Stud. Univ. Babeş-Bolyai Math. 69(2024), No. 3, 553–565
DOI: 10.24193/subbmath.2024.3.06

Monotone iterative technique for a sequential δ-Caputo fractional differential equations with nonlinear boundary conditions

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Abstract. In this article, we discuss the existence of extremal solutions for a class of nonlinear sequential δ–Caputo fractional differential equations involving nonlinear boundary conditions. Our results are founded on advanced functional analysis methods. To be more specific, we use the monotone iterative approach in conjunction with the upper and lower solution method to create adequate requirements for the existence of extremal solutions. As an application, we give an example to illustrate our results.

Mathematics Subject Classification (2010): 34A08, 26A33.

Keywords: Sequential δ–Caputo derivative, nonlinear boundary conditions, monotone iterative technique, upper and lower solutions.

1. Introduction

Nowadays, fractional differential equations appear in diverse fields such as physics, fluid mechanics, viscoelasticity, biology, control theory, chemistry, and so on, (see, for example, [19, 28, 31, 36]). For some fundamental results in the theory of fractional calculus and fractional differential equations, we suggest the monographs of several scientists [2, 3, 4, 20, 26, 29, 41, 42].

There are various techniques to defining fractional integrals and derivatives in the literature, such as Riemann–Liouville, Caputo, Caputo–Hadamard, Hilfer, δ–Caputo and δ–Hilfer. For more details, we refer the readers to [1, 5, 6, 7, 10, 11, 12, 14, 15, 17, 23, 32, 33, 34, 35, 37].

Received 30 December 2022; Accepted 08 May 2023.

On the other hand, much attention has been focused on the study of the existence and uniqueness of solutions for initial and boundary value problems involving sequential fractional differential equations, we refer the reader to [8, 9, 25] and the references cited therein. Additionally, it is well known that the monotone iterative technique [22] combined with the method of upper and lower solutions is used as a fundamental tool to prove the existence and approximation of solutions to many applied problems of nonlinear differential equations and integral equations. Moreover, this technique has more advantages, such as it not only proves the existence of solutions but also can provide computable monotone sequences that converge to the extremal solutions in a sector generated by upper and lower solutions. Recent results by means of the monotone iterative method are obtained in [13, 16, 18, 24, 27, 38, 39, 40] and the references therein. However, to the best of the author's knowledge, no results yet exist for the sequential fractional differential equations involving the δ–Caputo fractional derivative by using the monotone iterative technique.

Motivated by this fact together with recent works [5, 10, 18, 21, 24, 39], we investigate the existence of extremal solutions for the following boundary value problem of δ–Caputo sequential fractional differential equations involving nonlinear boundary conditions:

$$\begin{cases} \left({}^{c}\mathcal{D}^{\zeta+1;\delta}_{\kappa_1+} + \lambda\, {}^{c}\mathcal{D}^{\zeta;\delta}_{\kappa_1+}\right)\xi(\vartheta) = \Psi(\vartheta, \xi(\vartheta)), \ \ \vartheta \in \Theta := [\kappa_1, \kappa_2], \\ \Phi(\xi(\kappa_1), \xi(\kappa_2)) = 0, \quad \xi'(\kappa_1) = 0, \end{cases} \tag{1.1}$$

where ${}^{c}\mathcal{D}^{\zeta;\delta}_{\kappa_1+}$ is the δ–Caputo fractional derivative of order $\zeta \in (0, 1]$, $\Psi\colon [\kappa_1, \kappa_2] \times \mathbb{R} \longrightarrow \mathbb{R}$, $\Phi\colon \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ are both continuous functions and λ is a positive real number.

The following is how the paper is structured. We provide some essential definitions and lemmas in section 2. The major findings are discussed in section 3. Lastly, an illustration is provided to demonstrate the applicability of the generated results.

2. Preliminaries

In this part, we provide certain fractional calculus notations and concepts, as well as definitions and lemmas that will be used later in our proofs.

Definition 2.1 ([10, 20])**.** For $\zeta > 0$, the left-sided δ–Riemann-Liouville fractional integral of order ζ for an integrable function $\xi\colon \Theta \longrightarrow \mathbb{R}$ with respect to another function $\delta\colon \Theta \longrightarrow \mathbb{R}$ that is an increasing differentiable function such that $\delta'(\vartheta) \neq 0$, for all $\vartheta \in \Theta$ is defined as follows

$$\mathcal{I}^{\zeta;\delta}_{\kappa_1+}\xi(\vartheta) = \frac{1}{\Gamma(\zeta)} \int_{\kappa_1}^{\vartheta} \delta'(\varrho)(\delta(\vartheta) - \delta(\varrho))^{\zeta-1}\xi(\varrho)d\varrho. \tag{2.1}$$

Definition 2.2 ([10])**.** Let $\beta \in \mathbb{N}$ and let $\delta, \xi \in C^{\beta}(\Theta, \mathbb{R})$ be two functions such that δ is increasing and $\delta'(\vartheta) \neq 0$, for all $\vartheta \in \Theta$. The left-sided δ–Riemann–Liouville fractional derivative of a function ξ of order ζ is defined by

$$\mathcal{D}^{\zeta;\delta}_{\kappa_1+}\xi(\vartheta) = \left(\frac{1}{\delta'(\vartheta)}\frac{d}{d\vartheta}\right)^{\beta} \mathcal{I}^{\beta-\zeta;\delta}_{\kappa_1+}\xi(\vartheta)$$
$$= \frac{1}{\Gamma(\beta-\zeta)}\left(\frac{1}{\delta'(\vartheta)}\frac{d}{d\vartheta}\right)^{\beta}\int_{\kappa_1}^{\vartheta}\delta'(\varrho)(\delta(\vartheta)-\delta(\varrho))^{\beta-\zeta-1}\xi(\varrho)d\varrho,$$

where $\beta = [\zeta]+1$.

