Stud. Univ. Babeş-Bolyai Math. 69(2024), No. 3, 567–585
DOI: 10.24193/subbmath.2024.3.07

New results on asymptotic stability of time-varying nonlinear systems with applications

Abir Kicha, Hanen Damak and Mohamed Ali Hammami

Abstract. In this paper, we present a converse Lyapunov theorem for the new notion of global generalized practical uniform h-stability of nonlinear systems of differential equations. We derive some sufficient conditions which guarantee the global generalized practical uniform h-stability of time-varying perturbed systems. In addition, these results are used to study the practical h-stability of models of infectious diseases and vaccination.

Mathematics Subject Classification (2010): 35B40, 37B55, 34D20, 93D15, 92D30.

Keywords: Epidemic models, generalized practical uniform h-stability, Gronwall's inequalities, Lyapunov functions.

1. Introduction

The most important stability concept used in the qualitative theory of differential equations is the uniform exponential stability. In some situations, particularly, in the non-autonomous setting, the notion of uniform exponential stability is to restrictive and it is important to look for a more general behavior. In the last century, Manual Pinto (see [21, 20]) introduced a new notion of stability called h-stability for nonlinear differential equations on the Euclidean space $\mathbb{R}^n$, with the intention of obtaining results about stability for a weakly stable system (at least, weaker than those given exponential asymptotic stability) under some perturbations. Some important properties about h-stability for various differential systems and nonlinear differential systems are given. In [4], the authors investigated the h-stability properties for nonlinear differential systems using the notion of t_∞-similarity and Lyapunov functions. Goo and al. studied h-stability for the nonlinear Volterra integro-differential system

Received 31 March 2022; Accepted 26 September 2022.

(see [10]) and for the linear perturbed Volterra integro-differential systems (see [9]). An interesting and fruitful technique that gained increasing significance and has given decisive impetus for modern development of stability theory of differential equations is Lyapunov's method. The strength of this technique is that it is possible to ascertain stability without solving the underlying differential equation. This method states that if one can find an appropriate Lyapunov function, then the system has some stability property (see [6, 18]). However, a system might be stable or asymptotically stable in theory, nevertheless, it is actually unstable in practice because the stable domain or the domain of the desired attractor is not large enough. Thus, from an engineering point of view we need a notion of stability that more suitable in several situations than Lyapunov stability. Such a concept is called practical stability (see [2, 5]). The novelty of this paper is to present a new notion of stability called generalized practical uniform h-stability as an extension of the generalized exponential asymptotic stability in [17] and practical uniform h-stability in [6, 7, 8]. In recent years, mathematical models of infectious diseases have been studied by a numbers of authors, see [11, 12, 13, 15, 19, 22] and many others. For instance, Ito in [12] considered a variety of models of infectious diseases and vaccination through the language of iISS and ISS. The method of Lyapunov functions is widely used to establish global stability results for biological models (see [12, 13, 15]).
The remainder of this work is organized as follows. In Section 2, we recall a new concept of stability and some tools used in the proofs. In Section 3, under growth conditions on the perturbed term, we investigate the global practical uniform h-stability of a nonlinear perturbed system using the Nonlinear Gronwall Inequality. In addition, we propose sufficient conditions with the extended of a Lyapunov function to indicate the global generalized practical uniform h-stability of the nonlinear system. The main result is provided in Section 4 in which the generalization of converse Lyapunov theorem is established by requiring the existence of a continuously non-differentiable Lyapunov function that satisfying certain properties. Moreover, a practical approach is obtained of time-varying dynamical perturbed system using the indirect Lyapunov's method, the comparison principle and the Generalized Gronwall-Bellman Inequality. However, Section 5 employs the notion of practical h-stability to evaluate robustness of infectious diseases with respect to integrable perturbation of the newborn/immigration rate and time-varying death rate. Finally, our conclusion is proposed in Section 6.

2. Preliminaries

The notation used throughout this note is standard. $\mathbb{R}_+$ indicates the set of non-negative real numbers, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space and $\|\cdot\|$ stands for its Euclidean vector norm. Also, we denote by:

- $I, J \subset \mathbb{R}$ are two intervals that are not empty and not reduce to a singleton.
- $\mathcal{BC}(I, J)$ is the space of continuous bounded functions on I to J endowed with the norm $\|f\|_\infty = \sup_{t\in I} |f(t)|$.
- $\mathcal{C}(I, J)$ is the space of continuous functions on I to J.

- $\mathcal{C}^1(I, J)$ is the space of continuous differentiable functions on I to J.

We consider the nonlinear non-autonomous differential system:

$$\dot{x} = f(t, x), \qquad x(t_0) = x_0, \qquad t \geq t_0 \geq 0, \tag{2.1}$$

where $t \in \mathbb{R}_+$ is the time, $x \in \mathbb{R}^n$ is the state and $f \in \mathcal{C}(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$ is locally Lipschitz in x, uniformly in t.
Let $x(t) = x(t, t_0, x_0)$ be denoted by the unique solution of (2.1) through $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$.
We assume that the Jacobian matrix $f_x = \left[\dfrac{\partial f}{\partial x}\right]$ exists and continuous on $\mathbb{R}_+ \times \mathbb{R}^n$.
We consider also the associated variational system:

$$\dot{z}(t) = f_x(t, x(t, t_0, x_0))z(t), \qquad z(t_0) = z_0, \qquad t \geq t_0 \geq 0. \tag{2.2}$$

Theorem 2.1. (See [1]) *If f is differentiable in $\mathbb{R}^n$ for $t \in \mathbb{R}_+$ and $x(t, t_0, x_0)$ is in $\mathbb{R}^n$ for $t \in \mathbb{R}_+$, then $x(t, t_0, x_0)$ is differentiable with respect to x_0 and*

$$R(t, t_0, x_0) = \frac{\partial}{\partial x_0} x(t, t_0, x_0)$$

is the fundamental matrix of solutions of the variational system (2.2), *such that $R(t_0, t_0, x_0) = I$ is the identity matrix which is independent of x_0.*

A precise definition of the global generalized practical uniform h-stability will be given as follows.

Definition 2.2. Let $h \in \mathcal{BC}(\mathbb{R}_+, \mathbb{R}_+^\star)$.

- System (2.1) is called generalized practically uniformly h-stable if there exist $\eta \geq 0$ and a function $K \in \mathcal{BC}(\mathbb{R}_+, \mathbb{R}_+^\star)$, such that for any initial state x_0, with $\|x_0\| \leq r$ and for all $t \geq 0$, we have

$$\|x(t)\| \leq \eta + K(t_0)\|x_0\|h(t)h(t_0)^{-1}, \qquad \forall\, t \geq t_0. \tag{2.3}$$

- System (2.1) is said to be globally generalized practically uniformly h-stable if the previous definition is satisfied for any initial state $x_0 \in \mathbb{R}^n$.

Here, $h(t)^{-1} = \dfrac{1}{h(t)} \cdot$

Remark 2.3. Definition 2.2 generalizes the notions of h-stability (see [20]). More precisely, when $\eta = 0$ we obtained the definition of global generalized uniform h-stability. Moreover, for $\eta > 0$ and for some special cases of h, the generalized practical uniform h-stability coincides with known practical types of stability:

- If $K(t) = c > 0$, we say that the system (2.1) is globally practically uniformly h-stable (see [6]).
- The practical uniform exponential stability is a particular case of generalized practical h-stability by taking $K(t) = c > 0$ and $h(t) = e^{-\beta t}$ with $\beta > 0$ (see [2]).
- If $h(t) = \dfrac{1}{(1+t)^\gamma}$ with $\gamma > 0$, we say that the system (2.1) is generalized practically uniformly polynomially stable (see [6]).

