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q-Deformed and λ-parametrized A-generalized logistic function induced Banach space valued multivariate multi layer neural network approximations

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Abstract. Here we research the multivariate quantitative approximation of Banach space valued continuous multivariate functions on a box or $\mathbb{R}^N$, $N \in \mathbb{N}$, by the multivariate normalized, quasi-interpolation, Kantorovich type and quadrature type neural network operators. We investigate also the case of approximation by iterated multilayer neural network operators of the last four types. These approximations are achieved by establishing multidimensional Jackson type inequalities involving the multivariate modulus of continuity of the engaged function or its partial derivatives. Our multivariate operators are defined by using a multidimensional density function induced by a q-deformed and λ-parametrized A-generalized logistic function, which is a sigmoid function. The approximations are pointwise and uniform. The related feed-forward neural network are with one or multi hidden layers.

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1. Introduction

The author in [2] and [3], see chapters 2-5, was the first to establish neural network approximations to continuous functions with rates by very specifically defined

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neural network operators of Cardaliaguet-Euvrard and "Squashing" types, by employing the modulus of continuity of the engaged function or its high order derivative, and producing very tight Jackson type inequalities. He treats there both the univariate and multivariate cases. The defining these operators "bell-shaped" and "squashing" functions are assumed to be of compact support. Also in [3] he gives the Nth order asymptotic expansion for the error of weak approximation of these two operators to a special natural class of smooth functions, see chapters 4-5 there. The author started with [1].

Motivations for this work are the article [14] of Z. Chen and F. Cao, also by [4]-[13], [15], [16].

Here we perform a q-deformed and λ-parametrized, $q, \lambda > 0$, $A > 1$, A-generalized logistic sigmoid function based neural network approximations to continuous functions over boxes or over the whole $\mathbb{R}^N$, $N \in \mathbb{N}$ and also iterated, multi layer approximations. All convergences here are with rates expressed via the multivariate modulus of continuity of the involved function or its partial derivatives and given by very tight multidimensional Jackson type inequalities.

We come up with the "right" precisely defined multivariate normalized, quasi-interpolation neural network operators related to boxes or $\mathbb{R}^N$, as well as Kantorovich type and quadrature type related operators on $\mathbb{R}^N$. Our boxes are not necessarily symmetric to the origin. In preparation to prove our results we establish important properties of the basic multivariate density function induced by the q-deformed and λ-parametrized A-generalized logistic sigmoid function.

Feed-forward neural networks (FNNs) with one hidden layer, the only type of networks we deal with in this article, are mathematically expressed as

$$N_n(x) = \sum_{j=0}^{n} c_j \sigma\left(\langle a_j \cdot x \rangle + b_j\right), \quad x \in \mathbb{R}^s, \quad s \in \mathbb{N},$$

where for $0 \le j \le n$, $b_j \in \mathbb{R}$ are the thresholds, $a_j \in \mathbb{R}^s$ are the connection weights, $c_j \in \mathbb{R}$ are the coefficients, $\langle a_j \cdot x \rangle$ is the inner product of a_j and x, and σ is the activation function of the network. In many fundamental network models, the activation function is a kind of logistic sigmoid function. About neural networks read [17] - [19].

2. Preliminaries

We consider the q-deformed and λ-parametrized function

$$\varphi_{q,\lambda}(x) = \frac{1}{1 + qA^{-\lambda x}}, \quad x \in \mathbb{R}, \text{ where } q, \lambda > 0, \ A > 1. \tag{2.1}$$

This is an A-generalized logistic type function.

We easily observe that

$$\varphi_{q,\lambda}(+\infty) = 1, \ \ \varphi_{q,\lambda}(-\infty) = 0. \tag{2.2}$$

Furthermore we have

$$1 - \varphi_{\frac{1}{q},\lambda}(-x) = 1 - \frac{1}{1 + \frac{1}{q}A^{\lambda x}} = \frac{1 + \frac{1}{q}A^{\lambda x} - 1}{1 + \frac{1}{q}A^{\lambda x}}$$

$$= \frac{\frac{1}{q}A^{\lambda x}}{1+\frac{1}{q}A^{\lambda x}} = \frac{1}{\frac{1}{\frac{1}{q}A^{\lambda x}}+1} = \frac{1}{1+qA^{-\lambda x}} = \varphi_{q,\lambda}(x),$$

proving

$$\varphi_{q,\lambda}(x) = 1 - \varphi_{\frac{1}{q},\lambda}(-x). \quad (2.3)$$

We also have that

$$\varphi_{q,\lambda}(0) = \frac{1}{1+q}. \quad (2.4)$$

Consider the activation function

$$G_{q,\lambda}(x) := \frac{1}{2}\left(\varphi_{q,\lambda}(x+1) - \varphi_{q,\lambda}(x-1)\right), \quad x \in \mathbb{R}. \quad (2.5)$$

Then

$$G_{q,\lambda}(-x) = \frac{1}{2}\left(\varphi_{q,\lambda}(-x+1) - \varphi_{q,\lambda}(-x-1)\right)$$
$$= \frac{1}{2}\Big(1 - \varphi_{\frac{1}{q},\lambda}(x-1) - 1 + \varphi_{\frac{1}{q},\lambda}(x+1)\Big)$$
$$= \frac{1}{2}\Big(\varphi_{\frac{1}{q},\lambda}(x+1) - \varphi_{\frac{1}{q},\lambda}(x-1)\Big) = G_{\frac{1}{q},\lambda}(x). \quad (2.6)$$

That is

$$G_{q,\lambda}(-x) = G_{\frac{1}{q},\lambda}(x), \quad \forall\, x \in \mathbb{R}. \quad (2.7)$$

We have

$$\varphi'_{q,\lambda}(x) = \Big(\left(1+qA^{-\lambda x}\right)^{-1}\Big)'$$
$$= -1\left(1+qA^{-\lambda x}\right)^{-2} q(\ln A)A^{-\lambda x}(-\lambda) = q\lambda(\ln A)\left(1+qA^{-\lambda x}\right)^{-2} A^{-\lambda x} > 0. \quad (2.8)$$

So that $\varphi_{q,\lambda}$ is a strictly increasing function over $\mathbb{R}$.
Hence it holds

$$\varphi'_{q,\lambda}(x) = \frac{q\lambda(\ln A)}{\left(1+qA^{-\lambda x}\right)^2 A^{\lambda x}}$$
$$= \frac{q\lambda(\ln A)}{\left(1+q^2A^{-2\lambda x}+2qA^{-\lambda x}\right)A^{\lambda x}} = \frac{q\lambda(\ln A)}{\left(A^{\lambda x}+q^2A^{-\lambda x}+2q\right)}. \quad (2.9)$$

That is

$$\varphi'_{q,\lambda}(x) = q\lambda(\ln A)\left(A^{\lambda x}+q^2A^{-\lambda x}+2q\right)^{-1}. \quad (2.10)$$

Therefore it holds

$$\varphi''_{q,\lambda}(x) = q\lambda(\ln A)(-1)\left(A^{\lambda x}+q^2A^{-\lambda x}+2q\right)^{-2}\left((\ln A)A^{\lambda x}\lambda + q^2(\ln A)A^{-\lambda x}(-\lambda)\right)$$
$$= q\lambda^2(\ln A)^2\left(A^{\lambda x}+q^2A^{-\lambda x}+2q\right)^{-2}\left(q^2A^{-\lambda x}-A^{\lambda x}\right). \quad (2.11)$$

That is

$$\varphi''_{q,\lambda}(x) = q\lambda^2(\ln A)^2\left(A^{\lambda x}+q^2A^{-\lambda x}+2q\right)^{-2}\left(q^2A^{-\lambda x}-A^{\lambda x}\right) \in C(\mathbb{R}). \quad (2.12)$$

We have

$$q^2A^{-\lambda x} - A^{\lambda x} > 0, \text{ iff } q^2A^{-\lambda x} > A^{\lambda x}, \text{ iff } q^2 > A^{2\lambda x}, \text{ iff } q > A^{\lambda x},$$
$$\text{iff } \log_A q > \lambda x, \text{ iff } x < \frac{\log_A q}{\lambda}.$$

So, $\varphi''_{q,\lambda}(x) > 0$, for $x < \frac{\log_A q}{\lambda}$ and there $\varphi_{q,\lambda}$ is concave up.

When $x > \frac{\log_A q}{\lambda}$, we have $\varphi''_{q,\lambda}(x) < 0$ and $\varphi_{q,\lambda}$ is concave down.
Of course

$$\varphi''_{q,\lambda}\left(\frac{\log_A q}{\lambda}\right) = 0.$$

So, $\varphi_{q,\lambda}$ is a sigmoid function, see [12].
We have that

$$G'_{q,\lambda}(x) = \frac{1}{2}\left(\varphi'_{q,\lambda}(x+1) - \varphi'_{q,\lambda}(x-1)\right).$$

We got that $\varphi'_{q,\lambda}$ is strictly increasing for $x < \frac{\log_A q}{\lambda}$. Let $x < \frac{\log_A q}{\lambda} - 1$, then

$$x - 1 < x + 1 < \frac{\log_A q}{\lambda}.$$

Hence $\varphi'_{q,\lambda}(x+1) > \varphi'_{q,\lambda}(x-1)$. Thus $G'_{q,\lambda} > 0$, i.e. $G_{q,\lambda}$ is strictly increasing over $\left(-\infty, \frac{\log_A q}{\lambda} - 1\right)$.

Let now $x > \frac{\log_A q}{\lambda} + 1$, then $x+1 > x-1 > \frac{\log_A q}{\lambda}$, and $\varphi'_{q,\lambda}(x+1) < \varphi'_{q,\lambda}(x-1)$, by $\varphi'_{q,\lambda}$ being strictly decreasing over $\left(\frac{\log_A q}{\lambda}, +\infty\right)$. Hence $G'_{q,\lambda} < 0$, and $G_{q,\lambda}$ is strictly decreasing over $\left(\frac{\log_A q}{\lambda}, +\infty\right)$.

