of f are real positive numbers.

2. Main results

Theorem 2.1. *Suppose that* $f \in \mathrm{A}$ *given by (1.1). Then* $f \in \Theta^{m,q}_{\alpha,\beta}(\mu, \tau, N, M)$ *iff*

$$\sum_{\vartheta=2}^{\infty} \left[([\vartheta] + N) - (M [\vartheta] + 1) \right] \aleph^{m,q}_{\alpha,\beta}(\vartheta, \mu, \tau) a_\vartheta < N - M. \tag{2.1}$$

Proof. Let (2.1) holds. then from (1.9) we have

$$\begin{aligned} \left| \frac{\frac{\varsigma \partial_q (D^{m,q}_{\alpha,\beta}(\mu,\tau) f(\varsigma))}{D^{m,q}_{\alpha,\beta}(\mu,\tau) f(\varsigma)} - 1}{N - M \left(\frac{\varsigma \partial_q (D^{m,q}_{\alpha,\beta}(\mu,\tau) f(\varsigma))}{D^{m,q}_{\alpha,\beta}(\mu,\tau) f(\varsigma)} \right)} \right| &= \left| \frac{\sum\limits_{\vartheta=2}^{\infty} ([\vartheta]_q - 1) \aleph^{m,q}_{\alpha,\beta}(\vartheta, \mu, \tau) a_\vartheta \varsigma^\vartheta}{(N - M)\varsigma + \sum\limits_{\vartheta=2}^{\infty} (N - M [\vartheta]_q) \aleph^{m,q}_{\alpha,\beta}(\vartheta, \mu, \tau) a_\vartheta \varsigma^\vartheta} \right| \\ &\le \frac{\sum\limits_{\vartheta=2}^{\infty} ([\vartheta]_q - 1) \aleph^{m,q}_{\alpha,\beta}(\vartheta, \mu, \tau) a_\vartheta}{(N - M) - \sum\limits_{\vartheta=2}^{\infty} (N - M [\vartheta]_q) \aleph^{m,q}_{\alpha,\beta}(\vartheta, \mu, \tau) a_\vartheta} < 1, \end{aligned}$$

then from (1.2), (1.7), and (2.1) this completes the direct part.
Conversely, $f \in \Theta^{m,q}_{\alpha,\beta}(\mu, \tau, N, M)$ then from (1.9) and (1.7), hence

$$\left| \frac{\frac{\varsigma \partial_q (D^{m,q}_{\alpha,\beta}(\mu,\tau) f(\varsigma))}{D^{m,q}_{\alpha,\beta}(\mu,\tau) f(\varsigma)} - 1}{N - M \left(\frac{\varsigma \partial_q (D^{m,q}_{\alpha,\beta}(\mu,\tau) f(\varsigma))}{D^{m,q}_{\alpha,\beta}(\mu,\tau) f(\varsigma)} \right)} \right| = \left| \frac{\sum\limits_{\vartheta=2}^{\infty} ([\vartheta]_q - 1) \aleph^{m,q}_{\alpha,\beta}(\vartheta, \mu, \tau) a_\vartheta \varsigma^\vartheta}{(N - M)\varsigma + \sum\limits_{\vartheta=2}^{\infty} (N - M [\vartheta]_q) \aleph^{m,q}_{\alpha,\beta}(\vartheta, \mu, \tau) a_\vartheta \varsigma^\vartheta} \right| < 1.$$

Since $|\Re(\varsigma)| \le |\varsigma|$, we get

$$\Re \left(\frac{\sum\limits_{\vartheta=2}^{\infty} ([\vartheta]_q - 1) \aleph^{m,q}_{\alpha,\beta}(\vartheta, \mu, \tau) a_\vartheta \varsigma^\vartheta}{(N - M) + \sum\limits_{\vartheta=2}^{\infty} (N - M [\vartheta]_q) \aleph^{m,q}_{\alpha,\beta}(\vartheta, \mu, \tau) a_\vartheta \varsigma^\vartheta} \right) < 1. \tag{2.2}$$