Definition 2.3 ([10]). Let $\beta \in \mathbb{N}$ and let $\delta, \xi \in C^{\beta}(\Theta,\mathbb{R})$ be two functions such that δ is increasing and $\delta'(\vartheta) \neq 0$, for all $\vartheta \in \Theta$. The left-sided δ-Caputo fractional derivative of ξ of order ζ is defined by

$$^{c}\mathcal{D}^{\zeta;\delta}_{\kappa_1+}\xi(\vartheta) = \mathcal{I}^{\beta-\zeta;\delta}_{\kappa_1+}\left(\frac{1}{\delta'(\vartheta)}\frac{d}{d\vartheta}\right)^{\beta}\xi(\vartheta),$$

where $\beta = [\zeta]+1$ for $\zeta \notin \mathbb{N}$, $\beta = \zeta$ for $\zeta \in \mathbb{N}$.

In the sequel, we will employ the following:

$$\xi^{[\beta]}_{\delta}(\vartheta) = \left(\frac{1}{\delta'(\vartheta)}\frac{d}{d\vartheta}\right)^{\beta}\xi(\vartheta). \tag{2.2}$$

From the previous definition, it is clear that

$$^{c}\mathcal{D}^{\zeta;\delta}_{\kappa_1+}\xi(\vartheta) = \begin{cases} \int_{\kappa_1}^{\vartheta}\frac{\delta'(\varrho)(\delta(\vartheta)-\delta(\varrho))^{\beta-\zeta-1}}{\Gamma(\beta-\zeta)}\xi^{[\beta]}_{\delta}(\varrho)d\varrho & , \text{ if } \zeta \notin \mathbb{N}, \\ \xi^{[\beta]}_{\delta}(\vartheta) & , \text{ if } \zeta \in \mathbb{N}. \end{cases} \tag{2.3}$$

This generalization (2.3) yields the Caputo fractional derivative operator when $\delta(\vartheta) = \vartheta$. Moreover, for $\delta(\vartheta) = \ln\vartheta$, it gives the Caputo–Hadamard fractional derivative.

We note that if $\xi \in C^{\beta}(\Theta,\mathbb{R})$ the δ–Caputo fractional derivative of order ζ of ξ is determined as

$$^{c}\mathcal{D}^{\zeta;\delta}_{\kappa_1+}\xi(\vartheta) = \mathcal{D}^{\zeta;\delta}_{\kappa_1+}\left[\xi(\vartheta) - \sum_{\jmath=0}^{\beta-1}\frac{\xi^{[\jmath]}_{\delta}(\kappa_1)}{\jmath!}(\delta(\vartheta)-\delta(\kappa_1))^{\jmath}\right].$$

(For more details, see [10, Theorem 3]).

Lemma 2.4 ([12]). *Let $\zeta, \varkappa > 0$, and $\xi \in L^1(\Theta,\mathbb{R})$. Then,*

$$\mathcal{I}^{\zeta;\delta}_{\kappa_1+}\mathcal{I}^{\varkappa;\delta}_{\kappa_1+}\xi(\vartheta) = \mathcal{I}^{\zeta+\varkappa;\delta}_{\kappa_1+}\xi(\vartheta), \ a.e.\ \vartheta \in \Theta.$$

If $\xi \in C(\Theta,\mathbb{R})$, then $\mathcal{I}^{\zeta;\delta}_{\kappa_1+}\mathcal{I}^{\varkappa;\delta}_{\kappa_1+}\xi(\vartheta) = \mathcal{I}^{\zeta+\varkappa;\delta}_{\kappa_1+}\xi(\vartheta)$, $\vartheta \in \Theta$.

Next, we recall the property describing the composition rules for fractional δ-integrals and δ-derivatives.

Lemma 2.5 ([12]). *Let $\zeta > 0$, The following holds:*

- *If $\xi \in C(\Theta,\mathbb{R})$, then*

$$^{c}\mathcal{D}^{\zeta;\delta}_{\kappa_1+}\mathcal{I}^{\zeta;\delta}_{\kappa_1+}\xi(\vartheta) = \xi(\vartheta), \ \vartheta \in \Theta.$$

- *If* $\xi \in C^{\beta}(\Theta, \mathbb{R})$, $\beta - 1 < \zeta < \beta$, *then*

$$\mathcal{I}^{\zeta;\delta}_{\kappa_1+}\ {}^{c}\mathcal{D}^{\zeta;\delta}_{\kappa_1+}\xi(\vartheta) = \xi(\vartheta) - \sum_{\jmath=0}^{\beta-1} \frac{\xi^{[\jmath]}_{\delta}(\kappa_1)}{\jmath!}\left[\delta(\vartheta) - \delta(\kappa_1)\right]^{\jmath},\ \vartheta \in \Theta.$$

Lemma 2.6 **([12, 20]).** *Let* $\vartheta > \kappa_1$, $\zeta \geq 0$, *and* $\varkappa > 0$. *Then*