We are now in position to present the following lemmas which are important tools in the subsequent discussion.

Lemma 2.4. (See [16]) *Assume that $x(t,t_0,x_0)$ and $x(t,t_0,y_0)$ be any two solutions of system* (2.1) *through $(t_0,x_0)\in\mathbb{R}_+\times\mathbb{R}^n$ and $(t_0,y_0)\in\mathbb{R}_+\times\mathbb{R}^n$, respectively, existing for $t\geq t_0$, such that x_0 and y_0 belong to a convex subset D of $\mathbb{R}^n$. Then,*

$$x(t,t_0,x_0)-x(t,t_0,y_0)=\int_0^1 R\big(t,t_0,x_0+s(y_0-x_0)\big)ds(y_0-x_0), \qquad t\geq t_0\geq 0, \tag{2.4}$$

where $R(t,t_0,x_0)$ is the fundamental matrix solution of system (2.2).

Lemma 2.5. *The variational system* (2.2) *is globally generalized uniformly h-stable if and only if there exist functions $K\in\mathcal{BC}(\mathbb{R}_+,\mathbb{R}_+^\star)$ and $h\in\mathcal{BC}(\mathbb{R}_+,\mathbb{R}_+^\star)$, such that for all $x_0\in\mathbb{R}^n$ and all $t_0\in\mathbb{R}_+$, we have*

$$\|R(t,t_0,x_0)\|\leq K(t_0)h(t)h(t_0)^{-1}, \qquad \forall\ t\geq t_0.$$

Definition 2.6. (Lyapunov Functions)
We define the upper-right hand derivative Lyapunov functions of (2.1) as follows:

$$D^+V_{(2.1)}(t,x)=\limsup_{T\to 0^+}\frac{1}{T}\Big(V\big(t+T,x+Tf(t,x)\big)-V(t,x)\Big),$$

for $(t,x)\in\mathbb{R}_+\times\mathbb{R}^n$ and for the solution $x(t)=x(t,t_0,x_0)$ of (2.1),

$$D^+V(t,x(t))=\limsup_{T\to 0^+}\frac{1}{T}\Big(V\big(t+T,x(t+T)\big)-V(t,x)\Big).$$

Lemma 2.7. *Assume that the continuous function $V:\mathbb{R}_+\times\mathbb{R}^n\to\mathbb{R}$ is Lipschitz in x for a function $K\in\mathcal{BC}(\mathbb{R}_+,\mathbb{R}_+^\star)$, that is,*

$$\Big|V(t,x)-V(t,y)\Big|\leq K(t)|x-y|, \qquad \forall\ t\geq 0, \quad \forall\ x,y\in\mathbb{R}^n,$$

Then,

$$D^+V_{(2.1)}(t,x)=D^+V(t,x(t)).$$

Proof. We have,

$$\begin{aligned} V(t+T,x(t+T))-V(t,x) &= V(t+T,x+Tf(t,x)+\circ(T))-V(t,x)\\ &= \Big(V(t+T,x+Tf(t,x)+\circ(T))-\\ &\quad V(t+T,x+Tf(t,x))\Big)\\ &+ \Big(V(t+T,x+Tf(t,x))-V(t,x)\Big). \end{aligned}$$

Since $V(t,x)$ is Lipschitz in x for a continuous function $K(t)>0$ for all $t\in\mathbb{R}_+$, one easily sees that

$$V(t+T,x+Tf(t,x)+\circ(T))-V(t+T,x+Tf(t,x))=\circ(T).$$

Therefore, by Definition 2.6 we immediately deduce that

$$\begin{aligned} D^+V(t,x(t)) &= \limsup_{T\to 0^+}\frac{1}{T}\Big(V\big(t+T,x(t+T)\big)-V(t,x)\Big)\\ &= \limsup_{T\to 0^+}\frac{1}{T}\Big(V\big(t+T,x+Tf(t,x)\big)-V(t,x)\Big)\\ &= D^+V_{(2.1)}(t,x). \end{aligned}$$

□

Remark 2.8. If $V(t,x)\in\mathcal{C}^1(\mathbb{R}_+\times\mathbb{R}^n,\mathbb{R}^n)$, then

$$D^+V_{(2.1)}(t,x)=D^+V(t,x(t))=\dot{V}_{(2.1)}(t,x)=\frac{\partial V}{\partial t}(t,x)+\frac{\partial V}{\partial x}(t,x)f(t,x).$$

We use also the following lemmas to prove our results.

Lemma 2.9. (Nonlinear Gronwall Inequality)
Let $\mu:\mathbb{R}_+\to\mathbb{R}$ be a continuous function that satisfies the integral inequality

$$\mu(t)\le d+\int_{t_0}^{t} b(s)\mu^{\alpha}(s)ds,\quad d\ge 0,\qquad 0\le\alpha<1,\qquad t\ge t_0,$$

where b is a non-negative continuous function on $\mathbb{R}_+$. Then, we have

$$\mu(t)\le\left(d^{1-\alpha}+(1-\alpha)\int_{t_0}^{t} b(s)ds\right)^{\frac{1}{1-\alpha}},\qquad\forall\ t\ge t_0.$$

Proof. Let,

$$\varpi(t)=d+\int_{t_0}^{t} b(s)\varpi^{\alpha}(s)ds,\qquad 0\le\alpha<1,\qquad t\ge t_0.$$

Then,

$$\dot{\varpi}(t)=b(t)\varpi(t))^{\alpha},\qquad \varpi(t_0)=d,\qquad\forall\ t\ge t_0.$$

It is follows that,

$$\varpi^{1-\alpha}(t_0)=d^{1-\alpha}+(1-\alpha)\int_{t_0}^{t} b(s)ds.$$

Therefore,

$$\varpi(t)\le\left(d^{1-\alpha}+(1-\alpha)\int_{t_0}^{t} b(s)ds\right)^{\frac{1}{1-\alpha}}.$$

□

Lemma 2.10. (Generalized Gronwall-Bellman Inequality) (See [23])
Let $\rho,\ \varphi:\mathbb{R}_+\to\mathbb{R}$ be continuous functions and $\mu:\mathbb{R}_+\to\mathbb{R}_+$ is a function, such that

$$\dot{\mu}(t)\le\rho(t)\mu(t)+\varphi(t),\quad\forall\ t\ge t_0.$$

Then, for all $t_0\ge 0$, we have

$$\mu(t)\le\mu(t_0)\exp\Big(\int_{t_0}^{t}\rho(\tau)d\tau\Big)+\int_{t_0}^{t}\exp\Big(\int_{s}^{t}\rho(\tau)d\tau\Big)\varphi(s)ds,\quad\forall\ t\ge t_0.$$

3. Sufficient conditions for practical h-stability results

We start this section by studying the global practical uniform h-stability of a perturbed system under sufficient conditions on the perturbed term using the Nonlinear Gronwall Inequality. We need Alekseev formula to compare between the solutions of system (2.1) and the solutions of the following perturbed nonlinear system:

$$\dot{y} = f(t,y) + p(t,y), \qquad y(t_0) = y_0, \qquad t \geq t_0 \geq 0, \tag{3.1}$$

where $p \in \mathcal{C}(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$. Let $y(t) = y(t,t_0,y_0)$ represent the solution of the perturbed system passing through the point $(t_0,y_0) \in \mathbb{R}_+ \times \mathbb{R}^n$.
The next lemma is a generalization to nonlinear system of the variation of constants formula on account of Alekseev (see [3]).