We now select ς values along the real axis such that $\frac{\varsigma\partial_q(D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma))}{D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma)}$ is real. Then for letting $\varsigma \to 1^-$, we get (2.1). □

If we set $\alpha = 1$, $\beta = 2$, and $\tau = 0$ in Theorem 2.1 we have:

Corollary 2.2. *$f \in \Theta^{m,q}_{\alpha,\beta}(\mu, N, M)$ iff*

$$\sum_{\vartheta=2}^{\infty} \left[\left([\vartheta]_q + N \right) - \left(M [\vartheta]_q + 1 \right) \right] \left(1 + \mu \left([\vartheta]_q - 1 \right) \right)^m a_\vartheta < N - M.$$

Theorem 2.3. *Suppose that $f \in \Theta^{m,q}_{\alpha,\beta}(\mu, \tau, N, M)$. Therefore*

$$D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma) = \exp\left(\frac{\ln q}{q-1} \int_0^{\varsigma} \frac{1}{t} \left(\frac{1+N\varphi(t)}{1+M\varphi(t)} \right) d_q(t) \right),$$

where $|\varphi(t)| < 1$.

Proof. Let $f \in \Theta^{m,q}_{\alpha,\beta}(\mu, \tau, N, M)$ and putting

$$\frac{\varsigma\partial_q(D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma))}{D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma)} = \omega(\varsigma),$$

with

$$\omega(\varsigma) \prec \frac{1+N\varsigma}{1+M\varsigma},$$

equivalently, we can write

$$\left| \frac{\omega(\varsigma) - 1}{N - M\omega(\varsigma)} \right| < 1,$$

hence, there is

$$\frac{\omega(\varsigma) - 1}{N - M\omega(\varsigma)} = \varphi(\varsigma),$$

such that $|\varphi(\varsigma)| < 1$. Hence,

$$\frac{\partial_q(D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma))}{D^{m,q}_{\alpha,\beta}(\mu,\tau)f(\varsigma)} = \frac{1}{\varsigma} \left(\frac{1+N\varphi(t)}{1+M\varphi(t)} \right).$$

Using simple calculation we get the result. □

Theorem 2.4. *Let $f_j \in \Theta^{m,q}_{\alpha,\beta}(\mu, \tau, N, M)$ and*

$$f_j(\varsigma) = \varsigma + \sum_{\iota=1}^{\infty} a_{\iota,j}\varsigma^{\iota}, \qquad (j = 1, 2, 3, ..., \kappa).$$

Therefore $F \in \Theta^{m,q}_{\alpha,\beta}(\mu, \tau, N, M)$, such that

$$f(\varsigma) = \sum_{j=1}^{\kappa} c_j f_j(\varsigma) \quad \textit{with} \quad \sum_{j=1}^{\kappa} c_j = 1.$$

Proof. FromTheorem 2.1, hence

$$\sum_{\vartheta=2}^{\infty}\left\{\frac{\left[([\vartheta]_q+N)-(M[\vartheta]_q+1)\right]\aleph_{\alpha,\beta}^{m,q}(\vartheta,\mu,\tau)}{N-M}\right\}a_{\vartheta,j}<1.$$

Therefore, we get

$$f(\varsigma)=\sum_{j=2}^{\kappa}c_j(\varsigma+\sum_{\vartheta=2}^{\infty}a_{\vartheta,j}\varsigma^{\vartheta})=\varsigma+\sum_{j=2}^{\kappa}\sum_{\vartheta=2}^{\infty}c_ja_{\vartheta,j}\varsigma^{\vartheta}=\varsigma+\sum_{\vartheta=2}^{\infty}\left(\sum_{j=2}^{\kappa}c_ja_{\vartheta,j}\right)\varsigma^{\vartheta}.$$