- $\mathcal{I}^{\zeta;\delta}_{\kappa_1+}(\delta(\vartheta) - \delta(\kappa_1))^{\varkappa-1} = \frac{\Gamma(\varkappa)}{\Gamma(\varkappa+\zeta)}(\delta(\vartheta) - \delta(\kappa_1))^{\varkappa+\zeta-1}$,
- ${}^{c}\mathcal{D}^{\zeta;\delta}_{\kappa_1+}(\delta(\vartheta) - \delta(\kappa_1))^{\varkappa-1} = \frac{\Gamma(\varkappa)}{\Gamma(\varkappa-\zeta)}(\delta(\vartheta) - \delta(\kappa_1))^{\varkappa-\zeta-1}$,
- ${}^{c}\mathcal{D}^{\zeta;\delta}_{\kappa_1+}(\delta(\vartheta) - \delta(\kappa_1))^{\jmath} = 0$, *for all* $\jmath \in \{0, \ldots, \beta - 1\}, \beta \in \mathbb{N}$.

Now, we give the definitions of lower and upper solutions of problem (1.1).

Definition 2.7. A function $\xi_0 \in C(\Theta, \mathbb{R})$ is called a lower solution of problem (1.1), if it satisfies

$$\begin{cases} \left({}^{c}\mathcal{D}^{\zeta+1;\delta}_{\kappa_1+} + \lambda\ {}^{c}\mathcal{D}^{\zeta;\delta}_{\kappa_1+}\right)\xi_0(\vartheta) \leq \Psi(\vartheta, \xi_0(\vartheta)), \quad \vartheta \in (\kappa_1, \kappa_2], \\ \Phi(\xi_0(\kappa_1), \xi_0(\kappa_2)) \leq 0, \quad \xi_0'(\kappa_1) = 0. \end{cases} \tag{2.4}$$

Definition 2.8. $\sigma_0 \in C(\Theta, \mathbb{R})$ is an upper solution of problem (1.1), if it satisfies

$$\begin{cases} \left({}^{c}\mathcal{D}^{\zeta+1;\delta}_{\kappa_1+} + \lambda\ {}^{c}\mathcal{D}^{\zeta;\delta}_{\kappa_1+}\right)\sigma_0(\vartheta) \geq \Psi(\vartheta, \sigma_0(\vartheta)), \quad \vartheta \in (\kappa_1, \kappa_2], \\ \Phi(\sigma_0(\kappa_1), \sigma_0(\kappa_2)) \geq 0, \quad \sigma_0'(\kappa_1) = 0. \end{cases} \tag{2.5}$$

Lemma 2.9. *For any* $h \in C(\Theta, \mathbb{R})$, *the unique solution of the following sequential fractional differential equation,*

$$\left({}^{c}\mathcal{D}^{\zeta+1;\delta}_{\kappa_1+} + \lambda\ {}^{c}\mathcal{D}^{\zeta;\delta}_{\kappa_1+}\right)\xi(\vartheta) = h(\vartheta), \quad \vartheta \in \Theta = [\kappa_1, \kappa_2], \tag{2.6}$$

supplemented with the initial conditions

$$\xi(\kappa_1) = \xi_{\kappa_1}, \quad \xi'(\kappa_1) = 0, \tag{2.7}$$

is given by

$$\xi(\vartheta) = \xi_{\kappa_1} + \int_{\kappa_1}^{\vartheta} \delta'(\varrho)e^{-\lambda(\delta(\vartheta)-\delta(\varrho))}\left(\int_{\kappa_1}^{\varrho} \frac{\delta'(r)(\delta(\varrho) - \delta(r))^{\zeta-1}}{\Gamma(\zeta)}h(r)dr\right)d\varrho. \tag{2.8}$$

Proof. Applying the δ–Riemann–Liouville fractional integral of order ζ to both sides of (2.6) and using Lemma 2.5, we get

$$\xi^{[1]}_{\delta}(\vartheta) + \lambda\xi(\vartheta) = \mathcal{I}^{\zeta;\delta}_{\kappa_1+}h(\vartheta) + c_0, \quad c_0 \in \mathbb{R}.$$

Using the notation of $\xi^{[1]}_{\delta}$ given by Eq (2.2) we obtain

$$\xi'(\vartheta) + \delta'(\vartheta)\lambda\xi(\vartheta) = \delta'(\vartheta)\left(\mathcal{I}^{\zeta;\delta}_{\kappa_1+}h(\vartheta) + c_0\right). \tag{2.9}$$

By multiplying $e^{\lambda(\delta(\vartheta)-\delta(\kappa_1))}$ to both sides of (2.9), we can write

$$\left(\xi(\vartheta)e^{\lambda(\delta(\vartheta)-\delta(\kappa_1))}\right)' = \delta'(\vartheta)e^{\lambda(\delta(\vartheta)-\delta(\kappa_1))}\mathcal{I}^{\zeta;\delta}_{\kappa_1+}h(\vartheta) + c_0\delta'(\vartheta)e^{\lambda(\delta(\vartheta)-\delta(\kappa_1))}.$$