Lemma 3.1. *If $y_0 \in \mathbb{R}^n$, then for all $t \geq t_0$, $x(t,t_0,y_0) \in \mathbb{R}^n$ and $y(t,t_0,y_0) \in \mathbb{R}^n$, we have*

$$y(t,t_0,y_0) - x(t,t_0,y_0) = \int_{t_0}^{t} R(t,s,y(s))p(s,y(s,t_0,y_0))ds.$$

Let consider the following theorem.

Theorem 3.2. *We consider the perturbed system* (3.1) *with the perturbation $p \in \mathcal{C}(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$ is locally Lipschitz in x. Let the origin be globally uniformly h-stable of system* (2.1) *and $z = 0$ of system* (2.2) *is globally uniformly h-stable. Assume that $p(t,x)$ satisfies the following condition:*

$$\|p(t,y)\| \leq \vartheta(t)\|y\|^{\alpha} + \nu(t), \qquad 0 \leq \alpha < 1, \qquad \forall\, y \in \mathbb{R}^n, \qquad \forall\, t \geq 0, \tag{3.2}$$

where ϑ, ν are non-negative continuous functions on $\mathbb{R}_+$ and there exist positive constants M_1 and M_2, such that

$$\int_0^t \vartheta(s)h(s)^{-1}ds \leq M_1, \qquad \int_0^t \nu(s)h(s)^{-1}ds \leq M_2, \qquad \forall\, t \geq 0. \tag{3.3}$$

Then, the system (3.1) *is globally practically uniformly h-stable.*

Proof. Let $y(t) = y(t,t_0,y_0)$ and $x(t) = x(t,t_0,y_0)$ be solutions of systems (3.1) and (2.1), respectively, then by Lemma 3.1, we have

$$y(t) = x(t) + \int_{t_0}^{t} R(t,s,y(s))p(t,y(s))ds.$$

Thus, from the global uniform h-stability of system (2.1), there exists $c > 0$, such that

$$\|y(t)\| = c\|y_0\|h(t)h(t_0)^{-1} + ch(t)\int_{t_0}^{t} \vartheta(s)h(s)^{-1}\|y(s)\|^{\alpha}ds + ch(t)\int_{t_0}^{t} \nu(s)h(s)^{-1}ds.$$

Hence,

$$h(t)^{-1}\|y(t)\| \leq \Big(c\|y_0\|h(t_0)^{-1} + cM_2\Big) + c\int_{t_0}^{t} \vartheta(s)h(s)^{\alpha-1}\big(h(s)^{-1}\|y(s)\|\big)^{\alpha}ds.$$

Let, $\rho(t) = h(t)^{-1}\|y(t)\|$, then

$$\rho(t) \leq \Big(c\rho(t_0) + cM_2\Big) + c\int_{t_0}^{t} \vartheta(s)h(s)^{\alpha-1}\rho^{\alpha}(s)ds.$$

Applying the Nonlinear Gronwall Inequality and the fact that

$$(\lambda_1 + \lambda_2)^r \leq 2^{r-1}(\lambda_1^r + \lambda_2^r),$$

for all $\lambda_1,\ \lambda_2 \geq 0$ and $r \geq 1$, we get

$$\rho(t) \leq 2^{\frac{\alpha}{1-\alpha}}\big(c\rho(t_0) + cM_2\big) + 2^{\frac{\alpha}{1-\alpha}}\Big(cM_1(1-\alpha)\|h\|_{\infty}^{\alpha}\Big)^{\frac{1}{1-\alpha}},$$

with $\|h\|_{\infty} = \sup\limits_{t\geq 0}\{h(t)\}$. This yields, for all $y_0 \in \mathbb{R}^n$ and all $t \geq t_0$ the solution of system (3.1) satisfies:

$$\|y(t)\| \leq \eta + c_1\|y_0\|h(t)h(t_0)^{-1},$$

with $c_1 = 2^{\frac{\alpha}{1-\alpha}}c$ and $\eta = 2^{\frac{\alpha}{1-\alpha}}cM_2\|h\|_{\infty} + 2^{\frac{\alpha}{1-\alpha}}\Big(cM_1(1-\alpha)\|h\|_{\infty}\Big)^{\frac{1}{1-\alpha}}$.
Consequently, system (3.1) is globally practically uniformly h-stable. This completes the proof. □

The stability properties of the solutions of nonlinear differential equations can be studied using the Lyapunov functions and the theory of differential and integral inequalities. This interesting and useful technique is called Lyapunov's second method. The following theorem proves the global generalized practical uniform h-stability of solutions of system (2.1) by requiring the existence of a continuously non-differentiable Lyapunov function that satisfying sufficient conditions.

Theorem 3.3. *Suppose that h is a positive bounded continuously differentiable function on $\mathbb{R}_+$. Furthermore, assume that there exist $a > 0$, $b \geq 1$, $\varrho \geq 0$, a function $K \in \mathcal{BC}(\mathbb{R}_+, \mathbb{R}_+^{\star})$ and a continuously non-differentiable Lyapunov function $V(t,x)$ defined on $\mathbb{R}_+ \times \mathbb{R}^n$, such that the following conditions are hold.*

1. *$V(t,x)$ is Lipschitzian in x for a function $K \in \mathcal{BC}(\mathbb{R}_+, \mathbb{R}_+^{\star})$,*
2. $a\|x\|^b \leq V(t,x) \leq K(t)\|x\|^b, \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n,$
3. $D^+V_{(2.1)}(t,x) \leq h'(t)h(t)^{-1}V(t,x) - \varrho h'(t)h(t)^{-1}, \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n.$

Then, the system (2.1) *is globally generalized practically uniformly $h^{\frac{1}{b}}$-stable.*

Proof. One has,

$$D^+V_{(2.1)}(t,x) \leq h'(t)h(t)^{-1}V(t,x) - \varrho h'(t)h(t)^{-1}.$$

We apply the comparison principle (see [14]), where

$$\dot{u}(t) = h'(t)h(t)^{-1}u(t) - \varrho h'(t)h(t)^{-1}, \qquad u(t_0) = u_0, \qquad t \geq t_0 \geq 0, \qquad (3.4)$$

with $V(t_0,x_0) \leq u_0 \leq K(t_0)\|x_0\|^b$. Then, by using the Generalized Gronwall-Bellman Inequality, the maximal solution of the scalar equation (3.4) is as follows:

$$\begin{aligned} u(t) &\leq u_0 h(t)h(t_0)^{-1} - \varrho \int_{t_0}^{t} \exp\left(\int_s^t h'(\tau)h(\tau)^{-1}d\tau\right) h'(s)h(s)^{-1}ds \\ &= u_0 h(t)h(t_0)^{-1} - \varrho h(t)\int_{t_0}^{t} h'(s)(h(s)^{-1})^2 ds \\ &\leq \varrho + K(t_0)\|x_0\|^b h(t)h(t_0)^{-1}. \end{aligned}$$

Hence, for all $x_0 \in \mathbb{R}^n$ and all $t \geq t_0$, we have

$$\|x(t)\| \leq \left(\frac{\varrho}{a}\right)^{\frac{1}{b}} + \left(\frac{K(t_0)}{a}\right)^{\frac{1}{b}} \|x_0\| h(t)^{\frac{1}{b}} h(t_0)^{-\frac{1}{b}}.$$

Consequently, the system (2.1) is globally generalized practically uniformly $h^{\frac{1}{b}}$-stable. □

4. Converse theorem

The purpose of this section is to represent a converse Lyapunov result for nonlinear time-varying systems that are globally generalized practically uniformly h-stable.

