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# A class of harmonic univalent functions associated with modified $q-$Catas operator

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**Abstract.** Using the modified $q-$Catas operator, we define a class of harmonic univalent functions and obtain various properties for functions in this class.

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## 1. Introduction

A function $f = u + iv$ , continuous and defined in a simply connected complex domain $\mathcal{D}$ is called harmonic in $\mathcal{D}$ if both $u$ and $v$ are real harmonic in $\mathcal{D}$. If $h, g$ are analytic in $\mathcal{D}$, then $f$ can be written in the form

$$f = h + \overline{g}, \tag{1.1}$$

where, $h$ and $g$ are the analytic and co-analytic parts, respectively. The necessary and sufficient condition for $f$ to be locally univalent and sense-preserving in $\mathcal{D}$ is that $|h'| > |g'|$ in $\mathcal{D}$ (see [17]).

The class of harmonic, univalent, and orientation preserving functions, of the form (1.1) defined in $\mathcal{E} = \{z : |z| < 1\}$ is denoted by $\mathcal{H}$, for which $f(0) = f'(0)-1 = 0$.

Thus, for $f = h + \overline{g} \in \mathcal{H}$ , $h$ and $g$ can be expressed in the form:

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad g = \sum_{k=1}^{\infty} b_k z^k, \; |b_1| < 1,$$

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then $f$ is of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k + \sum_{k=1}^{\infty} \overline{b_k z^k}. \tag{1.2}$$

If the co-analytic part $g \equiv 0$, then $\mathcal{H}$ reduces to class $\mathcal{S}$ of normalized analytic univalent functions.

Let $\overline{\mathcal{H}}$ denotes the subclass of $\mathcal{H}$ consisting of functions $f = h + \overline{g}$ such that $h$ and $g$ given by

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad g = (-1)^n \sum_{k=1}^{\infty} b_k z^k, |b_1| < 1. \tag{1.3}$$

Recently, several researchers studied classes of harmonic functions (see Aouf [3], Aouf et al. [8, 10, 11], Dixit and Porwal [18], Porwal and Dixit [26, 27]).

For $f \in \mathcal{S}$, and $0 < q < 1$, the Jackson's $q-$derivative is given by [22] (see also [2, 4, 7, 12, 19, 20, 21, 28, 30, 31, 32]):

$$D_q f(z) = \begin{cases} \frac{f(z)-f(qz)}{(1-q)z} & z \neq 0 \\ 0 & z = 0 \end{cases} \ (z \in \mathcal{E}), \tag{1.4}$$

where

$$[k]_q = \frac{1-q^k}{1-q} \quad (0 < q < 1).$$

As $q \to 1^-, [k]_q \to k$ and, so $D_q f(z) = f'(z)$.

For $f(z) \in \mathcal{S}, \delta, l \geq 0$ and $q \in (0,1)$, Aouf and Madian [5, with $p = 1$] defined the $q-$Catas operator by:

$$\begin{aligned} I_q^0(\delta,l)f(z) &= f(z), \\ I_q^1(\delta,l)f(z) &= (1-\delta)f(z) + \frac{\delta}{[l+1]_q z^{l-1}} D_q(z^l f(z)) = I_q(\delta,l)f(z) \\ &= z + \sum_{k=2}^{\infty} \frac{[l+1]_q + \delta([k+l]_q - [l+1]_q)}{[l+1]_q} a_k z^k \\ &\vdots \\ I_q^n(\delta,l)f(z) &= (1-\delta)I_q^{n-1}(\delta,l)f(z) + \frac{\delta}{[l+1]_q z^{l-1}} D_q(z^l I_q^{n-1}(\delta,l)f(z)) \\ (n &\in \mathcal{N}, \mathcal{N} = \{1,2,...\}). \end{aligned}$$

That is

$$I_q^n(\delta,l)f(z) = z + \sum_{k=2}^{\infty} \sigma_{q,k}^n(\delta,l) a_k z^k \ .(n \in \mathcal{N}_0 = \mathcal{N} \cup \{0\}), \tag{1.5}$$

where

$$\sigma_{q,k}^n(\delta,l) = \left[ \frac{[l+1]_q + \delta([k+l]_q - [l+1]_q)}{[l+1]_q} \right]^n. \tag{1.6}$$

From (1.5) we have:

$$z\delta q^l D_q(I_q^n(\delta,l)f(z)) = [l+1]_q I_q^{n+1}(\delta,l)f(z) - \{(1-\delta)q^l + [l]_q\} I_q^n(\delta,l)f(z), \delta \neq 0.$$

Note that:

(i) $\lim_{q\to 1-} I_q^n(\delta,l)f(z) = I^n(\delta,l)f(z)$, (see [14]);

(ii) $I_q^n(1,0)f(z) = D_q^n f(z)$ (see Govindaraj and Sivasubramanian [21] and [9]);

(iii) $I_q^n(\delta,0)f(z) = D_{\delta,q}^n f(z)$ :

$$\left\{ f \in \mathcal{S} : D_{\delta,q}^n f(z) = z + \sum_{k=2}^{\infty} \left[1+\delta([k]_q - 1)\right]^n a_k z^k \right\},$$

which reduces to Al-Oboudi operator when $q \to 1-$, (see [1]) which is the Salagean operator when $\delta = 1$ (see [29] and [6]);

(iv) $I_q^n(1,l)f(z) = I_q^n(l)f(z)$ which when $q \to 1-$ reduces to $I_l^n f(z)$ (see Cho and Srivastava [15], see also [16]).

