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Existence of positive solutions to impulsive nonlinear differential systems of second order with two point boundary conditions

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Abstract. In this paper the authors consider the existence of positive solutions to a two point boundary value problem for nonlinear second-order impulsive systems. They use a vector version of Krasnosel'skii's fixed point theorem in cones in their proofs. Examples are provided to illustrate the results.

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1. Introduction

The existence of positive solutions to second order impulsive differential equations and systems has been studied by many authors such as in [7, 9, 10, 11, 12].

Liu *et al.* [10, 11, 12] studied the existence of one and multiple positive solutions to two point boundary value problems for systems of nonlinear second-order singular impulsive differential equations by using fixed point index theory. In [7], He investigated the existence of positive solutions to second order periodic boundary value problems with impulse actions by applying fixed point index theory.

The existence and location of positive solutions for ordinary differential systems has been studied in [4, 8, 13, 14] by using a technique based on a vector version of

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Krasnosel'skii's fixed point theorem in cones. In [8], Herlea considered the system of first order equations with integral boundary conditions

$$\begin{cases} u_1'(t) = f_1(t, u_1, u_2), \\ u_2'(t) = f_2(t, u_1, u_2), \\ u_1(0) - a_1 u_1(1) = g_1[u_1], \\ u_2(0) - a_2 u_2(1) = g_2[u_2], \end{cases}$$

where f_i, $f_2 \in C([0,1] \times \mathbb{R}_+^2, \mathbb{R}^+)$ and $g_i : C[0,1] \to \mathbb{R}$, $i = 1,2$, are linear functionals given by

$$g_i[u] = \int_0^1 u(s) d\gamma_i, \quad i = 1,2$$

with $g_i[1] < 1$ and $\gamma_i \in C^1[0,1]$ is increasing and satisfies $0 < a_i < 1 - g_i[1]$ for $i = 1,2$. Herlea obtained the existence and the location of positive solutions by using a vector version of Krasnosel'skii's fixed point theorem in cones.

Precup [14] also used the vector version of Krasnosel'skii's fixed point theorem to study the existence and localization of positive solutions of the nonlinear differential system

$$\begin{cases} u_1''(t) + f_1(t, u_1, u_2) = 0, \\ u_2'(t) + f_2(t, u_1, u_2) = 0, \\ u_1(0) = u_1(1) = 0, \\ u_2(0) = u_2(1) = 0. \end{cases}$$

Other authors have recently studied the existence of solution for system of impulsive differential equations using vector versions of fixed point theorems, such as in [1, 3, 2, 5, 6].

With this background in mind, in this paper we examine the existence and location of positive solutions of the two point boundary value problem for the system of nonlinear second-order impulsive differential equations

$$\begin{cases} -u_1''(t) = f_1(t, u_1(t), u_2(t)), & t \in J', \\ -u_2''(t) = f_2(t, u_1(t), u_2(t)), & t \in J', \\ -\Delta u_1' \mid_{t=t_k} = I_{1,k} u_1(t_k), & k = 1, 2, \cdots, m, \\ -\Delta u_2' \mid_{t=t_k} = I_{2,k} u_2(t_k), & k = 1, 2, \cdots, m, \\ \alpha u_1(0) - \beta u_1'(0) = 0, \ \alpha u_2(0) - \beta u_2'(0) = 0, & \\ \gamma u_1(1) + \delta u_1'(1) = 0, \ \gamma u_2(1) + \delta u_2'(1) = 0, & \end{cases} \tag{1.1}$$

where α, β, γ, $\delta \geq 0$, $\rho = \beta\gamma + \alpha\gamma + \alpha\delta > 0$, $J = [0,1]$, $0 < t_1 < t_2 < \cdots < t_m < 1$, $J' = J \setminus \{t_1, t_2, \cdots, t_m\}$, $f_i \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $I_{i,k} \in C(\mathbb{R}, \mathbb{R})$, $i = 1,2$, $k \in \{1, 2, \cdots, m\}$. Here, $\Delta u' \mid_{t=t_k} = u_1(t_k^+) - u_1(t_k^-)$ and $\Delta u_2' \mid_{t=t_k} = u_2(t_k^+) - u_2(t_k^-)$, where $u_1'(t_k^+)$ and $u_2'(t_k^+)$, ($u_1'(t_k^-)$ and $u_2'(t_k^-)$) denote the right (left) hand limits of $u_1'(t)$ and $u_2'(t)$ at $t = t_k$, respectively.