Theorem 4.1. *Assume that the system* (2.1) *is globally generalized practically uniformly h-stable and the solution $z = 0$ of system* (2.2) *is globally generalized uniformly h-stable. Suppose further that $h \in \mathcal{C}^1(\mathbb{R}_+, \mathbb{R}_+^\star)$ is a decreasing function. Then, there exist $K \in \mathcal{BC}(\mathbb{R}_+, \mathbb{R}_+^\star)$, a positive constant η and a continuously non-differentiable Lyapunov function $V(t,x)$, such that the following properties are hold.*

1. *$V(t,x)$ is Lipschitzian in x for a function $K \in \mathcal{BC}(\mathbb{R}_+, \mathbb{R}_+^\star)$,*
2. $\|x\| \leq V(t,x) \leq K(t)\|x\| + \eta, \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n,$
3. $D^+V_{(2.1)}(t,x) \leq h'(t)h(t)V(t,x) - \eta h'(t)h(t)^{-1}, \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n.$

Proof. Since system (2.1) is globally generalized practically uniformly h-stable, then there exist $\eta \geq 0$, functions $K \in \mathcal{BC}(\mathbb{R}_+, \mathbb{R}_+^\star)$ and $h \in \mathcal{BC}(\mathbb{R}_+, \mathbb{R}_+^\star)$, such that for all $t,\ \tau \in \mathbb{R}_+$ and all $x \in \mathbb{R}^n$, we have

$$\|x(t+\tau,t,x)\| \leq \eta + K(t)\|x\|h(t+\tau)h(t)^{-1},$$

We define the Lyapunov function $V : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ as

$$V(t,x) = \sup_{\tau \geq 0} \Big(h(t+\tau)^{-1}h(t)(\|x(t+\tau,t,x\| - \eta)\Big) + \eta,$$

where $x(t+\tau,t,x)$ is the solution of system (2.1) through $(t,x)\in\mathbb{R}_+\times\mathbb{R}^n$.
To prove the Lipschitzian of $V(t,x)$, let $(t,x),\ (t,y)\in\mathbb{R}_+\times\mathbb{R}^n$, then one has

$$\begin{aligned}\Big|V(t,x)-V(t,y)\Big| &= \Big|\sup_{\tau\geq 0}\Big(\|x(t+\tau,t,x)\|-\eta\Big)h(t+\tau)^{-1}h(t)\\ &- \sup_{\tau\geq 0}\Big(\|x(t+\tau,t,y)\|-\eta\Big)h(t+\tau)^{-1}h(t)\Big|\\ &\leq \sup_{\tau\geq 0}\Big\|x(t+\tau,t,x)-x(t+\tau,t,y)\Big\|h(t+\tau)^{-1}h(t).\end{aligned}$$

Since for each x and y in a convex subset $D\subset\mathbb{R}^n$, thus by Lemma 2.4, we obtain the following inequalities

$$\begin{aligned}\Big|V(t,x)-V(t,y)\Big| &\leq \|x-y\|h(t+\tau)^{-1}h(t)\sup_{\rho\in D}\|\phi(t+\tau,t,\rho)\|\\ &\leq K(t)h(t+\tau)^{-1}h(t)h(t)^{-1}h(t+\tau)\|x-y\|\\ &= K(t)\|x-y\|,\end{aligned}$$

where $K\in\mathcal{BC}(\mathbb{R}_+,\mathbb{R}_+^\star)$ and D is a convex subset of $\mathbb{R}^n$ containing x and y. Then, the first inequality holds.
We show next the continuity of $V(t,x)$. For that, let $T\geq 0$, then

$$\begin{aligned}\Big|V(t+T,\hat{x})-V(t,x)\Big| &\leq \Big|V(t+T,\hat{x})-V(t+T,x)\Big|\\ &+ \Big|V(t+T,x)-V\big(t+T,x(t+T,x(t+T,t,x))\big)\Big|\\ &+ \Big|V\big(t+T,x(t+T,x(t+T,t,x))\big)-V(t,x)\Big|.\end{aligned}$$

Since $V(t,x)$ is Lipschitzian in x and $x(t+T,t,x)$ is continuous in T, the first two terms on the right-hand side of the proceeding inequality are small when $\|x-\hat{x}\|$ and T are small.
Let us consider the third term. We have,

$$\begin{aligned}\Big|V\big(t+T,x(t+T,t,x)\big)-V(t,x)\Big| &= \Big|\sup_{\tau\geq 0}\Big(\|x(t+\tau+T,t+T,x(t+T,t,x))\|\\ &\quad -\eta\Big)h(t+\tau+T)^{-1}h(t+T)\\ &- \sup_{\tau\geq 0}\Big(\|x(t+\tau,t,x)\|-\eta\Big)h(t+\tau)^{-1}h(t)\Big|\\ &= \Big|\sup_{\tau\geq T}\Big(\|x(t+\tau,t,x)\|-\eta\Big)h(t+\tau)^{-1}\\ &\quad h(t+T)-\sup_{\tau\geq 0}\Big(\|x(t+\tau,t,x)\|-\eta\Big)\\ &\quad h(t+\tau)^{-1}h(t)\Big|.\end{aligned}$$

Put,

$$\alpha(T)=\sup_{\tau\geq T}\Big(\|x(t+\tau,t,x)\|-\eta\Big)h(t+\tau)^{-1}h(t+T).$$

We notice that, the function $\alpha(T)$ is non-decreasing and since $\Big(\|x(t+\tau,t,x)\| - \eta\Big)h(t+\tau)^{-1}h(t)$ is a bounded continuous function for all $\tau \geq 0$, then $\alpha(T) \to \alpha(0)$ as $T \to 0$. Hence,

$$\Big|V\big(t+T, x(t+T,t,x)\big) - V(t,x)\Big| = \Big|\alpha(T) - \alpha(0)\Big|$$

implies that the third term tends to zero as $T \to 0^+$. Consequently, the continuity of $V(t,x)$ is satisfied. On the other hand, we have

$$V(t,x) = \sup_{\tau\geq 0}\Big(h(t+\tau)^{-1}h(t)(\|x(t+\tau,t,x\| - \eta)\Big) + \eta \geq (\|x(t,t,x)\| - \eta) + \eta = \|x\|.$$

In addition,

$$V(t,x) \leq \Big(h(t+\tau)^{-1}h(t)\big(K(t)\|x\|h(t+\tau)h(t)^{-1} + \eta - \eta\big)\Big) + \eta = K(t)\|x\| + \eta.$$

Hence, the second property of the theorem is satisfied.
The last property can be proved using the uniqueness of solutions and the definition of generalized practical h-stability.

