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Some applications of a Wright distribution series on subclasses of univalent functions

Mallikarjun G. Shrigan, Sidharam D. Bhourgunde and
Girish D. Shelake

Abstract. The purpose of the present paper is to find the sufficient conditions for the subclasses of analytic functions associated with Wright distribution series to be in subclasses of univalent functions and inclusion relations for such subclasses in the open unit disk $\mathbb{D}$. Further, we consider the properties of integral operator related to Wright distribution series.

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1. Introduction

Let $\mathcal{A}$ denote the class of functions f of the form

$$f(z) = \sum_{n=1}^{\infty} a_n\, z^n; (a_1 := 1), \tag{1.1}$$

which are analytic in the open unit disk given by $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$.

A function $f \in \mathcal{A}$ is said to be starlike of order $\gamma(0 \leq \gamma < 1)$, if and only if $\operatorname{Re}(zf'(z)/f(z)) > \gamma$, which is denoted by $S^*(\gamma)$. We also write $S^*(\gamma) \subseteq S^*(0) := S^*$, where S^* denotes the class of functions $f \in \mathcal{A}$ that $f(\mathbb{U})$ is starlike with respect to the origin. Also, a function $f \in \mathcal{A}$ is said to be convex of order $\gamma(0 \leq \gamma < 1)$, if and only if $\operatorname{Re}(1 + (zf''(z)/f'(z))) > \gamma$. This function class is denoted by $\mathcal{K}(\gamma)$. We also write $\mathcal{K}(\gamma) \subseteq \mathcal{K}(0) := \mathcal{K}$, the well-known standard class of convex functions. It is an established fact that $f \in \mathcal{K}(\gamma) \Leftrightarrow zf' \in S^*(\gamma)$.

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A function $f \in \mathcal{A}$ is said to be starlike of reciprocal order $\gamma (0 \leq \gamma < 1)$, if and only if

$$\operatorname{Re}\left(\frac{f(z)}{zf'(z)}\right) > \gamma, \quad (z \in \mathbb{U}). \tag{1.2}$$

We denote the class of such functions by $S_r^*(\gamma)$. Also, a function $f \in \mathcal{A}$ is said to be convex of reciprocal order $\gamma(0 \leq \gamma < 1)$, if and only if

$$\operatorname{Re}\left(\frac{f'(z)}{f'z) + zf''(z)}\right) > \gamma, \quad (z \in \mathbb{U}). \tag{1.3}$$

This function class is denoted by $\mathcal{K}_r^*(\gamma)$. We also write $S_r^*(0) := S^*, \mathcal{K}_r^*(0) = \mathcal{K}$ and $f \in \mathcal{K}_r^*(\gamma) \Leftrightarrow zf' \in S_r^*(\gamma)$.

In 2002, Owa and Srivastava [24] studied the classes of p-valent starlike and p-valent convex functions of reciprocal order γ with $\gamma > p$, and further investigated by Polatoglu et al. [25]. Uyanik et al. [36] introduced the classes of p-valently spirallike and p-valently Robertson functions (cf. [31]). Frasin and Sabri [9] derived sufficient condition for starlikeness of reciprocal order. Ravichandran and Kumar [30] investigated the argument estimates for the analytic functions $f \in S_r^*(\gamma)$. Al-Hawar and Frasin [2] determine coefficient bounds and subordination results of analytic functions of reciprocal order by means of Hadamard product. For more related results of some associated classes, see [1, 4, 6, 13, 16, 20, 22, 32, 33, 37].

Frasin et al. [10] introduced the subclasses of analytic functions of reciprocal order as

Definition 1.1. [10] A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{G}^{-1}(\gamma)$ of order γ if and only if it satisfies the condition

$$\operatorname{Re}\left(\frac{f(z)}{zf'(z)}\right) > \gamma, \quad (z \in \mathbb{U}), \tag{1.4}$$

for some $\gamma > 1$.

Definition 1.2. [10] A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{H}^{-1}(\gamma)$ of order γ if and only if it satisfies the condition

$$\operatorname{Re}\left(\frac{f'(z)}{f'(z) + zf''(z)}\right) > \gamma, \quad (z \in \mathbb{U}), \tag{1.5}$$

for some $\gamma > 1$.