Let now $\frac{\log_A q}{\lambda} - 1 \leq x \leq \frac{\log_A q}{\lambda} + 1$. We have that

$$G''_{q,\lambda}(x) = \frac{1}{2}\left(\varphi''_{q,\lambda}(x+1) - \varphi''_{q,\lambda}(x-1)\right)$$

$$= \frac{q\lambda^2 (\ln A)^2}{2}\left[\frac{\left(q^2 A^{-\lambda(x+1)} - A^{\lambda(x+1)}\right)}{\left(A^{\lambda(x+1)} + q^2 A^{-\lambda(x+1)} + 2q\right)^2} - \frac{\left(q^2 A^{-\lambda(x-1)} - A^{\lambda(x-1)}\right)}{\left(A^{\lambda(x-1)} + q^2 A^{-\lambda(x-1)} + 2q\right)^2}\right] \tag{2.13}$$

$$= \frac{q\lambda^2 (\ln A)^2}{2}\left[\frac{\left(q^2 - A^{2\lambda(x+1)}\right)}{\left(A^{\lambda(x+1)} + q^2 A^{-\lambda(x+1)} + 2q\right)^2 A^{\lambda(x+1)}} - \frac{\left(q^2 - A^{2\lambda(x-1)}\right)}{\left(A^{\lambda(x-1)} + q^2 A^{-\lambda(x-1)} + 2q\right)^2 A^{\lambda(x-1)}}\right]$$

$$= \frac{q\lambda^2 (\ln A)^2}{2}\left[\frac{\left(q - A^{\lambda(x+1)}\right)\left(q + A^{\lambda(x+1)}\right)}{\left(A^{\lambda(x+1)} + q^2 A^{-\lambda(x+1)} + 2q\right)^2 A^{\lambda(x+1)}} - \frac{\left(q - A^{\lambda(x-1)}\right)\left(q + A^{\lambda(x-1)}\right)}{\left(A^{\lambda(x-1)} + q^2 A^{-\lambda(x-1)} + 2q\right)^2 A^{\lambda(x-1)}}\right].$$

By $\frac{\log_A q}{\lambda} \leq x + 1 \Leftrightarrow \log_A q \leq \lambda(x+1) \Leftrightarrow q \leq A^{\lambda(x+1)} \Leftrightarrow q - A^{\lambda(x+1)} \leq 0$.

By $x \leq \frac{\log_A q}{\lambda} + 1 \Leftrightarrow x - 1 \leq \frac{\log_A q}{\lambda} \Leftrightarrow \lambda(x-1) \leq \log_A q \Leftrightarrow A^{\lambda(x-1)} \leq q \Leftrightarrow q - A^{\lambda(x-1)} \geq 0$.

Clearly, when $\frac{\log_A q}{\lambda} - 1 \leq x \leq \frac{\log_A q}{\lambda} + 1$ by the above we get that $G''_{q,\lambda}(x) \leq 0$, that is $G''_{q,\lambda}$ is concave down there.

Clearly $G_{q,\lambda}$ is strictly concave down over $\left(\frac{\log_A q}{\lambda} - 1, \frac{\log_A q}{\lambda} + 1\right)$.

Overall $G_{q,\lambda}$ is a bell-shaped function over $\mathbb{R}$.

Of course it holds $G''_{q,\lambda}\left(\frac{\log_A q}{\lambda}\right) < 0$.

We have that

$$G'_{q,\lambda}\left(\frac{\log_A q}{\lambda}\right) = \frac{1}{2}\left(\varphi'_{q,\lambda}\left(\frac{\log_A q}{\lambda}+1\right) - \varphi'_{q,\lambda}\left(\frac{\log_A q}{\lambda}-1\right)\right)$$

$$= \frac{q\lambda(\ln A)}{2}\left[\frac{1}{A^{\lambda\left(\frac{\log_A q}{\lambda}+1\right)}+q^2A^{-\lambda\left(\frac{\log_A q}{\lambda}+1\right)}+2q} - \frac{1}{A^{\lambda\left(\frac{\log_A q}{\lambda}-1\right)}+q^2A^{-\lambda\left(\frac{\log_A q}{\lambda}-1\right)}+2q}\right] \tag{2.14}$$

$$= \frac{q\lambda(\ln A)}{2}\left[\frac{A^{\lambda\left(\frac{\log_A q}{\lambda}-1\right)}+q^2A^{-\lambda\left(\frac{\log_A q}{\lambda}-1\right)}-A^{\lambda\left(\frac{\log_A q}{\lambda}+1\right)}-q^2A^{-\lambda\left(\frac{\log_A q}{\lambda}+1\right)}}{\left(A^{\lambda\left(\frac{\log_A q}{\lambda}+1\right)}+q^2A^{-\lambda\left(\frac{\log_A q}{\lambda}+1\right)}+2q\right)\left(A^{\lambda\left(\frac{\log_A q}{\lambda}-1\right)}+q^2A^{-\lambda\left(\frac{\log_A q}{\lambda}-1\right)}+2q\right)}\right] \tag{2.15}$$

$$= \frac{q\lambda(\ln A)}{2}\left[\frac{qA^{-\lambda}+q^2q^{-1}A^{\lambda}-qA^{\lambda}-q^2q^{-1}A^{-\lambda}}{(qA^{\lambda}+q^2q^{-1}A^{-\lambda}+2q)(qA^{\lambda}+q^2q^{-1}A^{-\lambda}+2q)}\right]$$

$$= \frac{q\lambda(\ln A)}{2}\left[\frac{qA^{-\lambda}+qA^{\lambda}-qA^{\lambda}-qA^{-\lambda}}{(qA^{\lambda}+qA^{-\lambda}+2q)(qA^{\lambda}+qA^{-\lambda}+2q)}\right] = 0. \tag{2.16}$$

So $\frac{\log_A q}{\lambda}$ is the only critical number of $G_{q,\lambda}$ over $\mathbb{R}$. Therefore $G_{q,\lambda}\left(\frac{\log_A q}{\lambda}\right)$ is the maximum of $G_{q,\lambda}$.

We calculate it:

We have that

$$G_{q,\lambda}\left(\frac{\log_A q}{\lambda}\right) = \frac{1}{2}\left(\varphi_{q,\lambda}\left(\frac{\log_A q}{\lambda}+1\right) - \varphi_{q,\lambda}\left(\frac{\log_A q}{\lambda}-1\right)\right)$$

$$= \frac{1}{2}\left(\frac{1}{1+qA^{-\lambda\left(\frac{\log_A q}{\lambda}+1\right)}} - \frac{1}{1+qA^{-\lambda\left(\frac{\log_A q}{\lambda}-1\right)}}\right) \tag{2.17}$$

$$= \frac{1}{2}\left(\frac{1}{1+qq^{-1}A^{-\lambda}} - \frac{1}{1+qq^{-1}A^{\lambda}}\right) = \frac{1}{2}\left(\frac{1}{1+A^{-\lambda}} - \frac{1}{1+A^{\lambda}}\right)$$

$$= \frac{1}{2}\left(\frac{A^{\lambda}-A^{-\lambda}}{(1+A^{-\lambda})(1+A^{\lambda})}\right) = \frac{A^{\lambda}-1}{2(A^{\lambda}+1)}.$$

The global maximum of $G_{q,\lambda}$ is

$$G_{q,\lambda}\left(\frac{\log_A q}{\lambda}\right) = \frac{A^{\lambda}-1}{2(A^{\lambda}+1)}. \tag{2.18}$$

Finally we have that

$$\lim_{x\to+\infty} G_{q,\lambda}(x) = \frac{1}{2}\left(\varphi_{q,\lambda}(+\infty) - \varphi_{q,\lambda}(+\infty)\right) = 0, \tag{2.19}$$

and

$$\lim_{x\to-\infty} G_{q,\lambda}(x) = \frac{1}{2}\left(\varphi_{q,\lambda}(-\infty) - \varphi_{q,\lambda}(-\infty)\right) = 0. \tag{2.20}$$

Consequently the x-axis is the horizontal asymptote of $G_{q,\lambda}$. Of course $G_{q,\lambda}(x) > 0$, $\forall\, x \in \mathbb{R}$.

We need

Theorem 2.1. *It holds*

$$\sum_{i=-\infty}^{\infty} G_{q,\lambda}\left(x-i\right)=1, \quad \forall\ x\in\mathbb{R}, \forall\ q,\lambda>0,\ A>1. \tag{2.21}$$

Proof. We observe that

$$\sum_{i=-\infty}^{\infty}\left(\varphi_{q,\lambda}\left(x-i\right)-\varphi_{q,\lambda}\left(x-1-i\right)\right)$$

$$=\sum_{i=0}^{\infty}\left(\varphi_{q,\lambda}\left(x-i\right)-\varphi_{q,\lambda}\left(x-1-i\right)\right)+\sum_{i=-\infty}^{-1}\left(\varphi_{q,\lambda}\left(x-i\right)-\varphi_{q,\lambda}\left(x-1-i\right)\right).$$

Furthermore ($\lambda\in\mathbb{Z}^{+}$)

$$\sum_{i=0}^{\infty}\left(\varphi_{q,\lambda}\left(x-i\right)-\varphi_{q,\lambda}\left(x-1-i\right)\right) \tag{2.22}$$

$$=\lim_{\lambda\to\infty}\sum_{i=0}^{\lambda^{*}}\left(\varphi_{q,\lambda}\left(x-i\right)-\varphi_{q,\lambda}\left(x-1-i\right)\right)\quad\text{(telescoping sum)}$$

$$=\lim_{\lambda^{*}\to\infty}\left(\varphi_{q,\lambda}\left(x\right)-\varphi_{q,\lambda}\left(x-\left(\lambda^{*}+1\right)\right)\right)=\varphi_{q,\lambda}\left(x\right).$$

Similarly, it holds

$$\sum_{i=-\infty}^{-1}\left(\varphi_{q,\lambda}\left(x-i\right)-\varphi_{q,\lambda}\left(x-1-i\right)\right)=\lim_{\lambda^{*}\to\infty}\sum_{i=-\lambda^{*}}^{-1}\left(\varphi_{q,\lambda}\left(x-i\right)-\varphi_{q,\lambda}\left(x-1-i\right)\right)$$

$$=\lim_{\lambda*\to\infty}\left(\varphi_{q,\lambda}\left(x+\lambda^{*}\right)-\varphi_{q,\lambda}\left(x\right)\right)=1-\varphi_{q,\lambda}\left(x\right). \tag{2.23}$$

Therefore we derive

$$\sum_{i=-\infty}^{\infty}\left(\varphi_{q,\lambda}\left(x-i\right)-\varphi_{q,\lambda}\left(x-1-i\right)\right)=1,\quad\forall\ x\in\mathbb{R}, \tag{2.24}$$

and

$$\sum_{i=-\infty}^{\infty}\left(\varphi_{q,\lambda}\left(x+1-i\right)-\varphi_{q,\lambda}\left(x-i\right)\right)=1,\quad\forall\ x\in\mathbb{R}. \tag{2.25}$$