Motivated with the definition of modified Salagean operator introduced by Jahangiri et al. [23], Mostafa et al. [24], defined the modified Catas operator by

$$I^n(\delta,l)\,f(z) = I^n(\delta,l)\,h(z) + (-1)^n\,\overline{I^n(\delta,l)\,g(z)},$$

where

$$I^n(\delta,l)\,h(z) = z + \sum_{k=2}^{\infty} \left(\frac{l+1+\delta(k-1)}{l+1}\right)^n a_k z^k$$

and

$$I^n(\delta,l)\,g(z) = (-1)^n \sum_{k=1}^{\infty} \left(\frac{l+1+\delta(k-1)}{l+1}\right)^n b_k z^k.$$

Now, we define the modified $q-$Catas operator by:

$$I_q^n(\delta,l)\,f(z) = I_q^n(\delta,l)\,h(z) + (-1)^n\,\overline{I_q^n(\delta,l)\,g(z)}, \tag{1.7}$$

where

$$I_q^n(\delta,l)\,h(z) = z + \sum_{k=2}^{\infty} \sigma_{q,k}^n(\delta,l) a_k z^k$$

and

$$I_q^n(\delta,l)\,g(z) = \sum_{k=1}^{\infty} \sigma_{q,k}^n(\delta,l) b_k z^k.$$

For $1 \le \beta < \frac{4}{3}, n \in \mathcal{N}_0, \delta, l \ge 0, q \in (0,1)$ and for all $z \in \mathcal{E}$, let $\mathcal{G}_q^n(\delta,l,\beta)$ denote the family of harmonic functions $f$ of the form (1.2) and satisfying:

$$\operatorname{Re}\left\{ \frac{I_q^{n+1}(\delta,l)\,f(z)}{I_q^n(\delta,l)\,f(z)} \right\} < \beta. \tag{1.8}$$

Choosing different values of $n, l, \delta, \beta$ when $q \to 1-$, we obtain many subclasses of $\mathcal{G}_q^n(\delta,l,\beta)$ for example:

(1) Putting $\delta = 1$, then it reduces to the class $S_H(n,l,\beta)$ studied by Porwal [25].

(2) Putting $\delta = 1$ and $l = 0$, then it reduces to the class $S_H(n,\beta)$ studied by Porwal and Dixit [27];

(3) Putting $n = 0, l = 0$ and $\delta = 1$, then it reduces to the class $L_H(\beta)$ studied by Porwal and Dixit [26];

(4) Putting $n = 1, l = 0$ and $\delta = 1$, then it reduces to the class $M_H(\beta)$ studied by Porwal and Dixit [26];

(5) Putting $n = 0$ and $n = 1$ with $l = 0, \delta = 1$, $g \equiv 0$, then it reduces to the classes $\mathcal{N}(\beta)$ and $\mathcal{M}(\beta)$ studied by Uralegaddi et al. [33].

Also we can obtain the following subclasses:
i) $\mathcal{G}_q^n(\delta, 0, \beta) = \mathcal{G}_q^n(\delta, \beta)$ :

$$\operatorname{Re}\left\{\frac{D_q^{n+1}(\delta)\, f(z)}{D_q^n(\delta)\, f(z)}\right\} < \beta, D_q^n(\delta)\, f(z) = D_q^n(\delta)\, h(z) + (-1)^n\, \overline{D_q^n(\delta)\, g(z)};$$

ii) $\mathcal{G}_q^n(1, 0, \beta) = \mathcal{G}_q^n(\beta)$ :

$$\operatorname{Re}\left\{\frac{D_q^{n+1} f(z)}{D_q^n f(z)}\right\} < \beta, D_q^n f(z) = D_q^n h(z) + (-1)^n\, \overline{D_q^n g(z)};$$

iii) $\mathcal{G}_q^n(1, l, \beta) = \mathcal{G}_q^n(l, \beta)$ :

$$\operatorname{Re}\left\{\frac{I_q^{n+1}(l) f(z)}{I_q^n(l) f(z)}\right\} < \beta, I_q^n(l) f(z) = I_q^n(l) h(z) + (-1)^n\, \overline{I_q^n(l) g(z)}.$$

Let $\overline{\mathcal{G}_q}^n(\delta, l, \beta)$ be the subclass of $\mathcal{G}_q^n(\delta, l, \beta)$ consisting functions $f = h + \overline{g}$ such that $h$ and $g$ given by (1.3).

## 2. Main results

Unless otherwise mentioned, we assume in the reminder of this paper that, $1 \leq \beta < \frac{4}{3}, n \in \mathcal{N}_0, \delta, l \geq 0, q \in (0, 1)$, $\sigma_{q,k}^n(\delta, l)$ is given by (1.6) and $f$ is of the form (1.3).

**Theorem 2.1.** *Let $f = h + \overline{g}$ be given by (1.2). Furthermore, let*

$$\sum_{k=2}^{\infty} \frac{\sigma_{q,k}^n(\delta, l)\, \{[l+1]_q + \delta([k+l]_q - [l+1]_q) - \beta[l+1]_q\}}{[l+1]_q(\beta - 1)}\, |a_k| +$$

$$+ \sum_{k=1}^{\infty} \frac{\sigma_{q,k}^n(\delta, l)\, \{[l+1]_q + \delta([k+l]_q - [l+1]_q) + \beta[l+1]_q\}}{[l+1]_q(\beta - 1)}\, |b_k| \leq 1. \tag{2.1}$$

*Then $f\ (z)$ is sense-preserving, harmonic univalent in $\mathcal{E}$ and $f\ (z) \in \mathcal{G}_q^n(\delta, l, \beta)$.*

*Proof.* If $z_1 \neq z_2$, then

$$\begin{aligned} \left|\frac{f(z_1)-f(z_2)}{h(z_1)-h(z_2)}\right| &\geq 1-\left|\frac{g(z_1)-g(z_2)}{h(z_1)-h(z_2)}\right| = 1-\left|\frac{\sum\limits_{k=1}^{\infty} b_k\left(z_1^k-z_2^k\right)}{\left(z_1^k-z_2^k\right)+\sum\limits_{k=2}^{\infty} a_k\left(z_1^k-z_2^k\right)}\right| \\ &> 1-\frac{\sum\limits_{k=1}^{\infty} k\,|b_k|}{1-\sum\limits_{k=2}^{\infty} k\,|a_k|} \\ &\geq 1-\frac{\frac{\sigma_{q,k}^n(\delta,l)\{[l+1]_q+\delta([k+l]_q-[l+1]_q)+\beta[l+1]_q\}}{[l+1]_q(\beta-1)}\,|b_k|}{1-\frac{\sigma_{q,k}^n(\delta,l)\{[l+1]_q+\delta([k+l]_q-[l+1]_q)-\beta[l+1]_q\}}{[l+1]_q(\beta-1)}\,|a_k|} \geq 0, \end{aligned}$$