However,

$$\begin{aligned}&\sum_{\vartheta=2}^{\infty}\frac{\left[([\vartheta]_q+N)-(M[\vartheta]_q+1)\right]\aleph_{\alpha,\beta}^{m,q}(\vartheta,\mu,\tau)}{N-M}\left(\sum_{j=2}^{\kappa}c_ja_{\vartheta,j}\right)\\ =\ &\sum_{j=2}^{\kappa}\left\{\sum_{\vartheta=2}^{\infty}\frac{\left[([\vartheta]_q+N)-(M[\vartheta]_q+1)\right]\aleph_{\alpha,\beta}^{m,q}(\vartheta,\mu,\tau)}{N-M}a_{\vartheta,j}\right\}c_j\leq 1,\end{aligned}$$

then $F\in\Theta_{\alpha,\beta}^{m,q}(\mu,\tau,N,M)$ and the proof is complete. □

Theorem 2.5. *If $f,\hbar\in\Theta_{\alpha,\beta}^{m,q}(\mu,\tau,N,M)$, then $h_j(j\in\mathbb{N})$ is in $\Theta_{\alpha,\beta}^{m,q}(\mu,\tau,N,M)$, such that h_j denoted by*

$$h_j(\varsigma)=\frac{(1-j)f(\varsigma)+(1+j)\hbar(\varsigma)}{2}. \tag{2.3}$$

Proof. By (2.3), then

$$h_j(\varsigma)=\varsigma+\sum_{\vartheta=2}^{\infty}\left[\frac{(1-j)a_{\vartheta}+(1+j)b_{\vartheta}}{2}\right]\varsigma^{\vartheta}.$$

To prove $h_j(\varsigma)\in\Theta_{\alpha,\beta}^{m,q}(\mu,\tau,N,M)$, we need to show that

$$\sum_{\vartheta=2}^{\infty}\frac{\left[([\vartheta]_q+N)-(M[\vartheta]_q+1)\right]}{N-M}\left\{\frac{(1-j)a_{\vartheta}+(1+j)b_{\vartheta}}{2}\right\}\aleph_{\alpha,\beta}^{m,q}(\vartheta,\mu,\tau)<1.$$

For this, consider

$$\begin{aligned}&\sum_{\vartheta=2}^{\infty}\frac{\left[([\vartheta]_q+N)-(M[\vartheta]_q+1)\right]}{N-M}\left\{\frac{(1-j)a_{\vartheta}+(1+j)b_{\vartheta}}{2}\right\}\aleph_{\alpha,\beta}^{m,q}(\vartheta,\mu,\tau)\\ =\ &\frac{(1-j)}{2}\sum_{\vartheta=2}^{\infty}\frac{\left[([\vartheta]_q+N)-(M[\vartheta]_q+1)\right]}{N-M}\aleph_{\alpha,\beta}^{m,q}(\vartheta,\mu,\tau)a_{\vartheta}\\ &+\frac{(1+j)}{2}\sum_{\vartheta=2}^{\infty}\frac{\left[([\vartheta]_q+N)-(M[\vartheta]_q+1)\right]}{N-M}\aleph_{\alpha,\beta}^{m,q}(\vartheta,\mu,\tau)b_{\vartheta}\\ <\ &\frac{(1-j)}{2}+\frac{(1+j)}{2}=1,\end{aligned}$$

by using 2.1 we get the result. □

Theorem 2.6. *Let $f_j \in \Theta^{m,q}_{\alpha,\beta}(\mu,\tau,N,M)$ with $j = 1,2,.....\alpha(\alpha \in \mathbb{N})$. Then*

$$h(\varsigma) = \frac{1}{\alpha}\sum_{j=1}^{\alpha} f_j(\varsigma), \tag{2.4}$$

also is in the class $\Theta^{m,q}_{\alpha,\beta}(\mu,\tau,N,M)$.