Adding the last two equations we get

$$\sum_{i=-\infty}^{\infty}\left(\varphi_{q,\lambda}\left(x+1-i\right)-\varphi_{q,\lambda}\left(x-1-i\right)\right)=2,\ \forall x\in\mathbb{R}. \tag{2.26}$$

Since

$$G_{q,\lambda}\left(x\right)=\frac{1}{2}\left(\varphi_{q,\lambda}\left(x+1\right)-\varphi_{q,\lambda}\left(x-1\right)\right),$$

we have that

$$G_{q,\lambda}\left(x-i\right)=\frac{1}{2}\left[\varphi_{q,\lambda}\left(x+1-i\right)-\varphi_{q,\lambda}\left(x-1-i\right)\right], \tag{2.27}$$

giving

$$\sum_{i=-\infty}^{\infty} G_{q,\lambda}(x-i) = 1.$$

Thus

$$\sum_{i=-\infty}^{\infty} G_{q,\lambda}(nx-i) = 1, \forall\, n \in \mathbb{N}, \ \forall\, x \in \mathbb{R}. \tag{2.28}$$

Similarly, it holds

$$\sum_{i=-\infty}^{\infty} G_{\frac{1}{q},\lambda}(x-i) = 1, \forall\, x \in \mathbb{R}. \tag{2.29}$$

But $G_{\frac{1}{q},\lambda}(x-i) \overset{(2.7)}{=} G_{q,\lambda}(i-x)$, $\forall\, x \in \mathbb{R}$.
Hence

$$\sum_{i=-\infty}^{\infty} G_{q,\lambda}(i-x) = 1, \forall\, x \in \mathbb{R}, \tag{2.30}$$

and

$$\sum_{i=-\infty}^{\infty} G_{q,\lambda}(i+x) = 1, \forall\, x \in \mathbb{R}. \tag{2.31}$$

□

It follows

Theorem 2.2. *It holds*

$$\int_{-\infty}^{\infty} G_{q,\lambda}(x)\, dx = 1, \quad \lambda, q > 0, \ A > 1. \tag{2.32}$$

Proof. We observe that

$$\int_{-\infty}^{\infty} G_{q,\lambda}(x)\, dx = \sum_{j=-\infty}^{\infty} \int_{j}^{j+1} G_{q,\lambda}(x)\, dx = \sum_{j=-\infty}^{\infty} \int_{0}^{1} G_{q,\lambda}(x+j)\, dx \tag{2.33}$$

$$= \int_{0}^{1} \left(\sum_{j=-\infty}^{\infty} G_{q,\lambda}(x+j)\, dx \right) = \int_{0}^{1} 1dx = 1.$$

□

So that $G_{q,\lambda}$ is a density function on $\mathbb{R}$; $\lambda, q > 0, \ A > 1$.
We need the following result

Theorem 2.3. *Let* $0 < \alpha < 1$, *and* $n \in \mathbb{N}$ *with* $n^{1-\alpha} > 2$. *Then*

$$\sum_{\left\{ \begin{array}{c} k = -\infty \\ : |nx-k| \geq n^{1-\alpha} \end{array} \right.}^{\infty} G_{q,\lambda}(nx-k) < \max\left\{ q, \frac{1}{q} \right\} \frac{1}{A^{\lambda\left(n^{1-\alpha}-2\right)}} = \gamma A^{-\lambda\left(n^{1-\alpha}-2\right)}, \tag{2.34}$$

where $q, \lambda > 0, \ A > 1$; $\gamma := \max\left\{ q, \frac{1}{q} \right\}$.

Proof. Let $x \geq 1$. That is $0 \leq x - 1 < x + 1$. Applying the mean value theorem we obtain

$$G_{q,\lambda}(x) = \frac{1}{2}\left(\varphi_{q,\lambda}(x+1) - \varphi_{q,\lambda}(x-1)\right)$$
$$= \frac{1}{2} \cdot 2 \cdot \varphi'_{q,\lambda}(\xi) = q\lambda(\ln A)\frac{A^{-\lambda\xi}}{\left(1 + qA^{-\lambda\xi}\right)^2}, \tag{2.35}$$

where $0 \leq x - 1 < \xi < x + 1$.

Notice that

$$G_{q,\lambda}(x) < q\lambda(\ln A)A^{-\lambda\xi} < q\lambda(\ln A)A^{-\lambda(x-1)}, \quad \forall\, x \geq 1. \tag{2.36}$$

Thus, we observe that

$$\sum_{\substack{k=-\infty \\ : |nx-k| \geq n^{1-\alpha}}}^{\infty} G_{q,\lambda}(|nx-k|)$$
$$< q\lambda(\ln A)\sum_{\substack{k=-\infty \\ : |nx-k| \geq n^{1-\alpha}}}^{\infty} A^{-\lambda(|nx-k|-1)} \leq q\lambda(\ln A)\int_{n^{1-\alpha}-1}^{\infty} A^{-\lambda(x-1)}dx$$
$$= q\lambda(\ln A)\int_{n^{1-\alpha}-2}^{\infty} A^{-\lambda z}d(z) \overset{(y=\lambda z)}{=} q(\ln A)\int_{n^{1-\alpha}-2}^{\infty} A^{-y}dy \tag{2.37}$$
$$= (-q)\int_{n^{1-\alpha}-2}^{\infty}\left(-(\ln A)A^{-y}\right)dy = -q\left(\int_{n^{1-\alpha}-2}^{\infty} dA^{-y}\right) = (-q)\left(A^{-y}\Big|_{n^{1-\alpha}-2}^{\infty}\right)$$
$$= q\left(A^{-y}\Big|_{\infty}^{n^{1-\alpha}-2}\right) = q\left(A^{-\lambda z}\Big|_{\infty}^{n^{1-\alpha}-2}\right) = qA^{-\lambda\left(n^{1-\alpha}-2\right)} = \frac{q}{A^{\lambda\left(n^{1-\alpha}-2\right)}}.$$

We have proved that

$$\sum_{\substack{k=-\infty \\ : |nx-k| \geq n^{1-\alpha}}}^{\infty} G_{q,\lambda}(|nx-k|) < \frac{q}{A^{\lambda\left(n^{1-\alpha}-2\right)}}, \tag{2.38}$$

for $n^{1-\alpha} > 2$, $n \in \mathbb{N}$; $\lambda, q > 0$, $A > 1$.

If $(nx - k) > 0$, then

$$\sum_{\substack{k=-\infty \\ : |nx-k| \geq n^{1-\alpha}}}^{\infty} G_{q,\lambda}(nx-k) < \frac{q}{A^{\lambda\left(n^{1-\alpha}-2\right)}}. \tag{2.39}$$

Similarly, it holds

$$\sum_{\begin{cases} k=-\infty \\ : |nx-k| \geq n^{1-\alpha} \end{cases}}^{\infty} G_{\frac{1}{q},\lambda}\left(|nx-k|\right) < \frac{1}{qA^{\lambda\left(n^{1-\alpha}-2\right)}}, \quad \lambda, q > 0, \ A > 1. \tag{2.40}$$

Assume now that $nx - k \leq 0$, then

$$\sum_{\begin{cases} k=-\infty \\ : |nx-k| \geq n^{1-\alpha} \end{cases}}^{\infty} G_{q,\lambda}\left(nx-k\right) \overset{(2.7)}{=} \sum_{\begin{cases} k=-\infty \\ : |nx-k| \geq n^{1-\alpha} \end{cases}}^{\infty} G_{\frac{1}{q},\lambda}\left(-\left(nx-k\right)\right)$$

$$< \frac{1}{qA^{\lambda\left(n^{1-\alpha}-2\right)}}, \quad \lambda, q > 0, \ A > 1. \tag{2.41}$$

Therefore, it holds (by (2.39), (2.41))

$$\sum_{\begin{cases} k=-\infty \\ : |nx-k| \geq n^{1-\alpha} \end{cases}}^{\infty} G_{q,\lambda}\left(nx-k\right) < \max\left\{q, \frac{1}{q}\right\} \frac{1}{A^{\lambda\left(n^{1-\alpha}-2\right)}}, \tag{2.42}$$

where $q, \lambda > 0, \ A > 1$.

The claim is proved. □

Let $\lceil\cdot\rceil$ the ceiling of the number, and $\lfloor\cdot\rfloor$ the integral part of the number.

Theorem 2.4. *Let $x \in [a,b] \subset \mathbb{R}$ and $n \in \mathbb{N}$ so that $\lceil na\rceil \leq \lfloor nb\rfloor$. For $q > 0$, $\lambda > 0$, $A > 1$, we consider the number $\lambda_q > z_0 > 0$ with $G_{q,\lambda}\left(z_0\right) = G_{q,\lambda}\left(0\right)$ and $\lambda_q > 1$. Then*

$$\frac{1}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} G_{q,\lambda}\left(nx-k\right)} < max\left\{\frac{1}{G_{q,\lambda}\left(\lambda_q\right)}, \frac{1}{G_{\frac{1}{q},\lambda}\left(\lambda_{\frac{1}{q}}\right)}\right\} =: K\left(q\right). \tag{2.43}$$

Proof. By Theorem 2.1 we have

$$\sum_{i=-\infty}^{\infty} G_{q,\lambda}\left(x-i\right) = 1, \ \forall \ x \in \mathbb{R}, \ \forall \ \lambda, q > 0; \ A > 1,$$

and by (2.30), we have that

$$\sum_{i=-\infty}^{\infty} G_{q,\lambda}\left(i-x\right) = 1, \ \forall \ x \in \mathbb{R}, \ \forall \ \lambda, q > 0; \ A > 1. \tag{2.44}$$

Therefore we get

$$\sum_{i=-\infty}^{\infty} G_{q,\lambda}\left(|x-i|\right) = 1, \ \forall \ x \in \mathbb{R}, \ \forall \ \lambda, q > 0; \ A > 1. \tag{2.45}$$

Hence

$$1=\sum_{k=-\infty}^{\infty}G_{q,\lambda}\left(|nx-k|\right)>\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}G_{q,\lambda}\left(|nx-k|\right)>G_{q,\lambda}\left(|nx-k_0|\right), \tag{2.46}$$

$\forall\ k_0\in[\lceil na\rceil,\lfloor nb\rfloor]\cap\mathbb{Z}$.

We can choose $k_0\in[\lceil na\rceil,\lfloor nb\rfloor]\cap\mathbb{Z}$, such that $|nx-k_0|<1$.