which proves univalence. $f(z)$ is sense-preserving in $\mathcal{E}$ since

$$\begin{aligned} \left|h'(z)\right| &\geq 1-\sum_{k=2}^{\infty} k\,|a_k|\,|z|^{k-1} \\ &> 1-\sum_{k=2}^{\infty} k\,|a_k| \geq 1-\sum_{k=2}^{\infty}\frac{\sigma_{q,k}^n(\delta,l)\{[l+1]_q+\delta([k+l]_q-[l+1]_q)-\beta[l+1]_q\}}{[l+1]_q(\beta-1)}\,|a_k| \\ &\geq \sum_{k=1}^{\infty}\frac{\sigma_{q,k}^n(\delta,l)\{[l+1]_q+\delta([k+l]_q-[l+1]_q)+\beta[l+1]_q\}}{[l+1]_q(\beta-1)}\,|b_n| \geq \sum_{k=1}^{\infty} k\,|b_k| \\ &> \sum_{k=1}^{\infty} k\,|b_k|\left|z^{k-1}\right| \geq \left|g'(z)\right|. \end{aligned}$$

Now to show that $f \in \mathcal{G}_q^n(\delta, l; \beta)$, we may show that if (2.1) holds then (1.8) is satisfied. Using the fact that $Re\{w\} < \beta$ if and only if $|w-1| < |w+1-2\beta|$, it suffices to show that

$$\left|\frac{\dfrac{I_q^{n+1}(\delta,l)\,f(z)}{I_q^n(\delta,l)\,f(z)}-1}{\dfrac{I_q^{n+1}(\delta,l)\,f(z)}{I_q^n(\delta,l)\,f(z)}+1-2\beta}\right| < 1. \tag{2.2}$$

The L.H.S. of (2.2):

$$= \left|\frac{\left\{\begin{array}{c}\sum\limits_{k=2}^{\infty}\sigma_{q,k}^n(\delta,l)\{[l+1]_q+\delta([k+l]_q-[l+1]_q)-[l+1]_q\}\,a_k z^k \\ +(-1)^{n+1}\sum\limits_{k=1}^{\infty}\sigma_{q,k}^n(\delta,l)\{[l+1]_q+\delta([k+l]_q-[l+1]_q)+[l+1]_q\}\,\overline{b_k z^k}\end{array}\right.}{\left\{\begin{array}{c}2(1-\beta)z+\sum\limits_{k=2}^{\infty}\sigma_{q,k}^n(\delta,l)\{[l+1]_q+\delta([k+l]_q-[l+1]_q)+(1-2\beta)[l+1]_q\}\,a_k z^k \\ +(-1)^{n+1}\sum\limits_{k=2}^{\infty}\sigma_{q,k}^n(\delta,l)\{[l+1]_q+\delta([k+l]_q-[l+1]_q)-(1-2\beta)[l+1]_q\}\,\overline{b_k z^k}\end{array}\right.}\right|$$

$$\leq \frac{\left\{\begin{array}{c} \sum\limits_{k=2}^{\infty}\sigma_{q,k}^{n}(\delta,l)\left\{[l+1]_q+\delta([k+l]_q-[l+1]_q)-[l+1]_q\right\}|a_k|\,|z|^k \\ +\sum\limits_{k=1}^{\infty}\sigma_{q,k}^{n}(\delta,l)\left\{[l+1]_q+\delta([k+l]_q-[l+1]_q)+[l+1]_q\right\}|b_k|\,|z|^k \end{array}\right.}{\left\{\begin{array}{c} 2(\beta-1)-\sum\limits_{k=2}^{\infty}\sigma_{q,k}^{n}(\delta,l)\left\{[l+1]_q+\delta([k+l]_q-[l+1]_q)-(2\beta-1)[l+1]_q\right\}|a_k|\,|z|^k \\ -\sum\limits_{k=2}^{\infty}\sigma_{q,k}^{n}(\delta,l)\left\{[l+1]_q+\delta([k+l]_q-[l+1]_q)+(2\beta-1)[l+1]_q\right\}|b_k|\,|z|^k \end{array}\right.}$$

$$< \frac{\left\{\begin{array}{c} \sum\limits_{k=2}^{\infty}\sigma_{q,k}^{n}(\delta,l)\left\{[l+1]_q+\delta([k+l]_q-[l+1]_q)-[l+1]_q\right\}|a_k| \\ +\sum\limits_{k=1}^{\infty}\sigma_{q,k}^{n}(\delta,l)\left\{[l+1]_q+\delta([k+l]_q-[l+1]_q)+[l+1]_q\right\}|b_k| \end{array}\right.}{\left\{\begin{array}{c} 2(\beta-1)-\sum\limits_{k=2}^{\infty}\sigma_{q,k}^{n}(\delta,l)\left\{[l+1]_q+\delta([k+l]_q-[l+1]_q)-(2\beta-1)[l+1]_q\right\}|a_k| \\ -\sum\limits_{k=2}^{\infty}\sigma_{q,k}^{n}(\delta,l)\left\{[l+1]_q+\delta([k+l]_q-[l+1]_q)+(2\beta-1)[l+1]_q\right\}|b_k| \end{array}\right.},$$

which according to (1.8) is bounded by 1. The harmonic univalent function of the form

$$\begin{aligned} f(z) &= z+\sum_{k=2}^{\infty}\frac{(\beta-1)[l+1]_q}{\sigma_{q,k}^{n}(\delta,l)\left\{[l+1]_q+\delta([k+l]_q-[l+1]_q)-\beta[l+1]_q\right\}}x_k z^k \\ &\quad +\sum_{k=1}^{\infty}\frac{(\beta-1)[l+1]_q}{\sigma_{q,k}^{n}(\delta,l)\left\{[l+1]_q+\delta([k+l]_q-[l+1]_q)+\beta[l+1]_q\right\}}\overline{y_k z^k}, \quad (2.3) \end{aligned}$$

where $\sum\limits_{k=2}^{\infty}|x_k|+\sum\limits_{k=1}^{\infty}|y_k|=1$, shows that the coefficient bound given by (2.1) is sharp. It is worthy to note that the function of the form (2.3) belongs to the class $\mathcal{G}_q^n(\delta,l,\beta)$ for all $\sum\limits_{k=2}^{\infty}|x_k|+\sum\limits_{k=1}^{\infty}|y_k|\leq 1$ since (2.1) holds. □