$$\begin{aligned}
D^+V(t,x(t)) &= \limsup_{T\to 0^+}\frac{1}{T}\Big[V(t+T,x(t+T,t,x)) - V(t,x)\Big] \\
&= \limsup_{T\to 0^+}\frac{1}{T}\Big[\sup_{\tau\geq 0}\Big(h(t+\tau+T)^{-1}h(t+T) \\
&\quad \big(\|x(t+\tau+T,t+T,x(t+T,t,x))\| - \eta\big)\Big) \\
&- \sup_{\tau\geq 0}\Big(h(t+\tau)^{-1}h(t)\big(\|x(t+\tau,t,x)\| - \eta\big)\Big] \\
&= \limsup_{T\to 0^+}\frac{1}{T}\Big[\sup_{\tau\geq T}\Big(h(t+\tau)^{-1}h(t+T)\big(\|x(t+\tau,t,x)\| - \eta\big)\Big) \\
&- \sup_{\tau\geq 0}\Big(h(t+\tau)^{-1}h(t)\big(\|x(t+\tau,t,x)\| - \eta\big)\Big] \\
&\leq \limsup_{T\to 0^+}\frac{1}{T}\Big[\Big(h(t+T)h(t)^{-1} - 1\Big)\sup_{\tau\geq 0}\Big(h(t+\tau)^{-1}h(t) \\
&\quad \big(\|x(t+\tau,t,x)\| - \eta\big)\Big) \\
&+ \eta\big(h(t+T)h(t)^{-1} - 1\big) - \eta\big(h(t+T)h(t)^{-1} - 1\big)\Big] \\
&\leq h'(t)h(t)^{-1}V(t,x) - \eta h'(t)h(t)^{-1}.
\end{aligned}$$

Since, for small $T > 0$

$$\begin{aligned}
V(t+T,x+Tf(t,x)) - V(t,x) &\leq \big|V\big(t+T,x+Tf(t,x)\big) \\
&- V\big(t+T,x(t+T,t,x)\big)\big| \\
&+ \big|V\big(t+T,x(t+T,t,x)\big) - V(t,x)\big| \\
&\leq K(t)\|x+Tf(t,x) - x(t+T,t,x)\| \\
&+ \big|V\big(t+T,x(t+T,t,x)\big) - V(t,x)\big|,
\end{aligned}$$

therefore

$$D^+V_{(2.1)}(t,x) \leq h'(t)h(t)^{-1}V(t,x) - \eta h'(t)h(t)^{-1}.$$

This ends the proof. □

Next, we use Lyapunov's indirect method and the Generalized Gronwall-Bellman Inequality to show the global generalized practical uniform h-stability of perturbed systems.

Theorem 4.2. *Consider the perturbed system:*

$$\dot{x} = f(t,x) + p(t,x), \qquad x(t_0) = x_0, \qquad t \geq t_0 \geq 0, \tag{4.1}$$

where $p \in \mathcal{C}(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$ is locally Lipschitz in x and satisfies the following condition:

$$\|p(t,x)\| \leq \vartheta(t)\|x\| + \nu(t), \quad \forall\ x \in \mathbb{R}^n, \qquad \forall\ t \geq 0, \tag{4.2}$$

where ϑ and ν are non-negative continuous and integrable functions on $\mathbb{R}_+$. Let the origin be globally generalized practically uniformly h-stable of system (2.1) *with $h \in \mathcal{C}^1(\mathbb{R}_+, \mathbb{R}_+^\star)$ is a decreasing function and the solution $z = 0$ of system* (2.2) *is globally generalized uniformly h-stable. Then, the perturbed system* (4.1) *is globally generalized practically uniformly h-stable.*

Proof. From Theorem 4.1, there exists a Lyapunov function $V(t,x)$ satisfies the properties in that theorem. Then, we have

$$\begin{aligned} D^+V_{(4.1)}(t,x) &\leq D^+V_{(2.1)}(t,x) + K(t)\|p(t,x)\| \\ &\leq h'(t)h(t)^{-1}V(t,x) - \eta h'(t)h(t)^{-1} + K(t)\vartheta(t)\|x\| + K(t)\nu(t) \\ &= \Big(h'(t)h(t)^{-1} + K(t)\vartheta(t)\Big)V(t,x) + K(t)\nu(t) - \eta h'(t)h(t)^{-1}. \end{aligned}$$

By applying the comparison principle, where

$$\dot{u}(t) = \Big(h'(t)h(t)^{-1} + K(t)\vartheta(t)\Big)u(t) + K(t)\nu(t) - \eta h'(t)h(t)^{-1}, \quad u(t_0) = u_0, \tag{4.3}$$

for $t \geq t_0 \geq 0$, such that $V(t_0,x_0) \leq u_0 \leq K(t_0)\|x_0\| + \eta$ and using the Generalized Gronwall-Bellman Inequality, the maximal solution of (4.3) is given by:

$$\begin{aligned} u(t) &\leq u_0 h(t)h(t_0)^{-1}\exp\left(\int_{t_0}^t K(s)\vartheta(s)ds\right) + h(t)\int_{t_0}^t h(s)^{-1} \\ &\quad \exp\left(\int_s^t K(\tau)\vartheta(\tau)d\tau\right)\Big(K(s)\nu(s) - \eta h'(s)h(s)^{-1}\Big)ds \\ &\leq K(t_0)e^{\|K\|_\infty M_\vartheta}\|x_0\|h(t)h(t_0)^{-1} + \eta e^{\|K\|_\infty M_\vartheta}h(t)h(t_0)^{-1} + e^{\|K\|_\infty M_\vartheta} \\ &\quad \big(\|K\|_\infty M_\nu + \eta\big) - \eta e^{\|K\|_\infty M_\vartheta}h(t)h(t_0)^{-1} \\ &= e^{\|K\|_\infty M_\vartheta}\big(\|K\|_\infty M_\nu + \eta\big) + K(t_0)e^{\|K\|_\infty M_\vartheta}\|x_0\|h(t)h(t_0)^{-1}, \end{aligned}$$

where $\|K\|_\infty = \sup\limits_{t \in \mathbb{R}_+}\{K(t)\}$, $M_\vartheta = \displaystyle\int_0^\infty \vartheta(t)dt$ and $M_\nu = \displaystyle\int_0^\infty \nu(t)dt$.

Therefore, for all $x_0 \in \mathbb{R}^n$ and all $t \geq t_0$, the solution $x(t)$ of system (4.1) satisfies

$$\|x(t)\| \leq \eta_1 + K_1(t_0)\|x_0\|h(t)h(t_0)^{-1},$$

with $K_1(t_0) = K(t_0)e^{\|K\|_\infty M_\vartheta}$ and $\eta_1 = e^{\|K\|_\infty M_\vartheta}\big(\|K\|_\infty M_\nu + \eta\big)$. Consequently, the system (4.1) is globally generalized practically uniformly h-stable. □

Proposition 4.3. *Consider the perturbed system* (4.1)*. If the nonlinear system* (2.1) *is globally practically uniformly h-stable with $h \in \mathcal{C}^1(\mathbb{R}_+, \mathbb{R}_+^\star)$ is a decreasing function, the solution $z = 0$ of system* (2.2) *is globally uniformly h-stable and $p(t,x)$ satisfies the condition* (4.2) *where ϑ and ν are non-negative continuous and integrable functions on $\mathbb{R}_+$. Then, the system* (4.1) *is globally practically uniformly h-stable.*

A particular case of Theorem 4.2 is given in the following corollary.