Integrating from κ_1 to ϑ, we have

$$\xi(\vartheta) = c_1 e^{-\lambda(\delta(\vartheta)-\delta(\kappa_1))} + \frac{c_0}{\lambda} + \int_{\kappa_1}^{\vartheta} \delta'(\varrho) e^{-\lambda(\delta(\vartheta)-\delta(\varrho))} \mathcal{I}_{\kappa_1+}^{\zeta;\delta} h(\varrho) d\varrho, \tag{2.10}$$

where c_1 is an arbitrary constant. Differentiating (2.10), we obtain

$$\begin{aligned} \xi'(\vartheta) = & -\lambda c_1 \delta'(\vartheta) e^{-\lambda(\delta(\vartheta)-\delta(\kappa_1))} + \delta'(\vartheta) \mathcal{I}_{\kappa_1+}^{\zeta;\delta} h(\vartheta) \\ & - \lambda \delta'(\vartheta) \int_{\kappa_1}^{\vartheta} \delta'(\varrho) e^{-\lambda(\delta(\vartheta)-\delta(\varrho))} \mathcal{I}_{\kappa_1+}^{\zeta;\delta} h(\varrho) d\varrho. \end{aligned} \tag{2.11}$$

Using the initial conditions given by equation (2.7) together with equations (2.10) and (2.11), we obtain

$$c_0 = \lambda \xi_{\kappa_1}, \quad c_1 = 0.$$

Substituting the value of c_0, c_1 in (2.10) we get (2.8). The converse of the lemma follows by direct computation. This completes the proof. □

Now consider the following linear fractional initial value problem.

Lemma 2.10. *Let* $0 < \zeta \leq 1$ *and* $p, q \in C(\Theta, \mathbb{R})$*. Then the following linear fractional initial value problem*

$$\begin{cases} \left({}^c\mathcal{D}_{\kappa_1+}^{\zeta+1;\delta} + \lambda \, {}^c\mathcal{D}_{\kappa_1+}^{\zeta;\delta} \right) \xi(\vartheta) - p(\vartheta)\xi(\vartheta) = q(\vartheta), \quad \vartheta \in \Theta := [\kappa_1, \kappa_2], \\ \xi(\kappa_1) = \xi_{\kappa_1}, \quad \xi'(\kappa_1) = 0, \end{cases} \tag{2.12}$$

has a unique solution $\xi \in C(\Theta, \mathbb{R})$*, provided that*

$$\|p\| < \frac{\lambda \Gamma(\zeta+1)}{(\delta(\kappa_2) - \delta(\kappa_1))^{\zeta}}. \tag{2.13}$$

Proof. It follows from Lemma 2.9 that problem (2.12) is equivalent to the following integral equation:

$$\begin{aligned} \xi(\vartheta) = \xi_{\kappa_1} + & \int_{\kappa_1}^{\vartheta} \delta'(\varrho) e^{-\lambda(\delta(\vartheta)-\delta(\varrho))} \\ & \times \left(\int_{\kappa_1}^{\varrho} \frac{\delta'(r)(\delta(\varrho)-\delta(r))^{\zeta-1}}{\Gamma(\zeta)} \big(p(r)\xi(r) + q(r)\big) dr \right) d\varrho. \end{aligned}$$

Define the operator $\aleph \colon C(\Theta, \mathbb{R}) \longrightarrow C(\Theta, \mathbb{R})$ as follows

$$\begin{aligned} \aleph x(\vartheta) = \xi_{\kappa_1} + & \int_{\kappa_1}^{\vartheta} \delta'(\varrho) e^{-\lambda(\delta(\vartheta)-\delta(\varrho))} \\ & \times \left(\int_{\kappa_1}^{\varrho} \frac{\delta'(r)(\delta(\varrho)-\delta(r))^{\zeta-1}}{\Gamma(\zeta)} \big(p(r)\xi(r) + q(r)\big) dr \right) d\varrho, \ \vartheta \in \Theta. \end{aligned}$$

Now, we have to show that the operator $\aleph$ has a unique fixed point. To do this, we will prove that $\aleph$ is a contraction map.

Let $\xi, \sigma \in C(\Theta, \mathbb{R})$ and $\vartheta \in [\kappa_1, \kappa_2]$. Then, we have

$$\begin{aligned}
|\aleph x(\vartheta) - \aleph y(\vartheta)| &\leq \int_{\kappa_1}^{\vartheta} \delta'(\varrho) e^{-\lambda(\delta(\vartheta)-\delta(\varrho))} \\
&\quad \times \left(\int_{\kappa_1}^{\varrho} \frac{\delta'(r)(\delta(\varrho)-\delta(r))^{\zeta-1}}{\Gamma(\zeta)} |p(r)||\xi(r)-\sigma(r)| dr \right) d\varrho \\
&\leq \frac{(\delta(\kappa_2)-\delta(\kappa_1))^{\zeta}}{\Gamma(\zeta+1)} \|p\| \|\xi - \sigma\| \int_{\kappa_1}^{\vartheta} \delta'(\varrho) e^{-\lambda(\delta(\vartheta)-\delta(\varrho))} d\varrho \\
&\leq \frac{(\delta(\kappa_2)-\delta(\kappa_1))^{\zeta}}{\lambda \Gamma(\zeta+1)} \|p\| \|\xi - \sigma\|.
\end{aligned}$$

By (2.13) it follows that the operator $\aleph$ is a contraction. Consequently, by Banach's fixed point theorem, the operator $\aleph$ has a unique fixed point. That is, problem (2.12) has a unique solution. This completes the proof. □

The following result will play a very important role in this paper.