**Theorem 2.2.** *A function $f\in\overline{\mathcal{G}_q}^n(\delta,l,\beta)$ if and only if*

$$\begin{aligned} &\sum_{k=2}^{\infty}\sigma_{q,k}^{n}(\delta,l)\left\{[l+1]_q+\delta([k+l]_q-[l+1]_q)-\beta[l+1]_q\right\}|a_k| \quad (2.4) \\ &+\sum_{k=1}^{\infty}\sigma_{q,k}^{n}(\delta,l)\left\{[l+1]_q+\delta([k+l]_q-[l+1]_q)+\beta[l+1]_q\right\}|b_k| \\ \leq\; &(\beta-1)[l+1]_q. \end{aligned}$$

*Proof.* Since $\overline{\mathcal{G}_q}^n(\delta,l,\beta) \subset \mathcal{G}_q^n(\delta,l,\beta)$, we only need to prove the "only if" part. The condition (1.8) is equivalent to

$$\mathrm{Re}\left\{\frac{\begin{array}{c}(\beta-1)z-\sum\limits_{k=2}^{\infty}\sigma_{q,k}^n(\delta,l)\left\{[l+1]_q+\delta([k+l]_q-[l+1]_q)-\beta[l+1]_q\right\}|a_k|\,z^k\\ -(-1)^{2n-1}\sum\limits_{k=1}^{\infty}\sigma_{q,k}^n(\delta,l)\left\{[l+1]_q+\delta([k+l]_q-[l+1]_q)+\beta[l+1]_q\right\}|b_k|\,\overline{z^k}\end{array}}{z+\sum\limits_{k=2}^{\infty}\sigma_{q,k}^n(\delta,l)\,|a_k|\,z^k+\sum\limits_{k=1}^{\infty}(-1)^{2n-1}\sigma_{q,k}^n(\delta,l)\,|b_k|\,\overline{z^k}}\right\}>0.$$

The above condition must hold for all $z$, $|z|=r<1$. Choosing the values of $z$ on the positive real axis where $0\le r<1$, we must have

$$\mathrm{Re}\left\{\frac{\begin{array}{c}(\beta-1)-\sum\limits_{k=2}^{\infty}\sigma_{q,k}^n(\delta,l)\left\{[l+1]_q+\delta([k+l]_q-[l+1]_q)-\beta[l+1]_q\right\}|a_k|\,r^{k-1}\\ -\sum\limits_{k=2}^{\infty}\sigma_{q,k}^n(\delta,l)\left\{[l+1]_q+\delta([k+l]_q-[l+1]_q)+\beta[l+1]_q\right\}|b_k|\,r^{k-1}\end{array}}{1-\sum\limits_{k=2}^{\infty}\sigma_{q,k}^n(\delta,l)\,|a_k|\,r^{k-1}-\sum\limits_{k=1}^{\infty}\sigma_{q,k}^n(\delta,l)\,|b_k|\,r^{k-1}}\right\}\ge 0. \tag{2.5}$$

If condition (2.4) does not hold, then the numerator in (2.5) is negative for $r$ sufficiently close to 1. Hence there exist $z_0=r_0\in(0,1)$ for which the quotient in (2.5) is negative. This contradicts the required condition for $f(z)\in\overline{\mathcal{G}_q}^n(\delta,l,\beta)$. This completes the proof of Theorem 2.2. □

**Theorem 2.3.** *Let $f(z)\in\overline{\mathcal{G}_q}^n(\delta,l,\beta)$. Then for $|z|=r<1$,we have*

$$\begin{aligned}&(1+|b_1|)r-\tfrac{1}{\sigma_{q,2}^n(\delta,l)}\left(\tfrac{(\beta-1)[l+1]_q}{\sigma_{q,2}^n(\delta,l)\{[l+1]_q(1-\beta)+\delta q^{l+1}\}}-\tfrac{(\beta+1)[l+1]_q}{\sigma_{q,2}^n(\delta,l)\{[l+1]_q(1-\beta)+\delta q^{l+1}\}}\,|b_1|\right)r^2\\ \le\;& |f(z)|\le\\ &(1+|b_1|)r+\tfrac{1}{\sigma_{q,2}^n(\delta,l)}\left(\tfrac{(\beta-1)[l+1]_q}{\sigma_{q,2}^{m-n}(\delta,l)\{[l+1]_q(1-\beta)+\delta q^{l+1}\}}+\tfrac{(\beta+1)[l+1]_q}{\sigma_{q,2}^n(\delta,l)\{[l+1]_q(1-\beta)+\delta q^{l+1}\}}\,|b_1|\right)r^2.\end{aligned} \tag{2.6}$$

*The results are sharp with equality for $f(z)$ defined by*

$$f(z)=z\pm b_1\overline{z}\pm\tfrac{1}{\sigma_{q,2}^n(\delta,l)}\left(\tfrac{(\beta-1)[l+1]_q}{\sigma_{q,2}^{m-n}(\delta,l)\{[l+1]_q(1-\beta)+\delta q^{l+1}\}}-\tfrac{(\beta+1)[l+1]_q}{\sigma_{q,2}^n(\delta,l)\{[l+1]_q(1-\beta)+\delta q^{l+1}\}}\,|b_1|\right)\overline{z}^2, \tag{2.7}$$

where

$$\sigma_{q,2}^n(\delta,l)=\left[\frac{[l+1]_q+\delta q^{l+1}}{[l+1]_q}\right]^n. \tag{2.8}$$