Corollary 4.4. *Consider the perturbed system* (4.1)*. Assume that the system* (2.1) *is globally generalized practically uniformly h-stable with $h \in \mathcal{C}^1(\mathbb{R}_+, \mathbb{R}_+^\star)$ is a decreasing function, the solution $z = 0$ of system* (2.2) *is globally generalized h-stable. Suppose that the perturbed term $p(t,x)$ satisfies the condition:*

$$\|p(t,x)\| \leq \gamma(t), \qquad \forall\ t \geq 0, \tag{4.4}$$

where γ is a non-negative continuous and integrable function on $\mathbb{R}_+$. Hence, the system (4.1) *is globally generalized practically uniformly h-stable.*

5. Applications

In this section, the systematic method developed and applied to various diseases models to illustrate several aspects of these methods.

5.1. SIR model

We consider the solution $x(t) = (S(t), I(t), R(t))^T \in \mathbb{R}_+^3$ of the ordinary differential equation:

$$\begin{cases} \dot{S} = B(t) - \mu(t)S - \beta IS, \qquad t \geq t_0 \geq 0, \\ \\ \dot{I} = \beta IS - \nu I - \mu(t)I, \\ \\ \dot{R} = \nu I - \mu(t)R, \end{cases} \tag{5.1}$$

defined for any $x(t_0) = (S(t_0), I(t_0), R(t_0))^T \in \mathbb{R}_+^3$, any continuous function $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ and any continuous and integrable function $B : \mathbb{R}_+ \to \mathbb{R}_+$. Here, the variable $S(t)$ describes the number of susceptible population and $I(t)$ is the number of infected individuals, while $R(t)$ is of individuals recovered with immunity. $B(t)$ is the newborn/ immigration rate. $\mu(t)$ is the death rate. The positive number β and ν are parameters describing the contact rate and the recovery rate, respectively.
Select the appropriate state variable as $x_1 = S$, $x_2 = I$ and $x_3 = R$. Thus, the equations describing a SIR Model can be written as

$$\begin{cases} \dot{x}_1 = B(t) - \mu(t)x_1 - \beta x_2 x_1, \\ \dot{x}_2 = \beta x_2 x_1 - \nu x_2 - \mu(t)x_2, \\ \dot{x}_3 = \nu x_2 - \mu(t)x_3. \end{cases} \tag{5.2}$$

The state model (5.1) is equivalent to system (4.1), where $x = (x_1, x_2, x_3)^T \in \mathbb{R}^3_+$ and

$$f(t,x) = \begin{pmatrix} -\mu(t)x_1 - \beta x_2 x_1 \\ \beta x_2 x_1 - \nu x_2 - \mu(t)x_2 \\ \nu x_2 - \mu(t)x_3 \end{pmatrix}.$$

and $p(t,x) = (B(t), 0, 0)^T$. We consider the following Lyaunov function

$$V(t,x) = x_1(t) + x_2(t) + x_3(t).$$

The derivative of V in t along the solution of the system $\dot{x} = f(t,x)$ leads to

$$\begin{aligned} D^+V(t,x) &= \dot{x}_1 + \dot{x}_2 + \dot{x}_3 \\ &= -\mu(t)V(t,x). \end{aligned}$$

Then, the nominal system $\dot{x} = f(t,x)$ is uniformly h-stable with $K(t) = 1$ and

$$h(t) = \exp\left(-\int_0^t \mu(s)ds\right).$$

On the other hand, the perturbed term $p(t,x)$ satisfies the condition (4.4) with $\gamma(t) = B(t)$, which is non-negative, continuous and integrable function on $\mathbb{R}_+$. Thus, all assumptions of Corollary 4.4 are satisfied. We conclude that the SIR Model (5.1) is practically uniformly h-stable.
From Figure 1, we can see that the SIR Model (5.1) is practically uniformly h-stable with $h(t) = \dfrac{1}{1+t}$. In this case, for integrable newborn/immigration rate $B(t)$ the convergence of $I(t)$, $S(t)$ and $R(t)$ to a neighborhood of the origin are guaranteed where the initial values $S(0) = 600$, $I(0) = 100$ and $R(0) = 60$. The parameters of SIR Model (5.1) are $\beta = 0.0002$ and $\nu = 0.020$.

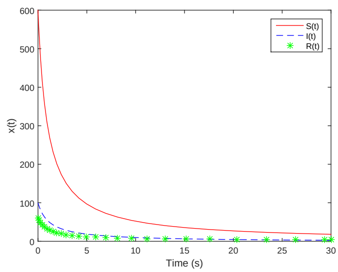

FIGURE 1. *Populations of the SIR model with* $B(t) = \frac{1}{1+t^2}$ *and* $\mu(t) = \frac{1}{1+t}$.

Remark 5.1. If we suppose that $\mu(t) = \tilde{\mu}$ is constant and $B(t)$ is a measurable and locally essentially bounded, then by using Theorem 3.3 the SIR Model (5.1) is practically uniformly h-stable with $h(t) = \exp(-\tilde{\mu}t)$.

5.2. SEIS model

Let $x(t) = (S(t), E(t), I(t))^T \in \mathbb{R}^3_+$ for

$$\begin{cases} \dot{S} = B(t) - \mu(t)S - \beta IS + \nu I, \qquad t \geq t_0 \geq 0, \\ \dot{E} = \beta IS - \varepsilon E - \mu(t)E, \\ \dot{I} = \varepsilon E - \nu I - \mu(t)I, \end{cases} \tag{5.3}$$

with $x(t_0) = (S(t_0), E(t_0), I(t_0))^T \in \mathbb{R}^3_+$, any continuous function $\mu : \mathbb{R}_+ \to \mathbb{R}_+$ and any continuous and integrable function $B : \mathbb{R}_+ \to \mathbb{R}_+$. The variable E individuals move to the class I at the rate ε. Model (5.3) is refereed to as the SEIS model [15] when $\mu(t)$ is constant. The SEIS model is known to be useful for describing diseases which have non-negligible incubation periods and also consider infections which do not give long lasting immunity and recovered individuals become susceptible again. Select the appropriate state variable as $x_1 = S$, $x_2 = E$ and $x_3 = I$. Thus, the equations describing a SEIS model can be written as:

$$\begin{cases} \dot{x}_1 = B(t) - \mu(t)x_1 - \beta x_3 x_1 + \nu x_3, \\ \dot{x}_2 = \beta x_3 x_1 - \varepsilon x_2 - \mu(t)x_2, \\ \dot{x}_3 = \varepsilon x_2 - \nu x_3 - \mu(t)x_3. \end{cases} \tag{5.4}$$

The state model (5.3) is equivalent to system (4.1), where $x = (x_1, x_2, x_3)^T \in \mathbb{R}^3_+$,

$$f(t,x) = \begin{pmatrix} -\mu(t)x_1 - \beta x_3 x_1 + \nu x_3 \\ \beta x_3 x_1 - \varepsilon x_2 - \mu(t)x_2 \\ \varepsilon x_2 - \nu x_3 - \mu(t)x_3 \end{pmatrix}$$

and $p(t,x) = (B(t), 0, 0)^T$. Let

$$V(t,x) = x_1(t) + x_2(t) + x_3(t).$$

The derivative of V in t along the solution of the nominal system $\dot{x} = f(t,x)$ leads to

$$\begin{aligned} D^+V(t,x) &= \dot{x}_1 + \dot{x}_2 + \dot{x}_3 \\ &= -\mu(t)V(t,x). \end{aligned}$$