Lemma 2.11 (Comparison result). *Assume that* $p \in C(\Theta, \mathbb{R}_+^*)$ *and satisfies* (2.13). *If* $\theta \in C(\Theta, \mathbb{R})$ *satisfies the following inequalities*

$$\begin{cases} \left({}^{c}\mathcal{D}_{\kappa_1^+}^{\zeta+1;\delta} + \lambda \, {}^{c}\mathcal{D}_{\kappa_1^+}^{\zeta;\delta} \right) \theta(\vartheta) \geq p(\vartheta)\theta(\vartheta), \quad \vartheta \in \Theta := [\kappa_1, \kappa_2], \\ \theta(\kappa_1) \geq 0, \quad \theta'(\kappa_1) = 0, \end{cases} \tag{2.14}$$

then $\theta(\vartheta) \geq 0$ *on* $[\kappa_1, \kappa_2]$.

Proof. Let

$$\left({}^{c}\mathcal{D}_{\kappa_1^+}^{\zeta+1;\delta} + \lambda \, {}^{c}\mathcal{D}_{\kappa_1^+}^{\zeta;\delta} \right) \theta(\vartheta) - p(\vartheta)\theta(\vartheta) = q(\vartheta),$$
$$\theta(\kappa_1) = \xi_{\kappa_1} \quad \text{and } \theta'(\kappa_1) = 0.$$

We know that

$$q(\vartheta) \geq 0, \ \xi_{\kappa_1} \geq 0.$$

Suppose that the inequality $\theta(\vartheta) \geq 0, \vartheta \in [\kappa_1, \kappa_2]$ is not true. It means that there exists at least a $\vartheta_0 \in [\kappa_1, \kappa_2]$ such that $\theta(\vartheta_0) < 0$. Without loss of generality, we assume $\theta(\vartheta_0) = \min_{\vartheta \in [\kappa_1, \kappa_2]} \theta(\vartheta)$. Then by Lemma 2.10 we have

$$\begin{aligned}
\theta(\vartheta) &= \xi_{\kappa_1} + \int_{\kappa_1}^{\vartheta} \delta'(\varrho) e^{-\lambda(\delta(\vartheta)-\delta(\varrho))} \\
&\quad \times \left(\int_{\kappa_1}^{\varrho} \frac{\delta'(r)(\delta(\varrho)-\delta(r))^{\zeta-1}}{\Gamma(\zeta)} \big(p(r)\theta(r) + q(r)\big) dr \right) d\varrho \\
&\geq \theta(\vartheta_0) \int_{\kappa_1}^{\vartheta} \delta'(\varrho) e^{-\lambda(\delta(\vartheta)-\delta(\varrho))} \\
&\quad \times \left(\int_{\kappa_1}^{\varrho} \frac{\delta'(r)(\delta(\varrho)-\delta(r))^{\zeta-1}}{\Gamma(\zeta)} p(r) dr \right) d\varrho.
\end{aligned}$$

For $\vartheta = \vartheta_0$, we can get

$$\theta(\vartheta_0) \geq \theta(\vartheta_0) \int_{\kappa_1}^{\vartheta_0} \delta'(\varrho) e^{-\lambda(\delta(\vartheta_0)-\delta(\varrho))} \left(\int_{\kappa_1}^{\varrho} \frac{\delta'(r)(\delta(\varrho)-\delta(r))^{\zeta-1}}{\Gamma(\zeta)} p(r) dr \right) d\varrho.$$

Therefore, keeping in mind that $\theta(\vartheta_0) < 0$, we have

$$1 \leq \int_{\kappa_1}^{\vartheta_0} \delta'(\varrho) e^{-\lambda(\delta(\vartheta_0)-\delta(\varrho))} \left(\int_{\kappa_1}^{\varrho} \frac{\delta'(r)(\delta(\varrho)-\delta(r))^{\zeta-1}}{\Gamma(\zeta)} p(r) dr \right) d\varrho.$$

Hence,

$$\|p\| \geq \frac{\lambda\Gamma(\zeta+1)}{(\delta(\kappa_2)-\delta(\kappa_1))^{\zeta}},$$

which is in contradiction to (2.13). Hence, $\theta(\vartheta) \geq 0$ for all $\vartheta \in [\kappa_1, \kappa_2]$. □

3. Main results

In this paper, we will apply the monotone iterative method to present a result on the existence of the solution of problem (1.1).

Theorem 3.1. *Let the function $\Psi \in C(\Theta \times \mathbb{R}, \mathbb{R})$. In addition assume that:*

(H_1) *There exist ξ_0, $\sigma_0 \in C(\Theta, \mathbb{R})$ such that ξ_0 and σ_0 are lower and upper solutions of problem (1.1), respectively, with $\xi_0(\vartheta) \leq \sigma_0(\vartheta), \vartheta \in \Theta$.*

(H_2) *There exists $p \in C(\Theta, \mathbb{R}^+)$ satisfies (2.13) such that*

$$\Psi(\vartheta, \varpi_2) - \Psi(\vartheta, \varpi_1) \geq p(\vartheta)(\varpi_2 - \varpi_1) \quad \textit{for} \quad \xi_0 \leq \varpi_1 \leq \varpi_2 \leq \sigma_0.$$

(H_3) *There exist $\Bbbk_1 > 0$ and $\Bbbk_2 \geq 0$, where for $\xi_0(\kappa_1) \leq u_1 \leq u_2 \leq \sigma_0(\kappa_1)$, $\xi_0(\kappa_2) \leq v_1 \leq v_2 \leq \sigma_0(\kappa_2)$,*

$$\Phi(u_2, v_2) - \Phi(u_1, v_1) \leq \Bbbk_1(u_2 - u_1) - \Bbbk_2(v_2 - v_1).$$