*Proof.* We only prove the right-hand inequality and the proof of the left-hand is similar and will be omitted. Let $f(z)\in\overline{\mathcal{G}}^n(\delta,l,\beta)$.Taking the absolute value of $f$ we have:

$$|f(z)| \leq (1+|b_1|)r + r^2 \sum_{k=2}^{\infty} (|a_k| + |b_k|)$$

$$= (1+|b_1|)r + \frac{(\beta-1)\,[l+1]_q}{\sigma_{q,2}^n(\delta,l)\{[l+1]_q(1-\beta)+\delta q^{l+1}\}} \sum_{n=2}^{\infty} \left( \frac{\sigma_{q,2}^n(\delta,l)\{[l+1]_q(1-\beta)+\delta q^{l+1}}{(\beta-1)\,[l+1]_q} |a_k| \right.$$

$$\left. + \frac{\sigma_{q,2}^n(\delta,l)\{[l+1]_q(1-\beta)+\delta q^{l+1}\}}{(\beta-1)[l+1]_q} |b_k| \right) r^2$$

$$\leq (1+|b_1|)r + \frac{(\beta-1)[l+1]_q}{\sigma_{q,2}^n(\delta,l)\{[l+1]_q(1-\beta)+\delta q^{l+1}\}} \sum_{k=2}^{\infty} \left( \frac{\sigma_{q,k}^n(\delta,l)\{[l+1]_q+\delta([k+l]_q-[l+1]_q)-\beta[l+1]_q\}}{(\beta-1)[l+1]_q} |a_k| \right.$$

$$\left. + \frac{\{[l+1]_q+\delta([k+l]_q-[l+1]_q)+\beta[l+1]_q\}}{(\beta-1)[l+1]_q} |b_k| \right) r^2$$

$$\leq (1+|b_1|)r + \frac{(\beta-1)[l+1]_q}{\sigma_{q,2}^n(\delta,l)\{[l+1]_q(1-\beta)+\delta q^{l+1}\}} \left(1 - \frac{(1+\beta)}{(\beta-1)} |b_1|\right) r^2$$

$$= (1+|b_1|)r + \left( \frac{(\beta-1)[l+1]_q}{\sigma_{q,2}^n(\delta,l)\{[l+1]_q(1-\beta)+\delta q^{l+1}\}} - \frac{(1+\beta)\,[l+1]_q}{\sigma_{q,2}^n(\delta,l)\{[l+1]_q(1-\beta)+\delta q^{l+1}\}} |b_1| \right) r^2.$$

This completes the proof of the Theorem 2.3. □

**Theorem 2.4.** *The function $f(z) \in \overline{\mathcal{G}_q}^n(\delta,l,\beta)$ if and only if*

$$f(z) = \sum_{k=1}^{\infty} (\gamma_k h_k(z) + \eta_k g_k(z)), \tag{2.9}$$

*where $h_1(z) = z$,*

$$h_k(z) = z + \frac{(\beta-1)[l+1]_q}{\sigma_{q,k}^n(\delta,l)\{[l+1]_q + \delta([k+l]_q - [l+1]_q) - \beta[l+1]_q\}} z^k, k = 2,3,... \tag{2.10}$$

*and*

$$g_k(z) = z + (-1)^{n-1} \frac{(\beta-1)[l+1]_q}{\sigma_{q,k}^n(\delta,l)\{[l+1]_q + \delta([k+l]_q - [l+1]_q) + \beta[l+1]_q\}} \overline{z}^k, \; k=1,2,..., \tag{2.11}$$

*$\gamma_k \geq 0, \eta_k \geq 0, \sum_{k=1}^{\infty} (\gamma_k + \eta_k) = 1$ and the extreme points of the class $\overline{\mathcal{G}_q}^n(\delta,l,\beta)$ are $\{h_k\}$ and $\{g_k\}$.*

*Proof.* Suppose that

$$\begin{aligned} f(z) &= \sum_{k=1}^{\infty} (\gamma_k h_k(z) + \eta_k g_k(z)) \\ &= z + \sum_{k=2}^{\infty} \frac{(\beta-1)[l+1]_q}{\sigma_{q,k}^n(\delta,l)\{[l+1]_q + \delta([k+l]_q - [l+1]_q) - \beta[l+1]_q\}} \gamma_k z^k \\ &\quad + (-1)^n \sum_{k=1}^{\infty} \frac{(\beta-1)[l+1]_q}{\sigma_{q,k}^n(\delta,l)\{[l+1]_q + \delta([k+l]_q - [l+1]_q) + \beta[l+1]_q\}} \eta_k \overline{z^k}. \end{aligned}$$

Then

$$\sum_{k=2}^{\infty} \frac{\sigma_{q,k}^{n}(\delta,l)\{[l+1]_q+\delta([k+l]_q-[l+1]_q)-\beta[l+1]_q\}}{(\beta-1)[l+1]_q} \left(\frac{(\beta-1)[l+1]_q}{\sigma_{q,k}^{n}(\delta,l)\{[l+1]_q+\delta([k+l]_q-[l+1]_q)-\beta[l+1]_q\}}\gamma_k\right)$$
$$+\sum_{k=1}^{\infty} \frac{\sigma_{q,k}^{n}(\delta,l)\{[l+1]_q+\delta([k+l]_q-[l+1]_q)+\beta[l+1]_q\}}{(\beta-1)[l+1]_q} \left(\frac{(\beta-1)[l+1]_q}{\sigma_{q,k}^{n}(\delta,l)\{[l+1]_q+\delta([k+l]_q-[l+1]_q)+\beta[l+1]_q\}}\eta_k\right)$$
$$= \sum_{k=2}^{\infty}\gamma_k+\sum_{k=1}^{\infty}\eta_k = 1-\gamma_1 \le 1$$

and so $f(z) \in \overline{\mathcal{G}_q}^{n}(\delta,l,\beta)$.