It can be seen from (1.4) and (1.5) that

$$f(z) \in \mathcal{H}^{-1}(\gamma) \text{ if and only if } zf'(z) \in \mathcal{G}^{-1}(\gamma).$$

Remark 1.3. Silverman [34], consider the condition

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1 - \gamma, \quad (z \in \mathbb{U}), \tag{1.6}$$

for the class $S^*(\gamma)$. This condition shows that the image of $\mathbb{U}$ by $\dfrac{zf'(z)}{f(z)}$ is inside of the circle with the center at 1 and the radius $1 - \gamma$, which is very small circle. If we consider the condition

$$\left|\frac{zf'(z)}{f(z)} - \frac{1}{2\gamma}\right| < \frac{1}{2\gamma}, \quad (z \in \mathbb{U}), \tag{1.7}$$

for $0 < \gamma < 1$ the condition (1.7) shows that

$$\mathrm{Re}\left(\frac{f(z)}{zf'(z)}\right) > \gamma, \quad (z \in \mathbb{U}),$$

which means that $f(z) \in S_r^*(\gamma)$. This condition (1.7) shows that the image of $\mathbb{U}$ by $\frac{zf'(z)}{f(z)}$ is inside of the circle with the center at $\frac{1}{2\gamma}$ and the radius $\frac{1}{2\gamma}$. Thus if $0 < \gamma < \frac{1}{2}$, the condition (1.7) is better than (1.6). This is the motivation to discuss of the classes $S_r^*(\gamma)$ and $\mathcal{K}_r^*(\gamma)$.

Example 1.4. The function $f(z) = \frac{z}{(1-z)^{2(1-\gamma)}}, (0 < \gamma < 1)$ is a starlike function of reciprocal order 0 in $\mathbb{U}$ ([23], Example 1).

Example 1.5. The function $f(z) = ze^{(1-\gamma)z}, (0 < \gamma < 1)$ is a starlike function of reciprocal order $1/(2-\gamma)$ in $\mathbb{U}$ ([23], Example 2).

In recent years, several interesting subclasses of analytic functions were introduced and investigated from different view points. Several researchers including Altinkaya and Yalçin [3], Eker et al. [8], El-Deeb and Bulboacă [7], Nazeer et al. [21], Porwal and Ahamad [26], Porwal and Kumar [27], Wanas and Khuttar [38], and many more have studied interesting results on certain classes of univalent functions for various distribution series (see also, [17, 32]).

In 1933, Wright [39] introduced a special function known as Wright functions, is given by:

$$W_{\lambda,\kappa}(z) = \sum_{n=0}^{\infty} \frac{1}{\Gamma(\lambda n + \kappa)} \frac{z^n}{n!}, \tag{1.8}$$

for $\lambda > -1, \kappa \in \mathbb{C}$ which is convergent for all $z \in \mathbb{C}$, while for $\lambda > -1$ this is absolutely convergent in $\mathbb{U}$. Gorenflo et al. [11] and Mustafa [18] gave insight of some characterizations and basic properties for the Wright functions. Prajapat [29] obtained certain geometric properties including univalency, starlikeness, convexity and close-to-convexity in the open unit disk $\mathbb{U}$ (see also, [15, 14, 19]). It is easy to see that the series (1.8) is not in normalized form so we normalized it as

$$\mathbb{W}_{\lambda,\kappa} = \Gamma(\kappa) z W_{\lambda,\kappa}(z)$$

$$\mathbb{W}_{\lambda,\kappa}(z) = \sum_{n=0}^{\infty} \frac{\Gamma(\kappa)}{\Gamma(\lambda n + \kappa)} \frac{z^{n+1}}{n!}, \tag{1.9}$$

for $\lambda > -1, \kappa > 0$ and $z \in \mathbb{U}$. Wright distribution recognized as a vitally important distribution in its own right, first we define the series

$$\mathbb{W}_{\lambda,\kappa}(s) = \sum_{n=0}^{\infty} \frac{\Gamma(\kappa)}{\Gamma(\lambda n + \kappa)} \frac{s^{n+1}}{n!},$$

which is convergent for all $\lambda, \kappa, s > 0$. The probability mass function of Wright distribution is given by

$$p(n) = \frac{\Gamma(\kappa)}{\Gamma(\lambda n + \kappa)\mathbb{W}_{\lambda,\kappa}(s)} \frac{s^{n+1}}{n!}, \quad \lambda, \kappa, s > 0; n = 0, 1, 2, \cdots .$$

The Wright distribution is a particular case of the familiar Poisson distribution which widely used as analysing traffic flow, fault prediction in electric cables, defects occurring in manufactured objects such as castings, email messages arriving at a computer and in the prediction of randomly occurring events or accidents.