Proof. By (2.4), therefore

$$h(\varsigma) = \frac{1}{\alpha}\sum_{j=1}^{\alpha}\left(\varsigma + \sum_{\vartheta=2}^{\infty} a_{\vartheta,j}\varsigma^{\vartheta}\right) = \varsigma + \sum_{\vartheta=2}^{\infty}\left(\frac{1}{\alpha}\sum_{j=1}^{\alpha} a_{\vartheta,j}\right)\varsigma^{\vartheta}. \tag{2.5}$$

Since $f_j \in \Theta^{m,q}_{\alpha,\beta}(\mu,\tau,N,M)$, Through (2.5) and (2.1), there will be

$$\begin{aligned}
&\sum_{\vartheta=2}^{\infty}\left[([\vartheta]_q + N) - (M[\vartheta]_q + 1)\right]\aleph^{m,q}_{\alpha,\beta}(\vartheta,\mu,\tau)\left(\frac{1}{\alpha}\sum_{j=1}^{\alpha} a_{\vartheta,j}\right)\\
= \;& \frac{1}{\alpha}\sum_{j=1}^{\alpha}\left(\sum_{\vartheta=2}^{\infty}\left[([\vartheta]_q + N) - (M[\vartheta]_q + 1)\right]\aleph^{m,q}_{\alpha,\beta}(\vartheta,\mu,\tau)a_{\vartheta,j}\right)\\
\leq \;& \frac{1}{\alpha}\sum_{j=1}^{\alpha}(N-M) = N-M,
\end{aligned}$$

the proof is completed. □

Theorem 2.7. *Suppose that $f \in \Theta^{m,q}_{\alpha,\beta}(\mu,\tau,N,M)$. Then $f \in S^*(\delta)$, for $|\varsigma| < r_1$, where*

$$r_1 = \left(\frac{(1-\delta)\left[([\vartheta]_q + N) - (M[\vartheta]_q + 1)\right]}{(\vartheta-\delta)(N-M)}\aleph^{m,q}_{\alpha,\beta}(\vartheta,\mu,\tau)\right)^{\frac{1}{\vartheta-1}}.$$

Proof. Let $f \in \Theta^{m,q}_{\alpha,\beta}(\mu,\tau,N,M)$. To show that $f \in S^*(\delta)$, we need

$$\left|\frac{\frac{\varsigma f'(\varsigma)}{f(\varsigma)} - 1}{\frac{\varsigma f'(\varsigma)}{f(\varsigma)} + 1 - 2\delta}\right| < 1.$$

By using (1.1) along with some simple computations we have

$$\sum_{\vartheta=2}^{\infty}\left(\frac{\vartheta-\delta}{1-\delta}\right)|a_\vartheta|\,|\varsigma|^{\vartheta-1} < 1. \tag{2.6}$$

Since $f \in \Theta^{m,q}_{\alpha,\beta}(\mu,\tau,N,M)$, from (2.1), there are

$$\sum_{\vartheta=2}^{\infty}\frac{\left[([\vartheta]_q + N) - (M[\vartheta]_q + 1)\right]}{N-M}\aleph^{m,q}_{\alpha,\beta}(\vartheta,\mu,\tau)\,|a_\vartheta| < 1. \tag{2.7}$$

And then, (2.6) is true, if

$$\sum_{\vartheta=2}^{\infty}\left(\frac{\vartheta-\delta}{1-\delta}\right)|a_\vartheta|\,|\varsigma|^{\vartheta-1}<\sum_{\vartheta=2}^{\infty}\frac{\left[([\vartheta]_q+N)-(M\,[\vartheta]_q+1)\right]}{N-M}\aleph_{\alpha,\beta}^{m,q}(\vartheta,\mu,\tau)\,|a_\vartheta|\,,$$

holds, which implies that

$$|\varsigma|^{\vartheta-1}<\frac{(1-\delta)\left[([\vartheta]_q+N)-(M\,[\vartheta]_q+1)\right]}{(\vartheta-\delta)\,(N-M)}\aleph_{\alpha,\beta}^{m,q}(\vartheta,\mu,\tau),$$

and thus we get the required result. □

If we set $\alpha=1$, $\beta=2$, and $\tau=0$ in Theorem 2.7 we get:

Corollary 2.8. *Suppose that $f\in\Theta_{\alpha,\beta}^{m,q}(\mu,N,M)$. Then $f\in S^*(\delta)$, for $|\varsigma|<r_1$, where*

$$r_1=\left(\frac{(1-\delta)\left[([\vartheta]_q+N)-(M\,[\vartheta]_q+1)\right]}{(\vartheta-\delta)\,(N-M)}\left(1+\mu\left([\vartheta]_q-1\right)\right)^m\right)^{\frac{1}{\vartheta-1}}.$$

Remark 2.9. For $q\to1^-$, $\mu=\tau=0$ and $m=1$, in the above results we get the results investigated by Wang et al. [30].

3. Conclusion

This study introduces a subclass $\Theta_{\alpha,\beta}^{m,q}(\mu,\tau,N,M)$ of analytic functions on q-analogue associated with a new linear extended multiplier q-Choi-Saigo-Srivastava operator $D_{\alpha,\beta}^{m,q}(\mu,\tau)$ in the open unit disk $\mathbb{D}$. We have obtain coefficient estimates, integral representation, linear combination, weighted and arithmetic means, and radius of starlikeness belonging to the class $\Theta_{\alpha,\beta}^{m,q}(\mu,\tau,N,M)$. Some of the earlier efforts of numerous writers are generalized by our findings.

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References

[1] Aldweby, H., Darus, M., *Some subordination results on q-analogue of Ruscheweyh differential operator*, Abstr. Appl. Anal., (2014), 958563, 6 pp.

[2] Ali, E.E., Bulboaca, T., *Subclasses of multivalent analytic functions associated with a q-difference operator*, Mathematics, **8**(2020), 2184.

[3] Ali, E.E., Lashin, A.M., Albalahi, A.M., *Coefficient estimates for some classes of bi-univalent function associated with Jackson q-difference operator*, J. Funct. Spaces, (2022), 2365918, 8 pp.

[4] Ali, E.E., Oros, G.I., Shah, S.A., Albalahi, A.M., *Applications of q-calculus multiplier operators and subordination for the study of particular analytic function subclasses*, Mathematics, **11**(2023), 2705.

[5] Al-Oboudi, F.M., *On univalent functions defined by a generalized Sălăgean operator*, Internat. J. Math. Math Sci., **27**(2004), 1429-1436.

[6] Al-Oboudi, F.M., Al-Amoudi, K.A., *On classes of analytic functions related to conic domain*, J. Math. Anal. Appl., **339**(2008), 655-667.

[7] Aouf, M.K., Mostafa, A.O., Elmorsy, R.E., *Certain subclasses of analytic functions with varying arguments associated with q-difference operator*, Afr. Mat., **32**(2021), 621–630.

[8] Bulboacă, T., *Differential Subordinations and Superordinations. Recent Results*, House of Scientific Book Publ., Cluj-Napoca, 2005.

[9] Cătaş, A., *On certain classes of p-valent functions defined by multiplier transformations*, in Proceedings of the International Symposium on Geometric Function Theory and Applications: GFTA 2007 Proceedings (Istanbul, Turkey; 20-24 August 2007) (S. Owa and Y. Polatog lu, Editors), pp. 241–250, TC Istanbul Kűltűr University Publications, Vol. 91, TC Istanbul Kűltűr University, Istanbul, Turkey, 2008.

[10] Cho, N.E., Srivastava, H.M., *Argument estimates of certain analytic functions defined by a class of multiplier transformations*, Math. Comput. Model., **37**(2003), 39–49.

[11] El-Ashwah, R.M., Aouf, M.K., Shamandy, A., Ali, E.E., *Subordination results for some subclasses of analytic functions*, Math. Bohem., **136**(2011), 311–331.