Motivated by the work mentioned above, here we study the existence and location of positive solution of the system (1.1) using the vector version of Krasnosel'skii's fixed point theorem in cones given in [13]. As we will see, this approach allows the nonlinear terms and impulses in the system to have different types of behaviors in their variables.

2. Preliminaries

In this paper we need the following concepts. For a normed linear space $(X, \|\cdot\|)$, a cone $K \subset X$ is a closed and convex set with $K \setminus \{0\} \neq \emptyset$, $\lambda K \subset K$ for all $\lambda \in R^+$, and $K \cap (-K) = \{0\}$. A cone K in X induces a partial order relation in X that we will denote by $\preceq$; we write $u \preceq v$ if and only if $v - u \in K$. We say that $u \prec v$ if $v - u \in K \setminus \{0\}$ and $u \not\prec v$ if $v - u \notin K \setminus \{0\}$. Finally, $u \succeq v$ means $v \preceq u$.

Consider two cones K_1 and K_2 in X and the corresponding cone $K := K_1 \times K_2$ in X^2. We use the same symbol $\preceq$ to denote the partial order relation induced by K in X^2 as we do for K_1 or K_2 in X. In X^2, $u = (u_1, u_2) \prec v = (v_1, v_2)$ means $u_i \prec v_i$ for $i = 1, 2$. For r, $R \in \mathbb{R}^2_+$ with $r = (r_1, r_2)$ and $R = (R_1, R_2)$, we will write $0 < r < R$ to mean $0 < r_1 < R_1$ and $0 < r_2 < R_2$. Also, we set

$$(K_i)_{r_i,R_i} := \{u \in K_i : r_i \leq \|u\| \leq R_i\},\ i = 1, 2,$$

$$K_{r,R} := \{u \in K : r_i \leq \|u_i\| \leq R_i \ for\ i = 1, 2\},$$

and we see that $K_{r,R} = (K_1)_{r_1,R_1} \times (K_2)_{r_2,R_2}$.

The following vector version of Krasnosel'skii's fixed point theorem in a cone [13, Theorem 2.1] will be used to obtain our main existence result.

Theorem 2.1. *Let $(X, \|\cdot\|)$ be a normed linear space, K_1, $K_2 \subset X$ be two cones in X, $K := K_1 \times K_2$, r, $R \in R^+$ with $0 < r < R$, and $N : K_{r,R} \to K$ given by $N = (N_1, N_2)$ be a compact map. Assume that for each $i \in \{1, 2\}$, one of the following conditions is satisfied in $K_{r,R}$:*

(a) $N_i(u) \not\prec u_i$ if $\|u_i\| = r_i$ and $N_i(u) \not\succ u_i$ if $\|u_i\| = R_i$;
(b) $N_i(u) \not\succ u_i$ if $\|u_i\| = r_i$ and $N_i(u) \not\prec u_i$ if $\|u_i\| = R_i$.

Then N has a fixed point u in K with $r_i \leq \|u_i\| \leq R_i$ for $i \in \{1, 2\}$.

3. Main Result

We first formulate problem (1.1) as a fixed point problem for a vector-valued mapping $N = (N_1, N_2)$. Then, $u := (u_1, u_2)$ will satisfy an operator system

$$\begin{cases} u_1 = N_1(u_1, u_2), \\ u_2 = N_2(u_1, u_2), \end{cases} \tag{3.1}$$

in the vector conical shell $K_{r,R}$ with $u \in K$ and

$$r_1 \leq \|u_1\| \leq R_1,\ r_2 \leq \|u_2\| \leq R_2.$$

We denote by $G(t, s)$ the Green's function for the boundary value problem

$$\begin{cases} -x''(t) = 0, \\ \alpha x(0) - \beta x'(0) = 0, \\ \gamma x(1) + \delta x'(1) = 0. \end{cases} \tag{3.2}$$

It is given explicitly by

$$G(t, s) = \frac{1}{\rho} \begin{cases} (\gamma + \delta - \gamma t)(\beta + \alpha s), & 0 \leq s \leq t \leq 1 \\ (\beta + \alpha t)(\gamma + \delta - \gamma s), & 0 \leq t \leq s \leq 1. \end{cases}$$