Then, the nominal system $\dot{x} = f(t,x)$ is uniformly h-stable with $K(t) = 1$ and $h(t) = \exp\left(-\int_0^t \mu(s)ds\right)$. On the other hand, the perturbed term $p(t,x)$ satisfies the condition (4.4) with $\gamma(t) = B(t)$ which is non-negative, continuous and integrable function on $\mathbb{R}_+$. We deduce that all hypothesis of Corollary 4.4 are satisfied. Therefore, the SEIS model (5.3) is practically uniformly h-stable.
From Figure 2, we can see that the SEIS model (5.3) is practically uniformly h-stable with $h(t) = \exp(-t^2)$. The parameters of SEIS model are $\varepsilon = 0.15$, $\beta = 0.002$ and $\nu = 0.032$ with the initial state is $\Big(S(0), E(0), I(0)\Big) = (100, 60, 200)$.

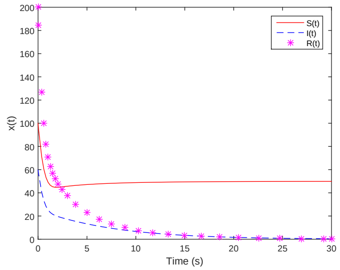

FIGURE 2. *Populations of the SEIS model with* $B(t) = \frac{1}{1+t^2}$ *and* $\mu(t) = t$.

5.3. SEIR model

Let $x(t) = (S(t), E(t), I(t), R(t)^T \in \mathbb{R}_+^4$ for

$$\begin{cases} \dot{S} = B(t) - \mu(t)S - \beta IS, \qquad t \geq t_0 \geq 0, \\ \dot{E} = \beta IS - \varepsilon E - \mu(t)E, \\ \dot{I} = \varepsilon E - \nu I - \mu(t)I, \\ \dot{R} = \nu I - \mu(t)R, \end{cases} \tag{5.5}$$

which called the SEIR model. The analysis of SEIR model is almost the same as the SIR Model.

Select the appropriate state variable as $x_1 = S$, $x_2 = E$, $x_3 = I$ and $x_4 = R$. Thus, the equations describing a SEIR Model can be written as:

$$\begin{cases} \dot{x}_1 = B(t) - \mu(t)x_1 - \beta x_3 x_1, \\ \dot{x}_2 = \beta x_3 x_1 - \varepsilon x_2 - \mu(t)x_2, \\ \dot{x}_3 = \varepsilon x_2 - \nu x_3 - \mu(t)x_3, \\ \dot{x}_4 = \nu x_3 - \mu(t)x_4. \end{cases} \tag{5.6}$$

The state model (5.5) is equivalent to system (4.1), where $x = (x_1, x_2, x_3, x_4)^T \in \mathbb{R}_+^4$ and

$$f(t,x) = \begin{pmatrix} -\mu(t)x_1 - \beta x_3 x_1 \\ \beta x_3 x_1 - \varepsilon x_2 - \mu(t)x_2 \\ \varepsilon x_2 - \nu x_3 - \mu(t)x_3 \\ \nu x_3 - \mu(t)x_4 \end{pmatrix}.$$

and $p(t,x) = (B(t), 0, 0, 0)^T$. Let

$$V(t,x) = x_1(t) + x_2(t) + x_3(t) + x_4(t).$$

The derivative of V in t along the solution of the system $\dot{x} = f(t,x)$ leads to

$$\begin{aligned} D^+V(t,x) &= \dot{x}_1 + \dot{x}_2 + \dot{x}_3 + \dot{x}_4 \\ &= -\mu(t)V(t,x). \end{aligned}$$

Then, the nominal system $\dot{x} = f(t,x)$ is uniformly h-stable with $K(t) = 1$ and $h(t) = \exp\left(-\int_0^t \mu(s)ds\right)$. On the other hand, the perturbed term $p(t,x)$ satisfies the condition (4.4) with $\gamma(t) = B(t)$ which is non-negative, continuous and integrable function on $\mathbb{R}_+$. Thus, all assumptions of Corollary 4.4 are satisfied. We conclude that the SEIR Model (5.5) is practically uniformly h-stable.

5.4. Vaccination models

One way of eradicating infections diseases is to vaccinate newborns and entering individuals. Let $P \in (0,1)$ the vaccination fraction. Considering a vaccine giving lifelong immunity [11], the SIR model can be modified as

$$\begin{cases} \dot{S} = B(t)(1-P) - \mu(t)S - \beta IS, \qquad t \geq t_0 \geq 0, \\ \\ \dot{I} = \beta IS - \nu I - \mu(t)I, \\ \\ \dot{R} = \nu I - \mu(t)R, \\ \\ \dot{A} = B(t)P - \mu(t)A, \end{cases} \tag{5.7}$$

where A is the number of vaccinated individuals.
Select the appropriate state variable as $x_1 = S$, and $x_2 = I$, $x_3 = R$ and $x_4 = A$. Thus, the equations describing a SIR Model can be written as:

$$\begin{cases} \dot{x}_1 = B(t)(1-P) - \mu(t)x_1 - \beta x_2 x_1, \\ \dot{x}_2 = \beta x_2 x_1 - \nu x_2 - \mu(t)x_2, \\ \dot{x}_3 = \nu x_2 - \mu(t)x_3. \\ \dot{x}_4 = B(t)P - \mu(t)x_4. \end{cases} \tag{5.8}$$

The state model (5.8) is equivalent to system (4.1), where $x = (x_1, x_2, x_3, x_4)^T \in \mathbb{R}^4_+$ and

$$f(t,x) = \begin{pmatrix} B(t)(1-P) - \mu(t)x_1 - \beta x_2 x_1 \\ \beta x_2 x_1 - \nu x_2 - \mu(t)x_2 \\ \nu x_2 - \mu(t)x_3 \\ B(t)P - \mu(t)x_4 \end{pmatrix}.$$

and $p(t,x) = (B(t), 0, 0, 0)^T$. Let

$$V(t,x) = x_1(t) + x_2(t) + x_3(t) + x_4(t).$$

The derivative of V in t along the solution of the system $\dot{x} = f(t,x)$ leads to

$$\begin{aligned} D^+V(t,x) &= \dot{x}_1 + \dot{x}_2 + \dot{x}_3 + \dot{x}_4(t) \\ &= -\mu(t)V(t,x). \end{aligned}$$

Then, the nominal system $\dot{x} = f(t,x)$ is uniformly h-stable with $K(t) = 1$ and $h(t) = \exp\left(-\int_0^t \mu(s)ds\right)$. Moreover, the perturbed term $p(t,x)$ satisfies the condition (4.4) with $\gamma(t) = B(t)$ which is non-negative, continuous and integrable function on $\mathbb{R}_+$. Thus, all assumptions of Corollary 4.4 are satisfied. We conclude that the SIR Model (5.8) is practically uniformly h-stable.