[12] Govindaraj, M., Sivasubramanian, S., *On a class of analytic functions related to conic domains involving q-calculus*, Anal. Math., **43**(2017), 475-487.

[13] Jackson, F.H., *On q-functions and a certain difference operator*, Trans. Roy. Soc. Edinburgh, **46**(1908), 253–281.

[14] Jackson, F.H., *On q-definite integrals*, Quart., J. Pure Appl. Math., **41**(1910), 193–203.

[15] Jafari, M., Motamednezad, A., Adegani, E.A., *Coefficient estimates for a subclass of analytic functions by Srivastava-Attiya operator*, Stud. Univ. Babes-Bolyai Math., **67**(2022), 739-747.

[16] Janowski, W., *Some extremal problems for certain families of analytic functions*, Ann. Polon. Math., **28**(1973), 297–326.

[17] Kanas, S., Raducanu, D., *Some class of analytic functions related to conic domains*, Math. Slovaca, **64**(2014), 1183–1196.

[18] Kota, W.Y., El-Ashwah, R.M., *Some application of subordination theorems associated with fractional q-calculus operator*, Math. Bohem., **148**(2023), No. 2, 131-148.

[19] Liu, M.-S., *On a subclass of p-valent close to convex functions of type α and order β*, J. Math. Study, **30**(1997), 102–104.

[20] Mahmood, S., Sokol, J., *New subclass of analytic functions in conical domain associated with Ruscheweyh q-differential operator*, Results Math., **71**(2017), 1–13.

[21] Mason, T.E., *On properties of the solution of linear q-difference equations with entire function coefficients*, Amer. J. Math., **37**(1915), 439-444.

[22] Miller, S.S., Mocanu, P.T., *Differential subordinations and univalent functions*, Michigan Math. J., **28**(1981), 157-171.

[23] Miller, S.S., Mocanu, P.T., *Differential Subordinations: Theory and Applications*, Series on Monographs and Texbooks in Pure and Applied Mathematics, Vol. 225, Marcel Dekker, New York and Basel, 2000.

[24] Mohammed, N.H., Cho, N.E., Adegani, E.A., Bulboaca, T., *Geometric properties of normalized imaginary error function*, Stud. Univ. Babeş-Bolyai Math., **67**(2022), 455-462.

[25] Noor, K.I., *On new classes of integral operator*, J. Natur. Geom., **16**(1999), 71–80.

[26] Noor, K.I., Noor, M.A., *On integral operators*, J. Math. Anal. Appl., **238**(1999), 341-352.

[27] Sălăgean, G.S., *Subclasses of univalent functions*, in Complex Analysis-Fifth Romanian-Finish Seminar, Part-I, Bucharest, 1981, in Lecture notes in Math., Vol. 1013, Springer, Berlin, 1983, pp. 362-372.

[28] Shareef, Z., Hussain, S., Darus, M., *Convolution operator in geometric functions theory*, J. Inequal. Appl., **213**(2012), 1-11.

[29] Srivastava, H.M., Khan, S., Ahmad, Q.Z., Khan, N., Hussain, S., *The Faber polynomial expansion method and its application to the general coefficient problem for some subclasses of bi-univalent functions associated with a certain q-integral operator*, Stud. Univ. Babecs-Bolyai Math., **63**(2018), 419–436.

[30] Wang, Z.G., Hussain, S., Naeem, M., Mahmood, T., Khan, S., *A subclass of univalent functions associated with $q-$analogue of Choi-Saigo-Srivastava operator*, Hacet. J. Math. Stat., **XX**(x)(2019), 1–9.

Ekram E. Ali
Department of Mathematics, College of Science,
University of Ha'il, Ha'il 81451, Saudi Arabia,
and
Department of Mathematics and Computer Science,
Faculty of Science, Port Said University, Port Said 42521, Egypt
e-mail: e.ahmad@uoh.edu.sa; ekram_008eg@yahoo.com