Consequently, there exist monotone iterative sequences $\{\xi_\beta\}$ and $\{\sigma_\beta\}$, which converge uniformly on Θ to the extremal solutions of (1.1) in $[\xi_0, \sigma_0]$, where

$$[\xi_0, \sigma_0] = \big\{\varpi \in C(\Theta, \mathbb{R}) : \xi_0(\vartheta) \leq \varpi(\vartheta) \leq \sigma_0(\vartheta), \quad \vartheta \in \Theta\big\}.$$

Proof. First, for any $\xi_0, \sigma_0 \in C(\Theta, \mathbb{R})$, consider:

$$\begin{cases} \big({}^{c}\mathcal{D}_{\kappa_1^+}^{\zeta+1;\delta} + \lambda\, {}^{c}\mathcal{D}_{\kappa_1^+}^{\zeta;\delta} \big)\xi_{\beta+1}(\vartheta) = \Psi\big(\vartheta, \xi_n(\vartheta)\big) + p(\vartheta)\big(\xi_{\beta+1}(\vartheta) - \xi_\beta(\vartheta)\big), \\ \xi_{\beta+1}(\kappa_1) = \xi_\beta(\kappa_1) - \frac{1}{\Bbbk_1}\Phi\big(\xi_\beta(\kappa_1), \xi_\beta(\kappa_2)\big), \quad \xi'_{\beta+1}(\kappa_1) = 0, \end{cases} \tag{3.1}$$

and

$$\begin{cases} \big({}^{c}\mathcal{D}_{\kappa_1^+}^{\zeta+1;\delta} + \lambda\, {}^{c}\mathcal{D}_{\kappa_1^+}^{\zeta;\delta} \big)\sigma_{\beta+1}(\vartheta) = \Psi\big(\vartheta, \sigma_n(\vartheta)\big) + p(\vartheta)\big(\sigma_{\beta+1}(\vartheta) - \sigma_n(\vartheta)\big), \\ \sigma_{\beta+1}(\kappa_1) = \sigma_\beta(\kappa_1) - \frac{1}{\Bbbk_1}\Phi\big(\sigma_\beta(\kappa_1), \sigma_\beta(\kappa_2)\big), \quad \sigma'_{\beta+1}(\kappa_1) = 0. \end{cases} \tag{3.2}$$

By Lemma 2.10, we know that (3.1) and (3.2) have a unique solutions in $C(\Theta, \mathbb{R})$. We will divide the proof in the following steps.

Step 1: We prove that $\xi_\beta, \sigma_\beta (\beta \geq 1)$ are lower and upper solutions of problem (1.1), respectively and

$$\xi_0(\vartheta) \leq \xi_1(\vartheta) \leq \cdots \leq \xi_\beta(\vartheta) \leq \cdots \leq \sigma_\beta(\vartheta) \leq \cdots \leq \sigma_1(\vartheta) \leq \sigma_0(\vartheta), \quad \vartheta \in \Theta. \tag{3.3}$$

First, we prove that

$$\xi_0(\vartheta) \leq \xi_1(\vartheta) \leq \sigma_1(\vartheta) \leq \sigma_0(\vartheta), \quad \vartheta \in \Theta. \tag{3.4}$$

Set $\theta(\vartheta) = \xi_1(\vartheta) - \xi_0(\vartheta)$. From (3.1) and Definition 2.7, we obtain

$$\begin{aligned}\left({}^{c}\mathcal{D}^{\zeta+1;\delta}_{\kappa_1^+} + \lambda\, {}^{c}\mathcal{D}^{\zeta;\delta}_{\kappa_1^+}\right)\theta(\vartheta) &= \Psi\big(\vartheta, \xi_0(\vartheta)\big) - \left({}^{c}\mathcal{D}^{\zeta+1;\delta}_{\kappa_1^+} + \lambda\, {}^{c}\mathcal{D}^{\zeta;\delta}_{\kappa_1^+}\right)\xi_0(\vartheta)\\ &\quad + p(\vartheta)\theta(\vartheta)\\ &\geq p(\vartheta)\theta(\vartheta).\end{aligned}$$

Again, since $\theta'(\kappa_1) = 0$ and

$$\theta(\kappa_1) = -\frac{1}{\Bbbk_1}\Phi\big(\xi_0(\kappa_1), \xi_0(\kappa_2)\big) \geq 0.$$

By Lemma 2.11, $\theta(\vartheta) \geq 0$, for $\vartheta \in \Theta$. That is, $\xi_0(\vartheta) \leq \xi_1(\vartheta)$. Also, we have

$$\sigma_1(\vartheta) \leq \sigma_0(\vartheta),\ \ \vartheta \in \Theta.$$

Now, let $\theta(\vartheta) = \sigma_1(\vartheta) - \xi_1(\vartheta)$. From (3.1), (3.2) and $(H2)$, we get

$$\begin{aligned}\left({}^{c}\mathcal{D}^{\zeta+1;\delta}_{\kappa_1^+} + \lambda\, {}^{c}\mathcal{D}^{\zeta;\delta}_{\kappa_1^+}\right)\theta(\vartheta) &= \Psi\big(\vartheta, \sigma_0(\vartheta)\big) - \Psi\big(\vartheta, \xi_0(\vartheta)\big)\\ &\quad + p(\vartheta)\big(\sigma_1(\vartheta) - \sigma_0(\vartheta)\big) - p(\vartheta)\big(\xi_1(\vartheta) - \xi_0(\vartheta)\big)\\ &\geq p(\vartheta)\big(\sigma_0(\vartheta) - \xi_0(\vartheta)\big) + p(\vartheta)\big(\sigma_1(\vartheta) - \sigma_0(\vartheta)\big)\\ &\quad - p(\vartheta)\big(\xi_1(\vartheta) - \xi_0(\vartheta)\big)\\ &= p(\vartheta)\theta(\vartheta).\end{aligned}$$