The function $G(t,s)$ is positive and satisfies the properties (see [10, p. 552], [11, p. 3775]):

$$\begin{aligned} G(t,s) &\leq G(s,s), \text{ for all } t,\ s\in[0,1], \qquad (3.3)\\ 0<\sigma G(s,s) &\leq G(t,s),\ t\in[a,b],\ s\in[0,1], \qquad (3.4)\end{aligned}$$

where $a\in[0,t_1],\ b\in[t_m,1]$ and $0\leq\sigma=\min\{\frac{(1-b)\gamma+\delta}{\gamma+\delta},\frac{a\alpha+\beta}{\alpha+\beta}\}\leq 1$.

In this paper, we consider the space

$$PC(J,\mathbb{R}^+)=\{x:[0,1]\longrightarrow\mathbb{R}^+\mid x_k\in C(J',\mathbb{R}), k=1,\dots,m,$$
$$x(t_k^-) \text{ and } x(t_k^+) \text{ exist},\ k=1,\dots,m, \text{ and } x(t_k^-)=x(t)\}.$$

We see that $PC(J,\mathbb{R}^+)$ is a Banach space with the norm

$$\|x\|_{PC}=\sup_{t\in J}|x(t)|.$$

Let P be the cone of all nonnegative functions in $PC([0,1],\mathbb{R}^+)$.

Definition 3.1. A pair $(u_1,u_2)\in PC(J,\mathbb{R}^+)\times PC(J,\mathbb{R}^+)$ is called a solution of system (1.1) if it satisfies system (1.1).

The following lemma is obvious.

Lemma 3.2. *The vector* $(u_1,u_2)\in PC(J,\mathbb{R}^+)\times PC(J,\mathbb{R}^+)$ *is a solution of the differential system* (1.1) *if and only if* $(u_1,u_2)\in PC^1(J,\mathbb{R}^+)\times PC^1(J,\mathbb{R}^+)$ *is a solution of the integral system*

$$\begin{cases} u_1(t)=\displaystyle\int_0^1 G(t,s)f_1(s,u_1(s),u_2(s))ds+\sum_{k=1}^m G(t,t_k)I_{1,k}(u_1(t_k)),\\ \\ u_2(t)=\displaystyle\int_0^1 G(t,s)f_2(s,u_1(s),u_2(s))ds+\sum_{k=1}^m G(t,t_k)I_{2,k}(u_2(t_k)).\end{cases} \qquad (3.5)$$

Let $N:P^2\to P^2$ be the completely continuous map $N=(N_1,N_2)$ given by

$$N_i(u(t))=\int_0^1 G(t,s)f_i(s,u(s),v(s))ds+\sum_{k=1}^m G(t,t_k)I_{i,k}(u_i(t_k)) \quad i=1,2.$$

Then (3.5) is equivalent to the fixed point problem

$$u=N(u),\quad u\in P^2.$$

If $v\in P$,

$$u_i(t):=\int_0^1 G(t,s)v(s)ds+\sum_{k=1}^m G(t,t_k)I_{i,k}(u_i(t_k)),$$

and if $u_i(t')=\|u_i\|_{PC}$, then in view of (3.4), for every $t\in[0,1]$, we have

$$u_i(t)\geq\sigma\int_0^1 G(s,s)v(s)ds+\sigma\sum_{k=1}^m G(t_k,t_k)I_{i,k}(u_i(t_k)).$$

If $t' \neq t_k$ for $k = 1, 2, \cdots, m$, then

$$\begin{aligned} u_i(t) &\geq \sigma \int_0^1 G(t', s)v(s)ds + \sigma \sum_{k=1}^{m} G(t_k, t_k) I_{i,k}(u_i(t_k)) \\ &\geq \sigma \int_0^1 G(t', s)v(s)ds + \sigma \sum_{k=1}^{m} G(t', t_k) I_{i,k}(u_i(t_k)) = \sigma u_i(t') = \sigma \|u\|_{PC}. \end{aligned}$$

If $t' = t_k$ for $k = 1, 2, \cdots, m$, then

$$\begin{aligned} u_i(t) &\geq \sigma \int_0^1 G(s, s)v(s)ds + \sigma \sum_{k=1}^{m} G(t', t_k) I_{i,k}(u_i(t_k)) \\ &\geq \sigma \int_0^1 G(t', s)v(s)ds + \sigma \sum_{k=1}^{m} G(t', t_k) I_{i,k}(u_i(t_k)) = \sigma u_i(t') = \sigma \|u\|_{PC}. \end{aligned}$$