Recently in 2022, Porwal et al. [28] invented Wright distribution series and gave a nice application of it on certain classes of univalent functions. Porwal et al. [28] introduce the Wright distribution series as follows

$$K_{\psi}(\lambda, \kappa, s, z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\kappa)s^n}{\Gamma(\lambda(n-1)+\kappa)(n-1)!\mathbb{W}_{\lambda,\kappa}(s)} z^n. \tag{1.10}$$

Porwal et al. [28] introduced the linear operator $\mathcal{I}(\lambda, \kappa, s) : \mathcal{A} \to \mathcal{A}$ defined by using the Hadamard (convolution) product as

$$\mathcal{I}(\lambda, \kappa, s)f(z) = K_{\psi}(\lambda, \kappa, s, z) * f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\kappa)s^n}{\Gamma(\lambda(n-1)+\kappa)(n-1)!\mathbb{W}_{\lambda,\kappa}(s)} a_n z^n. \tag{1.11}$$

To establish our main results, we need to recall the following lemmas due to Frasin et al. [10] and Dixit and Pal [5].

Lemma 1.6. [10] *If $f \in \mathcal{A}$ satisfies*

$$\sum_{n=2}^{\infty} (\gamma n - 1)|a_n| \leq \gamma - 1, \tag{1.12}$$

for some $\gamma > 1$, then $f(z) \in \mathcal{G}^{-1}(\gamma)$.

Lemma 1.7. [10] *If $f \in \mathcal{A}$ satisfies*

$$\sum_{n=2}^{\infty} n(\gamma n - 1)|a_n| \leq \gamma - 1, \tag{1.13}$$

for some $\gamma > 1$, then $f(z) \in \mathcal{H}^{-1}(\gamma)$.

Definition 1.8. A function $f \in \mathcal{A}$ is said to in the class $\mathcal{R}^{\tau}(\vartheta, \delta)$, if it satisfies the inequality

$$\left| \frac{(1-\vartheta)\frac{f(z)}{z} + \vartheta f'(z) - 1}{2\tau(1-\delta) + (1-\vartheta)\frac{f(z)}{z} + \vartheta f'(z) - 1} \right| < 1, \ \ (z \in \mathbb{D}),$$

where $\tau \in \mathbb{C} \setminus \{0\}, \delta < 1, 0 < \vartheta \leq 1$.The class $\mathcal{R}^{\tau}_{\delta}(\vartheta)$ was introduced by Swaminathan [35].

Lemma 1.9. [5] *If $f \in \mathcal{R}^{\tau}(\vartheta, \delta)$ is of the form (1.1) then*

$$|a_n| \leq \frac{|\tau|(\vartheta - \delta)}{n}, \ \ n \in \mathbb{N} \setminus \{1\}. \tag{1.14}$$

The result is sharp.

Motivated by the stated research works, we establish some sufficient conditions for the Wright distribution series $K_\psi(\lambda,\kappa,s,z)$ belonging to the classes $\mathcal{G}^{-1}(\gamma)$ and $\mathcal{H}^{-1}(\gamma)$. We also obtain inclusion relations for aforecited classes with $\mathcal{R}^\tau(C,D)$ by applying certain convolution operator $\mathcal{I}(\lambda,\kappa,s)$ defined by (1.11).

2. Main result

In this section, first we establish a sufficient condition for the function $f \in \mathcal{A}$ to be in the class $\mathcal{G}^{-1}(\lambda)$ and $\mathcal{H}^{-1}(\lambda)$.