Notice that $|nx-k_0|$ could be $\substack{\leq\\ = \\ >}\ \frac{\log_A q}{\lambda}$. If $0\leq|nx-k_0|<\frac{\log_A q}{\lambda}$, by down concavity of $G_{q,\lambda}$ over $\mathbb{R}$, we can choose $z\in[\frac{\log_A q}{\lambda},+\infty)$ such that $G_{q,\lambda}\left(|nx-k_0|\right)=G_{q,\lambda}\left(z\right)$. If $|nx-k_0|\geq\frac{\log_A q}{\lambda}$ we just set $z:=|nx-k_0|$. Next, we can choose large enough $\lambda_q>1$, and such that $\lambda_q>z_0>0$ where $G_{q,\lambda}\left(z_0\right)=G_{q,\lambda}\left(0\right)$. Clearly, it is $z\leq z_0<\lambda_q$.

Since $G_{q,\lambda}$ is decreasing over $[\frac{\log_A q}{\lambda},+\infty)$ we get that

$$G_{q,\lambda}\left(|nx-k_0|\right)\geq G_{q,\lambda}\left(\lambda_q\right).$$

Consequently,

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}G_{q,\lambda}\left(|nx-k|\right)>G_{q,\lambda}\left(\lambda_q\right),$$

and

$$\frac{1}{\sum\limits_{k=\lceil na\rceil}^{\lfloor nb\rfloor}G_{q,\lambda}\left(|nx-k|\right)}<\frac{1}{G_{q,\lambda}\left(\lambda_q\right)}, \tag{2.47}$$

$\forall\ \lambda,q>0;\ A>1$.

If $nx-k>0$, by (2.47), we get

$$\frac{1}{\sum\limits_{k=\lceil na\rceil}^{\lfloor nb\rfloor}G_{q,\lambda}\left(nx-k\right)}<\frac{1}{G_{q,\lambda}\left(\lambda_q\right)},\quad\forall\ \lambda,q>0;\ A>1. \tag{2.48}$$

We have also that

$$\frac{1}{\sum\limits_{k=\lceil na\rceil}^{\lfloor nb\rfloor}G_{\frac{1}{q},\lambda}\left(|nx-k|\right)}<\frac{1}{G_{\frac{1}{q},\lambda}\left(\lambda_{\frac{1}{q}}\right)},\quad\forall\ \lambda,q>0;\ A>1. \tag{2.49}$$

Let now $nx-k\leq0$, then

$$\frac{1}{\sum\limits_{k=\lceil na\rceil}^{\lfloor nb\rfloor}G_{q,\lambda}\left(nx-k\right)}\overset{(2.7)}{=}\frac{1}{\sum\limits_{k=\lceil na\rceil}^{\lfloor nb\rfloor}G_{\frac{1}{q},\lambda}\left(-\left(nx-k\right)\right)}\overset{(2.49)}{<}\frac{1}{G_{\frac{1}{q},\lambda}\left(\lambda_{\frac{1}{q}}\right)}, \tag{2.50}$$

$\forall\ \lambda,q>0;\ A>1$.

Consequently, it holds

$$\frac{1}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} G_{q,\lambda}(nx-k)} < \max\left\{\frac{1}{G_{q,\lambda}(\lambda_q)}, \frac{1}{G_{\frac{1}{q},\lambda}\left(\lambda_{\frac{1}{q}}\right)}\right\}, \tag{2.51}$$

$\forall\ \lambda, q > 0;\ A > 1$.

The claim is proved. $\square$

We make

Remark 2.5. (i) We also notice for $q \geq 1$ that

$$1 - \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} G_{q,\lambda}(nb-k) = \sum_{k=-\infty}^{\lceil na\rceil-1} G_{q,\lambda}(nb-k) + \sum_{k=\lfloor nb\rfloor+1}^{\infty} G_{q,\lambda}(nb-k)$$

$$> G_{q,\lambda}(nb - \lfloor nb\rfloor - 1) \tag{2.52}$$

(call $\varepsilon := nb - \lfloor nb\rfloor$, $0 \leq \varepsilon < 1$)

$$= G_{q,\lambda}(\varepsilon - 1) = G_{q,\lambda}(-(1-\varepsilon)) = G_{\frac{1}{q},\lambda}(1-\varepsilon)$$

($0 < \frac{1}{q} \leq 1$ and $0 < 1 - \varepsilon \leq 1$)

($G_{\frac{1}{q},\lambda}$ is decreasing on $[0, +\infty)$).

$$\geq G_{\frac{1}{q},\lambda}(1) > 0.$$

Therefore

$$\lim_{n\to\infty}\left(1 - \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} G_{q,\lambda}(nb-k)\right) > 0, \quad q \geq 1,\ \lambda > 0;\ A > 1. \tag{2.53}$$

(ii) Let now $0 < q \leq 1$, then we work as in (i), and we have

$$1 - \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} G_{q,\lambda}(nb-k) > G_{\frac{1}{q},\lambda}(1-\varepsilon) \tag{2.54}$$

($\varepsilon := nb - \lfloor nb\rfloor$, $0 \leq \varepsilon < 1$).

That is $\frac{1}{q} \geq 1$, and choose $\overline{\lambda} : 0 < 1 - \varepsilon \leq 1 < \overline{\lambda}$, where $\overline{\lambda} > \frac{\log_A \frac{1}{q}}{\lambda} = -\frac{\log_A q}{\lambda}$.

First assume that $1 - \varepsilon \in [-\frac{\log_A q}{\lambda}, +\infty)$. Hence

$$G_{\frac{1}{q},\lambda}(1-\varepsilon) > G_{\frac{1}{q},\lambda}\left(\overline{\lambda}\right) > 0, \tag{2.55}$$

by $G_{\frac{1}{q},\lambda}$ being decreasing on $[-\frac{\log_A q}{\lambda}, +\infty)$.

If $0 < 1 - \varepsilon < -\frac{\log_A q}{\lambda}$, then we use the concavity-bell shape of $G_{q,\lambda}$.

So, there exists $z_{\varepsilon} \in \left(-\frac{\log_A q}{\lambda}, +\infty\right)$ such that $G_{\frac{1}{q},\lambda}(1-\varepsilon) = G_{\frac{1}{q},\lambda}(z_{\varepsilon})$. We also consider $z_0 \in \left(-\frac{\log_A q}{\lambda}, +\infty\right)$ such that $G_{\frac{1}{q},\lambda}(z_0) = G_{\frac{1}{q},\lambda}(0)$. Clearly it holds $-\frac{\log_A q}{\lambda} < z_{\varepsilon} \leq z_0$ and we choose $\overline{\lambda} : z_0 < \overline{\lambda}$. Therefore, it holds

$$G_{\frac{1}{q},\lambda}(1-\varepsilon) \geq G_{\frac{1}{q},\lambda}(0) \geq G_{\frac{1}{q},\lambda}\left(\overline{\lambda}\right) > 0,$$

by $G_{\frac{1}{q},\lambda}$ being decreasing on $[-\frac{\log_A q}{\lambda}, +\infty)$.

Again it holds

$$\lim_{n\to\infty}\left(1 - \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} G_{q,\lambda}(nb-k)\right) > 0, \;\; 0 < q \leq 1,\; \lambda > 0,\; A > 1. \tag{2.56}$$

(iii) Similarly, $(q > 0)$

$$\begin{aligned} 1 - \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} G_{q,\lambda}(na-k) &= \sum_{k=-\infty}^{\lceil na\rceil-1} G_{q,\lambda}(na-k) + \sum_{k=\lfloor nb\rfloor+1}^{\infty} G_{q,\lambda}(na-k) \\ &> G_{q,\lambda}(na - \lceil na\rceil + 1) \\ &\text{(call } \eta := \lceil na\rceil - na,\; 0 \leq \eta < 1) \\ &= G_{q,\lambda}(1-\eta), \quad \text{etc.} \end{aligned} \tag{2.57}$$

Acting as in (i), (ii) we derive that

$$\lim_{n\to+\infty}\left(1 - \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} G_{q,\lambda}(na-k)\right) > 0. \tag{2.58}$$

Conclusion: (i) We have that

$$\lim_{n\to+\infty} \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} G_{q,\lambda}(nx-k) \neq 1, \quad \text{for at least some } x \in [a,b], \tag{2.59}$$

where $\lambda, q > 0$.

(ii) Let $[a,b] \subset \mathbb{R}$. For large n we always have $\lceil na\rceil \leq \lfloor nb\rfloor$. Also $a \leq \frac{k}{n} \leq b$, iff $\lceil na\rceil \leq k \leq \lfloor nb\rfloor$. In general it holds

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} G_{q,\lambda}(nx-k) \leq 1. \tag{2.60}$$

We make

Remark 2.6. We introduce

$$Z_{q,\lambda}(x_1,\ldots,x_N) := Z_{q,\lambda}(x) := \prod_{i=1}^{N} G_{q,\lambda}(x_i), \quad x = (x_1,\ldots,x_N) \in \mathbb{R}^N, \tag{2.61}$$

$\lambda, q > 0,\; A > 1,\; N \in \mathbb{N}$.

It has the properties:

(i) $Z_{q,\lambda}(x) > 0, \;\; \forall\, x \in \mathbb{R}^N$,

(ii)

$$\sum_{k=-\infty}^{\infty} Z_{q,\lambda}(x-k) := \sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\cdots\sum_{k_N=-\infty}^{\infty} Z_{q,\lambda}(x_1-k_1,\ldots,x_N-k_N) = 1, \tag{2.62}$$

where $k := (k_1,\ldots,k_n) \in \mathbb{Z}^N$, $\forall\, x \in \mathbb{R}^N$,
hence

(iii)

$$\sum_{k=-\infty}^{\infty} Z_{q,\lambda}(nx-k) = 1, \tag{2.63}$$

$\forall\, x \in \mathbb{R}^N$; $n \in \mathbb{N}$,
and

(iv)

$$\int_{\mathbb{R}^N} Z_{q,\lambda}(x)\,dx = 1, \tag{2.64}$$

that is Z_q is a multivariate density function.

Here denote $\|x\|_\infty := \max\{|x_1|,\ldots,|x_N|\}$, $x \in \mathbb{R}^N$, also set $\infty := (\infty,\ldots,\infty)$, $-\infty := (-\infty,\ldots,-\infty)$ upon the multivariate context, and

$$\begin{aligned}\lceil na\rceil &:= (\lceil na_1\rceil,\ldots,\lceil na_N\rceil),\\ \lfloor nb\rfloor &:= (\lfloor nb_1\rfloor,\ldots,\lfloor nb_N\rfloor),\end{aligned} \tag{2.65}$$

where $a := (a_1,\ldots,a_N)$, $b := (b_1,\ldots,b_N)$.