Define the cones K_i in P by

$$K_i = \{u_i \in P : u_i(t) \geq \sigma \|u_i\|_{PC} \text{ for all } t \in [a, b]\}, \ i = 1, 2,$$

and the product cone $K = K_1 \times K_2$ in X^2. Then $N(K) \subset K$. Before we state our main result we introduce the following notations. For any α_i, $\beta_i > 0$ with $\alpha_i \neq \beta_i$, let $r_i = \min\{\alpha_i, \beta_i\}$, $R_i = \max\{\alpha_i, \beta_i\}$, and

$$\gamma_1 = \min\{f_1(t, u_1, u_2) : \ a \leq t \leq b, \ \sigma\beta_1 \leq u_1 \leq \beta_1, \ \sigma r_2 \leq u_2 \leq R_2\},$$

$$\gamma_2 = \min\{f_2(t, u_1, u_2) : \ a \leq t \leq b, \ \sigma r_1 \leq u_1 \leq R_1, \ \sigma\beta_2 \leq u_2 \leq \beta_2\},$$

$$\Gamma_1 = \max\{f_1(t, u_1, u_2) : \ 0 \leq t \leq 1, \ \sigma\alpha_1 \leq u_1 \leq \alpha_1, \ \sigma r_2 \leq u_2 \leq R_2\},$$

$$\Gamma_2 = \max\{f_2(t, u_1, u_2) : \ 0 \leq t \leq 1, \ \sigma r_1 \leq u_1 \leq R_1, \ \sigma\alpha_2 \leq u_2 \leq \alpha_2\}.$$

Also, let

$$B = \max\{G(t, s) : 0 \leq t \leq 1, \ 0 \leq s \leq 1\},$$

$$A = \min\{G(t, s) : a \leq t \leq b, \ a \leq s \leq b\},$$

$$\lambda_1 = \min_{1 \leq k \leq m} \{\min\{I_{1,k}(u_1) : \sigma\beta_1 \leq u_1 \leq \beta_1\}\},$$

$$\lambda_2 = \min_{1 \leq k \leq m} \{\min\{I_{2,k}(u_2) : \sigma\beta_2 \leq u_2 \leq \beta_2\}\},$$

$$\Lambda_1 = \max_{1 \leq k \leq m} \{\max\{I_{1,k}(u_1) : \sigma\alpha_1 \leq u_1 \leq \alpha_1\}\},$$

$$\Lambda_2 = \max_{1 \leq k \leq m} \{\max\{I_{2,k}(u_2) : \sigma\alpha_2 \leq u_2 \leq \alpha_2\}\}.$$

Theorem 3.3. *Assume that there exist α_i, $\beta_i > 0$ with $\alpha_i \neq \beta_i$, $i = 1, 2$, such that*

$$\begin{aligned} &B(\Gamma_1 + \Lambda_1 m) \leq \alpha_1, \quad A(\gamma_1(b - a) + \lambda_1 m) \geq \beta_1, \\ &B(\Gamma_2 + \Lambda_2 m) \leq \alpha_2, \quad A(\gamma_2(b - a) + \lambda_2 m) \geq \beta_2. \end{aligned} \tag{3.6}$$

Then (1.1) has a positive solution $u = (u_1, u_2)$ with $r_i \leq \|u_i\|_{PC} \leq R_i$, $i = 1, 2$, where $r_i = \min\{\alpha_i, \beta_i\}$, $R_i = \max\{\alpha_i, \beta_i\}$. Moreover, the corresponding orbit of u is included in the rectangle $[\sigma r_1, R_1] \times [\sigma r_2, R_2]$.