Figure 3 is the simulation result of the SIR Model (5.8) with the initial values $S(0) = 600$, $I(0) = 150$, $R(0) = 70$ and $A(0) = 50$. The parameters of SIR Model are $\beta = 0.0002$, $\nu = 0.035$ and $P = 0.5$.
From the simulation, we see that the states trajectories converge eventually to a small neighborhood of the origin.

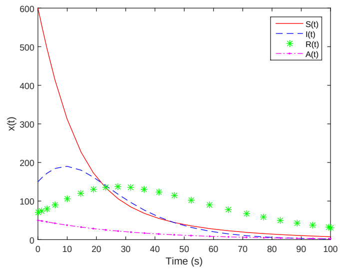

FIGURE 3. *Populations of Vaccination model with* $B(t) = \exp(-t)$ *and* $\mu = 0.0015$.

Another way to model the newborn vaccination within the SIR model is

$$\begin{cases} \dot{S} = B(t)(1-P) - \mu(t)S - \beta IS, \qquad t \geq t_0 \geq 0, \\ \dot{I} = \beta IS - \nu I - \mu(t)I, \\ \dot{R} = \nu I - \mu(t)R + B(t)P. \end{cases} \tag{5.9}$$

In the same way as of the model (5.9), we have the newborn vaccination model is practically uniformly h-stable with $h(t) = \exp\left(-\int_0^t \mu(s)ds\right)$.
If non-newborns/non-immigrants are vaccinated [19, 22] with a continuous function $\mu : \mathbb{R}_+ \to \mathbb{R}_+$, a way to modifier the SIR model is

$$\begin{cases} \dot{S} = B(t) - \rho S - \mu(t)S - \beta IS, \qquad t \geq t_0 \geq 0, \\ \dot{I} = \beta IS - \nu I + \mu(t)I, \\ \dot{R} = \nu I - \mu(t)R, \\ \dot{A} = \rho S - \mu(t)A, \end{cases} \tag{5.10}$$

where $\rho \in \mathbb{R}_+$ is the vaccination rate. The analysis is the same as of the model (5.9), the model (5.10) also is practically uniformly h-stable with

$$h(t) = \exp\left(-\int_0^t \mu(s)ds\right).$$

6. Conclusion

Non-differentiable Lyapunov-like function is proposed for obtaining the global generalized practical uniform h-stability of the nonlinear system. Sufficient conditions are given to study the practical approach of nonlinear time-varying perturbed systems using Lyapunov's indirect method, the comparison principle and some generalizations of Gronwall's inequality. This results can be viewed as an extension of [4] and [17]. The models considered in this paper are practically uniformly h-stable. This conclusion is valid for non-autonomous systems.

References

[1] Bebernes, J., *Differential Equations: Stability, Oscillations, Time Lags (Halanay, A.)*, Society for Industrial and Applied Mathematics, 1966.

[2] Benabdallah, A., Ellouze, I., Hammami, M.A., *Practical stability of nonlinear time-varying cascade system*, J. Dyn. Control Syst., **15**(2009), no. 1, 45-62.

[3] Brauer, F., *Perturbations of nonlinear systems of differential equations II*, J. Math. Anal. Appl., **17**(1967), no. 3, 418-434.

[4] Choi, S.K., Koo, N.J., Ryu, H.S., *h-stability of differentiable systems via t_∞-similarity*, Bull. Korean Math. Soc., **34**(1997), no. 3, 371-383.

[5] Damak, H., Hammami, M.A., Kalitine, B., *On the global uniform asymptotic stability of time-varying systems*, Differ. Equ. Dyn. Syst., **22**(2014), no. 2, 113-124.

[6] Damak, H., Hammami, M.A., Kicha, A., *A converse theorem on practical h-stability of nonlinear systems*, Mediterr. J. Math., **17**(2020), no. 3, 1-18.

[7] Damak, H., Hammami, M.A., Kicha, A., *On the practical h-stabilization of nonlinear time-varying systems: Application to separately excited DC motor*, COMPEL-Int. J. Comput. Math. Electr. Electron. Eng., **40**(2021), no. 4, 888-904.

[8] Damak, H., Taieb, N.H., Hammami, M.A., *A practical separation principle for nonlinear non-autonomous systems*, Internat. J. Control, (2021), https://doi.org/10.1080/00207179.2021.1986640.

[9] Goo, Y.H., Ji, M.H., Ry, D.H., *h-stability in certain integro-differential equations*, J. Chungcheong Math. Soc., **22**(2009), no. 1, 81-88.

[10] Goo, Y.H., Ry, D.H., *h-stability for perturbed integro-differential systems*, J. Chungcheong Math. Soc., **21**(2008), no. 4, 511-517.

[11] Hethcote, H.W., *The mathematics of infectious diseases*, SIAM Rev., **42**(2000), no. 4, 599-653.

[12] Ito, H., *Interpreting models of infectious diseases in terms of integral input-to-state stability*, Math. Control Signals Systems, **32**(2020), no. 4, 611-631.

[13] Ito, H., *Input-to-state-stability and Lyapunov functions with explicit domains for SIR model of infectious diseases*, Discrete Contin. Dyn. Syst. Ser. B, **26**(2021), no. 9, 5171.

[14] Khalil, H.K., *Nonlinear Systems*, Prentice-Hall, 2002.

[15] Korobeinikov, A., *Lyapunov functions and global properties for SEIR and SEIS epedemic models*, Math. Med. Biol., **21**(2004), no. 2, 75-83.

[16] Lakshmikantham, V., Deo, S.G., *Method of Variation of Parameters for Dynamic Systems*, Gordon and Breach Science Publishers, **1**, 1998.

[17] Lakshmikantham, V., Leela, S., *Differential and Integral Inequalities*, Academic Press New York and London, **I**, 1969.

[18] Li, X., Guo, Y., *A converse Lyapunov theorem and robustness with respect to unbounded perturbations for exponential dissipativity*, Adv. Differential Equations, **2010**(2010), 1-15.

[19] Ögren, P., Martin, C.F., *Vaccination strategies for epedemics in highly mobile populations*, Appl. Math. Comput., **127**(2002), no. 2-3, 261-276.

[20] Pinto, M., *Perturbations of asymptotically stable differential systems*, Analysis, **4**(1984), no. 1-2, 161-175.

[21] Pinto, M., *Stability of nonlinear differential systems*, Appl. Anal., **43**(1992), no. 1-2, 1-20.

[22] Zaman, G., Kang, Y.H., Jung, I.H., *Stability analysis and optimal vaccination of an SIR epidemic model*, BioSystems, **93**(2008), no. 3, 240-249.

[23] Zhou, B., *Stability analysis of nonlinear time-varying systems by Lyapunov functions with indefinite derivatives*, IET Control Theory Appl., **11**(2017), no. 9, 1434-1442.

Abir Kicha
University of Jijel, Faculty of Exact Sciences and Computer Sciences,
B.P. 98, Ouled Aissa, 18000 Jijel, Algeria
e-mail: abir.kicha@univ-jijel.dz

Hanen Damak
University of Sfax, Faculty of Sciences of Sfax,
Road Soukra BP1171, 3000 Sfax, Tunisia
e-mail: hanen.damak@yahoo.fr

Mohamed Ali Hammami
University of Sfax, Faculty of Sciences of Sfax,
Road Soukra BP1171, 3000 Sfax, Tunisia
e-mail: mohamedali.hammami@fss.rnu.tn