Theorem 2.1. *Let $\lambda,\kappa,s>0$ and for some $\gamma(\gamma>1)$. Then $K_\psi(\lambda,\kappa,s,z)\in\mathcal{G}^{-1}(\gamma)$ if*

$$\gamma\Gamma(\kappa)\mathbb{W}_{\lambda,\kappa+\lambda}(s)\le(\gamma-1)\Gamma(\kappa+\lambda). \tag{2.1}$$

Proof. To prove that $K_\psi(\lambda,\kappa,s,z)\in\mathcal{G}^{-1}(\gamma)$, according to Lemma 1.6, we must show that

$$\sum_{n=2}^{\infty}(\gamma n-1)\frac{\Gamma(\kappa)s^n}{\Gamma(\lambda(n-1)+\kappa)(n-1)!\mathbb{W}_{\lambda,\kappa}(s)}\le\gamma-1$$

Now

$$\begin{aligned}
&\sum_{n=2}^{\infty}(\gamma n-1)\frac{\Gamma(\kappa)s^n}{\Gamma(\lambda(n-1)+\kappa)(n-1)!\mathbb{W}_{\lambda,\kappa}(s)}\\
&=\sum_{n=2}^{\infty}\{\gamma(n-1)+\gamma-1\}\frac{\Gamma(\kappa)s^n}{\Gamma(\lambda(n-1)+\kappa)(n-1)!\mathbb{W}_{\lambda,\kappa}(s)}\\
&=\frac{1}{\mathbb{W}_{\lambda,\kappa}(s)}\left[\sum_{n=2}^{\infty}\frac{\gamma\Gamma(\kappa)s^n}{\Gamma(\lambda(n-1)+\kappa)(n-2)!}+\sum_{n=2}^{\infty}\frac{(\gamma-1)\Gamma(\kappa)s^n}{(\lambda(n-1)+\kappa)(n-1)!}\right]\\
&=\frac{1}{\mathbb{W}_{\lambda,\kappa}(s)}\left[\gamma s\frac{\Gamma(\kappa)}{\Gamma(\kappa+\lambda)}\mathbb{W}_{\lambda,\kappa+\lambda}(s)+(\gamma-1)\{\mathbb{W}_{\lambda,\kappa}(s)-s\}\right]\\
&\le\gamma-1,\ \text{by the given hypothesis.}
\end{aligned}$$

This concludes the proof of Theorem 2.1. □

Theorem 2.2. *Let $\lambda,\kappa,s>0$ and for some $\gamma(\gamma>1)$. Then $K_\psi(\lambda,\kappa,s,z)\in\mathcal{H}^{-1}(\gamma)$ if*

$$\gamma s\frac{\Gamma(\kappa)}{\Gamma(\kappa+2\lambda)}\mathbb{W}_{\lambda,\kappa+2\lambda}(s)+(3\gamma-1)\frac{\Gamma(\kappa)}{\Gamma(\kappa+\lambda)}\mathbb{W}_{\lambda,\kappa+\lambda}(s)\le\gamma-1. \tag{2.2}$$

Proof. The proof is similar to Theorem 2.1. Therefore, we omit the details involved. □

Theorem 2.3. *Let $\lambda,\kappa,s>0, f\in\mathcal{R}^\tau(\vartheta,\delta)$ and for some $\gamma(\gamma>1)$. Then $\mathcal{I}(\lambda,\kappa,s)f\in\mathcal{H}^{-1}(\gamma)$ if*

$$\frac{(\vartheta-\delta)|\tau|}{\mathbb{W}_{\lambda,\kappa}(s)}\left[\gamma s\frac{\Gamma(\kappa)}{\Gamma(\kappa+\lambda)}\mathbb{W}_{\lambda,\kappa+\lambda}(s)+(\gamma-1)\{\mathbb{W}_{\lambda,\kappa}(s)-s\}\right]\le\gamma-1. \tag{2.3}$$