We obviously see that

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z_{q,\lambda}(nx-k) = \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\left(\prod_{i=1}^{N} G_{q,\lambda}(nx_i-k_i)\right)$$

$$= \sum_{k_1=\lceil na_1\rceil}^{\lfloor nb_1\rfloor}\cdots\sum_{k_N=\lceil na_N\rceil}^{\lfloor nb_N\rfloor}\left(\prod_{i=1}^{N} G_{q,\lambda}(nx_i-k_i)\right) = \prod_{i=1}^{N}\left(\sum_{k_i=\lceil na_i\rceil}^{\lfloor nb_i\rfloor} G_{q,\lambda}(nx_i-k_i)\right). \tag{2.66}$$

For $0<\beta^*<1$ and $n \in \mathbb{N}$, a fixed $x \in \mathbb{R}^N$, we have that

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z_{q,\lambda}(nx-k)$$

$$= \sum_{\begin{cases} k=\lceil na\rceil \\ \left\|\frac{k}{n}-x\right\|_\infty \le \frac{1}{n^{\beta^*}}\end{cases}}^{\lfloor nb\rfloor} Z_{q,\lambda}(nx-k) + \sum_{\begin{cases} k=\lceil na\rceil \\ \left\|\frac{k}{n}-x\right\|_\infty > \frac{1}{n^{\beta^*}}\end{cases}}^{\lfloor nb\rfloor} Z_{q,\lambda}(nx-k). \tag{2.67}$$

In the last two sums the counting is over disjoint vector sets of k's, because the condition $\left\|\frac{k}{n}-x\right\|_\infty > \frac{1}{n^{\beta^*}}$ implies that there exists at least one $\left|\frac{k_r}{n}-x_r\right| > \frac{1}{n^{\beta^*}}$, where $r \in \{1,\ldots,N\}$.

(v) By Theorem 2.3 and as in [10], pp. 379-380, we derive that

$$\sum_{\substack{k=\lceil na\rceil \\ \left\|\frac{k}{n}-x\right\|_{\infty}>\frac{1}{n^{\beta^*}}}}^{\lfloor nb\rfloor} Z_{q,\lambda}(nx-k) < \gamma A^{-\lambda\left(n^{1-\beta^*-2}\right)}, \ \ 0<\beta^*<1, \tag{2.68}$$

with $n\in\mathbb{N}: n^{1-\beta^*}>2$, $x\in\prod_{i=1}^{N}[a_i,b_i]$.

(vi) By Theorem 2.4 we get that

$$0<\frac{1}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z_{q,\lambda}(nx-k)}<(K(q))^{N}, \tag{2.69}$$

$\forall\, x\in\left(\prod_{i=1}^{N}[a_i,b_i]\right)$, $n\in\mathbb{N}$.

It is also clear that

(vii)

$$\sum_{\substack{k=-\infty \\ \left\|\frac{k}{n}-x\right\|_{\infty}>\frac{1}{n^{\beta^*}}}}^{\infty} Z_{q,\lambda}(nx-k) < \gamma A^{-\lambda\left(n^{1-\beta^*-2}\right)}, \tag{2.70}$$

$0<\beta^*<1$, $n\in\mathbb{N}: n^{1-\beta^*}>2$, $x\in\mathbb{R}^N$.

Furthermore it holds

$$\lim_{n\to\infty}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z_{q,\lambda}(nx-k)\neq 1, \tag{2.71}$$

for at least some $x\in\left(\prod_{i=1}^{N}[a_i,b_i]\right)$.

Here $\left(X,\|\cdot\|_{\gamma}\right)$ is a Banach space.

Let $f\in C\left(\prod_{i=1}^{N}[a_i,b_i],X\right)$, $x=(x_1,\ldots,x_N)\in\prod_{i=1}^{N}[a_i,b_i]$, $n\in\mathbb{N}$ such that $\lceil na_i\rceil\le\lfloor nb_i\rfloor$, $i=1,\ldots,N$.

We introduce and define the following multivariate linear normalized neural network operator ($x:=(x_1,\ldots,x_N)\in\left(\prod_{i=1}^{N}[a_i,b_i]\right)$):

$$A_n(f,x_1,\ldots,x_N):=A_n(f,x):=\frac{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} f\left(\frac{k}{n}\right)Z_{q,\lambda}(nx-k)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z_{q,\lambda}(nx-k)}$$

$$=\frac{\sum_{k_1=\lceil na_1\rceil}^{\lfloor nb_1\rfloor}\sum_{k_2=\lceil na_2\rceil}^{\lfloor nb_2\rfloor}\cdots\sum_{k_N=\lceil na_N\rceil}^{\lfloor nb_N\rfloor} f\left(\frac{k_1}{n},\ldots,\frac{k_N}{n}\right)\left(\prod_{i=1}^{N}G_{q,\lambda}(nx_i-k_i)\right)}{\prod_{i=1}^{N}\left(\sum_{k_i=\lceil na_i\rceil}^{\lfloor nb_i\rfloor}G_{q,\lambda}(nx_i-k_i)\right)}. \tag{2.72}$$

For large enough $n\in\mathbb{N}$ we always obtain $\lceil na_i\rceil\le\lfloor nb_i\rfloor$, $i=1,\ldots,N$. Also $a_i\le\frac{k_i}{n}\le b_i$, iff $\lceil na_i\rceil\le k_i\le\lfloor nb_i\rfloor$, $i=1,\ldots,N$.

When $g \in C\left(\prod_{i=1}^{N}[a_i,b_i]\right)$ we define the companion operator

$$\widetilde{A}_n(g,x) := \frac{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} g\left(\frac{k}{n}\right) Z_{q,\lambda}(nx-k)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z_{q,\lambda}(nx-k)}. \tag{2.73}$$

Clearly $\widetilde{A}_n$ is a positive linear operator. We have that

$$\widetilde{A}_n(1,x) = 1, \ \ \forall\, x \in \left(\prod_{i=1}^{N}[a_i,b_i]\right).$$

Notice that $A_n(f) \in C\left(\prod_{i=1}^{N}[a_i,b_i], X\right)$ and $\widetilde{A}_n(g) \in C\left(\prod_{i=1}^{N}[a_i,b_i]\right)$.

Furthermore it holds

$$\|A_n(f,x)\|_\gamma \le \frac{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} \left\|f\left(\frac{k}{n}\right)\right\|_\gamma Z_{q,\lambda}(nx-k)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z_{q,\lambda}(nx-k)} = \widetilde{A}_n\left(\|f\|_\gamma, x\right), \tag{2.74}$$

$\forall\, x \in \prod_{i=1}^{N}[a_i,b_i]$.

Clearly $\|f\|_\gamma \in C\left(\prod_{i=1}^{N}[a_i,b_i]\right)$.

So, we have that

$$\|A_n(f,x)\|_\gamma \le \widetilde{A}_n\left(\|f\|_\gamma, x\right), \tag{2.75}$$

$\forall\, x \in \prod_{i=1}^{N}[a_i,b_i]$, $\forall\, n \in \mathbb{N}$, $\forall\, f \in C\left(\prod_{i=1}^{N}[a_i,b_i], X\right)$.

Let $c \in X$ and $g \in C\left(\prod_{i=1}^{N}[a_i,b_i]\right)$, then $cg \in C\left(\prod_{i=1}^{N}[a_i,b_i], X\right)$.

Furthermore it holds

$$A_n(cg,x) = c\widetilde{A}_n(g,x), \ \ \forall\, x \in \prod_{i=1}^{N}[a_i,b_i]. \tag{2.76}$$

Since $\widetilde{A}_n(1) = 1$, we get that

$$A_n(c) = c, \ \ \forall\, c \in X. \tag{2.77}$$

We call $\widetilde{A}_n$ the companion operator of A_n.

For convenience we call

$$A_n^*(f,x) := \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} f\left(\frac{k}{n}\right) Z_{q,\lambda}(nx-k)$$

$$= \sum_{k_1=\lceil na_1\rceil}^{\lfloor nb_1\rfloor} \sum_{k_2=\lceil na_2\rceil}^{\lfloor nb_2\rfloor} \cdots \sum_{k_N=\lceil na_N\rceil}^{\lfloor nb_N\rfloor} f\left(\frac{k_1}{n},\ldots,\frac{k_N}{n}\right)\left(\prod_{i=1}^{N} G_{q,\lambda}(nx_i-k_i)\right), \tag{2.78}$$

$\forall\, x \in \left(\prod_{i=1}^{N}[a_i,b_i]\right)$.

That is

$$A_n(f,x) := \frac{A_n^*(f,x)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z_{q,\lambda}(nx-k)}, \tag{2.79}$$

$\forall\ x \in \left(\prod_{i=1}^{N}[a_i,b_i]\right)$, $n \in \mathbb{N}$.

Hence

$$A_n(f,x) - f(x) = \frac{A_n^*(f,x) - f(x)\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z_{q,\lambda}(nx-k)\right)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z_{q,\lambda}(nx-k)}. \tag{2.80}$$

Consequently we derive

$$\|A_n(f,x) - f(x)\|_\gamma \overset{(2.69)}{\leq} (K(q))^N \left\| A_n^*(f,x) - f(x) \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z_{q,\lambda}(nx-k)\right\|_\gamma, \tag{2.81}$$

$\forall\ x \in \left(\prod_{i=1}^{N}[a_i,b_i]\right)$.

We will estimate the right hand side of (2.81).

For the last and others we need

Definition 2.7. ([11], p. 274) Let M be a convex and compact subset of $\left(\mathbb{R}^N, \|\cdot\|_p\right)$, $p \in [1,\infty]$, and $\left(X, \|\cdot\|_\gamma\right)$ be a Banach space. Let $f \in C(M,X)$. We define the first modulus of continuity of f as

$$\omega_1(f,\delta) := \sup_{\substack{x,y \in M:\\ \|x-y\|_p \leq \delta}} \|f(x) - f(y)\|_\gamma, \quad 0 < \delta \leq diam(M). \tag{2.82}$$

If $\delta > diam(M)$, then

$$\omega_1(f,\delta) = \omega_1(f, diam(M)). \tag{2.83}$$

Notice $\omega_1(f,\delta)$ is increasing in $\delta > 0$. For $f \in C_B(M,X)$ (continuous and bounded functions) $\omega_1(f,\delta)$ is defined similarly.