Conversely, if $f(z) \in \overline{\mathcal{G}_q}^{n}(\delta,l,\beta)$, then

$$|a_k| \le \frac{(\beta-1)[l+1]_q}{\sigma_{q,k}^{n}(\delta,l)\{[l+1]_q+\delta([k+l]_q-[l+1]_q)-\beta[l+1]_q\}}$$

and

$$|b_k| \le \frac{(\beta-1)[l+1]_q}{\sigma_{q,k}^{n}(\delta,l)\{[l+1]_q+\delta([k+l]_q-[l+1]_q)+\beta[l+1]_q\}}.$$

Setting

$$\gamma_k = \frac{\sigma_{q,k}^{n}(\delta,l)\{[l+1]_q+\delta([k+l]_q-[l+1]_q)-\beta[l+1]_q\}}{(\beta-1)[l+1]_q}|a_k| \ \ (k=2,3,...)$$

and

$$\eta_k = \frac{\sigma_{q,k}^{n}(\delta,l)\{[l+1]_q+\delta([k+l]_q-[l+1]_q)+\beta[l+1]_q\}}{(\beta-1)[l+1]_q}|b_k| \ \ (k=1,2,...),$$

we have $0 \le \gamma_k \le 1$ $(k=2,3,...)$ and $0 \le \eta_k \le 1$ $(k=1,2,...)$,

$$\gamma_1 = 1-\sum_{k=2}^{\infty}\gamma_k-\sum_{k=1}^{\infty}\eta_k \ge 0,$$

then, $f(z)$ can be expressed in the form (2.9).
For harmonic functions of the form:

$$f(z) = z+\sum_{k=2}^{\infty}|a_k|z^k+(-1)^{n-1}\sum_{k=1}^{\infty}|b_k|\overline{z^k} \tag{2.12}$$

and

$$G(z) = z+\sum_{k=2}^{\infty}|d_k|z^k+(-1)^{n-1}\sum_{k=1}^{\infty}|c_k|\overline{z^k}, \tag{2.13}$$

the convolution of $f$ and $G$ is given by

$$(f*G)(z) = (G*f)(z) = z+\sum_{k=2}^{\infty}|a_kd_k|z^k+\sum_{k=1}^{\infty}|b_kc_k|\overline{z^k}. \qquad \square$$

The next theorem shows that the class $\overline{\mathcal{G}_q}^{n}(\delta,l,\beta)$ is closed under convolution.

**Theorem 2.5.** *For $1 \le \beta \le \zeta < \frac{4}{3}$, let $f \in \overline{\mathcal{G}_q}^{n}(\delta,l,\beta)$ and $G \in \overline{\mathcal{G}_q}^{n}(\delta,l,\zeta)$. Then $f*G \in \overline{\mathcal{G}_q}^{n}(\delta,l,\beta) \subset \overline{\mathcal{G}_q}^{n}(\delta,l,\zeta)$.*

*Proof.* Since $G \in \overline{\mathcal{G}_q}^n(\delta, l, \zeta)$ then $|d_k| \leq 1$ and $|c_k| \leq 1$. For $f * G$ we have

$$\begin{aligned}
&\sum_{k=2}^{\infty} \tfrac{\sigma_{q,k}^n(\delta,l)\{[l+1]_q+\delta([k+l]_q-[l+1]_q)-\beta[l+1]_q\}}{(\beta-1)[l+1_q} \left|a_k d_k\right| z^k \\
&\quad + \sum_{k=1}^{\infty} \tfrac{\sigma_{q,k}^n(\delta,l)\{[l+1]_q+\delta([k+l]_q-[l+1]_q)+\beta[l+1]_q\}}{(\beta-1)[l+1_q} \left|b_k c_k\right| \overline{z^k} \\
\leq\; &\sum_{k=2}^{\infty} \tfrac{\sigma_{q,k}^n(\delta,l)\{[l+1]_q+\delta([k+l]_q-[l+1]_q)-\beta[l+1]_q\}}{(\beta-1)[l+1_q} \left|a_k\right| z^k \\
&\quad + \sum_{k=1}^{\infty} \tfrac{\sigma_{q,k}^n(\delta,l)\{[l+1]_q+\delta([k+l]_q-[l+1]_q)+\beta[l+1]_q\}}{(\beta-1)[l+1_q} \left|b_k\right| \overline{z^k} \\
\leq\; &\sum_{k=2}^{\infty} \tfrac{\sigma_{q,k}^n(\delta,l)\{[l+1]_q+\delta([k+l]_q-[l+1]_q)-\beta[l+1]_q\}}{(\beta-1)[l+1_q} \left|a_k\right| \\
&\quad + \sum_{k=1}^{\infty} \tfrac{\sigma_{q,k}^n(\delta,l)\{[l+1]_q+\delta([k+l]_q-[l+1]_q)+\beta[l+1]_q\}}{(\beta-1)[l+1_q} \left|b_k\right| \\
\leq\; & 1,
\end{aligned}$$

since $f \in \overline{\mathcal{G}_q}^n(\delta, l, \beta)$. Therefore by Theorem 2.1, $f * G \in \overline{\mathcal{G}}_q^n(\delta, l, \beta) \subset \overline{\mathcal{G}_q}^n(\delta, l, \zeta)$. □

The class $\overline{\mathcal{G}_q}^n(\delta, l, \beta)$ is closed under convex combinations by the following theorem.

**Theorem 2.6.** *The class* $\overline{\mathcal{G}_q}^n(\delta, l, \beta)$ *is closed under convex combination.*

*Proof.* For $i = 1, 2, 3, \ldots$, let $f_i \in \overline{\mathcal{G}_q}^n(\delta, l, \beta)$, where $f_i$ is given by

$$f_i = z + \sum_{k=2}^{\infty} \left|a_{k_i}\right| z^k + \sum_{k=1}^{\infty} \left|b_{k_i}\right| \overline{z^k}. \qquad \square$$

Then by using Theorem 2.1, we have

$$\sum_{k=2}^{\infty} \tfrac{\sigma_{q,k}^n(\delta,l)\{[l+1]_q+\delta([k+l]_q-[l+1]_q)-\beta[l+1]_q\}}{(\beta-1)[l+1]_q} \left|a_{k_i}\right|$$

$$+\sum_{k=1}^{\infty} \tfrac{\sigma_{q,k}^n(\delta,l)\{[l+1]_q+\delta([k+l]_q-[l+1]_q)+\beta[l+1]_q\}}{(\beta-1)[l+1]_q} \left|b_{k_i}\right| \leq 1. \tag{2.14}$$