Since, $\theta'(\kappa_1) = 0$ and

$$\begin{aligned}\theta(\kappa_1) &= \big(\sigma_0(\kappa_1) - \xi_0(\kappa_1)\big) - \frac{1}{\Bbbk_1}\left(\Phi\big(\sigma_0(\kappa_1), \sigma_0(\kappa_2)\big) - \Phi\big(\xi_0(\kappa_1), \xi_0(\kappa_2)\big)\right)\\ &\geq \frac{\Bbbk_2}{\Bbbk_1}\big(\sigma_0(\kappa_2) - \xi_0(\kappa_2)\big) \geq 0.\end{aligned}$$

By Lemma 2.11, we get $\xi_1(\vartheta) \leq \sigma_1(\vartheta),\ \vartheta \in \Theta$.

Next, we prove that $\xi_1(\vartheta), \sigma_1(\vartheta)$ are lower and upper solutions of (1.1), respectively. Since ξ_0 and σ_0 are lower and upper solutions of (1.1), by (H_2), it follows that

$$\left({}^{c}\mathcal{D}^{\zeta+1;\delta}_{\kappa_1^+} + \lambda\, {}^{c}\mathcal{D}^{\zeta;\delta}_{\kappa_1^+}\right)\xi_1(\vartheta) = \Psi\big(\vartheta, \xi_0(\vartheta)\big) + p(\vartheta)\big(\xi_1(\vartheta) - \xi_0(\vartheta)\big) \leq \Psi\big(\vartheta, \xi_1(\vartheta)\big),$$

also $\xi_1'(\kappa_1) = 0$ and

$$\begin{aligned}0 &= \Bbbk_1\big(\xi_1(\kappa_1) - \xi_0(\kappa_1)\big) + \Phi\big(\xi_0(\kappa_1), \xi_0(\kappa_2)\big)\\ &\geq \Phi\big(\xi_1(\kappa_1), \xi_1(\kappa_2)\big) + \Bbbk_2\big(\xi_1(\kappa_2) - \xi_0(\kappa_2)\big).\end{aligned}$$

Thus,

$$\Phi(\xi_1(\kappa_1), \xi_1(\kappa_2)) \leq 0.$$

Therefore, $\xi_1(\vartheta)$ is a lower solution of (1.1). Also, we get that $\sigma_1(\vartheta)$ is an upper solution of (1.1).

By induction, we demonstrate that $\xi_\beta(\vartheta), \sigma_\beta(\vartheta), (\beta \geq 1)$ are lower and upper solutions of problem (1.1), respectively and the following relation holds

$$\xi_0(\vartheta) \leq \xi_1(\vartheta) \leq \cdots \leq \xi_\beta(\vartheta) \leq \cdots \leq \sigma_\beta(\vartheta) \leq \cdots \leq \sigma_1(\vartheta) \leq \sigma_0(\vartheta), \quad \vartheta \in \Theta.$$

Step 2: The sequences $\{\xi_\beta(\vartheta)\}$, $\{\sigma_\beta(\vartheta)\}$ uniformly converge to their limit functions $\xi^*(\vartheta), \sigma^*(\vartheta)$.
Note that $\{\xi_\beta(\vartheta)\}$ is monotone nondecreasing and is bounded from above by $\sigma_0(\vartheta)$. Also, since the sequence $\{\sigma_\beta(\vartheta)\}$ is monotone nonincreasing and is bounded from below by $\xi_0(\vartheta)$, thus the pointwise limits ξ^* and σ^* exist. And, since $\{\xi_\beta(\vartheta)\}$, $\{\sigma_\beta(\vartheta)\}$ are sequences of continuous functions defined on $[\kappa_1, \kappa_2]$, hence by Dini's theorem [30], the convergence is uniform. This is

$$\lim_{\beta\to\infty} \xi_\beta(\vartheta) = \xi^*(\vartheta) \quad \text{and} \quad \lim_{\beta\to\infty} \sigma_\beta(\vartheta) = \sigma^*(\vartheta),$$

uniformly on $\vartheta \in \Theta$ and the limit functions ξ^*, σ^* satisfy problem (1.1). Furthermore, ξ^* and σ^* satisfy the relation

$$\xi_0 \le \xi_1 \le \cdots \le \xi_\beta \le \xi^* \le \sigma^* \le \cdots \le \sigma_\beta \le \cdots \le \sigma_1 \le \sigma_0.$$

Step 3: ξ^* and σ^* are extremal solutions of problem (1.1) in $[\xi_0, \sigma_0]$.
Let $\varpi \in [\xi_0, \sigma_0]$ be any solution of (1.1). We assume that the following relation holds for some $\beta \in \mathbb{N}$:

$$\xi_\beta(\vartheta) \le \varpi(\vartheta) \le \sigma_\beta(\vartheta), \quad \vartheta \in \Theta. \tag{3.5}$$