Lemma 2.8. *([11], p. 274) We have $\omega_1(f,\delta) \to 0$ as $\delta \downarrow 0$, iff $f \in C(M,X)$, where M is a convex compact subset of $\left(\mathbb{R}^N, \|\cdot\|_p\right)$, $p \in [1,\infty]$.*

Clearly we have also: $f \in C_U\left(\mathbb{R}^N, X\right)$ (uniformly continuous functions), iff $\omega_1(f,\delta) \to 0$ as $\delta \downarrow 0$, where ω_1 is defined similarly to (2.82). The space $C_B\left(\mathbb{R}^N, X\right)$ denotes the continuous and bounded functions on $\mathbb{R}^N$.

Let now $f \in C^m\left(\prod_{i=1}^{N}[a_i,b_i]\right)$, $m, N \in \mathbb{N}$. Here f_α denotes a partial derivative of f, $\alpha := (\alpha_1, \ldots, \alpha_N)$, $\alpha_i \in \mathbb{Z}_+$, $i = 1, \ldots, N$, and $|\alpha| := \sum_{i=1}^{N} \alpha_i = l$, where $l = 0, 1, \ldots, m$. We write also $f_\alpha := \frac{\partial^n f}{\partial x^n}$ and we say it is of order l.

We denote

$$\omega_{1,m}^{\max}(f_\alpha, h) := \max_{\alpha:|\alpha|=m} \omega_1(f_\alpha, h). \tag{2.84}$$

Call also

$$\|f_\alpha\|_{\infty,m}^{\max} := \max_{|\alpha|=m} \{\|f_\alpha\|_\infty\}, \tag{2.85}$$

where $\|\cdot\|_\infty$ is the supremum norm.

When $f \in C_B\left(\mathbb{R}^N, X\right)$ we define,

$$B_n(f,x) := B_n(f,x_1,\ldots,x_N) := \sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right) Z_{q,\lambda}(nx-k)$$

$$:= \sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\cdots\sum_{k_N=-\infty}^{\infty} f\left(\frac{k_1}{n},\frac{k_2}{n},\ldots,\frac{k_N}{n}\right)\left(\prod_{i=1}^{N} G_{q,\lambda}(nx_i-k_i)\right), \tag{2.86}$$

$n \in \mathbb{N}$, $\forall\, x \in \mathbb{R}^N$, $N \in \mathbb{N}$, the multivariate quasi-interpolation neural network operator.

Also for $f \in C_B\left(\mathbb{R}^N, X\right)$ we define the multivariate Kantorovich type neural network operator

$$C_n(f,x) := C_n(f,x_1,\ldots,x_N) := \sum_{k=-\infty}^{\infty}\left(n^N \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)\,dt\right) Z_{q,\lambda}(nx-k)$$

$$= \sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\cdots\sum_{k_N=-\infty}^{\infty}\left(n^N \int_{\frac{k_1}{n}}^{\frac{k_1+1}{n}}\int_{\frac{k_2}{n}}^{\frac{k_2+1}{n}}\cdots\int_{\frac{k_N}{n}}^{\frac{k_N+1}{n}} f(t_1,\ldots,t_N)\,dt_1\ldots dt_N\right)$$

$$\cdot\left(\prod_{i=1}^{N} G_{q,\lambda}(nx_i-k_i)\right), \tag{2.87}$$

$n \in \mathbb{N}$, $\forall\, x \in \mathbb{R}^N$.

Again for $f \in C_B\left(\mathbb{R}^N, X\right)$, $N \in \mathbb{N}$, we define the multivariate neural network operator of quadrature type $D_n(f,x)$, $n \in \mathbb{N}$, as follows.

Let $\theta = (\theta_1,\ldots,\theta_N) \in \mathbb{N}^N$, $r = (r_1,\ldots,r_N) \in \mathbb{Z}_+^N$, $w_r = w_{r_1,r_2,\ldots r_N} \geq 0$, such that $\sum\limits_{r=0}^{\theta} w_r = \sum\limits_{r_1=0}^{\theta_1}\sum\limits_{r_2=0}^{\theta_2}\cdots\sum\limits_{r_N=0}^{\theta_N} w_{r_1,r_2,\ldots r_N} = 1$; $k \in \mathbb{Z}^N$ and

$$\delta_{nk}(f) := \delta_{n,k_1,k_2,\ldots,k_N}(f) := \sum_{r=0}^{\theta} w_r f\left(\frac{k}{n}+\frac{r}{n\theta}\right)$$

$$= \sum_{r_1=0}^{\theta_1}\sum_{r_2=0}^{\theta_2}\cdots\sum_{r_N=0}^{\theta_N} w_{r_1,r_2,\ldots r_N} f\left(\frac{k_1}{n}+\frac{r_1}{n\theta_1},\frac{k_2}{n}+\frac{r_2}{n\theta_2},\ldots,\frac{k_N}{n}+\frac{r_N}{n\theta_N}\right), \tag{2.88}$$

where $\frac{r}{\theta} := \left(\frac{r_1}{\theta_1},\frac{r_2}{\theta_2},\ldots,\frac{r_N}{\theta_N}\right)$.

We set

$$D_n(f,x) := D_n(f,x_1,\ldots,x_N) := \sum_{k=-\infty}^{\infty} \delta_{nk}(f)\, Z_{q,\lambda}(nx-k) \tag{2.89}$$

$$= \sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\cdots\sum_{k_N=-\infty}^{\infty} \delta_{n,k_1,k_2,\ldots,k_N}(f)\left(\prod_{i=1}^{N} G_{q,\lambda}(nx_i-k_i)\right),$$

$\forall\, x \in \mathbb{R}^N$.

In this article we study the approximation properties of A_n, B_n, C_n, D_n neural network operators and as well of their iterates, that is acting with multilayer neural networks. Thus the quantitative pointwise and uniform convergence of these operators to the unit operator I.

3. Multivariate general Neural Network Approximations

Here we present several vectorial neural network approximations to Banach space valued functions given with rates.

We give

Theorem 3.1. *Let* $f \in C\left(\prod_{i=1}^{N}[a_i,b_i], X\right)$, $0<\beta^*<1$, $q,\lambda>0$, $A>1$, $x \in \left(\prod_{i=1}^{N}[a_i,b_i]\right)$, $N,n\in\mathbb{N}$ *with* $n^{1-\beta^*}>2$. *Then*

1)

$$\|A_n(f,x)-f(x)\|_\gamma$$
$$\leq (K(q))^N\left[\omega_1\left(f,\frac{1}{n^{\beta^*}}\right)+2\gamma A^{-\lambda\left(n^{1-\beta^*}-2\right)}\left\|\|f\|_\gamma\right\|_\infty\right]=:\lambda_1(n), \tag{3.1}$$

and

2)

$$\left\|\|A_n(f)-f\|_\gamma\right\|_\infty \leq \lambda_1(n). \tag{3.2}$$

We notice that $\lim_{n\to\infty} A_n(f) \overset{\|\cdot\|_\gamma}{=} f$, *pointwise and uniformly.*

Above ω_1 *is with respect to* $p=\infty$.

Proof. We observe that

$$\overline{\Delta}(x) := A_n^*(f,x)-f(x)\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z_{q,\lambda}(nx-k)$$
$$=\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} f\left(\frac{k}{n}\right)Z_{q,\lambda}(nx-k)-\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} f(x)Z_{q,\lambda}(nx-k)$$
$$=\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\left(f\left(\frac{k}{n}\right)-f(x)\right)Z_{q,\lambda}(nx-k). \tag{3.3}$$

Thus

$$\left\|\overline{\Delta}(x)\right\|_\gamma \leq \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\left\|f\left(\frac{k}{n}\right)-f(x)\right\|_\gamma Z_{q,\lambda}(nx-k)$$
$$=\sum_{\substack{k=\lceil na\rceil \\ \left\|\frac{k}{n}-x\right\|_\infty\leq\frac{1}{n^{\beta^*}}}}^{\lfloor nb\rfloor}\left\|f\left(\frac{k}{n}\right)-f(x)\right\|_\gamma Z_{q,\lambda}(nx-k)$$

$$+ \sum_{\left\{\begin{array}{c} k=\lceil na\rceil \\ \left\|\frac{k}{n}-x\right\|_{\infty}>\frac{1}{n^{\beta^*}}\end{array}\right.}^{\lfloor nb\rfloor} \left\| f\left(\frac{k}{n}\right)-f(x)\right\|_{\gamma} Z_{q,\lambda}(nx-k)$$

$$\overset{(2.63)}{\leq} \omega_1\left(f,\frac{1}{n^{\beta^*}}\right)+2\left\|\|f\|_{\gamma}\right\|_{\infty} \sum_{\left\{\begin{array}{c} k=\lceil na\rceil \\ \left\|\frac{k}{n}-x\right\|_{\infty}>\frac{1}{n^{\beta^*}}\end{array}\right.}^{\lfloor nb\rfloor} Z_{q,\lambda}(nx-k)$$

$$\overset{(2.68)}{\leq} \omega_1\left(f,\frac{1}{n^{\beta^*}}\right)+2\gamma A^{-\lambda\left(n^{1-\beta^*}-2\right)}\left\|\|f\|_{\gamma}\right\|_{\infty}. \tag{3.4}$$

So that

$$\left\|\overline{\Delta}(x)\right\|_{\gamma}\leq \omega_1\left(f,\frac{1}{n^{\beta^*}}\right)+2\gamma A^{-\lambda\left(n^{1-\beta^*}-2\right)}\left\|\|f\|_{\gamma}\right\|_{\infty}. \tag{3.5}$$

Now using (2.81) we finish the proof. □

When $X=\mathbb{R}$, next we discuss the high order of approximation.