Proof. If $u \in K_{r,R}$, then $r_1 \le \|u_1\|_{PC} \le R_1$ and $r_2 \le \|u_2\|_{PC} \le R_2$, so from the definition of K,

$$\sigma r_1 \le \|u_1\|_{PC} \le R_1 \quad \text{and} \quad \sigma r_2 \le \|u_2\|_{PC} \le R_2,$$

for $t \in [a,b]$, that is, for $t \in [a,b]$, $u(t) \in [\sigma r_1, R_1] \times [\sigma r_2, R_2]$. Also, if $\|u_1\|_{PC} = \alpha_1$, then $u_1(t) \le \alpha_1$ for $t \in [0,1]$, and

$$\sigma \alpha_1 \le u_1(t) \le \alpha_1 \text{ for all } t \in [a,b].$$

We wish to show that for every $u \in K_{r,R}$ and each $i \in \{1,2\}$, we have

$$\begin{aligned} \|u_i\|_{PC} = \alpha_i \text{ implies } u_i \not\prec N_i(u), \\ \|u_i\|_{PC} = \beta_i \text{ implies } u_i \not\succ N_i(u). \end{aligned} \tag{3.7}$$

If $\|u_1\|_{PC} = \alpha_1$ and $u_1 \prec N_1(u)$, then

$$u_1(t) < N_1(u)(t) \le B(\Gamma_1 + \Lambda_1 m) \le \alpha_1$$

for $t \in [0,1]$, which leads to the contradiction $\alpha_1 < \alpha_1$.

If $\|u_1\|_{PC} = \beta_1$ and $u_2 \succ N_2(u)$, then for $t \in [a,b]$, we obtain

$$\begin{aligned} u_1(t) > N_1(u)(t) &\ge \int_a^b G(t,s) f_1(s, u_1(s), u_2(s)) ds + \sum_{k=1}^{m} G(t,t_k) I_{1,k}(u_1(t_k)) \\ &\ge A(\gamma_1(b-a) + \lambda_1 m) \ge \beta_1, \end{aligned}$$

yielding the contradiction $\beta_1 > \beta_1$. Hence, (3.7) holds for $i = 1$. In a similar way we can show that (3.7) holds for $i = 2$. By Theorem 2.1, we see that N has at least one nonzero fixed point in K. Therefore, system (1.1) has at least one positive solution. This completes the proof of the theorem. □

Analogous to the discussion by Precup in [13] and [14], we examine the situation where f_1 and f_2 are independent of t, i.e., suppose $f_1 = f_1(u_1,u_2)$ and $f_2 = f_2(u_1,u_2)$. If f_1, f_2, $I_{1,k}$, and $I_{2,k}$, $k = 1, 2, \ldots.., m$, satisfy various monotonicity conditions, then we can obtain specific estimates for γ_1, γ_2, Γ_1, Γ_2, λ_1, λ_2, Λ_1, Λ_2. As examples, we have the following cases.

Case 1. If f_1 and f_2 are nondecreasing in u_1 and u_2, and $I_{1,k}$ and $I_{2,k}$ are nondecreasing respectively in u_1 and u_2 for $k = 1, 2, \ldots, m$, then

$$\begin{aligned} \Gamma_1 &= f_1(\alpha_1, R_2), & \gamma_1 &= f_1(\sigma\beta_1, \sigma r_2), \\ \Gamma_2 &= f_2(R_1, \alpha_2), & \gamma_2 &= f_2(\sigma r_1, \sigma\beta_2), \\ \Lambda_1 &= \max_{1\le k\le m}\{I_{1,k}(\alpha_1)\}, & \lambda_1 &= \min_{1\le k\le m}\{I_{1,k}(\sigma\beta_1)\}, \\ \Lambda_2 &= \max_{1\le k\le m}\{I_{2,k}(\alpha_2)\}, & \lambda_2 &= \min_{1\le k\le m}\{I_{2,k}(\sigma\beta_2)\}. \end{aligned}$$

Case 2. If f_1 is nondecreasing in u_1 and u_2, f_2 is nondecreasing in u_1 and non increasing in u_2, and on the other hand $I_{1,k}$ are nondecreasing in u_1 and $I_{2,k}$ are non increasing in u_2 for $k = 1, 2, \ldots, m$, then

$$\begin{aligned} \Gamma_1 &= f_1(\alpha_1, R_2), & \gamma_1 &= f_1(\sigma\beta_1, \sigma r_2),\\ \Gamma_2 &= f_2(R_1, \sigma\alpha_2), & \gamma_2 &= f_2(\sigma r_1, \beta_2),\\ \Lambda_1 &= \max_{1\le k\le m}\{I_{1,k}(\alpha_1)\}, & \lambda_1 &= \min_{1\le k\le m}\{I_{1,k}(\sigma\beta_1)\},\\ \Lambda_2 &= \max_{1\le k\le m}\{I_{2,k}(\sigma\alpha_2)\}, & \lambda_2 &= \min_{1\le k\le m}\{I_{2,k}(\beta_2)\}. \end{aligned}$$