For $\sum\limits_{k=1}^{\infty} \mu_i = 1$, $0 \leq \mu_i \leq 1$, the convex combination of $f_i$ may be written as

$$\sum_{i=1}^{\infty} \mu_i f_i(z) = z + \sum_{k=2}^{\infty} \left( \sum_{i=1}^{\infty} \mu_i \left|a_{k_i}\right| \right) z^k + \sum_{k=1}^{\infty} \left( \sum_{i=1}^{\infty} \mu_i \left|b_{k_i}\right| \right) \overline{z^k}. \tag{2.15}$$

Then by (2.14), we have

$$\begin{aligned}
&\sum_{k=2}^{\infty} \tfrac{\sigma_{q,k}^{n}(\delta,l)\{[l+1]_q+\delta([k+l]_q-[l+1]_q)-\beta[l+1]_q\}}{(\beta-1)[l+1]_q}\left(\sum_{i=1}^{\infty}\mu_i\left|a_{k_i}\right|\right)\\
&\quad+\tfrac{\sigma_{q,k}^{n}(\delta,l)\{[l+1]_q+\delta([k+l]_q-[l+1]_q)+\beta[l+1]_q\}}{(\beta-1)[l+1]_q}\sum_{k=1}^{\infty}\left(\sum_{i=1}^{\infty}\mu_i\left|b_{k_i}\right|\right)\\
&= \sum_{i=1}^{\infty}\mu_i\left(\sum_{k=2}^{\infty}\tfrac{\sigma_{q,k}^{n}(\delta,l)\{[l+1]_q+\delta([k+l]_q-[l+1]_q)-\beta[l+1]_q\}}{(\beta-1)[l+1]_q}\left|a_{k_i}\right|\right.\\
&\quad\left.+\sum_{k=1}^{\infty}\tfrac{\sigma_{q,k}^{n}(\delta,l)\{[l+1]_q+\delta([k+l]_q-[l+1]_q)+\beta[l+1]_q\}}{(\beta-1)[l+1]_q}\left|b_{k_i}\right|\right)\\
&\le \sum_{i=1}^{\infty}\mu_i = 1.
\end{aligned}$$

By Theorem 2.2, $\sum\limits_{i=1}^{\infty}\mu_i f_i(z)\in\overline{\mathcal{G}_q}^{n}(\delta,l,\beta)$.

Let $f(z)=h(z)+\overline{g(z)}$ be defined by (1.2) then $F(z)$ defined by

$$F(z)=\frac{c+1}{z^c}\int_0^z t^{c-1}h(t)dt+\overline{\frac{c+1}{z^c}\int_0^z t^{c-1}h(t)dt}(c>-1),$$

have the representation

$$F(z)=z+\sum_{k=2}^{\infty}\frac{c+1}{k+c}a_k z^k+(-1)^{n-1}\sum_{k=1}^{\infty}\frac{c+1}{k+c}b_k\overline{z}^k. \tag{2.16}$$

**Theorem 2.7.** *Let $f(z)=h(z)+\overline{g(z)}\in\overline{\mathcal{G}_q}^{n}(\delta,l,\beta)$, then $F(z)$ defined by (2.16) also belongs to $\overline{\mathcal{G}_q}^{n}(\delta,l,\beta)$.*

*Proof.* Since $f(z)\in\overline{\mathcal{G}_q}^{n}(\delta,l,\beta)$, then (2.1) is satisfied.

Now,

$$\begin{aligned}
&\sum_{k=2}^{\infty}\tfrac{\sigma_{q,k}^{n}(\delta,l)\{[l+1]_q+\delta([k+l]_q-[l+1]_q)-\beta[l+1]_q\}}{(\beta-1)[l+1]_q}\frac{c+1}{k+c}\left|a_k\right|\\
&\quad+\sum_{k=1}^{\infty}\tfrac{\sigma_{q,k}^{n}(\delta,l)\{[l+1]_q+\delta([k+l]_q-[l+1]_q)+\beta[l+1]_q\}}{(\beta-1)[l+1]_q}\frac{c+1}{k+c}\left|b_k\right|\\
&\le \sum_{k=2}^{\infty}\tfrac{\sigma_{q,k}^{n}(\delta,l)\{[l+1]_q+\delta([k+l]_q-[l+1]_q)-\beta[l+1]_q\}}{(\beta-1)[l+1]_q}\left|a_k\right|\\
&\quad+\sum_{k=1}^{\infty}\tfrac{\sigma_{q,k}^{n}(\delta,l)\{[l+1]_q+\delta([k+l]_q-[l+1]_q)+\beta[l+1]_q\}}{(\beta-1)[l+1]_q}\left|b_k\right|\\
&\le 1,
\end{aligned}$$

that is, $F(z)\in\overline{\mathcal{G}_q}^{n}(\delta,l,\beta)$. □

**Remark 2.8.** (i) Taking $\delta = 1$ and $q \to 1-$, in the above results, we obtain the results obtained by Porwal [25].

(ii) Specializing the parameters $\beta, l, \delta$ and $n$ in the above results, we obtain the corresponding results for the subclasses $\mathcal{G}_q^n(\delta, \beta), \mathcal{G}_q^n(\beta)$ and $\mathcal{G}_q^n(1, \beta)$.

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## References

[1] Al-Oboudi, F.M., *On univalent functions defined by a generalized Salagean operator*, Internat. J. Math. Math. Sci., **27**(2004), 1429-1436.

[2] Annby, M.H., Mansour, Z.S., *q−Fractional Calculus Equations*, Lecture Notes in Math., 2056, Springer-Verlag Berlin Heidelberg, 2012.

[3] Aouf, M.K., *A subclass of Salagean-type harmonic univalent functions*, Acta Univ. Apulensis Math. Inform., (2011), no. 25, 263-275.