Theorem 3.2. *Let* $f\in C^m\left(\prod_{i=1}^{N}[a_i,b_i]\right)$, $0<\beta^*<1$, $n,m,N\in\mathbb{N}$, $n^{1-\beta^*}\geq 3$, $A>1$, $\lambda>0$, $q>0$, $x\in\left(\prod_{i=1}^{N}[a_i,b_i]\right)$. *Then*

i)

$$\left|\widetilde{A}_n(f,x)-f(x)-\sum_{j=1}^{m}\left(\sum_{|\alpha|=j}\left(\frac{f_\alpha(x)}{\prod_{=1}^{N}\alpha_i!}\right)\widetilde{A}_n\left(\prod_{i=1}^{N}(\cdot-x_i)^{\alpha_i},x\right)\right)\right| \tag{3.6}$$

$$\leq (K(q))^N\left\{\frac{N^m}{m!n^{m\beta^*}}\omega_{1,m}^{\max}\left(f_\alpha,\frac{1}{n^{\beta^*}}\right)+\left(\frac{\|b-a\|_\infty^m\|f_\alpha\|_{\infty,m}^{\max}N^m}{m!}\right)2\gamma A^{-\lambda\left(n^{1-\beta^*}-2\right)}\right\}.$$

ii)

$$\left|\widetilde{A}_n(f,x)-f(x)\right|\leq (K(q))^N \tag{3.7}$$

$$\left\{\sum_{j=1}^{m}\left(\sum_{|\alpha|=j}\left(\frac{|f_\alpha(x)|}{\prod_{=1}^{N}\alpha_i!}\right)\left[\frac{1}{n^{\beta^*j}}+\left(\prod_{i=1}^{N}(b_i-a_i)^{\alpha_i}\right)\gamma A^{-\lambda\left(n^{1-\beta^*}-2\right)}\right]\right)\right.$$

$$\left.+\frac{N^m}{m!n^{m\beta^*}}\omega_{1,m}^{\max}\left(f_\alpha,\frac{1}{n^{\beta^*}}\right)+\left(\frac{\|b-a\|_\infty^m\|f_\alpha\|_{\infty,m}^{\max}N^m}{m!}\right)2\gamma A^{-\lambda\left(n^{1-\beta^*}-2\right)}\right\}.$$

iii)

$$\left\|\widetilde{A}_n(f)-f\right\|_\infty\leq (K(q))^N \tag{3.8}$$

$$\left\{ \sum_{j=1}^{m} \left(\sum_{|\alpha|=j} \left(\frac{\||f_\alpha|\|_\infty}{\prod_{=1}^{N} \alpha_i!} \right) \left[\frac{1}{n^{\beta^* j}} + \left(\prod_{i=1}^{N} (b_i - a_i)^{\alpha_i} \right) \gamma A^{-\lambda\left(n^{1-\beta^*}-2\right)} \right] \right) \right.$$

$$\left. + \frac{N^m}{m! n^{m\beta^*}} \omega_{1,m}^{\max} \left(f_\alpha, \frac{1}{n^{\beta^*}} \right) + \left(\frac{\|b-a\|_\infty^m \|f_\alpha\|_{\infty,m}^{\max} N^m}{m!} \right) 2\gamma A^{-\lambda\left(n^{1-\beta^*}-2\right)} \right\}.$$

iv) Assume $f_\alpha(x_0) = 0$, *for all* $\alpha : |\alpha| = 1, \ldots, m$; $x_0 \in \left(\prod_{i=1}^{N} [a_i, b_i] \right)$. *Then*

$$\left| \widetilde{A}_n (f, x_0) - f(x_0) \right| \tag{3.9}$$

$$\leq (K(q))^N \left\{ \frac{N^m}{m! n^{m\beta^*}} \omega_{1,m}^{\max} \left(f_\alpha, \frac{1}{n^{\beta^*}} \right) + \left(\frac{\|b-a\|_\infty^m \|f_\alpha\|_{\infty,m}^{\max} N^m}{m!} \right) 2\gamma A^{-\lambda\left(n^{1-\beta^*}-2\right)} \right\},$$

notice in the last the extremely high rate of convergence at $n^{-\beta^*(m+1)}$.

Proof. As similar to [10], pp. 389-391, is omitted. □

We continue with

Theorem 3.3. *Let* $f \in C_B\left(\mathbb{R}^N, X\right)$, $0 < \beta^* < 1$, $x \in \mathbb{R}^N$, $q > 0$, $\lambda > 0$, $A > 1$, $N, n \in \mathbb{N}$ *with* $n^{1-\beta^*} > 2$, ω_1 *is for* $p = \infty$. *Then*

1)

$$\|B_n(f, x) - f(x)\|_\gamma \leq \omega_1 \left(f, \frac{1}{n^{\beta^*}} \right) + 2\gamma A^{-\lambda\left(n^{1-\beta^*}-2\right)} \left\| \|f\|_\gamma \right\|_\infty =: \lambda_2(n), \tag{3.10}$$

2)

$$\left\| \|B_n(f) - f\|_\gamma \right\|_\infty \leq \lambda_2(n). \tag{3.11}$$

Given that $f \in \left(C_U\left(\mathbb{R}^N, X\right) \cap C_B\left(\mathbb{R}^N, X\right) \right)$, *we obtain* $\lim_{n\to\infty} B_n(f) = f$, *uniformly.*

Proof. We have that

$$B_n(f, x) - f(x) \overset{(2.63)}{=} \sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right) Z_{q,\lambda}(nx - k) - f(x) \sum_{k=-\infty}^{\infty} Z_{q,\lambda}(nx - k) \tag{3.12}$$

$$= \sum_{k=-\infty}^{\infty} \left(f\left(\frac{k}{n}\right) - f(x) \right) Z_{q,\lambda}(nx - k).$$

Hence

$$\|B_n(f, x) - f(x)\|_\gamma \leq \sum_{k=-\infty}^{\infty} \left\| f\left(\frac{k}{n}\right) - f(x) \right\|_\gamma Z_{q,\lambda}(nx - k)$$

$$= \sum_{\substack{k=-\infty \\ \left\| \frac{k}{n} - x \right\|_\infty \leq \frac{1}{n^{\beta^*}}}}^{\infty} \left\| f\left(\frac{k}{n}\right) - f(x) \right\|_\gamma Z_{q,\lambda}(nx - k)$$

$$+\sum_{\left\{\begin{array}{c}k=-\infty\\ \left\|\frac{k}{n}-x\right\|_{\infty}>\frac{1}{n^{\beta^*}}\end{array}\right.}^{\infty}\left\|f\left(\frac{k}{n}\right)-f(x)\right\|_{\gamma}Z_{q,\lambda}(nx-k)$$

$$\overset{(2.63)}{\leq}\omega_1\left(f,\frac{1}{n^{\beta^*}}\right)+2\left\|\|f\|_{\gamma}\right\|_{\infty}\sum_{\left\{\begin{array}{c}k=-\infty\\ \left\|\frac{k}{n}-x\right\|_{\infty}>\frac{1}{n^{\beta^*}}\end{array}\right.}^{\infty}Z_{q,\lambda}(nx-k)$$

$$\overset{(2.70)}{\leq}\omega_1\left(f,\frac{1}{n^{\beta^*}}\right)+2\gamma A^{-\lambda\left(n^{1-\beta^*}-2\right)}\left\|\|f\|_{\gamma}\right\|_{\infty}, \tag{3.13}$$

proving the claim. ∎

We give

Theorem 3.4. *Let $f\in C_B\left(\mathbb{R}^N,X\right)$, $0<\beta^*<1$, $x\in\mathbb{R}^N$, $q>0$, $\lambda>0$, $A>1$, $N,n\in\mathbb{N}$ with $n^{1-\beta^*}>2$, ω_1 is for $p=\infty$. Then*

1)

$$\|C_n(f,x)-f(x)\|_{\gamma}\leq\omega_1\left(f,\frac{1}{n}+\frac{1}{n^{\beta^*}}\right)+2\gamma A^{-\lambda\left(n^{1-\beta^*}-2\right)}\left\|\|f\|_{\gamma}\right\|_{\infty}=:\lambda_3(n), \tag{3.14}$$

2)

$$\left\|\|C_n(f)-f\|_{\gamma}\right\|_{\infty}\leq\lambda_3(n). \tag{3.15}$$

Given that $f\in\left(C_U\left(\mathbb{R}^N,X\right)\cap C_B\left(\mathbb{R}^N,X\right)\right)$, we obtain $\lim_{n\to\infty}C_n(f)=f$, uniformly.

Proof. We notice that

$$\int_{\frac{k}{n}}^{\frac{k+1}{n}}f(t)\,dt=\int_{\frac{k_1}{n}}^{\frac{k_1+1}{n}}\int_{\frac{k_2}{n}}^{\frac{k_2+1}{n}}\ldots\int_{\frac{k_N}{n}}^{\frac{k_N+1}{n}}f(t_1,t_2,\ldots,t_N)\,dt_1dt_2\ldots dt_N$$

$$=\int_0^{\frac{1}{n}}\int_0^{\frac{1}{n}}\ldots\int_0^{\frac{1}{n}}f\left(t_1+\frac{k_1}{n},t_2+\frac{k_2}{n},\ldots,t_N+\frac{k_N}{n}\right)dt_1\ldots dt_N=\int_0^{\frac{1}{n}}f\left(t+\frac{k}{n}\right)dt. \tag{3.16}$$

Thus it holds (by (2.87))

$$C_n(f,x)=\sum_{k=-\infty}^{\infty}\left(n^N\int_0^{\frac{1}{n}}f\left(t+\frac{k}{n}\right)dt\right)Z_{q,\lambda}(nx-k). \tag{3.17}$$