Case 3. If f_1 is nondecreasing in u_1 and non increasing in u_2, f_2 is non increasing in u_1 and nondecreasing in u_2, and on the other hand $I_{1,k}$ are non increasing in u_1 and $I_{2,k}$ are nondecreasing in u_2 for $k = 1, 2, \ldots.., m$, then

$$\begin{aligned} \Gamma_1 &= f_1(\alpha_1, \sigma r_2), & \gamma_1 &= f_1(\sigma\beta_1, R_2),\\ \Gamma_2 &= f_2(\sigma r_1, \alpha_2), & \gamma_2 &= f_2(R_1, \sigma\beta_2),\\ \Lambda_1 &= \max_{1\le k\le m}\{I_{1,k}(\sigma\alpha_1)\}, & \lambda_1 &= \min_{1\le k\le m}\{I_{1,k}(\beta_1)\},\\ \Lambda_2 &= \max_{1\le k\le m}\{I_{2,k}(\alpha_2)\}, & \lambda_2 &= \min_{1\le k\le m}\{I_{2,k}(\sigma\beta_2)\}. \end{aligned}$$

Case 4. If f_1 and f_2 are nondecreasing in u_1 and nonincreasing in u_2, and $I_{1,k}$ are nondecreasing in u_1 and $I_{2,k}$ are nonincreasing in u_2 for $k = 1, 2, \ldots, m$, then

$$\begin{aligned} \Gamma_1 &= f_1(\alpha_1, \sigma r_2), & \gamma_1 &= f_1(\sigma\beta_1, R_2),\\ \Gamma_2 &= f_2(R_1, \sigma\alpha_2), & \gamma_2 &= f_2(\sigma r_1, \beta_2),\\ \Lambda_1 &= \max_{1\le k\le m}\{I_{1,k}(\alpha_1)\}, & \lambda_1 &= \min_{1\le k\le m}\{I_{1,k}(\sigma\beta_1)\},\\ \Lambda_2 &= \max_{1\le k\le m}\{I_{2,k}(\sigma\alpha_2)\}, & \lambda_2 &= \min_{1\le k\le m}\{I_{2,k}(\beta_2)\}. \end{aligned}$$

4. Examples

We conclude this paper with two examples to illustrate Theorem 3.3 in the Cases 1 and 4 above.

Example 4.1. Consider the second-order impulsive system

$$\begin{cases} u_1''(t) + u_1^\theta + u_2^\varepsilon = 0, & 0 < \theta < \varepsilon < 1, \quad t \neq \frac{1}{4}, \quad 0 \le t \le 1,\\ u_2''(t) + u_1^\varepsilon + u_2^\theta = 0, & 0 < \theta < \varepsilon < 1, \quad t \neq \frac{1}{4}, \quad 0 \le t \le 1,\\ -\Delta u_1' \mid_{t=\frac{1}{4}} = c\sqrt{u_1\left(\frac{1}{4}\right)}, & c > 0,\\ -\Delta u_2' \mid_{t=\frac{1}{4}} = d\sqrt{u_2(\frac{1}{4})}, & d > 0,\\ u_1(0) - u_1'(0) = 0, \quad u_1(1) - u_1'(1) = 0, & \\ u_2(0) + u_2'(0) = 0, \quad u_2(1) + u_2'(1) = 0. & \end{cases} \tag{4.1}$$

We can establish that system (4.1) has at least one positive solution $u = (u_1, u_2)$. Here,

$$f_1(u_1, u_2) = u_1^\theta + u_2^\varepsilon, \quad f_2(u_1, u_2) = u_1^\varepsilon + u_2^\theta,$$

$$I_{1,1}\left(u_1\left(\frac{1}{4}\right)\right) = c\sqrt{u_1\left(\frac{1}{4}\right)}, \quad I_{2,1}\left(u_2\left(\frac{1}{4}\right)\right) = d\sqrt{u_2\left(\frac{1}{4}\right)}.$$