[4] Aouf, M.K. Darwish, H.E., Salagean, G.S., *On a generalization of starlike functions with negative coefficients*, Math. Tome 43, **66**(2001), no. 1, 3-10.

[5] Aouf, M.K., Madian, S.M., *Neighborhoods properties for certain multivalent analytic functions associated with q − p−valent Catas operator*, J. Taibah Univ. for Science, **14**(2020), no. 1, 1226-1232.

[6] Aouf, M.K., Mostafa, A.O., *Some properties of a subclass of uniformly convex function with negative coefficients*, Demonstr. Math., **41**(2008), no. 2, 1-18.

[7] Aouf, M.K., Mostafa, A.O., *Subordination results for analytic functions associated with fractional q-calculus operators with complex order*, Afr. Mat. **31**(2020), 1387–1396.

[8] Aouf, M.K., Mostafa, A.O., Adwan, E.A., *Subclass of multivalent harmonic functions defined by Wright generalized hypergeometric functions*, J. Complex Anal. Volume 2013, Article ID 397428, 1-7.

[9] Aouf, M.K., Mostafa, A.O., Al-Quhali, F.Y., *A class of β−uniformly univalent functions defined by Salagean type q−difference operator*, Acta Univ. Apulensis Math. Inform., (2019), no. 60. 19-35.

[10] Aouf, M.K., Mostafa, A.O., Shamandy, A., Adwan, E.A., *Subclass of harmonic univalent functions defined by Dziok-Srivastafa operator*, LE Matimatiche, **68**(2013), no. 1, 165–177.

[11] Aouf, M.K., Mostafa, A.O., Shamandy, A., Wagdy, A.K., *A study on certain class of harmonic functions of complex order associated with convolution*, LE Matimatiche, **67**(2012), no. 2, 169–183.

[12] Aouf, M.K., Seoudy, T.M., *Convolution properties for classes of bounded analytic functions with complex order defined by q−derivative operator*, Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Math., **113**(2019), no. 2, 1279-1288.

[13] Aral, A, Gupta, V., Agarwal, R.P., *Applications of q−Calculus in Operator Theory*, Springer, New York, USA, 2013.

[14] Catas, A., Oros, G.I., Oros, G., *Differential subordinations associated with multiplier transformations*, Abstract Appl. Anal., **2008**(2008), ID 845724, 1-11.

[15] Cho, N.E., Kim, T.H., *Multiplier transformations and strongly close-to-convex functions*, Bull. Korean Math. Soc., **40** (2003), no. 3, 399-410.

[16] Cho, N.E., Srivastava, H.M., *Argument estimates of certain analytic functions defined by a class of multiplier transformations*, Math. Comput. Modelling, **37**(2003), no. 1-2, 39-49.

[17] Clunie, J., Shell-Small, T., *Harmonic univalent functions*, Ann. Acad. Sci. Fenn. Ser. A. I. Math., **9**(1984), 3-25.

[18] Dixit, K.K., Porwal, S., *A subclass of harmonic univalent functions with positive coefficients*, Tamkang J. Math., **41**(3) (2010), 261-269.

[19] Frasin, B.A., Murugusundaramoorthy, G., *A subordination results for a class of analytic functions defined by q-differential operator*, Ann. Univ. Paedagog. Crac. Stud. Math., **19**(2020), 53-64.

[20] Gasper, G., Rahman, M., *Basic Hypergeometric Series*, Combridge Univ. Press, Cambrididge, U. K., 1990.

[21] Govindaraj, M., Sivasubramanian, S., *On a class of analytic function related to conic domains involving $q-$calculus*, Anal. Math., **43**(3)(2017), no. 5, 475-487.

[22] Jackson, F.H., *On q-functions and a certain difference operator*, Trans. Royal. Soc. Edinburgh, **46**(1908), 253-281.

[23] Jahangiri, J.M., Murugusundaramoorthy, G., Vijaya, K., *Salagean-type harmonic univalent functions*, Southwest J. Pure Appl. Math., **2**(2002), 77-82.

[24] Mostafa, A.O., Aouf, M.K., Shamandy, A., Adwan, E.A., *Subclass of harmonic univalent functions defined by modified Catas operator*, Acta Univ. Apulensis Math. Inform., (2012), no. 32, 1-12.

[25] Porwal, S., *On a new subclass of harmonic univalent functions defined by multiplier transformation*, Mathematica Moravica, **19-2**(2015), 75-87.

[26] Porwal, S., Dixit, K.K., *New subclasses of harmonic starlike and convex functions*, Kyungpook Math. J., **53**(2013), 467-478.

[27] Porwal, S., Dixit, K.K., *On a new subclass of Salagean-type harmonic univalent functions*, Indian J. Math., **54**(2)(2012), 199-210.

[28] Ramachandran, C., Soupramanien, T., Frasin, B.A., *New subclasses of analytic function associated with $q-$difference operator*, Eur. J. Pure Appl. Math, **10**(2017), no. 2, 348-362.

[29] Salagean, G., *Subclasses of univalent functions*, Lect. Notes in Math., (Springer Verlag), **1013**(1983), 362-372.

[30] Seoudy, T.M., Aouf, M.K., *Coefficient estimates of new classes of $q-$starlike and $q-$convex functions of complex order*, J. Math. Ineq., **10**(2016), no. 1, 135-145.

[31] Srivastava, H.M., Aouf, M.K., Mostafa, A.O., *Some properties of analytic functions associated with fractional $q-$ calculus operators*, Miskolc Math. Notes, **20**(2019), no. 2, 1245–1260.

[32] Srivastava, H.M., Mostafa, A.O., Aouf, M.K., Zayed, H.M., *Basic and fractional $q-$calculus and associated Fekete-Szego problem for $p-$valently $q-$starlike functions and $p-$valently $q-$convex functions of complex order*, Miskolc Math. Notes, **20**(2019), no. 1, 489-509.

[33] Uralegaddi, B.A., Somanatha, C., *Certain Classes of Univalent Functions*, in Current Topics in Analytic Function Theory, 371-374, World Sci. Publishing, River Edge, New Jersey, London, Hong Kong, 1992, 371-374.

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