We observe that

$$\|C_n(f,x)-f(x)\|_{\gamma}$$

$$=\left\|\sum_{k=-\infty}^{\infty}\left(n^N\int_0^{\frac{1}{n}}f\left(t+\frac{k}{n}\right)dt\right)Z_{q,\lambda}(nx-k)-\sum_{k=-\infty}^{\infty}f(x)Z_{q,\lambda}(nx-k)\right\|_{\gamma}$$

$$=\left\|\sum_{k=-\infty}^{\infty}\left(\left(n^N\int_0^{\frac{1}{n}}f\left(t+\frac{k}{n}\right)dt\right)-f(x)\right)Z_{q,\lambda}(nx-k)\right\|_{\gamma}$$

$$= \left\| \sum_{k=-\infty}^{\infty} \left(n^N \int_0^{\frac{1}{n}} \left(f\left(t + \frac{k}{n} \right) - f(x) \right) dt \right) Z_{q,\lambda}(nx - k) \right\|_\gamma \tag{3.18}$$

$$\leq \sum_{k=-\infty}^{\infty} \left(n^N \int_0^{\frac{1}{n}} \left\| f\left(t + \frac{k}{n} \right) - f(x) \right\|_\gamma dt \right) Z_{q,\lambda}(nx - k)$$

$$= \sum_{\substack{k=-\infty \\ \left\| \frac{k}{n} - x \right\|_\infty \leq \frac{1}{n^{\beta^*}}}}^{\infty} \left(n^N \int_0^{\frac{1}{n}} \left\| f\left(t + \frac{k}{n} \right) - f(x) \right\|_\gamma dt \right) Z_{q,\lambda}(nx - k)$$

$$+ \sum_{\substack{k=-\infty \\ \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^{\beta^*}}}}^{\infty} \left(n^N \int_0^{\frac{1}{n}} \left\| f\left(t + \frac{k}{n} \right) - f(x) \right\|_\gamma dt \right) Z_{q,\lambda}(nx - k)$$

$$\leq \sum_{\substack{k=-\infty \\ \left\| \frac{k}{n} - x \right\|_\infty \leq \frac{1}{n^{\beta^*}}}}^{\infty} \left(n^N \int_0^{\frac{1}{n}} \omega_1 \left(f, \|t\|_\infty + \left\| \frac{k}{n} - x \right\|_\infty \right) dt \right) Z_{q,\lambda}(nx - k)$$

$$+2 \left\| \|f\|_\gamma \right\|_\infty \left(\sum_{\substack{k=-\infty \\ \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^{\beta^*}}}}^{\infty} Z_{q,\lambda}(|nx - k|) \right)$$

$$\leq \omega_1 \left(f, \frac{1}{n} + \frac{1}{n^{\beta^*}} \right) + 2\gamma A^{-\lambda\left(n^{1-\beta^*-2}\right)} \left\| \|f\|_\gamma \right\|_\infty, \tag{3.19}$$

proving the claim. □

We also present

Theorem 3.5. *Let $f \in C_B\left(\mathbb{R}^N, X\right)$, $0 < \beta^* < 1$, $x \in \mathbb{R}^N$, $q > 0$, $\lambda > 0$, $A > 1$, $N, n \in \mathbb{N}$ with $n^{1-\beta^*} > 2$, ω_1 is for $p = \infty$. Then*

1)

$$\|D_n(f, x) - f(x)\|_\gamma \leq \omega_1 \left(f, \frac{1}{n} + \frac{1}{n^{\beta^*}} \right) + 2\gamma A^{-\lambda\left(n^{1-\beta^*-2}\right)} \left\| \|f\|_\gamma \right\|_\infty = \lambda_4(n), \tag{3.20}$$

2)

$$\left\| \|D_n(f) - f\|_\gamma \right\|_\infty \leq \lambda_4(n). \tag{3.21}$$

Given that $f \in \left(C_U\left(\mathbb{R}^N, X\right) \cap C_B\left(\mathbb{R}^N, X\right)\right)$, we obtain $\lim_{n\to\infty} D_n(f) = f$, uniformly.

Proof. Similar to the proof of Theorem 3.4, as such is omitted. □

Definition 3.6. Let $f \in C_B\left(\mathbb{R}^N, X\right)$, $N \in \mathbb{N}$, $q > 0$, $\lambda > 0$, $A > 1$, where $\left(X, \|\cdot\|_\gamma\right)$ is a Banach space. We define the general neural network operator

$$F_n(f,x) := \sum_{k=-\infty}^{\infty} l_{nk}(f)\, Z_{q,\lambda}(nx-k) =$$

$$\begin{cases} B_n(f,x), & \text{if } l_{nk}(f) = f\left(\frac{k}{n}\right), \\ C_n(f,x), & \text{if } l_{nk}(f) = n^N \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)\,dt, \\ D_n(f,x), & \text{if } l_{nk}(f) = \delta_{nk}(f). \end{cases} \tag{3.22}$$

Clearly $l_{nk}(f)$ is an X-valued bounded linear functional such that

$$\|l_{nk}(f)\|_\gamma \leq \left\| \|f\|_\gamma \right\|_\infty .$$

Hence $F_n(f)$ is a bounded linear operator with $\left\| \|F_n(f)\|_\gamma \right\|_\infty \leq \left\| \|f\|_\gamma \right\|_\infty$.

We need

Theorem 3.7. *Let $f \in C_B\left(\mathbb{R}^N, X\right)$, $N \geq 1$, $\lambda, q > 0$, $A > 1$. Then*

$$F_n(f) \in C_B\left(\mathbb{R}^N, X\right).$$

Proof. It is very lengthy and very similar to [13], pp. 167-171. As such is omitted. □

Remark 3.8. By (2.72) it is obvious that $\left\| \|A_n(f)\|_\gamma \right\|_\infty \leq \left\| \|f\|_\gamma \right\|_\infty < \infty$, and $A_n(f) \in C\left(\prod_{i=1}^N [a_i, b_i], X\right)$, given that $f \in C\left(\prod_{i=1}^N [a_i, b_i], X\right)$.

Call L_n any of the operators A_n, B_n, C_n, D_n.

Clearly then

$$\left\| \|L_n^2(f)\|_\gamma \right\|_\infty = \left\| \|L_n(L_n(f))\|_\gamma \right\|_\infty \leq \left\| \|L_n(f)\|_\gamma \right\|_\infty \leq \left\| \|f\|_\gamma \right\|_\infty, \tag{3.23}$$

etc.

Therefore we get

$$\left\| \|L_n^k(f)\|_\gamma \right\|_\infty \leq \left\| \|f\|_\gamma \right\|_\infty, \quad \forall\, k \in \mathbb{N}, \tag{3.24}$$

the contraction property.

Also we see that

$$\left\| \|L_n^k(f)\|_\gamma \right\|_\infty \leq \left\| \|L_n^{k-1}(f)\|_\gamma \right\|_\infty \leq \ldots \leq \left\| \|L_n(f)\|_\gamma \right\|_\infty \leq \left\| \|f\|_\gamma \right\|_\infty. \tag{3.25}$$

Here L_n^k are bounded linear operators.

Notation 3.9. Here $q > 0$, $\lambda > 0$, $A > 1$, $N \in \mathbb{N}$, $0 < \beta^* < 1$. Denote by

$$c_N := \begin{cases} (K(q))^N, & \text{if } L_n = A_n, \\ 1, & \text{if } L_n = B_n, C_n, D_n, \end{cases} \tag{3.26}$$

$$\varphi(n) := \begin{cases} \frac{1}{n^{\beta^*}}, & \text{if } L_n = A_n, B_n, \\ \frac{1}{n} + \frac{1}{n^{\beta^*}}, & \text{if } L_n = C_n, D_n, \end{cases} \tag{3.27}$$

$$\Omega := \begin{cases} C\left(\prod_{i=1}^{N}[a_i,b_i],X\right), & \text{if } L_n = A_n, \\ C_B\left(\mathbb{R}^N,X\right), & \text{if } L_n = B_n, C_n, D_n, \end{cases} \tag{3.28}$$

and

$$Y := \begin{cases} \prod_{i=1}^{N}[a_i,b_i], & \text{if } L_n = A_n, \\ \mathbb{R}^N, & \text{if } L_n = B_n, C_n, D_n. \end{cases} \tag{3.29}$$

We give the following combined result.

Theorem 3.10. *Let $f \in \Omega$, $0 < \beta^* < 1$, $x \in Y$; $q > 0$, $\lambda > 0$, $A > 1$, n, $N \in \mathbb{N}$ with $n^{1-\beta^*} > 2$. Then*

(i)

$$\|L_n(f,x) - f(x)\|_\gamma \leq c_N\left[\omega_1(f,\varphi(n)) + 2\gamma A^{-\lambda\left(n^{1-\beta^*}-2\right)}\left\|\|f\|_\gamma\right\|_\infty\right] =: \tau(n), \tag{3.30}$$

where ω_1 is for $p = \infty$,
x and

(ii)

$$\left\|\|L_n(f) - f\|_\gamma\right\|_\infty \leq \tau(n) \to 0, \text{ as } n \to \infty. \tag{3.31}$$

For f uniformly continuous and in Ω we obtain

$$\lim_{n\to\infty} L_n(f) = f,$$

pointwise and uniformly.

Proof. By Theorems 3.1, 3.3, 3.4, 3.5. □

Next we talk about iterated multilayer neural network approximation (see also [9]).

We give

Theorem 3.11. *All here as in Theorem 3.10 and $r \in \mathbb{N}$, $\tau(n)$ as in (3.30). Then*

$$\left\|\|L_n^r f - f\|_\gamma\right\|_\infty \leq r\tau(n). \tag{3.32}$$

So that the speed of convergence to the unit operator of L_n^r is not worse than of L_n.

Proof. As similar to [13], pp. 172-173, is omitted. □

We also present the more general

Theorem 3.12. *Let $f \in \Omega$; $q > 0$, $\lambda > 0$, $A > 1$, $N, m_1, m_2, \ldots, m_r \in \mathbb{N} : m_1 \leq m_2 \leq \ldots \leq m_r$, $0 < \beta^* < 1$; $m_i^{1-\beta^*} > 2$, $i = 1, \ldots, r$, $x \in Y$, and let $(L_{m_1}, \ldots, L_{m_r})$ as $(A_{m_1}, \ldots, A_{m_r})$ or $(B_{m_1}, \ldots, B_{m_r})$ or $(C_{m_1}, \ldots, C_{m_r})$ or $(D_{m_1}, \ldots, D_{m_r})$, $p = \infty$. Then*

$$\left\|L_{m_r}\left(L_{m_{r-1}}\left(\ldots L_{m_2}\left(L_{m_1}f\right)\right)\right)(x) - f(x)\right\|_\gamma$$
$$\leq \left\|\left\|L_{m_r}\left(L_{m_{r-1}}\left(\ldots L_{m_2}\left(L_{m_1}f\right)\right)\right) - f\right\|_\gamma\right\|_\infty$$

$$\leq \sum_{i=1}^{r} \left\| \left\| L_{m_i} f - f \right\|_{\gamma} \right\|_{\infty}$$

$$\leq c_N \sum_{i=1}^{r} \left[\omega_1 \left(f, \varphi \left(m_i \right) \right) + 2\gamma A^{-\lambda\left(n^{1-\beta^*-2}\right)} \left\| \left\| f \right\|_{\gamma} \right\|_{\infty} \right]$$

$$\leq r c_N \left[\omega_1 \left(f, \varphi \left(m_1 \right) \right) + 2\gamma A^{-\lambda\left(n^{1-\beta^*-2}\right)} \left\| \left\| f \right\|_{\gamma} \right\|_{\infty} \right]. \quad (3.33)$$

Clearly, we notice that the speed of convergence to the unit operator of the multiply iterated operator is not worse than the speed of L_{m_1}.

Proof. As similar to [13], pp. 173-175, is omitted. □

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