System (4.1) is equivalent to the integral system

$$\begin{cases} u_1(t) = \int_0^1 G(t,s)[u_1(s)^\theta + u_2(s)^\varepsilon]ds + cG\left(t, \frac{1}{4}\right)\sqrt{u_1\left(\frac{1}{4}\right)}, \\ u_2(t) = \int_0^1 G(t,s)[u_1(s)^\varepsilon + u_2(s)^\theta]ds + dG\left(t, \frac{1}{4}\right)\sqrt{u_2\left(\frac{1}{4}\right)}. \end{cases}$$

where $G(t,s)$ is the Green's function

$$G(t,s) = \frac{1}{3}\begin{cases} (2-t)(1+s), & 0 \le s \le t \le 1 \\ (2-s)(1+t), & 0 \le t \le s \le 1 \end{cases}$$

Clearly $B = \frac{9}{4}$ and $A = \sigma$. In this case $f_1(u_1,u_2)$ and $f_2(u_1,u_2)$ are both nondecreasing in u_1 and u_2, while $I_{1,1}$ and $I_{2,1}$ are nondecreasing respectively in u_1 and u_2 for u_1, $u_2 \in R^+$, so we are in Case 1. We choose $\alpha_1 = \alpha_2 =: \alpha^*$ and $\beta_1 = \beta_2 =: \beta^*$, with $\beta^* < \alpha^*$, and so $r_1 = r_2 = \beta^*$, $R_1 = R_2 = \alpha^*$, and $\gamma_i = f_i(\sigma\beta^*, \sigma\beta^*)$, $\Gamma_i = f_i(\alpha^*, \alpha^*)$, $\Lambda_i = I_{i,1}(\alpha^*)$, $\lambda_i = I_{i,2}(\sigma\beta^*)$ for $i = 1, 2$. The values of α^* and β^* will be made precise in what follows. Since

$$\lim_{x\to\infty} \frac{f_i(x,x)}{x} = 0, \ \lim_{x\to 0} \frac{f_i(x,x)}{x} = \infty,$$

$$\lim_{x\to\infty} \frac{I_{1,1}(x)}{x} = 0 \ \text{ and } \ \lim_{x\to 0} \frac{I_{2,1}(x)}{x} = \infty,$$

we may find β^* small enough and α^* large enough that the conditions

$$\frac{f_i(\alpha^*,\alpha^*)}{\alpha^*} \le \frac{1}{2B}, \qquad \frac{f_i(\sigma\beta^*,\sigma\beta^*)}{\sigma\beta^*} \ge \frac{1}{2\sigma A(b-a)},$$
$$\frac{I_{i,1}(\alpha^*)}{\alpha^*} \le \frac{1}{2Bm}, \qquad \frac{I_{i,1}(\sigma\beta^*)}{\sigma\beta^*} \ge \frac{1}{2\sigma Am},$$

$i \in \{1,2\}$, are satisfied. Thus, condition (3.6) holds. Hence, system (4.1) has at least one positive solution (u_1,u_2) with $\beta^* \le \|u_i\|_{PC} \le \alpha^*$ for $i \in \{1,2\}$.

Example 4.2. Consider the second-order impulsive system

$$\begin{cases} u_1''(t) + \dfrac{u_1^{\frac{1}{4}}}{u_2+1} = 0, & t \ne \frac{1}{2}, \quad 0 \le t \le 1, \\ u_1''(t) + \dfrac{u_1}{u_2+1} = 0, & t \ne \frac{1}{2}, \quad 0 \le t \le 1, \\ -\Delta u_1' \mid_{t=\frac{1}{2}} = u_1^{\frac{1}{3}}(\frac{1}{2}), \\ -\Delta u_2' \mid_{t=\frac{1}{2}} = e^{-u_2(\frac{1}{2})}, \\ u_1(0) - u_1'(0) = 0, \quad u_1(1) - u_1'(1) = 0, \\ u_2(0) + u_2'(0) = 0, \quad u_2(1) + u_2'(1) = 0. \end{cases} \tag{4.2}$$

Here we have

$$f_1(u_1,u_2) = \frac{u_1^{\frac{1}{4}}}{u_2+1}, \ f_2(u_1,u_2) = \frac{u_1}{u_2+1},$$

$$I_{1,1}\left(u_1\left(\frac{1}{2}\right)\right) = u_1^{\frac{1}{3}}(\frac{1}{2}) \text{ and } I_{2,1}\left(u_2\left(\frac{1}{2}\right)\right) = e^{-u_2(\frac{1}{2})}.$$