Let $\theta(\vartheta) = \varpi(\vartheta) - \xi_{\beta+1}(\vartheta)$. We have

$$\begin{aligned}\big({}^c\mathcal{D}^{\zeta+1;\delta}_{\kappa_1^+} + \lambda\, {}^c\mathcal{D}^{\zeta;\delta}_{\kappa_1^+}\big)\theta(\vartheta) &= \Psi\big(\vartheta, \varpi(\vartheta)\big) - \Psi\big(\vartheta, \xi_\beta(\vartheta)\big) \\ &\quad - p(\vartheta)\big(\xi_{\beta+1}(\vartheta) - \xi_\beta(\vartheta)\big) \\ &\ge p(\vartheta)\big(\varpi(\vartheta) - \xi_\beta(\vartheta)\big) - p(\vartheta)\big(\xi_{\beta+1}(\vartheta) - \xi_\beta(\vartheta)\big) \\ &= p(\vartheta)\theta(\vartheta).\end{aligned} \tag{3.6}$$

Furthermore, $\theta'(\kappa_1) = 0$ and

$$\begin{aligned} 0 &= \Phi\big(\varpi(\kappa_1), \varpi(\kappa_2)\big) - \Phi\big(\xi_\beta(\kappa_1), \xi_\beta(\kappa_2)\big) + \mathbb{k}_1\big(\xi_{\beta+1}(\kappa_1) - \xi_\beta(\kappa_1)\big) \\ &\ge \mathbb{k}_1\big(\varpi(\kappa_1) - \xi_\beta(\kappa_1)\big) - \mathbb{k}_2\big(\varpi(\kappa_2) - \xi_\beta(\kappa_2)\big) + \mathbb{k}_1\big(\xi_{\beta+1}(\kappa_1) - \xi_\beta(\kappa_1)\big) \\ &= \mathbb{k}_1\theta(\kappa_1) - \mathbb{k}_2\big(\varpi(\kappa_2) - \xi_\beta(\kappa_2)\big).\end{aligned}$$

That is,

$$\theta(\kappa_1) \ge \frac{\mathbb{k}_2}{\mathbb{k}_1}\big(\varpi(\kappa_2) - \xi_\beta(\kappa_2)\big) \ge 0.$$

By Lemma 2.11, we obtain $\theta(\vartheta) \ge 0,\ \vartheta \in \Theta$, which means

$$\xi_{\beta+1}(\vartheta) \le \varpi(\vartheta),\ \vartheta \in \Theta.$$

Using the same method, we can show that

$$\varpi(\vartheta) \le \sigma_{\beta+1}(\vartheta),\ \vartheta \in \Theta.$$

Hence, we have

$$\xi_{\beta+1}(\vartheta) \le \varpi(\vartheta) \le \sigma_{\beta+1}(\vartheta),\ \vartheta \in \Theta.$$

Therefore, (3.5) holds on Θ for all $\beta \in \mathbb{N}$. Taking the limit as $\beta \to \infty$ on (3.5), we obtain

$$\xi^* \le \varpi \le \sigma^*.$$

Consequently, ξ^* and σ^* are the extremal solutions of (1.1) in $[\xi^*, \sigma^*]$. □

Example 3.2. Consider the following boundary value problem:

$$\begin{cases} \left({}^{c}\mathcal{D}^{\frac{3}{2}}_{\kappa_1^+} + \frac{2}{\sqrt{\pi}}\, {}^{c}\mathcal{D}^{\frac{1}{2}}_{\kappa_1^+} \right)\xi(\vartheta) = \sin(\vartheta)(\xi - 1) + e^{-\vartheta}, \quad \vartheta \in \Theta := [0,1], \\ \xi(0) = 1, \quad \xi'(0) = 0. \end{cases} \tag{3.7}$$

Note that, this problem is a particular case of BVP (1.1), where

$$\zeta = \frac{1}{2}, \quad \lambda = \frac{2}{\sqrt{\pi}}, \quad \delta(\vartheta) = \vartheta,$$
$$\Psi(\vartheta, \xi) = \sin(\vartheta)(\xi - 1) + e^{-\vartheta}, \quad \Phi(\xi, \sigma) = \xi - 1.$$

Obviously, $\Psi \in C([0,1] \times \mathbb{R}, \mathbb{R})$, $\Phi \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$. On the other hand, taking $\xi_0(\vartheta) = 1$ and $\sigma_0(\vartheta) = 1 + \vartheta\sqrt{\vartheta}$, it is not difficult to verify that ξ_0, σ_0 are lower and upper solutions of (3.7), respectively, and $\xi_0 \leq \sigma_0$. So condition (H_1) holds.

Moreover, for $\xi_0 \leq \xi \leq \sigma \leq \sigma_0$ we have

$$\Psi(\vartheta, \sigma) - \Psi(\vartheta, \xi) \geq \sin\vartheta(\sigma - \xi). \tag{3.8}$$

And if $\xi_0(\kappa_1) \leq u_1 \leq u_2 \leq \sigma_0(\kappa_1)$, $\xi_0(\kappa_2) \leq v_1 \leq v_2 \leq \sigma_0(\kappa_2)$, we have

$$\Phi(u_2, v_2) - \Phi(u_1, v_1) \leq (u_2 - u_1). \tag{3.9}$$

In view of (3.8) and (3.9), we can choose $p(\vartheta) = \sin\vartheta, \Bbbk_1 = 1$ and $\Bbbk_2 = 0$ in Theorem 3.1. At last, by a simple computation, we have

$$\frac{(\delta(\kappa_2) - \delta(\kappa_1))^{\zeta}}{\lambda\Gamma(\zeta + 1)}\|p\| < 1.$$

Hence, all conditions of Theorem 3.1 are satisfied and consequently the problem (3.7) has extremal solutions on $[\xi_0, \sigma_0]$.

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