Proof. To prove that $\mathcal{I}(\lambda,\kappa,s)f \in \mathcal{H}^{-1}(\gamma)$, according to Lemma 1.7, we must show that
$$\sum_{n=2}^{\infty} n(\gamma n-1)\frac{\Gamma(\kappa)s^n|a_n|}{\Gamma(\lambda(n-1)+\kappa)(n-1)!\mathbb{W}_{\lambda,\kappa}(s)} \leq \gamma-1.$$
Since $f \in \mathcal{R}^{\tau}(\vartheta,\delta)$, from Lemma 1.9, we have $|a_n| \leq \frac{|\tau|(\vartheta-\delta)}{n}$.
Now
$$\begin{aligned}
&\sum_{n=2}^{\infty} n(\gamma n-1)\frac{\Gamma(\kappa)s^n|a_n|}{\Gamma(\lambda(n-1)+\kappa)(n-1)!\mathbb{W}_{\lambda,\kappa}(s)}\\
&= \frac{(\vartheta-\delta)|\tau|}{\mathbb{W}_{\lambda,\kappa}(s)}\sum_{n=2}^{\infty}(\gamma n-1)\frac{\Gamma(\kappa)s^n|a_n|}{\Gamma(\lambda(n-1)+\kappa)(n-1)!}\\
&= \frac{(\vartheta-\delta)|\tau|}{\mathbb{W}_{\lambda,\kappa}(s)}\left[\sum_{n=2}^{\infty}\frac{\gamma\Gamma(\kappa)s^n}{\Gamma(\lambda(n-1)+\kappa)(n-2)!}+\sum_{n=2}^{\infty}\frac{(\gamma-1)\Gamma(\kappa)s^n}{(\lambda(n-1)+\kappa)(n-1)!}\right]\\
&= \frac{(\vartheta-\delta)|\tau|}{\mathbb{W}_{\lambda,\kappa}(s)}\left[\gamma s\frac{\Gamma(\kappa)}{\Gamma(\kappa+\lambda)}\mathbb{W}_{\lambda,\kappa+\lambda}(s)+(\gamma-1)\left\{\mathbb{W}_{\lambda,\kappa}(s)-s\right\}\right]\\
&\leq \gamma-1, \text{ by the given hypothesis.}
\end{aligned}$$
This concludes the proof of Theorem 2.3. □

3. An integral operator

Theorem 3.1. *If the function $\mathcal{G}(\lambda,\kappa,s,z)$ is given by*
$$\mathcal{G}(\lambda,\kappa,s,z) = \int_0^z \frac{K_{\psi}(\lambda,\kappa,s,t)}{t}dt, \quad z \in \mathbb{U} \tag{3.1}$$
then $\mathcal{G}(\lambda,\kappa,s,z) \in \mathcal{H}^{-1}(\gamma)$ if
$$\gamma\Gamma(\kappa)\mathbb{W}_{\lambda,\kappa+\lambda}(s) \leq (\gamma-1)\Gamma(\kappa+\lambda). \tag{3.2}$$

Proof. Since
$$\mathcal{G}(\lambda,\kappa,s,z) = z+\sum_{n=2}^{\infty}\frac{\Gamma(\kappa)s^n}{n!\Gamma(\lambda(n-1)+\kappa)\mathbb{W}_{\lambda,\kappa}(s)}z^n$$
by Lemma 1.7, we need only to show that
$$\sum_{n=2}^{\infty} n(\gamma n-1)\frac{\Gamma(\kappa)s^n|a_n|}{n!\Gamma(\lambda(n-1)+\kappa)\mathbb{W}_{\lambda,\kappa}(s)} \leq \gamma-1.$$
or, consistently
$$\sum_{n=2}^{\infty}(\gamma n-1)\frac{\Gamma(\kappa)s^n|a_n|}{(n-1)!\Gamma(\lambda(n-1)+\kappa)\mathbb{W}_{\lambda,\kappa}(s)} \leq \gamma-1.$$
The enduring part of the proof of Theorem 3.1 is parallel to that of Theorem 2.1, and so we omit the details. □

4. Conclusions

In this paper we have considered the subclasses $\mathcal{G}^{-1}(\lambda)$ and $\mathcal{H}^{-1}(\lambda)$ of reciprocal order related with Wright distribution series. We obtained sufficient condition, inclusion relation and properties related to integral operator for functions of these subclasses related to Wright distribution series.

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Mallikarjun G. Shrigan
Department of Mathematics,
School of Computational Sciences and Mathematics,
JSPM University Pune,
Pune, India
e-mail: mgshrigan@gmail.com

Sidharam D. Bhourgunde
Department of Mathematics,
Bhivrabai Sawant Institute of Technology and Research,
Pune, India
e-mail: sdbhourgunde@gmail.com

Girish D. Shelake
Department of Mathematics,
Willingdon College,
Sangli, India
e-mail: shelakegd@gmail.com