System (4.2) is equivalent to the integral system

$$\begin{cases} u_1(t) = \int_0^1 G(t,s)\dfrac{u_1(s)^{\frac{1}{4}}}{u_2(s)+1}ds + G\left(t,\dfrac{1}{2}\right)u_1^{\frac{1}{3}}\left(\dfrac{1}{2}\right), \\ u_2(t) = \int_0^1 G(t,s)\dfrac{u_1(s)}{u_2(s)+1}ds + G\left(t,\dfrac{1}{2}\right)e^{-u_2(\frac{1}{2})}. \end{cases}$$

The Green function $G(t,s)$ is the same as in Example 4.1. In this case $f_1(u_1,u_2)$ and $f_2(u_1,u_2)$ are nondecreasing in u_1 and nonincreasing in u_2. Also, $I_{1,1}$ is nondecreasing in u_1 and $I_{2,1}$ is nonincreasing in u_2 for u_1, $u_2 \in R^+$, so we are in Case 4. We choose $\alpha_1 = \alpha_2 =: \alpha^*$, $\beta_1 = \beta_2 =: \beta^*$, with $\beta^* < \alpha^*$. Then $r_1 = r_2 = \beta^*$, $R_1 = R_2 = \alpha^*$ and $\Gamma_1 = f_1(\alpha^*, \sigma\beta^*)$, $\Gamma_2 = f_2(\alpha^*, \sigma\alpha^*)$, $\gamma_1 = f_1(\sigma\beta^*, \alpha^*)$, $\gamma_2 = f_2(\sigma\beta^*, \beta^*)$, $\Lambda_1 = I_{1,1}(\alpha^*)$, $\lambda_1 = I_{1,1}(\sigma\beta^*)$, $\Lambda_2 = I_{2,1}(\sigma\alpha^*)$, $\lambda_2 = I_{2,1}(\beta^*)$, where α^* and β^* will be made precise below. Since

$$\lim_{x\to\infty}\frac{f_1(x,0)}{x} = 0, \ \lim_{y\to\infty}\frac{f_2(x,\sigma y)}{y} = 0,$$

$$\lim_{x\to\infty}\frac{I_{1,1}(x)}{x} = 0, \quad \text{and} \quad \lim_{y\to\infty}\frac{I_{2,1}(\sigma y)}{y} = 0,$$

we may find $\alpha^* > 0$ large enough so that

$$\frac{f_1(\alpha^*,0)}{\alpha^*} \le \frac{1}{2B}, \quad \frac{f_2(\alpha^*,\sigma\alpha^*)}{\alpha^*} \le \frac{1}{2B},$$
$$\frac{I_{1,1}(\alpha^*)}{\alpha^*} \le \frac{1}{2Bm}, \quad \frac{I_{1,2}(\sigma\alpha^*)}{\alpha^*} \le \frac{1}{2Bm}.$$

Since

$$\frac{f_1(\alpha^*,\sigma\beta^*)}{\alpha^*} \le \frac{f_1(\alpha^*,0)}{\alpha^*},$$

we have

$$\frac{f_1(\alpha^*,\sigma\beta^*)}{\alpha^*} \le \frac{1}{2B}.$$

And since

$$\lim_{x\to 0}\frac{f_1(\sigma x,y)}{x} = \infty, \quad \lim_{y\to 0}\frac{f_2(x,y)}{y} = \infty,$$
$$\lim_{x\to 0}\frac{I_{1,1}(\sigma x)}{x} = \infty, \quad \lim_{y\to 0}\frac{I_{2,1}(y)}{y} = \infty,$$

with α fixed as above, we can choose β small enough that

$$\frac{f_1(\sigma\beta^*,\alpha^*)}{\beta^*} \ge \frac{1}{2A(b-a)}, \quad \frac{f_2(\sigma\beta^*,\beta^*)}{\beta^*} \ge \frac{1}{2A(b-a)},$$
$$\frac{I_{1,1}(\sigma\beta^*)}{\beta^*} \ge \frac{1}{2Am}, \quad \frac{I_{1,2}(\beta^*)}{\beta^*} \ge \frac{1}{2Am}.$$

Conditions (3.6) are satisfied, hence system (4.2) has at least one positive solution $u = (u_1, u_2)$.

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